Integrable Extensions and Discretizations of Classical Gaudin Models

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PhD THESIS

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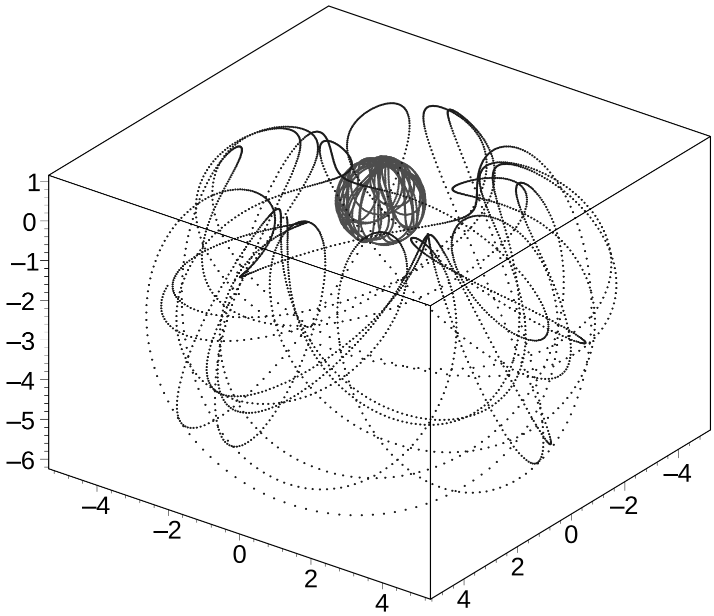
To Vadim B. Kuznetsov (1963–2005)



Integrable discrete-time evolution of the vectors $\mathbf{z}_1$ and $\mathbf{z}_1 - \mathbf{z}_2$ generated by:

$$\begin{cases} \dot{\mathbf{z}}_0 = [\,\mathbf{p}, \mathbf{z}_1\,], \\ \dot{\mathbf{z}}_1 = [\,\mathbf{z}_0, \mathbf{z}_1\,] + [\,\mathbf{p}, \mathbf{z}_2\,], \\ \dot{\mathbf{z}}_2 = [\,\mathbf{z}_0, \mathbf{z}_2\,]. \end{cases}$$

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Introduction

The field of integrable systems is born together with classical mechanics, with a quest for exact solutions of Newton's equations of motion. It turned out that apart from Kepler problem, which was solved by Newton himself, after two centuries of hard investigations, only a handful of other cases were found (harmonic oscillator, Euler top (1758), Lagrange top (1788), geodesic motion on the ellipsoid). In 1855 Liouville finally provided a general framework characterizing the cases where the equations of motion are "solvable by quadratures". All examples previously found indeed pertained to this setting and some other integrable cases were found (particle on the sphere under a quadratic potential (1859), Kirchoff top (1870), Kowalesvki top (1889)). The subject stayed dormant until the second half of the twentieth century, when Gardner, Greene, Kruskal and Miura invented the *classical inverse scattering method* for the famous Korteveg-de Vries equation. Soon afterwards, the *Lax formulation* was discovered and the connection with integrability was unveiled by Faddeev, Zakharov and Gardner. This was the signal for a revival of the domain leading to an enormous amount of results, and truly general structures emerged which organized the subject. More recently, the extension of these results to quantum mechanics led to remarkable results and is still a very active field of research.

1.1 A synopsis of finite-dimensional classical integrable systems

Almost all known integrable systems possess a *Lax representation*. In the situation of systems described by ordinary differential equations, a Lax representation for a given system means that there exist two maps $L : \mathcal{P} \to \mathfrak{g}$ and $M : \mathcal{P} \to \mathfrak{g}$, from the system's phase space $\mathcal{P}$ into some Lie algebra $\mathfrak{g}$, such that the equations of motion are equivalent to

$$\dot{L} = [\, L, M \,]. \tag{1.1}$$

The map L is called *Lax matrix*, while the map M is called *auxiliary matrix* of the Lax representation. The pair (L, M) is called *Lax pair*. Finding a Lax representation for a given system usually implies its integrability, due to the fact the Ad-invariant functions on $\mathfrak{g}$ are integrals of motion of the system (1.1), and therefore the values of such functions composed with the map L deliver functions on $\mathcal{P}$ serving as integrals of motion of the original system.

In the Hamiltonian context, there remains something to be done in order to establish the complete integrability, namely to show that the number of functionally independent integrals thus found is large enough, and that they are in involution. There exists an approach which incorporates the involutivity property in the construction of Lax equations, namely the *r-matrix approach*.

It is based on the following observation: usually the auxiliary matrix M in Eq. (1.1) may be presented as

$$M = R\,(f(L)), \tag{1.2}$$

where $R : \mathfrak{g} \to \mathfrak{g}$ is a linear operator, and $f : \mathfrak{g} \to \mathfrak{g}$ is an Ad-covariant function. This observation is related to the fact that integrable systems appear not separately, but are organized in *hierarchies*: to every covariant function f there corresponds a Lax equation of the form (1.1) with M given in Eq. (1.2). The linear operator R is called the *R-operator governing* *the hierarchy*. If R depends on the points of $\mathcal{P}$ it is called *dynamical*, otherwise it is called *constant*. In the following we shall deal with constant R-operators.

A remarkable feature of Eq. (1.1), is that they often can be included into an abstract framework of Hamiltonian equations on $\mathfrak{g}$. Precisely, there can be defined suitable Poisson structures on $\mathfrak{g}$, called *r-matrix algebras*, such that the corresponding Hamiltonian equations have the form (1.1) and the map L is Poisson. There exist several variants of r-matrix structures. The most important ones are the so called *linear r-matrix brackets* and *quadratic r-matrix brackets*. The linear r-matrix brackets are certain Lie-Poisson brackets on $\mathfrak{g}^*$, where the dual space $\mathfrak{g}^*$ is identified with $\mathfrak{g}$ by means of an invariant scalar product. The definition of the quadratic r-matrix brackets requires the introduction of suitable associative algebras, so that somewhat more than just a Lie commutator is needed. In the following we shall deal just with linear r-matrix structures.

The aim of this Section is to briefly introduce some of the standard techniques used in the modern theory of classical finite-dimensional integrable systems. We shall present just some useful notions and concepts in order to insert the contents of our Thesis in a well-defined context.

There exist several excellent textbooks covering all the material reviewed in this Section. Our presentation is based mainly on [4, 91]. A complete treatment of the subject can be found also in [9, 29, 59, 79].

1.1.1 Poisson brackets and Hamiltonian flows

Let $\mathcal{F}(\mathcal{P})$ be the set of smooth real-valued functions on a smooth manifold $\mathcal{P}$.

Definition 1.1 *A* Poisson bracket *on $\mathcal{P}$ is a bilinear operation on the set $\mathcal{F}(\mathcal{P})$, denoted with $\{\cdot,\cdot\}$ and possessing the following properties:*

1. *skew-symmetry:*
$$\{f,g\} = -\{g,f\}, \qquad \forall\, f,g \in \mathcal{F}(\mathcal{P});$$
2. *Leibniz rule:*
$$\{f,g\,h\} = h\,\{f,g\} + g\,\{f,h\}, \qquad \forall\, f,g,h \in \mathcal{F}(\mathcal{P});$$
3. *Jacobi identity:*
$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0, \qquad \forall\, f,g,h \in \mathcal{F}(\mathcal{P}).$$

The pair $(\mathcal{P},\{\cdot,\cdot\})$ is called a Poisson manifold.

Definition 1.2 *Let $(\mathcal{P},\{\cdot,\cdot\})$ be a Poisson manifold. A* Hamiltonian vector field X_H *corresponding to the function $H \in \mathcal{F}(\mathcal{P})$, is the unique vector field on $\mathcal{P}$ satisfying*
$$X_H \cdot f \doteq \{H,f\}, \qquad \forall\, f \in \mathcal{F}(\mathcal{P}).$$
The function H is called a Hamilton function *of X_H. The flow $\phi^t : \mathcal{P} \to \mathcal{P}$, $t \in \mathbb{R}$, of the Hamiltonian vector field X_H is called a* Hamiltonian flow *of the Hamilton function H.*

Proposition 1.1 *Let $\phi^t : \mathcal{P} \to \mathcal{P}$ be the Hamiltonian flow with the Hamilton function H. Then*
$$H \circ \phi^t = H,$$
and
$$\frac{d}{dt}(F \circ \phi^t) = \{H, F \circ \phi^t\}.$$

In particular, a function $f \in \mathcal{F}(\mathcal{P})$ is an *integral of motion* of the flow ϕ^t if and only if $\{H, f\} = 0$, that is, if H and f are in involution.

A remarkable feature of the Hamiltonian flows is that each of the map constituting such flows preserves the Poisson brackets.

Definition 1.3 *Let* $(\mathcal{P}, \{\cdot,\cdot\}_{\mathcal{P}})$ *and* $(\mathcal{Q}, \{\cdot,\cdot\}_{\mathcal{Q}})$ *be two Poisson manifolds, and let* $\phi : \mathcal{P} \to \mathcal{Q}$ *be a smooth map. Is is called a* Poisson map *if*

$$\{f, g\}_{\mathcal{Q}} \circ \phi = \{f \circ \phi, g \circ \phi\}_{\mathcal{P}}, \qquad \forall\, f, g \in \mathcal{F}(\mathcal{Q}).$$

Proposition 1.2 *If* $\phi^t : \mathcal{P} \to \mathcal{P}$ *is a Hamiltonian flow on* $\mathcal{P}$*, then for each* $t \in \mathbb{R}$*, the map* ϕ^t *is Poisson.*

1.1.2 Symplectic manifolds and symplectic leaves

A more traditional approach to Hamiltonian mechanics is based on another choice of the fundamental structure, namely the symplectic manifold.

Definition 1.4 *A* symplectic structure *on a manifold* $\mathcal{P}$ *is a non-degenerate closed two-form* Ω *on* $\mathcal{P}$*. The pair* $(\mathcal{P}, \Omega)$ *is called a* symplectic manifold.

This structure is a particular case of the Poisson bracket structure. One can define Hamiltonian vector fields with respect to a symplectic structure.

Definition 1.5 *Let* $(\mathcal{P}, \Omega)$ *be a symplectic manifold. A* Hamiltonian vector field X_H *corresponding to the function* $H \in \mathcal{F}(\mathcal{P})$*, is the unique vector field on* $\mathcal{P}$ *satisfying* [1]

$$\Omega\,(\xi, X_H(Q)) = \langle\, \nabla H(Q), \xi\,\rangle, \qquad \forall\, \xi \in T_Q\mathcal{P}.$$

The function H *is called a* Hamilton function *of* X_H*. The flow* $\phi^t : \mathcal{P} \to \mathcal{P}$*,* $t \in \mathbb{R}$*, of the Hamiltonian vector field* X_H *is called a* Hamiltonian flow *of the Hamilton function* H*.*

A symplectic structure yields a vector bundle isomorphism between $T^*\mathcal{P}$ and $T\mathcal{P}$. Indeed, to any vector $\eta \in T_Q\mathcal{P}$ there corresponds a one-form $\omega_\eta \in T^*_Q\mathcal{P}$ defined as

$$\omega_\eta(\xi) \doteq \Omega(\xi, \eta), \qquad \forall\, \xi \in T_Q\mathcal{P}.$$

Actually the correspondence $\eta \mapsto \omega_\eta$ is an isomorphism between $T^*_Q\mathcal{P}$ and $T_Q\mathcal{P}$. Denote by $J : T^*_Q\mathcal{P} \to T_Q\mathcal{P}$ the inverse isomorphism. Then Definition 1.5 implies that

$$X_H = J\,(\nabla H).$$

At any point $Q \in \mathcal{P}$, the tangent space $T_Q\mathcal{P}$ is spanned by Hamiltonian vector fields at Q.

Definition 1.6 *Let* $(\mathcal{P}_1, \Omega_1)$ *and* $(\mathcal{P}_2, \Omega_2)$ *be two symplectic manifolds. A smooth map* $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ *is called* symplectic *if*

$$\Omega_1(\xi, \eta) = \Omega_2\,(T_Q\phi(\xi), T_Q\phi(\eta)), \qquad \forall\, \xi, \eta \in T_Q\mathcal{P},$$

the form Ω_1 *being taken in an arbitrary point* $Q \in \mathcal{P}_1$*, and the form* Ω_2 *being taken in the corresponding* $\phi(Q) \in \mathcal{P}_2$*.*

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[1] For a function $f \in \mathcal{F}(\mathcal{P})$ we define its gradient $\nabla f : \mathcal{P} \to T^*\mathcal{P}$ in the usual way: let $Q \in \mathcal{P}$, then $\nabla f(Q)$ is an element of $T^*_Q\mathcal{P}$ such that

$$\langle\, \nabla f(Q), \dot{Q}\,\rangle \doteq \frac{d}{d\varepsilon}\, f(Q(\varepsilon))\Big|_{\varepsilon=0}, \qquad \forall\, \dot{Q} \in T_Q\mathcal{P},$$

where $Q(\varepsilon)$ stands for an arbitrary curve in $\mathcal{P}$ through $Q(0) = Q$, with the tangent vector $\dot{Q}(0) = \dot{Q}$.

Let $\mathfrak{X}(\mathcal{P})$ the set of vector fields on $\mathcal{P}$.

Proposition 1.3 *The flow ϕ^t of a vector field $X \in \mathfrak{X}(\mathcal{P})$ on a symplectic manifold $(\mathcal{P}, \Omega)$ consists of symplectic maps if and only if this field is locally Hamiltonian, i.e. if there exists locally a function $H \in \mathcal{F}(\mathcal{P})$ such that $X = X_H = J(\nabla H)$.*

Let us show how to include the symplectic Hamiltonian mechanics into the Poisson bracket formalism. The following proposition holds.

Proposition 1.4 *The flow ϕ^t of a vector field $X \in \mathfrak{X}(\mathcal{P})$ on a symplectic manifold $(\mathcal{P}, \Omega)$ consists of symplectic maps if and only if this field is locally Hamiltonian, i.e. if there exists locally a function $H \in \mathcal{F}(\mathcal{P})$ such that $X = X_H = J(\nabla H)$.*

Proposition 1.5 *Let $(\mathcal{P}, \Omega)$ be a symplectic manifold. Then it can be made into a Poisson manifold by defining a Poisson bracket via the formula*

$$\{f, g\} \doteq \Omega(X_f, X_g) = \Omega(J(\nabla f), J(\nabla g)).$$

We can now characterize symplectic manifolds as a subclass of Poisson manifolds. Let $(\mathcal{P}, \{\cdot,\cdot\})$ be a N-dimensional Poisson manifold. Let $Q \in \mathcal{P}$ and consider the local coordinates $\{x_i\}_{i=1}^N$ in the neighborhood of Q. We have

$$\{f, g\} \doteq \sum_{i,j=1}^{N} P_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \qquad \forall\, f, g \in \mathcal{F}(\mathcal{P}),$$

where the skew-symmetric $N \times N$ matrix $P_{i,j} \doteq \{x_i, x_j\}$, is a coordinate representation of an intrinsic object called *Poisson tensor*.

Definition 1.7 *The rank of the matrix $P_{i,j}$, $1 \le i, j \le N$, is called the* rank of the Poisson structure*.*

Of course, the rank of $P_{i,j}$ does not depend on the system of local coordinates $\{x_i\}_{i=1}^N$. In a more invariant way, we can say that there is an antisymmetric tensor $\pi \in \Lambda^2(T^*\mathcal{P})$ such that

$$\{f, g\} \doteq \pi(\nabla f, \nabla g).$$

Proposition 1.6 *A Poisson manifold $(\mathcal{P}, \{\cdot,\cdot\})$ is symplectic if the rank of the Poisson structure is everywhere equal to the dimension of the manifold $\mathcal{P}$.*

Since the matrix $P_{i,j}$, $1 \le i, j \le N$, is skew-symmetric, it can have full rank only if N is even. Hence the dimension of a symplectic manifold is always an even number.

Definition 1.8 *Let $(\mathcal{P}, \{\cdot,\cdot\}_{\mathcal{P}})$ be a Poisson manifold. A submanifold $\mathcal{M} \subset \mathcal{P}$ is called a* Poisson submanifold *of $\mathcal{P}$, if there exists a Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ on $\mathcal{M}$ such that the inclusion map $i : \mathcal{P} \to \mathcal{M}$ is Poisson. Such a bracket is unique if it exists and then it is called the* induced Poisson bracket *on $\mathcal{M}$.*

Proposition 1.7 *A submanifold $\mathcal{M} \subset \mathcal{P}$ is Poisson if and only if an arbitrary Hamiltonian vector field $\{H, \cdot\}$ on $\mathcal{P}$ in the points of $\mathcal{M}$ is tangent to $\mathcal{M}$.*

So, Poisson submanifolds are integral manifolds for arbitrary Hamiltonian vector fields. Minimal Poisson submanifolds are the so-called *symplectic leaves*. A degeneration of a Poisson structure is related to the existence of functions which are in involution with an arbitrary function on $\mathcal{P}$.

Definition 1.9 *A function $C \in \mathcal{F}(\mathcal{P})$ is called a* Casimir function *of a Poisson manifold $(\mathcal{P}, \{\cdot, \cdot\})$ if*

$$\{C, f\} = 0, \qquad \forall\, f \in \mathcal{F}(\mathcal{P}).$$

In other words, the Casimir functions generate trivial Hamiltonian equations of motion and they are constant on symplectic leaves.

1.1.3 Complete integrability

We now present the key notion of integrability of a given Hamiltonian system. The following theorem, called *Arnold-Liouville theorem*, tells how many integrals of motion assure integrability of a given system, and describes the motion on the common level set of these integrals.

Theorem 1.1 *Let $(\mathcal{P}, \{\cdot, \cdot\})$ be a $2N$-dimensional symplectic manifold. Suppose that there exist N functions $f_1, ..., f_N \in \mathcal{F}(\mathcal{P})$, such that*

1. *$f_1, ..., f_N$ are functionally independent, i.e. the gradients ∇f_i's are linearly independent everywhere on $\mathcal{P}$;*
2. *$f_1, ..., f_N$ are in involution, i.e. $\{f_i, f_j\} = 0$, $1 \leq i, j \leq N$.*

Let W be a connected component of a common level set:

$$W \doteq \{Q \in \mathcal{P} \,:\, f_i(Q) = c_i,\, 1 \leq i \leq N\}.$$

Then W is diffeomorphic to $\mathbb{T}^d \times \mathbb{R}^{N-d}$, with $0 \leq d \leq N$, being $\mathbb{T}^d$ a d-dimensional torus.

If W is compact, then it is diffeomorphic to $\mathbb{T}^N$. In some neighborhood $W \times V$, where $V \subset \mathbb{R}^N$ is an open ball, there exist coordinates $\{I_i, \theta_i\}_{i=1}^N$, $\{I_i\}_{i=1}^N \in V$, $\{\theta_i\}_{i=1}^N \in \mathbb{T}^N$, called action-angle coordinates*, with the following properties:*

1. *$I_i = I_i(f_1, ..., f_N)$, $1 \leq i \leq N$;*
2. *$\{I_i, I_j\} = \{\theta_i, \theta_j\} = 0$, $\{I_i, \theta_j\} = \delta_{i,j}$, $1 \leq i, j \leq N$;*
3. *For an arbitrary Hamilton function $H = H(f_1, ..., f_N)$, the Hamiltonian equations of motion on $\mathcal{P}$ read*

$$\dot{I}_i = 0, \qquad \dot{\theta}_i = \omega_i(I_1, ..., I_N), \qquad 1 \leq i \leq N;$$

4. *For an arbitrary symplectic map $\Phi : \mathcal{P} \to \mathcal{P}$ admitting $f_1, ..., f_N$ as integrals of motion, the equations of motions in the coordinates $\{I_i, \theta_i\}_{i=1}^N$ take the form:*

$$\widetilde{I}_i = I_i, \qquad \widetilde{\theta}_i = \theta_i + \Omega_i(I_1, ..., I_N), \qquad 1 \leq i \leq N.$$

Definition 1.10 *Hamiltonian flows and Poisson maps on $2N$-dimensional symplectic manifolds possessing N functionally independent and involutive integrals of motion, are called* completely integrable *(in the Arnold-Liouville sense).*

1.1.4 Lie-Poisson brackets

Let $\mathfrak{g}$ be a finite-dimensional (complex) Lie algebra with Lie bracket $[\cdot, \cdot]$, and let $\mathfrak{g}^*$ its dual space. Thus, to $X \in \mathfrak{g}$ we associate a linear function X^* on the dual vector space $\mathfrak{g}^*$, which is defined by

$$X^* : \mathfrak{g}^* \to \mathbb{C} : L \mapsto \langle\, L, X \,\rangle,$$

with $L \in \mathfrak{g}^*$. The vector space of linear functions on $\mathfrak{g}^*$ forms a Lie algebra, isomorphic to $\mathfrak{g}$, by setting $[X^*, Y^*]_{\mathfrak{g}^*} \doteq ([X, Y])^*$. It follows that $\mathfrak{g}^*$ admits a Poisson structure $\{\cdot, \cdot\}$ whose structure functions are linear functions with the structure constants of $\mathfrak{g}$ as coefficients. We give the following definition.

Definition 1.11 *The* Lie-Poisson bracket *on* $\mathfrak{g}^*$ *is defined by the formula*

$$\{f, g\}(L) \doteq \langle L, [\nabla f(L), \nabla g(L)] \rangle, \qquad \forall f, g \in \mathcal{F}(\mathfrak{g}^*). \tag{1.3}$$

It is easy to verify that the bracket defined in Eq. (1.3) satisfies the Jacobi identity. The above definition implies that the dual of $\mathfrak{g}$ carries a natural Poisson structure, called *Lie-Poisson structure on* $\mathfrak{g}^*$.

The Hamiltonian equations of motion on $\mathfrak{g}^*$ with respect to the Lie-Poisson bracket read

$$\dot{L} = \{H, L\} = \mathrm{ad}^*_{\nabla H(L)} L, \tag{1.4}$$

where $H \in \mathcal{F}(\mathfrak{g}^*)$ is a Hamilton function. The Casimir functions are exactly those that generate a trivial dynamics. They are characterized by the equation

$$\mathrm{ad}^*_{\nabla C(L)} L = 0,$$

and they are called *coadjoint invariants*.

An important case, in which the above formulae simplify significantly, is the following one. Let $\mathfrak{g}$ be equipped with a non-degenerate scalar product $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, invariant in the sense that $\langle \xi, [\eta, \zeta] \rangle = \langle [\xi, \eta], \zeta \rangle$, $\forall \xi, \eta, \zeta \in \mathfrak{g}$. This is the case, for example, for semi-simple Lie algebras (in particular for the matrix Lie algebras), where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form. In such a case, $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ by means of this scalar product, and its invariance means that $\mathrm{ad}^* = -\mathrm{ad}$, so that the notion of coadjoint invariant functions on $\mathfrak{g}^*$ coincides with the notion of adjoint invariant functions on $\mathfrak{g}$. Hence, Eq. (1.4) admits the following *Lax form*:

$$\dot{L} = \{H, L\} = [L, \nabla H(L)].$$

The Casimir functions are defined by the equation

$$[L, \nabla C(L)] = 0.$$

Let us now choose an arbitrary basis $\{X^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$. The commutation relations between the elements of the basis read

$$\left[X^\alpha, X^\beta\right] = C_\gamma^{\alpha\beta} X^\gamma, \qquad 1 \le \alpha, \beta \le \dim \mathfrak{g},$$

being $\{C_\gamma^{\alpha\beta}\}_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}}$ the set of structure constants of the Lie algebra $\mathfrak{g}$. Hereafter we shall use the convention of summing over repeated greek indices: they shall always run from 1 to $\dim \mathfrak{g}$.

Let $\{X_\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ be a basis of $\mathfrak{g}^*$, with pairing $\langle X^\alpha, X_\beta \rangle = \delta^\alpha_\beta$. Under the assumption that $\mathfrak{g}$ has a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ invariant under the adjoint action, we can take the Cartan-Killing form as $\langle \cdot, \cdot \rangle$, thus identifying $\mathfrak{g}$ with $\mathfrak{g}^*$. If $\mathfrak{g}$ is simple and represented as a matrix Lie algebra we may assume that the Cartan-Killing form is given by $g^{\alpha\beta} \doteq \mathrm{tr}[X^\alpha X^\beta]$.

If $L \in \mathfrak{g}^*$ then $L = y^\beta X_\beta = g_{\alpha\beta} X^\alpha y^\beta$, where the y^β's are the coordinate functions on $\mathfrak{g}^*$ and $g_{\alpha\beta}$ is the inverse of the Cartan-Killing metric.

The gradient of a coordinate function y^α is given by $\nabla y^\alpha = X^\alpha$. Using Eq. (1.3) we get

$$\left\{y^\alpha, y^\beta\right\}(L) = \langle L, \left[X^\alpha, X^\beta\right] \rangle = C_\gamma^{\alpha\beta} \langle L, X^\gamma \rangle = C_\gamma^{\alpha\beta} y^\gamma, \tag{1.5}$$

with $1 \leq \alpha, \beta \leq \dim \mathfrak{g}$.

It is possible to give a compact representation for the Lie-Poisson bracket (1.5). Let us introduce the so-called *tensor Casimir* $\Pi \doteq g_{\alpha\beta}\, X^\alpha \otimes X^\beta \in \mathfrak{g} \otimes \mathfrak{g}$. Defining

$$\{L \otimes \mathbb{1}, \mathbb{1} \otimes L\} \doteq g_{\alpha\beta}\, g_{\gamma\delta}\, \left\{y^\beta, y^\delta\right\} X^\alpha \otimes X^\delta,$$

we immediately get

$$\{L \otimes \mathbb{1}, \mathbb{1} \otimes L\} + [\,\Pi, L \otimes \mathbb{1} + \mathbb{1} \otimes L\,] = 0.$$

1.1.5 Linear r-matrix structure

The linear r-matrix structure on $\mathfrak{g}^*$ is, in principle, nothing but a special case of the Lie-Poisson structure, corresponding to an alternative Lie bracket on $\mathfrak{g}$. To define it, the following ingredients are necessary: a Lie algebra $\mathfrak{g}$ with Lie bracket $[\,\cdot,\cdot\,]$ and a linear operator $R : \mathfrak{g} \to \mathfrak{g}$.

Definition 1.12 *A linear operator $R : \mathfrak{g} \to \mathfrak{g}$ is called an* R-operator, *if the following bracket defines on $\mathfrak{g}$ a new structure of Lie algebra:*

$$[\,\xi, \eta\,]_R \doteq \frac{1}{2}\left([\,R(\xi), \eta\,] + [\,\xi, R(\eta)\,]\right), \qquad \forall\, \xi, \eta \in \mathfrak{g}. \tag{1.6}$$

A sufficient condition for the bracket in Eq. (1.6) to be indeed a Lie bracket is given by the following statement.

Proposition 1.8 *If $R : \mathfrak{g} \to \mathfrak{g}$ satisfies the* modified Yang-Baxter equation, *that is given by*

$$[\,R(\xi), R(\eta)\,] - R\,([\,R(\xi), \eta\,] + [\,\xi, R(\eta)\,]) = -\alpha\,[\,\xi, \eta\,], \qquad \forall\, \xi, \eta \in \mathfrak{g}, \quad \alpha \in \mathbb{C}, \tag{1.7}$$

then $[\,\xi, \eta\,]_R$ is a Lie bracket. If $\alpha = 0$, Eq. (1.7) is called the classical Yang-Baxter equation

If the R-operator satisfies Eq. (1.7), then a new Lie-Poisson bracket on $\mathfrak{g}^*$ is defined:

$$\{f, g\}(L) \doteq \frac{1}{2}\langle\, [\,R(\nabla f(L)), \nabla g(L)\,] + [\,\nabla f(L), R(\nabla g(L))\,], L\,\rangle, \qquad \forall\, f, g \in \mathcal{F}(\mathfrak{g}^*). \tag{1.8}$$

This bracket is called the *linear r-matrix bracket* corresponding to the operator R.

The most important general feature of this construction is given in the following statement.

Theorem 1.2 *1) Let $H \in \mathcal{F}(\mathfrak{g}^*)$ be a coadjoint invariant of $\mathfrak{g}$. Then the Hamiltonian equations on $\mathfrak{g}^*$ with respect to the Lie-Poisson bracket (1.8) read*

$$\dot{L} = \mathrm{ad}^*_M\, L, \qquad M \doteq \frac{1}{2} R\,(\nabla H(L));$$

2) Coadjoints invariants of $\mathfrak{g}$ are in involution with respect to the Lie-Poisson bracket (1.8).

Let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the matrix (called *r-matrix*), canonically corresponding to the operator R. Notice that the tensor Casimir Π is nothing but the r-matrix canonically corresponding to the identity operator on $\mathfrak{g}$.

Let us give the following definition.

Definition 1.13 *Let $(\mathcal{P}, \{\cdot,\cdot\})$ be a Poisson manifold and let $\mathfrak{g}$ be a Lie algebra, equipped with a Lie-Poisson bracket (1.8). An element $L : \mathcal{P} \to \mathfrak{g}$ is called a Lax operator with r-matrix r if L is a Poisson map.*

Proposition 1.9 *Let $L : \mathcal{P} \to \mathfrak{g}$ be a Lax operator with r-matrix r. Then*

$$\{L \otimes \mathbb{1}, \mathbb{1} \otimes L\} + [\,r, L \otimes \mathbb{1} + \mathbb{1} \otimes L\,] = 0.$$

We now specialize the above construction to the case in which $\mathfrak{g} \equiv \mathfrak{gl}(V)$, where V is a finite-dimensional vector space. The relevance of r-matrices and their Lie-Poisson brackets for the theory of integrable systems is given by the following theorem.

