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Thesis:

Dynamics of fields and particles in a 5-dimensional scenario: problems and perspectives of the Kaluza-Klein theory

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Introduction

This work is devoted to a deep analysis of the features of the fields and particles dynamics in the 5D scenario provided by a Kaluza-Klein model ([18]). With this respect, the framework of our analysis concerns the 5D Kaluza-Klein model in its first historical formulation, where the cylindricity and compactification hypotheses are assumed. Such a model was formulated at first in 1921 by Kaluza ([1]), and thereafter improved by Klein ([2]), and, among others, by Jordan ([9]) and Thirry ([10]). It provides a coupling between gravity and electromagnetism, in a geometrical picture. However, despite it looks like a brilliant unification picture, and although it yields some striking outcomes, like, for instance, an elegant explanation for the discretization of the charge, the compactified model shows some unsatisfactory features, especially as far as the matter dynamics is concerned ([95]), which make the model not consistent with respect to the observed phenomenology. Anyway, the idea that it would be possible to achieve the unification of interactions via a geometrical picture, which is outlined in the Kaluza-Klein model, has always been fascinating; extension to more than five dimension have been developed ([13]); String theory and Supergravity ([19], [20]), indeed, move from the original Kaluza idea, and, at today, there is a wide variety of Kaluza-Klein-like approaches, like brane models ([44]) or induced-matter theory ([32]), where the original hypotheses are in some way relaxed or manipulated, in order to overcome the difficulties of the original model. Our aim is to enter in the debate about the meaning of the extra-dimension by re-analyzing, with a new point of view ([139], [140])- which we will develop in this work- the problems of the compactified approach. This will provide us new tools to check the physical value of the model, either as a viable step in the search of a unification scheme, either as a theory able to deal with effects beyond the General Relativity, like, for instance, the problem of the frozen formalism ([71], [73]) of the Hamiltonian or the departures with respect to the Friedmann dynamics. The analysis which we are going to present stands in two main research paths: the hamiltonian reformulation of the dynamics ([136]) and a revised approach to the motion ([139]), which moves from a critical analysis of the outcome of the geodesic approach, which is the scheme usually applied to face the test-particles motion. Indeed, these two issues are closely inherent to the mainly debated problems of the Kaluza-Klein model, which are the presence of an additional scalar degree of freedom among the metric fields, and the problem of the proper definition of coupling between matter and fields. The physical meaning of the scalar field is indeed a not well established issue; although its presence is appealing in order to look for effects beyond GR, it has been noted how its presence could lead to violation of the Free Falling Universality of the particles ([96]); in principle, it is possible to set such a field as a constant, but this assumption must be taken as a starting condition in the Action governing the dynamics, in order to discard the associated scalar equation : otherwise, assuming this condition after the variational procedure leads to some unphysical restrictions on the allowable solutions. Hence, to ensure consistency, it is necessary to add a further restriction on the model, whose physical meaning becomes therefore weakened. On the other side, the problem related to matter concerns the correct identification of the charge-mass ratio for a test particle: such a ratio arises with an unphysical upper bound, which requires, to be satisfied, a value for the mass beyond the Planck scale: such a picture, when we examine the related quantum dynamics, leads to the well known problem of the tower of huge massive modes. Actually, a third issue concerns the role of the cylindricity hypothesis; in the past it has been criticized and it has been suggested that if we want to seriously consider the extra dimension we need to take into account the dependence on the fifth coordinate ([32]). In this model, however, we work with a compactified extra dimension: therefore we do not assume the cylindricity as a real condition but rather just as low order approximation. We will focus on the scalar field and on the matter dynamics. Anyway, we will see that some outcomes of this work will provide insights on the meaning of the cylindricity and that there is a close interplay between the unobservability of the extra dimension and the features of the revised approach to matter, which we are going to discuss.

In the first part of the work we will face the problem of the hamiltonian reformulation of the dynamics. This will require to develop the space-time slicing of the model through the ADM reformulation ([62]). We will check the compatibility of the ADM approach to the Kaluza-Klein model ([136]): the problems related to this approach will be outlined and therefore solved providing the commutation of the ADM slicing procedure with respect to the Kaluza-Klein reduction procedure. At the end, we will be able to write the Hamiltonian of the Kaluza-Klein model. Such an analysis will provide insight about the generation of the U(1) gauge symmetry as a particular case of the diffeomorphism invariance. Furthermore, it will provide us a tool to start an analysis concerning the role of the scalar field as a suitable time variable, in the relational point of view ([77]). We will see how, from a quantum point of view, such a field deparametrizes from the hamiltonian constraint, via a Brown-Kuchar approach ([78]), and how, from a classical point of view, its behaviour its strictly related to the behaviour of the spatial scale factor. In the second part, we will face the problem of the matter dynamics; we will review the longstanding problem of the charge-mass ratio for test particle and we will enforce such analysis by using the hamiltonian scheme given in the first part. Such an effort will end with a well grounded criticism versus the geodesic approach: we will point out that the geodesic approach, as far as the KK model is concerned, is not able to face the proper definition of a particle and therefore it provides misdealing results. We will propose a revised approach to face the definition of a point-like particle, based on the generalization of the Papapetrou multipole expansion ([107]). The most striking feature of such an approach is that the particle turns out to be delocalized within the extra dimension, in agreement to the demand of unobservability. Within this revised scheme, we will recover a new equation for the particle motion which is not affected by bounds on its coupling factor; at the same time, the revised Klein-Gordon dynamics associated to this behaviour does not present the spectrum of huge massive modes. An interplay between the properties of the scalar field and the outcomes of this revised approach to matter will be recognized. We will see that the presence of matter will allow to consider the hypothesis that the scalar filed is a constant avoiding inconsistencies in the fields equation; at the same time, allowing consistently the scalar filed in the dynamics will provide some novelty with respect to the GR; for instance, we will deal with a variable rest mass, whose behaviour seems appealing in order to deal with the dark matter problem ([116]). Indeed, at the end of the work we will able to perform a complete and simultaneously reduction of matter and geometry, taking into account a 5D matter tensor, providing a model without inconsistencies related to the scalar field or matter terms. It will appear as a modified Einstein-Maxwell system, where two additional scalar degrees of freedom are taken into account. The first is the usual scalar field of Kaluza-Klein, while the second is a novelty, due to this approach, and it is related to the scalar degree of freedom of the reduced tensor. While setting as a constant the Kaluza-Klein scalar field reproduces exactly the Einstein-Maxwell dynamics - and therefore enforces the characterization of the theory as a viable step in view of an unification theory -, the general model offers suitable and promising scenarios to be compared with dark matter models ([111]).

The work is organized as follows. In the first chapter we give a review of the Kaluza-

Klein model in vacuum, outlining some observations of interest for the following analysis. In chapter (2) we discuss the hamiltonian formulation of the dynamics; section (2.1) is devoted to a review of the ADM splitting procedure which will provide us the tools for the hamiltonian formulation. In section (2.2) we face the ADM splitting of the KK model, while the analysis of the hamiltonian dynamics takes place in section (2.3). The most important results concerning the scalar field are in (2.3.3) and (2.4).

In chapter (3) we consider the geodesic approach to motion. We start with the standard review of the procedure (3.1), adding an original contribute in (3.1.2). Some emphasis is devoted to the analysis of the charge-mass puzzle in (3.1.1). In section (3.2) and (3.3) we present respectively our analysis based on the hamiltonian reformulation and the Klein-Gordon approach.

Chapter (4) should be considered the core of the work. After a brief review of the Papapetrou approach in (4.1.1) we apply this procedure to the KK model; in (4.2) we consider the classical dynamics and find a new equation for the motion; in (4.3) we consider the related Klein-Gordon approach and we find that the tower of huge massive modes is removed.

Finally, in chapter (5) we perform the complete reduction of matter and geometry. The procedure is outlined in (5.1) while in (5.2) some promising scenarios are discussed.

Concluding remarks and perspectives follow in the end.

List of publications

Peer reviewed journals:

- V. Lacquaniti, G. Montani, *On the ADM decomposition of the 5-D Kaluza-Klein model*, *Int. J. Mod. Phys.D*, **1**, N. 4, 2006, 559-5815, [gr-qc/0601101].
- V. Lacquaniti, G. Montani, *Hamiltonian Formulation of 5D Kaluza-Klein Theory*, in Proceedings of the I Stueckelberg Workshop, Pescara 2006, *Il Nuovo Cimento B*, Vol 122 , [gr-qc/0702007] 2007 .
- V. Lacquaniti, G.Montani, *Matter Coupling in 5D KK theory* , in Proceedings of the Second Stueckelberg Workshop, Pescara, 2007, *Int. J. Mod .Phys. A*, **23**, 1105-1293, (2008) [gr-qc/0803.2807v1].
- V. Lacquaniti, G. Montani, *Dynamics of Matter in a Compactified 5D Kaluza-Klein Model* , in press, *Int. J. Mod .Phys. D*, 2008.
- V. Lacquaniti, G. Montani, *Geometry and Matter Reduction in a 5D Kaluza-Klein Framework*, submitted to *Int. J. Mod .Phys. D*, 2008.
- V. Lacquaniti, G. Montani, F. Vietri, *Dimensional Reduction of the 5D Kaluza-Klein Geodesic Deviation Equation*, submitted to *Gen. Rel. Grav.*, 2008.
- V. Lacquaniti, G. Montani, S. Zonetti, *Deparameterization of the 5D Kaluza-Klein scalar field in the Kuchar-Brown framework*, submitted to *Europhys. Lett.*, 2008.

Proceedings:

- V. Lacquaniti and G. Montani, *Hamiltonian formulation of the 5D Kaluza-Klein model and test-particles motion* in Proceeding of XI Marcel Grossman Meeting [gr-qc/0612118] .

- F.Cianfrani, V.Lacquaniti and G. Montani, *Particles and fields within a unification scheme*, in Proceedings of the X Italian-Korean Symposium on Relativistic Astrophysics, Pescara (Italy) , June 25-30 2007 [gr-qc/0805-3963].

- V.Lacquaniti, G.Montani , *Recent Developement in Particle and Field Motion within the Kaluza-Klein Picture* , in Proceedings of the III Stueckelberg Workshop, Pescara, 2008.

- V. Lacquaniti, G. Montani, F. Vietri, *Geodesic Deviation on a Kaluza-Klein Background*, in Proceedings of the III Stueckelberg Workshop, Pescara, 2008.

Notations

In this work we adopt the following notations.

Indexes related to the full 5D manifold:

$$A = 0, 1, 2, 3, 5 = \mu, 5$$

Indexes related to the 4D submanifold given by the product of the usual 3D space-like submanifold plus the extra-dimension:

$$\hat{I}, \hat{J} = 1, 2, 3, 5 = i, 5$$

Indexes related to the usual 4D space-time:

$$\mu = 0, 1, 2, 3$$

In this manifold we define $g = -det\, g_{\mu\nu}$.

Indexes related to the usual 3D space-like submanifold:

$$i = 1, 2, 3$$

We work with units such that $c = 1$.

Chapter 1

Kaluza-Klein model in vacuum

A common approach in the quest for a unification theory of interactions is to adopt a multidimensional background. The idea is that the extra degrees of freedom provided by extra-dimensions could be linked to Yang-Mills fields. What makes a differences, among various approaches, is the kind of the restrictive hypotheses we have to assume in order to break the multidimensional gravity into the ordinary gravity plus its coupling with Yang-Mills fields. The choice we make, with respect to such hypotheses, will affect the physical meaning we address to extra degrees of freedom, the influence of the extra dimensions on our 4D phenomenology, the definitions of the dynamical couplings of the model. The first formulation of the 5D model is due to Kaluza ([1]); in the following Klein ([2], [3]) and Mandel ([4]) separately obtained the same theory, in some way considered also by Einstein ([5]). It was a 5D model which unified gravity and electromagnetism; it can be regarded as a low-energy effective theory, in which the 5D Poincaré symmetry is broken, as well as the 5D Equivalence Principle (PE), and U(1) gauge symmetry appears. In such a scheme the main hypothesis is that the extra dimension is compactified at a very low length scale, such that it results to be unobservable. It has to be reminded, anyway, that the first attempt to build a geometrical picture of unification was due to Nordstrom ([6]), whose model was not properly correct, however, because he considered just a scalar gravity. Moving from the original KK theory, many authors recovered the Klein-Gordon equation (Klein [2], Schroedinger [7], Fock [8]) and this increased the interest of physicists in the idea of multidimensional backgrounds (Jordan [9], Thirry [10]). The 5D model can be used to achieve a formulation of QED ([11], [12]), as well as a further extension to more than five dimension allows to consider a geometrical picture of Yang-Mills fields ([13]-[19]). The assumption of a multidimensional background was indeed a leading tool in the formulation of String and Supergravity theories ([20], [21], [19]). At the same time, the idea that the extra dimensions, although unobservable, could have a real physical meaning stimulated the search of a suitable mechanism for a compactification scenario ([22]-[31]). At today, starting from the Kaluza idea, there is a variety of Kaluza-Klein like models, where the original hypotheses concerning the compactification are in some way manipulated or relaxed ([32]). The projective approach ([33], [34], [35]) gave up the idea that the extra dimensions have a real physical meaning and just assume them as a mathematical tool; other approaches gave up with the compactification idea, and deal with large or non compactified dimension; such approach eventually led to induced matter theory ([36]- [43]) and brane models ([44], [45]).

In this work we will concern about the compactified 5D Kaluza-Klein (KK) model; the leading assumption, as it is stressed above, is that the fifth dimension is compactified at a small scale, and it therefore results to be unobservable. The problem of dealing with a scenario for such a compactification goes over the aim of our work; we will just focus on problems and perspectives of this approach and try to get a physical evaluation of the model analyzing its outcomes. We will consider the model as a classical, low energy, effective theory . At the end of the dimensional reduction procedure, we could consider it as a modified theory of gravity, which, in some scenarios, restores General Relativity (GR). In this chapter we give an axiomatic review of the 5D KK model in vacuum.

1.1 5D Dimensional reduction procedure

The mathematical framework of GR can be easily generalized to spaces with dimensionality greater than four. At first, we define a 5D metric tensor J_{AB}, therefore we deal with a metrics with 15 degrees of freedom. Hence, being 10 degrees saved for gravity, we have five degrees allowable to describe others interactions: we will be able to link them to a scalar field plus a $U(1)$ gauge field. Given the 5D metrics J_{AB}, the 5D generalization of GR is easily provided as follows:

$$ds_5^2 = J_{AB}dx^A dx^B \tag{1.1}$$

$$S_5 = -\frac{1}{16\pi G_{(5)}} \int d^5x \sqrt{J}\ {}^5R \tag{1.2}$$

With obvious formalism we have defined the 5D line element, the 5D Newton constant, the 5D Einstein-Hilbert Action and the 5D scalar curvature. The dynamics of the model is driven by three fundamental hypotheses (KK hypotheses):

- (*Compactification hyp.*) We assume to deal with a 5D manifold that is the direct product $\mathcal{M}^4 \otimes S^1$, where $\mathcal{M}^4$ is the ordinary space-time and S^1 is a space-like loop. Assuming a space-like extra dimension we avoid problems related to the causality violation ([46]). In order to make the extra dimension unobservable, we assume its size to be below our observational bound ([47]): $L_5 = \int \sqrt{J_{55}}\, dx^5 \ < 10^{-18} cm.$

- (*Cylindricity hyp.*) All components of the metric tensor are not depending on the extra coordinate x^5, therefore we have an invariant Killing vector with components $(0, 0, 0, 0, 1)$. We can figure this statement as the zero-order approximation of the Fourier-expansion on the extra coordinate.

- (*Scalar hyp.*) The J_{55} component of the metrics is a scalar.

1.1.1 Reduction of the metrics

Now, in order to ensure that the second and the last hypotheses hold, we have to put a restriction on allowable diffeomorphism. The most general class of transformation (KK diffeomorphisms) we can consider is the following:

$$\begin{cases} x^5 = x^{5'} + ek\Psi(x^{\mu'}) \\ x^\mu = x^\mu(x^{\nu'}) \end{cases} \tag{1.3}$$

Here Ψ is an arbitrary scalar function and ek an appropriate dimensional factor to be fixed later. KK model is therefore characterized by its invariance with respect to the above class of transformations.

- As we see, KK transformations leave freedom in 4D transformations but admit only translations along the fifth dimension. In consequence of this, we emphasize that the 5D general covariance is broken, but the 4D one holds, and GR is still safe.

The breaking of the 5D covariance is the mechanism that allows us to take into account the appearance of the gauge field and the scalar field. The metric tensor J_{AB} admits the following representation:

$$\begin{cases} J_{55} = -\phi^2 \\ J_{5\mu} = -\phi^2(ek)A_\mu \\ J_{\mu\nu} = g_{\mu\nu} - \phi^2(ek)^2 A_\mu A_\nu \end{cases} \tag{1.4}$$

$$J_{AB} \Rightarrow \begin{pmatrix} g_{\mu\nu} - \phi^2(ek)^2 A_\mu A_\nu & -\phi^2(ek)A_\mu \\ -\phi^2(ek)A_\mu & -\phi^2 \end{pmatrix} \tag{1.5}$$

Actually, until now we have done nothing but a rearrangement of degrees of freedom: the physical relevance of new variables is provided when we identify $g_{\mu\nu}$ with gravity tensor, A_μ with a $U(1)$ gauge vector and ϕ with an extra scalar field. This identification is consistent because we can recognize that, with respect to KK diffeomorphism (1.3), ϕ

is a scalar, $g_{\mu\nu}$ is a 4D tensor, A_μ is a 4D vector containing an additional gauge term. Indeed we have:

$$\begin{cases} \phi' = \phi \\ g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x^{\mu'}} \frac{\partial x^\sigma}{\partial x^{\nu'}} \\ A'_\mu = A_\nu \frac{\partial x^\nu}{\partial x^{\mu'}} + \frac{\partial \Psi}{\partial x^{\mu'}} \end{cases} \tag{1.6}$$

As we can see, A_μ transforms like a vector in curved manifold with respect to ordinary diffeomorphisms, which does not involve the extra dimension, while it transforms like a gauge vector in flat space, with respect to a translation along the fifth dimension. Hence, such a translation produces a $U(1)$ gauge transformation for the vector field and, granted the general invariance of the model with respect to KK diffeomorphisms, we expect the dynamics to be invariant with respect to the gauge transformations of A_μ. The set of equations (1.5) therefore gives us the so-called KK reduction of the metrics. The following point should be now remarked:

- As well as the general 5D covariance is broken, the same happens to the five dimensional Equivalence Principle (PE). Indeed, let us consider the following reasoning: in standard GR , provided the PE, we identify inertial mass with gravitational one. This allows us to have the geometrical picture of gravity and it is implemented in the fact that we are always free to find a local Minkowskyan space ([48]), usually by a local, non linear, transformations of coordinates. But in KK model, we note that, due to the law of transformations of metrics components (that does not admit non-linear transformations as far as x^5 is concerned), there is no way to find a local transformations which could bring us to a Minkowsky 5D space (except for the case $A_\mu = 0$, $\phi = 1$, which is not interesting us). This means that the 5D PE is broken. The 4D PE is however still safe, because we can always find a 4D Mynkowskian space.

1.1.2 Reduction of the dynamics

Now, starting from the above reduction of metrics , we want to consider the reduction of the dynamical equations. In order to accomplish this task we need to use the 5-bein projection technique ([16], [11], [49], [50]). We omit various steps of the calculus and only provide formulas which will be useful in what follows. According to equations (1.5), we first evaluate the reduction of the line element and this yields:

$$ds_5^2 = ds^2 - \phi^2(ekA_\mu dx^\mu + dx^5)^2 \tag{1.7}$$

Then, we calculate 5-bein basis vectors $e_A^{(A)}$; given the 5-bein metrics

$$\eta_{(A)(B)} = diag(1,-1,-1,-1,-1)\,,$$

we have:

$$\begin{cases} e_A^{(5)} = \phi(ekA_\mu, 1) \\ \\ e_A^{(\mu)} = (u_\mu^{(\mu)}, 0) \end{cases} \begin{cases} e_{(5)}^A = (0,0,0,0,\frac{1}{\phi}) \\ \\ e_{(\mu)}^A = (u_{(\mu)}^\mu, -ekA_\mu u_{(\mu)}^\mu) \end{cases} \tag{1.8}$$

where 5-bein indexes are labelled by brackets and $u_{(\mu)}^\mu$ is the tetrad basis set for the 4D metrics $g_{\mu\nu}$. Now, we need to evaluate anolomies $\lambda_{(A)(B)(C)}$. Non-zero components are:

$$\begin{cases} \lambda_{(\mu)(\nu)(\lambda)} = {}^4\lambda_{(\mu)(\nu)(\lambda)} \\ \lambda_{(5)(\mu)(\nu)} = -ek\phi F_{\mu\nu} u_{(\mu)}^\mu u_{(\nu)}^\nu \\ \lambda_{(5)(5)(\mu)} = -\frac{1}{\phi}(\partial_\mu\phi) u_{(\mu)}^\mu \end{cases} \tag{1.9}$$

where ${}^4\lambda_{(\mu)(\nu)(\lambda)}$ is the anolonomy tensor associated to the metrics $g_{\mu\nu}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor. Finally, we can evaluate the 5D scalar curvature, via the formula

$$\begin{aligned} {}^5R &= -2\partial_{(C)}\lambda_{(B)}{}^{(B)(C)} - \frac{1}{4}\lambda_{(A)(B)(C)}\lambda^{(A)(B)(C)} - \frac{1}{2}\lambda_{(C)(B)(A)}\lambda^{(C)(B)(A)} \\ &\quad -\lambda^{(A)}{}_{(A)(C)}\lambda^{(B)}{}_{(B)}{}^{(C)} \end{aligned} \tag{1.10}$$

Therefore, the splitting of ${}^5R$ yields

$${}^5R = R - \frac{2}{\phi} g^{\mu\nu}\nabla_\mu\partial_\nu\phi + \frac{1}{4}(ek)^2\phi^2 F_{\mu\nu}F^{\mu\nu}\,, \tag{1.11}$$

where R is now the 4D curvature scalar related to the metrics $g_{\mu\nu}$. Finally, by using the identity $\sqrt{J} = \phi\sqrt{g}$, after KK splitting, the Action reads

$$S_5 = -\frac{1}{16\pi G_5{}'}\int d^4x\sqrt{g}\,(\phi R - 2\nabla_\mu\partial_\nu\phi + \frac{1}{4}(ek)^2\phi^3 F_{\mu\nu}F^{\mu\nu})\,, \tag{1.12}$$

where, by making use of the compactification and cylindricity hypotheses we have ruled out the integration over the extradimension and we define:

$$\frac{1}{G'_5} = \frac{1}{G_5}\int dx^5\,.$$

.

Analysis of the dynamics

Let us consider at first a scenario where we have a constant scalar field; without loosing generality we can consider $\phi = 1$. In such a scenario, dynamical terms depending on ϕ vanish and we are able to reproduce exactly the Action for the Einstein-Maxwel (EM) theory. We just need to define as follows the coupling terms:

$$G = G_5{}' \qquad \frac{4G}{c^4} = (ek)^2 \tag{1.13}$$

Thus, our previous identification of $g_{\mu\nu}$ and A_μ with gravitational field and U(1) gauge field (namely the electromagnetic field) respectively, is now justified. In the general case , with no hypothesis on ϕ, this identification still holds but we have to deal with an extra scalar degree of freedom and its couplings. The task of assigning a well-understood physical meaning to this field is not a trivial one. Actually, in the original KK model, it was postulated $\phi = 1$, but in modern researches, motivated by the presence of scalar fields in modified theories of gravity, its presence is allowed. As pointed out by some authors, from a quantum viewpoint, ϕ can be viewed as a massless boson of zero spin ([16], [11], [18], [51]) , while from a classical point of view we can recognize, for instance, the interplay of this model with Brans-Dicke theory. In principle, instead of requiring

$\phi = 1$, we could adsorb the effects of ϕ by assuming that it is, at least in this cosmological era, a slow varying, quasi-static, scalar field. This way, we neglect derivatives terms of ϕ and re-scale observed coupling terms as follows :

$$G(\phi) = \phi^{-1}{G_5}' \qquad \frac{4G(\phi)}{c^4} = (ek)^2\phi^2 \tag{1.14}$$

Hence, so doing we restore the Einstein-Maxwell theory as an effective one, in the contest of varying fundamental constant models ([52], [53], [54]); of course, we have to require a behaviour of ϕ that would fit with present observed values and their limits.

However, we don't enter now in this debate: actually, we allow the presence of ϕ in our model: we will refer to the definition (1.13) and therefore we consider the outcome of the dimensional reduction as a modified Einstein-Maxwell theory. We focus to some effects of ϕ in the dynamics; this will eventually offer other issues to the above mentioned debate.

Anyway, regardless the presence of ϕ , the point that should be remarked is the following: KK model reproduces an Einstein-Maxwell-like dynamics, as far as metric fields in vacuum are concerned. The notably feature is the identification of gauge symmetry with a particular case of diffeomorphism invariance , i.e. translations along the fifth dimensions. Therefore we can consider the unification of gravity and electromagnetism in a geometrical picture. The prize to pay, to achieve this scenario, is the breaking of the 5D covariance and 5D PE.

1.2 5D Fields equations

Now let us take a look on the fields equations of motion. In principle we have to choose between two allowable procedure.

- We could consider the 5D Einstein-Hilbert (EH) Action (1.2), its dimensional reduction into (1.12) and finally achieve the motion equations by performing the variational procedure with respect to the reduced variables $g_{\mu\nu}$,A_μ,ϕ.

- We could consider as the fundamental object to be generalized in 5D the Einstein tensor ; therefore we can postulate as viable motion equations the set ${}^5G_{AB} = 0$ and consider its dimensional reduction

The question arises because of the presence of the term $J_{AB}\delta({}^5G^{AB})$ in the variational procedure which we have to perform to achieve the equation ${}^5G_{AB} = 0$ from the unreduced Action (1.2). In the 4D theory, such a term is a surface one ([50]), giving the known Einstein equation from the EH Action. A key in the proof is the possibility to use a local flat space, what is actually forbidden in the 5D model. Therefore, in principle , the 5D equation ${}^5G_{AB} = 0$ could be not compatible with the Action (1.2). Actually, by making use of the cylindricity hypothesis and a long calculation, it is possible to probe such an equivalence; therefore both procedure leads to an equivalent dynamics. Anyway, such an issue should be considered as an advice of the care we need to employ when generalizing objects from 4 to 5 dimensions in the KK model: we will have to face similar tasks in this work, when we will have to consider the ADM formulation of the model or the choice of the right approach to motion to be generalized.

Hence, the dynamics of fields is given by 5D Einstein equations ${}^5G_{AB} = 0$; performing the dimensional splitting, after some rearrangements, we get the following set, previously discussed by Jordan-Thirry ([9], [10]):

$$\begin{aligned} &G^{\mu\nu} = \frac{1}{\phi}\nabla^\mu\partial^\nu\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\phi^2 T^{\mu\nu}_{electrom.} \\ &\nabla_\mu\left(\phi^3F^{\mu\nu}\right) = 0 \\ &\Box\phi = -\frac{1}{4}\phi^3(ek)^2F^{\mu\nu}F_{\mu\nu}\,. \end{aligned} \tag{1.15}$$

Here $G^{\mu\nu}$ is the usual 4D Einstein tensor, $T^{\mu\nu}_{electrom.}$ the energy-momentum tensor related to the electromagnetic field, and $\Box$ is the Laplace-Beltrami operator: $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$. The

scalar equation is obtained combining the ${}^5G_{55} = 0$ equation with the trace of the tensor equation.
Now let us consider the scenario $\phi = 1$. As observed in the previous section, we expect to recover the Einstein-Maxwell (EM) theory; but an additional problem now arises. Indeed, first two equations reduce to Einstein-Maxwell dynamics while the last turns into the inconsistent result $F_{\mu\nu}F^{\mu\nu} = 0$. Therefore the hypothesis $\phi = 1$ can be considered only if we request it from the beginning, as an additional request of our model. This way, we loose the scalar degree of freedom in the Action and the last equation of (1.15) does not appear. Hence, if we want to consider exactly the EM theory, we are forced to add an additional restriction on our model and we cannot consider, for instance, the existence of a natural behaviour which leads to an asymptotically flat scalar field. In principle the presence of a matter term in the Einstein equation could be modify such a scenario; we will turn later on this point.

1.3 Extension to non-abelian models

In this section we will discuss the extension of the Kaluza-Klein approach to multidimensional models; such an extension allows us, following the guidelines we have seen in the framework of the 5D model, to consider the geometrization of a generic, non-abelian, Yang-Mills gauge group ([32], [55]-[61]). Rather than giving an exhaustive review we will just stress the key points of such generalization, with some emphasis on analogies and differences with respect to the 5D approach.