Theorem 1.3 *Let $L : \mathcal{P} \to \mathfrak{g}$ be a Lax operator with r-matrix r and $\mathfrak{g} \equiv \mathfrak{gl}(V)$. Then the functions $H_i \doteq \mathrm{tr} L^i \in \mathcal{F}(\mathfrak{g}^*)$, $i \in \mathbb{N}\backslash\{0\}$, are in involution with respect to Lie-Poisson brackets induced by r. Moreover, the Hamiltonian vector field associated to H_i has the Lax form*

$$X_{H_i} \cdot L = [\,i\,\mathrm{tr}_2(\mathbb{1} \otimes L^{i-1}\, r), L\,].$$

The fact that the traces of powers of L are in involution can also be restated by saying that the coefficients of the characteristic polynomial of L, which are elements of $\mathcal{F}(\mathfrak{g}^*)$, are in involution, or by saying that the eigenvalues of L are in involution.

Suppose that $L \in \mathfrak{g}$ is a Lax operator. If there exist functions $a, b \in \mathfrak{g} \otimes \mathfrak{g}$ such that

$$\{L \otimes \mathbb{1}, \mathbb{1} \otimes L\} = [\,\mathbb{1} \otimes L, a\,] - [\,L \otimes \mathbb{1}, b\,], \tag{1.9}$$

one can proves that the traces of the powers of L are also in involution. The following theorem, called *Babelon-Viallet theorem*, assures that, if the traces of L are in involution, then - under some genericity assumption on L - there exist functions $a, b \in \mathfrak{g} \otimes \mathfrak{g}$ such that Eq. (1.9) holds.

Theorem 1.4 *Let $L : \mathcal{P} \to \mathfrak{g}$ be a Lax operator with r-matrix r and $\mathfrak{g} \equiv \mathfrak{gl}(V)$. Suppose that:*

1. *There exists an open subset $\mathcal{U}$ of $\mathcal{P}$ such that L is diagonalizable for all $Q \in \mathcal{P}$;*
2. *The coefficients of the characteristic polynomial of L are in involution.*

Then there exist an open subset $\mathcal{V} \subseteq \mathcal{U}$ of $\mathcal{P}$, and smooth functions $a, b : \mathcal{V} \to \mathfrak{g} \otimes \mathfrak{g}$, such that

$$\{L \otimes \mathbb{1}, \mathbb{1} \otimes L\} = [\,\mathbb{1} \otimes L, a\,] - [\,L \otimes \mathbb{1}, b\,].$$

A very important role in the theory of integrable systems is played by the so-called *loop algebras* (or *affine Lie algebras*), i.e. Lie algebras of Laurent polynomials over some finite-dimensional Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,] \doteq \left\{X(\lambda) = \textstyle\sum_{p\in\mathbb{Z}} X_p\, \lambda^p,\ X_p \in \mathfrak{g}\right\},$$

with Lie bracket given by $[\,X, Y\,](\lambda) \doteq [\,X(\lambda), Y(\lambda)\,] = \lambda^{i+j}\,[\,X, Y\,]$. There exist an infinite family of non-degenerate scalar products in $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,]$, enumerated by $\sigma \in \mathbb{Z}$:

$$\langle\, X(\lambda), Y(\lambda)\,\rangle_\sigma \doteq (\langle\, X(\lambda), Y(\lambda)\,\rangle)_\sigma,$$

i.e. the coefficient by λ^σ of the Laurent polynomial on the right hand side; this Laurent coefficient is nothing but the pointwise scalar product in $\mathfrak{g}$.

All the above constructions can be generalized to the case of loop algebras. For further details see [9, 29, 59, 79, 91]. We now give two explicit examples of R-operators and r-matrices.

Example 1.1 *Let $\mathfrak{g} \equiv \mathfrak{gl}(N)$, with the non-degenerate invariant scalar product $\langle\, X, Y\,\rangle \doteq \mathrm{tr}(X\,Y)$. A generic element of $\mathfrak{g}$ is written*

$$L = \sum_{j,k=1}^{N} \ell_{jk}\, E_{jk},$$

where the matrices E_{jk} form a basis of $\mathfrak{gl}(N)$: the only non-vanishing entry of E_{jk} is the unit on the intersection of the j-th row anf the k-th column. The dual basis is given by $E^*_{jk} = E_{kj}$. The functions $L \mapsto \ell_{jk}$ form a functional basis of $\mathcal{F}(\mathfrak{g})$ and their gradients are given by $\nabla \ell_{jk} = E^*_{jk} = E_{kj}$.

The pairwise Lie-Poisson brackets of the coordinate functions read, see Eq. (1.8),

$$\{\ell_{ij}, \ell_{kl}\} = \frac{1}{2} \langle\, [\, R(E_{ji}), E_{lk}\,] + [\, E_{ji}, R(E_{lk})\,], L\,\rangle. \tag{1.10}$$

Introduce N^4 coefficients $r_{ij,mn}$ according to the formula

$$R(E_{ji}) = \sum_{m,n=1}^{N} r_{ij,mn}\, E_{mn} \quad \Leftrightarrow \quad r_{ij,mn} = \langle\, R(E_{ji}), E_{nm}\,\rangle.$$

Hence Eq. (1.10) may be rewritten as

$$\{\ell_{ij}, \ell_{kl}\} = \frac{1}{2}\left(\sum_{m=1}^{N} r_{ij,ml}\, \ell_{kn} - \sum_{n=1}^{N} r_{ij,kn}\, \ell_{nl} + \sum_{n=1}^{N} r_{kl,in}\, \ell_{nj} - \sum_{m=1}^{N} r_{kl,mj}\, \ell_{im} \right).$$

The $N^2 \times N^2$ r-matrix, canonically corresponding to the linear operator R is given by

$$r \doteq \sum_{i,j,k,l=1}^{N} r_{ij,kl}\, E_{ij} \otimes E_{kl}.$$

This is the matrix a appearing in Eq. (1.9), i.e. $r = a$, while the matrix b is given by

$$b \doteq \sum_{i,j,k,l=1}^{N} r_{kl,ij}\, E_{ij} \otimes E_{kl}.$$

Notice that $b = \Pi\, a\, \Pi$, where $\Pi \doteq \sum_{j,k=1}^{N} E_{jk} \otimes E_{kj}$ is the permutation operator.

Example 1.2 *Let us consider the loop algebra $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,]$ with $\mathfrak{g} \equiv \mathfrak{gl}(N)$, with the non-degenerate invariant scalar product $\langle\, X(\lambda), Y(\lambda)\,\rangle_{-1} \doteq (\langle\, X(\lambda), Y(\lambda)\,\rangle)_{-1}$. In this case we may write a generic element of $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,]$ as*

$$L(\lambda) = \sum_{p\in\mathbb{Z}} \sum_{j,k=1}^{N} \ell^{(p)}_{jk}\, \lambda^p\, E_{jk},$$

and the gradients of the functions $L \mapsto \ell^{(p)}_{jk}$ are given by $\nabla \ell^{(p)}_{jk} = \lambda^{-p-1}\, E_{kj}$. Consider now the following two subalgebras of $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,]$:

$$\mathfrak{g}_{\geq 0} \doteq \bigoplus_{p\geq 0} \lambda^p\, \mathfrak{g}, \qquad \mathfrak{g}_{<0} \doteq \bigoplus_{p<0} \lambda^p\, \mathfrak{g}.$$

Obviuosly, as a vector space, $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,] = \mathfrak{g}_{\geq 0} \oplus \mathfrak{g}_{<0}$. Let $P_{\geq 0}$ and $P_{<0}$ stand for the projection operators from $\mathfrak{g}\,[\,\lambda, \lambda^{-1}\,]$ onto the corresponding subspace $\mathfrak{g}_{\geq 0}$, $\mathfrak{g}_{<0}$ along the complementary one. It is easily shown that the skew-symmetric operator $R \doteq P_{\geq 0} - P_{<0}$ satisfies the modified Yang-Baxter equation (1.7). Let us compute the corresponding r-matrix. We get:

$$r^{(p,q)}_{ij,kl} = \langle\, R(\lambda^{-p-1}\, E_{ji}), \lambda^{-q-1}\, E_{lk}\,\rangle_{-1} = \left\{ \begin{array}{cc} + & p < 0 \\ - & p \geq 0 \end{array} \right\} \delta_{p,-q-1}\, \delta_{j,k}\, \delta_{i,l},$$

so that

$$r(\lambda,\mu)=\sum_{(p,q)\in\mathbb{Z}^2}\sum_{i,j,k,l=1}^{N} r^{(p,q)}_{kl,ij}\,E_{ij}\otimes E_{kl}=\left(\sum_{p<0}-\sum_{p\geq 0}\right)\lambda^p\,\mu^{-p-1}\sum_{i,j}^{N}E_{ij}\otimes E_{ji}.$$

Performing the formal summation we get

$$r(\lambda,\mu)=a(\lambda,\mu)=\frac{2}{\lambda-\mu}\Pi=-b(\lambda,\mu).$$

Therefore, Eq. (1.9) reads

$$\{L(\lambda)\otimes\mathbb{1},\mathbb{1}\otimes L(\mu)\}=\frac{1}{\lambda-\mu}[\,\mathbb{1}\otimes L+L\otimes\mathbb{1},\Pi\,].$$

1.1.6 The problem of integrable discretization

The importance of the problem of integrable discretization is evident for anyone who agrees with the following wisdom [91]:

- Differential equations form an extremely useful instrument in the sciences;
- In order to extract quantitative informations from the models governed by differential equations, it is often necessary to solve them numerically, with the help of various discretization methods;
- By investigation of long-term dynamics, i.e. the features of dynamical systems on very long time intervals, their qualitative features become of first rank importance;
- It is therefore crucial to assure that the discretized models exhibit the same qualitative features of the dynamics as their continuous counterparts.

To assure the coincidence of the qualitative properties of discretized models with that of the continuous ones becomes one of the central ideas of modern numerical analysis, which therefore gets into a close interplay with different aspects of dynamical systems theory.

We mention here some of the most important approaches to integrable discretization [91]:

1. Any integrable system possesses a zero *curvature representation*, i.e. a representation as a compatibility condition of two auxiliary linear problems. Realization of this led naturally to a discretization of these linear problems [1, 29]
2. One of the most intriguing and universal approaches to integrable systems is the *Hirota's bilinear method* (1973). It seems to be able to produce discrete versions of the majority of solitons equations, but it still remains somewhat mysterious, and the mechanism behind is yet to be fully understood.
3. A fruitful method is based on the *direct linearization* [69]. Its basic idea is to derive integrable non-linear differential and difference equations which are satisfied by the solutions of certain linear integral equations. A large variety of continuous and discrete soliton equations has been obtained in this way.
4. Differential equations describing various geometric problems (surfaces of constant Gaussian or mean curvature, motion of a curve in the space, etc.) turn out to be integrable. Correspondently, a discretization of geometric notions naturally leads to discrete integrable equations. The area of *discrete differential geometry* has fluorished in recent years, see for instance [17, 18, 20, 26, 27, 49].

5. Considering stationary and restricted flows of soliton hierarchies, closely related to the "non-linearization" of spectral problems, often leads to interesting discrete equations [76, 77, 78].

6. An approach based on the *discrete variational principle* combined with matrix factorization, was pushed forward by Veselov and Moser [63, 99, 100, 101]. A set of examples with similar properties, which also belong to the most beautiful ones, but were derived without a systematical approach, was given by Suris, see references in the book [91]. Although this approach is heavily based on a *guesswork*, historically it was the work of Veselov that consolidated the subject of integrable discretizations into a separate branch of the theory of integrable systems.

7. Finally, we mention the method of integrable discretization based on the notion of *Bäcklund transformations* for finite-dimensional integrale systems. This approach has been developed by Kuznetsov and Sklyanin [55, 56, 87] and led to several discrete integrable systems [31, 41, 53].

In the present Thesis we shall apply the techniques 6. and 7. in order to construct the discrete-time counterparts of certain finite-dimensional integrable systems related to classical Gaudin models. Henceforth we shall give later some further details about these approaches to integrable discretization. Let us now present a more precise definition of integrable discretization of a give integrable system.

Let $(\mathcal{P}, \{\cdot,\cdot\})$ be a Poisson manifold. Let H be a Hamilton function of a completely integrable flow on $\mathcal{P}$:

$$\dot{x} = f(x) = \{H, x\}. \tag{1.11}$$

It is supposed that this flow possesses sufficiently many functionally independent integrals $I_k(x)$ in involution.

The problem of integrable discretization consists in finding a family of diffeomorphisms $\phi_\varepsilon : \mathcal{P} \to \mathcal{P}$,

$$\widehat{x} = \phi_\varepsilon(x), \tag{1.12}$$

depending smoothly on a small parameter $\varepsilon > 0$, and satisfying the following properties:

- The maps (1.12) approximate the flow (1.11) in the following sense:

$$\phi_\varepsilon(x) = x + \varepsilon\, f(x) + O(\varepsilon^2).$$

- The maps (1.12) are Poisson with respect to $\{\cdot,\cdot\}$ on $\mathcal{P}$ or with respect to some deformation, $\{\cdot,\cdot\}_\varepsilon = \{\cdot,\cdot\} + O(\varepsilon)$.

- The maps (1.12) are integrable, i.e. they possess the necessary number of independent integrals in involution $I_k(x,\varepsilon)$, approximating the integrals of the original system: $I_k(x,\varepsilon) = I_k(x) + O(\varepsilon)$.

1.2 Some remarks on the Yang-Baxter equation (YBE)

At an early stage, the Yang-Baxter equation (YBE) appeared in several different works in literature and sometimes its solutions even preceded the equation. One can trace three streams of ideas from which the YBE has emerged: the Bethe Ansatz, commuting transfer matrices in statistical mechanics and factorizable S matrices in field theory. For general references about the first two topics see [13, 36].

One of the first occurence of YBE can be found in the study of a one-dimensional quantum mechanical many-body problem with δ function interaction. By building the Bethe-type wave-functions, McGuire and others [16, 60] discovered that the N-particle S-matrix factorized into the product of two-particle ones. C.N. Yang treated the case of arbitrary statistics of particles by introducing the nested Bethe Ansatz [104, 105]. The YBE appears here in the present form as the consistency condition for the factorization.

The significance of the YBE in statistical mechanics lies in that it imples the existence of a commuting family of transfer matrices. Baxter's solution of the eight vertex model [12] uses this property to derive equations that determine the eigenvalues of the transfer matrix.

The topics concerning the YBE began to be studied thoroughly in the 80's also by mathematicians like A.A. Belavin, V.G. Drinfeld, P.P. Kulish, E.K. Sklyanin, L.D. Faddeev, M.A. Semenov-Tian-Shansky and others. This study was motivated by the multitude of applications that the YBE has in different areas of mathematics and physics: classical and quantum integrable systems, inverse scattering problems, group theory, algebraic geometry and statistical mechanics.

The classical Yang-Baxter equation (CYBE) was firstly introduced by E.K. Sklyanin [85]. Compared to the YBE, the CYBE represents an important case, since it can be formulated in the language of Lie algebras. The form of the CYBE is the following one [9, 15]:

$$[\,r_{12}(\lambda,\mu), r_{13}(\lambda,\zeta)\,] + [\,r_{12}(\lambda,\mu), r_{23}(\mu,\zeta)\,] + [\,r_{13}(\lambda,\zeta), r_{23}(\mu,\zeta)\,] = 0, \tag{1.13}$$

where r is a $\mathfrak{g}\otimes\mathfrak{g}$ valued function of two complex parameters and $r_{12} \doteq r\otimes \mathbb{1}$, $r_{23} \doteq \mathbb{1}\otimes r$, etc., are the natural imbeddings of $r(\lambda)$ from $\mathfrak{g}\otimes\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$, being $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ (an associative algebra with unit). In the theory of classical integrable systems Eq. (1.13) assures the Jacobi identity for the Lie-Poisson bracket induced by the r-matrix r.

One of the directions of study in this domain is the classification of solutions in the case of a simple complex finite-dimensional Lie algebra. Usually one consider solutions with the following additional conditions:

1. $r_{12}(\lambda,\mu) = r_{21}(\mu,\lambda)$ (*unitarity condition*);
2. $r_{12}(\lambda,\mu) = r_{12}(\lambda-\mu)$. In this case Eq. (1.13) can be written as

$$[\,r_{13}(\lambda), r_{23}(\mu)\,] + [\,r_{12}(\lambda-\mu), r_{13}(\lambda) + r_{23}(\mu)\,] = 0, \tag{1.14}$$

   and the unitarity condition implies $r_{12}(\lambda) = -r_{21}(-\lambda)$.

In [15] the authors investigate the non-degenerate solutions of the CYBE, proving that the poles of such solutions form a discrete group of the additive group of complex numbers. Moreover, they give a classification of non-degenerate solutions: elliptic, trigonometric and rational. Concerning the first class of solutions, the authors reduce the problem of finding non-degenerate elliptic solutions to the one of describing triples $(\mathfrak{g}, A, B)$, where A and B are commuting automorphisms of finite order of $\mathfrak{g}$ not having common fixed nonzero vectors. Moreover they prove that if such triples exist then there is an isomorphism $\mathfrak{g}\cong\mathfrak{sl}(n)$. Concerning the second class of solutions, they give a complete classification using the data from the Dynkin diagrams. Regarding the rational solutions, in [15] there are given several examples associated with Frobenius subalgebras of $\mathfrak{g}$. In [92] the author reduces the problem of listing rational solutions to the classification of quasi-Frobenius subalgebras of $\mathfrak{g}$, which are related to the so-called maximal orders in the loop algebra corresponding to the extended Dynkin diagrams.

1.3 A brief history of Gaudin models

The Gaudin models were introduced in 1976 by M. Gaudin [35] and attracted considerable interest among theoretical and mathematical physicists, playing a distinguished role in the realm of integrable systems. Their peculiar properties, holding both at the classical and at the quantum level, are deeply connected with the long-range nature of the interaction described by its commuting Hamiltonians, which in fact yields a typical "mean field" dynamic.

Precisely, the Gaudin models describe completely integrable classical and quantum long-range spin chains. Originally [35] the Gaudin model was formulated as a spin model related to the Lie algebra $\mathfrak{sl}(2)$. Later it was realized [36, 44], that one can associate such a model with any semi-simple complex Lie algebra $\mathfrak{g}$ and a solution of the corresponding classical Yang-Baxter equation [15, 85]. An important feature of Gaudin models is that they can be formulated in the framework of the r-matrix approach. In particular they admit a linear r-matrix structure, that characterizes both the classical and the quantum models, and holds whatever be the dependence (rational (XXX), trigonometric (XXZ), elliptic (XYZ)) on the spectral parameter. In this context, it is possible to see Gaudin models as limiting cases of the integrable Heisenberg magnets [87, 92], which admit a quadratic r-matrix structure.

In the 80's, the rational Gaudin model was studied by Sklyanin [86] and Jurco [44] from the point of view of the quantum inverse scattering method. Precisely, Sklyanin studied the $\mathfrak{su}(2)$ rational Gaudin models, diagonalizing the commuting Hamiltonians by means of separation of variables and underlining the connection between his procedure and the functional Bethe Ansatz. In [34] the separation of variables in the rational Gaudin model was interpreted as a geometric Langlands correspondence. On the other hand, the algebraic structure encoded in the linear r-matrix algebra allowed Jurco to use the algebraic Bethe Ansatz to simultaneously diagonalize the set of commuting Hamiltonians in all cases when $\mathfrak{g}$ is a generic classical Lie algebra. We have here to mention also the the work of Reyman and Semenov-Tian-Shansky [79]. Classical Hamiltonian systems associated with Lax matrices of the Gaudin-type were widely studied by them in the context of a general group-theoretic approach.

Some others relevants paper on the separability property of Gaudin models are [3, 28, 30, 38, 46, 51, 92]. In particular, the results in [28], see also [34], are based on the interpretation of the corresponding Gaudin models as conformal field theoretical models (Wess-Zumino-Witten (WZW) models). As a matter of fact, elliptic Gaudin models played an important role in establishing the integrability of the Seiberg-Witten theory [93] and in the study of isomonodromic problems and Knizhnik-Zamolodchikov systems [32, 39, 68, 81, 95].

Let us mention some important recent works on (classical and quantum) Gaudin models:

- In [30] it is discussed the bi-Hamiltonian formulation of $\mathfrak{sl}(n)$ rational Gaudin models. The authors obtained a pencil of Poisson brackets that recursively define a complete set of integrals of motion, alternative to the one associated with the standard Lax representation. The constructed integrals coincide, in the $\mathfrak{sl}(2)$ case, with the Hamiltonians of the bending flows in the moduli space of polygons in the euclidean space introduced in [47].

- In [41] it is proposed an integrable time-discretization of $\mathfrak{su}(2)$ rational Gaudin models. The approach to discrete-time mechanics used here is the one through Bäcklund transformations proposed by Sklyanin and Kuznetsov [49].

- The algebraic richness and robustness of Gaudin models allowed the construction of several integrable extensions of them. We mention the papers of Ragnisco, Ballesteros and Musso [11] where integrable q-deformations of Gaudin models are considered in the framework of the coalgebric approach. Also the superalgebra extensions of the Gaudin systems have been worked out, see for instance [22, 50, 67].

- Recently, in [94], the quantum eigenvalue problem for the $\mathfrak{gl}(n)$ rational Gaudin model has been widely studied and a construction for the higher Hamiltonians has been proposed.

Finally, we would like to mention the fact that recently a certain interest in Gaudin models arose in the theory of condensed matter physics. In fact, it has been noticed [5, 84, 82] that the BCS model, describing the superconductivity in metals, and the $\mathfrak{sl}(2)$ Gaudin models are closely related. In particular, in [5], this relation allowed to translate the results of Sklyanin on correlation functions of the $\mathfrak{sl}(2)$ Gaudin models [89] to the BCS model, obtaining the exact correlation functions in the canonical ensemble.

1.4 Outline of the Thesis

The present Thesis consists of a short Introduction and two Chapthers. In the following we briefly summarize their contents.

- **Chapter 2. Integrable extensions of Gaudin models**. In the first two Sections of this Chapter we give an essential review of the Drinfeld-Belavin solutions to the CYBE [15], explaining how one can associate - under some genericity conditions - with a non-degenerate solution $r(\lambda)$ a proper Lax matrix, thus defining the (N-site) *classical Gaudin models* associated with a finite-dimensional simple Lie algebra $\mathfrak{g}$ [35, 36, 44, 79, 86]. This general construction allow us to present the Lax matrices of the elliptic, trigonometric and rational Gaudin models and to give an r-matrix formulation in terms of *linear r-matrix brackets*. The explicit form of the integrals of motion is given in the case of $\mathfrak{g} \equiv \mathfrak{su}(2)$.

  In the remaining three Sections we present a general algebraic construction, based on Inönü-Wigner contractions (or equivalently *Leibniz extensions*) performed on the Lie algebra $\oplus^N \mathfrak{g}$ underlying the model. We shall prove that the linear r-matrix structure is not affected by such contractions. Suitable algebraic and pole-coalescence procedures performed on the N-pole Gaudin Lax matrices, enable us to construct one-body and many-body hierarchies of integrable models sharing the same (linear) r-matrix structure of the ancestor models. We remark that this technique can be applied for any simple Lie algebra $\mathfrak{g}$ and whatever be the dependence (rational, trigonometric, elliptic) on the spectral parameter.

  Fixing $\mathfrak{g} \equiv \mathfrak{su}(2)$, we construct the so called $\mathfrak{su}(2)$ hierarchies. For instance, assuming $N = 2$ and a rational dependence on the spectral parameter, we obtain the standard Lagrange top associated with $\mathfrak{e}^*(3)$, in the one-body case, and a homogeneous long-range integrable chain of interacting Lagrange tops, in the many-body one. This latter system has been called *Lagrange chain*. For an arbitrary order N of the Leibniz extension - where N is also the number of sites of the ancestor model - the one-body hierarchy consists of a family of generalized Lagrange tops. They provide an interesting example of integrable rigid body dynamics described by a Lagrange top with $N-2$ interacting heavy satellites.

  In this context, our main goal is the *derivation* of integrable systems: we say practically nothing about *solving* them. We do not discuss such methods of obtaining solutions, as the inverse scattering method with its numerous variants or algebro-geometric techniques. However, we always have in mind one of the motivations of integrable discretization, namely the possibility of applying integrable Poisson maps for actual numerical computations. Chapter 3 is devoted to this topic.

The results presented in Chapter 2, Sections 2.3, 2.4, 2.5, are already published. They can be found in [64, 65]. Some mistakes contained in [64, 65] are now corrected.

We have here to compare our results with the ones known in literature. Actually, the integrable systems considered in Chapter 2, are not new. In fact, they have been considered before in several works, but without using a systematic reduction of Gaudin models. For instance, to the best of our knowledge, the Lax matrix and the r-matrix formulation of the Lagrange chain has been introduced in [80] and then investigated in [79], even if the explicit form of the Hamiltonians and of the equations of motion is not given.

The basic tool of our construction is the notion of generalized Inönü-Wigner contraction of a Lie algebra. In our case, these contractions are equivalent to certain extensions of Lie algebras, called *Leibniz extensions* in [71]. In [71], the authors introduce these new algebraic structures in the context of current algebras and they do not use them to construct integrable systems. The first application of *Leibniz extensions* of Lie algebras (also called *jet-extensions*) to (finite-dimensional) integrable systems appears in [40], where just few examples are considered and without using a general construction.

The first systematic approach to these integrable extensions of Gaudin models appears indipendently in our paper [64] and in the work of Yu.B. Chernyakov [24]. Nevertheless we remark here that in [24] the author consider just $\mathfrak{sl}(n)$ Gaudin models and he does not give an r-matrix interpretation of these systems. For instance, the integrability property is not explicitly proven.

Let us now recall some papers where some elements of the one-body and many-body $\mathfrak{su}(2)$ hierarchies are considered.

The first (rational) extension of the Lagrange top has been introduced, in a different framework, in [96] and it is here called the *twisted Lagrange top*. The authors study this model in the spirit of the dynamical systems theory, so that they do not use a Lax pair and an r-matrix approach. They obtain this new kind of integrable top adding a cocycle to the Lie-Poisson structure for the two-field top, thus breaking its semidirect product structure. We remark that in [96] the so-called twisted top remains a mathematical construction without a physical interpretation. Later on, in [102], the author constructs a Lax matrix for such system, called here *generalized Lagrange top*. The integrability is proven by direct inspection since an r-matrix approach is not used, and the author provides a complete study of the spectral curve through the algebro-geometric techniques. The main goal in [102] is the proof that the generalized Lagrange top has monodromy, as well as the standard Lagrange top, so that it does not admit global action-angle variables.

We finally recall the papers [46, 51] where the problem of separation of variables is investigated. As a matter of fact, the contracted models inherit the separability property of the ancestor model. In [46, 51] the authors consider $\mathfrak{so}(2,1)$ and $\mathfrak{so}(3)$ rational Gaudin models: suitable contraction procedures on the separation coordinates lead to new separable integrable models.

Therefore, the novelty of the results contained in Chapter 2 consists in: i) *a general and systematic reduction of Gaudin models, preserving the r-matrix formulation*; ii) *a complete construction of the Hamiltonians and a physical interpretation of the $\mathfrak{su}(2)$ hierarchies.*

- **Chapter 3. Integrable discretizations of $\mathfrak{su}(2)$ extended rational Gaudin models**. This Chapter is devoted to the construction of Poisson integrable discretizations of the rational $\mathfrak{su}(2)$ Gaudin model and its Leibniz extensions. We study the problem of integrable discretization for such models using two different approaches:

    1. *The technique of Bäcklund transformations (BTs) for finite-dimensional integrable systems* [55, 56, 87]. Using the method of BTs developed by V.B. Kuznetsov, E.K. Sklyanin and P. Vanhaecke we construct integrable Poisson maps for the first Leibniz extension of the Lagrange top and for the rational Lagrange chain. An explicit construction of BTs for the standard Lagrange top can be found in our paper [53]:

actually this result is the specialization of BTs for the rational Lagrange chain. As explained in Section 3.1 these special maps discretize a family of Hamiltonian flows of the integrable system (and not a particular one).

2. *The machinery of discrete-time mechanics* [63, 91, 99, 100, 101]. Starting from a well-known time-discretization of the Lagrange top obtained by Yu.B. Suris and A. Bobenko in [19], we are able to construct an explicit Poisson integrable map for the rational $\mathfrak{su}(2)$ Gaudin model. The obtained discretization is different from the one considered in [41] through BTs. In particular we focus our attention just on a special Hamiltonian flow of the system, finding its discrete-time version and proving its integrability and Poisson property.

   Then, using the machinery presented in Chapter 2, we are able to perform the contraction procedure on the discrete-time rational $\mathfrak{su}(2)$ Gaudin model, thus obtaining integrable discretizations for the contracted systems. In particular, we shall present the discrete-time extended Lagrange tops and an alternative discretization of the rational Lagrange chain.