1.3.1 Reduction of the metrics

As we have seen in the first section, in the $5D$ approach we add a space-like loop to the usual $4D$ manifold and assume the cylindricity hypothesis. The first step in order to generalize this setup is to define our manifold as follows:

$$M = M^4 \otimes B^K. \tag{1.16}$$

Here M^4 represents the usual space-time while B^K is a K-dimensional extra space. In the following we will denote as x^μ the internal coordinates of M^4 and as y^m $(m = 1, 2, ..., K)$ those belonging to the extra-space; the set of coordinates X^A $(A = \mu, m)$ will refer the the whole manifold M. The added space B^K is characterized by the following statements, which replace and generalize the assumptions we made in the 5D approach:

- B^K is a compact manifold and its characteristic length scale is smaller than the ones we can probe in current experiments (approximatively $10^{-18} cm$). Therefore we assume that B^K is not observable.

- B^K is homogeneous: it means that there exist some Killing vectors $\xi^m_{\bar{M}} = \xi^m_{\bar{M}}(y)$, where at least we require $\bar{M} = K$, which are characterized by the following relation:

$$\xi^n_{\bar{N}} \partial_n \xi^m_{\bar{M}} - \xi^n_{\bar{M}} \partial_n \xi^m_{\bar{N}} = C^{\bar{P}}_{\bar{N}\bar{M}} \xi^m_{\bar{P}} \, , \tag{1.17}$$

  where $C^{\bar{P}}_{\bar{N}\bar{M}} = -C^{\bar{P}}_{\bar{M}\bar{N}}$ are constants. It can be proved that the above relation implies that any point of the manifold can be mapped into any other point via an isometry.

The presence of the constants $C^{\bar{P}}_{\bar{N}\bar{M}}$ envisages how, at the end of the reduction procedure we will deal with a non-abelian algebra. An important remark should be noted at this stage: the presence of a non-vanishing $C^{\bar{P}}_{\bar{N}\bar{M}}$ could be achieved only if we allow a dependence of the Killing vectors on the extra coordinates y^m. Hence, we are giving up with the cylindricity hypothesis, in order to deal with a non-abelian structure.

The next step is to define the breaking of the invariance we need in order to define internal gauge symmetries; within the 5D approach we admitted just translations along the fifth dimension, which were indeed the only kind of isometries we were able to consider. Therefore we simply assume now that the model is characterized by invariance with respect to generic 4D diffeomorphism and isometries of the extra space, i.e. we have the following allowable class of transformations:

$$\begin{cases} x'^{\mu} = x'^{\mu}(x^{\nu}) \\ y'^{m} = y^{m} + \omega^{\bar{N}}(x^{\nu})\xi^{m}_{\bar{N}}(y^{n}) \end{cases} . \tag{1.18}$$

In the above formulas $\omega^{\bar{N}}(x^{\nu})$ are infinitesimal functions. It is worth noting that, with respect to such a class of transformations, the relation (1.17) is invariant. A difference, however, arises with respect to the setup of the 5D model. While the Killing vector of the extra space, in the 5D case, was an invariant vector, because of the cylindricity hypothesis, in the present approach the functional form of the Killing vector change as a consequence of the transformations concerning y^m as follows:

$$\xi'^{m}_{\bar{M}}(y') = \xi^{m}_{\bar{M}}(y') + \xi^{m}_{\bar{P}}(y')C^{\bar{P}}_{\bar{M}\bar{Q}}\omega^{\bar{Q}}$$

In a recent proposal ([15]) it has been argued that, in order to reconcile this situation with the request of unobservability we must assume that we are not able to detect any change in the Killing vector because, being the extra space not observable, we cannot probe transformations of quantities defined on it. Therefore, despite the above formulas, following this reasoning we assume that we are not able to distinguish between $\xi'^{m}_{\bar{M}}(y')$ and $\xi^{m}_{\bar{M}}(y')$.

Now, following our procedure we deal with the reparametrization of the metrics. The generalization of the 5D setup provide the following formulation:

$$j_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} - \phi^2\gamma_{mn}\xi^{m}_{\bar{M}}\xi^{n}_{\bar{N}}A^{\bar{M}}_{\mu}A^{\bar{N}}_{\nu} & -\phi^2\gamma_{mn}\xi^{m}_{\bar{M}}A^{\bar{M}}_{\mu} \\ \hline -\phi^2\gamma_{mn}\xi^{n}_{\bar{N}}A^{\bar{N}}_{\nu} & -\phi^2\gamma_{mn} \end{array}\right) \tag{1.19}$$

Here we have $g_{\mu\nu} = g_{\mu\nu}(x^{\rho})$ and $\phi = \phi(x^{\rho})$, while $\gamma_{mn} = \gamma_{mn}(y^{r})$ depends only on y^m. It is worth remarking that the dependance on coordinates in the extra dimensional sector of the metrics is factorized. In view of the geometrization, $g_{\mu\nu}$ and $\phi^2\gamma_{mn}$ are going to be interpreted as 4- and extra-dimensional metric tensors, respectively, while $A_\mu^{\bar{M}}$ will be addressed to the Yang-Mills fields. It can be proved that, with respect to our class of transformations, $g_{\mu\nu}$ behaves like a 4D tensor, ϕ like a scalar while, about the extra dimensional metrics, we have by construction $\gamma'_{mn}(y') = \gamma_{mn}(y')$. The most interesting case, however, which deserves a deep investigation, is the behaviour of the vector fields $A_\mu^{\bar{M}}$. Indeed, implementing the effect of a coordinates transformations on the components $J_{\mu m}$, we are able to write the following formula:

$$\xi'^{\,m}_{\bar{M}}(y')A'^{\bar{M}}_{\mu} = \xi'^{\,m}_{\bar{M}}(y')(A^{\bar{M}}_{\mu} - \partial_\mu \omega^{\bar{M}}) \tag{1.20}$$

This suggest to assume a vector field transforming like an abelian boson field, i.e. :

$$A'^{\bar{M}}_{\mu} = A^{\bar{M}}_{\mu} - \partial_\mu \omega^{\bar{M}}$$

In some sense this result shows how the definition of a non-abelian model is an issue more subtle than the setup of the abelian one. It is possible to recover the transformation law for a non-abelian field, addressing the point of view discussed in ([15]) where the unobservability of the extra-space is properly taken into account; indeed, assuming that we are not able to detect any change in the Killing vector, it has been suggested that the physically relevant equation to deal with, for the analysis of the vector field , is the following:

$$\xi'^{\,m}_{\bar{M}}(y')A'^{\bar{M}}_{\mu} = \xi^m_{\bar{M}}(y')(A^{\bar{M}}_{\mu} + C^{\bar{M}}_{\bar{P}\bar{Q}}A^{\bar{P}}_{\mu}\omega^{\bar{Q}} - \partial_\mu \omega^{\bar{M}}). \tag{1.21}$$

The above equation is equivalent with the previous one, from a mathematical point of view, but allows us to get the desired transformation law; performing an average on the space B^K in order to rule out the dependence on the extra coordinate, i.e.

$$\frac{1}{V^K}\int_{B^K} d^K y' \sqrt{\gamma}\xi'^{\,m}_{\bar{M}}(y')A'^{\bar{M}}_{\mu} = \frac{1}{V^K}\int_{B^K} d^K y' \sqrt{\gamma}\xi^m_{\bar{M}}(y')(A^{\bar{M}}_{\mu} + C^{\bar{M}}_{\bar{P}\bar{Q}}A^{\bar{P}}_{\mu}\omega^{\bar{Q}} - \partial_\mu \omega^{\bar{M}})$$

we finally get the right transformation properties for non-abelian gauge bosons:

$$A'^{\bar{M}}_{\mu} = A^{\bar{M}}_{\mu} + C^{\bar{M}}_{\bar{P}\bar{Q}} A^{\bar{P}}_{\mu} \omega^{\bar{Q}} - \partial_{\mu}\omega^{\bar{M}}. \tag{1.22}$$

1.3.2 Reduction of the dynamics

As we have seen for the 5D case the reduction of the dynamics is straightforward carried on starting from the multidimensional Einstein-Hilbert Action, i.e. :

$$S = -\frac{1}{16\pi G_{4+K}} \int_{V^4 \otimes B^K} \sqrt{j}^{(4+K)} R d^4x d^K y. \tag{1.23}$$

Adopting the vielbein technique of reduction it is possible to get the reduced formula for the scalar curvature:

$$^{(4+K)}R = R - \frac{1}{4}\gamma_{rs}\xi^r_{\bar{M}}\xi^s_{\bar{N}} F^{\bar{M}}_{\mu\nu} F^{\bar{N}}_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - 2g^{\mu\nu}\frac{\nabla_\mu \partial_\nu \phi}{\phi} + \frac{1}{\phi^2} R_K \tag{1.24}$$

where

$$F^{\bar{M}}_{\mu\nu} = \partial_\nu A^{\bar{M}}_{\mu} - \partial_\mu A^{\bar{M}}_{\nu} + C^{\bar{M}}_{\bar{N}\bar{P}} A^{\bar{N}}_{\mu} A^{\bar{P}}_{\nu} \tag{1.25}$$

and R and R_K are curvatures associated with $g_{\mu\nu}$ and γ_{mn}, respectively. Finally, after the integration on B^K, granted the following relations,

$$\int_{B^K} \sqrt{\gamma}[\gamma_{rs}\xi^r_{\bar{M}}\xi^s_{\bar{N}}] d^k y = -\delta_{\bar{M}\bar{N}} V^K, \qquad V^K = \int_{B^K} \sqrt{\gamma} d^K y, \tag{1.26}$$

we can rewrite the Action as:

$$S = -\frac{1}{16\pi G}\int_{V^4} \sqrt{g}\phi^K \left[R + \frac{1}{4}\phi^2 \sum_{\bar{M}} F^{\bar{M}}_{\mu\nu} F^{\bar{M}\mu\nu} + \phi^{-2} R'_K - 2g^{\mu\nu}\frac{\nabla_\mu\partial_\nu\phi}{\phi} \right] \tag{1.27}$$

The Newton coupling constant reads now

$$G = \frac{G_{4+K}}{V^K}, \tag{1.28}$$

and the quantity R'_K is $R'_K = \frac{1}{V^K}\int_{B^K} \sqrt{\gamma} R_K d^k y$.

Thus, at the end of the procedure we deal with an Einstein-like dynamics plus coupling with the scalar field ϕ and the Yang-Mills fields, where, as it happens in the 5D case, the gauge symmetries are viewed as particular cases of more general diffeomorphism. We stress that the key point, with respect to the 5D formulation, is the implementation of the phenomenological viewpoint for which we are not able to probe extra dimensions. Indeed, this issue is supported by another reason ([15]): when we was dealing with the 5D dynamics we asked if the reduction of the dynamics performed on the Action commutes with the reduction performed on the 5D Einstein equation. Indeed, these procedure lead to equivalent result, because of the presence of the cylindricity hypothesis, which is a tool we can no more employ in the non-abelian model. Hence, in the framework of multidimensions the equations of motion will contain functions depending on y^m; but, since we assumed that a 4D observer cannot probe such coordinates, we must adopt an averaging procedure on the extra space, in order to rule out such dependence. It has been shown that the resulting equations coincide with those arising from the reduced Action; therefore the consistency of the model is recovered. Hence, the correct implementation of the unobservability, via such an averaging procedure, is a crucial key to recover at the same time the proper behaviour of the non-abelian field as well as a consistent set of motion equations.

Another distinctive feature of non abelian model is the presence of the term $\phi^{-2}R'_K$ in the Action which is related to the curvature of the extra-space. In the framework of compact multidimensional model this term cannot be avoided, because S^1 is the only compact space with a non zero curvature. If the scalar field is taken as a constant, then this term simply acts as a cosmological constant.

So far, we have discussed the more simple (but still non-abelian) kind of generalization of the 5D model. Indeed we can consider an extra space B^K which is itself a direct product of more different compact spaces, each of them with its characteristic scalar field ϕ, i.e.

$$B^K = B^{K_1} \otimes B^{K_2} \otimes \dots \qquad (\phi^1, \phi^2, \dots). \tag{1.29}$$

After the dimensional reduction, we finally deal with Einstein dynamics with coupling to various Yang-Mills fields, where each of them is associated to the proper gauge symmetry induced by the isometries of a given B^{K_i}, plus couplings to each scalar fields. The scalar field sector reads then:

$$S_\phi = -\frac{1}{16\pi G}\int_{V^4}\sqrt{g}\Pi_i(\phi^i)^{K_i}\left[2g^{\mu\nu}\sum_i \frac{\nabla_\mu\partial_\nu\phi^i}{\phi^i} + g^{\mu\nu}\sum_{i\neq j}\frac{\partial_\mu\phi^i}{\phi^i}\frac{\partial_\nu\phi^j}{\phi^j} + \sum_i\left(\frac{1}{\phi^i}\right)^2 R'_{K_i}\right]d^4x. \tag{1.30}$$

In recent papers ([13], [14], [16]), the geometrization of the electroweak model is considered, and, via the averaging procedure previously discussed, also the geometrization of spinor fields has been considered, either in a 7- and in 8- dimensional scenario. To end the discussion of this section we give as final example the starting setup of the 7-dimensional case. Indeed, 7 is the minimum dimensionality required to achieve the geometrization of the $U(1)\otimes SU(2)$ group. Being 4 dimension saved for the usual gravity, the addiction of the fifth dimension allows us to reproduce as usual a $U(1)$ symmetry, while the addiction of another two dimension is the least suitable extension to build three extra Killing vectors to describe the $SU(2)$ group. Therefore, we assume in this case:

$$V^7 = V^4 \otimes S^1 \otimes S^2$$

Here S^1 is a space-like loop and S^2 is a space-like spherical surface. S^2 is maximally symmetric and posses three Killing vectors whose constants $C^{\bar{P}}_{\bar{N}\bar{M}}$ are equal to the total skew-symmetric tensor $\epsilon^{\bar{P}}_{\bar{N}\bar{M}}$. The total metrics then reads:

$$j_{AB} = \left(\begin{array}{c|c|c} g_{\mu\nu} - \phi'^2 B_\mu B_\nu - \gamma_{mn}\xi^m_{\bar{M}}\xi^n_{\bar{N}}\phi^2 W^{\bar{M}}_\mu W^{\bar{N}}_\nu & -\phi'^2 B_\mu & -\phi^2\gamma_{mr}\xi^r_{\bar{M}}W^{\bar{M}}_\mu \\ \hline -\phi'^2 B_\nu & -\phi'^2 & 0 \\ \hline -\phi^2\gamma_{nr}\xi^r_{\bar{M}}W^{\bar{M}}_\nu & 0 & -\phi^2\gamma_{mn} \end{array}\right), \tag{1.31}$$

B_μ represents the gauge field associated to the hypercharge symmetry, while $W_\mu^{\bar{M}}$ ($\bar{M} = 1, 2, 3$) are bosons associated to the $SU(2)$ symmetry.

Chapter 2

Hamiltonian formulation of Kaluza-Klein model

In this chapter we analyze the hamiltonian formulation of KK model (Lacquaniti-Montani '06 [136], [137]). This will give us insight concerning the generation of the U(1) gauge symmetry and the role of the field ϕ. As it happens in GR, to consider the hamiltonian formulation of the 5D EH dynamics, it will be convenient to perform the ADM splitting ([62]) of the involved variables. However, as observed in the previous chapter (1.2), the generalization in 5D KK of a 4D procedure could be not straightforward; therefore we will care to probe the consistency of the ADM approach to KK model . We start this chapter with a brief review of the ADM procedure, therefore we will consider the ADM splitting of the KK model and finally its hamiltonian dynamics.

2.1 ADM formulation

Now, for sake of simplicity, we examine the rules of ADM splitting for a 4D manifold (i.e. the usual space-time). From a geometrical viewpoint this procedure can be easily extended in every number of dimensions. Indeed, the problem of the compatibility of such an approach to KK does not depend by the addiction of the extra dimension but by the role played by the KK hypotheses.

2.1.1 ADM splitting of the metrics

Let us consider a manifold $\mathcal{M}^4$, with a tensor metric $g_{\mu\nu}$ and internal coordinates u^μ ($\mu = 0,1,2,3$). At a frozen time we can identify a spatial 3-D hypersurface Σ^3, by a parameterization like $u^\mu = u^\mu(x^i)$, where x^i ($i = 1,2,3$) defines a set of local coordinates on Σ^3 ([63], [64], [65]). The possibility to always identify such a hypersurface is due to a theorem of Geroch ([66]) which holds for homogeneous and isotropic manifold of hyperbolic signature. By the definition of a spatial hypersurface, the three tangent vectors e^μ_i ($e^\mu_i = \frac{\partial u^\mu}{\partial x^i}$) and the normal vector η^μ must satisfy the following orthogonal and completeness ([67]) relations:

$$\begin{cases} \eta^\mu \eta^\nu g_{\mu\nu} = 1 \\ \eta^\mu e^\nu_i g_{\mu\nu} = 0 \\ e^\mu_i e^\nu_j g_{\mu\nu} = -\vartheta_{ij} \end{cases} \tag{2.1}$$

$$\eta^\mu \eta^\nu - \vartheta^{ij} e^\mu_i e^\nu_j = g^{\mu\nu} \tag{2.2}$$

In the above formulas, ϑ_{ij} is a positive-defined metrics and represents the 3D induced metrics on the spatial manifold.

Now let us define x^0 as a time parameter and associate to every value of x^0 a different spatial manifold; then, letting x^0 vary with continuity, we get a family of hypersurfaces defined by the general parameterization

$$u^\mu = u^\mu(x^i; x^0)\,, \tag{2.3}$$

that represents a complete transformation of coordinates ([65]). The so-called "time-deformation" vector $e^\mu_0 = \frac{\partial u^\mu}{\partial x^o}$ links points with same spatial coordinates on two surfaces separated by an infinitesimal distance dx_0. The set e^μ_0, e^μ_i rebuilds the Jacobian of the transformations (2.3). By completeness, the time-deformation vector can be expressed in terms of the tangent and normal vector to the surface, as follows:

$$e^\mu_0 = N\eta^\mu + S^i e^\mu_i \tag{2.4}$$

The components N and S^i of the time deformation vector in this picture are called *Lapse* and *Shift* functions respectively, and play a fundamental role in the ADM splitting. Finally, by using (2.4) and (2.1), we can recast the tensor metric according to the diffeomorphism (2.3) and we get:

$$g_{\mu\nu} \Rightarrow \begin{pmatrix} N^2 - \vartheta_{ij} S^i S^j & -\vartheta_{ij} S^j \\ \\ -\vartheta_{ij} S^j & -\vartheta_{ij} \end{pmatrix} \tag{2.5}$$

The splitting for the inverse metrics yields:

$$g^{\mu\nu} \Rightarrow \begin{pmatrix} \frac{1}{N^2} & -\frac{S^i}{N^2} \\ \\ -\frac{S^i}{N^2} & \frac{S^i S^j}{N^2} - \vartheta^{ij} \end{pmatrix} \tag{2.6}$$

It can be shown that N, S_i, ϑ_{ij}, are scalar, vector and tensor respectively, with respect to pure spatial diffeomorphisms. This is, by the way, the most important feature of this picture. The above formulas represent the ADM splitting of the metrics ([63], [64], [65]), and it is showed how we can redefine the ten degrees of freedom of the metrics $g_{\mu\nu}$ in terms of objects with well-defined properties of transformation under spatial diffeomorphisms. The procedure we outlined in this section still holds if a manifold with extra space-like dimensions is considered; while we usually refer to the ADM splitting as the $3+1$ space-time slicing, we simply refer to it, in a multidimensional background, as a $n+1$ slicing. No relevant modifications occurs in the concerned formulas, except, of course, some changes, due to the number of the involved dimensions, in the calculation of the traces. Anyway, we will aware the reader when some other relevant change occurs.

2.1.2 Projection rules

As well as we considered the reduction of metrics, now we concern on the splitting rules for a generic vector or tensor. Generally speaking, we have to be aware that the set of spatial components of any tensor does not correspond exactly to the correct spatial part of the tensor itself; the equivalence holds, as we'll see, only in the covariant component picture. Let us examine what happens for vector objects. We define at first a 3D vector in a covariant picture, namely ${}^3A_i$, and therefore a 4D vector whose spatial components coincide with the spatial vector ${}^3A_i$, i.e. we have $A_\mu = (A_0, {}^3A_i)$. Therefore, the covariant partner ${}^3A^i$, defined on the spatial surface, reads ${}^3A^i = \vartheta^{ij3}A_j$, whereas for the spatial components of the vector A^μ we now have $A^i = g^{i\mu}A_\mu \neq {}^3A^i$. Due to the presence of $g^{i\mu}$ these components depend also on A_0 and on *Shift* functions; this means that although A^i is a spatial vector (i.e. the components A^i transform like a vector under pure spatial diffeomorphisms), it is not the complete spatial part of the vector itself (the exact equivalence hold only when the time deformation vector is normal to the surface, i.e. a synchronous reference where *Shift* functions vanish). The most general way to split any vector or tensor according to ADM rules is to use the projection tensor, implicitly defined by the completeness relation (2.2)

$$\begin{cases} \eta^\mu\eta^\nu + q^{\mu\nu} = g^{\mu\nu} \\ \\ q^{\mu\nu} = -\vartheta^{ij}e^\mu_i e^\nu_j \end{cases} \tag{2.7}$$

It's clear that $q^{\mu\nu}$ acts as a projector on the spatial hypersurface and the spatial metrics itself is simply the projection tensor expressed in terms of the spatial coordinates ([67]). Thus, given a generic vector B^μ we can immediately obtain its spatial part $A^\mu_{\Sigma^3}$ (i.e. $\eta_\mu A^\mu_{\Sigma^3} \equiv 0$) from its contraction with the projection tensor:

$$A^\mu_{\Sigma^3} = q^\mu_\nu A^\nu$$

Then, in order to perform the ADM splitting we only have to recast the spatial part of the vector in the spatial coordinates picture; we have:

$$\begin{cases} {}^3A^i = \frac{\partial x^i}{\partial u^\mu} A^\mu_{\Sigma^3} \\ \\ {}^3A_i = \frac{\partial u^\mu}{\partial x^i} g_{\mu\nu} A^\nu_{\Sigma^3} = \frac{\partial u^\mu}{\partial x^i} A_\mu = A_i \end{cases} \tag{2.8}$$

The equivalence ${}^3A_i = A_i$, that arises from the second of these relations is due to the definition of tensor projection itself and to the orthogonal relations (2.1). It shows that the projecton tensor acts as the identity operator on the covariant spatial components. Also, from the first of (2.8), we could show the following relations:

$$\begin{cases} A^0 = \frac{1}{N^2}(A_0 - {}^3A_i S^i) \\ A^i = -{}^3A^i - S^i A^0 \end{cases} \qquad i, j = 1, 2, 3 \tag{2.9}$$

We can obtain the same result by simply considering a generic vector in covariant picture and its contraction with the inverse metrics $g^{\mu\nu}$ recast in term of ADM variables (see 2.6). The above rules for a vector can be easily extended to a tensor of any order ([67]). For instance, as far a tensor $T^{\mu\nu}$ is concerned, we have:

$$T^{\mu\nu}_{\Sigma^3} = q^\mu_\rho q^\nu_\sigma T^{\rho\sigma}$$

$$\begin{cases} T^{ij} = \frac{\partial x^i}{\partial u^\mu}\frac{\partial x^j}{\partial u^\nu} T^{\mu\nu}_{\Sigma^3} \\ T_{ij} = \frac{\partial u^\mu}{\partial x^i}\frac{\partial u^\nu}{\partial x^j} T_{\mu\nu} \end{cases} \tag{2.10}$$

2.1.3 Hamiltonian formulation of General Relativity

The advantage of the ADM splitting is revealed when we consider the hamiltonian formulation of the GR ([63], [68], [69], [70]). Indeed, a remarkable feature of the EH Action is the presence of second time-derivatives of the metrics ([50]). This fact does not affect the lagrangian formulation that leads to Einstein equations because , in the variational calculus, the term containing second time-derivatives is only a global surface term, but becomes a problem in the hamiltonian approach, when we have to calculate the conjugate momenta of the metrics. Therefore, we need a procedure to identify , in the hamiltonian approach, that part of the Action which is not truly dynamical. This task is accomplished by the ADM splitting. Indeed, the difference between lagrangian and hamiltonian formulation relies in the properties of Lagrangian and Hamiltonian under transformation of coordinates. While the former is manifestly 4D covariant, the latter is covariant only for pure spatial transformations (due to definition of conjugates momenta in terms of first time derivatives); hence we need to recast our degrees of freedom in order to correctly identify spatial or temporal objects, and this is obtained by making use of the ADM splitting.

Now, if we want to to consider the EH action in terms of ADM variables, we need the splitting of the curvature scalar. This is provided by the Gauss-Codacci formula (that comes from the splitting of Riemann tensor according to (2.5) ([64], [67], [69],):

$$R = (K^2 - K_{ij}K^{ij} - {}^3R) + 2\nabla_\nu(\eta^\mu \nabla_\mu \eta^\nu - \eta^\nu K) \tag{2.11}$$

In the above formula, ${}^3R$ is the curvature scalar related to the spatial hypersurface (depending only on ϑ_{ij} and its spatial derivatives) and K_{ij} is the extrinsic curvature which is defined as follows ([67], [69]):

$$K_{ij} = \frac{1}{2N}(D_i S_j + D_j S_i - \partial_0 \vartheta_{ij}) \tag{2.12}$$

Here D_i is the 3D covariant derivative defined on the spatial surface (therefore is ϑ_{ij}-compatible, i.e. $D_i\vartheta_{jk} = 0$). The lowering/raising of indexes that appears in K^{ij}, in its trace $K = K^{ij}\vartheta_{ij}$, and in S_i is of course made via ϑ_{ij}. The second term that appears in the right side of the Gauss-Codacci equation is not expressed in ADM variables, but it doesn't matter at this step because, when considering the Action, it becomes a global surface term. We will see, however, how an analogue term will become relevant in the splitting of the KK model. Finally, by observing that we have $\sqrt{g} = N\sqrt{\vartheta}$, we can give an expression for the Lagrangian in ADM variables:

$$L = bN\sqrt{\vartheta}(K^2 - K_{ij}K^{ij} - {}^3R) + S.T. \tag{2.13}$$

where $b = -\frac{1}{16\pi G}$. As we can see, now the time derivative appears only at first order, so we can easily compute the conjugate momenta. Moreover, we have no time derivative for *Lapse* and *Shift* functions; this means that they are not true dynamical variables and that the real dynamics is carried out only by the six degrees of freedom of the spatial metrics

ϑ_{ij}. Therefore, for the conjugate momenta (π and π^i for N, S_i and π^{ij} for ϑ_{ij}) and their inverse relations we have:

$$\begin{cases} \pi^{ij} = -b\sqrt{\vartheta}(K\vartheta^{ij} - K^{ij}) \\ \dot{\vartheta}^{ij} = (D_i S_j + D_j S_i) - 2NK_{ij} \\ \pi \equiv \pi^i \equiv 0 \end{cases} \tag{2.14}$$

It can be shown that we can put the Hamiltonian in the following representation:

$$H = NH^N + S_i H^i \tag{2.15}$$

Here H^N and H^i, called Super-hamiltonian and Super-momenta respectively, does not depend on the *Lapse* or *Shift* functions, as we can see by their explicitly expressions:

$$\begin{cases} H^N = -\frac{1}{b} G_{ijkl}\pi^{ij}\pi^{kl} + b\sqrt{\vartheta}{}^3R \\ H^i = -2D_j\pi^{ij} \end{cases} \tag{2.16}$$

G_{ijkl} is called Supermetrics and takes the following expression:

$$G_{ijkl} = \frac{1}{2\sqrt{\vartheta}}(\vartheta_{ik}\vartheta_{jl} + \vartheta_{il}\vartheta_{jk} - \vartheta_{ij}\vartheta_{kl})$$

The reader should be aware that the calculation of the Supermetrics, being involved the trace of the reduced spatial metrics, depends on the number of spatial dimensions. Therefore, if we consider a manifold with n spatial dimensions the following modifications occurs:

$$G_{ijkl} = \frac{1}{2\sqrt{h}}(h_{ik}h_{jl} + h_{il}h_{jk} - \frac{2}{n-1}h_{ij}h_{kl}) \qquad i, j = 1, 2, ..., n$$

Hence, from a mathematical point of view, the ADM splitting allows us to avoid the problem of second time derivatives. Furthermore it shows a remarkable structure of the dynamics. The conjugate momenta to *Lapse* and *Shift* are identically zero as well as their time derivatives vanish; thus, by examining the Poisson's brackets we can see that the dynamics is constrained by the following relations:

$$H^N \equiv 0 \tag{2.17}$$

$$H^i \equiv 0 \tag{2.18}$$

Then N and S_i play a role as lagrangian multipliers. This is a direct consequence of diffeomorphism invariance of the theory and is , indeed, the most interesting feature shown by ADM splitting (from the point of view of classical dynamics).

Let us consider now the canonical quantization; it can be shown that the dynamics can be described simply by the first of the constrained equations (2.17); thus, in order to have a quantum dynamics we have to upgrade this equation to a quantum one, replacing conjugates momenta with appropriate derivatives operators. We therefore have such an equation (Wheeler-DeWitt) ([71], [72]) :

$$\hat{H}^N \Psi = 0$$

Unfortunately, as an effect of the constraints, the equation we found is not evolutionary, hence its interpretation, as well as the interpretation of the wave function Ψ, is a not well understood issue ([73], [74], [72], [64]). At the same time there are difficulties in order to construct a well-defined Hilbert space ([71], [72], [73]). The real question concerns the meaning of time at quantum level and this is, indeed, still a matter of debate ([75], [74]). A valuable approach to the WdW equation is its reformulation in terms of Ashtekar variables ([76]), which leads to Loop Quantum Gravity theory, where a non perturbative approach to quantum gravity is considered and it is suggested to define a time in the relational point of view ([77]). Indeed, the effort of finding a relational time by adopting a suitable degree of freedom is not exclusive of Loop theory; other schemes like the multitime ([73]) approach - which consists in the quantization of the polarization degrees of freedom, through an appropriate reformulation of variables- or the

Brown-Kuchar ([78]) approach deal with the same issue. A common point in such efforts is the introduction of scalar fields or matter terms, from which we are able to obtain an unfrozen formalism ([79],[80],[81],[82]). From the above consideration arises our interest in the dynamics of the field ϕ, which is indeed an extra scalar degree of freedom provided by the KK model.

2.2 ADM approach to the Kaluza-Klein model

Before facing the problem of the ADM formulation of the KK model, let us consider the following observations. From a physical point of view, the KK reduction and the ADM reformulation have a different meaning; however, from a mathematical point of view they both consist in a splitting of the metrics which results in a reparemetrization of the involved degrees of freedom. Therefore, the ADM reformulation of KK model relies in performing in a row these two splitting procedures, starting from a generic (unreduced) 5-D manifold. At a first glance, it seems that we only have to impose KK conditions and then develop the ADM procedure, taking into account the extra (space-like) dimension. But, actually, we have to choose between two suitable ways, depending on which kind of splitting, KK or ADM, we want to perform as first step. Let us examine what we expect from these procedures.

- Let us suppose to perform at first the KK reduction ; this leads to the KK model we have seen in the previous chapter. As a second step, we have to perform the usual $(3+1)$ ADM slicing on the reduced variables we have in this model : the 4-D metrics and the gauge-vector. However, in this way the space-time slicing is not complete, because the extra dimension is not included in the splitting procedure.

- Let us suppose instead to do as first the ADM splitting; this corresponds to a $(4+1)$ slicing and leads us to consider a 4D spatial metrics and a set of four *Shift* functions; now the metrics and the set of *Shift* functions both contains degrees of freedom related to the extra dimension. As a second step of this procedure we perform the KK reduction and examine the reduction of the spatial metrics. By this procedure we find only a 3D spatial gauge-vector and the dynamics lacks an explicit time component for the gauge vector , therefore we cannot be sure that this procedure, where the space-time slicing is however complete, restores the correct KK model.

Hence, the question is to check if the KK reduction procedure commute with the ADM splitting procedure. Such a question is not without physical relevance. Indeed, as we've stressed in section (4.37) KK reduction implies a breaking in the symmetry of the 5-D space ; therefore this could be not consistent with the hypotheses on which ADM splitting is constructed and the two procedures could lead to different dynamics. Moreover, it seems that both procedures lead to some unsatisfactory features. Hence, what is really in trial now is the whole consistency of the application of ADM splitting to KK model.

Therefore, we now examine in details the above mentioned procedures, starting with the splitting of the metrics.

2.2.1 KK-ADM metrics

Let us consider a 5D tensor metric J_{AB}. As previously seen, (eq. 1.5) after the KK reduction the metrics reads as follows:

$$J_{AB} \Rightarrow \begin{pmatrix} g_{\mu\nu} - \phi^2(ek)^2 A_\mu A_\nu & -\phi^2(ek)A_\mu \\ \\ -\phi^2(ek)A_\mu & -\phi^2 \end{pmatrix}$$

We have now to recast $g_{\mu\nu}$ and A_μ in terms of ADM variables. For the space-time metrics this is easily done by using (2.5), while for the gauge vector we employ the splitting formulas we have discussed in (2.1.2).

By observing that we are dealing with a gauge-vector in a covariant picture, we finally have the splitting of the metrics, which yields :

$$J_{AB} \Rightarrow \begin{pmatrix} N^2 - S_iS^i - (ek)^2\phi^2 A_0^2 & -S_i - (ek)^2\phi^2 A_0 A_i & -ek\phi^2 A_0 \\ \\ -S_i - (ek)^2\phi^2 A_0 A_i & -\vartheta_{ij} - (ek)^2\phi^2 A_i A_j & -ek\phi^2 A_i \\ \\ -ek\phi^2 A_0 & -ek\phi^2 A_i & -\phi^2 \end{pmatrix} \tag{2.19}$$

where $A_i = {}^3A_i$.

What about transformation laws ? Let us consider pure spatial KK diffeomorphism:

$$\begin{cases} x^4 = x^{4'} + ek\Psi(x^{i'}) \\ x^0 = x^{0'} \\ x^i = x^i(x^{i'}) \end{cases} \tag{2.20}$$

The variables N, S_i and ϑ_{ij} arise from the splitting of $g_{\mu\nu}$, which is a tensor without gauge component, therefore they are scalar, 3D vector and 3D tensor respectively. A_μ presents a gauge-component, and so A_i does as well, while for pure spatial transformations A_0 is a scalar. Indeed we have:

$$\begin{cases} {A_0}' = A_0 \\ \\ {A_i}' = A_j \frac{\partial x^j}{\partial x^{i'}} + \frac{\partial \Psi}{\partial x^{i'}} \qquad\qquad (A_i = {}^3A_i) \\ \\ {S_i}' = S_j \frac{\partial x^j}{\partial x^{i'}} \end{cases} \tag{2.21}$$

Let us now examine also the inverse metrics J^{AB}: its KK reduction, as it can be easily verified by a direct calculus from (1.5), gives the result

$$J^{AB} \Rightarrow \begin{pmatrix} g^{\mu\nu} & -ekA^\mu \\ \\ -ekA^\mu & (ek)^2 A_\mu A^\mu - \frac{1}{\phi^2} \end{pmatrix}$$

where $A^\mu = g^{\mu\nu}A_\nu$. Applying the ADM splitting, according to (2.6), we get:

$$J^{AB} \Rightarrow \begin{pmatrix} \frac{1}{N^2} & -\frac{S^i}{N^2} & -ekA^0 \\ -\frac{S^i}{N^2} & \frac{S^iS^j}{N^2} - \vartheta^{ij} & -ekA^i \\ -ekA^0 & -ekA^i & (ek)^2(N^2{A^0}^2 - A_iA_j\vartheta^{ij}) - \frac{1}{\phi^2} \end{pmatrix} \qquad (2.22)$$

where now A^i and A^0 are given by (2.9). But now we have to notice that, according to (2.9), A^0 is not a scalar and A^i is not just a gauge-vector. We have:

$$\begin{cases} A^{0\prime} = A^0 - \frac{S^i}{N^2}\frac{\partial\Psi}{\partial x^i} \\ A^{i\prime} = A^j\frac{\partial x^{i\prime}}{\partial x^j} + \frac{\partial x^{i\prime}}{\partial x^j}(\frac{S^iS^k}{N^2} - \vartheta^{jk})\frac{\partial\Psi}{\partial x^j} \qquad (A^i \neq {}^3A^i) \\ S^{i\prime} = S^j\frac{\partial x^{i\prime}}{\partial x^j} \end{cases} \qquad (2.23)$$

It can be now recognized, from the step we've followed , that the space-time slicing is not complete, because we didn't take into account the extra-dimension; we only performed the splitting of the ordinary 4-D space-time and of the 4-D gauge vector; in other words we only have defined a 3-d spatial hypersurface , not including the extra dimension . As we will see, this will involve second time derivatives in the lagrangian formulation of the dynamics. However, before studying the Lagrangian, we will see the metrics that arises from the other procedure.

2.2.2 ADM-KK metrics

Now we start with a $(4+1)$ ADM splitting of the 5D metric tensor. So we split our manifold into the direct product of a 4D spatial manifold ${}^4\mathcal{V}$, including the extra dimension, times a one dimensional time-like manifold. Thus, we get a 4D spatial metrics $h_{\hat{I}\hat{J}}$ and four *Shift* functions $N_{\hat{I}}$ according to the following splitting:

$$J_{AB} \Rightarrow \begin{pmatrix} N^2 - N^{\hat{I}}N_{\hat{I}} & -N_{\hat{I}} \\ -N_{\hat{I}} & -h_{\hat{I}\hat{J}} \end{pmatrix} \qquad \hat{I},\hat{J} = 1,2,3,5 \qquad (2.24)$$

Now, the components of the above parametrized metrics shows covariant properties with respect to the class of pure spatial transformations involving the usual 3D space plus the extra dimension. The next step is to examine the effects of KK reduction on the 4D pure spatial manifold ${}^4\mathcal{V}$. Hence, among the pure spatial diffeomorphisms allowed in ${}^4\mathcal{V}$, the restricted KK diffeomorphisms are simply:

$$\begin{cases} x^5 = x^{5'} + ek\Psi(x^{i'}) \\ \\ x^i = x^i(x^{i'}) \end{cases} \tag{2.25}$$

Spatial metrics (positively defined) is split in a similar way to the complete 5D metrics (1.5) , with only a signature change :

$$\begin{cases} h_{55} = \phi^2 \\ h_{5i} = \phi^2(ek)A_i \\ h_{ij} = \vartheta_{ij} + \phi^2(ek)^2 A_i A_j \end{cases} \tag{2.26}$$

In the above formulas, ϕ is still a scalar field , A_i is a 3D spatial gauge vector and ϑ_{ij} is the 3-D spatial metrics. It is worth examining the 4-bein picture of this metrics, which mimics the 5-bein KK representation. For the basis set we have:

$$\begin{cases} e^{(5)}_{\hat{I}} = \phi(ekA_i, 1) \\ \\ e^{(j)}_{\hat{I}} = (u^{(j)}_i, 0) \end{cases} \tag{2.27}$$

$$\begin{cases} e^{\hat{I}}_{(5)} = (0, 0, 0, \frac{1}{\phi}) \\ \\ e^{\hat{I}}_{(j)} = (u^i_{(j)}, -ekA_i u^i_{(j)}) \end{cases} \tag{2.28}$$

Clearly $u^i_{(j)}$ and its inverse define the 3D spatial metrics ϑ_{ij} and have no gauge components in its transformation rules. Finally, we have

$$J_{AB} \Rightarrow \begin{pmatrix} N^2 - h_{\hat{I}\hat{J}}N^{\hat{I}}N^{\hat{J}} & -N_i & -N_5 \\ \\ -N_i & -\vartheta_{ij} - (ek)^2\phi^2 A_i A_j & -ek\phi^2 A_i \\ \\ -N_5 & -ek\phi^2 A_i & -\phi^2 \end{pmatrix} \tag{2.29}$$

where explicitly results

$$h_{\hat{I}\hat{J}}N^{\hat{I}}N^{\hat{J}} = (\phi N^5)^2 + 2ek\phi^2 A_i N^i N^5 + \vartheta_{ij}N^i N^j + (ek)^2\phi^2 A_i A_j N^i N^j$$

In a same way we can also obtain the ADM-KK splitting for the inverse metrics and we have:

$$J^{AB} \Rightarrow \begin{pmatrix} \frac{1}{N^2} & -\frac{N^i}{N^2} & -\frac{N^5}{N^2} \\ & & \\ -\frac{N^i}{N^2} & -\vartheta^{ij} - \frac{N^i N^j}{N^2} & \frac{N^i N^5}{N^2} + ekA_j\vartheta^{ij} \\ & & \\ -\frac{N^5}{N^2} & \frac{N^i N^5}{N^2} + ekA_j\vartheta^{ij} & \frac{N^{5\,2}}{N^2} - \frac{1}{\phi^2} - (ek)^2 A_i A_j \vartheta^{ij} \end{pmatrix} \tag{2.30}$$

The most striking feature of this procedure is, as we previously observed, that this picture lacks an explicit time component for the gauge vector. However, our analysis of the effects of KK reductions is not yet finished. Indeed, the controvariant components $N^{\hat{I}}$ that appear in the inverse metrics are still linked to covariant ones by the 4D metrics $h_{\hat{I}\hat{J}}$, as it arises from the $(4+1)$ splitting of the inverse metrics. Hence, we need to identify the correct 3D spatial part of *Shift* functions, whose indexes must be lowered/raised with the 3D spatial metrics ϑ_{ij}, that is the only metrics we want to take into account after the KK reduction.

To solve this problem, let us consider the *Shift* functions before the symmetry breaking that arises from KK reduction. $N_{\hat{I}}$ and $N^{\hat{I}}$ transform as 4D spatial vector. By imposing the KK restrictions, we deal with the following transformation rules:

$$\begin{cases} N_i{}' = ekN_5\frac{\partial\Psi}{\partial x^{i'}} + N_j\frac{\partial x^j}{\partial x^{i'}} \\ \\ N_5{}' = N_5 \end{cases} \tag{2.31}$$

$$\begin{cases} N^{i'} = N^j\frac{\partial x^{i'}}{\partial x^j} \\ \\ N^{5'} = N^5 - ekN^j\frac{\partial\Psi}{\partial x^j} \end{cases} \tag{2.32}$$

Then, we see that N_5 is a scalar and N^i a vector, while in the other components gauge-terms appear that must be related to a dependence on the gauge-vector. In order to examine in more details the structure of the *Shift* functions, we can express them in the 4-bein picture. We have:

$$\begin{cases} N_5 = N_{(5)}\phi \\ \\ N_i = N_{(l)}u_i^{(l)} + ekN_5A_i \end{cases} \tag{2.33}$$

$$\begin{cases} N^5 = \frac{N^{(5)}}{\phi} - ekA_iN^{(l)}u^i_{(l)} \\ \\ N^i = N^{(l)}u^i_{(l)} \end{cases} \tag{2.34}$$

This picture agrees with the transformation rules above outlined, but, moreover, it also reveals how *Shift* functions are a mix of a gravitational (pure vector) part and a gauge one. We define S_i as the vector part of N_i, and by its definition we have $S^i = \vartheta^{ij}S_j$, so we can rewrite:

$$\begin{cases} N_5 = N_{(5)}\phi \\ \\ N_i = S_i + ekN_5A_i \end{cases} \tag{2.35}$$

$$\begin{cases} N^5 = \frac{N^{(5)}}{\phi} - ekA_iS^i \\ \\ N^i = S^i \end{cases} \tag{2.36}$$

The above formulas allows us, in principle to write the ADM-KK metrics just in terms of objects that are scalar or described by 3D properties of transformation

2.2.3 Conversion formulas

Now let us make a comparison between the ADM-KK and KK-ADM metrics; we start by focusing on the elements J_{05} and J_{0i}. If the metrics were equivalent, the following expression should be satisfied.

$$\begin{cases} N_i = S_i + (ek)^2\phi^2A_0A_i \\ \\ N_5 = ek\phi^2A_0 \end{cases} \tag{2.37}$$

This suggests to define A_0 in terms of N_5, as in the second equation. Is this definition acceptable ? Indeed, the second members of these equations verify the transformations rules (2.31), as we can see by (2.21) as well as they satisfy also the 4-bein structure. Moreover, they lead also to an equivalence for the J_{00} component of the metrics, so the (2.37), that we label "conversion formulas" appear correct. Then, if we accept as true the (2.37), we can say that the vector part of N_i corresponds to the 3D *Shift* vector, and we are able to identify a time-component for the gauge-vector. However, we have to be sure , in defining a 4D gauge vector, that the conversion formulas (2.37) rebuild the particular relationships which link covariant and controvariant components, as we have seen during the ADM splitting procedure (see eq. 2.9). Thus, we need to examine the inverse metrics too. By comparison of the J^{05} and J^{0i} elements of the ADM-KK and KK-ADM metrics, we find the requirement:

$$ekA^0 = \frac{N^5}{N^2} \tag{2.38}$$

$$-ekA^i = \frac{N^i N^5}{N^2} + ek^3A^i \tag{2.39}$$

By using these equations with (2.37), (2.35), (2.36) we find:

$$A^0 = \frac{1}{N^2}(A_0 - S^i A_i) \tag{2.40}$$

$$A^i = -S^i A^0 - {}^3A^i \tag{2.41}$$

Hence, conversion formulas rebuild exactly the ADM formulas for the splitting of a vector. Moreover, it can be shown that conversion formulas lead to the equivalence of all others elements of the metrics.

Therefore, also in the ADM-KK procedure (that is the one with a complete space-time slicing) we can rebuild a complete 4D gauge vector. From a mathematical point of view, we can say that the two metrics gained with the ADM-KK and KK-ADM procedure are equivalent; at the same way, from this point of view, conversion formulas shows how, by the KK condition, the 4D *Shift* vector is split into a pure vector part and in a gauge one. It is remarkable that A_0 is linked to the extra dimension *Shift* component. We will return later on this point. Anyway, to make us sure that these formulas have a physical meaning, we have to examine the ADM-KK and KK-ADM Lagrangian, and see if they commute.

Conversion formulas Following expression summarize the conversion formulas:

$$\begin{cases} N_i = S_i + (ek)^2\phi^2 A_0 A_i \\ \\ N_5 = ek\phi^2 A_0 \end{cases} \qquad \begin{cases} N^i = S^i \\ \\ N^5 = ekN^2 A^0 \end{cases} \tag{2.42}$$

$$S^i = \vartheta^{ij} S_j \tag{2.43}$$

2.2.4 Lagrangian formulations

At this step of our analysis. the most useful properties of the conversion formulas is that they allows us to compute the Lagrangian arising from ADM-KK and KK-ADM procedure and put them in the same set of dynamical variables all with well-defined transformation properties with respect to the class of pure spatial KK diffeomorphisms. Therefore we now calculate the Lagrangians with respect the following set of variables:

- ϑ_{ij}, tensor; S_i vector; N, scalar
- A_0, scalar; A_i gauge-vector
- ϕ scalar

KK-ADM Lagrangian

Let us consider the action of the reduced 5D KK model:

$$S = b \int d^4x \sqrt{g} \quad (\phi R - 2\nabla_\mu \partial^\mu \phi + \frac{1}{4}(ek)^2 \phi^3 F_{\mu\nu} F^{\mu\nu}) \qquad b = -\frac{1}{16\pi G}$$

The second term in brackets is a total surface term , therefore from now on we can omit it. Therefore we now focus on the splitting of the electromagnetic term and the curvature term.

According to the projection rules (2.10), it can be shown that the electromagnetic term is split as follows:

$$F_{\mu\nu}F^{\mu\nu} = F_{ij}{}^{3}F^{ij} - \frac{2}{N^2}\vartheta^{ij}M_iM_j \, ,$$

where $M_i = F_{i0} - S^i F_{ij}$ and ${}^3F^{ij}$ is gained from F_{ij} via ϑ^{ij}. For the curvature term we can employ the Gauss-Codacci formula (2.11), but now, because of the presence of ϕ, we must take into account the term involving the derivative of the normal vector to the 3D manifold, which previously we treated as a surface term:

$$2b\sqrt{g}\,\phi\nabla_\nu(\eta^\mu\nabla_\mu\eta^\nu - \eta^\nu K)$$

In order to split this term we need again to use the projection tensor, but, by the presence of the covariant derivative, the projection rules is slightly changed; it can be shown that the correct formula is the following: given a generic vector B_ν we have:

$$q^{\mu\nu}(\nabla_\mu B_\nu) = q^{\mu\nu}D_\mu B_\nu + B^\eta K$$

where B^η reads $B^\eta = B_\mu\eta^\mu$ and D_μ is the 4-D covariant derivative compatible with $q_{\mu\nu}$ that, when recasted in the 3D spatial picture, provides the 3-D covariant derivative constructed via ϑ_{ij}. Thus, setting $B_\mu = \nabla_\mu\phi$ in the above formula, after some algebraic arrangement we get our desired splitting of the extra gravitational term :

$$\phi\nabla_\nu(\eta^\mu\nabla_\mu\eta^\nu - \eta^\nu K) = -\partial_\eta\partial_\eta\phi + \vartheta^{ij}D_i\partial_j\phi + S.T. \, . \tag{2.44}$$

Here D_i is the 3-D covariant derivatives and we have:

$$\partial_\eta = \frac{1}{N}(\partial_0 - S^i\partial_i) \tag{2.45}$$

Finally , Lagrangian takes the following expression:

$$\begin{aligned}\mathcal{L}_{KK-ADM} &= b\sqrt{\vartheta}\,N\phi(K^2 - K_{ij}K^{ij} - {}^3R) + 2b\sqrt{\vartheta}\,N(D_i\partial^i\phi - \partial_\eta\partial_\eta\phi) + \\ &+\frac{b}{4}\sqrt{\vartheta}\,N(ek)^2\phi^3\left(F_{ij}{}^3F^{ij} - \frac{2}{N^2}M_iM^i\right) \end{aligned} \quad (2.46)$$

- $b = \frac{-1}{16\pi G}$
- $\partial_\eta\phi = \frac{1}{N}(\partial_0\phi - S^i\partial_i\phi)$
- $M_i = F_{i0} - S^jF_{ij}$
- $K_{ij} = \frac{1}{2N}(D_iS_j + D_jS_i - \partial_0\vartheta_{ij})$

As we can see, the term $\partial_\eta\partial_\eta\phi$ contains a second time derivative and this is due to a not complete space-time slicing. To stress this effect we put:

$$\mathcal{L}_{KK-ADM} = \widetilde{\mathcal{L}} - 2b\sqrt{\vartheta}\,N\partial_\eta\partial_\eta\phi \quad (2.47)$$

with $\widetilde{\mathcal{L}}$ implicitly defined by (2.46).

ADM-KK Lagrangian

Now, we use at first ADM splitting to gain a complete $(4+1)$ space-time slicing. We start from the 5D action and use the Gauss-Codacci formula, omitting the surface term. After this, we impose KK conditions; via the compactification and the cylindricity conditions, we perform the dimensional reduction and finally the Action takes the following form:

$$S = b\int d^4x N\sqrt{h}({}^4K^2 - {}^4K_{\hat{I}\hat{J}}K^{\hat{I}\hat{J}} - {}^4R) \qquad \hat{I},\hat{J} = 1,2,3,5$$

where $h_{\hat{I}\hat{J}}$ is the 4D spatial metrics we defined within the ADM-KK procedure of reduction. In the above formula, ${}^4R$ and ${}^4K$ are respectively the curvature scalar and the extrinsic curvature related to $h_{\hat{I}\hat{J}}$, and we need their KK reduction. This can be achieved employing the 4-bein projection technique associated to the basis set (2.28, 2.27), as well as it is applied to achieve the dimensional reduction within the 5D manifold. For the intrinsic curvature term we have :

$$ {}^4R = {}^3R - \frac{2}{\phi}\vartheta^{ij}D_i\partial_j\phi - \frac{1}{4}(ek)^2\phi^2 F_{ij}{}^3F^{ij}\,, \qquad i,j=1,2,3 \tag{2.48} $$

where ϑ_{ij} is the 3D spatial metrics, ${}^3R$ and D_i are curvature scalar and covariant derivative related to ϑ_{ij} and F_{ij} is the spatial Faraday tensor constructed with the 3D gauge-vector A_i. In a same way, the splitting of extrinsic curvature terms give us the following result:

$$ {}^4K^2 - {}^4K_{\hat{I}\hat{J}}{}^4K^{\hat{I}\hat{J}} = [K^2 - K_{ij}K^{ij}] - \frac{2}{\phi^2}K\phi\partial_\eta\phi - \frac{1}{2N^2}\phi^2 M_i M^i \tag{2.49} $$

where K_{ij} is the 3-D extrinsic curvature and ∂_η, M_i are the same objects we have defined in the KK-ADM procedure. Finally, we have the following expression for the Lagrangian:

$$ \begin{aligned} \mathcal{L}_{ADM-KK} &= b\sqrt{\vartheta}\,N\phi(K^2 - K_{ij}K^{ij} - {}^3R) + 2b\sqrt{\vartheta}\,N(D_i\partial^i\phi - K\partial_\eta\phi) + \\ &\quad + \frac{b}{4}\sqrt{\vartheta}\,N(ek)^2\phi^3(F_{ij}{}^3F^{ij} - \frac{2}{N^2}M_iM^i) \end{aligned} \tag{2.50} $$

It is worth noting that we have only a term containing a time derivative of the scalar field. We can rewrite the Lagrangian as follows, with obvious definition for $\widetilde{\mathcal{L}}$:

$$ \mathcal{L}_{ADM-KK} = \widetilde{\mathcal{L}} - 2b\sqrt{\vartheta}\,NK\partial_\eta\phi \tag{2.51} $$

Comparison of the dynamics

By a comparison between the two Lagrangians , we can see how the term $\widetilde{\mathcal{L}}$ is the same in both procedures. Thus, let us examine terms which gives a difference: they are the terms concerning the time derivatives of the scalar field:

- ADM-KK $\Rightarrow$ $\qquad -2b\sqrt{\vartheta}\,NK\partial_\eta\phi$
- KK-ADM $\Rightarrow$ $\qquad -2b\sqrt{\vartheta}\,N\partial_\eta\partial_\eta\phi$

Now we show how these terms are equivalent apart from a total surface term. For a generic matrix $M(x)$ and its inverse we have the identity:

$$Tr[M^{-1}(x)\partial_i M(x)] = \partial_i[ln(detM(x))] \tag{2.52}$$

By (2.52) we can express K in a useful form;

$$\begin{aligned} K = \vartheta^{ij}K_{ij} &= \frac{\vartheta^{ij}}{2N}(D_iS_j + D_jS_i - \partial\vartheta_{ij}) \\ &= \frac{1}{N}D_iS^i - \frac{1}{2N}\vartheta^{ij}\partial_0\vartheta_{ij} \\ &= \frac{1}{N}D_iS^i - \frac{1}{N}\frac{1}{\sqrt{\vartheta}}\partial_0\sqrt{\vartheta} \end{aligned} \tag{2.53}$$

Thus, by (2.53), we can rewrite the ADM-KK term as follows:

- ADM-KK $\quad N\sqrt{\vartheta}\,K\partial_\eta\phi = \sqrt{\vartheta}\,(D_iS^i)(\partial_\eta\phi) - (\partial_o\sqrt{\vartheta})(\partial_\eta\phi)$

Now let us turn our attention to KK-ADM term: making use of the definition of ∂_η (2.45) we have:

$$N\sqrt{\vartheta}\,\partial_\eta\partial_\eta\phi = \sqrt{\vartheta}\partial_0\partial_\eta\phi - \sqrt{\vartheta}S^i\partial_i\partial_\eta\phi \tag{2.54}$$

For the first term of the right side of (2.54) we can write:

$$\sqrt{\vartheta}\partial_0\partial_\eta\phi = \partial_o(\sqrt{\vartheta}\partial_\eta\phi) - (\partial_0\sqrt{\vartheta})(\partial_\eta\phi) \tag{2.55}$$

Hence we can observe how a surface term appears, due the presence of a total time derivative. For the second term of the righ side of (2.54), observing that we have $\partial_\eta\phi = \eta^\mu\partial_\mu\phi$ and then $D_i\partial_\eta\phi = \partial_i\partial_\eta\phi$, we have:

$$\sqrt{\vartheta}S^i\partial_i\partial_\eta\phi = \sqrt{\vartheta}D_i(S^i\partial_\eta\phi) - \sqrt{\vartheta}(D_iS^i)(\partial_\eta\phi \tag{2.56}$$

First term of the right side of the above formula is a surface term, too, due to Gauss theorem in curved space. Hence we have:

- KK-ADM $\quad N\sqrt{\vartheta}\,\partial_\eta\partial_\eta\phi = \sqrt{\vartheta}\,(D_iS^i)(\partial_\eta\phi) - (\partial_o\sqrt{\vartheta})(\partial_\eta\phi) + S.T.$

It follows that, apart from surface term, the two Lagrangians are equivalent. Therefore, it means that, adopting suitable boundary conditions, both ADM-KK and KK-ADM procedures lead to the same dynamics. Such a scenario leads to the following conclusions:

- We can consider the reduction of the dynamics regardless the ordering of the procedures. As far as the physical outcomes of the model are concerned the two procedures commute.
- The conversion formulas (2.42) have a real physical meaning which deserve some effort to be pursued in order to achieve a better understanding.
- Being the Lagrangians equivalent, we are have freedom in the choice of the starting Lagrangian to perform the Hamiltonian formulation.

2.2.5 The role of cylindricity hypothesis

Two last issues yet deserve to be investigated, in order to provide an enforcement about the proof of the commutation between the ADM-KK and KK-ADM procedures.