The results presented in Chapter 3, Section 3.1, are already published in [53, 65, 66], while the results in Section 3.2 are not.

2

Integrable extensions of Gaudin models

2.1 Drinfeld-Belavin solutions to CYBE

Let V be a complex finite-dimensional vector space and let $\mathfrak{g} \doteq (V, [\,\cdot\,,\cdot\,]) \doteq \mathrm{span}\{X^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ be a simple Lie algebra. The commutation relations between the elements of the basis read

$$\left[\, X^\alpha, X^\beta \,\right] = C_\gamma^{\alpha\beta}\, X^\gamma, \qquad 1 \leq \alpha, \beta \leq \dim \mathfrak{g}, \tag{2.1}$$

where $\{C_\gamma^{\alpha\beta}\}_{\alpha,\beta,\gamma=1}^{\dim \mathfrak{g}}$ denote the set of structure constants of the Lie algebra $\mathfrak{g}$. Recall that we are using the convention of summing over repeated greek indices: they shall always run from 1 to $\dim \mathfrak{g}$.

The classical Yang-Baxter equation (CYBE) is given by the following functional equation, see Eq. (1.14):

$$[\, r_{13}(\lambda), r_{23}(\mu)\,] + [\, r_{12}(\lambda - \mu), r_{13}(\lambda) + r_{23}(\mu)\,] = 0, \tag{2.2}$$

where $r(\lambda)$ is a $\mathfrak{g} \otimes \mathfrak{g}$ valued function of a complex parameter λ (called *spectral parameter*) and $r_{12} \doteq r \otimes \mathbb{1}$, $r_{23} \doteq \mathbb{1} \otimes r$, etc., are the natural imbeddings of $r(\lambda)$ from $\mathfrak{g} \otimes \mathfrak{g}$ into $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, being $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$ (an associative algebra with unit). If $r(\lambda)$ is a solution of Eq. (2.2) and (ρ_i, V_i), $i = 1, 2, 3$, is a fundamental representation of $\mathfrak{g}$ then $(\rho_i \otimes \rho_j)\, r(\lambda)$ gives a matrix solution of Eq. (2.2). Hereafter we shall consider always matrix solutions of the CYBE. In other words we shall identify $\mathfrak{g}$ with $\rho(\mathfrak{g})$, being ρ a fundamental representation of $\mathfrak{g}$.

The structure of the CYBE is well understood and a classification of non-degenerate solutions related to simple Lie algebras was given in [15]. It is close to the classification of the Dynkin diagrams and their automorphisms. Such a solutions is a meromorphic function that has a pole of first order at $\lambda = 0$ with residue

$$\mathrm{res}_{\lambda=0}\, r(\lambda) = g_{\alpha\beta}\, X^\alpha \otimes X^\beta, \tag{2.3}$$

where $g_{\alpha\beta}$ is the inverse of the Cartan-Killing metric related to the basis $\{X^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$. Recall that, under the assumption that $\mathfrak{g}$ has a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ invariant under the adjoint action, we can take the Cartan-Killing form as $\langle \cdot, \cdot \rangle$. If $\mathfrak{g}$ is simple and represented as a matrix Lie algebra we may assume $g^{\alpha\beta} \doteq \mathrm{tr}[\, X^\alpha\, X^\beta]$.

Remark 2.1 *Two remarkable properties of non-degenerate solutions $r(\lambda)$ of Eq. (2.2) are [15]:*

1. *$r(\lambda)$ fulfils the so-called unitarity condition*

$$r_{12}(-\lambda) = -r_{21}(\lambda); \tag{2.4}$$

2. *if $\gamma \in \Gamma$, Γ being the set of discrete poles of $r(\lambda)$, then*

$$r(\lambda + \gamma) = (A_\gamma \otimes \mathbb{1})\, r(\lambda) = (\mathbb{1} \otimes A_\gamma^{-1})\, r(\lambda), \tag{2.5a}$$

$$(A_\gamma \otimes A_\gamma)\, r(\lambda) = r(\lambda), \tag{2.5b}$$

where A_γ is an automorphism of $\mathfrak{g}$ (its form is given in [15]).

The remarkable feature of the CYBE (2.2) is that it allows averaging over a lattice in the complex λ-plane. In fact, according to their dependence on the complex parameter λ the solutions of the CBYE are called rational, trigonometric or elliptic. We give here the following fundamental result [15, 29, 80].

Theorem 2.1 *If $r(\lambda)$ is a rational, trigonometric or elliptic solution of Eq. (2.2), then its principal part is respectively given by*

$$r(\lambda) = \frac{1}{\lambda}\, g_{\alpha\beta}\, X^\alpha \otimes X^\beta$$
$$r(\lambda) = \sum_{\omega\in\Lambda_t} \frac{(A^n \otimes \mathbb{1})\, g_{\alpha\beta}\, X^\alpha \otimes X^\beta}{\lambda-\omega},$$
$$r(\lambda) = \sum_{\omega\in\Lambda_e} \frac{(A^n B^m \otimes \mathbb{1})\, g_{\alpha\beta}\, X^\alpha \otimes X^\beta}{\lambda-\omega},$$

where

$$\Lambda_t \doteq \{\omega = n\,\omega_1,\, (n,\omega_1)\in\mathbb{Z}\times\mathbb{C}\},$$

and

$$\Lambda_e \doteq \{\omega = n\,\omega_1 + m\,\omega_2,\, (n,m)\in\mathbb{Z}^2,\, \mathfrak{Im}\,(\omega_1/\omega_2)>0\}.$$

Here A and B are two finite order commuting automorphisms of $\mathfrak{g}$ not having a common fixed vector.

Remark 2.2 *Since $\mathfrak{sl}(n)$ is the only simple Lie algebra possessing such two automorphisms it follows that the elliptic solution can be defined only for $\mathfrak{g}\equiv\mathfrak{sl}(n)$.*

For our purposes we are interested in those solutions of Eq. (2.2) that can be written in the form

$$r(\lambda) = g_{\alpha\beta}\, X^\alpha \otimes X^\beta\, f^{\alpha\beta}(\lambda), \tag{2.7}$$

where the $f^{\alpha\beta}(\lambda)$'s, $1\le\alpha,\beta\le\dim\mathfrak{g}$ are meromorphic functions such that Eq. (2.3) holds. In [14, 15, 74, 75] it is shown that rational, trigonometric and elliptic solutions of the form (2.7) always exist for any choice of the simple Lie algebra $\mathfrak{g}$.

We now show that the CYBE (2.2) can be rewritten as a system of functional equations for the functions $f^{\alpha\beta}(\lambda)$ if $r(\lambda)$ is of the form (2.7) .

Proposition 2.1 *If $r(\lambda)$ is a solution of the form (2.7) of the CYBE (2.2) then the functions $f^{\alpha\beta}(\lambda)$, $1\le\alpha,\beta\le\dim\mathfrak{g}$, satisfy the following system of functional equations*

$$\sum_{\beta,\delta}\big[\, g_{\alpha\beta}\, g_{\gamma\delta}\, C^{\beta\delta}_{\eta}\, f^{\alpha\beta}(\lambda)\, f^{\gamma\delta}(\mu) + g_{\delta\gamma}\, g_{\beta\eta}\, C^{\delta\beta}_{\alpha}\, f^{\delta\gamma}(\lambda-\mu) f^{\beta\eta}(\lambda) +$$
$$+\, g_{\alpha\beta}\, g_{\delta\eta}\, C^{\beta\delta}_{\gamma}\, f^{\alpha\beta}(\lambda-\mu) f^{\delta\eta}(\mu)\,\big] = 0, \tag{2.8}$$

for all $1\le\alpha,\gamma,\eta\le\dim\mathfrak{g}$.

Remark 2.3 *Note that in Eq. (2.8) the summation is not on all the repeated indices, so that the sum is explicitly indicated.*

Proof: By means of a direct computation we get:

$$[\, r_{13}(\lambda), r_{23}(\mu)\,] = g_{\alpha\beta}\, g_{\gamma\delta}\, f^{\alpha\beta}(\lambda)\, f^{\gamma\delta}(\mu)\, \big[\, X^\alpha\otimes\mathbb{1}\otimes X^\beta, \mathbb{1}\otimes X^\gamma\otimes X^\delta\,\big] =$$
$$= g_{\alpha\beta}\, g_{\gamma\delta}\, C^{\beta\delta}_{\eta}\, f^{\alpha\beta}(\lambda)\, f^{\gamma\delta}(\mu)(X^\alpha\otimes X^\gamma\otimes X^\eta). \tag{2.9}$$

On the other hand we have:

$$
\begin{aligned}
&[\, r_{12}(\lambda-\mu), r_{13}(\lambda)+r_{23}(\mu)\,] = g_{\alpha\beta}\, g_{\gamma\eta}\, f^{\alpha\beta}(\lambda-\mu)\ \times \\
&\qquad \times \left[\, X^{\alpha}\otimes X^{\beta}\otimes \mathbb{1}, (X^{\gamma}\otimes \mathbb{1}\otimes X^{\eta})\, f^{\gamma\eta}(\lambda) + (\mathbb{1}\otimes X^{\gamma}\otimes X^{\eta})\, f^{\gamma\eta}(\mu)\,\right] = \\
&= g_{\alpha\beta}\, g_{\gamma\eta}\, f^{\alpha\beta}(\lambda-\mu)\ \times \\
&\qquad \times \left[C^{\alpha\gamma}_{\delta}\,(X^{\delta}\otimes X^{\beta}\otimes X^{\eta})\, f^{\gamma\eta}(\lambda) + C^{\beta\gamma}_{\delta}\,(X^{\alpha}\otimes X^{\delta}\otimes X^{\eta})\, f^{\gamma\eta}(\mu) \right] = \\
&= \left[\, g_{\delta\gamma}\, g_{\beta\eta}\, C^{\delta\beta}_{\alpha}\, f^{\delta\gamma}(\lambda-\mu) f^{\beta\eta}(\lambda) + g_{\alpha\beta}\, g_{\delta\eta}\, C^{\beta\delta}_{\gamma}\, f^{\alpha\beta}(\lambda-\mu) f^{\delta\eta}(\mu)\,\right] (X^{\alpha}\otimes X^{\gamma}\otimes X^{\eta}), \quad (2.10)
\end{aligned}
$$

where the last expression is obtained from the previous one swapping the indices $\gamma \leftrightarrow \beta, \alpha \leftrightarrow \delta$ in the first term and $\gamma \leftrightarrow \delta$ in the second one.

Equating Eqs. (2.9) and (2.10) we obtain the system (2.8).

□

2.2 Definition of classical Gaudin models

To each matrix $r(\lambda)$ of the form (2.7) satisfying the CYBE (2.2) we can associate a Lax matrix whose entries are functions on the Lie-Poisson manifold associated with $\mathfrak{g}$ in the following way [4, 35, 36, 44, 64, 79, 86, 91].

We introduce the vector space dual to $\mathfrak{g}$, namely $\mathfrak{g}^* \doteq \mathrm{span}\{X_\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ with pairing given by $\langle\, X^\alpha, X_\beta\,\rangle = \delta^\alpha_\beta$. If $L \in \mathfrak{g}^*$ then $L = y^\beta\, X_\beta = g_{\alpha\beta}\, X^\alpha\, y^\beta$, where the y^β's are the coordinate functions on $\mathfrak{g}^*$. Since the Cartan-Killing form is a non-degenerate invariant scalar product we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. The commutation relations on $\mathfrak{g}$ given in Eq. (2.1) are associated with the following Lie-Poisson brackets on $\mathfrak{g}^*$:

$$\left\{y^\alpha, y^\beta\right\} = C^{\alpha\beta}_{\gamma} y^\gamma, \qquad 1 \le \alpha, \beta \le \dim \mathfrak{g}.$$

We now introduce the following Lax matrix

$$\ell(\lambda) \doteq g_{\alpha\beta}\, X^\alpha\, y^\beta\, f^{\alpha\beta}(\lambda)\ \in \mathfrak{g}\,[\,\lambda, \lambda^{-1}], \tag{2.11}$$

where the meromorphic functions $f^{\alpha\beta}(\lambda)$, $1 \le \alpha, \beta \le \dim \mathfrak{g}$, satisfy Eq. (2.8). Hence the Lax matrix (2.11) can admit a rational, trigonometric or elliptic dependence on the spectral parameter. Notice that we have denoted with $\mathfrak{g}\,[\,\lambda, \lambda^{-1}]$ the Lie algebra of Laurent polynomials on λ, λ^{-1} with coefficients in $\mathfrak{g}$.

The following statement holds.

Proposition 2.2 *The Lax matrix (2.11) satisfies the linear r-matrix algebra*

$$\{\ell(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes \ell(\mu)\} + [\, r(\lambda-\mu), \ell(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes \ell(\mu)\,] = 0, \qquad \forall\,(\lambda,\mu)\in \mathbb{C}^2, \tag{2.12}$$

with $r(\lambda)$ given in Eq. (2.7).

Proof: The proposition can be proven by means of a direct computation:

$$
\begin{aligned}
\{\ell(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes \ell(\mu)\} &= g_{\alpha\beta}\, g_{\gamma\delta}\,(X^\alpha\otimes X^\gamma)\, f^{\alpha\beta}(\lambda)\, f^{\gamma\delta}(\mu)\,\left\{y^\beta, y^\delta\right\} = \\
&= g_{\alpha\beta}\, g_{\gamma\delta}\,(X^\alpha\otimes X^\gamma)\, f^{\alpha\beta}(\lambda)\, f^{\gamma\delta}(\mu)\, C^{\beta\delta}_{\eta}\, y^\eta. \qquad (2.13)
\end{aligned}
$$

On the other hand we have:

$$\begin{aligned}
&[\, r(\lambda-\mu), \ell(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes \ell(\mu)\,] = g_{\alpha\beta}\, g_{\gamma\eta}\, f^{\alpha\beta}(\lambda-\mu) \times \\
&\qquad \times \left[\, X^\alpha \otimes X^\beta, (X^\gamma \otimes \mathbb{1})\, y^\eta\, f^{\gamma\eta}(\lambda) + (\mathbb{1}\otimes X^\gamma)\, y^\eta\, f^{\gamma\eta}(\mu)\,\right] = \\
&= g_{\alpha\beta}\, g_{\gamma\eta}\, f^{\alpha\beta}(\lambda-\mu) \times \\
&\qquad \times \left[C^{\alpha\gamma}_{\delta}\,(X^\delta \otimes X^\beta)\, y^\eta\, f^{\gamma\eta}(\lambda) + C^{\beta\gamma}_{\delta}\,(X^\alpha \otimes X^\delta)\, y^\eta\, f^{\gamma\eta}(\mu) \right] = \\
&= \left[\, g_{\delta\gamma}\, g_{\beta\eta}\, C^{\delta\beta}_{\alpha}\, f^{\delta\gamma}(\lambda-\mu) f^{\beta\eta}(\lambda) + g_{\alpha\beta}\, g_{\delta\eta}\, C^{\beta\delta}_{\gamma}\, f^{\alpha\beta}(\lambda-\mu) f^{\delta\eta}(\mu)\,\right]\, y^\eta (X^\alpha \otimes X^\gamma), \qquad (2.14)
\end{aligned}$$

where the last expression is obtained from the previous one swapping the indices $\gamma \leftrightarrow \beta, \alpha \leftrightarrow \delta$ in the first term and $\gamma \leftrightarrow \delta$ in the second one.

Summing Eqs. (2.13) and (2.14) we immediately see that the resulting equation vanishes thanks to the system (2.8).

□

The Lax matrix (2.11) can be seen as the local Lax matrix of the integrable classical Gaudin model. In fact the definition of the classical Gaudin model requires the introduction of the Lie-Poisson manifold associated with the direct sum of N copies of the Lie algebra $\mathfrak{g}$.

Let us define $\mathfrak{G}_N \doteq \oplus^N \mathfrak{g}$. A basis of $\mathfrak{G}_N$ is given by $\{X_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \le i \le N$ with $\{X_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ denoting the basis $\{X^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$ on the i-th copy of $\mathfrak{g}$. The commutation relations for the elements of the basis of $\mathfrak{G}_N$ read

$$\left[\, X_i^\alpha, X_j^\beta \,\right] = \delta_{i,j}\, C^{\alpha\beta}_{\gamma}\, X_i^\gamma, \qquad 1 \le i, j \le N. \tag{2.15}$$

The classical N-site Gaudin model is defined on the Lie-Poisson manifold associated with $\mathfrak{G}_N^*$. We will denote by $\{y_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \le i \le N$, the set of the (time-dependent) coordinate functions relative on the i-th copy of $\mathfrak{g}^*$. Consequently, the Lie-Poisson brackets on $\mathfrak{G}_N^*$ read

$$\left\{ y_i^\alpha, y_j^\beta \right\}_{P_N^{\mathfrak{g}}} = \delta_{i,j}\, C^{\alpha\beta}_{\gamma}\, y_i^\gamma, \qquad 1 \le i, j \le N. \tag{2.16}$$

Here we have denoted with $P_N^{\mathfrak{g}}$ the tensor associated with the Lie-Poisson structure defined in Eq. (2.16). Such Lie-Poisson tensor can be written in the following block matrix form:

$$(P_N^{\mathfrak{g}})_{i,j} \doteq \delta_{i,j}\, Y_i, \qquad (Y_i)_{\alpha,\beta} \doteq C^{\alpha\beta}_{\gamma}\, y_i^\gamma, \qquad 1 \le i, j \le N. \tag{2.17}$$

The involutive Hamiltonians of the N-site Gaudin model are given by the spectral invariants of the Lax matrix

$$\mathcal{L}_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{N} \ell_i(\lambda - \lambda_i), \tag{2.18}$$

where the λ_i's, with $\lambda_i \neq \lambda_k, 1 \le i, k \le N$, are complex parameters of the model, and

$$\ell_i(\lambda) \doteq g_{\alpha\beta}\, X^\alpha\, y_i^\beta\, f^{\alpha\beta}(\lambda) \in \mathfrak{g}\,[\,\lambda, \lambda^{-1}]. \tag{2.19}$$

The Lax matrix given in Eq. (2.18) admits a linear r-matrix formulation, which ensures that all the spectral invariants of the Lax matrix form a family of involutive functions.

Proposition 2.3 *The Lax matrix (2.18) satisfies the linear r-matrix algebra*

$$\{\mathcal{L}_{\mathcal{G}}(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes \mathcal{L}_{\mathcal{G}}(\mu)\}_{P_N^{\mathfrak{g}}} + [\, r(\lambda-\mu), \mathcal{L}_{\mathcal{G}}(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes \mathcal{L}_{\mathcal{G}}(\mu)\,] = 0, \qquad \forall\, (\lambda,\mu) \in \mathbb{C}^2, \tag{2.20}$$

with $r(\lambda)$ given in Eq. (2.7).

Proof: Taking into account the locality of the Lax matrices $\ell_i(\lambda)$ (2.19), $1 \leq i \leq N$, namely the direct sum structure encoded in the Lie-Poisson brackets (2.16), and the fact that r-matrix depends just on the difference $\lambda - \mu$ we can perform a computation similar to the one given in Proposition 2.2, thus proving Eq. (2.20).

□

The involutive Hamiltonians can be computed studying the eigenvector equation $(\mathcal{L}_{\mathcal{G}}(\lambda) - \mu\,\mathbb{1})\,\Psi(\lambda,\mu) = 0$, where $\Psi(\lambda,\mu)$ is the eigenvector with eigenvalue μ [4, 9, 79]. The characteristic equation for this eigenvalue problem is the following algebraic curve in $\mathbb{C}^2$:

$$\Gamma: \ \det(\mathcal{L}_{\mathcal{G}}(\lambda) - \mu\,\mathbb{1}) = 0.$$

Notice that, if n is the dimension of the Lax matrix, the equation of this curve is of the form

$$(-\mu)^n + \sum_{j=0}^{n-1} h_j(\lambda)\,\mu^j = 0,$$

where the $h_j(\lambda)$'s are (time-independent) polynomials in the matrix elements of $\mathcal{L}_{\mathcal{G}}(\lambda)$ and therefore have poles at λ_i. The involutive Hamiltonians are given by $\mathrm{res}_{\lambda=\lambda_i}\, h_j(\lambda) \in \mathcal{F}(\mathfrak{G}_N^*)$, where $\mathcal{F}(\mathfrak{G}_N^*)$ denotes the space of differentiable functions on $\mathfrak{G}_N^*$. Obviously they are not all indipendent, and their involutivity is ensured thanks to the r-matrix formulation (2.20).

Few remarks on rational Gaudin models

In the case of the rational Gaudin model we have $f^{\alpha\beta}(\lambda) = \lambda^{-1}$, $1 \leq \alpha,\beta \leq \dim\mathfrak{g}$. It is convenient to write down the Lax matrix (2.18) in the form

$$\mathcal{L}^r_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{N} \frac{\mathcal{Y}_i}{\lambda - \lambda_i}, \qquad \mathcal{Y}_i \doteq g_{\alpha\beta}\, X^\alpha\, y_i^\beta \in \mathfrak{g}. \tag{2.21}$$

In the above formula we have used the index r to explicitly denote the rational dependence on the spectral parameter. The quadratic integrals, obtained from $\mathrm{tr}\,(\mathcal{L}^r_{\mathcal{G}}(\lambda))^2$, have the following form:

$$H_i^r \doteq \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\mathrm{tr}\,(\mathcal{Y}_i\,\mathcal{Y}_j)}{\lambda_i - \lambda_j}, \qquad \sum_{i=1}^{N} H_i^r = 0, \tag{2.22}$$

so that they are not all indipendent. It is possible to obtain a remarkable linear combination of the integrals (2.22), that is called *rational Gaudin Hamiltonian*, namely

$$\mathcal{H}^r_{\mathcal{G}} \doteq \sum_{i=1}^{N} \eta_i\, H_i^r = \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\eta_i - \eta_j}{\lambda_i - \lambda_j}\, \mathrm{tr}\,(\mathcal{Y}_i\,\mathcal{Y}_j)\,, \tag{2.23}$$

where the η_i's, with $\eta_i \neq \eta_k, 1 \leq i,k \leq N$, are arbitrary complex parameters. A relevant member of the family of rational Gaudin Hamiltonians (2.23) is obtained fixing $\eta_i \equiv \lambda_i$, $1 \leq i \leq N$. The resulting Hamiltonian (2.23) is indipendent from the parameters λ_i, η_i and it defines the so called *homogeneous rational Gaudin model*. The remarkable feature of such a system is that it is superintegrable [30].

Hamiltonian systems defined on direct products of Lie-Poisson manifolds with Lax matrices of the form (2.21) have been widely studied by Reyman and Semenov-Tian-Shansky [79]. From their results it follows that the spectral invariants of the Lax matrix (2.21) are not enough to provide the complete integrability of the model. To recover the missing integrals one has to notice that the spectral invariants of (2.21) are invariant under the global action of the Lie group associated to $\mathfrak{g}$. In particular, if $\tau \in \mathfrak{g}$, then $\phi_\tau \doteq \mathrm{tr}\,(\sum_{i=1}^N \mathcal{Y}_i\, \tau)$ defines a function Poisson commuting with all the spectral invariants of (2.21). Varying $\tau \in \mathfrak{g}$ the functions ϕ_τ span a not abelian Poisson algebra isomorphic to $\mathfrak{g}$: hence one has to extract from the functions ϕ_τ a maximal abelian subalgebra.

A different way to recover the missing integrals is to add "a posteriori" a constant term $\mathcal{P} \in \mathfrak{g}$ with simple spectrum to $\mathcal{L}^r_{\mathcal{G}}(\lambda)$, namely considering the Lax matrix

$$\mathcal{L}^r_{\mathcal{G}}(\lambda, \mathcal{P}) \doteq \mathcal{P} + \sum_{i=1}^N \frac{\mathcal{Y}_i}{\lambda - \lambda_i}. \tag{2.24}$$

In the case $\mathfrak{g} \equiv \mathfrak{su}(n)$, the constant term $\mathcal{P}$ has a natural physical interpretation. It is equivalent to adding to the Hamiltonian (2.23) a term describing the interaction of the $\mathfrak{su}(n)$ spins, described by the $\mathcal{Y}_i$'s, with an external constant in time and homogeneous field. In fact the rational Gaudin Hamiltonian is now defined by

$$\mathcal{H}^r_{\mathcal{G}}(\mathcal{P}) \doteq \sum_{i=1}^N \eta_i\, H^r_i(\mathcal{P}) = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N \frac{\eta_i - \eta_j}{\lambda_i - \lambda_j}\, \mathrm{tr}\,(\mathcal{Y}_i\, \mathcal{Y}_j) + \sum_{i=1}^N \eta_i\, \mathrm{tr}\,(\mathcal{P}\, \mathcal{Y}_i)\,. \tag{2.25}$$

We know from [79] that the spectral invariants of the Lax matrix (2.24) define a completely integrable system on $\mathfrak{G}^*_N \doteq \oplus^N \mathfrak{g}^*$, being $\mathfrak{g}$ one of the classical Lie algebras.

2.2.1 The continuous-time $\mathfrak{su}(2)$ *Gaudin models*

We now describe the main features of the (continuous-time) $\mathfrak{su}(2)$ Gaudin models, see for instance [30, 35, 36, 41, 44, 45, 64, 79, 86, 88, 92].

Let us fix $\mathfrak{g} \equiv \mathfrak{su}(2)$. We choose the following basis of the linear space $\mathfrak{su}(2)$:

$$\sigma_1 \doteq \frac{1}{2} \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \sigma_2 \doteq \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 \doteq \frac{1}{2} \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}.$$

It is well-known that the correspondence

$$\mathbb{R}^3 \ni \mathbf{a} \doteq (a^1, a^2, a^3)^T \longleftrightarrow \mathbf{a} \doteq \frac{1}{2} \begin{pmatrix} -\mathrm{i}\, a^3 & -\mathrm{i}\, a^1 - a^2 \\ -\mathrm{i}\, a^1 + a^2 & \mathrm{i}\, a^3 \end{pmatrix} \in \mathfrak{su}(2),$$

is an isomorphism between $(\mathfrak{su}(2), [\,\cdot\,,\cdot\,])$ and the Lie algebra $(\mathbb{R}^3, \times)$, where $\times$ stands for the vector product. This allows not to distinguish between vectors from $\mathbb{R}^3$ and matrices from $\mathfrak{su}(2)$. We supply $\mathfrak{su}(2)$ with the scalar product $\langle\,\cdot\,,\cdot\,\rangle$ induced from $\mathbb{R}^3$, namely

$$\langle\, \mathbf{a}, \mathbf{b}\,\rangle = -2\,\mathrm{tr}\,(\mathbf{a}\,\mathbf{b}) = 2\,\mathrm{tr}\,(\mathbf{b}\,\mathbf{a}^\dagger), \qquad \forall\, \mathbf{a}, \mathbf{b} \in \mathfrak{su}(2).$$

This scalar product allows us to identify the dual space $\mathfrak{su}^*(2)$ with $\mathfrak{su}(2)$, so that the coadjoint action of the algebra becomes the usual Lie bracket with minus, i.e. $\mathrm{ad}^*_\eta\, \xi = [\,\xi, \eta\,] = -\mathrm{ad}_\eta\, \xi$, $\xi, \eta \in \mathfrak{su}(2)$.

The Lie-Poisson algebra of the model is precisely (minus) $\oplus^N \mathfrak{su}^*(2)$, see Eq. (2.16):

$$\left\{ y^\alpha_i, y^\beta_j \right\}_{P^{\mathfrak{su}(2)}_N} = -\delta_{i,j}\, \varepsilon_{\alpha\beta\gamma}\, y^\gamma_i, \tag{2.26}$$

with $1 \le i, j \le N$. Here $\varepsilon_{\alpha\beta\gamma}$ is the skew-symmetric tensor with $\varepsilon_{123} \equiv 1$. The brackets (2.26) are degenerate: they possess the N Casimir functions

$$C_i \doteq \frac{1}{2}\langle \mathbf{y}_i, \mathbf{y}_i \rangle, \qquad 1 \le i \le N, \tag{2.27}$$

that provide a trivial dynamics.

The continuous-time $\mathfrak{su}(2)$ rational, trigonometric and elliptic Gaudin models are governed respectively by the following Lax matrices, see Eqs. (2.18) and (2.24):

$$\mathcal{L}^r_{\mathcal{G}}(\lambda, \mathbf{p}) \doteq \sigma_\alpha\, p^\alpha + \sum_{i=1}^{N} \frac{\sigma_\alpha\, y_i^\alpha}{\lambda - \lambda_i} = \mathbf{p} + \sum_{i=1}^{N} \frac{\mathbf{y}_i}{\lambda - \lambda_i} \;\in \mathfrak{su}(2)[\,\lambda, \lambda^{-1}], \tag{2.28a}$$

$$\mathcal{L}^t_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{N} \left[\frac{\sigma_1\, y_i^1 + \sigma_2\, y_i^2}{\sin(\lambda - \lambda_i)} + \cot(\lambda - \lambda_i)\, \sigma_3\, y_i^3 \right] \in \mathfrak{su}(2)[\,\lambda, \lambda^{-1}], \tag{2.28b}$$

$$\mathcal{L}^e_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{N} \left[\sigma_1\, y_i^1 \frac{\mathrm{dn}(\lambda - \lambda_i)}{\mathrm{sn}(\lambda - \lambda_i)} + \frac{\sigma_2\, y_i^2}{\mathrm{sn}(\lambda - \lambda_i)} + \sigma_3\, y_i^3 \frac{\mathrm{cn}(\lambda - \lambda_i)}{\mathrm{sn}(\lambda - \lambda_i)} \right] \in \mathfrak{su}(2)[\,\lambda, \lambda^{-1}], \tag{2.28c}$$

where $\mathrm{cn}(\lambda), \mathrm{dn}(\lambda), \mathrm{sn}(\lambda)$ are the elliptic Jacobi functions of modulus k. In Eq. (2.28a) $\mathbf{p}$ is a constant vector in $\mathbb{R}^3$.