- Although the observations we discussed at the beginning of this section were suggesting us to expect a discrepancy in the outcomes concerning our procedures, we found them to be equivalent, regardless the fact that the extra-dimension were involved, or not involved, in the slicing. Therefore we would like to understand how this fact does not play a role in the final dynamics.
- Although the adopted slicing procedures lead to the same dynamics, in principle two different definitions of time are employed; this can be recognized by looking at formula (2.4) , which depends on the number of *Shift* functions and therefore depends on the number of spatial dimension. Hence, it is worth performing a final check about the equivalence of these two definitions.

As we will see soon, there is an interplay between these issues.

Space-ambient embedding

To perform this analysis, we would like to employ the so-called "space-ambient embedding " picture of a curved manifold; then we now review briefly the main features of this technique, which can be considered like a simplified version of the Cartan formalism ([83]). Let us consider a Minkowskian vectorial space, $\mathcal{V}^{n+1}$, $n+1$-dimensional, and its canonical basis of orthonormal vectors provided by the set $\{\overrightarrow{n}_i\}$;

- $\{\overrightarrow{n}_i\}$ i=0,1,...,n
- $\overrightarrow{n}_i \cdot \overrightarrow{n}_j = \eta_{ij}$

where η_{ij} is the Minkowsky matrix. In the following we will refer to $\mathcal{V}^{n+1}$ as the "Space-Ambient". A generic manifold $\mathcal{M}^n$, n-dimensional and C^∞, embedded in the Space-Ambient can be described by a parameterization such as

$$\overrightarrow{Y} = \overrightarrow{Y}(u^\alpha) = Y^i(u^\alpha)\overrightarrow{n}_i\,,$$

where $\overrightarrow{Y}$ belongs to $\mathcal{V}^{n+1}$ and the n parameters $u^\alpha(\alpha = 0, 1, ..., n-1)$ define local coordinates on the manifold. For an assigned parameterization, the hyperplane tangent on the manifold is defined, point by point, through the n tangent vectors

$$\overrightarrow{f}_\alpha = \frac{\partial \overrightarrow{Y}}{\partial u^\alpha} = \frac{\partial Y^i}{\partial u^\alpha}\overrightarrow{n}_i\,.$$

We simply say that a vector belonging to $\mathcal{M}^n$ is a vector belonging to such a hyperplane. In this way we can employ the usual vector representation and have a local basis for the manifold. Hence, for a manifold's vector we have two possible representations:

- $\overrightarrow{V} = V^\alpha \overrightarrow{f}_\alpha$
- $\overrightarrow{V} = V^i \overrightarrow{n}_i$

where $V^i = V^\alpha \frac{\partial Y^i}{\partial u^\alpha}$. It can be shown that the components V^α are controvariant for a change of the parameterization (that is the same of a coordinates transformation). Moreover, we can define the components of the metric tensor of the manifold as the scalar products of local basis vectors

$$g_{\alpha\beta} = \overrightarrow{f}_\alpha \cdot \overrightarrow{f}_\beta \tag{2.57}$$

The covariant component of a vector can also be defined in terms of scalar product and we have

$$V_\alpha = g_{\alpha\beta} V^\beta = \overrightarrow{V} \cdot \overrightarrow{f}_\alpha \tag{2.58}$$

Within this picture we can easily rebuild all the typical relationships of General Relativity, like covariant derivation rules and so on. The most relevant advantage we take from this picture, is that it allows us to keep a vector representation also in curved space.

Equivalence of time definitions

Let us consider now a 5D manifold, embedded in a 6D Minkowskian space-ambient, with a set of five basis vectors $\{\overrightarrow{f}_\mu, \overrightarrow{f}_5\}$.

In the ADM-KK procedure we perform a complete space-time slicing providing four space-like vectors and a normal time-like one, plus a time deformation vectors:

$$\begin{cases} \overrightarrow{f}_0 = N\overrightarrow{\eta} + N_i \overrightarrow{f}_i + N_5 \overrightarrow{f}_5 \\ \overrightarrow{\eta} \cdot \overrightarrow{f}_i \equiv 0 \\ \overrightarrow{\eta} \cdot \overrightarrow{f}_5 \equiv 0 \end{cases} \tag{2.59}$$

In the KK-ADM procedure we first perform KK reduction; hence, according to (2.57), the basis vectors must satisfy the following relations:

$$\begin{cases} \overrightarrow{f}_5 \cdot \overrightarrow{f}_5 = -\phi^2 \\ \\ \overrightarrow{f}_5 \cdot \overrightarrow{f}_\mu = -(ek)\phi^2 A_\mu \\ \\ \overrightarrow{f}_\mu \cdot \overrightarrow{f}_\nu = g_{\mu\nu} - (ek)^2\phi^2 A_\mu A_\nu \end{cases} \tag{2.60}$$

Moreover, according to KK allowable diffeomorphisms, we have the following transformation rule:

$$\begin{cases} \overrightarrow{f}_5{}' = \overrightarrow{f}_5 \\ \\ \overrightarrow{f}_\mu{}' = \overrightarrow{f}_\mu \frac{\partial x^\mu}{\partial x^{\mu'}} + \frac{d\psi}{\partial x^{\mu'}} \end{cases} \tag{2.61}$$

At this point, in KK-ADM procedure we have to perform the splitting of the 4-D space-time starting by the four vector $\overrightarrow{f}_\mu$ and defining a new set of three space-like vectors and its normal time-like one;

$$\{\overrightarrow{f}_\mu\} \rightarrow \{\overrightarrow{e}_i, \overrightarrow{\eta}\}$$
$$\begin{cases} \overrightarrow{e}_0 = N\overrightarrow{\eta} + S_i \overrightarrow{e}_i \\ \overrightarrow{\eta} \cdot \overrightarrow{e}_i \equiv 0 \end{cases} \tag{2.62}$$

In this case we have in general $\overrightarrow{f}_5 \cdot \overrightarrow{\eta} \neq 0$. So, at a first glance, in our procedures we have two different relations of orthogonality between $\overrightarrow{\eta}$ and $\overrightarrow{f}_5$; these conditions are scalar, and, furthermore, as we see in the first of (2.61), $\overrightarrow{f}_5$ sets a preferred direction in space, as a direct consequence of the cylindricity hypothesis. Finally, we have also two different definitions of the time deformation vectors that could lead to different time variables and time derivatives. Thus, the space-ambient picture shows in a striking way the problems that made us doubt about the consistency of ADM reformulation. But let us continue such an analysis. We can see that, in the KK-ADM procedure, as we presented it at this point, the ADM splitting is not well defined: we turn our attention to $\overrightarrow{e}_i$;

$$\overrightarrow{e}_i = e_i^\mu \overrightarrow{f}_\mu$$

As we can see by transformation rules (2.61) a change in the local coordinates also change the basis vector , because of the presence of the gauge-terms. In order to have consistency we cannot do the splitting by starting from the vectors $\overrightarrow{f}_\mu$. Consider now the third equation of the (2.60); we define two new sets of vectors satisfying the following relations:.

- $\overrightarrow{f}_\mu = \overrightarrow{u}_\mu + \overrightarrow{a}_\mu$

- $\overrightarrow{u}_\mu \cdot \overrightarrow{a}_\nu \equiv 0$
- $\overrightarrow{u}_\mu \cdot \overrightarrow{u}_\nu = g_{\mu\nu}$
- $\overrightarrow{a}_\mu \cdot \overrightarrow{a}_\nu = -\phi^2(ek)^2 A_\mu A_\nu$

These definitions still agree with the third equation of (2.60) but now the vectors $\overrightarrow{u}_\mu$ doesn't have gauge-components. By this re-definition for $\overrightarrow{f}_\mu$, consider now the second of (2.60). We can assume as true the condition

$$\overrightarrow{f}_5 \cdot \overrightarrow{u}_\mu \equiv 0 \tag{2.63}$$

Indeed, if this condition were not true we would have a pure vectorial term that, really, is not present in our equation. The set $\{\overrightarrow{u}_\mu\}$ defines moreover the real space-time metrics so that it is correct to do with this set the ADM splitting. If we define $\overrightarrow{\eta}$ from this set we have, by (2.63) the condition $\overrightarrow{\eta} \cdot \overrightarrow{f}_5 \equiv 0$. Hence, while the cylindricity conditions sets the existence of a preferred direction, is the cylindricity itself that always sets $\overrightarrow{f}_5 \cdot \overrightarrow{\eta} = 0$. Moreover, as a final remark, let's examine the time deformation vector and the time derivative that arise from it in both procedure; we have:

- ADM-KK $\overrightarrow{f}_0 = N\overrightarrow{\eta} + N_i \overrightarrow{f}_i + N_5 \overrightarrow{f}_5 \Rightarrow \partial_0 = N\partial\eta + N^i\partial_i + N^5\partial_5$
- KK-ADM $\overrightarrow{e}_0 = N\overrightarrow{\eta} + S_i \overrightarrow{e}_i \Rightarrow \partial_0 = N\partial\eta + S^i\partial_i$

As we can see, reminding that from conversion formulas we have $N^i = S^i$ and observing that from cylindricity ∂_5 has no effect , we have the equivalence of the two definitions.

2.3 Hamiltonian formulation of the dynamics

Starting from the Lagrangian we have provided , by Legendre's transformation, we can calculate the Hamiltonian for the KK 5D model (Lacquaniti-Montani '06 [136], [137]) . The algebraic work underlying this task it is not quite simple, and present some difficulties, especially as far as the inversion of conjugate momenta with respect to lagrangian velocities is considered. Anyway, we solved the Legendre transformations, via some mathematical technicalities, which do not have, however, a particular physical relevance. Here we provide and discuss the final outcome, which mimics, in some way, the outcome we have for the hamiltonian formulation of GR. Indeed, the conjugates momenta to N, S_i and A_0 are identically zero: the real dynamics is carried out by the spatial metrics ϑ_{ij}, as in the 4D case, and by A_i and ϕ. The Hamiltonian function reads :

$$\mathcal{H} = NH^N + S_i H^i + A_0 H^0 \tag{2.64}$$

We have:

$$\begin{aligned} H^N &= b\sqrt{\vartheta}\phi R - 2b\sqrt{\vartheta}D^i\partial_i\phi - \frac{1}{2b\sqrt{\vartheta}\phi}T_{ijkl}\Sigma^{ij}\Sigma^{kl} - \frac{1}{6b\sqrt{\vartheta}}\pi_\phi^2\phi + \\ &+ \frac{1}{3b\sqrt{\vartheta}}\pi_\phi\Sigma^{ij}\vartheta_{ij} - \frac{1}{4}b\sqrt{\vartheta}(ek)^2\phi^3F_{ij}F^{ij} - \frac{2}{b\sqrt{\vartheta}(ek)^2\phi^3}\pi^i\pi^j\vartheta_{ij} \qquad (2.65) \\ H^i &= -2D_j\Sigma^{ij} + \pi_\phi\partial^i\phi - \pi^j F^i{}_j \qquad (2.66) \\ H^0 &= -D_i\pi^i \qquad (2.67) \end{aligned}$$

where $\Sigma^{ij}, \pi^i, \pi_\phi$ are the conjugates momenta to $\vartheta_{ij}, A_i, \phi$ respectively and T_{ijkl} is defined as follows:

$$T_{ijkl} = (-\frac{2}{3}\vartheta_{ij}\vartheta_{kl} + \vartheta_{ik}\vartheta_{jl} + \vartheta_{il}\vartheta_{jk}) \tag{2.68}$$

As we have seen in the 4D case, N, S_i, A_0 are not true dynamical variables but they play the role of Lagrange's multipliers, providing the following constraints on the dynamics:

$$H^N \equiv H^i \equiv H^0 \equiv 0$$

These constraints can be used instead of the complete equations for the study of the dynamics. Really, a remarkable feature of this Hamiltonian is given by the relation we have seen between A_0 and N_5 (2.42). Indeed, before the symmetry breaking due to KK conditions, we have a Lagrangian and an Hamiltonian analogues the those of the 4D theory, except for the dimension and the number of *Shift* functions, which provide constraints for the dynamics, due to the diffeomorphism invariance. After the KK reduction, we have the electromagnetism coupling and the presence of the constraint related to A_0, due to gauge-invariance. Hence, the relation between A_0 and N_5 shows how the electromagnetic constraint arises as a particular case of the diffeomorphism invariance. Moreover this mean that the hamiltonian formalism we have developed holds at the same way before and after the KK symmetry breaking and can itself take into account the alteration provided by the KK conditions.

Another feature deserves to be remarked; let us define

$$\tilde{G}_{ijkl} = \frac{1}{2\sqrt{\vartheta}} T_{ijkl} \, .$$

$\tilde{G}_{ijkl}$ plays the role of the Supermetrics we defined in (2.1.3); but, as we can see by comparing them, there is difference for a numerical factor. This effects is due to the presence of the conjugate momentum π_ϕ which affects the procedure of inversion of momenta with respect to lagrangian velocities. If we put $\phi = 1$ in the lagrangian, and hence we do not deal with its conjugate momentum, then we have $\tilde{G}_{ijkl} = G_{ijkl}$. Therefore, we can imagine such a discrepancy as an effect which can be in principle probed by observations in order to check the existence of the scalar field.

2.3.1 Canonical quantization

Let us examine now the canonical quantization. For sake of simplicity we consider the case of a vanishing electromagnetic field. In order to go from the classical equation $H^N = 0$ to the quantum one $\hat{H}^N \Psi = 0$, we simply upgrade the momenta to derivatives operators:

$$\begin{cases} \pi_\phi = -\hbar\frac{\delta}{\delta\phi} \\ \\ \Sigma^{ij} = -\hbar\frac{\delta}{\delta\vartheta_{ij}} \end{cases} \tag{2.69}$$

As it happens in the usual 4D theory, in the Wheeler-DeWitt equation, by the presence of the constraints, we do not have a time derivative and the resulting equation is not evolutionary. At this point of our analysis we should deal with models that try to overcome the problem of the frozen formalism. Actually, the proposal of a detailed scenario to solve the frozen formalism goes over the goal of this work. We just focus of some toy models, in order to extract some insight on the role of the scalar field and its possible role as a surrogate time variable from the viewpoint of the relational time formalism.

2.3.2 WKB approach

Let us now consider the semiclassical approach ([74],[84]); we start from the equation $\hat{H}^N\Psi = 0$ and assume for the wave function the following form:

$$\Psi = Me^{\frac{i}{\hbar}S}$$

Here M is the amplitude of the function and S is the Action. The quantum Hamiltonian reads

$$\hat{H}^N = \frac{\hbar^2}{2b\sqrt{\vartheta}\phi}T_{ijkl}\frac{\delta}{\delta\vartheta_{ij}}\frac{\delta}{\delta\vartheta_{kl}} + \frac{\hbar^2}{6b\sqrt{\vartheta}}\phi\frac{\delta^2}{\delta\phi^2} - \frac{\hbar^2}{3b\sqrt{\vartheta}}\vartheta_{ij}\frac{\delta}{\delta\phi}\frac{\delta}{\delta\vartheta_{ij}} + U(\phi,\vartheta_{ij})\,, \tag{2.70}$$

where $U(\phi,\vartheta_{ij})$ contains all terms without derivatives operators ; moreover we have chosen the most simple ordering for the derivatives operators, by not including the metrics fields in the derivation. Now, as required by the semiclassical approach, we write the complete equation; after this we separate the real part and the imaginary one, where in the real part we neglect terms quadratic in $\hbar$. Finally we get two equations:

Real part (zero-order in $\hbar$)

$$\frac{1}{3b\sqrt{\vartheta}}\vartheta_{ij}\frac{\delta S}{\delta\vartheta_{ij}}\frac{\delta S}{\delta\phi} - \frac{1}{6b\sqrt{\vartheta}}\phi(\frac{\delta S}{\delta\phi})^2 - \frac{1}{2b\sqrt{\vartheta}\phi}T_{ijkl}\frac{\delta S}{\delta\vartheta_{ij}}\frac{\delta S}{\delta\vartheta_{kl}} + U = 0 \tag{2.71}$$

Imaginary part (first order in $\hbar$)

$$-\frac{1}{3b\sqrt{\vartheta}}\left[\frac{\delta M}{\delta\vartheta_{ij}}\frac{\delta S}{\delta\phi} + \frac{\delta M}{\delta\phi}\frac{\delta S}{\delta\vartheta_{ij}} + M\frac{\delta}{\delta\phi}\frac{\delta}{\delta\vartheta_{ij}}S\right]\vartheta_{ij} + \frac{\phi}{6b\sqrt{\vartheta}}\left[2\frac{\delta M}{\delta\phi}\frac{\delta S}{\delta\phi} + M\frac{\delta^2 S}{\delta\phi^2}\right] \tag{2.72}$$

$$+\frac{1}{2b\phi\sqrt{\vartheta}}[2\frac{\delta M}{\delta\vartheta_{ij}}\frac{\delta S}{\delta\vartheta_{kl}} + M\frac{\delta}{\delta\vartheta_{ij}}\frac{\delta}{\delta\vartheta_{kl}}S]T_{ijkl} = 0 \tag{2.73}$$

The interpretation of these equations is not a simple task; in the first we could recognize the Hamilton-Jacobi reformulation of the classical dynamics, because it is exactly the classical Superhamiltonian H^N where the momenta are recasted in form of the Action derivatives. However we do not have the term $\frac{\partial S}{\partial t}$ that we must have in such an equation, because of the not evolutionarity of the dynamics. As an arbitrary hypothesis, let us suppose now that terms containing $\frac{\partial s}{\partial\phi}$ are very small. Then, in the equation for the real part we neglect the term of second order in $\frac{\partial s}{\partial\phi}$; in the equation for the imaginary part we put $\frac{\partial s}{\partial\phi} = 0$ and multiply for $2M$. We get:

$$\frac{1}{3b\sqrt{\vartheta}}\Sigma^{ij}\vartheta_{ij}\frac{\delta}{\delta\phi}M^2 - \frac{1}{b\sqrt{\vartheta}\phi}T_{ijkl}\frac{\delta}{\delta\vartheta_{ij}}[M^2\Sigma^{kl}] = 0 \tag{2.74}$$

$$\frac{1}{3b\sqrt{\vartheta}}\Sigma^{ij}\vartheta_{ij}\frac{\delta}{\delta\phi}S - \frac{1}{2b\sqrt{\vartheta}\phi}T_{ijkl}\Sigma^{ij}\Sigma^{kl} + U = 0 \tag{2.75}$$

Now we define a new operator:

$$\frac{\partial}{\partial t*} = \frac{1}{3b\sqrt{\vartheta}}\Sigma^{ij}\vartheta_{ij}\frac{\delta}{\delta\phi}$$

If this new operator could be viewed as a time-evolution operator, in the first equation we could recognize a continuity equation, and the second becomes the HJ equation for the gravitational part of the Hamiltonian. If the identification of this time-operator were correct we could rebuild a Schrodinger-like equation for the gravitational part of the dynamics. Of course, the dynamics it is not so simple, and our hypotheses, at this step of our study are arbitraries. However it can be suggested that the idea of using ϕ as a surrogate time-variables is not a simple artifice. A more detailed study in this view, starting from the Hamiltonian we have provided, seems to be interesting.

2.3.3 Deparametrization of the scalar field

In order to better explore a viable definition of the scalar field as a suitable relational time variable we now face another scenario (Lacquaniti-Montani-Zonetti '08 [142]) which deals with the Brown-Kuchař (BK) approach to the problem of time.

Time before the quantization

The idea of Brown-Kuchař ([78]) lies in the generic framework of those models which try to a achieve a time variable before the quantization procedure ([75]). Such scenarios need the addition of some extra degrees of freedom in the dynamics, usually addressed to the presence of matter or scalar fields. The common procedure of such models is the following:

- the constraints are solved classically,
- a functional T of the configuration variables is identified with the time parameter,
- the Hamiltonian associated with T is quantized.

A key point of these models is the proper choice of the time-functional. Indeed, two facts should be considered. At first, the definition of the time functional is not unique because the separation between the time and other variables can be obtained adopting different procedures ; moreover the time functional should satisfy some physical requests :

- T must be a monotonically increasing function, at least locally.
- After the quantization, T has to provide a well-defined and conserved probability density.

A suitable procedure is the above cited BK approach. The procedure ([78], [85], [86]) is based on the introduction of a dust, whose world-line identifies a preferred time-like direction without violating General Covariance. This direction plays the role of time. In terms of constraints, the super-Hamiltonian and the super-momentum are therefore modified by terms due to the matter field, i.e.

$$\mathcal{H}^{\mathcal{N}\prime} = \mathcal{H}^{\mathcal{N}} + \mathcal{H}^{D} \qquad \mathcal{H}_i' = \mathcal{H}_i + \mathcal{H}_i^{D}. \tag{2.76}$$

Brown and Kuchař demonstrated that, by using the new super-momentum constraint, the super-Hamiltonian can be rewritten as follows

$$\pi + h(h_{ij}; \pi^{ij}) = 0\,. \tag{2.77}$$

T and π are the proper time of dust flow lines and its conjugate momentum, respectively, while for h they got the expression

$$h = -\sqrt{G} \qquad G = \mathcal{H}^{\mathcal{N}2} - h^{ij}\mathcal{H}_i\mathcal{H}_j. \tag{2.78}$$

Therefore, by taking P as a time parameter, an evolution described by the hamiltonian h follows. This hamiltonian turns out to be positive-definite and, starting from the corresponding Schrodinger equation, a quantum description for the system can be given, together with a definition for an inner product (but it could be only formal).
The BK approach relays on the dualism existing between time and matter ([87], [81]) in GR. In this respect, the properties which a matter field should have to be a good clock, are still under investigation.
The Brown-Kuchař approach can be generalized , and in principle it is possible to consider models where the extra degree of freedom we need to define the time functional is searched among different kinds of matter or scalar fields ([88], [89], [90], [91], [92]). Therefore, with respect to this approach it is worth to analyze the role of the field ϕ which is naturally provided by KK model.

Brown-Kuchař approach in KK model

Let us consider again the KK Hamiltonian we provide at the beginning of sec. (2.3), together with the constraint relations; we wrote:

$$H = NH^N + S_i H^i + A_0 H^0 \,, \tag{2.79}$$

where

$$\begin{aligned} H^N &= b\sqrt{\theta}\phi R - 2b\sqrt{\theta}D^i\partial_i\phi - \frac{T_{ijkl}\Sigma^{ij}\Sigma^{kl}}{2b\sqrt{\theta}\phi} - \frac{\phi\pi_\phi^2}{6b\sqrt{\theta}} + & (2.80)\\ &+ \frac{\pi_\phi\Sigma_{ij}\theta^{ij}}{3b\sqrt{\theta}} - \frac{1}{4}b\sqrt{\theta}(ek)^2\phi^3F_{ij}F^{ij} - \frac{2\pi^i\pi^j\theta_{ij}}{b\sqrt{\theta}(ek)^2\phi^3} & (2.81)\\ H^i &= -2D_j\Sigma^{ij} + \pi_\phi\partial^i\phi - \pi^j F^i_j & (2.82)\\ H^0 &= -D_i\pi^i & (2.83)\end{aligned}$$

In this section too, we assume to work in a scenario where the electromagnetic variables and their conjugate momenta are vanishing. Defining the auxiliary variables $T_{ijkl}\Sigma^{ij}\Sigma^{kl} = T$ and $\Sigma_{ij}\theta^{ij} = \Sigma$, and omitting the subscript ϕ in the momentum π_ϕ we have:

$$\begin{aligned} H^N &= b\sqrt{\theta}\phi R - 2b\sqrt{\theta}D^i\partial_i\phi - \frac{T}{2b\sqrt{\theta}\phi} - \frac{\phi\pi^2}{6b\sqrt{\theta}} + \frac{\pi\Sigma}{3b\sqrt{\theta}} & (2.84)\\ H^i &= -2D_j\Sigma^{ij} + \pi\partial^i\phi & (2.85)\end{aligned}$$

Via the eq. (2.85) we can rule out the spatial gradient of ϕ from eq. (2.84); then, multiplying by $b\sqrt{\theta}$ we get a new constraint equivalent to the superhamiltonian one $H^N = 0$, i.e.

$$\tilde{H}^N[x] = b^2\theta\phi R[x] - 2b^2\theta D_i\left(\frac{H_i}{\pi}\right)[x] - \frac{T}{2\phi}[x] - \frac{\pi^2\phi}{6}[x] + \frac{\pi\Sigma}{3}[x] = 0 \tag{2.86}$$

Now, we upgrade $\tilde{H}^N$ to an operator acting on some functions space; therefore the constraint becomes $\tilde{H}^N(f) = 0$ for any given function $f(x)$. Thereafter we consider its integral on a probe function f; in such a way, using integration by parts, we can shift the derivative operator D_i to such a function:

$$\tilde{H}^N(f) = \int d^3x \left[\left(b^2\theta\phi R - \frac{T}{2\phi} - \frac{\pi^2\phi}{6} + \frac{\pi\Sigma}{3} \right) [x] f[x] + 2b^2\theta \left(\frac{H_i}{\pi} \right) [x] D_i f[x] \right] = \tilde{H}^N \cdot f = 0 \tag{2.87}$$

Hence, if the operator $\tilde{H}$ yields zero on every f, then also the operator $\pi\tilde{H}$ gives the same result. Thus we can multiply within the integral by π and perform again an integration by parts:

$$\int d^3x \left[\pi \left(b^2\theta\phi R - \frac{T}{2\phi} - \frac{\pi^2\phi}{6} + \frac{\pi\Sigma}{3} \right) - 2b^2\theta \left(D^i H_i \right) \right] [x] f[x] = 0 \tag{2.88}$$

Therefore, the operator in square brackets within the integral is equivalent to the Superhamiltonian. At this step we can now try to solve with respect to π. There exist three solutions, but only the following one does not contain imaginary parts:

$$\pi = \frac{1}{3\phi} \left(-2\Sigma + \frac{\Xi}{\phi \left(\Gamma \pm \sqrt{\Xi^3 + \Gamma^2} \right)^{1/3}} - \phi \left(\Gamma \pm \sqrt{\Xi^3 + \Gamma^2} \right)^{1/3} \right) \tag{2.89}$$

where:

$$\begin{aligned} \Xi &= -9T - 4\Sigma^2 + 18b^2 R\theta\phi^2 = \\ &= 2\Sigma^2 - 18 \left(\Sigma_{ij}\Sigma^{ij} + b^2 R\theta\phi^2 \right) \\ \Gamma &= 27T\Sigma + 8\Sigma^3 - 54b^2\theta(3D^i H_i + R\Sigma)\phi^2 = \\ &= -10\Sigma^3 + 54 \left(\Sigma_{ij}\Sigma^{ij}\Sigma - b^2\theta\phi^2(3D^i H_i + R\Sigma^2) \right) \end{aligned} \tag{2.90}$$

It is worth noting now that the new constraint (2.89) fulfils the first request needed by the BK procedure: the member on the right side does not depend on spatial derivatives of ϕ, which is indeed the field conjugate to π. Clearly, to claim that this procedure yields a successful BK scheme other checks are needed: it must be shown whether the constraint here derived satisfies the same algebraic properties of the Superhamiltonian and, together with the others constraints, act as a generator for diffeomorphisms (this is actually an expected result), and moreover we have to study the sign of the new Hamiltonian and the probability flow induced by π. The fact , however, that the field ϕ deparametrizes in such a way that it leads to the relation (2.89) - which was anyway a not so guaranteed result - is an interesting issue that deserves further investigations and represents, indeed, the first step in the definition of a well defined BK scheme in the Kaluza-Klein model.

2.4 Homogeneous solutions

Now we study the motion equation for the fields in the most simple scenarios: homogeneous background. The Lagrangian in the ADM reformulation, which we have previously discussed will be very useful to implement the homogeneity conditions. Then we start from:

$$\begin{aligned}\mathcal{L}_{KK-ADM} &= b\sqrt{\vartheta}\,N\phi(K^2 - K_{ij}K^{ij} - {}^3R) + 2b\sqrt{\vartheta}\,N(D_i\partial^i\phi - \partial_\eta\partial_\eta\phi) + \\ &+\frac{b}{4}\sqrt{\vartheta}\,N(ek)^2\phi^3(F_{ij}{}^3F^{ij} - \frac{2}{N^2}M_iM^i) \end{aligned} \tag{2.91}$$

Now we are going to make some assumptions, that are reasonable at a cosmological scale and that will simplify our model.

- We omit the electromagnetic term: at a cosmological scale we don not measure great electromagnetic field.
- Because of the spatial homogeneity and isotropy of the universe at a large scale we can put ourselves in a synchronous system ([93]): this mean that we can put $S^i = 0$ and $N = 1$.
- At a same way, for sake of simplicity, we assume the flatness of universe, as suggested by recent observations ([94]), and we put ${}^3R = 0$.