The Lax matrices (2.28a), (2.28b) and (2.28c) describe complete integrable systems on the Lie-Poisson manifold associated with $\oplus^N \mathfrak{su}^*(2)$. According to Eq. (2.20) they satisfy a linear r-matrix algebra with r-matrix given by

$$r(\lambda) \doteq -f^\alpha(\lambda)\, \sigma_\alpha \otimes \sigma_\alpha, \tag{2.29}$$

where the functions $f^\alpha(\lambda)$, $\alpha = 1, 2, 3$, are defined by

$$f^1(\lambda) \doteq \begin{cases} \dfrac{1}{\lambda} & \text{rational case,} \\[2ex] \dfrac{1}{\sin(\lambda)} & \text{trigonometric case,} \\[2ex] \dfrac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} & \text{elliptic case,} \end{cases} \tag{2.30}$$

$$f^2(\lambda) \doteq \begin{cases} \dfrac{1}{\lambda} & \text{rational case,} \\[2ex] \dfrac{1}{\sin(\lambda)} & \text{trigonometric case,} \\[2ex] \dfrac{1}{\mathrm{sn}(\lambda)} & \text{elliptic case,} \end{cases} \tag{2.31}$$

$$f^3(\lambda) \doteq \begin{cases} \dfrac{1}{\lambda} & \text{rational case,} \\[2ex] \cot(\lambda) & \text{trigonometric case,} \\[2ex] \dfrac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} & \text{elliptic case.} \end{cases} \tag{2.32}$$

Remark 2.4 *As a consequence of the condition (2.5a), the trigonometric Lax matrix (2.28b) must satisfy the quasi-periodicity condition* $\mathcal{L}^t_{\mathcal{G}}(\lambda+\omega_1) = A\,\mathcal{L}^t_{\mathcal{G}}(\lambda)$*, where A is the inner automorphism of* $\mathfrak{su}(2)$ *defined by* $A\,\xi \doteq \sigma_3\,\xi\,\sigma_3,\ \forall\,\xi \in \mathfrak{su}(2)$*. The elliptic Lax matrix (2.28c) must satisfy the quasi-periodicity conditions* $\mathcal{L}^e_{\mathcal{G}}(\lambda+\omega_1) = A\,\mathcal{L}^e_{\mathcal{G}}(\lambda)$*,* $\mathcal{L}^e_{\mathcal{G}}(\lambda+\omega_2) = B\,\mathcal{L}^e_{\mathcal{G}}(\lambda)$*, where* A, B *are the inner automorphisms of* $\mathfrak{su}(2)$ *defined by* $A\,\xi \doteq \sigma_3\,\xi\,\sigma_3$*,* $B\,\xi \doteq \sigma_1\,\xi\,\sigma_1$*,* $\forall\,\xi \in \mathfrak{su}(2)$*.*

The complete set of integrals of the $\mathfrak{su}(2)$ rational, trigonometric and elliptic Gaudin models can be constructed computing the residues in $\lambda = \lambda_i$ of $\det(\mathcal{L}^{r,t,e}_{\mathcal{G}}(\lambda) - \mu\,\mathbb{1}) = 0$ (or equivalently $\mu^2 \doteq -(1/2)\,\mathrm{tr}[\,(\mathcal{L}^{r,t,e}_{\mathcal{G}}(\lambda))^2])$. Let us give the following results [30, 35, 36, 41, 45, 64, 86, 88, 92, 95].

Proposition 2.4 *The hyperelliptic, genus* $N-1$*, curve* $\det(\mathcal{L}^r_{\mathcal{G}}(\lambda,\mathbf{p}) - \mu\,\mathbb{1}) = 0$*,* $(\lambda,\mu) \in \mathbb{C}^2$*, with* $\mathcal{L}^r_{\mathcal{G}}(\lambda,\mathbf{p})$ *given in Eq. (2.28a), provides a set of* $2N$ *independent involutive integrals of motion given by*

$$
\begin{aligned}
&H^r_i(\mathbf{p}) \doteq \langle\,\mathbf{p},\mathbf{y}_i\,\rangle + \sum_{\substack{j=1\\ j\neq i}}^{N} \frac{\langle\,\mathbf{y}_i,\mathbf{y}_j\,\rangle}{\lambda_i-\lambda_j}, \qquad \sum_{i=1}^{N} H^r_i(\mathbf{p}) = \sum_{i=1}^{N}\langle\,\mathbf{p},\mathbf{y}_i\,\rangle, \qquad (2.33)\\
&C_i \doteq \frac{1}{2}\langle\,\mathbf{y}_i,\mathbf{y}_i\,\rangle,\\
&\left\{H^r_i(\mathbf{p}),H^r_j(\mathbf{p})\right\}_{P^{\mathfrak{su}(2)}_N} = \left\{H^r_i(\mathbf{p}),C_j\right\}_{P^{\mathfrak{su}(2)}_N} = \left\{C_i,C_j\right\}_{P^{\mathfrak{su}(2)}_N} = 0.
\end{aligned}
$$

The integrals $\{H^r_i(\mathbf{p})\}_{i=1}^N$ *are first integrals of motion and the integrals* $\{C_i\}_{i=1}^N$ *are the Casimir functions given in Eq. (2.27).*

Proposition 2.5 *The curve* $\det(\mathcal{L}^t_{\mathcal{G}}(\lambda) - \mu\,\mathbb{1}) = 0$*,* $(\lambda,\mu) \in \mathbb{C}^2$*, with* $\mathcal{L}^t_{\mathcal{G}}(\lambda)$ *given in Eq. (2.28b), provides a set of* $2N$ *independent involutive integrals of motion given by*

$$
\begin{aligned}
&H^t_i \doteq \sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{y^1_i\,y^1_j + y^2_i\,y^2_j}{\sin(\lambda_i-\lambda_j)} + \cot(\lambda_i-\lambda_j)\,y^3_i\,y^3_j\right], \qquad \sum_{i=1}^{N} H^t_i = 0,\\
&H^t_0 \doteq \left(\sum_{i=1}^{N} y^3_i\right)^2,\\
&C_i \doteq \frac{1}{2}\langle\,\mathbf{y}_i,\mathbf{y}_i\,\rangle,\\
&\left\{H^t_i,H^t_j\right\}_{P^{\mathfrak{su}(2)}_N} = \left\{H^t_i,C_j\right\}_{P^{\mathfrak{su}(2)}_N} = \left\{C_i,C_j\right\}_{P^{\mathfrak{su}(2)}_N} = 0.
\end{aligned}
$$

The integrals $\{H^t_i\}_{i=0}^N$ *are first integrals of motion and the integrals* $\{C_i\}_{i=1}^N$ *are the Casimir functions given in Eq. (2.27).*

Proposition 2.6 *The curve* $\det(\mathcal{L}^e_{\mathcal{G}}(\lambda) - \mu\,\mathbb{1}) = 0$*,* $(\lambda,\mu) \in \mathbb{C}^2$*, with* $\mathcal{L}^e_{\mathcal{G}}(\lambda)$ *given in Eq.*

(2.28c), provides a set of $2N$ independent involutive integrals of motion given by

$$H_i^e \doteq \sum_{\substack{j=1\\ j\neq i}}^{N} \left[y_i^1\, y_j^1\, \frac{\mathrm{dn}(\lambda_i-\lambda_j)}{\mathrm{sn}(\lambda_i-\lambda_j)} + \frac{y_i^2\, y_j^2}{\mathrm{sn}(\lambda_i-\lambda_j)} + y_i^3\, y_j^3\, \frac{\mathrm{cn}(\lambda_i-\lambda_j)}{\mathrm{sn}(\lambda_i-\lambda_j)} \right], \qquad \sum_{i=1}^{N} H_i^e = 0,$$

$$H_0^e \doteq \frac{1}{2}\sum_{i,j=1}^{N} \left[y_i^1\, y_j^1\, \frac{\theta_{11}'\,\theta_{10}'(\lambda_i-\lambda_j)}{\theta_{10}\,\theta_{11}(\lambda_i-\lambda_j)} + y_i^2\, y_j^2\, \frac{\theta_{11}'\,\theta_{00}'(\lambda_i-\lambda_j)}{\theta_{00}\,\theta_{11}(\lambda_i-\lambda_j)} + y_i^3\, y_j^3\, \frac{\theta_{11}'\,\theta_{01}'(\lambda_i-\lambda_j)}{\theta_{01}\,\theta_{11}(\lambda_i-\lambda_j)} \right],$$

$$C_i \doteq \frac{1}{2}\langle\, \mathbf{y}_i, \mathbf{y}_i\,\rangle,$$

$$\left\{H_i^e, H_j^e\right\}_{P_N^{\mathfrak{su}(2)}} = \left\{H_i^e, C_j\right\}_{P_N^{\mathfrak{su}(2)}} = \left\{C_i, C_j\right\}_{P_N^{\mathfrak{su}(2)}} = 0,$$

where $\theta_{\alpha\beta}(\lambda)$, $\alpha,\beta = 0,1$, is the theta function[1]*, and $\theta_{\alpha\beta} \doteq \theta_{\alpha\beta}(0)$, $\theta_{\alpha\beta}' \doteq (d/d\lambda)_{\lambda=0}\theta_{\alpha\beta}(\lambda)$. The integrals $\{H_i^e\}_{i=0}^N$ are first integrals of motion and the integrals $\{C_i\}_{i=1}^N$ are the Casimir functions given in Eq. (2.27).*

In the next Chapter we shall focus our attention on the time-discretization of certain Hamiltonian flows of the $\mathfrak{su}(2)$ *rational* Gaudin model and its integrable extensions, obtained in the remaining part of this Chapter. Therefore we give some further details in such a case.

The Hamiltonian (2.25) takes the form

$$\mathcal{H}_{\mathcal{G}}^r(\mathbf{p}) \doteq \sum_{i=1}^{N} \eta_i\, H_i^r(\mathbf{p}) = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \frac{\eta_i-\eta_j}{\lambda_i-\lambda_j}\,\langle\, \mathbf{y}_i, \mathbf{y}_j\,\rangle + \sum_{i=1}^{N} \eta_i\,\langle\, \mathbf{p}, \mathbf{y}_i\,\rangle. \tag{2.36}$$

Moreover we note that there is one linear integral given by $\sum_{i=1}^N H_i^r(\mathbf{p}) = \sum_{i=1}^N \langle\, \mathbf{p}, \mathbf{y}_i\,\rangle$.

We shall consider the discrete-time version of the Hamiltonian flow generated by the integral (2.36) with the particular choice $\eta_i \equiv \lambda_i$, $1 \le i \le N$, namely

$$\mathcal{H}_{\mathcal{G}}^r(\mathbf{p}, \eta_i \equiv \lambda_i) = \frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N} \langle\, \mathbf{y}_i, \mathbf{y}_j\,\rangle + \sum_{i=1}^{N} \lambda_i\,\langle\, \mathbf{p}, \mathbf{y}_i\,\rangle. \tag{2.37}$$

A direct computation leads to the following proposition.

Proposition 2.7 *The equations of motion w.r.t. the Hamiltonian (2.37) are given by*

$$\dot{\mathbf{y}}_i = \left[\, \lambda_i\,\mathbf{p} + \textstyle\sum_{j=1}^N \mathbf{y}_j\,,\, \mathbf{y}_i \right], \qquad 1 \le i \le N, \tag{2.38}$$

where $\dot{\mathbf{y}}_i \doteq d\mathbf{y}_i/dt$. Eqs. (2.38) admit the following Lax representation:

$$\dot{\mathcal{L}}_{\mathcal{G}}^r(\lambda, \mathbf{p}) = \left[\, \mathcal{L}_{\mathcal{G}}^r(\lambda, \mathbf{p}), \mathcal{M}_{\mathcal{G}}^r(\lambda)\,\right],$$

with the matrix $\mathcal{L}_{\mathcal{G}}^r(\lambda, \mathbf{p})$ given in Eq. (2.28a) and

$$\mathcal{M}_{\mathcal{G}}^r(\lambda) \doteq \sum_{i=1}^{N} \frac{\lambda_i\,\mathbf{y}_i}{\lambda-\lambda_i}. \tag{2.39}$$

[1]We use the following notation: $\theta_{\alpha\beta}(\lambda) \doteq \theta_{\alpha\beta}(\lambda,\tau) = \sum_{n\in\mathbb{Z}} \exp\left[\pi\,\mathrm{i}\,(n+\frac{\alpha}{2})^2\,\tau + 2\,\pi\,\mathrm{i}\,(n+\frac{\alpha}{2})\left(n+\frac{\beta}{2}\right)\right]$, $\alpha,\beta = 0,1$, where τ is a complex number in the upper half plane.

2.3 Leibniz extensions of Gaudin models

The concept of limiting process between Lie algebras was proposed by I. Segal in 1951 [83]. The most known example concerning this concept is given by the connection between relativistic and classical mechanics with their underlying Poincaré and Galilei symmetry algebras. The other well-known example is the limit process from quantum mechanics to the classical one, which corresponds to the contraction of the Heisenberg algebra to an abelian algebra.

These ideas were developed in the works of E. Inönü and E. Wigner [43]. They introduced the so called Inönü-Wigner contractions, which, in spite of their simplicity, were applied to a wide range of physical and mathematical problems.

Later on, a large number of contractions have been introduced and the relation between contractions and deformations (or expansions) have been widely investigated [58].

2.3.1 Generalized Inönü-Wigner contractions and Leibniz extensions

Let us recall the notion of generalized Inönü-Wigner contractions of a finite dimensional Lie algebra $\mathfrak{g}$ [43, 103]. The following definition holds also for infinite-dimensional Lie algebras.

Let V be a complex finite-dimensional vector space. Let $\mathfrak{g} \doteq (V, \mu)$ be a Lie algebra with Lie multiplication $\mu : V \times V \to V$. The analytic notion of *continuous* contraction can be described by a continuous family of maps

$$U(\vartheta) : V \to V, \qquad 0 \leq \vartheta \leq 1, \quad U(1) = 1,$$

which are nonsingular for $0 < \vartheta \leq 1$ and singular for $\vartheta = 0$. The new Lie bracket on V,

$$\mu_\vartheta(X^\alpha, X^\beta) = U^{-1}(\vartheta)\, \mu\, (U(\vartheta)\, X^\alpha, (\vartheta)\, X^\beta), \qquad (X^\alpha, X^\beta) \in V \times V, \quad 0 < \vartheta \leq 1,$$

corresponds to a change of basis given by $U(\vartheta)$, and leads to the Lie algebra $\mathfrak{g}_\vartheta \doteq (V, \mu_\vartheta)$ isomorphic to $\mathfrak{g}$. If $\widetilde{\mu}(X^\alpha, X^\beta) = \lim_{\vartheta\to 0} \mu_\vartheta(X^\alpha, X^\beta)$ exists for all $(X^\alpha, X^\beta) \in V \times V$, we call $\widetilde{\mathfrak{g}} \doteq (V, \widetilde{\mu})$ the *contraction* of $\mathfrak{g}$ by $U(\vartheta)$. Obviuosly the contracted algebra $\widetilde{\mathfrak{g}}$ is not isomorphic to $\mathfrak{g}$.

Let us assume a direct sum structure for the vector space V, namely

$$V = \bigoplus_{i=0}^{N} V^{(i)}.$$

Definition 2.1 *A* generalized Inönü-Wigner contraction *of* $\mathfrak{g}$ *is defined through the family of maps*

$$U(\vartheta)|_{V^{(i)}} = \vartheta^{n_i} \mathbb{1}|_{V^{(i)}}, \qquad 0 \leq n_0 < n_1 < ... < n_N, \quad n_i \in \mathbb{R}, \quad 0 \leq i \leq N,$$

such that

$$\mu\, (V^{(j)}, V^{(k)}) \subset \bigoplus_i V^{(i)}, \qquad n_i \leq n_j + n_k.$$

The contracted Lie algebra $\widetilde{\mathfrak{g}} \doteq (V, \widetilde{\mu})$ *has the following Lie multiplication:*

$$\widetilde{\mu}(V^{(j)}, V^{(k)}) \subset V^{(i)}, \qquad n_i = n_j + n_k.$$

For our purposes we can consider the Lie product μ as the standard commutator $[\cdot,\cdot]$. In fact, our aim is to perform a suitable Inönü-Wigner contraction on the Lie algebra $\mathfrak{G}_N \doteq \oplus^N \mathfrak{g} \doteq \mathrm{span}\{X_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \leq i \leq N$, introduced in Section 2.2. We recall that the commutation relations for the elements of the basis of $\mathfrak{G}_N$ are, see Eq. (2.15),

$$\left[\, X_i^\alpha, X_j^\beta \,\right] = \delta_{i,j}\, C_\gamma^{\alpha\beta}\, X_i^\gamma, \qquad 1 \leq i, j \leq N. \tag{2.40}$$

Proposition 2.8 *The isomorphism* $\eta_\vartheta : \mathfrak{G}_N \to \mathfrak{G}_N$ *defined by*

$$\eta_\vartheta : X_k^\alpha \mapsto \widetilde{X}_i^\alpha \doteq \vartheta^i \sum_{j=1}^N \nu_j^i X_j^\alpha, \qquad 1 \le k \le N, \quad 0 \le i \le N-1, \tag{2.41}$$

with $\nu_j \in \mathbb{C}$, $\nu_j \neq \nu_k$, $1 \le j,k \le N$ *and* $0 < \vartheta \le 1$ (contraction parameter), *defines, in the limit* $\vartheta \to 0$, *a generalized Inönü-Wigner contraction of* $\mathfrak{G}_N$. *The Lie brackets of the contracted Lie algebra* $\widetilde{\mathfrak{G}}_N \doteq \mathrm{span}\{\widetilde{X}_i^\alpha\}_{\alpha=1}^{\dim\mathfrak{g}}$, $0 \le i \le N-1$ *are given by:*

$$\left[\widetilde{X}_i^\alpha, \widetilde{X}_j^\beta\right] = \begin{cases} C_\gamma^{\alpha\beta}\, \widetilde{X}_{i+j}^\gamma & i+j<N, \\ 0 & i+j \ge N, \end{cases} \tag{2.42}$$

with $0 \le i,j \le N-1$, *being* $\{C_\gamma^{\alpha\beta}\}_{\alpha,\beta,\gamma=1}^{\dim\mathfrak{g}}$ *the set of structure constants of the Lie algebra* $\mathfrak{g}$.

Proof: Using the isomorphism (2.41) and the commutations relations (2.40) we get:

$$\begin{aligned}\left[\widetilde{X}_i^\alpha, \widetilde{X}_j^\beta\right]_\vartheta &= \vartheta^{i+j} \sum_{n,m=1}^N \nu_n^i\, \nu_m^j \left[X_n^\alpha, X_m^\beta\right] = \\ &= C_\gamma^{\alpha\beta}\, \vartheta^{i+j} \sum_{n=1}^N \nu_n^{i+j}\, X_n^\gamma = \begin{cases} C_\gamma^{\alpha\beta}\, \widetilde{X}_{i+j}^\gamma & i+j<N, \\ O(\vartheta^N) & i+j \ge N. \end{cases}\end{aligned}$$

Performing the limit $\vartheta \to 0$, i.e. the contraction procedure, we obtain the Lie brackets (2.42). It is easy to check directly that the multiplication in Eq. (2.42) is antysymmetric and satisfies the Jacobi identity.

□

Remark 2.5 *Looking at the Lie brackets (2.42) we immediately see that the contraction procedure breaks the direct sum structure and the contracted Lie algebra does not have, in general, a semidirect structure.*

Example 2.1 *Let us give an illustrative (and well-known) example. Consider the direct sum of two copies of* $\mathfrak{su}(2)$, *namely a Lie algebra isomorphic to* $\mathfrak{o}(4)$, *spanned by* $\{X_i^\alpha\}_{\alpha=1}^3$, $i = 1,2$*:*

$$\left[X_1^\alpha, X_1^\beta\right] = \varepsilon_{\alpha\beta\gamma} X_1^\gamma, \quad \left[X_2^\alpha, X_2^\beta\right] = \varepsilon_{\alpha\beta\gamma} X_2^\gamma, \quad \left[X_1^\alpha, X_2^\beta\right] = 0.$$

Then we define $\widetilde{X}_0^\alpha \doteq X_1^\alpha + X_2^\alpha$ *and* $\widetilde{X}_1^\alpha \doteq \vartheta(\nu_1 X_1^\alpha + \nu_2 X_2^\alpha)$, $\nu_1 \neq \nu_2$. *After contraction, the resulting Lie algebra spanned by* $\widetilde{X}_0^\alpha, \widetilde{X}_1^\alpha$ *is* $\mathfrak{e}(3)$, *namely the six-dimensional real euclidean Lie algebra:*

$$\left[\widetilde{X}_0^\alpha, \widetilde{X}_0^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_0^\gamma, \quad \left[\widetilde{X}_0^\alpha, \widetilde{X}_1^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_1^\gamma, \quad \left[\widetilde{X}_1^\alpha, \widetilde{X}_1^\beta\right] = 0.$$

Considering the direct sum of three copies of $\mathfrak{su}(2)$ *the contraction procedure provides the following nine-dimensional Lie algebra:*

$$\left[\widetilde{X}_0^\alpha, \widetilde{X}_0^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_0^\gamma, \quad \left[\widetilde{X}_0^\alpha, \widetilde{X}_1^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_1^\gamma, \quad \left[\widetilde{X}_0^\alpha, \widetilde{X}_2^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_2^\gamma,$$

$$\left[\widetilde{X}_1^\alpha, \widetilde{X}_1^\beta\right] = \varepsilon_{\alpha\beta\gamma} \widetilde{X}_2^\gamma, \quad \left[\widetilde{X}_1^\alpha, \widetilde{X}_2^\beta\right] = 0, \quad \left[\widetilde{X}_2^\alpha, \widetilde{X}_2^\beta\right] = 0.$$

The new Lie algebra spanned by $\widetilde{X}_0^\alpha, \widetilde{X}_1^\alpha, \widetilde{X}_2^\alpha$ *can be written as* $\mathfrak{su}(2) \oplus_s \mathfrak{h}$ *where the six-dimensional algebra* $\mathfrak{h}$, *although including the abelian proper subalgebra* $\mathbb{R}^3$ *spanned by* $\widetilde{X}_2^\alpha$, *does not have a semidirect structure. In fact it is possible to show that* $\mathfrak{su}(2) \oplus_s \mathfrak{h}$ *can be obtained adding a cocycle to* $\mathfrak{e}(3,2) = \mathfrak{su}(2) \oplus_s (\mathbb{R}^3 \oplus \mathbb{R}^3)$. *Hence this is an algebraic extension in the usual sense [42, 96, 97].*

Remark 2.6 *The Lie algebras $\widetilde{\mathfrak{G}}_N$ defined by the commutation relations given in Eq. (2.42) have been introduced in [71] (1976) in the framework of current algebras and they have been called* Leibniz extensions *of $\mathfrak{g}$ of order N.*

The Inönü-Wigner contraction performed in Proposition 2.8 allows one to establish a deep relationship between algebraic contractions and algebraic extensions. As a matter of fact we have constructed an extension of $\mathfrak{g}$ in the usual sense of extensions of Lie algebras [42]. In fact defining the new vector spaces $W^{(i)} \doteq \bigoplus_{j=i}^{N-1} V^{(j)}$, $0 \le i \le N-1$, $W^{(0)} \simeq \widetilde{\mathfrak{G}}_N$, we obtain a flag of ideals $[W^{(i)}, W^{(j)}] \subseteq W^{(j)}$, $j \ge i$. In particular $[\widetilde{\mathfrak{G}}_N, W^{(1)}] \subseteq W^{(1)}$. So we have the exact sequence: $0 \to W^{(1)} \xrightarrow{i} \widetilde{\mathfrak{G}}_N \xrightarrow{\pi} \mathfrak{g} \to 0$, where i is the inclusion and π is the projection on the quotient $\widetilde{\mathfrak{G}}_N/W^{(1)}$; moreover $\widetilde{\mathfrak{G}}_N/W^{(1)} \simeq V^{(0)} \simeq \mathfrak{g}$.

2.3.2 Leibniz extensions of Gaudin Lax matrices

The isomorphism $\eta_\vartheta : \mathfrak{G}_N \to \mathfrak{G}_N$ defined in Eq. (2.41) naturally induces a dual map $\eta^*_\vartheta : \mathfrak{G}^*_N \to \mathfrak{G}^*_N$.

Proposition 2.9 *The isomorphism $\eta_\vartheta : \mathfrak{G}_N \to \mathfrak{G}_N$ induces the following map on the coordinates $\{y_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \le i \le N$, on $\widetilde{\mathfrak{G}}^*_N$:*

$$\widetilde{y}_i^\alpha \doteq \vartheta^i \sum_{j=1}^{N} \nu_j^i \, y_j^\alpha, \qquad 0 \le i \le N-1, \tag{2.43}$$

with $\nu_j \in \mathbb{C}$, $\nu_j \neq \nu_k$, $1 \le j,k \le N$ and $0 < \vartheta \le 1$. In the limit $\vartheta \to 0$, the Lie-Poisson brackets (2.16) on $\mathfrak{G}^*_N$ are mapped by (2.43) into the following Lie-Poisson brackets on $\widetilde{\mathfrak{G}}^*_N$:

$$\left\{ \widetilde{y}_i^\alpha, \widetilde{y}_j^\beta \right\}_{\widetilde{P}_N^{\mathfrak{g}}} = \begin{cases} C_\gamma^{\alpha\beta}\, \widetilde{y}_{i+j}^\gamma & i+j < N, \\ 0 & i+j \ge N, \end{cases} \tag{2.44}$$

with $0 \le i,j \le N-1$. Here

$$(\widetilde{P}_N^{\mathfrak{g}})_{i,j} \doteq \widetilde{Y}_{i+j-2}, \qquad (\widetilde{Y}_k)_{\alpha,\beta} \doteq C_\gamma^{\alpha\beta}\, \widetilde{y}_k^\gamma, \tag{2.45}$$

with $0 \le k \le N-1$, $1 \le i,j \le N$ and $\widetilde{Y}_i \equiv 0$, $i \ge N$.

Proof: Eq. (2.43) is a plane consequence of Eq. (2.41) in Proposition 2.8.