Moreover, by the hypothesis $N = 1$ the term containing spatial derivatives of ϕ becomes a surface term so we can omit it. Finally, the last two hypotheses allow us to write the spatial metrics as a Minkowskyan conformal metrics; hence we have

$$\vartheta_{ij} = a(t)^2 \delta_{ij}$$

where $a(t)$ is the scale factor and δ_{ij} the 3-d identity matrix. With this form for the spatial metrics and by using the other hypotheses is very simple to calculate the explicit expression for the extrinsic curvature and the Lagrangian takes the following expression:

$$\mathcal{L} = 6b(\phi \dot{a}^2 a + \dot{\phi} a^2 \dot{a}) \tag{2.92}$$

We will refer to this as the Friedmann-like model. The corresponding Eulero-Lagrange equations are:

$$\begin{cases} \frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = 0 \\ \\ \frac{\partial^2}{\partial t^2}(\phi a) = 0 \end{cases} \tag{2.93}$$

The first is the same equation for the RDE case of the 4D Friedmann universe, while the second show the link we have between ϕ and the scale factor. We can easily solve these equations and get the following solutions:

$$\begin{cases} a(t) = a_0 t^{\frac{1}{2}} \\ \\ \phi = \beta t^{\frac{1}{2}} + \gamma t^{-\frac{1}{2}} \end{cases} \tag{2.94}$$

Here β and γ unknown constants. We don't know the initial condition of the evolution, so we cannot find the value of the constants; however, we can see the possible existence of a Kasner's solution for a collapsing extra-dimension.

We can also consider an alternative model, by adding a cosmological constant term in the Lagrangian, so our model becomes a DeSitter-like one. Due to the introduction af a cosmological term ($\Lambda > 0$) the Lagrangian takes the expression

$$\mathcal{L} = 6b(\phi \dot{a}^2 a + \dot{\phi} a^2 \dot{a} + a^3 \phi \Lambda) \tag{2.95}$$

and we have the following equation:

$$\begin{cases} \frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 = \Lambda \\ \\ \frac{\partial^2}{\partial t^2}(\phi a) = \phi a \Lambda \end{cases} \tag{2.96}$$

Now the scale factor admits an exponential solution , increasing in time, as we require for our expanding universe:

$$a(t) = a_0 e^{\sqrt{\frac{\Lambda}{2}}\, t}$$

Scalar field too admits an exponential solutions but now we have no reason to consider only the solution increasing in time; then we have two possible kinds of solution:

$$\phi = \beta \frac{e^{-\sqrt{\Lambda}\, t}}{e^{\sqrt{\frac{\Lambda}{2}}\, t}} \tag{2.97}$$

$$\phi = \beta \frac{e^{\sqrt{\Lambda}\, t}}{e^{\sqrt{\frac{\Lambda}{2}}\, t}} \tag{2.98}$$

Hence ϕ admits increasing and decreasing exponential solutions . However the most remarkable feature of this study is that ϕ is strongly linked to the scalar factor and, indeed, they have a similar behaviours in both Friedmaan or Desitter like model. Moreover the exponential behaviour is typical of the inflationary theory so it seems reasonable that ϕ can have this feature during the inflation, leading to a great expansion/contraction for the extra-dimension. Clearly we have seen only vacuum solutions; the introduction of matter can lead us to find other solutions and learn more about the features of the scalar field. Anyway, the possibility that, as well as the scale factor $a(t)$, the field ϕ plays the role of of a cosmological clock for the universe is an issue that deserves further investigations.

Chapter 3

Geodesic approach to motion

In the first chapter we gave a review of the Kaluza-Klein approach, just focusing on the dynamics we have for metric fields in vacuum. Indeed, the dimensional reduction procedure we discussed is a well understood issue only without presence of matter. Actually, the description of matter dynamics, especially as far as test particles are involved, is a longstanding unsolved task in the framework of our compactified model. Two striking problems arise, which are indeed known from a long time ([95]): from the viewpoint of a classical analysis of the motion, when we consider the reduction of a 5D free test particle into an interacting particle, we have to deal with an inconsistent definition for the coupling factor of the particle, namely the charge-mass ratio, which results to be upper bounded in such a way that any kind of particle could satisfy this bound; from the point of view of a quantum analysis, namely a study of the Klein-Gordon dynamics, we find a tower of huge massive modes, beyond the Planck scale. These features are very unsatisfactory from a physical view point and they lead to a well-grounded criticism versus the Kaluza-Klein model. In more details, we have that these outcomes are strictly linked to the cylindricity and compactification hypotheses; therefore the main part of such a criticism is addressed to such assumptions, and various effort, in order to solve the matter puzzle, lead to model where the original KK hypotheses are in some way changed or relaxed. In this work, we will try to add a contribute to such a debate, and this will be the issue discussed in the next chapter. To develop all the tools needed for such a discussion, here we review the standard approach to motion in KK. In the first section we review the geodesic dimensional reduction, and discuss the outcome concerning the problem of the q/m. In the following sections we perform a deeper analysis of the geodesic approach (Lacquaniti-Montani '08 [139], [138]), grounded on the hamiltonian formulation we developed in the previous chapter. This will allow us to show how the q/m puzzle is strictly linked to the tower of huge massive modes and to extract useful insight for the following of the work.

3.1 Reduction of the 5D geodesic equation

In the standard GR theory, the motion of a test particles is described following the geodesic approach. At first, granted the Equivalence Principle, we assume the Free Falling Universality (FFU): all particles in free fall within the gravitational field follow the same trajectory, namely the geodesic trajectory. Therefore we assume that the motion of the particle is governed by an Action proportional to the geodesic Action:

$$S_{particle} \propto \int ds$$

The resulting equation, $\frac{D}{Ds}u^{\mu} = 0$, does not depend on the mass, as requested by the FFU condition. In order to fix the proportionality constant, and therefore to rigourously define the Action $S_{particle}$, we need a further ansatz; we assume that such a constant parameter (in $c = 1$ units) is given, apart from the algebraic sign, by the mass m of the particle: by doing this we are able to reproduce the dispersion relation $P_{\mu}P^{\mu} = m^2$ which is known from special relativity. Hence, the Action reads:

$$S_{particle} = -m \int ds$$

Now, we face the problem of motion in the 5D KK model. The aim is to reproduce the dynamics of a 4D particle interacting with the electromagnetic field (plus, eventually, terms due to the presence of the scalar field). The usual approach consists in the generalization of the above described procedure: the Action for a free test particle is provided, apart from a constant factor, by the infinitesimal line-element ds_5 . In the following, we will refer to this procedure as the geodesic approach. Therefore, our ansatz for the Action of a 5D free test particle gives us:

$$S_5 = -\hat{m} \int ds_5 \tag{3.1}$$

Here $\hat{m}$ is an unknown mass parameter whose physical meaning has to fixed by analyzing the outcomes of the procedure. The KK reduction of ds_5, as we have seen in formula (1.7), gives the following formula:

$$ds_5^2 = ds^2 - \phi^2(ekA_\mu dx^\mu + dx^5)^2 \tag{3.2}$$

Hence, is easily recognized that the 5D Action contains a term related to a free 4D particle, ds^2, plus interacting terms, and we expect that, via the dimensional reduction, our 5D test particle would become a 4D interacting particle. Moreover, due to the presence of the Killing vector $(0, 0, 0, 0, 1)$, provided by the cylindricity hypothesis, we expect to find a conserved quantity, which would play a role in the definition of coupling factors. In order to get the equation of motion of the reduced particle, we have to insert the reduced line element into the our Action and then perform the variational procedure. Actually, this is not the only procedure we could consider. A valuable alternative way is to consider at first the 5D motion equation that arises from (3.1): it is nothing else than the 5D geodesic equation

$$\frac{{}^5D}{Ds_5} w^A = 0$$

where ${}^5D$ is the 5D covariant derivative and we defined $w^A = \frac{dx^A}{ds_5}$. Given the above equation we could consider its dimensional reduction via the 5-bein projection technique. Anyway, these two procedure give the same result; in the following we will refer to the variational procedure. Hence, after variational procedure, we have the following Eulero-Lagrange equations:

$$\begin{cases} \frac{d}{ds_5} w_5 = 0 \\ \frac{d}{ds_5} w^\mu + \Gamma^\mu_{\lambda\nu} w^\lambda w^\nu = ekg^{\mu\nu} F_{\nu\lambda} w^\lambda w_5 - \frac{1}{2} g^{\mu\nu} \frac{\partial_\nu \phi^2}{\phi^4} w_5^2 \end{cases} \tag{3.3}$$

In these expressions $\Gamma^\mu_{\lambda\nu}$ is the Chrystoffel symbols related to the ordinary metrics $g_{\mu\nu}$ and $w^A = (w^\mu, w^5) = \frac{dx^A}{ds_5}$ is the 5D velocity as previously defined.

The first equation of (3.3) defines our expected constant of motion. Explicitly, it reads as follows:

$$w_5 = -\phi^2 (ekA_\mu w^\mu + w^5) \tag{3.4}$$

A direct calculus would show that w_5 is a scalar with respect to KK transformations[1]. Hence, the KK conditions allow us to identify a kinematical quantity that is scalar and conserved.

Now, let us examine the second equation of (3.3). If we set, for hypothesis, $\phi = 1$ and define the charge to be proportional to w_5, our equation becomes very similar to the Lorentz equation. Actually, we cannot do this comparison because in our equation we are still dealing with the 5D line element, ds_5, instead of the 4D one, ds, and with w^μ: they are the space-time components of the 5D velocities but they are not the components of the 4D velocity . Therefore, we need to address the reparametrization from ds_5 to the ordinary line-element ds. Labelling by α the reparametrization factor, we have:

$$\alpha = \frac{ds}{ds_5} \tag{3.5}$$

$$w^\mu = \alpha u^\mu \tag{3.6}$$

where $u^\mu = \frac{dx^\mu}{ds}$ is now the proper 4D velocity.

[1]Actually, this is a generic feature of KK model: for a given vector in covariant representation its fifth component is always a scalar

Now we evaluate explicitly the factor α. By using the definition of ds, from (3.5) we get

$$\alpha = \sqrt{g_{\mu\nu} w^\mu w^\nu} \tag{3.7}$$

Moreover, by definition itself of w^A, we must have the constraint $J_{AB} w^A w^B = 1$; by using the explicit expression of J_{AB} and the formula (3.4), our constraint becomes

$$g_{\mu\nu} w^\nu w^\mu = 1 + \frac{w_5^2}{\phi^2} \tag{3.8}$$

Therefore, recalling (3.7), we get the expression for α

$$\alpha = \sqrt{1 + \frac{w_5^2}{\phi^2}} \tag{3.9}$$

Alternatively, we could use directly equation (1.7), together with (3.4), to get the same result. We note that formula (3.9), together with (3.6) , while inserted in (3.8) yields

$$g_{\mu\nu} u^\nu u^\mu = 1$$

as it must be for the proper 4D velocity of a test particle.

Finally, we need to evaluate the term $\frac{d\alpha}{ds_5}$ arising from the term $\frac{d}{ds_5} w^\mu$ in (3.3). We have:

$$\frac{d\alpha}{ds_5} = -\frac{1}{2\phi^4} w_5^2 u^\mu \partial_\mu \phi^2 \tag{3.10}$$

Now, provided the formulas (3.6, 3.10), the second of equations (3.3) becomes

$$\frac{D}{Ds} u^\mu = ekF^{\mu\nu} u_\nu \left(\frac{w_5}{\sqrt{1 + \frac{w_5^2}{\phi^2}}} \right) + \frac{1}{\phi^3} (u^\mu u^\nu - g^{\mu\nu}) \partial_\nu \phi \left(\frac{w_5^2}{\sqrt{1 + \frac{w_5^2}{\phi^2}}^2} \right) \tag{3.11}$$

where $\frac{D}{Ds}$ now denotes the usual covariant derivative along the path, where we included Christoffel symbols.

This is the final expression for the motion equation and is now suitable for physical considerations.

As expected, our equation is suitable for the description of a 4D interacting test particle. It is expressed in terms of the 4D line element and the 4D velocity. Moreover, it matches with the usual request $g_{\mu\nu}u^{\nu}\frac{D}{Ds}u^{\mu} = 0$, as it is shown by a direct calculus. Our 4D particle interacts with vectorial field A_{μ} and scalar field ϕ and coupling factors are both defined by mean of the constant of motion w_5. Coupling factors of electrodynamical term and ϕ-dynamical term are proportional to $(\frac{w_5}{\alpha})$ and $(\frac{w_5}{\alpha})^2$ respectively. Indeed, this is a peculiarity of the geodesic approach: ϕ-dynamical term shows a coupling factor that is exactly the square of the elecrodynamical term. Thus, it looks like that in this model ϕ and A_{μ} are manifestations at different order of the same field, which is *a priori* a not justified feature. Anyway let us see now the main problem due to the definition of the electrodynamical coupling.

3.1.1 The charge-mass puzzle

We just focus on the electrodynamical term. Comparing our equation to the Lorenz one we define as follows the charge-mass ratio of our reduced particle:

$$\frac{q}{m} = ek\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}} \tag{3.12}$$

Let us consider now at first the scenario $\phi = 1$. This way, recalling also (1.13) we have:

$$\frac{q^2}{4Gm^2} = \frac{w_5^2}{1+w_5^2} \tag{3.13}$$

In principle, the scenario ϕ allows us to restore exactly the electrodynamics, because the ϕ-dynamical term vanishes and from the above definition we recover a conserved charge-mass ratio. But, looking at (3.13) we recognize that the ratio q/m is always upper bounded and we have:

$$\frac{q^2}{4Gm^2} < 1$$

Any known elementary particles does not satisfy such a bound. For instance, if we consider the electron (which is the most favourable case) we have $\frac{e^2}{4Gm_e^2} \simeq (10^{42})$. Therefore we have to deal with an inconsistent outcome. Let us consider now the general case, without restriction on the scalar field. This way, we deal with equation (3.12), which gives us, moreover, a q/m which is not conserved. The presence of ϕ modify the above discussed bound , which becomes:

$$\frac{q^2}{4Gm^2} < \phi^2$$

Hence, now is in principle possible to satisfy the bound by requiring $\phi > 10^{21}$. This appears actually a too strong constraint on the model. Strictly speaking, the numerical value of the scalar field alone has a little significance, without its comparison to the numerical value of the coordinate length $l_5 = \int dx^5$: the physical meaning has to be addressed to the physical length $L_5 = \phi \int dx^5$, where l_5 is, in principle, still a free parameter. But, by doing this , we need now a fine tuning on l_5, in order to fulfil the request of unobservability, i.e. the request of a compactification below the scale of our observational bound. Moreover, a huge scalar field does not seems the best case to work with, if we want to look for scenario where its effects become unobservable and we can just focus to the electrodynamical effects. Furthermore, there is another kind of criticism versus the presence of the scalar field: as noted by some authors ([96], [97], [98], [99]), we can resolve the presence of ϕ in the q/m ratio by defining a varying fine structure constant $\alpha(\phi)$. But, by doing this , considering for instance extended object bounded by electromagnetic interaction, we have that such a modified $\alpha(\phi)$ influences the definition of mass leading to a violation of the FFU. Therefore, this scenario too leads to inconsistent outcomes. Another scenario is given, as noted in eq. (1.14) by the possibility of including ϕ in the rescaling of the Newton constant. This allow us to define (in $c = 1$ units):

$$4G_{now} = (ek)^2 \phi^2_{now} \tag{3.14}$$

where G_{now} stands for the present (known) value of the Newton constant and ϕ_{now} for the (unknown) present value of the scalar field. Unfortunately, by doing this the q/m

puzzle is still unsolved because the bound now becomes:

$$\frac{q^2}{4G_{now}m^2} < 1$$

Therefore we still deal with an inconsistent outcome. In principle one could assume that the KK procedure for the motion description of matter only stands for macroscopic objects: but this is an unsatisfactory answer to the problem. The aim of the KK reduction is at least to entirely reproduces the electrodynamics in all of its feature, where we have no problem on the allowable values for charge and mass. Thus, we are forced to conclude that this approach is not able to reproduces electrodynamics.

3.1.2 Geodesic deviation equation in 5D background

For sake of completeness , before ending this section, we consider the topic of the geodesic deviation equation (Lacquaniti-Montani-Vietri '08 [141]). We will start by considering the 5D geodesic deviation equation and then we will implement the KK reduction. In a previous work ([100]) such a topic is faced setting from the beginning as a constant the scalar field; here we allow in our dynamics the presence of the scalar field - this was indeed a task suggested in the above cited paper-. The aim of the analysis is to get some insight with respect to the definitions of coupling that arise in the geodesic approach. The equation which we are going to search in our dimensional reduction procedure is the following:

$$\frac{D^2\delta x^\alpha}{Ds^2} = -R^\alpha_{\beta\gamma\lambda}u^\beta\delta x^\gamma u^\lambda + \frac{q}{m}\delta x^\nu\nabla_\nu\left(F^{\alpha\beta}u_\beta\right) . \tag{3.15}$$

In the framework of the usual 4D theory it describes the behaviour of the geodesic deviation parameter δ^α between two close test-particles, with same charge q and mass m, in presence of an electromagnetic field described by the Faraday tensor $F^{\mu\nu}$; δ^α is the infinitesimal displacement separating two close geodesic line, while, as usual, ∇_μ is the covariant derivative and $\frac{D}{Ds}$ is the derivative along the path.

The starting point of our analysis is the 5D geodesic deviation equation, which we obtain extending to the 5D background the usual geodesic deviation equation. We have:

$$\frac{D^2\delta x^A}{Ds_5^2} = -{}^5R^A_{BCD}w^B\delta x^C w^D \,, \tag{3.16}$$

where ${}^5R^A_{BCD}$ is the 5D Riemann tensor and w^A is the 5D velocity.

Constant scalar field

Let us start with the simplest scenario: $\phi = 1$. To simplify the notations in the following we assume a units system such that $ek = 1$. After the dimensional reduction and the reparametrization in terms of derivatives with respect to ds, eq. (3.16) spreads into the following set:

$$\frac{D^2\delta x^\alpha}{Ds^2} = -R^\alpha_{\beta\gamma\lambda}u^\beta\delta x^\gamma u^\lambda + \frac{w_5}{\sqrt{1+w_5^2}}\delta x^\nu\nabla_\nu\left(F^{\alpha\beta}u_\beta\right) - F^\alpha_\nu u^\nu\left(\frac{d\delta x_5}{ds} - F_{\mu\nu}\delta x^\mu u^\nu\right) \tag{3.17}$$

$$\frac{d}{ds}\left(\frac{d\delta x_5}{ds} - F_{\mu\nu}\delta x^\mu u^\nu\right) = 0 \tag{3.18}$$

Let us now define the new quantity δQ:

$$\frac{d\delta x_5}{ds} - F_{\mu\nu}\delta x^\mu u^\nu = \delta Q \tag{3.19}$$

Such quantity represents the difference in the q/m ratio for two particles following the geodesic lines separated by the infinitesimal displacement δx^μ. This factor arises because we are defining the coupling q/m by mean of the kinematical object w_5 which is conserved in a given geodesic but is allowed to vary between the two geodesic. Therefore the eq. (3.18) gives us the conservation of δQ between our closed geodesic lines. Using this definition, and recalling (3.12), we now rewrite eq. (3.17) :

$$\frac{D^2\delta x^\alpha}{Ds^2} = -R^\alpha_{\beta\gamma\lambda}u^\beta\delta x^\gamma u^\lambda + \frac{q}{m}\delta x^\nu\nabla_\nu\left(F^{\alpha\beta}u_\beta\right) - F^\alpha_\nu u^\nu\delta Q \tag{3.20}$$

The above equation, when compared to eq. (3.15), describes the geodesic deviation between two closed lines for particles with a charge-mass ratio given respectively by q/m and $q/m + \delta Q$. If we take as a starting condition $\delta Q = 0$, eq. (3.18) ensures us that this condition holds during the motion and therefore we formally reproduces the equation (3.15). This was indeed the result obtained by Kerner et al. in ([100]).

Variable scalar field

Now let us turn our attention to the full model. Via the dimensional reduction, after various algebraic steps, the 4D part of equation (3.16) reads as follows:

$$
\begin{aligned}
\frac{D^2\delta x^\alpha}{Ds^2} - \left(\frac{w_5^2}{1+\frac{w_5^2}{\phi^2}}\right)\frac{1}{\phi^3}\frac{d\phi}{ds}\frac{D\delta x^\alpha}{ds} &= -R^\alpha_{\beta\gamma\lambda}u^\beta\delta x^\gamma u^\lambda + \\
+\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\delta x^\nu\nabla_\nu\left[F^{\alpha\beta}u_\beta\right] + \left(\frac{w_5^2}{1+\frac{w_5^2}{\phi^2}}\right)\delta x^\nu\nabla_\nu\left(\frac{\partial^\alpha\phi}{\phi^3}\right) - \\
-\left(F^\alpha_\nu u^\nu - 2\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{\partial^\alpha\phi}{\phi^3}\right)\left(\frac{d\delta x_5}{ds} - 2\frac{\delta x_5}{\phi}\frac{d\phi}{ds} - \phi^2F_{\mu\nu}\delta x^\mu u^\nu + 2\frac{w_5}{\sqrt{1+\frac{u_5^2}{\phi^2}}}\frac{\delta x^\rho\partial_\rho\phi}{\phi}\right)
\end{aligned}
\tag{3.21}
$$

The fifth component reads:

$$
\begin{aligned}
\frac{d}{ds}\left[\sqrt{1+\frac{w_5^2}{\phi^2}}\left(\frac{d\delta x_5}{ds} - 2\frac{\delta x_5}{\phi}\frac{d\phi}{ds} - \phi^2F_{\mu\nu}\delta x^\mu w^\nu + 2\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{\delta x^\rho\partial_\rho\phi}{\phi}\right)\right] &= \\
= -w_5\frac{\delta x^\nu\partial_\nu\phi}{\phi^3}\frac{d\phi}{ds}
\end{aligned}
\tag{3.22}
$$

Now, we can manipulate the left side of (3.22), in order to write it in terms of the reparametrization factor between ds_5 and ds. We have:

$$\frac{d\sqrt{1+\frac{w_5^2}{\phi^2}}}{ds} = -\frac{1}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{w_5^2}{\phi^3}\frac{d\phi}{ds}.$$

$$\delta\left(\sqrt{1+\frac{w_5^2}{\phi^2}}\right) = \frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\left(\frac{1}{\phi^2}\frac{d\delta x_5}{ds} - \frac{w_5}{\phi^3}\delta x^\nu \partial_\nu \phi\right).$$

By using the above formulas we can rewrite the equation (3.22) in a more meaningful way; we have

$$w_5\frac{\delta x^\nu \partial_\nu}{\phi^3}\frac{d\phi}{ds} = \left[\left(1 + \frac{w_5^2}{\phi^2}\right)\delta\left(\sqrt{1+\frac{w_5^2}{\phi^2}}\right)\frac{\phi^3}{w_5^3} - \left(\sqrt{1+\frac{w_5^2}{\phi^2}}\right)\frac{\phi}{w_5^2}\frac{d\delta x_5}{ds}\right]\left(\frac{d\sqrt{1+\frac{w_5^2}{\phi^2}}}{ds}\right), \tag{3.23}$$

and, finally:

$$\frac{d}{ds}\left[\left(\frac{d\delta x_5}{ds} - 2\frac{\delta x_5}{\phi}\frac{d\phi}{ds} - \phi^2 F_{\mu\nu}\delta x^\mu w^\nu + 2\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{\delta x^\rho \partial_\rho \phi}{\phi}\right)\right] =$$
$$\left[\left(\sqrt{1+\frac{w_5^2}{\phi^2}}\right)\frac{\phi}{w_5^2}\frac{d\delta x_5}{ds} - \left(1 + \frac{w_5^2}{\phi^2}\right)\delta\left(\sqrt{1+\frac{w_5^2}{\phi^2}}\right)\frac{\phi^3}{w_5^3}\right]\left(\frac{d\sqrt{1+\frac{w_5^2}{\phi^2}}}{ds}\right) -$$
$$-\left(\frac{d\delta x_5}{ds} - 2\frac{\delta x_5}{\phi}\frac{d\phi}{ds} - \phi^2 F_{\mu\nu}\delta x^\mu w^\nu + 2\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}}\frac{\delta x^\rho \partial_\rho \phi}{\phi}\right)\left(\frac{d\sqrt{1+\frac{w_5^2}{\phi^2}}}{ds}\right). \tag{3.24}$$

The argument of the derivative operator $\frac{d}{ds}$ reduces to the previously considered δQ when $\phi = 1$, and we therefore assume it is representing the generalization of the charge-mass deviation between the two geodesic lines. But, now, due to the presence of ϕ it turns out that this factor is no more conserved during the motion and it means that even two particle with the same mass and charge suffer an additive deviation factor during their motion. The last equation, written in term of the reparameterization factor envisages how this results is an unavoidable outcome of the geodesic procedure, where we are forced to take into account the reduction of the line element ds_5.

3.2 Hamiltonian analysis of the dynamics

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In our analysis of the geodesic approach we were not able to obtain a definition for charge or mass separately; we defined only their ratio. This was due to the fact that we studied the Eulero-Lagrange equations, where the mass parameter $\hat{m}$ we put in the Action (3.1) does not appear, and we at end had to deal with an equation which only contains the final coupling factor, i.e. the ratio q/m. Thus, in order to be able to separately consider q and m and get deeper insights we now consider the hamiltonian formulation which will lead us to the analysis of the dispersion relation (Lacquaniti-Montani '08 [139], [138]) .

Therefore, working again in $c = 1$ unit, we start from the following Action, which we previously discussed.

$$S_5 = -\hat{m} \int ds_5 \tag{3.25}$$

Here we recall the expression of the KK reduced line element:

$$ds_5^2 = g_{\mu\nu} dx^\mu dx^\nu - \phi^2 (exA_\mu dx^\mu + dx^5)^2$$

At first, in order to get the Lagrangian, let us define a time-like coordinate q^0 and perform the following reparameterization:

$$S_5 = -\hat{m} \int dq^0 \frac{ds_5}{dq^0} \qquad \rightarrow L = -\hat{m} \frac{ds_5}{dq^0} \tag{3.26}$$

To achieve such a time-like coordinate and consider the hamiltonian reformulation we need to perform the ADM splitting of our variables. We apply the ADM splitting to the variables that appear in the reduced line element. It is given by the reduction of the metrics, and we have already give the proof that the metrics is insensitive to the ordering of the KK reduction and the ADM splitting procedures. Therefore we do not need to care about the ordering and go on with the $3+1$ ADM splitting. In doing this we follow the procedure given by the formula (2.5) and by the projection rules given in section (2.1.2).

Hence, we define:

$$g_{\mu\nu} = \begin{pmatrix} N^2 - h_{ij}N^iN^j & -h_{ij}N^j \\ -h_{ij}N^j & -h_{ij} \end{pmatrix} \tag{3.27}$$

$$A_\mu = (A_0, A_i) \tag{3.28}$$

$$dx_\mu = (dq_0, dq_i) \tag{3.29}$$

$$dx^\mu = (dq^0, -N^i dq^0 - dq^i) \tag{3.30}$$

where N, N^i, h_{ij} are the *Lapse* function, the *Shift* functions , the 3D induced spatial metrics respectively and we have: $dq_i = h_{ij}dq^j$, $dq^0 = \frac{1}{N^2}(dq_0 - dq_iN^i)$.

Let us finally define a set of Lagrangian coordinates and velocities:

$$\begin{aligned} q^\mu &= (q^0, q^i) \quad q^5 = -x^5 \\ \dot{q}^i &= \frac{dq^i}{dq^0} \quad \dot{q}^5 = \frac{dq^5}{dq^0} \end{aligned} \tag{3.31}$$

The minus sign which links q^5 to x^5 is given in analogy with the sign linking dx^i to dq^i (it is related to the fact that, from the point of view of the 4D observer, the 3D submanifold has a global negative euclidean signature).

By substituting these formulas into the line element ds_5, we can finally evaluate the Lagrangian (3.26) and we obtain:

$$L = -\hat{m}\left[N^2 - h_{ij}\left(\dot{q}^i\dot{q}^j\right) - \phi^2\left(-\dot{q}^5 + ekA_0 - ekA_jN^j - ekA_j\dot{q}^j\right)^2\right]^{\frac{1}{2}} \tag{3.32}$$

The next step is now to calculate conjugate momenta. We have:

$$P_5 = \frac{\partial L}{\partial \dot{q}^5} = -\frac{\hat{m}\phi^2}{\sqrt{X}}\omega \tag{3.33}$$

$$P_i = \frac{\partial L}{\partial \dot{q}^i} = \Pi_i + ekA_i P_5 \tag{3.34}$$

where we have defined the auxiliary variables

$$\omega = \left(-\dot{q}^5 + ekA_0 - ekA_j N^J - ekA_j \dot{q}^j\right) \tag{3.35}$$

$$\Pi_i = \dot{q}^i \frac{\hat{m}}{\sqrt{X}} \tag{3.36}$$

$$\sqrt{X} = \left[N^2 - h_{ij}\left(\dot{q}^i\dot{q}^j\right) - \phi^2\left(-\dot{q}^5 + ekA_0 - ekA_j N^j - ekA_j\dot{q}^j\right)^2\right]^{\frac{1}{2}} \tag{3.37}$$

We notice that P_5 is a conserved momentum, because q^5 is a cyclic coordinate. By combining (3.35, 3.36, 3.37), we are able to write

$$\dot{q}^i = \Pi^i N \frac{1}{\sqrt{\frac{P_5^2}{\phi^2} + \hat{m}^2 + \Pi_j \Pi^j}}\,, \tag{3.38}$$

where $\Pi^i = h^{ij}\Pi_j$. Once we get $\dot{q}^i$ as a function of conjugate momenta, we would be able to do the same for $\dot{q}^5$, by help of (3.33, 3.35, 3.38) and therefore we have all tools to calculate Legendre transformation and gain the Hamiltonian. Our result is:

$$H = N\sqrt{\left(\frac{P_5^2}{\phi^2} + \hat{m}^2\right) + \Pi_j\Pi^j} + ekA_0P_5 + ekA_jN^JP_5 \tag{3.39}$$

Moreover, being $w^A = \frac{dx^A}{ds_5}$ the 5D velocities, we have:

$$\Pi_i = \hat{m}w_i \tag{3.40}$$

$$P_5 = \hat{m}w_5 \tag{3.41}$$

In order to have a better feeling with eq. (3.39), let us suppose to work in a synchronous frame (or even better in a flat space); this simplifies our interpretation, because in this case we have $N = 1$, $N^i = 0$. Within this scenario, we can straightforward write:

$$\left(H - ekA_0P_5\right)^2 = \Pi_i\Pi^i + \left(\frac{P_5^2}{\phi^2} + \hat{m}^2\right) \tag{3.42}$$

We can then identify relation (3.42) with the dispersion relation of a test particle with charge q and mass m in an external electromagnetic field ([101]), where we define

$$q = ekP_5 \tag{3.43}$$

$$m^2 = \left(\frac{P_5^2}{\phi^2} + \hat{m}^2\right) \tag{3.44}$$

We notice explicitly that, via (3.41), we rebuild the same expression we previously for q/m ratio (3.12), that is

$$\frac{q}{m} = ek\frac{w_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}} \tag{3.45}$$

But, while previously we were not able to identify exactly the separate contribution of q and m to the total ratio, now equations (3.43, 3.44) give us a precious insight: indeed, charge is still conserved (as correct in a U(1) gauge invariant model); the mass term m only carries into its defining formula the factor $\frac{P_5^2}{\phi^2}$ which at the same time the cause of the violation of the conservation and of the upper bound. The striking feature of this out come is that the effective mass m of the particle does not coincide with the mass parameter we put in the Action, which is $\hat{m}$. Such a feature becomes more evident when we analyze both the 4D and 5D dispersion relation of the particle. Let us consider again eq. (3.42): if we identify H with the time component of the momentum we can write a 4D covariant relation which holds in every frame:

$$g^{\mu\nu}\Pi_\mu\Pi_\nu = \hat{m}^2 + \frac{P_5^2}{\phi^2}, \tag{3.46}$$

where $\Pi_0 = P_0 - ekP_5A_0$. Such a 4D relation is in a one-to-one correspondence with an analogue 5D dispersion relation, which can be rebuild in the following way: in agreement with eq. (3.40, 3.41) we define

$$P^A = \hat{m}w^A \tag{3.47}$$

. It yields:

$$P^A = (\Pi^\mu, P^5) \tag{3.48}$$

$$P_A = (\Pi_\mu + P_5 A_\mu, P_5) \tag{3.49}$$

$$\Pi^\mu = g^{\mu\nu}\Pi_\nu \qquad P_5 = -\phi^2(A_\mu \Pi^\mu + P^5). \tag{3.50}$$

Here we have $P^A = J^{AB}P_B$.

With such a definition we can reproduce the relation (3.46), and therefore relation (3.42), by the following 5D dispersion relation :

$$P_A P^A = \hat{m}^2. \tag{3.51}$$

The last is nothing else than the relation we obtain from the geodesic constraint $w_A w^B = 1$ by using definition (3.47).

Therefore, in concluding this section: we see clearly that the mass parameter $\hat{m}$ we put in the Action, coincides, as expected, with the parameter that appears in the 5D dispersion relation and so it plays the role of the proportionality factor between the 5D velocities and the 5D momentum. But, at the same time, it does not coincide with the physical mass m of the reduced particle, as we can see , for instance, via the relation (3.46). Hence, while in GR the formulation of the test particle Action takes for granted that the mass parameter which appears in the Action represents the physical mass, now we are facing a different scenario. This feature, in our opinion, plays an important role in the failure of the model to describe the test particles motion. We will turn back on this point later.

3.3 Klein-Gordon approach

Now let us take a look at the Klein-Gordon scenario that arises from the geodesic approach (Lacquaniti-Montani '08 [139], [138]). To do this we will develop a toy model, involving a complex scalar field invariant with respect to $U(1)$ gauge transformations. So far, we have seen that, within the geodesic procedure, the 5D velocity of the particle must satisfy the identity

$$J^{AB} w_A w_B = 1 \tag{3.52}$$

Introducing the mass parameter $\hat{m}$ and define linear momentum $P_A = \hat{m} w_A$, we gain the 5D dispersion relation (3.51)

$$J^{AB} P_A P_B = \hat{m}^2 \,, \tag{3.53}$$

which is a direct outcome of the Action (3.1). Such a relation provides the dispersion relation (3.42); actually, we could consider it as a more general formulation of eq.(3.42). We now consider the canonical quantization of this relation, upgrading the momenta to quantum operators, as far as a complex scalar field $\zeta(x^\mu, x^5)$ is concerned, and we get a 5D KG equation in curved background. The associated Lagrangian density reads:

$$\mathcal{L} = J^{AB} \partial_A \zeta (\partial_B \zeta)^+ - \hat{m}^2 \zeta \zeta^+ \tag{3.54}$$

Now we implement the KK reduction. To be consistent with the unobservability request concerning the extra dimension, we assume that our scalar field depends on x^5 only through a phase factor:

$$\zeta(x^\mu, x^5) = \eta(x^\mu) e^{iP_5 x^5} \tag{3.55}$$

The momentum P_5 is scalar and conserved. Moreover, recalling KK transformations (1.3), we notice that our field transforms as follows, with respect to a traslation along the fifth dimension:

$$\zeta = \zeta' e^{i(ekP_5)\Psi} \tag{3.56}$$

Therefore, we are dealing correctly with a U(1) gauge transformation. Then, implementing the reduction of the inverse metrics J^{AB} in (3.54), we finally can consider the reduction of the Lagrangian and this yields :

$$\mathcal{L} = g^{\mu\nu}(-i\partial_\mu - P_5 ekA_\mu)\eta[(-i\partial_\nu - P_5 ekA_\nu)\eta]^+ - \left(\hat{m}^2 + \frac{P_5^2}{\phi^2}\right)\eta\eta^+ \tag{3.57}$$

We recognize a U(1) gauge invariant lagrangian where the reduced field η acquires a charge ekP_5 and a mass term $m^2 = (\hat{m}^2 + \frac{P_5^2}{\phi^2})$; the ratio q/m fits to the result previously obtained for the motion of the test particle. Moreover, by using now the compactness of the fifth dimension, we get the quantization of P_5, i.e. $P_{5(n)} = 2\pi n/L_5$. Thus, the discretization of P_5 gives rise to a tower of modes for the mass term m; in the simplest case, with $\phi = 1$, we simply have:

$$m_{(n)}^2 = \hat{m}^2 + P_{5(n)}^2 \tag{3.58}$$

Fixing the minimum, non-zero, value of P_5 via the elementary charge e, we get an evaluation for the extra dimension size: we have indeed a value well below our observational limit, i.e. $L_5 \simeq 10^{-31}\ cm$. The discretization of the charge and this evaluation for the extra-dimension length are indeed nice outcomes of the model.

But, at the same way, looking at (3.58), this evaluation provides huge massive modes beyond the Planck scale, regardless the value of $\hat{m}^2$. Such a feature is a direct consequence of the choice of the Action (3.1). Therefore, within the geodesic approach, we see that the problem of the charge-mass ratio is strictly linked to the problem of the massive modes spectrum ([95], [32] pag. 23).

Chapter 4

Revised approach to motion

As we have seen in the previous chapter, the geodesic approach is not able to provide a physical description of matter dynamics without occurring in some inconsistencies. Various effort has been done to overcome this problem. In the framework of the compactified model it has been suggested ([61]) to identify light particles like the electron with the n = 0 mode, invoking then the spontaneous symmetry breaking mechanism to provide mass; this approach has however the problem of explaining how the n = 0 modes can have nonzero charge and to solve such additional feature it is necessary to go to higher dimensions. Other approach deal with non compact space and embedding procedure ([102], [103], [104], [105], [106]) which at the end leads to the brane scheme. Therefore, the problem of the inconsistent description of matter is addresses to the starting assumptions of the KK model we have considered so far, like the number of the involved extra dimensions or their characterization. In this chapter, which has to be considered the core of this work, we would like to propose another point view (Lacquaniti-Montani '08 [139], [138]), adding a contribute to the debate about matter in KK. In our opinion, the problem of inconsistencies does not stand in the statement of the compactified KK model, but in the assumptions on which is based the geodesic approach.

4.1 Criticism versus the geodesic approach

Let us consider again the assumptions underlying the geodesic approach , which we discussed at the beginning of sec. (3.1). Indeed, when we state that it is possible to describe a particle via the Action (3.1), we are implicitly taking for granted three assumptions, borrowing them from GR:

- $\hat{m}$ represents the rest mass.
- $\hat{m}$ is a constant .
- It is possible to consider a 5D point-like particle.

Let us look at the first two assumption: in GR they are supported by the PE; actually, the physical equivalence between the geodesic Action, namely $\int ds$ and the particle Action, namely $-m\int ds$, is an alternative statement of the PE. But, we know that in KK the 5D PE is violated, therefore in 5D such assumptions are not well grounded. Without PE, the meaning of the parameter $\hat{m}$ is quite ambiguous and in principle the 5D mass could be not constant: at least, we have seen that it does not represents the rest mass of the reduced particle. Hence, we are legitimated to take the hypothesis that the 5D geodesic equation only represents the trajectory of minimum path between two given points in 5D, but does not deal with the proper equation of motion of our test particle. Now we look at the third assumption with the following reasoning: in order to correctly define a 5D test particle we have to assume that is possible to localize it. But, if we require that the size of the fifth dimension is $L_5 \sim O(10^{-17})\,cm$ or smaller , to have a particle localized in the extra dimension we have to assume it to have an enormous amount of energy, beyond the scale of Tev , i.e. $E \sim O\left(\frac{1}{L}\right)$. Hence, we suppose that the particle is not localized around the extra dimension and the definition itself of a 5D test particle is ambiguous. Now, let us examine how these assumptions work in GR and how we can give them a proof. In

GR, indeed, we have a well grounded procedure which allows us to:

- face a rigourously definition of rest mass
- prove that mass is scalar and constant
- prove that the motion of a particle is really the geodesic one when we deal with the point-like approximation for the particle.

This procedure is the multipole expansion, and is due to Papapetrou ([107]). We now briefly review this approach.

4.1.1 Papapetrou expansion

In GR a matter system is associated to a symmetric energy-momentum tensor $T^{\mu\nu}$; the dynamics of matter is constrained by the following equation:

$$\nabla_\mu T^{\mu\nu} = 0 \tag{4.1}$$

To define a test particle, from a physical viewpoint we demand a scenario where it is possible to neglect the back-reaction of the particle. From a mathematical point of view, this means that we now look for a procedure that will allow us to define a test particle as a point-like object. Hence, we assume that the dimensions of the particle are very small when compared to a characteristic length scale of the background field. For instance, in the case of a Schwarzschild background, we assume the particle to be at a great distance from the central body. Hence, the motion of the particle will be associated to a thin tube in the 4D space time. Let us now arbitrarily choose a world-line L within this tube, identified by a set of coordinate X^μ. Thus, in agreement with the hypotheses we made, the tensor $T^{\mu\nu}$ will be non zero, at any given time t, only in a sphere centred in X^i with a small radius R, concerning the dimension of the particle. Taking the limit $R \to 0$ the arbitrariness in the choice of L disappear. Therefore, the matter tensor becomes now peaked in X^μ; we expect it to be associated to a Dirac delta, i.e.

$$T^{\mu\nu} \infty \int ds \alpha^{\mu\nu} \delta(x^\rho - X^\rho)$$

where s is an affine parameter associated the the world line L and $\alpha^{\mu\nu}$ a suitable function. Let us now turn back to the equation (4.1). At first we use the identity $\nabla_\rho L^{\mu\rho} = \frac{1}{\sqrt{g}} \partial_\rho \sqrt{g} L^{\mu\rho} + \Gamma^\mu_{\rho\lambda} L^{\rho\lambda}$; after we consider the derivative of $x^\nu \sqrt{g} T^{\mu\lambda}$. Defining $\hat{T}^{\mu\nu} = \sqrt{g} T^{\mu\nu}$ we obtain the following set:

$$\partial_\rho \hat{T}^{\mu\rho}(x^\alpha) + \Gamma^\mu_{\rho\lambda}(x^\alpha) \hat{T}^{\rho\lambda}(x^\alpha) = 0 \tag{4.2}$$

$$\partial_\lambda \left(x^\nu \hat{T}^{\mu\lambda}(x^\alpha) \right) = \hat{T}^{\mu\nu}(x^\alpha) - x^\nu \Gamma^\mu_{\rho\lambda}(x^\alpha) \hat{T}^{\rho\lambda}(x^\alpha) \tag{4.3}$$

In the above equation we are stressing explicitly the dependence of the involved objects on the generic coordinates x^α of the manifold. Now we integrate over the 3D space and use the Gauss theorem . This yields:

$$\frac{d}{dx^0} \int d^3x \, \hat{T}^{0\mu}(x^\alpha) = - \int d^3x \, \Gamma^\mu_{\lambda\rho}(x^\alpha) \hat{T}^{\lambda\rho}(x^\alpha) \tag{4.4}$$

$$\frac{d}{dx^0} \int d^3x \, x^\mu \hat{T}^{0\nu}(x^\alpha) = \int d^3x \, \hat{T}^{\mu\nu}(x^\alpha) - \int d^3x \, x^\mu \Gamma^\nu_{\rho\lambda}(x^\alpha) \hat{T}^{\rho\lambda}(x^\alpha) \tag{4.5}$$

Now let us consider the infinitesimal displacement :

$$\delta x^\alpha = x^\alpha - X^\alpha$$

We develop now $\Gamma^\mu_{\rho\lambda}(x^\alpha)$ in a Taylor series in powers of δx^μ, with centre in X^μ:

$$\Gamma^\mu_{\rho\lambda}(x^\alpha) = \Gamma^\mu_{\rho\lambda}(X^\alpha) + \left(\partial_\sigma \Gamma^\mu_{\rho\lambda}(x^\alpha) \right) |_{x^\alpha = X^\alpha} \delta x^\sigma + ...$$

As previously discussed, X^μ identifies the trajectory of the moving body; therefore it is a time- like vector and it is associated to the affine parameter s via the definition:

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu \tag{4.6}$$

After the Taylor expansion, we neglect integrals like $\int d^3x\,\hat{T}^{\mu\nu}(x^\alpha)\delta x^\rho$, $\int d^3x\,\hat{T}^{\mu\nu}(x^\alpha)\delta x^\rho \delta x^\sigma$. Indeed we are taking only the lowest order of such approximation, which is called the single-pole. The point like particle will turn be as described by the single-pole equation of Papapetrou. For instance, the higher order integrals will be eventually used as successive steps of the approximation and will lead to the pole-dipole case ([108]) and higher multipoles. Indeed, this procedure outlines, as it happens when we deal with an extended charge distribution in electrostatics, that higher multipoles become less important when the particle becomes small. Therefore, within the zero-order approximation, equations (4.4, 4.5) respectively become:

$$\frac{d}{dx^0}\left(\int d^3x\,\hat{T}^{\mu 0}(x^\alpha)\right) + \Gamma^\mu_{\rho\lambda}(X^\alpha)\int d^3x\,\hat{T}^{\rho\lambda}(x^\alpha) = 0 \tag{4.7}$$

$$\frac{dX^\nu}{dx^0}\int d^3x\,\hat{T}^{\mu 0}(x^\alpha) = \int d^3x\,\hat{T}^{\mu\nu}(x^\alpha) \tag{4.8}$$

Let us now define the auxiliary tensor $M^{\mu\nu} = u^0 \int d^3x\,\hat{T}^{\mu\nu}$, being $u^\mu = \frac{dx^\mu}{ds}$. Omitting now the explicit dependence on coordinates which is no longer necessary, equations (4.7, 4.8) yield :

$$\frac{d}{ds}\left(\frac{M^{\mu 0}}{u^0}\right) + \Gamma^\mu_{\rho\lambda}M^{\rho\lambda} = 0 \tag{4.9}$$

$$M^{\mu\nu} = \frac{u^\nu}{u^0}M^{\mu 0} \tag{4.10}$$

Now, from (4.10), we get immediately:

$$M^{0\nu} = \frac{u^\nu}{u^0}M^{00} \Rightarrow M^{\mu\nu} = u^\nu u^\mu \frac{M^{00}}{(u^0)^2}$$

This allows us to define the quantity m:

$$m = \frac{M^{00}}{(u^0)^2} = \frac{1}{u^0}\int d^3x\,\hat{T}^{00} \tag{4.11}$$

$$\tag{4.12}$$

Finally, coming back to eq. (4.9), the equation of motion reads:

$$\frac{d}{ds}\left(mu^\mu\right) + \Gamma^\mu_{\rho\lambda}u^\rho u^\lambda = 0 \tag{4.13}$$

The above equation accounts for the description of the single pole of the matter distribution. It does not contain only an equation for the motion but also provide a condition for the mass term m. Indeed, by requiring that the identity $u_\mu \frac{D}{Ds} u^\mu = 0$ - which is contained in (4.6) - holds, from the above equation we get the condition:

$$\frac{d}{ds} m = 0 \tag{4.14}$$

A further calculation can shows that m is a scalar ([107]). Therefore, the single pole equation, together with the condition on m, provides the geodesic equation for a test particle of mass m. We could obtain the same result for the motion equation if we define as follows the effective matter tensor for particles :

$$\sqrt{g} T^{\mu\nu} = \int ds \, m \, \delta^4(x^\alpha - X^\alpha) u^\mu u^\nu$$

Hence, the effective matter tensor fits to our beginning hypothesis of localization.

Thus, Papapetrou approach gives the proof that in standard GR the motion is effectively the geodesic one as far as the point like particle approximation is concerned. The mass parameter for the particle arises naturally during the procedure, and it turn to be a constant parameter. The Action $S = -m \int ds$ can be considered then as the effective Action standing for a localized, point-like, body. Therefore, our point of view is the following. The proper procedure to be generalized from 4D to 5D is not the geodesic one : we have to consider the general dynamical equation, as in the Papapetrou procedure, involving a 5D matter tensor, and only after this, if we want to analyze the motion of a test particle, address the procedure of multipole expansion , which is the only one able to correctly define the rest mass of the particle.

4.2 Papapetrou approach to motion

In this approach we define a generic, symmetric, 5D matter tensor T^{AB}; we assume that its dynamics is given by the following equations:

$$D_A T^{AB} = 0 \tag{4.15}$$

$$\partial_5 T^{AB} = 0 \tag{4.16}$$

Here D_A is now the 5D covariant derivative. The first equation is the generalization of the conservation law we have in GR, namely $\nabla_\mu T^{\mu\nu} = 0$. The second equation is given for consistency with respect to the field dynamics and the cylindricity hypothesis. In order to simplify the identification of matter tensor components with corresponding 4D objects, we will write our equations in terms of following components:

$$T^{\mu\nu} \rightarrow 4D\ tensor$$

$$T^{\mu}_{5} \rightarrow 4D\ vector$$

$$T_{55} \rightarrow\ scalar$$

4.2.1 Dimensional reduction

The 5-bein projection of equation (4.15) yields:

$$\begin{aligned}
e^{(A)}_{A} D_B T^{AB} &= D_B(e^{(A)}_{A} T^{AB}) - (D_B e^{(A)}_{A}) T^{AB} \\
&= D_B T^{(A)B} - (D_B e^{(A)}_{A}) T^{(M)(N)} e^{A}_{(M)} e^{B}_{(N)} \\
&= D_B T^{(A)B} - \gamma^{(A)}{}_{(B)(C)} T^{(B)(C)} \\
&= D_B (T^{(A)(B)} e^{B}_{(B)}) - \gamma^{(A)}{}_{(B)(C)} T^{(B)(V)} \\
&= e^{B}_{(B)} \partial_B T^{(A)(B)} + (\nabla_B e^{B}_{(B)}) T^{(A)(B)} - \gamma^{(A)}{}_{(B)(C)} T^{(B)(C)} \\
&= \partial_{(B)} T^{(A)(B)} - \gamma^{(A)}{}_{(B)(C)} T^{(B)(C)} - \gamma^{(C)}{}_{(B)(C)} T^{(A)(B)}
\end{aligned}$$

where $\gamma^{(A)}{}_{(B)(C)} = (D_B e^{(A)}_A) e^A_{(B)} e^B_{(C)}$ are 5D Ricci coefficients. Finally, we have:

$$D_B T^{AB} = e^A_{(A)}[\partial_{(B)} T^{(A)(B)} - \gamma^{(A)}{}_{(B)(C)} T^{(B)(C)} - \gamma^{(C)}{}_{(B)(C)} T^{(A)(B)}] = 0 \tag{4.17}$$

Now, we express $\gamma_{(A)(B)(C)}$ in terms of $\lambda_{(A)(B)(C)}$[1] in (4.17) and this yields:

$$D_B T^{AB} = e^A_{(A)}[\partial_{(B)} T^{(A)(B)} - \eta^{(M)(A)} \lambda_{(B)(C)(M)} T^{(B)(C)} - \eta^{(M)(C)} \lambda_{(M)(B)(C)} T^{(A)(B)}] = 0 \tag{4.18}$$

Our following step is to substitute explicit expressions for anolomies (1.9) and 5-bein vectors (1.8); after a little machinery we have the following set of equations:

$$5) \quad \rightarrow \frac{1}{\phi}[\partial_{(\beta)} T^{(5)(\beta)} - \lambda^{(\gamma)}{}_{(\beta)(\gamma)} T^{(5)(\beta)} + 2\frac{\partial_\rho \phi}{\phi} T^{(5)\rho}] = 0 \tag{4.19}$$

$$\mu) \quad \rightarrow \nabla_\rho T^{\mu\rho} = -(\frac{\partial_\rho \phi}{\phi}) T^{\mu\rho} - g^{\mu\rho}(\frac{\partial_\rho \phi}{\phi}) T^{(5)(5)} - g^{\mu\sigma} ek\phi F_{\sigma\rho} T^{(5)\rho} \tag{4.20}$$

Now we care at first of equation (4.19). First two terms in brackets are simply the tetrad projection of the term $\nabla_\beta T^{(5)\beta}$. Then, apart from multiplying factor, we have:

$$\nabla_\beta T^{(5)\beta} + 2\frac{\partial_\rho \phi}{\phi} T^{(5)\rho} = 0$$

Again, apart from a common factor ϕ^2, we can recast this equation in a more compact and intriguing way:

$$\nabla_\beta (\phi^2 T^{(5)\beta}) = 0 \tag{4.21}$$

By using the projection rule $T^{(5)\rho} = -\frac{1}{\phi} T_5{}^\rho$, we also have as suitable equation:

$$\nabla_\beta (\phi T_5{}^\beta) = 0 \tag{4.22}$$

Finally, we also use the rule $T_{55} = \phi^2 T^{(5)(5)}$, and at the end we can write down the following set of equations:

[1] We have $\gamma_{(A)(B)(C)} = \frac{1}{2}(\lambda_{(A)(B)(C)} + \lambda_{(C)(B)(A)} + \lambda_{(B)(C)(A)})$

$$5) \quad \rightarrow \nabla_\mu \left(\phi T_5^\mu\right) = 0 \tag{4.23}$$

$$\mu) \quad \rightarrow \nabla_\rho T^{\mu\rho} = -\left(\frac{\partial_\rho \phi}{\phi}\right) T^{\mu\rho} - g^{\mu\rho}\left(\frac{\partial_\rho \phi}{\phi^3}\right) T_{55} + ekF^\mu_{\ \rho} T_5^\rho \tag{4.24}$$

From the first of the above equations we recognize a conserved vectorial current, i.e.

$$j^\mu = ek\phi T_5^\mu \tag{4.25}$$

Therefore we rewrite:

$$5) \quad \rightarrow \nabla_\mu j^\mu = 0 \tag{4.26}$$

$$\mu) \quad \rightarrow \nabla_\rho \left(\phi T^{\mu\rho}\right) = -g^{\mu\rho}\left(\frac{\partial_\rho \phi}{\phi^2}\right) T_{55} + F^{\mu\rho} j_\rho \tag{4.27}$$

where $j_\mu = g_{\mu\nu} j^\nu$. It is worth noting that, if we take $\phi = 1$ from the beginning, our set reduces to

$$5) \quad \rightarrow \nabla_\mu j^\mu = 0 \tag{4.28}$$

$$\mu) \quad \rightarrow \nabla_\rho T^{\mu\rho} = F^{\mu\rho} j_\rho \tag{4.29}$$

Therefore, if we identify $T^{\mu\nu}$ with the matter tensor of the reduced 4D matter, we are able to take into account the coupling of matter with the conserved current associated to a U(1) gauge invariant field, namely the electromagnetic field. Any hypothesis hasn't been made on the kind of the 4D matter and the current is only defined in terms of components of the 5D matter tensor, without any connection with the kinematics of the matter. So, at this step, no definition of mass has been employed and there is not any kind of restriction on the value of the current.

4.2.2 Multipole expansion

Now, in order to describe in a consistent way the properties of a test particle and neglect the back-reaction, we have to adopt a localized matter tensor and follow the Papapetrou procedure.

Papapetrou procedure: Let us start from equations (4.26): we rewrite it via the identity $\nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}V^\mu)$ and moreover we perform the derivative $\partial_\mu(x^\nu\sqrt{g}j^\mu)$. Defining $\hat{j}^\mu = \sqrt{g}j^\mu$ we obtain the set

$$\begin{cases} \partial_\mu \hat{j}^\mu = 0 \\ \partial_\mu(x^\nu \hat{j}^\mu) = \hat{j}^\nu \end{cases} \tag{4.30}$$

Now we integrate these equations over the 3D space and use Gauss theorem. At this step we use the localization hypothesis: the current $\hat{j}^\mu$ is peaked on a world line X^μ and negligible outside: therefore we perform a Taylor expansion $x^\mu = X^\mu + \delta x^\mu$ and retain only the lowest order. In such an approximation our set becomes

$$\frac{d}{dx^0}\int d^3x\, \hat{j}^0 = 0 \tag{4.31}$$

$$\left(\frac{dX^\nu}{dx^0}\right)\int d^3x\, \hat{j}^0 = \int d^3x\, \hat{j}^\nu \tag{4.32}$$

Now , we define $q = \int d^3x\, \hat{j}^0$ and $u^\mu = \frac{dX^\mu}{ds}$, where $ds^2 = g_{\mu\nu}dX^\mu dX^\nu$. Therefore we have:

$$\begin{cases} \frac{d}{ds}q = 0 \\ \\ u^\mu q = u^0 \int d^3x\, \hat{j}^\mu \end{cases} \tag{4.33}$$

Hence, we have defined the charge of the test particle. Finally we can recognize that in such point-like approximation the effective current can be written as follows:

$$\sqrt{g}j^\mu = \int ds\, q u^\mu \delta(x^\mu - X^\mu)$$

Now we turn our attention to eq.(4.27) and generalize the above procedure: at first we have to define objects that we want to localize, as far as factors containing powers of the scalar field are concerned; for the tensor component, as it happens for the vector one, the most simple choice is to consider $\phi T^{\mu\nu}$ which is indeed the argument of the ∇_μ operator in (4.27). For the scalar component there is no particular reason driving us in some choice with respect to any others; we just assume, for the sake of symmetry, that that scalar component has to be rescaled in the same way of the tensor and the vector ones.