□

Example 2.2 *According to the Example 2.1 we have that $\widetilde{P}_2^{\mathfrak{su}(2)}$ is the Lie-Poisson tensor associated with the (minus) Lie-Poisson algebra $\mathfrak{e}^*(3) = \mathfrak{su}^*(2) \oplus_s \mathbb{R}^3$. In fact the contraction maps the Lie-Poisson tensor $P_2^{\mathfrak{su}(2)}$ in the following way, see Eqs. (2.17) and (2.45),*

$$P_2^{\mathfrak{su}(2)} \doteq \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \longmapsto \widetilde{P}_2^{\mathfrak{su}(2)} \doteq \begin{pmatrix} \widetilde{Y}_0 & \widetilde{Y}_1 \\ \widetilde{Y}_1 & 0 \end{pmatrix},$$

namely we get the Lie-Poisson brackets on (minus) $\mathfrak{e}^*(3)$:

$$\left\{ \widetilde{y}_0^\alpha, \widetilde{y}_0^\beta \right\}_{\widetilde{P}_2^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_0^\gamma, \qquad \left\{ \widetilde{y}_0^\alpha, \widetilde{y}_1^\beta \right\}_{\widetilde{P}_2^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_1^\gamma, \qquad \left\{ \widetilde{y}_1^\alpha, \widetilde{y}_1^\beta \right\}_{\widetilde{P}_2^{\mathfrak{su}(2)}} = 0.$$

If $N = 3$ we have

$$P_3^{\mathfrak{su}(2)} \doteq \begin{pmatrix} Y_1 & 0 & 0 \\ 0 & Y_2 & 0 \\ 0 & 0 & Y_3 \end{pmatrix} \longmapsto \widetilde{P}_3^{\mathfrak{su}(2)} \doteq \begin{pmatrix} \widetilde{Y}_0 & \widetilde{Y}_1 & \widetilde{Y}_2 \\ \widetilde{Y}_1 & \widetilde{Y}_2 & 0 \\ \widetilde{Y}_2 & 0 & 0 \end{pmatrix},$$

namely we get the Lie-Poisson brackets:

$$\left\{\widetilde{y}_0^\alpha, \widetilde{y}_0^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_0^\gamma, \quad \left\{\widetilde{y}_0^\alpha, \widetilde{y}_1^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_1^\gamma, \quad \left\{\widetilde{y}_0^\alpha, \widetilde{y}_2^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_2^\gamma,$$
$$\left\{\widetilde{y}_1^\alpha, \widetilde{y}_1^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = -\varepsilon_{\alpha\beta\gamma}\, \widetilde{y}_2^\gamma, \quad \left\{\widetilde{y}_1^\alpha, \widetilde{y}_2^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = 0, \quad \left\{\widetilde{y}_2^\alpha, \widetilde{y}_2^\beta\right\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} = 0.$$

Note that the Lie-Poisson bracket between two smooth functions $f(\widetilde{\mathbf{y}}_0, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2), g(\widetilde{\mathbf{y}}_0, \widetilde{\mathbf{y}}_1, \widetilde{\mathbf{y}}_2)$ *is given by*

$$\begin{aligned} \{f, g\}_{\widetilde{P}_3^{\mathfrak{su}(2)}} &= \langle\, \widetilde{\mathbf{y}}_0, [\,\nabla_{\widetilde{\mathbf{y}}_0} f, \nabla_{\widetilde{\mathbf{y}}_0} g]\,\rangle + \langle\, \widetilde{\mathbf{y}}_1, [\,\nabla_{\widetilde{\mathbf{y}}_0} f, \nabla_{\widetilde{\mathbf{y}}_1} g] + [\,\nabla_{\widetilde{\mathbf{y}}_1} f, \nabla_{\widetilde{\mathbf{y}}_0} g]\,\rangle + \\ &+ \langle\, \widetilde{\mathbf{y}}_2, [\,\nabla_{\widetilde{\mathbf{y}}_0} f, \nabla_{\widetilde{\mathbf{y}}_2} g] + [\,\nabla_{\widetilde{\mathbf{y}}_2} f, \nabla_{\widetilde{\mathbf{y}}_0} g]\,\rangle + \langle\, \widetilde{\mathbf{y}}_2, [\,\nabla_{\widetilde{\mathbf{y}}_1} f, \nabla_{\widetilde{\mathbf{y}}_1} g]\,\rangle, \end{aligned}$$

where ∇ *is a gradient with respect to its subscript. The non-semidirect structure lies just in the last term of the above equation: this term does not exist in the* $\mathfrak{e}^*(3,2)$ *Lie-Poisson algebra.*

Proposition 2.10 *Let* $H, G \in \mathcal{F}(\mathfrak{G}_N^*)$ *be two involutive functions w.r.t. the Lie-Poisson brackets (2.16). If* $\widetilde{H}, \widetilde{G} \in \mathcal{F}(\widetilde{\mathfrak{G}}_N^*)$ *are the correspondent functions obtained from* H, G *by applying the map (2.43) in the contraction limit* $\vartheta \to 0$*, then they are in involution w.r.t. the contracted Lie-Poisson brackets (2.44).*

Proof: In the local coordinates $\{y_i^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \le i \le N$, we have:

$$\begin{aligned} 0 = \{H, G\}_{P_N^{\mathfrak{g}}} &= \sum_{i,j=1}^{N} \frac{\partial H}{\partial y_i^\alpha} \frac{\partial G}{\partial y_j^\beta} \left\{y_i^\alpha, y_j^\beta\right\}_{P_N^{\mathfrak{g}}} = C_\gamma^{\alpha\beta} \sum_{i=1}^{N} \frac{\partial H}{\partial y_i^\alpha} \frac{\partial G}{\partial y_i^\beta}\, y_i^\gamma = \\ &= C_\gamma^{\alpha\beta} \sum_{i=1}^{N} \sum_{n,m=0}^{N-1} \frac{\partial \widetilde{H}}{\partial \widetilde{y}_n^\alpha} \frac{\partial \widetilde{G}}{\partial \widetilde{y}_m^\beta}\, \vartheta^{n+m}\, \nu_i^{n+m}\, y_i^\gamma = \\ &= C_\gamma^{\alpha\beta} \sum_{\substack{n,m=0 \\ n+m<N}}^{N-1} \frac{\partial \widetilde{H}}{\partial \widetilde{y}_n^\alpha} \frac{\partial \widetilde{G}}{\partial \widetilde{y}_m^\beta}\, \widetilde{y}_{n+m}^\gamma + O(\vartheta^N), \end{aligned}$$

where the first term does not depend explicitly on the contraction parameter ϑ. Performing the limit $\vartheta \to 0$ we get $\{\widetilde{H}, \widetilde{G}\}_{\widetilde{P}_N^{\mathfrak{g}}} = 0$.

□

Our aim is now to apply the map (2.43), in the contraction limit $\vartheta \to 0$, to the Lax matrix of the Gaudin models, given in Eq. (2.18), in order to get a new Lax matrix. This purely algebraic procedure shall be performed together with a *pole coalescence*. Our main goal is to prove that the resulting Lax matrix preserves the same linear r-matrix structure of the ancestor one, with the same r-matrix.

Proposition 2.11 *The isomorphism (2.43), maps, in the limit* $\vartheta \to 0$*, the Lax matrix* $\mathcal{L}_{\mathcal{G}}(\lambda)$*, defined in Eq. (2.18), with* $\lambda_i \equiv \vartheta\, \nu_i$, $1 \le i \le N$ (pole coalescence)*, into the new Lax matrix*

$$\mathcal{L}_N(\lambda) \doteq \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, \widetilde{y}_i^\beta\, f_i^{\alpha\beta}(\lambda), \tag{2.47}$$

where the functions $f_i^{\alpha\beta}(\lambda)$ *are defined by*

$$f_i^{\alpha\beta}(\lambda) \doteq \frac{(-1)^i}{i!}\, \partial_\lambda^i\, f_0^{\alpha\beta}(\lambda), \tag{2.48}$$

with $\partial_\lambda \doteq d/d\lambda$*, being* $f_0^{\alpha\beta}(\lambda) \equiv f^{\alpha\beta}(\lambda)$ *the meromorphic functions appearing in the* r*-matrix (2.7).*

Proof: Let us consider the pole coalescence $\lambda_i \equiv \vartheta\,\nu_i,\ 1 \le i \le N$, in the Lax matrix (2.18) and a formal expansion of the functions $f^{\alpha\beta}(\lambda) \equiv f_0^{\alpha\beta}(\lambda)$:

$$\begin{aligned}\mathcal{L}_{\mathcal{G}}(\lambda) &= \sum_{j=1}^{N} g_{\alpha\beta}\, X^\alpha\, y_j^\beta f_0^{\alpha\beta}(\lambda - \vartheta\,\nu_j) = \sum_{j=1}^{N} g_{\alpha\beta}\, X^\alpha\, y_j^\beta \sum_{i\ge 0} \frac{(-\vartheta\,\nu_j)^i}{i!}\,\partial_\lambda^i\, f_0^{\alpha\beta}(\lambda) = \\ &= \sum_{j=1}^{N}\sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\,(\vartheta\,\nu_j)^i\, y_j^\beta\, \frac{(-1)^i}{i!}\,\partial_\lambda^i\, f_0^{\alpha\beta}(\lambda) + O(\vartheta^N).\end{aligned} \qquad (2.49)$$

Using the map (2.43), the definition (2.48) and the limit $\vartheta \to 0$ in Eq. (2.49) we readily get the Lax matrix given in Eq. (2.47).

□

Remark 2.7 *We notice that the Lax matrix (2.47) describes a one-body dynamical system. Moreover,* N *is the order of the Leibniz extension of the Lie algebra* $\mathfrak{g}$ *and it coincides with the number of degrees of freedom of the model.*

We shall show that it is possible to extend such a matrix to the many-body case. For these reason we shall distinguish between the so called extended one-body hierarchy *and the* extended many-body hierarchy*.*

We finally give the announced result.

Theorem 2.2 *The Lax matrix (2.47) satisfies the linear* r*-matrix algebra*

$$\{\mathcal{L}_N(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes\mathcal{L}_N(\mu)\}_{\widetilde{P}_N^{\mathfrak{g}}} + [\,r(\lambda-\mu), \mathcal{L}_N(\lambda)\otimes\mathbb{1} + \mathbb{1}\otimes\mathcal{L}_N(\mu)\,] = 0, \qquad \forall\,(\lambda,\mu)\in\mathbb{C}^2, \qquad (2.50)$$

with $r(\lambda)$ *given in Eq. (2.7).*

Proof: Let us compute the first term in Eq. (2.50). According to Eq. (2.44) we set $\widetilde{y}_{i+j}^{\alpha} = 0$ if $i+j > N$. We get:

$$\begin{aligned}\{\mathcal{L}_N(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes\mathcal{L}_N(\mu)\}_{\widetilde{P}_N^{\mathfrak{g}}} &= g_{\alpha\beta}\, g_{\gamma\delta}\,(X^\alpha\otimes X^\gamma)\sum_{i,j=0}^{N-1} f_i^{\alpha\beta}(\lambda)\, f_j^{\gamma\delta}(\mu)\left\{\widetilde{y}_i^{\beta}, \widetilde{y}_j^{\delta}\right\}_{\widetilde{P}_N^{\mathfrak{g}}} = \\ &= g_{\alpha\beta}\, g_{\gamma\delta}\,(X^\alpha\otimes X^\gamma)\, C_\eta^{\beta\delta}\sum_{i,j=0}^{N-1} f_i^{\alpha\beta}(\lambda)\, f_j^{\gamma\delta}(\mu)\,\widetilde{y}_{i+j}^{\eta} = \\ &= g_{\alpha\beta}\, g_{\gamma\delta}\,(X^\alpha\otimes X^\gamma)\, C_\eta^{\beta\delta}\sum_{i=0}^{N-1}\widetilde{y}_i^{\eta}\sum_{j=0}^{i} f_j^{\alpha\beta}(\lambda)\, f_{i-j}^{\gamma\delta}(\mu).\end{aligned} \qquad (2.51)$$

On the other hand we have:

$$
\begin{aligned}
&[\, r(\lambda-\mu), \mathcal{L}_N(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes \mathcal{L}_N(\mu)\,] = \\
&= g_{\alpha\beta}\, g_{\gamma\eta}\, f_0^{\alpha\beta}(\lambda-\mu) \left[X^\alpha \otimes X^\beta, (X^\gamma\otimes \mathbb{1}) \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\gamma\eta}(\lambda) + (\mathbb{1}\otimes X^\gamma) \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\gamma\eta}(\mu) \right] = \\
&= g_{\alpha\beta}\, g_{\gamma\eta}\, f_0^{\alpha\beta}(\lambda-\mu) \left[C_\delta^{\alpha\gamma}\, (X^\delta \otimes X^\beta) \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\gamma\eta}(\lambda) + C_\delta^{\beta\gamma}\, (X^\alpha \otimes X^\delta) \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\gamma\eta}(\mu) \right] = \\
&= \left[g_{\delta\gamma}\, g_{\beta\eta}\, f_0^{\delta\gamma}(\lambda-\mu)\, C_\alpha^{\delta\beta} \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\beta\eta}(\lambda) + \right. \\
&\quad \left. + g_{\alpha\beta}\, g_{\delta\eta}\, f_0^{\alpha\beta}(\lambda-\mu)\, C_\gamma^{\beta\delta} \sum_{i=0}^{N-1} \widetilde{y}_i^\eta\, f_i^{\delta\eta}(\lambda) \right] (X^\alpha \otimes X^\gamma), \qquad (2.52)
\end{aligned}
$$

where the last expression is obtained from the previous one swapping the indices $\gamma \leftrightarrow \beta, \alpha \leftrightarrow \delta$ in the first term and $\gamma \leftrightarrow \delta$ in the second one.

Considering Eqs. (2.51) and (2.52) we obtain

$$
\begin{aligned}
&\{\mathcal{L}_N(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes \mathcal{L}_N(\mu)\}_{\widetilde{P}_N^{\mathfrak{g}}} + [\, r(\lambda-\mu), \mathcal{L}_N(\lambda)\otimes \mathbb{1} + \mathbb{1}\otimes \mathcal{L}_N(\mu)\,] = \\
&= \sum_{i=0}^{N-1} \widetilde{y}_i^\eta \left[\sum_{j=0}^{i} g_{\alpha\beta}\, g_{\gamma\delta}\, C_\eta^{\beta\delta}\, f_j^{\alpha\beta}(\lambda)\, f_{i-j}^{\gamma\delta}(\mu) + g_{\delta\gamma}\, g_{\beta\eta}\, C_\alpha^{\delta\beta}\, f_0^{\delta\gamma}(\lambda-\mu) f_i^{\beta\eta}(\lambda) + \right. \\
&\quad \left. + g_{\alpha\beta}\, g_{\delta\eta}\, C_\gamma^{\beta\delta}\, f_0^{\alpha\beta}(\lambda-\mu) f_i^{\delta\eta}(\mu) \right] (X^\alpha \otimes X^\gamma). \qquad (2.53)
\end{aligned}
$$

Let us look in more detail at the functional part enclosed in the square brackets in Eq. (2.53). Using Eq. (2.48) we get:

$$
\begin{aligned}
\sum_{j=0}^{i} f_j^{\alpha\beta}(\lambda)\, f_{i-j}^{\gamma\delta}(\mu) &= \sum_{j=0}^{i} \frac{(-1)^j}{j!} \partial_\lambda^j\, f_0^{\alpha\beta}(\lambda) \frac{(-1)^{i-j}}{(i-j)!} \partial_\mu^{i-j}\, f_0^{\gamma\delta}(\mu) = \\
&= \frac{(-1)^i}{i!} \sum_{j=0}^{i} \binom{i}{j} \partial_\lambda^j\, \partial_\mu^{i-j}\, f_0^{\alpha\beta}(\lambda)\, f_0^{\gamma\delta}(\mu) = \\
&= \frac{(-1)^i}{i!} \left(\partial_\lambda + \partial_\mu\right)^i f_0^{\alpha\beta}(\lambda)\, f_0^{\gamma\delta}(\mu), \qquad (2.54)
\end{aligned}
$$

and

$$
\begin{aligned}
&g_{\delta\gamma}\, g_{\beta\eta}\, C_\alpha^{\delta\beta}\, f_0^{\delta\gamma}(\lambda-\mu) f_i^{\beta\eta}(\lambda) + g_{\alpha\beta}\, g_{\delta\eta}\, C_\gamma^{\beta\delta}\, f_0^{\alpha\beta}(\lambda-\mu) f_i^{\delta\eta}(\mu) = \\
&= \frac{(-1)^i}{i!} \left[g_{\delta\gamma}\, g_{\beta\eta}\, C_\alpha^{\delta\beta}\, f_0^{\delta\gamma}(\lambda-\mu)\, \partial_\lambda^i\, f_0^{\beta\eta}(\lambda) + g_{\alpha\beta}\, g_{\delta\eta}\, C_\gamma^{\beta\delta}\, f_0^{\alpha\beta}(\lambda-\mu)\, \partial_\mu^i\, f_0^{\delta\eta}(\mu) \right] = \\
&= \frac{(-1)^i}{i!} \left(\partial_\lambda + \partial_\mu\right)^i \left[g_{\delta\gamma}\, g_{\beta\eta}\, C_\alpha^{\delta\beta}\, f_0^{\delta\gamma}(\lambda-\mu)\, f_0^{\beta\eta}(\lambda) + g_{\alpha\beta}\, g_{\delta\eta}\, C_\gamma^{\beta\delta}\, f_0^{\alpha\beta}(\lambda-\mu)\, f_0^{\delta\eta}(\mu) \right]. \qquad (2.55)
\end{aligned}
$$

Inserting Eqs. (2.54) and (2.55) in Eq. (2.53) we obtain Eq. (2.50) thanks to the functional equations given in Eq. (2.8).

□

Remark 2.8 *The fact that $\widetilde{P}_N^{\mathfrak{g}}$ is indeed a Lie-Poisson tensor can be seen as a plain consequence of Eq. (2.50).*

The involutive Hamiltonians of the one-body hierarchy governed by the Lax matrix $\mathcal{L}_N(\lambda)$ given in Eq. (2.47) can be obtained in two equivalent ways. One can directly compute the spectral invariants of $\mathcal{L}_N(\lambda)$, and their involutivity is ensured thanks to the r-matrix formulation (2.50). Otherwise we can apply the contraction procedure to the involutive uncontracted Hamiltonians. In particular, if the parameters λ_i's appear explicitly in these integrals we have also to perform the pole coalescence $\lambda_i \equiv \vartheta\, \nu_i$, $1 \le i \le N$, in order to construct the proper contracted Hamiltonians. Their involutivity is ensured by Proposition 2.10.

A more general extended rational Lax matrix

In Theorem 2.2 we have shown that the system of functional equations in Eq. (2.53) is satisfied if the functions $f_i^{\alpha\beta}(\lambda)$, $0 \le i \le N-1$, are given by

$$f_i^{\alpha\beta}(\lambda) \doteq \frac{(-1)^i}{i!}\, \partial_\lambda^i\, f_0^{\alpha\beta}(\lambda), \tag{2.56}$$

where the $f_0^{\alpha\beta}(\lambda)$'s are the meromorphic functions appearing in the r-matrix (2.7). Nevertheless we can construct a more general analytic solution to the system in Eq. (2.53) if

$$r(\lambda) = \frac{1}{\lambda}\, g_{\alpha\beta}\, X^\alpha \otimes X^\beta.$$

In such a case the functions $f_i^{\alpha\beta}(\lambda)$, $0 \le i \le N-1$, do not depend on α, β and the system of functional equations appearing in Eq. (2.53) takes the following simple form:

$$\sum_{i=0}^{k} f_i(\lambda) f_{k-i}(\mu) = \frac{f_k(\lambda) - f_k(\mu)}{\mu - \lambda}, \qquad 0 \le k \le N-1. \tag{2.57}$$

Our solution is based on the following conjecture.

Conjecture 2.1 *Let $k \in \mathbb{N}$ and let $\{c_i\}_{i=1}^k$ be a set arbitrary complex constants. Let us denote with $\{\mathbf{q}_i\}$ the set of vectors satisfying the Diophantine equation:*

$$\{\mathbf{q}_i\} \doteq \{\mathbf{q} \in \mathbb{N}^i \ : \ q_1 + 2\, q_2 + \cdots + i\, q_i = i\}, \qquad 1 \le i \le N-1.$$

Then the following polynomial identity holds:

$$\sum_{i=0}^{k} \left(\sum_{\{\mathbf{q}_i\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_i^{q_i}}{q_i!} \lambda^{|\mathbf{q}_i|} \sum_{\{\mathbf{q}_{k-i}\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_{k-i}^{q_{k-i}}}{q_{k-i}!} \mu^{|\mathbf{q}_{k-i}|} \right) = \sum_{\{\mathbf{q}_k\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_k^{q_k}}{q_k!} (\lambda + \mu)^{|\mathbf{q}_k|},$$

where $|\mathbf{q}_i| \doteq \sum_{k=1}^i q_k$ and $(\lambda, \mu) \in \mathbb{C}^2$.

Using a MAPLE program we have tested this conjecture for $N \le 25$.

Proposition 2.12 *If the Conjecture 2.1 holds for any $N \in \mathbb{N}$, then the functions*

$$f_i(\lambda) = \sum_{\{\mathbf{q}_i\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_i^{q_i}}{q_i!}\, \partial_\lambda^{|\mathbf{q}_i|}\, \frac{1}{\lambda}, \qquad 1 \le i \le N-1, \tag{2.58}$$

satisfy the system of functional equations given in Eq. (2.57) for any $N \in \mathbb{N}$.

Proof: Let us fix $f_0(\lambda) \doteq \lambda^{-1}$. Inserting the functions (2.58) into Eq. (2.57) and using the Conjecture 2.1 we get:

$$
\begin{aligned}
&\sum_{i=0}^{k} f_i(\lambda)\, f_{k-i}(\mu) = \\
&\quad = \sum_{i=0}^{k} \left[\sum_{\{\mathbf{q}_i\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_i^{q_i}}{q_i!}\, \partial_\lambda^{|\mathbf{q}_i|}\, f_0(\lambda) \sum_{\{\mathbf{q}_{k-i}\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_{k-i}^{q_{k-i}}}{q_{k-i}!}\, \partial_\mu^{|\mathbf{q}_{k-i}|}\, f_0(\mu) \right] = \\
&\quad = \sum_{\{\mathbf{q}_k\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_k^{q_k}}{q_k!}\, (\partial_\lambda + \partial_\mu)^{|\mathbf{q}_k|}\; f_0(\lambda)\, f_0(\mu) = \\
&\quad = \sum_{\{\mathbf{q}_k\}} \frac{c_1^{q_1}}{q_1!} \cdots \frac{c_k^{q_k}}{q_k!}\, (\partial_\lambda + \partial_\mu)^{|\mathbf{q}_k|} \left[\frac{f_0(\lambda) - f_0(\mu)}{\mu - \lambda} \right] = \\
&\quad = \frac{f_k(\lambda) - f_k(\mu)}{\mu - \lambda}.
\end{aligned}
$$

□

Example 2.3 *Let us fix $N = 4$ in Eq. (2.58). We get the following three functions:*

$$
\begin{aligned}
f_1(\lambda) &= -\frac{c_1}{\lambda^2}, \\
f_2(\lambda) &= \frac{c_1^2}{\lambda^3} - \frac{c_2}{\lambda^2}, \\
f_3(\lambda) &= -\frac{c_1^3}{\lambda^4} + \frac{2\, c_1\, c_2}{\lambda^3} - \frac{c_3}{\lambda^2}.
\end{aligned}
$$

In general, the solutions (2.56) can be recovered fixing in Eq. (2.58) the values of the constants c_i's as $c_1 \equiv -1$ and $c_i \equiv 0$, $2 \le i \le N-1$.

Remark 2.9 *The fact that solutions (2.58) provide the general analytic solution to (2.57) can be argued taking the limit $\lambda \to \mu$ in the functional equations (2.57), yielding the system of ordinary differential equations:*

$$
\sum_{i=0}^{k} f_i(\lambda) f_{k-i}(\lambda) = -\partial_\lambda\, f_k(\lambda), \qquad 0 \le k \le N-1. \tag{2.59}
$$

As the system (2.59) is triangular, for any given i the functions $f_i(\lambda)$ can be found solving a system of $i+1$ ordinary differential equations and therefore depend at most upon $i+1$ arbitrary parameters. This is exactly the number of arbitrary parameters entering our solutions (2.58).

Henceforth, in the rational case, a more general form of the extended Lax matrix (2.47) is given by

$$
\mathcal{L}_N^r(\lambda) \doteq \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, \widetilde{y}_i^{\beta}\, f_i(\lambda), \tag{2.60}
$$

where the functions $f_i(\lambda)$ are given in Eq. (2.58).

A multi-Hamiltonian formulation of the rational extended models

It is possible to prove the integrability of the dynamical systems described by the Lax matrices (2.47), with a rational dependence on the spectral parameter, also using the multi-Hamiltonian approach [79]. We prefer to omit a complete analysis of this technique and we shall just present the family of compatible Lie-Poisson tensors associated with such systems.

Let us consider the Lax matrix (2.47), with $f_0^{\alpha\beta}(\lambda) = \lambda^{-1}$, in the following form:

$$\mathcal{L}_N^r(\lambda, \mathcal{P}) \doteq \mathcal{P} + \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, \widetilde{y}_i^\beta\, f_i(\lambda), \qquad f_i(\lambda) \doteq \frac{(-1)^i}{i!}\, \partial_\lambda^i\, \frac{1}{\lambda} = \frac{1}{\lambda^{i+1}}, \tag{2.61}$$

where we have added a constant matrix $\mathcal{P} \in \mathfrak{g}$ (with simple spectrum). The rational Lax matrix (2.61) can be easily mapped into a polynomial one, namely

$$\mathcal{L}_N^p(\lambda, \mathcal{P}) \doteq \lambda^N\, \mathcal{P} + \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, \widetilde{y}_i^\beta\, \lambda^{N-1-i}. \tag{2.62}$$

For the space of polynomials pencils of matrices a family of mutually compatible Lie-Poisson tensors are defined [79], and the spectral invariants are involutive functions with respect to these Poisson tensors, called *Reyman-Semenov-Tian-Shansky tensors*. As a plain consequence of the multi-Hamiltonian formulation for these Lax matrices we get a multi-Hamiltonian formulation for the matrices in (2.62). The following proposition holds.

Proposition 2.13 *Consider the system governed by the Lax matrix $\mathcal{L}_N^p(\lambda, \mathcal{P})$ given in Eq. (2.62). Then there exist $N+1$ compatible Reyman-Semenov-Tian-Shansky tensors given by*

$$\Pi_k \doteq \begin{pmatrix} A_k & 0 \\ 0 & B_k \end{pmatrix}, \qquad 0 \le k \le N,$$

with

$$\begin{aligned} (A_k)_{i,j} &\doteq -\widetilde{Y}_{i+j-k-2} \qquad 1 \le i, j \le k, \\ (B_k)_{i,j} &\doteq \widetilde{Y}_{i+j+k-2} \qquad 1 \le i, j \le N-k, \end{aligned}$$

and $\widetilde{Y}_{-1} \equiv P$, $\widetilde{Y}_i \equiv 0$ for $i < -1$ and $i \ge N$. Here $(\widetilde{Y}_i)^{\alpha\beta} \doteq C_\gamma^{\alpha\beta}\, \widetilde{y}_i^\gamma$ and $(P)^{\alpha\beta} \doteq C_\gamma^{\alpha\beta}\, p^\gamma$ is a constant matrix.

Example 2.4 *Let us show the three compatible Reyman-Semenov-Tian-Shansky tensors in the case $N = 2$ for $\mathfrak{g} \equiv \mathfrak{su}(2)$, i.e. $\mathfrak{e}(3)$. According to Proposition 2.13 they are:*

$$\Pi_0 \doteq \begin{pmatrix} \widetilde{Y}_0 & \widetilde{Y}_1 \\ \widetilde{Y}_1 & 0 \end{pmatrix}, \quad \Pi_1 \doteq \begin{pmatrix} -P & 0 \\ 0 & \widetilde{Y}_1 \end{pmatrix}, \quad \Pi_2 \doteq \begin{pmatrix} 0 & -P \\ -P & \widetilde{Y}_0 \end{pmatrix}. \tag{2.63}$$

Notice that $\Pi_0 = \widetilde{P}_2^{\mathfrak{su}(2)}$.

2.3.3 Many-body systems associated with Leibniz extensions of Gaudin models

In Section 2.2 we have shown how to extend the local Lax matrix (2.11) to the N-body case, namely considering the Lax matrix (2.18) of the Gaudin models associated to the Lie-Poisson algebra $\mathfrak{G}_N^*$.

A similar procedure, based on the r-matrix approach, works for the extended models. Precisely we can consider the Lax matrix, see Eq. (2.47),

$$\mathcal{L}_N(\lambda) \doteq \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, \widetilde{y}_i^\beta\, f_i^{\alpha\beta}(\lambda), \qquad f_i^{\alpha\beta}(\lambda) \doteq \frac{(-1)^i}{i!}\, \partial_\lambda^i\, f_0^{\alpha\beta}(\lambda),$$

as the local Lax matrix of a long-range chain constructed as the direct sum of M copies of $\widetilde{\mathfrak{G}}_N^*$. Let us define $\widetilde{\mathfrak{H}}_{M,N}^* \doteq \oplus^M \widetilde{\mathfrak{G}}_N^*$. To do this it is convenient to label the coordinates functions $\widetilde{y}_i^\alpha$'s on $\widetilde{\mathfrak{G}}_N^*$ with an additional index in order to define the local coordinates on $\widetilde{\mathfrak{H}}_{M,N}^*$. Let us introduce the set $\{(\widetilde{y}_i^\alpha)_n\}_{\alpha=1}^{\dim \mathfrak{g}}$, $0 \le i \le N-1, 1 \le n \le M$ as the set of the coordinate functions relative on the k-th copy of $\widetilde{\mathfrak{G}}_N^*$.

Referring to Eq. (2.44), we obtain that the Lie-Poisson brackets on $\widetilde{\mathfrak{H}}_{M,N}^*$ read

$$\Big\{(\widetilde{y}_i^\alpha)_n, (\widetilde{y}_j^\beta)_m\Big\}_{\widetilde{P}_{M,N}^{\mathfrak{g}}} = \begin{cases} \delta_{n,m}\, C_\gamma^{\alpha\beta}\, (\widetilde{y}_{i+j}^\gamma)_n & i+j < N, \\ 0 & i+j \ge N, \end{cases} \tag{2.64}$$

with $0 \le i, j \le N-1$ and $1 \le n, m \le M$. Here $\widetilde{P}_{M,N}^{\mathfrak{g}}$ is the diagonal Lie-Poisson tensor associated with the direct sum of M copies of the Leibniz extension of order N of the ancestor Lie algebra $\mathfrak{g}$.

The M-body Lax matrix of the extended Gaudin models read

$$\mathcal{L}_{M,N}(\lambda) \doteq \sum_{k=1}^{M} \sum_{i=0}^{N-1} g_{\alpha\beta}\, X^\alpha\, (\widetilde{y}_i^\beta)_k\, f_i^{\alpha\beta}(\lambda - \mu_k), \tag{2.65}$$

where the complex numbers μ_k's, with $\mu_k \ne \mu_j, 1 \le k, j \le M$ are local parameters of the model.

Taking into account the direct sum structure encoded in the Lie-Poisson brackets (2.64), and the fact that r-matrix depends just on the difference $\lambda - \mu$ we can perform a computation similar to the one given in Theorem 2.2, thus proving the following proposition.

Proposition 2.14 *The Lax matrix (2.65) satisfies the linear r-matrix algebra*

$$\{\mathcal{L}_{M,N}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}_{M,N}(\mu)\}_{\widetilde{P}_{M,N}^{\mathfrak{g}}} + [\, r(\lambda-\mu), \mathcal{L}_{M,N}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}_{M,N}(\mu)\,] = 0,$$

for all $(\lambda, \mu) \in \mathbb{C}^2$, with $r(\lambda)$ given in Eq. (2.7).

An alternative construction of the many-body hierarchy

We now present an alternative, but equivalent, way to construct the integrable chain described by the Lax matrix (2.65) without using the r-matrix approach as in the previuos Subsection.

It is possible to obtain the Lie-Poisson brackets (2.64) on $\widetilde{\mathfrak{H}}_{M,N}^*$ directly from the Lie-Poisson bracktes (2.16) on $\mathfrak{G}_{MN}^*$, being these latter ones the brackets associated with the Gaudin model with MN sites.

Proposition 2.15 *The map on the coordinates $\{y_k^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \le k \le MN$ on $\mathfrak{G}_{MN}^*$ defined by*

$$(\widetilde{y}_i^\alpha)_n \doteq \vartheta^i \sum_{j=1}^{N} \nu_{Nn-j+1}^i\, y_{Nn-j+1}^\alpha, \qquad 1 \le n \le M, \quad 0 \le i \le N-1, \tag{2.66}$$

with $\nu_j \in \mathbb{C}$, $\nu_j \neq \nu_k$, $1 \leq j,k \leq N$ and $0 < \vartheta \leq 1$, maps, in the limit $\vartheta \to 0$, the Lie-Poisson brackets (2.16) on $\mathfrak{G}^*_{MN}$ into the Lie-Poisson brackets on $\widetilde{\mathfrak{H}}^*_{M,N}$ given by Eq. (2.64):

$$\left\{(\widetilde{y}^{\alpha}_{i})_n, (\widetilde{y}^{\beta}_{j})_m\right\}_{\widetilde{P}^{\mathfrak{g}}_{M,N}} = \begin{cases} \delta_{n,m}\, C^{\alpha\beta}_{\gamma}\, (\widetilde{y}^{\gamma}_{i+j})_n & i+j < N, \\ 0 & i+j \geq N, \end{cases} \tag{2.67}$$

with $0 \leq i,j \leq N-1$ and $1 \leq n,m \leq M$.

Proof: Let us construct the Lie-Poisson brackets given in Eq. (2.67) using the map (2.66) and Eq. (2.16):

$$\begin{aligned}\left\{(\widetilde{y}^{\alpha}_{i})_n, (\widetilde{y}^{\beta}_{j})_m\right\}_{\vartheta} &= \sum_{l,k=1}^{N} \vartheta^{i+j}\, \nu^{i}_{Nn-l+1}\, \nu^{j}_{Nm-k+1} \left\{y^{\alpha}_{Nn-l+1}, y^{\beta}_{Nm-k+1}\right\}_{P^{\mathfrak{g}}_{MN}} = \\ &= \sum_{l,k=1}^{N} \vartheta^{i+j}\, \nu^{i}_{Nn-l+1}\, \nu^{j}_{Nm-k+1}\, \delta_{Nn-l+1,Nm-k+1}\, y^{\alpha}_{Nn-l+1}.\end{aligned}$$

Notice that $\delta_{Nn-l+1,Nm-k+1} = 1$ iff $N\,(n-m) = l-k$, but $1 \leq l-k \leq N-1$, so that we have to require $n = m$. Hence we have:

$$\begin{aligned}\left\{(\widetilde{y}^{\alpha}_{i})_n, (\widetilde{y}^{\beta}_{j})_m\right\}_{\vartheta} &= \delta_{n,m}\, C^{\alpha\beta}_{\gamma} \sum_{l=1}^{N} \vartheta^{i+j}\, \nu^{i+j}_{Nn-l+1}\, y^{\alpha}_{Nn-l+1} = \\ &= \begin{cases} \delta_{n,m}\, C^{\alpha\beta}_{\gamma}\, (\widetilde{y}^{\gamma}_{i+j})_n & i+j < N, \\ O(\vartheta^N) & i+j \geq N. \end{cases}\end{aligned}$$

Performing the contraction limit $\vartheta \to 0$ we get the Lie-Poisson brackets (2.67).

A computation similar to the one done in Proposition 2.10 leads to the following statement.

Proposition 2.16 *Let $H, G \in \mathcal{F}(\mathfrak{G}^*_{MN})$ be two involutive functions w.r.t. the Lie-Poisson brackets (2.16). If $\widetilde{H}, \widetilde{G} \in \mathcal{F}(\widetilde{\mathfrak{H}}^*_{M,N})$ are the correspondent functions obtained from H, G by applying the map (2.66) in the contraction limit $\vartheta \to 0$, then they are in involution w.r.t. the contracted Lie-Poisson brackets (2.67).*

Example 2.5 *Let us fix $M = 1$ in Eq. (2.66). We get (omitting the index $n = 1$)*

$$\widetilde{y}^{\alpha}_{i} \doteq \vartheta^{i} \sum_{j=1}^{N} \nu^{i}_{N-j+1}\, y^{\alpha}_{N-j+1} = \vartheta^{i} \sum_{k=1}^{N} \nu^{i}_{k}\, y^{\alpha}_{k}, \qquad 0 \leq i \leq N-1,$$

namely we recover the map (2.43).

According to the procedure performed in the case of the one-body hierarchy, we have also to apply a suitable pole coalescence. Later we shall give an explicit example in the case of the rational Lagrange chain.

2.4 Leibniz extensions of $\mathfrak{su}(2)$ rational Gaudin models: the one-body hierarchy

Let us fix $\mathfrak{g} \equiv \mathfrak{su}(2)$. We shall use the notation introduced in Subsection 2.2.1 for the coordinate functions of the $\mathfrak{su}(2)$ rational Gaudin model. But, for practical computations, it is convenient to introduce a different notation to denote te contracted variables $\{\widetilde{y}^{\alpha}_{i}\}^{3}_{\alpha=1}$, $0 \leq i \leq N-1$, namely

$$\widetilde{\mathbf{y}}_i \doteq \mathbf{z}_i, \qquad 0 \leq i \leq N-1.$$

We shall always use the above notation in the case of $\mathfrak{su}(2)$ Leibniz extensions.

The $\mathfrak{su}(2)$ rational one-body hierarchy is governed by the following Lax matrix, see Eq. (2.61):

$$\mathcal{L}_N^r(\lambda) \doteq \mathbf{p} + \sum_{i=0}^{N-1} \frac{\mathbf{z}_i}{\lambda^{i+1}}. \tag{2.68}$$

The $3N$ coordinate functions z_i^α's obey to the following Lie-Poisson brackets, see Eq. (2.44):

$$\left\{ z_i^\alpha, z_j^\beta \right\} = \left\{ \begin{array}{cc} -\varepsilon_{\alpha\beta\gamma}\, z_{i+j}^\gamma & i+j < N, \\ 0 & i+j \geq N. \end{array} \right. \tag{2.69}$$

Proposition 2.17 *The Lie-Poisson brackets (2.69) are degenerate. They possess the following N Casimir functions:*

$$C_k^{(N)} \doteq \frac{1}{2} \sum_{i=k}^{N-1} \langle\, \mathbf{z}_i, \mathbf{z}_{N+k-i-1} \,\rangle, \qquad 0 \leq k \leq N-1. \tag{2.70}$$

Proof: Let us compute the Lie bracket $\{C_k^{(N)}, z_j^\beta\}$ for an arbitrary coordinate function z_j^β:

$$\begin{aligned} \left\{ C_k^{(N)}, z_j^\beta \right\} &= \sum_{i=k}^{N-1} \left\{ z_i^\alpha\, z_{N+k-i-1}^\alpha, z_j^\beta \right\} = \\ &= \sum_{i=k}^{N-1} \left[z_i^\alpha \left\{ z_{N+k-i-1}^\alpha, z_j^\beta \right\} + \left\{ z_i^\alpha, z_j^\beta \right\} z_{N+k-i-1}^\alpha \right] = \\ &= 2 \sum_{i=k}^{N-1} \varepsilon_{\alpha\beta\gamma}\, z_{i+j}^\gamma\, z_{N+k-i-1+j}^\alpha . \end{aligned}$$

Now, if $i+j \geq N$ then $\{C_k^{(N)}, z_j^\beta\}$=0 thanks to (2.69). Let us consider $i+j<N$:

$$\left\{ C_k^{(N)}, z_j^\beta \right\} = \varepsilon_{\alpha\beta\gamma} \left[\sum_{i=k}^{N-1} z_{i+j}^\gamma\, z_{N+k-i-1+j}^\alpha + \sum_{i'=k}^{N-1} z_{i'+j}^\alpha\, z_{N+k-i'-1+j}^\gamma \right] = 0,$$

where $i' = N+k-i-1$.

□

Example 2.6 *If $N=2$ we obtain from (2.70) the Casimir functions of $\mathfrak{e}^*(3)$, namely*

$$C_0^{(2)} \doteq \langle\, \mathbf{z}_0, \mathbf{z}_1 \,\rangle, \qquad C_1^{(2)} \doteq \frac{1}{2} \langle\, \mathbf{z}_1, \mathbf{z}_1 \,\rangle. \tag{2.71}$$

According to Theorem 2.2 the Lax matrix (2.68) satisfies the linear r-matrix algebra (2.50) with r-matrix given by

$$r(\lambda) \doteq -\frac{1}{\lambda}\, \sigma_\alpha \otimes \sigma_\alpha,$$

namely $r(\lambda) = -\Pi/(2\,\lambda)$, where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Now our aim is to compute the spectral curve associated to the Lax matrix (2.68) in order to explicitly get the involutive Hamiltonians of the $\mathfrak{su}(2)$ rational one-body hierarchy.

Proposition 2.18 *The hyperelliptic curve* $\Gamma_N^r\,:\,\det(\mathcal{L}_N^r(\lambda)-\mu\,\mathbb{1})=0$, $(\lambda,\mu)\in\mathbb{C}^2$, *with* $\mathcal{L}_N^r(\lambda)$ *given in Eq. (2.68) reads*

$$\Gamma_N^r\,:\;-\mu^2=\frac{1}{4}\langle\,\mathbf{p},\mathbf{p}\,\rangle+\frac{1}{2}\sum_{k=0}^{N-1}\frac{H_k^{(N)}}{\lambda^{k+1}}+\frac{1}{2}\sum_{k=0}^{N-1}\frac{C_k^{(N)}}{\lambda^{k+N+1}},\tag{2.72}$$

where

$$H_k^{(N)}\doteq\langle\,\mathbf{p},\mathbf{z}_k\,\rangle+\frac{1}{2}\sum_{i=0}^{k-1}\langle\,\mathbf{z}_i,\mathbf{z}_{k-i-1}\,\rangle,\tag{2.73a}$$

$$C_k^{(N)}\doteq\frac{1}{2}\sum_{i=k}^{N-1}\langle\,\mathbf{z}_i,\mathbf{z}_{N+k-i-1}\,\rangle.\tag{2.73b}$$

The N *independent integrals* $\{H_k^{(N)}\}_{k=0}^{N-1}$ *are involutive first integrals of motion and the* N *integrals* $\{C_k^{(N)}\}_{k=0}^{N-1}$ *are the Casimir functions given in Eq. (2.70).*

Proof: A straightforward computation.

□

Remark 2.10 *Assigning a proper degree to the generators* $\mathbf{z}_i$*'s and to the constant field* $\mathbf{p}$ *we can immediately see that the integrals of motions are homogeneous polynomials in such degree. For instance, if* $\deg(\mathbf{z}_i)\doteq i$ *and* $\deg(\mathbf{p})\doteq -1$ *then* $\deg H_k^{(N)}=k-1$ *and* $\deg C_k^{(N)}=N+k-1$.

Remark 2.11 *It is possible to obtain the integrals given in Eq. (2.73a) using the map (2.43), in the contraction limit* $\vartheta\to 0$, *and the pole coalescence* $\lambda_i\equiv\vartheta\,\nu_i$, $1\le i\le N$, *on the integrals* $\{H_k^r\}_{k=1}^N$ *given in Eq. (2.33) of the* $\mathfrak{su}(2)$ *rational Gaudin model:*

$$H_k^r\doteq\langle\,\mathbf{p},\mathbf{y}_k\,\rangle+\sum_{\substack{j=1\\ j\neq k}}^{N}\frac{\langle\,\mathbf{y}_k,\mathbf{y}_j\,\rangle}{\lambda_k-\lambda_j},\qquad \sum_{k=1}^{N}H_k^r=\sum_{k=1}^{N}\langle\,\mathbf{p},\mathbf{y}_k\,\rangle.$$

Let us fix i *such that* $0\le i\le N-1$. *We get*

$$\begin{aligned}\sum_{k=1}^{N}\vartheta^i\,\nu_k^i\,H_k^r &= \sum_{k=1}^{N}\vartheta^i\,\nu_k^i\langle\,\mathbf{p},\mathbf{y}_k\,\rangle+\frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N}\vartheta^{i-1}\frac{\nu_k^i-\nu_j^i}{\nu_k-\nu_j}\langle\,\mathbf{y}_k,\mathbf{y}_j\,\rangle=\\ &= \sum_{k=1}^{N}\vartheta^i\,\nu_k^i\langle\,\mathbf{p},\mathbf{y}_k\,\rangle+\frac{1}{2}\sum_{m=0}^{i-1}\sum_{\substack{j,k=1\\ j\neq k}}^{N}(\vartheta\,\nu_k)^m(\vartheta\,\nu_j)^{i-m-1}\langle\,\mathbf{y}_k,\mathbf{y}_j\,\rangle=\\ &= \langle\,\mathbf{p},\mathbf{z}_i\,\rangle+\frac{1}{2}\sum_{m=0}^{i-1}\langle\;\mathbf{z}_m,\mathbf{z}_{i-m-1}\,\rangle=H_i^{(N)}.\end{aligned}$$

In the above computation we have taken into account the polynomial identity

$$\nu_k^i-\nu_j^i=(\nu_k-\nu_j)\sum_{m=0}^{i-1}\nu_k^m\,\nu_j^{i-m-1}.$$

Notice that the contracted version of the Gaudin Hamiltonian (2.37) is given by $H_1^{(N)}$, *while the contracted version of the linear integral* $\sum_{k=1}^N H_k^r=\sum_{k=1}^N\langle\,\mathbf{p},\mathbf{y}_k\,\rangle$ *is given by* $H_0^{(N)}$.

In the following we shall refer to the Hamiltonian flow generated by the integral $H_1^{(N)}$:

$$H_1^{(N)} \doteq \langle\, \mathbf{p}, \mathbf{z}_1 \,\rangle + \frac{1}{2} \langle\, \mathbf{z}_0, \mathbf{z}_0 \,\rangle. \tag{2.74}$$

A direct verification leads to the following proposition.

Proposition 2.19 *The equations of motion w.r.t. the Hamiltonian (2.74) are given by*

$$\dot{\mathbf{z}}_i = [\, \mathbf{z}_0, \mathbf{z}_i \,] + [\, \mathbf{p}, \mathbf{z}_{i+1} \,], \qquad 0 \le i \le N-1, \quad \mathbf{z}_N \equiv \mathbf{0}. \tag{2.75}$$

Eqs. (2.75) admit the following Lax representation:

$$\dot{\mathcal{L}}_N^r(\lambda) = \left[\, \mathcal{L}_N^r(\lambda), \mathcal{M}_N^{r,-}(\lambda) \,\right] = -\left[\, \mathcal{L}_N^r(\lambda), \mathcal{M}_N^{r,+}(\lambda) \,\right],$$

with the matrix $\mathcal{L}_N(\lambda)$ given in Eq. (2.68) and

$$\mathcal{M}_N^{r,-}(\lambda) \doteq \sum_{i=1}^{N-1} \frac{\mathbf{z}_i}{\lambda^i}, \qquad \mathcal{M}_N^{r,+}(\lambda) \doteq \mathbf{p} + \lambda\, \mathbf{z}_0. \tag{2.76}$$

Remark 2.12 *Proposition 2.19 can be proved or by direct verification, namely considering the contracted model, either performing the contraction procedure and the pole coalescence $\lambda_i \equiv \vartheta\, \nu_i$, $1 \le i \le N$, on Eqs. (2.38) and (2.39). We get, see Eq. (2.38):*

$$\dot{\mathbf{z}}_i = \sum_{j=1}^{N} \vartheta^i\, \nu_j^i\, \dot{\mathbf{y}}_j = \sum_{j=1}^{N} [\, \mathbf{p}, \vartheta^{i+1}\, \nu_j^{i+1}\, \mathbf{y}_j \,] + \sum_{j=1}^{N} \left[\sum_{k=1}^{N} \mathbf{y}_k, \vartheta^i\, \nu_j^i\, \mathbf{y}_j \right] = [\, \mathbf{z}_0, \mathbf{z}_i \,] + [\, \mathbf{p}, \mathbf{z}_{i+1} \,],$$

and, see Eq. (2.39):

$$\mathcal{M}_{\mathcal{G}}(\lambda) = \sum_{j=1}^{N} \frac{\vartheta\, \nu_j\, \mathbf{y}_j}{\lambda - \vartheta\, \nu_j} = \sum_{j=1}^{N} \sum_{i=0}^{N-2} \left(\frac{\vartheta\, \nu_j}{\lambda} \right)^{i+1} \mathbf{y}_j + O(\vartheta^N) \xrightarrow{\vartheta \to 0} \sum_{i=0}^{N-2} \frac{\mathbf{z}_{i+1}}{\lambda^{i+1}} = \mathcal{M}_N^{r,-}(\lambda).$$

2.4.1 $N = 2$, the Lagrange top

Fixing $N = 2$ in the formulae of the previous Subsection we recover the well-kwown dynamics of the three-dimensional Lagrange top described in the rest frame [4, 8, 19, 37, 53, 59, 79, 91]. In other words we can say that *the Lagrange top is the Leibniz extension of order two of the $\mathfrak{su}(2)$ rational Gaudin model*. This result is contained in our paper [53], but in a less general framework.

The Lagrange case of the rigid body motion around a fixed point in a homogeneous field is characterized by the following data: the inertia tensor is given by $J \doteq \mathrm{diag}(1, 1, \alpha)$, $\alpha \in \mathbb{R}$, which means that the body is rotationally symmetric with respect to the third coordinate axis, and the fixed point lies on the symmetry axis.

The equations of motion (in the rest frame) are given by:

$$\begin{cases} \dot{\mathbf{z}}_0 = [\, \mathbf{p}, \mathbf{z}_1], \\ \\ \dot{\mathbf{z}}_1 = [\, \mathbf{z}_0, \mathbf{z}_1], \end{cases} \tag{2.77}$$

where $\mathbf{z}_0 \in \mathbb{R}^3$ is the vector of kinetic momentum of the body, $\mathbf{z}_1 \in \mathbb{R}^3$ is the vector pointing from the fixed point to the center of mass of the body and $\mathbf{p} \doteq (0, 0, p)$ is the constant vector along the external field. Note that Eqs. (2.77) are just the special case $N = 2$ of Eqs. (2.75).

An external observer is mainly interested in the motion of the symmetry axis of the top on the surface $\langle\, \mathbf{z}_1, \mathbf{z}_1 \,\rangle$=constant. For an actual integration of this flow in terms of theta functions see [37].

A remarkable feature of the equations of motion (2.77) is that they do not depend explicitly on the anistropy parameter α of the inertia tensor [19]. Moreover they are Hamiltonian equations with respect to the Lie-Poisson brackets of (minus) $\mathfrak{e}^*(3)$:

$$\left\{ z_0^\alpha, z_0^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_0^\gamma, \qquad \left\{ z_0^\alpha, z_1^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_1^\gamma, \qquad \left\{ z_1^\alpha, z_1^\beta \right\} = 0. \tag{2.78}$$

As we noticed in Examples 2.2 the Lie-Poisson algebra $\mathfrak{e}^*(3)$ is the Leibniz extension of order two of $\mathfrak{su}^*(2)$. The Hamiltonian function that generates the equations of motion (2.77) is given by

$$H_1^{(2)} \doteq \langle\, \mathbf{p}, \mathbf{z}_1 \,\rangle + \frac{1}{2} \langle\, \mathbf{z}_0, \mathbf{z}_0 \,\rangle, \tag{2.79}$$

and the complete integrability of the model is ensured by the second integral of motion $H_0^{(2)} \doteq \langle\, \mathbf{p}, \mathbf{z}_0 \,\rangle$. These involutive Hamiltonians can be obtained using Eq. (2.73a) with $N = 2$, namely considering the spectral invariants of the Lax matrix, see Eq. (2.68),

$$\mathcal{L}_2^r(\lambda) \doteq \mathbf{p} + \frac{\mathbf{z}_0}{\lambda} + \frac{\mathbf{z}_1}{\lambda^2}. \tag{2.80}$$

The remaining two spectral invariants are given by the Casimir functions of the Lie-Poisson brackets of $\mathfrak{e}^*(3)$, see Eq. (2.71).

We finally recall that the Lagrange top admits a tri-Hamiltonian formulation in terms of the compatible Reyman-Semenov-Tian-Shansky tensors given in Eq. (2.63), see [79].

Remark 2.13 *In Proposition 2.19 we have focussed our attention on the Hamiltonian flow generated by the integral $H_1^{(N)}$. The reason for this choice can be understood looking at the Lagrange case. As a matter of fact $H_1^{(N)}$, that is the contracted version of the N-site $\mathfrak{su}(2)$ Gaudin Hamiltonian (2.37), is the generalization of the integral (2.79) to the N-th Leibniz extension of $\mathfrak{su}(2)$.*

Remark 2.14 *If we refer to the generalized Lax matrix (with $N = 2$) given in Eq. (2.60) we get, see Example 2.3:*

$$\mathcal{L}_2^r(\lambda) \doteq \mathbf{p} + \frac{\mathbf{z}_0}{\lambda} - c_1 \frac{\mathbf{z}_1}{\lambda^2}.$$

The above Lax matrix provides the same spectral invariants of the Lax matrix (2.80), up to a rescale of the intensity of the field $\mathbf{p}$ in the Hamiltonian $H_1^{(2)}$.

2.4.2 $N = 3$, the first extension of the Lagrange top

Let us now consider the dynamical system governed by the Lax matrix (2.68) with $N = 3$ and $\mathbf{p} \doteq (0, 0, p)$. We shall call such a model the *first extension of the Lagrange top* or equivalently *the Leibniz extension of order three of the $\mathfrak{su}(2)$ rational Gaudin model.* The Lie-Poisson brackets are explicitly given in Example 2.2. They read

$$\left\{ z_0^\alpha, z_0^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_0^\gamma, \quad \left\{ z_0^\alpha, z_1^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_1^\gamma, \quad \left\{ z_0^\alpha, z_2^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_2^\gamma, \tag{2.81a}$$

$$\left\{ z_1^\alpha, z_1^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_2^\gamma, \quad \left\{ z_1^\alpha, z_2^\beta \right\} = 0, \quad \left\{ z_2^\alpha, z_2^\beta \right\} = 0. \tag{2.81b}$$

According to Proposition 2.18 the integrals of motion of the model are given by, see Eqs. (2.73a) and (2.33),

$$H_0^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{z}_0 \,\rangle, \quad H_1^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{z}_1 \,\rangle + \frac{1}{2}\langle\, \mathbf{z}_0, \mathbf{z}_0 \,\rangle, \quad H_2^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{z}_2 \,\rangle + \langle\, \mathbf{z}_0, \mathbf{z}_1 \,\rangle, \tag{2.82a}$$

$$C_0^{(3)} \doteq \langle\, \mathbf{z}_0, \mathbf{z}_2 \,\rangle + \frac{1}{2}\langle\, \mathbf{z}_1, \mathbf{z}_1 \,\rangle, \quad C_1^{(3)} \doteq \langle\, \mathbf{z}_1, \mathbf{z}_2 \,\rangle, \quad C_2^{(3)} \doteq \frac{1}{2}\langle\, \mathbf{z}_2, \mathbf{z}_2 \,\rangle. \tag{2.82b}$$

Looking at the brackets given in Eqs. (2.81a) and (2.81b) and taking into account that $\mathbf{z}_0$ and $\mathbf{z}_2$ span respectively $\mathfrak{su}^*(2)$ and $\mathbb{R}^3$, we may interpret them as the total angular momentum of the system and the vector pointing from a fixed point (which we shall take as $(0,0,0) \in \mathbb{R}^3$) to the centre of mass of a Lagrange top. Let us remark that $\mathbf{z}_0$ does not coincide with the angular momentum of the top due to the presence of the vector $\mathbf{z}_1$. We think of $\mathbf{z}_1$, whose norm is not constant, as the position of the moving centre of mass of the system composed by the Lagrange top and a satellite , whose position is described by $\mathbf{z}_1 - \mathbf{z}_2$. Here we are assuming that both bodies have unitary masses. The link between these two systems is given by the Casimir functions given in Eq. (2.82b). If we think of a canonical realization of the brackets (2.81a) and (2.81b) in terms of three canonical coordinates and their conjugated momenta we can immediately argue that the vector $\mathbf{z}_1$ must depend on momenta.

If we look at the integral $H_1^{(3)}$, see Eq. (2.82a), we immediately see that it formally coincides with the physical Hamiltonian of the Lagrange top (2.79) where now the vector $\mathbf{z}_0$ is the angular momentum of system and the vector $\mathbf{z}_1$ describes the motion of the total centre of mass.

The Hamiltonian equations of motion with respect to $H_1^{(3)}$ are given by, see Eq. (2.75):

$$\begin{cases} \dot{\mathbf{z}}_0 = [\, \mathbf{p}, \mathbf{z}_1 \,], \\ \dot{\mathbf{z}}_1 = [\, \mathbf{z}_0, \mathbf{z}_1 \,] + [\, \mathbf{p}, \mathbf{z}_2 \,], \\ \dot{\mathbf{z}}_2 = [\, \mathbf{z}_0, \mathbf{z}_2 \,]. \end{cases}$$

We immediately see that the vector $\mathbf{z}_1$ does not rotate rigidly, though $\mathbf{z}_2$ does. Obviously, since the euclidean norm of $\mathbf{z}_1$ is not preserved, the integral $C_1^{(3)}$ does not imply that the angle between $\mathbf{z}_1$ and $\mathbf{z}_2$ is constant.

We now provide a canonical realization of the Lie-Poisson algebra given in Eqs. (2.81a) and (2.81b). We will use three Euler angles $\theta \in [\,0, 2\pi)$, $\phi \in [\,0, 2\pi)$ and $\psi \in [\,0, \pi)$ with their canonical conjugate momenta p_θ, p_ϕ and p_ψ.

Our canonical description is restricted to the following symplectic leaf:

$$\mathcal{O} \doteq \Big\{ (\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2) \in \mathbb{R}^9 |\; C_0^{(3)} \equiv 0,\; C_1^{(3)} \equiv 0,\; C_2^{(3)} \equiv 1/2 \Big\}. \tag{2.83}$$

Proposition 2.20 *A canonical realization restricted to the symplectic leaf (2.83) of the Lie-Poisson algebra given in Eqs. (2.81a) and (2.81b) is given by:*

$$\mathbf{z}_0 = \left(\sin\phi\, p_\theta + \cot\theta\, \cos\phi\, p_\phi - \frac{\cos\phi}{\sin\theta}\, p_\psi, -\cos\phi\, p_\theta + \cot\theta\, \sin\phi\, p_\phi - \frac{\sin\phi}{\sin\theta}\, p_\psi, p_\phi \right),$$

$$\mathbf{z}_1 = \sqrt{2\, p_\psi}\; (\sin\psi\, \sin\phi - \cos\theta\, \cos\psi\, \cos\phi, -\sin\psi\, \cos\phi - \cos\theta\, \cos\psi\, \sin\phi, -\sin\theta\, \cos\psi)\,,$$

$$\mathbf{z}_2 = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta)\,.$$

Proof: It is well-known that the canonical realization of $\mathfrak{e}^*(3)$ spanned by the vectors $\mathbf{z}_0$ and $\mathbf{z}_2$ such that $\langle\, \mathbf{z}_2, \mathbf{z}_2 \,\rangle = 1$ is given by [59]:

$$\mathbf{z}_0 = \left(\sin\phi\, p_\theta + \cot\theta\, \cos\phi\, p_\phi - \frac{\cos\phi}{\sin\theta}\, p_\psi, -\cos\phi\, p_\theta + \cot\theta\, \sin\phi\, p_\phi - \frac{\sin\phi}{\sin\theta}\, p_\psi, p_\phi \right),$$

$$\mathbf{z}_2 = (\sin\theta\, \cos\phi, \sin\theta\, \sin\phi, \cos\theta),$$

where $q \doteq (\theta, \phi, \psi)$ are the standard Euler angles and $p \doteq (p_\theta, p_\phi, p_\psi)$ are their canonical conjugated momenta. Recall that if $f(q,p)$ and $g(q,p)$ are two arbitrary smooth functions then

$$\{f, g\} \doteq \sum_{i=1}^{3} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right].$$

We now require that the Lie-Poisson brackets (2.81a) and (2.81b) restricted to the symplectic leaf (2.83) hold. A straightforward computation leads to

$$\mathbf{z}_1 = \sqrt{2\, p_\psi}\, (\sin\psi\, \sin\phi - \cos\theta\, \cos\psi\, \cos\phi, -\sin\psi\, \cos\phi - \cos\theta\, \cos\psi\, \sin\phi, -\sin\theta\, \cos\psi).$$

□

As we have previously mentioned we have obtained that the vector $\mathbf{z}_1$ is described in terms of the canonical coordinates θ, ϕ, ψ and the conjugated momentum p_ψ. In particular, we have $\langle\, \mathbf{z}_1, \mathbf{z}_1 \,\rangle = 2\, p_\psi > 0$.