Then, we define $\tilde{T}^{\mu\nu} = T^{\mu\nu}\phi$, $\tilde{T}_{55} = T_{55}\phi$ and apply the localization hypothesis on these objects. Now, we use the identity $\nabla_\rho L^{\mu\rho} = \frac{1}{\sqrt{g}}\partial_\rho\sqrt{g}L^{\mu\rho} + \Gamma^\mu_{\rho\lambda}L^{\rho\lambda}$; moreover we consider the derivative of $x^\nu\sqrt{g}T^{\mu\lambda}$. Defining $\hat{T}^{\mu\nu} = \sqrt{g}\tilde{T}^{\mu\nu}$ we get the set of equations:

$$\partial_\rho\hat{T}^{\mu\rho} + \Gamma^\mu_{\rho\lambda}\hat{T}^{\rho\lambda} = -g^{\mu\rho}\left(\frac{\partial_\rho\phi}{\phi^3}\right)\hat{T}_{55} + F^{\mu\rho}\hat{j}_\rho \tag{4.34}$$

$$\partial_\lambda(x^\nu\hat{T}^{\mu\lambda}) = \hat{T}^{\mu\nu} - x^\nu\Gamma^\mu_{\rho\lambda}\hat{T}^{\rho\lambda} + x^\nu\left[-g^{\mu\rho}\left(\frac{\partial_\rho\phi}{\phi^3}\right)\hat{T}_{55} + F^{\mu\rho}\hat{j}_\rho\right] \tag{4.35}$$

Now we integrate over the 3D space, use the Gauss theorem and take into account the localization hypothesis. We assume that $\hat{T}^{\mu\nu}$ and $\hat{T}_{55}$, as well as $\hat{j}_\mu$, are peaked on a world line X^μ. Hence, the localization is performed only in the 4D space rather than in the 5D one ; this happens because the matter tensor does not depend on the fifth coordinate: such a scenario is consistent with the unobservability of the extra dimension and with the phenomenological request that we observe trajectories only in our 4D space. Given the displacement $\delta x^\mu = x^\mu - X^\mu$, we perform a Taylor expansion in powers of δx^μ, with centre X^μ, as far as the Chrystoffel symbols and the metric fields ϕ and A^μ (and their derivatives) are concerned. Thus, we obtain an expansion in terms of integrals like $\int d^3x\,\delta x^\alpha\hat{T}^{\rho\lambda}$, $\int d^3x\,\delta x^\alpha\delta x^\beta\hat{T}^{\rho\lambda}$ and so on, while metric fields and Chrystoffel (and their derivatives) are carried out from the integration being estimated on X^μ. The test particle motion is described by the lowest order of this approximation (single pole approximation), where we neglect all integrals with terms of order $\delta x^\mu\hat{T}^{\rho\lambda}$ and greater. Therefore, within the zero-order approximation, equations (4.34, 4.35) respectively become

$$\begin{aligned}\frac{d}{dx^0}\left(\int d^3x\,\hat{T}^{\mu0}\right) + \Gamma^\mu_{\rho\lambda}\int d^3x\,\hat{T}^{\rho\lambda} = -g^{\mu\rho}\left(\frac{\partial_\rho\phi}{\phi^3}\right)\int d^3x\,\hat{T}_{55} \\ + F^{\mu\rho}\int d^3x\,\hat{j}_\rho\end{aligned} \tag{4.36}$$

$$\frac{dX^\nu}{dx^0}\int d^3x\,\hat{T}^{\mu0} = \int d^3x\,\hat{T}^{\mu\nu} \tag{4.37}$$

where now metric fields and their derivatives are evaluated on X^μ. Now we define the auxiliary tensor $M^{\mu\nu} = u^0 \int d^3x \, \hat{T}^{\mu\nu}$, then equations (4.36, 4.37) become :

$$\begin{aligned} \frac{d}{ds}\left(\frac{M^{\mu 0}}{u^0}\right) + \Gamma^\mu_{\rho\lambda} M^{\rho\lambda} &= -g^{\mu\rho}\left(\frac{\partial_\rho \phi}{\phi^3}\right) u^0 \int d^3x \, \hat{T}_{55} \\ &+ F^{\mu\rho} u^0 \int d^3x \, \hat{j}_\rho \end{aligned} \tag{4.38}$$

$$M^{\mu\nu} = \frac{u^\nu}{u^0} M^{\mu 0} \tag{4.39}$$

Now, from (4.39), we get immediately:

$$M^{0\nu} = \frac{u^\nu}{u^0} M^{00} \Rightarrow M^{\mu\nu} = u^\nu u^\mu \frac{M^{00}}{(u^0)^2}$$

We define the scalars A and m:

$$m = \frac{M^{00}}{(u^0)^2} = \frac{1}{u^0} \int d^3x \, \hat{T}^{00} \tag{4.40}$$

$$A = u^0 \int d^3x \, \hat{T}_{55} \tag{4.41}$$

Finally, recalling eq. (4.33), the equation of motion (4.38) reads:

$$\frac{D}{Ds} m u^\mu = -g^{\mu\rho}\left(\frac{\partial_\rho \phi}{\phi^3}\right) A + q F^{\mu\rho} u_\rho \tag{4.42}$$

Now we require that the identity $u_\mu \frac{D}{Ds} u^\mu = 0$ holds; from the above equation we get the condition:

$$\frac{d}{ds} m = -\frac{A}{\phi^3}\frac{d\phi}{ds} \tag{4.43}$$

Finally, by using this new condition, we can rewrite the equation of motion as follows:

$$m \frac{Du^\mu}{Ds} = \left(u^\mu u^\rho - g^{\mu\rho}\right) \frac{\partial_\rho \phi}{\phi^3} A + q F^{\mu\rho} u_\rho \tag{4.44}$$

Coupling factors are explicitly defined as follows:

$$m = \frac{1}{u^0} \int d^3x \, \sqrt{g} \, \phi T^{00} \tag{4.45}$$

$$q = ek \int d^3x \, \sqrt{g} \, \phi T^0_5 \tag{4.46}$$

$$A = u^0 \int d^3x \, \sqrt{g} \, \phi T_{55} \tag{4.47}$$

Quantities m, q, A result to be scalar objects. We could obtain the same result for the motion equation and for coupling factors if we define as follows the effective matter tensor for particles :

$$\phi\sqrt{g}T^{\mu\nu} = \int ds\, m\, \delta^4(x - X)u^\mu u^\nu$$
$$ek\phi\sqrt{g}T^\mu_5 = \int ds\, q\, \delta^4(x - X)u^\mu$$
$$\phi\sqrt{g}T_{55} = \int ds\, A\, \delta^4(x - X)$$

Hence, the effective matter tensor fits to our beginning hypothesis of localization.

At first we note that, if we take from the beginning $\phi = 1$, then m becomes a constant and our equation reduces to the Lorenz equation, once we identify q with the charge and m with the rest mass of the reduced particle. Let us now compare the old equation (3.11) to this new one (4.44) in the general case. We recognize that:

- they show the same dynamical structure but coupling factors are not the same .
- in (4.44) we have the three factors m, q, A that are defined in terms of independent degrees of freedom of the matter tensor, therefore are not correlated each to other, while in the geodesic approach q and m were both defined in terms of P_5 (so giving the upper bound): therefore now no bound arises.
- q is conserved, due to the presence of a conserved current (gauge theory); A is not constant, but in principle there is no symmetry requiring its conservation.
- mass is not conserved (4.43) and this is indeed a relevant feature of this new equation. Anyway, there is no reason to require in principle the conservation of m : the 5D PE is broken and therefore mass is not necessarily a constant. However, in the scenario with $\phi = 1$ we deal with a purely Einstein-Maxwell system, so the PE is restored and mass turns to be a constant.

Papapetrou approach shows that the particle motion is not the geodesic one and that the idea of a 5D test particle is misdealing, being the particle localized only in the observed 4D space, and tells how charge and mass have to be defined correctly. In the following we will enforce this point of view showing how this new approach is able to remove the KK tower of massive modes.

Multipole expansion vs Dimensional reduction

A last point deserves to be better clarified before going on. In our derivation we have done as first step the dimensional reduction of the equation (4.15), taking into account in this procedure the condition (4.16), and only after the multipole expansion. One could ask what happens if we invert these procedures. The question is not trivial: in principle, by doing at first the multipole expansion on the eq. (4.15), as it happens in 4D theory, we expect to recover the 5D geodesic equation, for a test particle described in terms of its 5D velocity u^A , with a 5D mass $m = \frac{1}{w^0}\int\sqrt{J}\,T^{00}$, whose dynamics is governed by the equation $\frac{Dw^A}{Ds_5} = 0$. Given this equation we should then consider its dimensional reduction: but the dimensional reduction of this equation is nothing else that the same procedure underlying the geodesic approach. Therefore it leads to an outcome that we have demonstrated to be unphysical. In other words , it looks like that the dimensional reduction procedure and multipole expansion does non commute in order to describe the motion of a test particle. Actually, the reason relies in the fact that the multipole expansion is sensible to the hypotheses of KK. Let us consider now the set of equations suitable for multipole expansion in an unreduced 5D ambient:

$$\partial_A\hat{T}^{AB} + \Gamma^B_{AC}\hat{T}^{AC} = 0 \tag{4.48}$$

$$\partial_A(x^C\hat{T}^{AB}) = \hat{T}^{CB} - x^C\Gamma^B_{AC}\hat{T}^{AC} \tag{4.49}$$

where $\hat{T}^{AB} = \sqrt{J}T^{AB}$ and Γ^A_{BC} is now the 5D Chrystoffel symbol. We can delay the dimensional reduction after the end of multipole expansion, but, if we are in a Kaluza-

Klein background we have to remind the unobservability hypothesis : the matter tensor does not depend on the extra dimension and the extra dimension is compactified. Indeed, let us consider now the integration of equations (4.48, 4.49) over the 4D volume d^3xdx^5.. The integration over dx^5 only results in factor like l_5, l_5^2 ($l_5 = \int dx^5$) that will be ruled out. Equation (4.48) yields:

$$\frac{d}{dx^0}\int d^3x\,\hat{T}^{0B} + \int d^3x\,\Gamma^B_{AC}\hat{T}^{AC} = 0 \tag{4.50}$$

While, as far as equation (4.49) is concerned only its space-time part survives:

$$\frac{d}{dx^0}\int d^3x\,\hat{T}^{0B}x^\mu = \int d^3x\,\hat{T}^{\mu B} - \int d^3x\,x^\mu\Gamma^B_{AC}\hat{T}^{AC} \tag{4.51}$$

Now, what happens if we deal with these equations for the multipole expansions ? At first, we note that we have only integrals over the 3D volume d^3x, while for a covariant definition of the 5D geodesic equation we need integrals over d^4x. As second, and more important, point, we have the disappearance of x^5 and this will result in the impossibility to define w^5. Neither the Taylor expansion of Chrystoffel or metric fields will provide this factor, due to cylindricity hypothesis. Hence, the 5D geodesic equation can be obtained only if we consider a dynamical system which is not reduced with respect to KK starting conditions. This fact is in agreement with our previous observation concerning the localization of the particle. Indeed, due to request of unobservability, we cannot localize our particle with respect to the extra dimension and this causes the impossibility to define w_5 as an object concerning the dynamics of the particle, that is localized only in the usual 4D space.

4.3 Klein-Gordon dynamics

Let us consider now the KG dynamics that arises from the Papapetrou revised approach. At first, it is possible to prove that the new equation (4.44) can be derived from the following Action:

$$S = -\int m\,d\,s + q\left(A_\mu dx^\mu + \frac{dx^5}{ek}\right) \tag{4.52}$$

In agreement with equation (4.43), the parameter m has to be regarded as a variable function, not depending on x^5, whose derivatives are given by the following formula :

$$\frac{\partial m}{\partial x^\mu} = -\frac{\partial_\mu \phi}{\phi^3} A \tag{4.53}$$

A direct calculus of Eulero-Lagrange equations would give the proof that this Action leads to eq. (4.44). Moreover, it is worth noting that, if $\phi = 1$, being m a constant in such a scenario, we have exactly the Lorenz Action plus the charge conservation law, expressed in term of a Lagrange multiplier. From the above Action, we can calculate Lagrangian, conjugate momenta and Hamiltonian, in order to analyze the dispersion relation.

Hamiltonian formulation: Starting from the Action (4.52), we consider the ADM reformulation ([62], [64], [67], [136]), as well as we have done in the section (3.2): introducing the same set of Lagrangian coordinates q^0, q^i, q^5 we get the following Lagrangian:

$$L = -m[N^2 - h_{ij}(\dot{q}^i \dot{q}^j)]^{\frac{1}{2}} - q(-\frac{\dot{q}^5}{ek} + A_0 - A_j N^j - A_j \dot{q}^j)$$

From the above expression we calculate the Hamiltonian H and the conjugate momenta P_i, P_5. In a synchronous frame we have:

$$(H - qA_0)^2 = h^{ij}\Pi_i \Pi_j + m^2, \tag{4.54}$$

where

$$P_5 = \frac{q}{ek} \tag{4.55}$$

$$\Pi_i = P_i - qA_i = mu_i \qquad u^\mu = \frac{dx^\mu}{ds}. \tag{4.56}$$

Identifying the Hamiltonian with the time component of the momentum we can rewrite equation (4.54) in a 4D manifestly covariant expression:

$$\Pi_\mu \Pi^\mu = m^2, \tag{4.57}$$

where $\Pi_0 = P_0 - qA_0$. Finally, we can rebuild a 5D dispersion relation: the calculus of P_μ, P_5 we got from the Lagrangian, uniquely defines a 5D vector P^A such that we have:

$$P^A = (\Pi^\mu, P^5) \tag{4.58}$$

$$P_A = (\Pi_\mu + P_5 A_\mu, P_5) \tag{4.59}$$

$$\Pi^\mu = g^{\mu\nu}\Pi_\nu \qquad P_5 = -\phi^2(A_\mu \Pi^\mu + P^5). \tag{4.60}$$

where $P_5 = \frac{q}{ek}$, $\Pi^\mu = mu^\mu$. With such a definition, relations (4.54, 4.57) are provided by the following 5D dispersion relation :

$$P_A P^A = m^2 - \frac{q^2}{(ek)^2\phi^2} \tag{4.61}$$

Therefore, looking at equations (4.54, 4.57), we finally recognize the dispersion relation for a test particle with mass m and charge q interacting with the electromagnetic field. Now m is exactly the mass parameter we put in the Action, and at the same time is the scalar which defines the 4D dispersion relation (4.57) and the proportionality relation between usual 4D conjugate momenta and velocities. Thus, this result enforces our interpretation concerning the Papapetrou procedure.

Removal of the KK tower

Moreover , it is worth noting that there is now no link between P_5 and the velocity. Therefore, the quantization of charge, i.e. the quantization of P_5, does not affect the definition of the mass. Indeed, let us calculate the 5D KG equation corresponding to the dispersion relation (4.54). The formula (4.61) , when we consider its dimensional reduction, leads to the relation (4.54), therefore we assume that it represents the 5D dispersion relation of the particle. Hence, the 5D KG equation associated to our particle has to be defined via the quantization of relation (4.61). When we perform the canonical quantization of eq. (4.61), repeating the procedure of sec. (3.3), in the resulting Klein-Gordon equation we now have the counter term $-\frac{q^2}{(ek)^2\phi^2}$ that rules out the huge massive modes. The final Lagrangian for the reduced field η reads

$$\mathcal{L} = g^{\mu\nu}(-i\partial_\mu - qA_\mu)\eta[(-i\partial_\nu - qA_\nu)\eta]^+ - m^2\eta\eta^+, \tag{4.62}$$

and the tower of massive modes does not appear.

Chapter 5

Geometry and matter reduction

The Papapetrou procedure we outlined in the previous chapter offers the possibility to deal consistently with matter. Therefore now we will extend the results obtained within the Papapetrou approach to define a generic Kaluza-Klein model provided of a matter term and we consider a full reduction of matter and geometry (Lacquaniti-Montani '08 [140]). We will outline at the end some scenarios of interest.

5.1 Kaluza-Klein model with source

To introduce matter, let us define a 5D matter tensor $\mathcal{T}^{AB}$. The dynamics of fields and matter will be given by the 5D Einstein equation in presence of matter

$$^{5}G^{AB} = 8\pi G_5 \mathcal{T}^{AB} \,, \tag{5.1}$$

where G_5 is the 5D Newton constant. Such an equation could be derived from the 5D Lagrangian density

$$S_5 = -\frac{1}{16\pi G_5}\sqrt{J}^5 R + \sqrt{J}^5 \mathcal{L}$$

where the first term is the 5D Einstein-Hilbert Lagrangian and the second is an unknown Lagrangian for the 5D matter. Our 5D matter tensor is defined in term of this Lagrangian as follows:

$$\mathcal{T}^{AB} = 2\left(\frac{\delta^5 \mathcal{L}}{\delta J^{AB}} - \frac{1}{2} J_{AB}{}^5\mathcal{L}\right)$$

Moreover , given that the Einstein tensor does not depend on the fifth coordinate, we assume that the cylindricity hypothesis also holds for the matter tensor. Furthermore, we observe that the 5D reduced Bianchi identities hold ([41]), therefore the divergence of the tensor vanishes, giving us a conservation equation, i.e.

$$D_A {}^5G^{AB} = 0 \rightarrow D_A {}^5\mathcal{T}^{AB} = 0\,,$$

where D_A stands for the 5D covariant derivative. Actually, given the breaking of the 5D covariance that we have in KK, the proof of the validity of the 5B Bianchi identities is not a trivial task, and requires much more calculations than the analogue proof in 4D. However, apart from algebraic difficulties, it is possible to prove that 5D Bianchi identities stand. Indeed, from a physical point of view, the 5D covariance is broken, but still holds the invariance with respect to 5D translations, and we could consider the Bianchi identities and the conservation law for the 5D tensor as an implementation of such an invariance. Given the coordinate length of the extra dimension, $l_5 = \int dx^5$, we now define $G = G_5 l_5^{-1}$, $T^{AB} = \mathcal{T}^{AB} l_5$, being G the usual Newton constant. Finally, we deal with the following model:

$$D_A T^{AB} = 0 \qquad \partial_5 T^{AB} = 0 \tag{5.2}$$

$${}^5G^{AB} = 8\pi G T^{AB} \tag{5.3}$$

The first of the above considered equations reproduces the model we employed for the Papapetrou procedure (4.15). The dimensional reduction of eq. (5.2), as we have seen in the previous chapter yields the following set:

$$5) \quad \rightarrow \nabla_\mu \left(\phi T^\mu_5\right) = 0 \tag{5.4}$$

$$\mu) \quad \rightarrow \nabla_\rho(\phi T^{\mu\rho}) = -g^{\mu\rho}\left(\frac{\partial_\rho \phi}{\phi^2}\right) T_{55} + ekF^\mu_{\ \rho}\phi T^\rho_5 \tag{5.5}$$

looking at the arguments of the ∇_μ operator, it is convenient define:

$$T^{\mu\nu}_{matter} = \phi T^{\mu\nu} \qquad j^\mu = ek\phi T^\mu_5 \tag{5.6}$$

Indeed, we are borrowing such a definitions from the Papapetrou procedure: $\phi T^{\mu\nu}$ and ϕT^{μ}_{5} were the objects on which we addressed the localization hypothesis. Therefore, at this step, we are doing nothing else than a generalization of what we have learned from the approach to the test particles motion. There is, however, an heuristic argument we could consider to provide an enforcement to this definitions. Let us compare the definition of the starting tensor $\mathcal{T}^{AB}$ with respect to the definition of the matter tensor $T^{\mu\nu}_{matter}$ in GR; let $\mathcal{T}^{x}$ be one of the components of $\mathcal{T}^{AB}$, and T^{y}_{matter} a component of $T^{\mu\nu}_{matter}$; the requirements we address to the definition of the matter tensor give us that the physical dimension of such components should be a density of energy with respect to the unity of volume. Indeed, the hypothetic 5D observer, will perceive a 4D spatial volume, including the contribute of the extra dimensions, therefore will have these two definitions:

$$5D \quad \rightarrow \quad E^{x} = \int dx^{5} d^{3}x \sqrt{J} \mathcal{T}^{x} \tag{5.7}$$

$$4D \quad \rightarrow \quad E^{y} = \int d^{3}x \sqrt{g} T^{y}_{matter} \tag{5.8}$$

Now, using the cylindricity hypothesis , the compactification, and the reduction formula $\sqrt{J} = \phi \sqrt{g}$ we have:

$$5D \rightarrow E^{x} = \int d^{3}x \sqrt{g} \left(\phi l_{5} \mathcal{T}^{x} \right) \tag{5.9}$$

Then, comparing now the equations, this suggests indeed to define

$$T^{\mu\nu}_{matter} = l_{5} \phi \mathcal{T}^{\mu\nu} = \phi T^{\mu\nu}$$

Such a procedure does not provide a proof, but we consider it just a hint that, in looking for the 4D objects that should represent the 4D matter tensor, the above definition would provide a good candidate to be analyzed. Extending the procedure to the T_{55} component we also define, as well as we did within the Papapetrou approach,

$$\vartheta_{55} = \phi T_{55}$$

Taking now into account these definitions the reduction of eq. (5.3) leads to the set

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\phi^2 T^{\mu\nu}_{electrom.} + 8\pi G\frac{T^{\mu\nu}_{matter}}{\phi} \tag{5.10}$$

$$\nabla_{\nu}\left(\phi^3 F^{\nu\mu}\right) = 4\pi j^{\mu} \tag{5.11}$$

$$\frac{1}{2}R + \frac{3}{8}\phi^2(ek)^2 F^{\mu\nu}F_{\mu\nu} = 8\pi G\frac{\vartheta_{55}}{\phi^3}$$

Arranging the last equation with the trace of the first we have:

$$\Box\phi = -\frac{1}{4}\phi^3(ek)^2 F^{\mu\nu}F_{\mu\nu} + \frac{8}{3}\pi G\left(T_{matter} + 2\frac{\vartheta_{55}}{\phi^2}\right) \tag{5.12}$$

This set of equations is supported by eq. (5.5, 5.4) which now we rewrite in terms of new variables:

$$\nabla_{\rho}(T^{\mu\rho}_{matter}) = -g^{\mu\rho}\left(\frac{\partial_{\rho}\phi}{\phi^3}\right)\vartheta_{55} + F^{\mu}_{\ \rho}j^{\rho} \tag{5.13}$$

$$\nabla_{\mu}j^{\mu} = 0 \tag{5.14}$$

Equations (5.10, 5.11, 5.12, 5.13, 5.14) therefore describe the KK dynamics in presence of matter source terms.

The above considered procedure therefore allows to consider a simultaneously reduction of matter and geometry terms: the 5D metric tensor is split into its tensor component, $G^{\mu\nu}$, its vector component A_{μ}, and its scalar component ϕ; at the same time the 5D matter tensor is reduced into a tensor component $T^{\mu\nu}_{matter}$, a vector component j^{μ} and a scalar component ϑ_{55}. With respect to this procedure, and also recalling the Papapetrou procedure, an established physical meaning is assigned to $T^{\mu\nu}_{matter}$ and j^{μ}: they are the matter tensor and the electromagnetic current, respectively, of the observed 4D matter. The scalar source term ϑ_{55} is, indeed, a novelty of this model, as well as it is its integral partner A with respect to test particles dynamics and its physical meaning will deserve further investigations.

Furthermore, we observe at this step that the problem concerning the scenario $\phi = 1$ is now removed. In such a scenario eq. (5.13) leaves ϑ_{55} undetermined, eq. (5.10, 5.11) describe a Einstein-Maxwell system, and finally eq. (5.12) turns into a condition which uniquely fixes ϑ_{55} for a given matter + fields background. Hence, it is possible to consider a scenario for metric fields without including the scalar field and avoiding inconsistency. It is worth noting the existence of a curious feature; in principle it seem that is not forbidden to have charged massless matter, due to presence of *phi* in the concerned equations. Such a scenario is considered for instance, in ([109]). However, being the radiation addressed to the electromagnetic tensor $T^{\mu\nu}_{electrom}$, one could ask if $T^{\mu\nu}$ is suitable for describing massless matter. Anyway, setting $\phi = 1$ we recover the constraint that charged matter cannot be massless. Indeed, we notice that setting $\phi = 1$ eq. (5.13) gives the standard EM coupling between matter and fields. In this scenario we now have:

$$
\begin{aligned}
& G^{\mu\nu} = 8\pi G\,(T^{\mu\nu}_{electrom.} + T^{\mu\nu}_{matter}) \\
& \nabla_\nu\,(F^{\nu\mu}) = 4\pi j^\mu \\
& \nabla_\rho(T^{\mu\rho}_{matter}) = F^\mu{}_\rho j^\rho \\
& \nabla_\mu j^\mu = 0
\end{aligned}
$$

Therefore, we finally deal with the Einstein-Maxwell dynamics with source term. With this respect, having solved the inconsistency problems related to the introduction of matter, we can consider the KK model as a first step in the search for a unification theory.

If we consider the presence of the extra scalar terms ϕ and ϑ_{55}, then we can consider the outcome of the KK model as a modified gravity theory: such terms can be considered as valuable degrees of freedom to deal with dark matter and dark energy in models of modified gravity ([110], [111], [112], [113], [114], [115],)

5.2 Suitable scenarios

In the following we outline some simple scenarios. For sake of simplicity, from now on we just assume $j_\mu = 0$, $F_{\mu\nu} = 0$.

5.2.1 Free Falling Universality scenarios

Putting together the fields dynamics with the test particles dynamics we now consider some interesting scenario which in our opinion deserve some effort to be pursued. The equations we are interested in this section (5.12, 4.44, 4.43) :

$$\Box\phi = \frac{8}{3}\pi G\left(T_{matter} + 2\frac{\vartheta_{55}}{\phi^2}\right)$$
$$\frac{dm}{ds} = -\frac{A}{\phi^3}\frac{d\phi}{ds}$$
$$m\frac{Du^\mu}{Ds} = A(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho\phi}{\phi^3}$$

Now let us consider some hypothesis on ϑ_{55}

- $2\vartheta_{55} = -\phi^2 T_{matter}$

  This is the simplest case: we have as a suitable solution $\phi = 1$; therefore we restore $m = cost$, and the motion equation becomes $\frac{Du^\mu}{Ds} = 0$: we rebuild the Free Falling Universality (FFU) of the particle. Therefore we have exactly the GR theory.

- $\vartheta_{55} = 0$

  Within this scenario we have $m = cost$, being $A = 0$: and, again, we have $\frac{Du^\mu}{Ds} = 0$, therefore we have again FFU. But, in principle, ϕ is still variable and we can look for effects of physics beyond GR.

- $A = \alpha m\phi^2$

  In such a case we still have FFU, because m cancels from the motion equation, i.e.

we have

$$\frac{Du^\mu}{Ds} = \alpha(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho \phi}{\phi}$$

But now we have ϕ variable as well as m. Noticeably, equation (4.43) can be now easily integrated and we have a scaling law for the mass:

$$m = m_0 \left(\frac{\phi}{\phi_0}\right)^{-\alpha} \tag{5.15}$$

Such kind of equation is very interesting for various model of dark matter, which consider, for instance test particles of varying mass ([116]).

5.2.2 Homogeneous solutions

An interesting task to deal with it is the search for homogeneous and isotropic solutions. We faced this topic in the section (2.4), in Friedmann and DeSitter like models, where we was able to write down some solutions concerning the 3D scale factor $a(t)$ and the scalar field ϕ. In that case, we dealt with solutions in vacuum, but now we are able to take into account the presence of matter. Therefore , it will be an interesting issue to perform the Friedmann/DeSitter approach to the full model including matter. This will allow, in principle to find more detailed solutions for the behaviour of the scalar field and, moreover to search for effects beyond the GR , due to the presence of two extra scalar degrees of freedom, which can be compared to dark matter /energy models . Indeed , to deal with scalar fields in order to provide scenarios for dark matter it is a common line of researching. The scalar field ϕ could be linked to dark energy effects, while the source term ϑ_{55} can be viewed as a dark matter source term. For instance, as a first step we could check simple scenario of the form

$$\vartheta_{55} = \phi^k \left(\alpha\rho + \beta p\right) , \tag{5.16}$$

where k, α, β are free parameter (apart from further constraint eventually requested in order to give physical solutions) and ρ and p are the energy density and the pressure given by the energy momentum tensor of a perfect fluid, which is indeed adopted in homogeneous models ([93]):

$$T^{\mu\nu}_{matter} = (\rho + p)u^{\mu}u^{\nu} - g^{\mu\nu}p \tag{5.17}$$

Usually, in homogeneous models one adopts as a viable equation of state the following condition:

$$p = \gamma\rho$$

The values of the parameter γ then are linked to various era of the universe. Therefore, eq. (5.16) should be considered as a viable generalization of the above equation of state. With respect to this topic two last points deserve a remark. At first, we remind that a key point in the Friedmann dynamics is the possibility to define comoving coordinates for the observer ([93]). Indeed, given the hypotheses of homogeneity and isotropy of the space, and assuming an observer at rest with respect to the homogeneous matter background, it is possible to adopt a synchronous metric $g^{\mu\nu}$; such a metric is characterized by having vanishing *Shift* function and a unitary *Lapse* function, i.e. we have

$$g^{00} = 1 \qquad g^{0i} = 0 \tag{5.18}$$

With respect to this metrics, the so-called comoving coordinates of the rest observer read

$$u^{\mu} = (1, 0, 0, 0)\,, \tag{5.19}$$

which are moreover geodesic coordinates, i.e. they solve the equation $\frac{Du^{\mu}}{Ds} = 0$ with respect to a metrics constrained by (5.18).