For the sake of completeness we write the three Hamiltonians, see Eq. (2.82a), using the above canonical description:

$$\begin{aligned}
H_0^{(3)} &= p\, p_\phi, \\
H_1^{(3)} &= \frac{p_\theta^2}{2} + \frac{p_\psi^2 + p_\phi^2 - 2\, p_\psi\, p_\phi\, \cos\theta}{2 \sin^2\theta} - p\, \sqrt{2\, p_\psi} \sin\theta \cos\psi, \\
H_2^{(3)} &= \sqrt{2\, p_\psi}\, [\, p_\theta\, \sin\psi + (p_\psi - p_\phi\, \cos\theta) \cot\theta\, \cos\psi - p_\phi\, \sin\theta\, \cos\psi\,] + p\, \cos\theta.
\end{aligned}$$

Notice that the variable ϕ is cyclic as in the Lagrange case, while ψ explicitly enters in the potential term.

Remark 2.15 *If we refer to the generalized Lax matrix (with $N = 3$) given in Eq. (2.60) we get, see Example 2.3:*

$$\mathcal{L}_3^r(\lambda) \doteq \mathbf{p} + \frac{\mathbf{z}_0}{\lambda} - \frac{c_1\, \mathbf{z}_1 + c_2\, \mathbf{z}_2}{\lambda^2} + c_1^2\, \frac{\mathbf{z}_2}{\lambda^3}.$$

The above Lax matrix provides the following Hamiltonians:

$$\begin{aligned}
H_0^{(3)} &\doteq \langle\, \mathbf{p}, \mathbf{z}_0 \,\rangle, \\
H_1^{(3)}(c_1, c_2) &\doteq -c_1\, \langle\, \mathbf{p}, \mathbf{z}_1 + c_2\, \mathbf{z}_2 \,\rangle + \frac{1}{2} \langle\, \mathbf{z}_0, \mathbf{z}_0 \,\rangle, \\
H_2^{(3)}(c_1, c_2) &\doteq c_1\, (c_1\, \langle\, \mathbf{p}, \mathbf{z}_2 \,\rangle - \langle\, \mathbf{z}_0, \mathbf{z}_1 + c_2\, \mathbf{z}_2 \,\rangle)\,.
\end{aligned}$$

Remark 2.16 *For an arbitrary order N of the Leibniz extension - where N is also the number of sites of the Gaudin model - the $\mathfrak{su}(2)$ rational one-body hierarchy consists of a family of generalized Lagrange tops. They provide an example of integrable rigid body dynamics described by a Lagrange top with $N - 2$ interacting heavy satellites.*

Few remarks on the trigonometric and elliptic Lagrange top

The aim of this short paragraph is just to show the Hamiltonians of the systems governed by the Lax matrix (2.47) with $\mathfrak{g} \equiv \mathfrak{su}(2)$ and $N = 2$ in the case of a trigonometric and elliptic dependence on the spectral parameter: *they are the Leibniz extensions of order two of the $\mathfrak{su}(2)$ trigonometric and elliptic Gaudin models.*

According to Eqs. (2.30), (2.31) and (2.32) and Eq. (2.48) we consider the following Lax matrices:

$$\mathcal{L}_2^t(\lambda) \doteq \frac{\sigma_1\, z_0^1 + \sigma_2\, z_0^2}{\sin(\lambda)} + \cot(\lambda)\, \sigma_3\, z_0^3 + \frac{\cot(\lambda)}{\sin(\lambda)} (\sigma_1\, z_1^1 + \sigma_2\, z_1^2) + (\cot^2(\lambda) + 1)\, \sigma_3\, z_1^3, \tag{2.85a}$$

$$\begin{aligned} \mathcal{L}_2^e(\lambda) \doteq\; & \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)}\, \sigma_1\, z_0^1 + \frac{1}{\mathrm{sn}(\lambda)}\, \sigma_2\, z_0^2 + \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)}\, \sigma_3\, z_0^3 + \\ & + \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}^2(\lambda)}\, \sigma_1\, z_1^1 + \frac{\mathrm{cn}(\lambda)\,\mathrm{dn}(\lambda)}{\mathrm{sn}^2(\lambda)}\, \sigma_2\, z_2^1 + \frac{\mathrm{dn}^2(\lambda)}{\mathrm{sn}^2(\lambda)}\, \sigma_3\, z_3^1, \end{aligned} \tag{2.85b}$$

where the six variables z_0^α, z_1^α obey to the Lie-Poisson brackets of (minus) $\mathfrak{e}^*(3)$, see Eq. (2.78). The Lax matrices (2.85a) and (2.85b) satisfy the linear r-matrix structure (2.50) with r-matrix given respectively by, see Eq. (2.29),

$$r(\lambda) \doteq -\frac{1}{2} \begin{pmatrix} \cot(\lambda) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sin(\lambda)} & 0 \\ 0 & \frac{1}{\sin(\lambda)} & 0 & 0 \\ 0 & 0 & 0 & \cot(\lambda) \end{pmatrix},$$

$$r(\lambda) \doteq -\frac{1}{4} \begin{pmatrix} \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} & 0 & 0 & \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} - \frac{1}{\mathrm{sn}(\lambda)} \\ 0 & -\frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} & \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} + \frac{1}{\mathrm{sn}(\lambda)} & 0 \\ 0 & \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} + \frac{1}{\mathrm{sn}(\lambda)} & -\frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} & 0 \\ \frac{\mathrm{dn}(\lambda)}{\mathrm{sn}(\lambda)} - \frac{1}{\mathrm{sn}(\lambda)} & 0 & 0 & \frac{\mathrm{cn}(\lambda)}{\mathrm{sn}(\lambda)} \end{pmatrix}.$$

The system described by the Lax matrix (2.85a) has the following two involutive integrals of motion:

$$I_0^t \doteq (z_0^3)^2, \tag{2.86a}$$

$$I_1^t \doteq \frac{1}{2}\left[\, (z_0^1)^2 + (z_0^2)^2 + (z_0^3)^2 \,\right] + \frac{1}{2}(z_1^3)^2. \tag{2.86b}$$

The system described by the Lax matrix (2.85b) has the following two involutive integrals of motion:

$$I_0^e \doteq \frac{1}{2}\left[\, (z_0^1)^2 + (z_0^2)^2 \,\right] - \frac{k^2}{2}\left[\, (z_0^1)^2 + (z_1^3)^2 \,\right] + \frac{1}{2}(z_1^3)^2, \tag{2.87a}$$

$$I_1^e \doteq \frac{1}{2}\left[\, (z_0^1)^2 + (z_0^2)^2 + (z_0^3)^2 \,\right] - \frac{k^2}{2}\left[\, (z_1^2)^2 + (z_1^3)^2 \,\right] + \frac{1}{2}(z_1^3)^2, \tag{2.87b}$$

where k is the modulus of the elliptic Jacobi functions. Notice that fixing $k = 0$ in Eqs. (2.87a-2.87b) we obviously recover the integrals (2.86a-2.86b).

2.5 Leibniz extensions of $\mathfrak{su}(2)$ rational Gaudin models: the many-body hierarchy

The rational $\mathfrak{su}(2)$ many-body hierarchy is governed by the Lax matrix (2.65) with $\mathfrak{g} \equiv \mathfrak{su}(2)$ and assuming a rational dependence on the spectral parameter, namely:

$$\mathcal{L}^r_{M,N}(\lambda) \doteq \mathbf{p} + \sum_{k=1}^{M} \sum_{i=0}^{N-1} \frac{(\mathbf{z}_i)_k}{(\lambda - \mu_k)^{i+1}},$$

where the complex numbers μ_k's, with $\mu_k \neq \mu_j, 1 \leq k, j \leq M$ are local parameters of the model. The Lie-Poisson brackets are, see Eq. (2.67),

$$\left\{(z_i^\alpha)_n, (z_j^\beta)_m\right\} = \begin{cases} -\delta_{n,m}\, \varepsilon_{\alpha\beta\gamma}\, (z^\gamma_{i+j})_n & i+j < N, \\ 0 & i+j \geq N, \end{cases}$$

with $0 \leq i, j \leq N-1$ and $1 \leq n, m \leq M$.

Our aim is now to study the case $N = 2$, namely we shall consider an integrable long-range homogeneuous chain of interacting Lagrange tops, that we call *rational Lagrange chain.*

2.5.1 The rational Lagrange chain

The rational Lagrange chain is an integrable systems (with $2M$ degrees of freedom) associated with the Lie-Poisson algebra $\oplus^M \mathfrak{e}^*(3)$. Since we have just two generators, i.e. $\mathbf{z}_0$ and $\mathbf{z}_1$, for each copy of the Lie algebra, it is convenient to simplify the notation using the following definitions:

$$(\mathbf{z}_0)_k \doteq \mathbf{m}_k, \qquad (\mathbf{z}_1)_k \doteq \mathbf{a}_k, \qquad 1 \leq k \leq M. \tag{2.88}$$

Hence $\mathbf{m}_k \doteq (m_k^1, m_k^2, m_k^3) \in \mathbb{R}^3$ and $\mathbf{a}_k \doteq (a_k^1, a_k^2, a_k^3) \in \mathbb{R}^3$ describe respectively the angular momentum and the the vector pointing from the fixed point to the center of mass of the k-th top. The Lie-Poisson brackets on $\oplus^M \mathfrak{e}^*(3)$ are:

$$\left\{m_k^\alpha, m_j^\beta\right\} = -\delta_{k,j}\, \varepsilon_{\alpha\beta\gamma}\, m_k^\gamma, \quad \left\{m_k^\alpha, a_j^\beta\right\} = -\delta_{k,j}\, \varepsilon_{\alpha\beta\gamma}\, a_k^\gamma, \quad \left\{a_k^\alpha, a_j^\beta\right\} = 0, \tag{2.89}$$

with $1 \leq k, j \leq M$. The above brackets are degenerate: they possess the following $2M$ Casimir functions:

$$C_k^{(1)} \doteq \langle\, \mathbf{m}_k, \mathbf{a}_k\,\rangle, \qquad C_k^{(2)} \doteq \frac{1}{2}\langle\, \mathbf{a}_k, \mathbf{a}_k\,\rangle, \qquad 1 \leq k \leq M. \tag{2.90}$$

Using the notation introduced in Eq. (2.88), the Lax matrix of the rational Lagrange chain reads

$$\mathcal{L}^r_{M,2}(\lambda) \doteq \mathbf{p} + \sum_{i=1}^{M} \left[\frac{\mathbf{m}_i}{\lambda - \mu_i} + \frac{\mathbf{a}_i}{(\lambda - \mu_i)^2}\right], \tag{2.91}$$

where $\mathbf{p} \doteq (0, 0, p)$ as in the Lagrange case.

According to Proposition 2.14 the Lax matrix (2.91) satisfies a linear r-matrix algebra with r-matrix given by $r(\lambda) = -\Pi/(2\,\lambda)$, where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The complete set of integrals of the model can be obtained in the usual way. In fact, a straightforward computation leads to the following statement.

Proposition 2.21 *The hyperelliptic curve* $\Gamma^r_{M,2} : \det(\mathcal{L}^r_{M,2}(\lambda) - \mu\,\mathbb{1}) = 0$, $(\lambda, \mu) \in \mathbb{C}^2$, *with* $\mathcal{L}^r_{M,2}(\lambda)$ *given in Eq. (2.91) reads*

$$\Gamma^r_{M,2} : \; -\mu^2 = \frac{1}{4}\langle\, \mathbf{p}, \mathbf{p}\,\rangle + \frac{1}{2}\sum_{k=1}^{M}\left[\frac{R_k^r}{\lambda - \mu_k} + \frac{S_k^r}{(\lambda - \mu_k)^2} + \frac{C_k^{(1)}}{(\lambda - \mu_k)^3} + \frac{C_k^{(2)}}{(\lambda - \mu_k)^4}\right], \tag{2.92}$$

where

$$R^r_k \doteq \langle\, \mathbf{p}, \mathbf{m}_k \,\rangle + \sum_{\substack{j=1\\ j\neq k}}^{M} \left[\frac{\langle\, \mathbf{m}_k, \mathbf{m}_j \,\rangle}{\mu_k - \mu_j} + \frac{\langle\, \mathbf{m}_k, \mathbf{a}_j \,\rangle - \langle\, \mathbf{m}_j, \mathbf{a}_k \,\rangle}{(\mu_k-\mu_j)^2} - 2\,\frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\mu_k-\mu_j)^3} \right], \tag{2.93a}$$

$$S^r_k \doteq \langle\, \mathbf{p}, \mathbf{a}_k \,\rangle + \frac{1}{2}\langle\, \mathbf{m}_k, \mathbf{m}_k \,\rangle + \sum_{\substack{j=1\\ j\neq k}}^{M} \left[\frac{\langle\, \mathbf{a}_k, \mathbf{m}_j \,\rangle}{\mu_k - \mu_j} + \frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\mu_k-\mu_j)^2} \right]. \tag{2.93b}$$

The $2M$ independent integrals $\{R^r_k\}_{k=1}^M$ and $\{S^r_k\}_{k=1}^M$ are involutive first integrals of motion and the integrals $\{C^{(1)}_k\}_{k=1}^M$ and $\{C^{(2)}_k\}_{k=1}^M$ are the Casimir functions given in Eq. (2.90).

Remark 2.17 *Obviously, the curve (2.92) with $M=1$ coincides with the curve (2.72) with $N=2$.*

Notice that, as in the $\mathfrak{su}(2)$ rational Gaudin model, there is a linear integral given by $\sum_{k=1}^M R^r_k = \sum_{k=1}^M \langle\, \mathbf{p}, \mathbf{m}_k \,\rangle$. A possible choice for a physical Hamiltonian describing the dynamics of the model can be constructed considering a linear combination of the Hamiltonians $\{R_k\}_{k=1}^M$ and $\{S_k\}_{k=1}^M$ similar to the one considered for the rational Gaudin model, see Eq. (2.37):

$$\mathcal{H}^r_{M,2} \doteq \sum_{k=1}^M (\mu_k\, R^r_k + S^r_k) = \sum_{k=1}^M \langle\, \mathbf{p}, \mu_k\, \mathbf{m}_k + \mathbf{a}_k \,\rangle + \frac{1}{2}\sum_{i,k=1}^M \langle\, \mathbf{m}_i, \mathbf{m}_k \,\rangle. \tag{2.94}$$

If $M=1$ the Hamiltonian (2.94) gives the sum of the two integrals of motion of the Lagrange top. Our aim is now to find the Hamiltonian flow generated by $\mathcal{H}^r_{M,2}$.

Proposition 2.22 *The equations of motion w.r.t. the Hamiltonian (2.94) are given by*

$$\begin{cases} \dot{\mathbf{m}}_i = [\, \mathbf{p}, \mathbf{a}_i \,] + \left[\, \mu_i\, \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{m}_i \,\right], \\[2ex] \dot{\mathbf{a}}_i = \left[\, \mu_i\, \mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{a}_i \,\right], \end{cases} \tag{2.95}$$

with $1 \le i \le M$. Eqs. (2.95) admit the following Lax representation:

$$\dot{\mathcal{L}}^r_{M,2}(\lambda) = \left[\, \mathcal{L}^r_{M,2}(\lambda), \mathcal{M}^r_{M,2}(\lambda) \,\right],$$

with the matrix $\mathcal{L}^r_{M,2}(\lambda)$ given in Eq. (2.91) and

$$\mathcal{M}^r_{M,2}(\lambda) \doteq \sum_{i=1}^M \frac{1}{\lambda-\mu_i}\left[\mu_i\, \mathbf{m}_i + \frac{\lambda\, \mathbf{a}_i}{\lambda-\mu_i} \right]. \tag{2.96}$$

Proof: A direct calculation.

□

2.5.2 An alternative construction of the rational Lagrange chain

We now use the procedure described in Proposition 2.15 in order to recover the results obtained in the previuos Subsection.

Let us consider a $\mathfrak{su}(2)$ rational Gaudin model with $2M$ sites (i.e. $N = 2$, where N is the order of the Leibniz extension, see the notation used in Subsection 2.3.3). We have to apply the map defined in Eq. (2.66) to the coordinates $\{y_k^\alpha\}_{\alpha=1}^{\dim \mathfrak{g}}$, $1 \leq k \leq 2M$. According to the notation introduced in Eq. (2.88) we get:

$$(\mathbf{z}_0)_i \doteq \mathbf{m}_i \doteq \sum_{j=1}^{2} \mathbf{y}_{2i-j+1} = \mathbf{y}_{2i} + \mathbf{y}_{2i-1}, \tag{2.97a}$$

$$(\mathbf{z}_1)_i \doteq \mathbf{a}_i \doteq \vartheta \sum_{j=1}^{2} \nu_{2i-j+1}\, \mathbf{y}_{2i-j+1} = \vartheta\, (\nu_{2i}\, \mathbf{y}_{2i} + \nu_{2i-1}\, \mathbf{y}_{2i-1}), \tag{2.97b}$$

with $1 \leq i \leq M$. Moreover we define the following pole coalescence:

$$\lambda_{2i} \equiv \vartheta\, \nu_{2i} + \mu_i, \qquad \lambda_{2i-1} \equiv \vartheta\, \nu_{2i-1} + \mu_i, \qquad 1 \leq i \leq M, \tag{2.98}$$

where the λ_i's are the $2M$ parameters of the Gaudin model and the μ_i's are M distinct new parameters. We recall here some features of the $2M$-site $\mathfrak{su}(2)$ rational Gaudin model. The Lax matrix is, see Eq. (2.28a):

$$\mathcal{L}^r_{\mathcal{G}}(\lambda) \doteq \mathbf{p} + \sum_{i=1}^{2M} \frac{\mathbf{y}_i}{\lambda - \lambda_i}. \tag{2.99}$$

The $2M$ involutive Hamiltonians are, see Eq. (2.33):

$$H^r_i \doteq \langle\, \mathbf{p}, \mathbf{y}_i \,\rangle + \sum_{\substack{j=1 \\ j \neq i}}^{2M} \frac{\langle\, \mathbf{y}_i, \mathbf{y}_j \,\rangle}{\lambda_i - \lambda_j}, \qquad \sum_{i=1}^{2M} H^r_i = \sum_{i=1}^{2M} \langle\, \mathbf{p}, \mathbf{y}_i \,\rangle. \tag{2.100}$$

The equations of motion with respect to the Gaudin Hamiltonian $\sum_{i=1}^{2M} \lambda_i\, H^r_i$ are given by, see Eq. (2.38):

$$\dot{\mathbf{y}}_i = \left[\, \lambda_i\, \mathbf{p} + \textstyle\sum_{j=1}^{2M} \mathbf{y}_j \,,\, \mathbf{y}_i \,\right], \qquad 1 \leq i \leq 2M, \tag{2.101}$$

and the auxiliary matrix appearing in the Lax representation is, see Eq. (2.39):

$$\mathcal{M}^r_{\mathcal{G}}(\lambda) \doteq \sum_{i=1}^{2M} \frac{\lambda_i\, \mathbf{y}_i}{\lambda - \lambda_i}. \tag{2.102}$$

Proposition 2.23 *The isomorphism defined in Eqs. (2.97a) and (2.97b) and the pole coalescence (2.98) maps, in the contraction limit $\vartheta \to 0$:*

1. *the Lax matrix (2.99) into the Lax matrix (2.91);*
2. *the Hamiltonians (2.100) into the Hamiltonians (2.93a-2.93b);*
3. *the equations of motion (2.101) into the equations of motion (2.95);*
4. *the auxiliary matrix (2.102) into the auxiliary matrix (2.96).*

Proof: All results are obtained by a direct computation.

1. Considering Eq. (2.99) we obtain:

$$
\begin{aligned}
\mathcal{L}^r_{\mathcal{G}}(\lambda) &\doteq \mathbf{p} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i-1}}{\lambda - \lambda_{2i-1}} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i}}{\lambda - \lambda_{2i}} = \\
&= \mathbf{p} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i-1}}{\lambda - \mu_i - \vartheta\, \nu_{2i-1}} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i}}{\lambda - \mu_i - \vartheta\, \nu_{2i}} = \\
&= \mathbf{p} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i-1}}{\lambda - \mu_i} \left(1 + \frac{\vartheta\, \nu_{2i-1}}{\lambda - \mu_i} + O(\vartheta^2) \right) + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i}}{\lambda - \mu_i} \left(1 + \frac{\vartheta\, \nu_{2i}}{\lambda - \mu_i} + O(\vartheta^2) \right) = \\
&= \mathbf{p} + \sum_{i=1}^{M} \frac{\mathbf{y}_{2i-1} + \mathbf{y}_{2i}}{\lambda - \mu_i} + \sum_{i=1}^{M} \frac{\vartheta\, (\nu_{2i}\, \mathbf{y}_{2i} + \nu_{2i-1}\, \mathbf{y}_{2i-1})}{(\lambda - \mu_i)^2} + O(\vartheta^2).
\end{aligned}
$$

   Hence:

$$
\mathcal{L}^r_{\mathcal{G}}(\lambda) \xrightarrow{\vartheta \to 0} \mathbf{p} + \sum_{i=1}^{M} \left[\frac{\mathbf{m}_i}{\lambda - \mu_i} + \frac{\mathbf{a}_i}{(\lambda - \mu_i)^2} \right].
$$

2. The formulae giving the integrals $\{R^r_i\}_{i=1}^M$ and $\{S^r_i\}_{i=1}^M$, see Eqs. (2.93a-2.93b), are:

$$
\begin{aligned}
R^r_i &= \lim_{\vartheta \to 0} (H^r_{2i} + H^r_{2i-1}), \\
S^r_i &= \lim_{\vartheta \to 0} [\, \vartheta\, (\nu_{2i}\, H^r_{2i} + \nu_{2i-1}\, H^r_{2i-1})].
\end{aligned}
$$

We get:

$$\begin{aligned}
H_{2i}^{r} &+ H_{2i-1}^{r} = \langle\, \mathbf{p}, \mathbf{y}_{2i} + \mathbf{y}_{2i-1} \,\rangle + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i}, \mathbf{y}_{2j-1} \,\rangle}{\mu_i - \mu_j} \left[1 - \frac{\vartheta\,(\nu_{2i} - \nu_{2j-1})}{\mu_i - \mu_j} + \frac{\vartheta^2\,(\nu_{2i} - \nu_{2j-1})^2}{(\mu_i - \mu_j)^2} \right] + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i}, \mathbf{y}_{2j} \,\rangle}{\mu_i - \mu_j} \left[1 - \frac{\vartheta\,(\nu_{2i} - \nu_{2j})}{\mu_i - \mu_j} + \frac{\vartheta^2\,(\nu_{2i} - \nu_{2j-1})^2}{(\mu_i - \mu_j)^2} \right] + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i-1}, \mathbf{y}_{2j-1} \,\rangle}{\mu_i - \mu_j} \left[1 - \frac{\vartheta\,(\nu_{2i-1} - \nu_{2j-1})}{\mu_i - \mu_j} + \frac{\vartheta^2\,(\nu_{2i} - \nu_{2j-1})^2}{(\mu_i - \mu_j)^2} \right] + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i-1}, \mathbf{y}_{2j} \,\rangle}{\mu_i - \mu_j} \left[1 - \frac{\vartheta\,(\nu_{2i-1} - \nu_{2j})}{\mu_i - \mu_j} + \frac{\vartheta^2\,(\nu_{2i} - \nu_{2j-1})^2}{(\mu_i - \mu_j)^2} \right] + O(\vartheta^3) = \\
&= \langle\, \mathbf{p}, \mathbf{y}_{2i} + \mathbf{y}_{2i-1} \,\rangle + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i} + \mathbf{y}_{2i-1}, \mathbf{y}_{2j} + \mathbf{y}_{2j-1} \,\rangle}{\mu_i - \mu_j} + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \mathbf{y}_{2i} + \mathbf{y}_{2i-1}, \vartheta\,(\nu_{2j}\,\mathbf{y}_{2j} + \nu_{2j-1}\,\mathbf{y}_{2j-1}) \,\rangle}{(\mu_i - \mu_j)^2} - \\
&- \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1}), \mathbf{y}_{2j} + \mathbf{y}_{2j-1} \,\rangle}{(\mu_i - \mu_j)^2} + \\
&+ 2\sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1}), \vartheta\,(\nu_{2j}\,\mathbf{y}_{2j} + \nu_{2j-1}\,\mathbf{y}_{2j-1}) \,\rangle}{(\mu_i - \mu_j)^3} + O(\vartheta^3),
\end{aligned}$$

with $1 \leq i \leq M$. Using the map (2.97a-2.97b) and performing the limit $\vartheta \to 0$ we obtain the integrals given in Eq. (2.93a).

For the integrals $\{S_i^r\}_{i=1}^M$ we obtain:

$$\begin{aligned}
\vartheta\,(\nu_{2i}\,H_{2i}^r \;&+\; \nu_{2i-1}\,H_{2i-1}^r) = \langle\,\mathbf{p}, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1})\,\rangle + \langle\,\mathbf{y}_{2i}, \mathbf{y}_{2i-1}\,\rangle + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta\,\nu_{2i}\,\langle\,\mathbf{y}_{2i}, \mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i}-\nu_{2j-1})}{\mu_i-\mu_j}\right]+ \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta\,\nu_{2i}\,\langle\,\mathbf{y}_{2i}, \mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i}-\nu_{2j})}{\mu_i-\mu_j}\right]+ \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta\,\nu_{2i-1}\,\langle\,\mathbf{y}_{2i-1}, \mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i-1}-\nu_{2j-1})}{\mu_i-\mu_j}\right]+ \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\vartheta\,\nu_{2i-1}\,\langle\,\mathbf{y}_{2i-1}, \mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i-1}-\nu_{2j})}{\mu_i-\mu_j}\right]+O(\vartheta^3) = \\
&= \langle\,\mathbf{p}, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1})\,\rangle + \langle\,\mathbf{y}_{2i}, \mathbf{y}_{2i-1}\,\rangle + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1}), \mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j} + \\
&+ \sum_{\substack{j=1\\ j\neq i}}^{M} \frac{\langle\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1}), \vartheta\,(\nu_{2j}\,\mathbf{y}_{2j} + \nu_{2j-1}\,\mathbf{y}_{2j-1})\,\rangle}{(\mu_i-\mu_j)^2} + O(\vartheta^3),
\end{aligned}$$

with $1 \le i \le M$. Notice that

$$\langle\,\mathbf{y}_{2i}, \mathbf{y}_{2i-1}\,\rangle = \frac{1}{2}\,\langle\,\mathbf{y}_{2i}+\mathbf{y}_{2i-1}, \mathbf{y}_{2i}+\mathbf{y}_{2i-1}\,\rangle - C_{2i} - C_{2i-1},$$

where the functions $C_i \doteq \langle\,\mathbf{y}_i, \mathbf{y}_i\,\rangle$, $1 \le i \le 2M$, are Casimirs for $\oplus^{2M}\mathfrak{su}^*(2)$. Using the map (2.97a-2.97b) and performing the limit $\vartheta \to 0$ we obtain the integrals given in Eq. (2.93b). Notice that the Hamiltonians (2.93a-2.93b) are in involution w.r.t. the Lie-Poisson brackets (2.89) thanks to Proposition 2.16.