Now we show that also in this KK model , the comoving coordinates (5.19) solve the motion equation (5.15). We assume a 4D metrics constrained by (5.18) and we suppose that the scalar field depends only on time t. Therefore the motion equation (

5.15) yields the set

$$m\frac{Du^0}{Ds} = A\left[\left(u^0\right)^2 - g^{00}\right]\frac{\dot{\phi}}{\phi^3} \tag{5.20}$$

$$m\frac{Du^i}{Ds} = A\left[u^0u^i - g^{0i}\right]\frac{\dot{\phi}}{\phi^3} \tag{5.21}$$

Now, taking into account the conditions (5.18) if we insert as a probe solution the (5.19) into the right member then it leaves us with the geodesic equation $\frac{Du^\mu}{Ds}$, which, as noticed before, admits (5.19) as its solution. Hence, this shows us that, although we are dealing with a modified motion equation, we can still address to the comoving observer the 4D velocity (5.19) as a suitable solution of the motion equation.

Such a statement leads us to another interesting observation. Let us consider for instance a dust matter tensor; in such a case we have $p = 0$ and therefore the parameterization (5.16) reads:

$$\vartheta_{55} = \phi^k \alpha \rho$$

Thus, it is worth noting that, setting $k = 2$, assuming ϑ_{55} and ρ localized in order to define a test particle, adopting comoving coordinates, such a parameterization give us, for a test particle (4.45, 4.47) the following relation:

$$A = \alpha\phi^2 m$$

The above equation is indeed one of the scenarios (5.15) we previously considered in order to restore the FFU of test particle.

Therefore, the framework of homogeneous background provide a promising arena where to test this model. Moreover, in addition to a comparison with respect to the homogeneous dynamics, it looks like interesting to calculate the growth of perturbations and compare the outcomes with some cosmic observable, like, for instance the luminosity distance from Supernovae IA ([117]).

5.2.3 Spherical solutions

Another interesting task to be addressed is the search for spherical solutions. By "spherical solutions" we mean those solutions of the 5D reduced Einstein equations which have a 4D metrics $g^{\mu\nu}$ and a scalar field ϕ isotropic which respect to rotations in the 3D space. In general, these solutions do not coincide with the 5D spherical solutions, which are given by a metrics J_{AB} isotropic with respect to rotation into the 4D space given by the ordinary one plus the extra dimensions. Our spherical solutions, therefore have to be characterized by a 4D Schwarzschild-like metrics. There are various reason to have interest in this topic. From a theoretical point of view, we are interested in the properties of matter, and especially in the physical meaning of the definition of mass, which , in this model, turns out to be modified by the presence of scalar terms. Thus, the analysis of spherical objects, as it happens in the 4D theory, is the natural extension of our study, in order to deal with the gravitational mass. From the point of view of the suitable applications of this model, as well as we discussed for the homogeneous solutions, it is interesting to look for effects of new physics, in order to provide prediction which can be, hopefully, verified or falsified by experimental test, and to compare the outcome with some dark matter scenarios. In particular, with respect to the problem of dark matter, an interesting scenario which deserves a mention concerns the so-called Small Mass Black Holes (SMBH) ([118]). It consists in searching an extra-dimensional scenario for the formation of such objects. The SMBH are supposed to have mass $m \sim 10^{17}g$ and a Schwarzschild radius between the size of the proton and of the atom. The SMBH model is strictly linked to an experimental evidence of $511\,keV$ annihilation line in the Galactic Centre. These objects, with the mechanism of their accretion disk, could reproduce this emission. But, more important, they can be a good candidate for dark matter. Their mass is too small to be detected by micro-lensing and too big to evaporate by Hawking scenario. The problem is that the usual 4D fluctuation scenario seems not able to explain these objects; it provides a continuous spectrum, depending on the mass, while it would be preferable to have a peak on the supposed mass that fits with the observed emission. Therefore, a suitable scheme is to apply the KK model, that provides extra parameters (*i.e.*, scalar fields) and check how these new degrees of freedom could affect the mass of the black holes and its formation scenario. Let us now turn back to the general topic concerning spherical solutions. The set of equations which has to be solved is the following:

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\frac{T^{\mu\nu}_{matter}}{\phi} \tag{5.22}$$

$$\Box\phi = \frac{8}{3}\pi G\left(T_{matter} + 2\frac{\vartheta_{55}}{\phi^2}\right) \tag{5.23}$$

$$\nabla_{\rho}(T^{\mu\rho}_{matter}) = -g^{\mu\rho}\left(\frac{\partial_{\rho}\phi}{\phi^3}\right)\vartheta_{55} \tag{5.24}$$

The usual first step is to deal as first with the solution in vacuum, which has to provide the "exterior" solution, describing the behaviour of metric fields outside the spherical objects which is supposed to be at the centre of the geometry, acting as the source. Therefore we have:

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi \tag{5.25}$$

$$\Box\phi = 0 \tag{5.26}$$

It is worth noting that, in order to deal with vacuum, it is enough to assume $T^{\mu\nu}_{matter} = 0$: indeed, eq.(5.24), apart from the case $\phi = cost$ which now we are not interested in, gives us the condition $\vartheta_{55} = 0$. Moreover, it could be shown, as it happens in the 4D theory, that these equations are not all independent equations. From a mathematical point of view, hence, solving the KK equations in vacuum is equivalent to solve 4D Einstein equations coupled to a tensor $T^{\mu\nu}_{\phi}$, where we define

$$T^{\mu\nu}_{\phi} = \frac{1}{8\pi G}\left[\frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi\right] \tag{5.27}$$

As given by eq. (5.26) such a tensor is traceless, therefore we expect it to act like a radiation source term, as it was indeed noticed by some authors ([119]). However, we consider such an equivalence just as a mathematical one. The field ϕ is a geometry field, as well as $g^{\mu\nu}$, and its source has to be looked for in the matter terms.

In principle, what we have to do now is just to put in the above system a Schwarzschild-like metrics $g^{\mu\nu}$ and start with the algebraic machinery. Unfortunately, the addition of an extra dimension strongly increases the mathematical complexity of the equations, with respect to what happens in the 4D theory. For instance, it could be proved that the Birkhoff theorem does not hold ([120], [121], [122], [43]), therefore a spherical solution is not necessarily static; at the same time, a Schwarzschild-like solution is not necessarily the unique solution.

Some spherical solutions of the vacuum model are known ; they are due to Sorkin ([123]), Gross-Perry ([120]), Davison-Owen ([124]) and have been thereafter re-examined by Wesson et al. ([121], [41], [42], [125]). The prototype solution has the following parametrization: we assume $\phi = \phi(r)$, being r a 3D radial coordinate, and a metrics $g^{\mu\nu}$ such that

$$ds^2 = A^2(r)^2 dt^2 - B^2(r)\left(dr^2 + r^2 d\Omega^2\right)$$

The solution reads:

$$\begin{cases} A(r) = \left(\frac{ar-1}{ar+1}\right)^{\epsilon k} \\ B(r) = \frac{1}{a^2 r^2}\frac{(ar+1)^{\epsilon(k-1)+1}}{(ar-1)^{\epsilon(k-1)-1}} \\ \phi(r) = \left(\frac{ar+1}{ar-1}\right)^{\epsilon} \end{cases} \tag{5.28}$$

where a is an integration constant related to the mass of the central object and the parameters ϵ and k are constrained by the following relation:

$$\epsilon^2(k^2 - k + 1) = 1$$

The above mentioned solution is a static one; there exist generalizations of such a solution which deal with a factorized dependence on time of the form $A, B, \phi = f(r)g(t)$.

Various efforts have been done to extract a physical meaning from these solutions (see, for instance, the report [32]). At first, let us consider the following limit:

$$\epsilon \to 0\,, k \to \infty\,, \epsilon k \to 1$$

In such a case, by defining $a = \frac{2}{GM}$ we obtain:

$$ds^2 = \left(\frac{1 - GM/2r}{1 + GM/2r}\right)^2 dt^2 - \left(1 + \frac{GM}{2r}\right)^4 \left(dr^2 + r^2 d\Omega^2\right) \tag{5.29}$$

$$\phi = 1 \tag{5.30}$$

Therefore, we can recognize the usual exterior Schwarzschild solution written in isotropic coordinates; depending on the value of the mass M we will eventually deal with a black hole solution. Such an outcome is achieved in the limit $\phi = 1$ and hence, is in agreement with the general requirement that in such a case KK model would reproduce GR. For this reason, the general solution above considered is called the Generalized Schwarzschild Solution (GSS). Let us turn our attention to the feature of the general solution. The most striking feature is that, although it posses a limit where a black hole dynamics is recovered, it does not represents in general a black hole. Actually, the above considered limit is the only scenario for this class of solutions where we deal with a black hole. To define a black hole we at first require it to have an horizon event. Let us consider the horizon event: it is usually defined as the surface where the norm of the timelike Killing vector vanishes. In our case the Killing vector simply reads (1, 0, 0, 0), therefore its norm vanishes where g_{00} does. Looking at (5.28), we can see that this happens on the surface $r = \frac{1}{a}$, in the limit $r \to \frac{1}{a^+}$ for $k > 0$ and $\epsilon > 0$. After this, we have to check that the singularity given by the horizon event is not a physical one; we require that the physical singularity only stands at the centre of the geometry. Hence, let us now examine the fundamental 4D invariant $C = R^{\mu\nu} R_{\mu\nu}$; given (5.28) it yields:

$$C = \frac{1}{(a^2 r^2 - 1)^8} \left(\frac{ar - 1}{ar + 1}\right)^{2\epsilon(k-1)} p(r)\,,$$

where $p(r)$ is a regular polynomial , which is non zero when $r = \frac{1}{a}$. Given the constraint on ϵ and k , C diverges when $r = \frac{1}{a}$: therefore $\frac{1}{a}$ represents a physical singularity. To better understand what happens in such a surface, let us consider the surface area of 2-shells: it varies as $(ar-1)^{1-\epsilon(k-1)}$ and therefore vanishes when r approaches to $\frac{1}{a}$ (given $k > 0$). It means that the manifold ends at $r = \frac{1}{a}$ and that the point $r = 0$ is not included in the manifold. Indeed, the centre of the geometry has to be considered the point $r = \frac{1}{a}$. Thus, it happens that GSS solutions show an horizon event but the surface of such an horizon is shrinked to a point which coincides to the physical singularity of the model at the centre of the manifold. Therefore, correctly speaking, the GSS should be considered naked singularities or solitons, as noted by many authors ([125], [126], [127], [128], [129]). Hence, it could be debated if such objects, in the vacuum model, could be realized , accordingly to the censorship hypothesis, which forbids the existence of naked singularity unprotected by an horizon event. This is an intriguing issue, because, as noticed by Wesson ([130]), if these objects exist, then they could be an interesting candidate for dark matter. Until now, we have discussed the vacuum solution; indeed, taking into account the presence of matter, which is the novelty of our work, we should be able to consider GSS solutions where the singularity at the centre of the geometry becomes now dressed with matter. Therefore, taking into account the presence of matter we should be able, in principle, to find the interior solution concerning the GSS scenario. Within this hypothetic scenario these objects loose their characterization as solitons and it could be argued that they are not black holes due to the presence of the scalar field which acts as a radiation field, providing a pressure which works against the gravitational collapse. Indeed , in ([131]) it is considered a black holed dressed by a massive scalar field. The study of the interior solution is of great importance also for the analysis of the meaning of the mass. For instance, in the framework of the vacuum model, the solutions (5.28) allow us, via the Tolman-Whittaker ([132], [133], [134], [135], [125]) procedure, to rebuild a behaviour for the gravitational mass , as it is perceived in the exterior region:

$$M_g(r) = 2ek \left(\frac{ar-1}{ar+1} \right)^{\epsilon} = 2ek\phi^{-1}$$

We can see that this behaviour fits with the equation (5.15) for the case $\alpha = -1$. Unfortunately, the presence of matter in the fields equations provides us a system of differential equations of great mathematical complexity. The resolution of such a system goes beyond the aim of this work, but it represents a valuable task to be addressed. At the same time, it looks like interesting to search for solution of the form $A, B, \phi = f(r, t)$; given that GSS solutions does not represents black holes, it will be interesting to check if black holes modified solutions exist in such a parameterization.

5.2.4 Compactification hypothesis

We end this chapter by presenting , just as a hint which deserves further investigations, a purpose to link the features of matter we have outlined in this work with a suitable compactification scenario. Indeed, the idea that the compactification should be related to some matter behaviour is envisaged in ([22], [23]). As we have seen by analyzing the Klein-Gordon dynamics, the cylindricity condition is implemented in the assumption that the wave function of an observable depends on the extra coordinate only via a phase factor, i.e.

$$\Psi = \eta(x^{\mu})e^{ip_5 x^5}$$

Therefore we can figure that, if the wave function is naturally constrained to collapse in an eigenstate of the operator $\hat{p}_5$ we recover the cylindricity as an effective classical limit. Let us figure now a behaviour for the scalar field such that , during its cosmological evolution it decreases, leading to the shrinking of the physical length L_5 of the extra dimension. Therefore, let us suppose that the extra dimension collapses below the bound of our observations and, at the same time it shrinks below a typical scale length of the wave function, let us say its indetermination Δx^5. With this respect, taking to the extreme the limit $\Delta x^5 >> L_5$ we can describe the particle by adopting the condition $\Delta x^5 = \infty$. Hence, a particle described by such a wave function turns out to be delocalized within the extradimension, as pointed out during the Papapetrou procedure, and the wave function collapses in an eigenstate of the operator $\hat{p}_5$, leading therefore to the cylindricity conditions. We discussed similar behaviour for the scalar field when we analyzed the vacuum solutions; a more detailed study in this sense, with respect to the model containing matter, acquires even more interest.

Conclusions

Let us now summarize the results of this work. At first we discussed the hamiltonian formulation of the dynamics and we faced the problem of the ADM reformulation of the Kaluza-Klein model: it was related to the identification of the correct application order of the ADM slicing procedure with respect to the dimensional reduction procedure. Our result, in this respect, is that the dynamics is insensitive to the choice of the order, granted the assumption of suitable boundary condition ([136], [137]). Such a result is enforced by the proof that, regardless the procedure order, the same definition for the time variable arises. In some sense, a different result would lead to a weakening of the physical meaning of the model. Moreover, we have seen how the electromagnetic constraint on the dynamics arises as a particular case of the more general hamiltonian constraints ([136], [137]), which are related to the *Shift* functions and due to the diffeomorphism invariance. Such a result is in agreement with the statement that the gauge symmetry is, in multidimensional scenarios, a particular case of diffeomorphism invariance, and thus enforces the physical characterization of the KK model as a viable step in the search of a unification theory. Provided the ADM reformulation, we considered the Hamiltonian formulation of the model, which can be an important tool for further analysis. In this work we just focused on some properties of the dynamics of the scalar field. We provided some solutions in homogeneous background which envisage a close relation of the scalar field to the 3D scale factor and, more important, we faced a Brown-Kuchar approach. With this respect the obtained result is that the scalar field ϕ deparametrizes from the hamiltonian constraint and it is therefore possible to write down a Schroedinger-like equation, where the conjugate momentum to ϕ would play the role of a relational time ([142]). Actually, our analysis deserves further investigations, but it is indeed a first step toward a full exploration of the suitable role of ϕ as a time variable. To achieve a better comprehension, with respect to this scheme, it would be necessary to consider the full dynamics of the model, involving matter. This was indeed the topic faced in the second part of the work.

Af first we re-analyzed from a critical point of view the usual geodesic approach, either reviewing the Eulero-Lagrange equations of the motion, either adding a new analysis based on the hamiltonian formulation ([139], [138]). At the end of this analysis, we made the conclusion that the unphysical outcomes that arise, when dealing with the test particles motion in Kaluza-Klein theory, should not be addressed to the Kaluza-Klein approach, or in more detail to the compactification hypothesis, but they are rather the consequence of an incorrect generalization to the 5D scenario of the statements underlying the 4D geodesic approach. Indeed, we pointed out that in the Kaluza-Klein model the 5D Equivalence Principle is broken and, at the same time, due to the unobservability of the extra dimension, the generalization of the geodesic approach is misdealing in order to achieve the proper definition of a test particle. Therefore, we discussed a revised approach, grounded on a rigourous procedure able to face the definition of a point like particle starting from a generic matter tensor ([139], [138]). Within this scheme, we were able to write a new equation for the particle motion, which is indeed very similar to the geodesic equation, but it differs in the definition of the coupling factor, which now are not affected by bounds on their value. A key point in this procedure was the assumption of a cylindrical matter tensor; such assumption is in agreement with the cylindricity hypothesis we addressed to the dynamics of matter field or, generally speaking, to the demand that, at the present allowable energy scale, our observable does not depend on the fifth coordinate. Indeed, the striking feature that arises from this revised approach is that the particles turns out to be delocalized within the extra dimension and therefore, the localization of matter is achieved just in the usual 4D space time. This reason explains why the 5D geodesic approach is not able to describe the particle motion. If we consider the minimal scenario $\phi = 1$, such a new approach reproduces exactly the electrodynamics, either for the test particles scenario, either for a generic matter tensor. If we consider the presence of the scalar field, then a variable rest mass is recovered, as well as it is recognized the presence of a new coupling between matter and the scalar field; such features are indeed a novelty of this model and further analysis is needed in order to better understand their role. In principle, it is possible to consider minimal scenarios where their effects vanish, but it is as well interesting to compare them with the features of models where a modified gravity is considered. Such an interpretation is enforced by the study of the related Klein-Gordon dynamics. Indeed, during our analysis, we recognized that the generation of the Kaluza-Klein tower of huge massive modes is a direct consequence of the quantum dynamics associated to the effective geodesic Action. Therefore, we calculated the effective Action corresponding to the revised equation and we got the related Klein-Gordon dynamics. Within this new framework, we recognized that the massive tower disappears, due to the presence of a proper counterterm in the dispersion relation. Such a result is achieved regardless the presence of the scalar field; the removal of the Kaluza-Klein tower is probably the most original result we got within this approach. It should be stressed that such a result is a direct consequence of the fact that, in this scheme, due to the delocalization of the particle within the fifth dimension, the mass is no more related to the momentum conjugate to the extra-dimensional degree of freedom. The outcomes obtained with this revised approach to motion allowed us to define a complete reduction for matter and geometry, as far as a Kaluza-Klein model with source is concerned. This was the topic faced in the ending part of the work ([140]).

Using the consistent approach to matter provided by our revised procedure, we were able to write down the equations for the full Kaluza-Klein dynamics involving fields and source terms. This was done considering the dimensional reduction of a generic 5D matter tensor. Borrowing the results obtained for the test particles dynamics, and generalizing them, we were able to separate a tensor source, a vector source, and a scalar one, which we identified respectively with the usual 4D matter tensor, with the charged current and, finally with a new scalar term, related to the extra dimension, which is a novelty provided by this scheme ([140]). As it was recognized for the particles dynamics, the minimal scenario $\phi = 1$ reproduces the Einstein-Maxwell dynamics, while the general case looks like a theory of gravity modified by two additional degrees of freedom. But, within this scenario, and due to the presence of matter, is now possible to consider the hypothesis $\phi = 1$ into the motion equation, without occurring in inconsistencies.

Therefore the picture of the compactified model we discussed in this work represents a new point of view, with respect to the problem of matter in Kaluza-Klein theories, and in our opinion it deserves further investigations. If we restrict to the minimal scenario $\phi =$, which is now allowed as a suitable solutions of the reduced 5D Einstein equations, then we reproduce the Einstein-Maxwell dynamics, either for fields, either for matter, providing consistent definitions for the involved coupling factors. With respect to such a scenario, the Kaluza-Klein model should be considered as a valuable toy-model in order to develop a unifying geometrical picture of interactions. In this sense, it should be stressed that the revised approach we discussed allows us to keep in the extradimensional scenario the presence of an extra dimension compactified at a very low scale; this is relevant, because it provides us an elegant explanation for the discretization of the charge and for the generation of the local U(1) gauge symmetry. With respect to this scenario, the perspective of an extension of this approach to multidimensional models with more than five dimension should be considered in order to face the coupling of the other interactions. As far as just the 5D model and its phenomenological implications are considered, however, the most promising perspectives involve the presence of the scalar field in the dynamics. Indeed, there is a close interplay between the features of the scalar field and the possibility to consider the presence of matter in this model. As far as the behaviour of the field ϕ is considered, the presence of matter in the fields equations allows us, apart from mathematical difficulties, to search for solutions of the motion more meaningful than the solutions we considered in vacuum. This is particular relevant as far as the Kuchar-Brown approach is taken into account and the possibility to define a relational time is considered. Other promising tasks concern the general solutions of the fields equation; we have seen a close link, in vacuum, between the scalar field and the scale factor $a(t)$, therefore it is worth to examine the impact of ϕ on the cosmological behaviour, especially near the chaos regime or the inflationary era. Moreover, we have suggested that a decreasing behaviour of the field ϕ could be responsible of the shrink of the fifth dimension, giving a close link between a suitable compactification scenario and the delocalization of particles within the extra dimension; with this respect, an analysis of the suitable behaviours of the scalar filed should be recommended. Another interesting task to be addressed is the analysis of the path deviation equation in the revised framework; we discussed this topic within the scheme of the geodesic approach ([141]); therefore it is worth performing a comparison between these two approaches. A natural arena where is possible to test the outcomes of this model is the framework of modified gravity theories. Indeed, among various approaches to the dark matter/energy puzzle, a common one is to deal with modification addressed to the presence of scalar field ([110]-[117]). With this respect, it appears interesting to develop the model in some simplified scenario, like the framework of an homogeneous background or spherical symmetry. A comparison with theory of extended gravity indeed, would give insight about the possibility of this model to deal with dark matter scenario and provide constraint on the scalar degree of freedom (which in principle could be ruled out). It can be stressed, for instance, that the model we discussed is able to provide a physical ground for the existence of modified spherical solutions, which otherwise should be considered solitons or naked singularities ([123]-[130], [41]-[43]), and, in more detail, to provide a scenario for the search of modified black holes ([118]). It is worth noting, finally, that at the end of the work we outlined some scenarios where, although some modified effects are still taken into account, the Free Falling Universality of particles is recovered. Among these minimal scenarios, the most promising one is that where we chose to parametrize the extra scalar source in terms of a power of the field ϕ times a linear combination of density and pressure; depending on the parameter of this picture, such a representation fulfils at the same time the Free Falling Universality of the particles motion, plus some constraint derived in the framework of homogeneous and spherical background. Therefore, it appears as a good candidate for a physical check of the model, either with respect to a Friedmann/Schwarschild-like framework either for a study involving perturbations with respect to an assigned homogeneous background.

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Finally, there is something last i would like to tell to my close friends and relatives, and i do it in my mother tongue.

Non é stata una passeggiata, ma é stato bello. Non é stato semplice, ma é stato entusiasmante. In questo lavoro confluisce molto di quello che ho vissuto in questi tre anni, sia per la mia crescita scientifica sia per la mia crescita personale. C'é dentro tanto tabacco e tanto caffé, notti insonni sui calcoli e albe davanti al pc, sia al freddo invernale di Annecy sia all'afa estiva di Roma; ci sono le ansie e le incertezze, ci sono i " ma chi me lo ha fatto fare....??", ma c'é anche tanta gioia: lo stupore infantile davanti a una formula inaspettata, l'orgoglio di dire "questo l'ho fatto io !!! ", la felicitá di trovare nel fondo di un portapenne il brillio di un'intuizione. Ci sono calcoli fatti e rifatti piú volte, tentativi riusciti bene e altri meno bene, ci sono litigi e discussioni, e perfino un aereo perso e un treno preso sí, di corsa, ma sotto una tormenta di neve ! C'é un pezzo significativo della mia vita, che mi ha formato e mi ha reso diverso, ed é stata una cosa buona avere attorno l'affetto delle persone a cui tengo. C'é chi mi ha potuto accompagnare solo per un pezzo: per scelte, o perché il suo tempo era finito, ma la loro presenza non é stata meno importante.
A mia madre,
A mio padre,
A mia sorella,
Alla mia compagna,
vorrei che sapeste che sono lieto, al termine di tutto questo, e che ne é valsa la pena: non per la scienza ma per la mia felicitá.

Attachments

On the ADM decomposition of the 5-D Kaluza-Klein model

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Abstract

Our purpose is to recast KK model in terms of ADM variables. We examine and solve the problem of the consistency of this approach, with particular care about the role of the cylindricity hypothesis. We show in details how the KK reduction commutes with the ADM slicing procedure and how this leads to a well defined and unique ADM reformulation. This allows us to consider the hamiltonian formulation of the model and can be the first step for the Ashtekar reformulation of the KK scheme. Moreover we show how the time component of the gauge vector arises naturally from the geometrical constraints of the dynamics; this is a positive check for the autoconsistency of the KK theory and for an hamiltonian description of the dynamics which wants to take into account the compactification scenario: this result enforces the physical meaning of KK model.

Hamiltonian Formulation of 5-dimensional Kaluza-Klein Theory

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Abstract

We analyze the consistency of the ADM approach to KK model; we prove that KK reduction commute with ADM splitting. This leads to a well defined Hamiltonian; we provide the outcome. The electromagnetic constraint is derived from a geometrical one and this result enforces the physical meaning of KK model. Moreover we study the role of the extra scalar field ϕ we have in our model; classical hints from geodesic motion and cosmological solutions suggest that ϕ can be an alternative time variable in the relational point of view.

On matter coupling in 5D Kaluza-Klein model

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Abstract

We analyze some unphysical features of the geodesic approach to matter coupling in a compactified Kaluza-Klein scenario, like the q/m puzzle and the huge massive modes. We propose a new approach, based on Papapetrou multipole expansion, that provides a new equation for the motion of a test particle. We show how this equation provides right couplings and does not generate huge massive modes.

Dynamics of Matter in a Compactified Kaluza-Klein Model

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Abstract

A longstanding problem in Kaluza-Klein models is the description of matter dynamics. Within the 5D model, the dimensional reduction of the geodesic motion for a 5D free test particle formally restores electrodynamics, but the reduced 4D particle shows a charge-mass ratio that is upper bounded, such that it cannot fit to any kind of elementary particle. At the same time, from the quantum dynamics viewpoint, there is the problem of the huge massive modes generation. We present a criticism against the 5D geodesic approach and face the hypothesis that in Kaluza-Klein space the geodesic motion does not deal with the real dynamics of test particle. We propose a new approach: starting from the conservation equation for the 5D matter tensor, within the Papapetrou multipole expansion, we prove that the 5D dynamical equation differs from the 5D geodesic one. Our new equation provides right coupling terms without bounding and in such a scheme the tower of massive modes is removed.

Geometry and Matter Reduction in a 5D Kaluza-Klein Framework

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Abstract

In this paper we consider the Kaluza-Klein fields equations in presence of a generic 5D matter tensor which is governed by a conservation equation due to 5D Bianchi identities. Following a previous work, we provide a consistent approach to matter where the problem of huge massive modes is removed; therefore we perform the dimensional reduction either for metric fields and for matter identifying a pure 4D tensor term, a 4D vector term and a scalar one. Hence we are able to write down a consistent set of equations for the complete dynamics of matter and fields ; with respect to the pure Einstein-Maxwell system we now have two additional scalar field: the usual dilaton one plus a scalar source term. Some simple scenarios involving these terms are discussed and perspectives for cosmological application are suggested.

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Dimensional Reduction of the 5D Kaluza-Klein Geodesic Deviation Equation

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Abstract

In a work of Kerner et al. (2000) the problem of the geodesic deviation in 5D KK is faced. The 4D space-time projection of the resulting equation coincides with the usual geodesic deviation equation in the presence of the Lorenz force, provided that the fifth component of the deviation vector satisfies an extra constraint which takes into account the q/m conservation along the path. The analysis was performed setting as a constant the scalar field. Here we focus on the extension of such a work to the model where the presence of the scalar field is considered. Our result coincides with that of Kerner et al. when the minimal case $\Phi = 1$ is considered, while it shows some departures in the general case. The novelty due to the presence of ϕ is that the variation of the q/m between the two geodesic line is not conserved during the motion; an exact law for such a behavior has been derived.

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Deparameterization of the 5D Kaluza-Klein scalar field in the Kuchar-Brown framework

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Abstract

Following a previous work, we consider the ADM splitting of the 5D Kaluza-Klein model and we give the Hamiltonian formulation of the dynamics which involves the extra scalar field and its conjugate momentum. In such a framework it is possible to see, via the Kuchar-Brown approach, that the momentum de-parametrizes from the hamiltonian constraints and we are therefore able to write down a Schroedinger-like equation. This is a hint toward the interpretation of the scalar field as a time variable in the relational point of view.

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