3. Considering Eqs. (2.101) we obtain:

$$\begin{aligned}
\dot{\mathbf{m}}_i \;&\doteq\; \dot{\mathbf{y}}_{2i} + \dot{\mathbf{y}}_{2i-1} = \\
&= \left[\,(\vartheta\,\nu_{2i}+\mu_i)\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\,\mathbf{y}_{2i}\,\right] + \left[\,(\vartheta\,\nu_{2i-1}+\mu_i)\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\,\mathbf{y}_{2i-1}\,\right] = \\
&= [\,\mathbf{p}, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1})\,] + \left[\,\mu_i\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\,\mathbf{y}_{2i-1}+\mathbf{y}_{2i}\,\right] = \\
&= [\,\mathbf{p}, \mathbf{a}_i\,] + \left[\,\mu_i\,\mathbf{p} + \textstyle\sum_{k=1}^{M}\mathbf{m}_k, \mathbf{m}_i\,\right],
\end{aligned}$$

and

$$\begin{aligned}
\dot{\mathbf{a}}_i \;&\doteq\; \vartheta\,(\nu_{2i}\,\dot{\mathbf{y}}_{2i} + \nu_{2i-1}\,\dot{\mathbf{y}}_{2i-1}) = \left[\,(\vartheta\,\nu_{2i}+\mu_i)\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\,\vartheta\,\nu_{2i}\,\mathbf{y}_{2i}\,\right] + \\
&+ \left[\,(\vartheta\,\nu_{2i-1}+\mu_i)\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\,\vartheta\,\nu_{2i-1}\,\mathbf{y}_{2i-1}\,\right] = \\
&= \left[\,\mu_i\,\mathbf{p} + \textstyle\sum_{j=1}^{2M}\mathbf{y}_j\,,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i} + \nu_{2i-1}\,\mathbf{y}_{2i-1})\,\right] + O(\vartheta^2) \xrightarrow{\vartheta\to 0} \left[\,\mu_i\,\mathbf{p} + \textstyle\sum_{k=1}^{M}\mathbf{m}_k, \mathbf{a}_i\,\right].
\end{aligned}$$

4. Considering Eq. (2.102) we obtain:

$$
\begin{aligned}
\mathcal{M}^r_{\mathcal{G}}(\lambda) &\doteq \sum_{i=1}^{M} \frac{\lambda_{2i-1}\,\mathbf{y}_{2i-1}}{\lambda-\lambda_{2i-1}} + \sum_{i=1}^{M} \frac{\lambda_{2i}\,\mathbf{y}_{2i}}{\lambda-\lambda_{2i}} = \\
&= \sum_{i=1}^{M} \frac{(\vartheta\,\nu_{2i-1}+\mu_i)\,\mathbf{y}_{2i-1}}{\lambda-\mu_i-\vartheta\,\nu_{2i-1}} + \sum_{i=1}^{M} \frac{(\vartheta\,\nu_{2i}+\mu_i)\,\mathbf{y}_{2i}}{\lambda-\mu_i-\vartheta\,\nu_{2i}} = \\
&= \sum_{i=1}^{M} \frac{(\vartheta\,\nu_{2i-1}+\mu_i)\,\mathbf{y}_{2i-1}}{\lambda-\mu_i} \left(1+\frac{\vartheta\,\nu_{2i-1}}{\lambda-\mu_i}+O(\vartheta^2)\right) + \\
&\quad + \sum_{i=1}^{M} \frac{(\vartheta\,\nu_{2i}+\mu_i)\,\mathbf{y}_{2i}}{\lambda-\mu_i} \left(1+\frac{\vartheta\,\nu_{2i}}{\lambda-\mu_i}+O(\vartheta^2)\right) = \\
&= \sum_{i=1}^{M} \frac{1}{\lambda-\mu_i}\left[\mu_i\,(\mathbf{y}_{2i-1}+\mathbf{y}_{2i}) + \frac{\lambda\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1})}{\lambda-\mu_i}\right] + O(\vartheta^2).
\end{aligned}
$$

Hence:

$$
\mathcal{M}^r_{\mathcal{G}}(\lambda) \xrightarrow{\vartheta\to 0} \sum_{i=1}^{M} \frac{1}{\lambda-\mu_i}\left[\mu_i\,\mathbf{m}_i + \frac{\lambda\,\mathbf{a}_i}{\lambda-\mu_i}\right].
$$

□

Few remarks on the trigonometric Lagrange chain

We now construct a trigonometric Lagrange chain, namely a long-range partially inhomogeneous integrable chain of interacting Lagrange tops. We shall use the notation introduced in Eq. (2.88) to denote the coordinates of the Lie-Poisson algebra $\oplus^M \mathfrak{e}^*(3)$.

According to Eqs. (2.65) and (2.85a), the Lax matrix governing the trigonometric Lagrange chain reads

$$
\begin{aligned}
\mathcal{L}^t_{M,2}(\lambda) &\doteq \sum_{i=1}^{M}\left[\frac{\sigma_1\,m_i^1+\sigma_2\,m_i^2}{\sin(\lambda-\mu_i)} + \cot(\lambda-\mu_i)\,\sigma_3\,m_i^3 + \right.\\
&\quad \left. + \frac{\cot(\lambda-\mu_i)}{\sin(\lambda-\mu_i)}\,(\sigma_1\,a_i^1+\sigma_2\,a_i^2) + (\cot^2(\lambda-\mu_i)+1)\,\sigma_3\,a_i^3\right]. \qquad (2.103)
\end{aligned}
$$

We know, from Proposition 2.14, that the above Lax matrix satisfies a linear r-matrix algebra with r-matrix given by, see Eq. (2.29),

$$
r(\lambda) \doteq -\frac{1}{2}\begin{pmatrix} \cot(\lambda) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sin(\lambda)} & 0 \\ 0 & \frac{1}{\sin(\lambda)} & 0 & 0 \\ 0 & 0 & 0 & \cot(\lambda) \end{pmatrix},
$$

Let us give the following statement. The proof is straightforward.

Proposition 2.24 *The curve* $\Gamma^t_{M,2} : \det(\mathcal{L}^t_{M,2}(\lambda) - \mu\,\mathbb{1}) = 0$, $(\lambda,\mu)\in\mathbb{C}^2$, *with* $\mathcal{L}^t_{M,2}(\lambda)$ *given in Eq. (2.103) reads*

$$
\begin{aligned}
\Gamma^t_{M,2}: \qquad -\mu^2 = H_0^t + \frac{1}{2}\sum_{k=1}^{M} \big[\, R_k^t\,\cot(\lambda-\mu_k) + S_k^t\,\cot^2(\lambda-\mu_k) + \\
+ C_k^{(1)}\,\cot^3(\lambda-\mu_k) + C_k^{(2)}\,\cot^4(\lambda-\mu_k)\big],
\end{aligned}
$$

where

$$H_0^t \doteq \frac{1}{2}\sum_{i=1}^{M}\left[\,(m_i^1)^2+(m_i^2)^2\,\right]-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{M} m_i^3\, m_j^3+\frac{1}{2}\left(\sum_{i=1}^{M} a_i^3\right)^2+$$
$$+\sum_{\substack{i,j=1\\ i\neq j}}^{M}\frac{1}{\sin(\mu_i-\mu_j)}\left[\,a_i^1\, m_j^1+a_i^2\, m_j^2+a_i^3\, m_j^3\,\cos(\mu_i-\mu_j)\,\right]+$$
$$+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{M}\frac{\cot(\mu_i-\mu_j)}{\sin(\mu_i-\mu_j)}\left[\,a_i^1\, a_j^1+a_i^2\, a_j^2+a_i^3\, a_j^3\,\cos(\mu_i-\mu_j)\,\right], \tag{2.104a}$$

$$R_k^t \doteq C_k^{(1)}+\frac{1}{2}\sum_{\substack{j=1\\ j\neq k}}^{M}\left(m_k^3\, a_j^3-m_j^3\, a_k^3\right)+$$
$$+\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{1}{\sin(\mu_k-\mu_j)}\left[\,m_k^1\, m_j^1+m_k^2\, m_j^2+m_k^3\, m_j^3\,\cos(\mu_k-\mu_j)\,\right]+$$
$$+\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{\cot(\mu_k-\mu_j)}{\sin(\mu_k-\mu_j)}\left[\,m_k^1\, a_j^1+m_k^2\, a_j^2+m_k^3\, a_j^3\,\cos(\mu_k-\mu_j)-\right.$$
$$\left.-\,m_j^1\, a_k^1-m_j^2\, a_k^2-m_j^3\, a_k^3\,\cos(\mu_k-\mu_j)\,\right]-$$
$$-\,2\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{1}{\sin^3(\mu_k-\mu_j)}\left[\,a_k^1\, a_j^1+a_k^2\, a_j^2+a_k^3\, a_j^3\,\cos(\mu_k-\mu_j)\,\right], \tag{2.104b}$$

$$S_k^t \doteq C_k^{(2)}+\frac{1}{2}\left[\,(m_k^1)^2+(m_k^2)^2+(m_k^3)^2\,\right]+\frac{1}{2}(a_k^3)^2+\frac{1}{2}\sum_{\substack{i,j=1\\ j\neq k}}^{M} a_i^3\, a_j^3+$$
$$+\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{1}{\sin(\mu_k-\mu_j)}\left[\,a_k^1\, m_j^1+a_k^2\, m_j^2+a_k^3\, m_j^3\,\cos(\mu_k-\mu_j)\,\right]+$$
$$+\sum_{\substack{j=1\\ j\neq k}}^{M}\frac{\cot(\mu_k-\mu_j)}{\sin(\mu_k-\mu_j)}\left[\,a_k^1\, a_j^1+a_k^2\, a_j^2+a_k^3\, a_j^3\,\cos(\mu_k-\mu_j)\,\right]. \tag{2.104c}$$

The integrals H_0^t, $\{R_k^t\}_{k=1}^M$, $\{S_k^t\}_{k=1}^M$ are involutive first integrals of motion (only $2M$ of them are independent). The integrals $\{C_k^{(1)}\}_{k=1}^M$ and $\{C_k^{(2)}\}_{k=1}^M$ are the Casimir functions given in Eq. (2.90).

Remark 2.18 *Let us fix $M=1$ in Eqs. (2.104a) and (2.104c). We get (omitting the lower index in the coordinates $\mathbf{m}_1, \mathbf{a}_1$):*

$$H_0^t \doteq \frac{1}{2}\left[\,(m^1)^2+(m^2)^2\,\right]+\frac{1}{2}\left(a^3\right)^2,$$
$$S^t \doteq \frac{1}{2}\langle\,\mathbf{a},\mathbf{a}\,\rangle+\frac{1}{2}\langle\,\mathbf{m},\mathbf{m}\,\rangle+\frac{1}{2}\left(a^3\right)^2,$$

where $\langle\, \mathbf{a}, \mathbf{a}\, \rangle$ *is a Casimir function of the Lie-Poisson algebra* $\mathfrak{e}^*(3)$*. Recalling the Hamiltonians (2.86a-2.86b) of the trigonometric Lagrange top, we immediately see that* $I_1^t = S^t$ *(up to a Casimir function) and* $I_0^t = S^t - H_0^t$ *(up to a Casimir function).*

3

Integrable discretizations of $\mathfrak{su}(2)$ extended rational Gaudin models

3.1 Integrable discretizations through Bäcklund transformations

The theory of integrable maps got a boost when Veselov developed a theory of Lagrange correspondences [99, 100, 101]. Roughly speaking, these maps are symplectic multi-valued transformations which have enough integrals of motion, this definition being a proper analog of the classical Liouville integrability. In the main examples, studied by him and later by other authors, the integrable maps are constructed as time-discretizations of classical integrable models (such as the Neumann system, the geodesic flow on an ellipsoid, the Euler-Manakov top, the Lagrange top, the Toda lattice, the Calogero-Moser systems and other families of integrable systems), see, for instance, [21, 23, 31, 41, 53, 54, 55, 56, 70, 77, 87, 99, 100, 101], the excellent book of Yu.B. Suris [91] and the references inside.

Moreover these correspondences associate with a given solution of an integrable system a new solution, a property reminescent of Bäcklund transformations (BTs) for soliton equations.

In the first part of this Chapter we shall apply the theory of BTs for finite-dimensional integrable systems, developed by V.B. Kuznetsov, E.K. Sklyanin and P. Vanhaecke in the papers [55, 56, 87]. Following this approach we look at BTs as special Poisson maps. It is possible to find an exhaustive list of the features of these BTs in [55, 56, 87]:

1. A BT is an integrable Poisson map that discretizes a family of flows of the integrable system (and not a particular one);

2. The discrete flow corresponds to an interpolating Hamiltonian $\mathcal{H}$ which is a multi-valued function of the involutive integrals of motion. A BT $\mathcal{B}$ acts on a point x of the phase space as [56]
$$\mathcal{B}:\ x \longmapsto \ \widehat{x} = x + \{\mathcal{H}, x\} + \frac{1}{2}\{\mathcal{H}, \{\mathcal{H}, x\}\} + ...$$
Nevertheless, although a BT is multi-valued, it leads to a single-valued map on any level manifold of the integrals of motion;

3. a BT can be constructed using a quite universal receipt. One has to consider the following similarity transform on the Lax matrix $\mathcal{L}(\lambda)$:
$$\mathcal{B}_\eta:\ \mathcal{L}(\lambda) \longmapsto \widehat{\mathcal{L}}_\eta(\lambda) \doteq \mathcal{M}_\eta(\lambda)\,\mathcal{L}(\lambda)\,\mathcal{M}_\eta^{-1}(\lambda), \qquad \forall\ \lambda \in \mathbb{C}, \quad \eta \in \mathbb{C}, \tag{3.1}$$
with some generically non-degenerate matrix $\mathcal{M}_\eta(\lambda)$, simply because a BT should preserve the spectrum of $\mathcal{L}(\lambda)$. The parameter η is called *Bäcklund parameter*. It is possible to consider BTs with several parameters: in particular, the number of zeros of $\det \mathcal{M}_\eta(\lambda)$ is the number of essential Bäcklund parameters;

4. Two BTs $\mathcal{B}_{\eta_1}$ and $\mathcal{B}_{\eta_2}$, $\eta_1 \neq \eta_2$, for a given integrable sistem commute, namely $\mathcal{B}_{\eta_1} \circ \mathcal{B}_{\eta_2} = \mathcal{B}_{\eta_2} \circ \mathcal{B}_{\eta_1}$. This commutation follows from the canonicity and the invariance of the integrals of motion. Indeed, any integrable canonical mapping acts on the Liouville torus as a collection of shifts of the angle variables [99, 100, 101];

5. When one searches for the simplest BT of an integrable system, then one finds a one-dimensional family $\{\mathcal{B}_\eta \,|\, \eta \in \mathbb{C}\}$ of them. The Bäcklund parameter η is canonically conjugate to μ, i.e. $\mu = -\partial F_\eta/\partial\eta$ with F_η generating function of $\{\mathcal{B}_\eta \,|\, \eta \in \mathbb{C}\}$. Here μ is bound to η by the equation of an algebraic curve (dependent on the integrals), which is exactly the characteristic curve that appears in the linearization of the integrable system. This property is called *spectrality* of the BT;

6. A direct consequence of the spectrality property is the explicitness of the constructed maps; see [56] for further details about the geometrical meaning of the spectrality property;

7. The explicit nature of a BT makes it purely iterative, so that it is very well suited as symplectic integrator for the underlying model. The Bäcklund parameter η plays the role of an adjustable discrete-time step.

Remark 3.1 *An important practical question arising in the theory of BTs is how to find, given a Lax matrix $\mathcal{L}(\lambda)$, the matrix $\mathcal{M}_\eta(\lambda)$ which would generate a BT. To check that a matrix $\mathcal{M}_\eta(\lambda)$ is admissible one needs, first, to verify that the system of equations resulting from Eq. (3.1) is self-consistent, and, second, to prove that the resulting transformation is canonical, i.e. it preserves the Lie-Poisson brackets.*

For instance, if $\mathcal{L}(\lambda)$ satisfies a linear r-matrix algebra, namely

$$\{\mathcal{L}(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}(\mu)\} + [\, r(\lambda-\mu), \mathcal{L}(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}(\mu)\,] = 0, \qquad \forall\, (\lambda,\mu) \in \mathbb{C}^2,$$

one has to prove that the same holds for the Lax matrix $\widehat{\mathcal{L}}_\eta(\lambda)$.

This problem is completely solved in the case of $\mathfrak{su}(2)$ Lax matrices associated with linear and quadratic r-matrix algebras with rational dependence on λ [31, 87] and a general ansatz for the matrix $\mathcal{M}_\eta(\lambda)$ exists. In particular, the matrix $\mathcal{M}_\eta(\lambda)$ should be a simple Lax operator of the quadratic algebra

$$\{\mathcal{M}_\eta(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{M}_\eta(\mu)\} + [\, r(\lambda-\mu), \mathcal{M}_\eta(\lambda) \otimes \mathcal{M}_\eta(\mu)\,] = 0, \qquad \forall\, (\lambda,\mu) \in \mathbb{C}^2, \tag{3.2}$$

with the same rational r-matrix associated with the Lax matrix $\mathcal{L}(\lambda)$. The fact that the right ansatz for the matrix $\mathcal{M}_\eta(\lambda)$ obeys to the algebra (3.2) guarantees that the resulting map will be Poisson; see [56, 87] for further details.

In [53, 73] we have constructed (complex and real) BTs for the standard Lagrange top. An interesting feature of these BTs, see our paper [53], is that they can be obtained performing a contraction and a pole coalescence on the BTs for the $\mathfrak{su}(2)$ two-body rational Gaudin model. We recall that the BTs for the $\mathfrak{su}(2)$ N-body rational Gaudin model have been constructed in [41].

We prefer to omit the complete analysis of the BTs for the Lagrange top, since they can be obtained from the BTs of the rational Lagrange chain fixing $M = 1$, being M the number of interacting Lagrange tops.

We now study the problem of constructing BTs for the first rational extension of the Lagrange top and for the rational Lagrange chain.

3.1.1 BTs for the first rational extension of the Lagrange top

The continuous-time first rational extension of the Lagrange top is described in the previuos Chapter, see Subsection 2.4.2. Since we have just three generators, i.e. $\mathbf{z}_0$, $\mathbf{z}_1$, $\mathbf{z}_2$, we prefer to use the following notation:

$$\mathbf{z}_0 \doteq \mathbf{m}, \qquad \mathbf{z}_1 \doteq \mathbf{a}, \qquad \mathbf{z}_2 \doteq \mathbf{b}.$$

According to this notation, the Lax matrix is obtained fixing $N = 3$ in Eq. (2.68):

$$\mathcal{L}_3(\lambda) \doteq \mathbf{p} + \frac{\mathbf{m}}{\lambda} + \frac{\mathbf{a}}{\lambda^2} + \frac{\mathbf{b}}{\lambda^3}, \tag{3.3}$$

where $\mathbf{p} \doteq (0, 0, p)$ as in the Lagrange case. The Lie-Poisson brackets read, see Eqs. (2.81a-2.81b):

$$\left\{m^\alpha, m^\beta\right\} = -\varepsilon_{\alpha\beta\gamma}\, m^\gamma, \qquad \left\{m^\alpha, a^\beta\right\} = -\varepsilon_{\alpha\beta\gamma}\, a^\gamma, \qquad \left\{m^\alpha, b^\beta\right\} = -\varepsilon_{\alpha\beta\gamma}\, b^\gamma, \tag{3.4a}$$

$$\left\{a^\alpha, a^\beta\right\} = -\varepsilon_{\alpha\beta\gamma}\, b^\gamma, \qquad \left\{a^\alpha, b^\beta\right\} = 0, \qquad \left\{b^\alpha, b^\beta\right\} = 0. \tag{3.4b}$$

Let us recall that the Lax matrix (3.3) satisfies the linear r-matrix algebra (2.50) with r-matrix given by $r(\lambda) = -\Pi/(2\,\lambda)$, being Π the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

It is convenient to write the Lax matrix (3.3) in the following way:

$$\mathcal{L}_3(\lambda) \doteq -\frac{\mathrm{i}}{2} \begin{pmatrix} u(\lambda) & v(\lambda) \\ w(\lambda) & -u(\lambda) \end{pmatrix}, \tag{3.5}$$

with

$$u(\lambda) \doteq p + \frac{m^3}{\lambda} + \frac{a^3}{\lambda^2} + \frac{b^3}{\lambda^3}, \qquad v(\lambda) \doteq \frac{m^-}{\lambda} + \frac{a^-}{\lambda^2} + \frac{b^-}{\lambda^3}, \qquad w(\lambda) \doteq \frac{m^+}{\lambda} + \frac{a^+}{\lambda^2} + \frac{b^+}{\lambda^3},$$

where we have introduced the complex variables $m^\pm \doteq m^1 \pm \mathrm{i}\, m^2$, $a^\pm \doteq a^1 \pm \mathrm{i}\, a^2$, $b^\pm \doteq b^1 \pm \mathrm{i}\, b^2$. In terms of such variables the non-trivial Lie-Poisson brackets (3.4a-3.4b) read

$$\{m^3, m^\pm\} = \pm\mathrm{i}\, m^\pm, \qquad \{m^+, m^-\} = 2\,\mathrm{i}\, m^3,$$

$$\{m^3, a^\pm\} = \{a^3, m^\pm\} = \pm\mathrm{i}\, a^\pm, \quad \{m^+, a^-\} = \{a^+, m^-\} = 2\,\mathrm{i}\, a^3,$$

$$\{m^3, b^\pm\} = \{b^3, m^\pm\} = \pm\mathrm{i}\, b^\pm, \quad \{m^+, b^-\} = \{b^+, m^-\} = 2\,\mathrm{i}\, b^3,$$

$$\{a^3, a^\pm\} = \pm\mathrm{i}\, b^\pm, \qquad \{a^+, a^-\} = 2\,\mathrm{i}\, b^3.$$

Finally we give the complete set of integrals of motion of the first extension of the Lagrange top, see Eqs. (2.82a-2.82b):

$$H_0^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{m}\,\rangle, \quad H_1^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{a}\,\rangle + \frac{1}{2}\langle\, \mathbf{m}, \mathbf{m}\,\rangle, \quad H_2^{(3)} \doteq \langle\, \mathbf{p}, \mathbf{b}\,\rangle + \langle\, \mathbf{m}, \mathbf{a}\,\rangle,$$

$$C_0^{(3)} \doteq \langle\, \mathbf{m}, \mathbf{b}\,\rangle + \frac{1}{2}\langle\, \mathbf{a}, \mathbf{a}\,\rangle, \quad C_1^{(3)} \doteq \langle\, \mathbf{a}, \mathbf{b}\,\rangle, \quad C_2^{(3)} \doteq \frac{1}{2}\langle\, \mathbf{b}, \mathbf{b}\,\rangle.$$

According to Proposition 2.18 they are the coefficients of the inverse powers of λ of the hyper-elliptic spectral curve $\Gamma_3 :\, \det(\mathcal{L}_3(\lambda) - \mu\, \mathbb{1}) = 0$, namely

$$\Gamma_3 :\; -\mu^2 = \frac{1}{4}\langle\, \mathbf{p}, \mathbf{p}\,\rangle + \frac{1}{2}\left(\frac{H_0^{(3)}}{\lambda} + \frac{H_1^{(3)}}{\lambda^2} + \frac{H_2^{(3)}}{\lambda^3} + \frac{C_0^{(3)}}{\lambda^4} + \frac{C_1^{(3)}}{\lambda^5} + \frac{C_2^{(3)}}{\lambda^6}\right). \tag{3.7}$$

One-point BTs

A one-point BT can be constructed performing the similarity transform given in Eq. (3.1) on our Lax matrix $\mathcal{L}_3(\lambda)$ (3.5):

$$\mathcal{B}_\eta :\; \mathcal{L}_3(\lambda) \longmapsto \widehat{\mathcal{L}}_3(\lambda;\eta) \doteq \mathcal{M}_\eta(\lambda)\, \mathcal{L}_3(\lambda)\, \mathcal{M}_\eta^{-1}(\lambda) = -\frac{\mathrm{i}}{2} \begin{pmatrix} \widehat{u}(\lambda) & \widehat{v}(\lambda) \\ \widehat{w}(\lambda) & -\widehat{u}(\lambda) \end{pmatrix}, \tag{3.8}$$

for all $\lambda \in \mathbb{C}$, being $\eta \in \mathbb{C}$ the Bäcklund parameter, and

$$\widehat{u}(\lambda) \doteq p + \frac{\widehat{m}^3}{\lambda} + \frac{\widehat{a}^3}{\lambda^2} + \frac{\widehat{b}^3}{\lambda^3}, \qquad \widehat{v}(\lambda) \doteq \frac{\widehat{m}^-}{\lambda} + \frac{\widehat{a}^-}{\lambda^2} + \frac{\widehat{b}^-}{\lambda^3}, \qquad \widehat{w}(\lambda) \doteq \frac{\widehat{m}^+}{\lambda} + \frac{\widehat{a}^+}{\lambda^2} + \frac{\widehat{b}^+}{\lambda^3}.$$

Here we use the $\widehat{\ }$-notation for the updated variables. The intertwining matrix $\mathcal{M}_\eta(\lambda)$ is given by [1][31, 41, 53, 87]:

$$\mathcal{M}_\eta(\lambda) \doteq \begin{pmatrix} \lambda - \eta + r\,q & r \\ q & 1 \end{pmatrix}, \qquad \det \mathcal{M}_\eta(\lambda) = \lambda - \eta. \tag{3.9}$$

The variables r and q are "a priori" indeterminate dynamical variables, but comparing the asymptotics in $\lambda \to \infty$ in both sides of Eq. (3.8) we readily get

$$r = \frac{m^-}{2\,p}, \qquad q = \frac{\widehat{m}^+}{2\,p}, \qquad \widehat{m}^3 = m^3. \tag{3.10}$$

Notice that the last equation in (3.10) immediately gives the conservation of the Hamiltonian $H_0^{(3)}$. If we want an explicit map from $\mathcal{L}_3(\lambda)$ to $\widehat{\mathcal{L}}_3(\lambda;\eta)$ we must express q in terms of the old variables. To overcome this problem one can use the spectrality of the BTs [55, 56]. Eq. (3.8) defines a map $\mathcal{B}_P$ parametrized by the point $P \doteq (\eta,\mu) \in \Gamma_3$, see Eq. (3.7) . Notice that there are two points on Γ_3, $P \doteq (\eta,\mu)$ and $Q \doteq (\eta,-\mu)$, corresponding to the same η and sitting one above the other because of the hyperelliptic involution:

$$(\eta,\mu) \in \Gamma_3 : \quad \det(\mathcal{L}_3(\eta) - \mu\,\mathbb{1}) = 0 \;\Leftrightarrow\; \mu^2 + \det(\mathcal{L}_3(\eta)) = 0.$$

This spectrality property provides an explicit formula allowing us to express q in terms of the old variables [41, 53]. Because $\det \mathcal{M}_\eta(\eta) = 0$, the matrix $\mathcal{M}_\eta(\eta)$ has a one-dimensional kernel:

$$\mathcal{M}_\eta(\eta)\,\Omega = \begin{pmatrix} r\,q & r \\ q & 1 \end{pmatrix} \Omega = 0 \qquad \Rightarrow \qquad \Omega = \begin{pmatrix} 1 \\ -q \end{pmatrix}.$$

The equality $\mathcal{M}_\eta(\eta)\,\mathcal{L}_3(\eta)\,\Omega = \widehat{\mathcal{L}}_3(\eta;\eta)\,\mathcal{M}_\eta(\eta)\,\Omega$ implies that $\mathcal{L}_3(\eta)\,\Omega \sim \Omega$, so that Ω is an eigenvector of $\mathcal{L}_3(\eta)$. Fixing the corresponding point of the spectrum as $P \doteq (\eta,\mu) \in \Gamma_3$, we get

$$\begin{pmatrix} u(\eta) - \mu & v(\eta) \\ w(\eta) & -u(\eta) - \mu \end{pmatrix} \begin{pmatrix} 1 \\ -q \end{pmatrix} = 0.$$

This gives us the formula for the variable q:

$$q = \frac{u(\eta) - \mu}{v(\eta)} = -\frac{w(\eta)}{u(\eta) + \mu}, \tag{3.11}$$

where η and μ are bounded by the algebraic curve (3.7).

[1] Fixing the simplest case of a linear function $\mathcal{M}_\eta(\lambda) \doteq \mathcal{M}_1\,\lambda + \mathcal{M}_0$ and taking the limit $\lambda \to \infty$ in Eq. (3.8) it is possible to show that $\mathcal{M}_1$ must be diagonal. Moreover, the most elementary one-point BT should correspond to the case when $\det \mathcal{M}_\eta(\lambda)$ has only one zero $\lambda = \eta$, which will lead to having only one Bäcklund parameter. So we can choose

$$\mathcal{M}_1 \doteq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \mathcal{M}_1 \doteq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

These two matrices produce similar BTs: they are different just in the direction of the discrete-time.

Proposition 3.1 *The similarity transform (3.8) defines a complex BT* $\mathcal{B}_\eta : \mathbb{R}^9 \to \mathbb{C}^9$ *that maps the variables* $(\mathbf{m}, \mathbf{a}, \mathbf{b}) \in \mathbb{R}^9$ *into* $(\widehat{\mathbf{m}}, \widehat{\mathbf{a}}, \widehat{\mathbf{b}}) \in \mathbb{C}^9$ *according to the following relations:*

$$
\begin{aligned}
\widehat{m}^3 &= m^3, \\
\widehat{m}^- &= a^- + (r\,q - \eta)\, m^- - 2\, r\, m^3, \\
\widehat{m}^+ &= 2\, q\, p, \\
\widehat{a}^3 &= a^3 + r\, m^+ - q\, a^- - q\,(r\,q - \eta)\, m^- + 2\, r\, q\, m^3, \\
\widehat{a}^- &= (2\, r\, q - \eta)\, a^- - 2\, r\, a^3 - r^2\, m^+ + r\, q\,(r\,q - \eta)\, m^- - 2\, r^2\, q\, m^3, \\
\widehat{a}^+ &= m^+ - \frac{q}{r}(r\,q - \eta)\, m^- + 2\, q\, m^3, \\
\widehat{b}^3 &= b^3 - q\, b^- + 2\, r\, q\, a^3 - r\, q^2\, a^- + r\, a^+ + \eta\, q\,(r\,q - \eta)\, m^- + r\,\eta\, m^+ + 2\,\eta\, r\, q\, m^3, \\
\widehat{b}^- &= (2\, r\, q - \eta)\, b^- - 2\, r\, b^3 - 2\, r^2\, q\, a^3 + r^2\, q^2\, a^- - r^2\, a^+ - \eta\, r\, q(r\,q - \eta) m^- - \\
&\quad - r^2\,\eta\, m^+ - 2\,\eta\, r^2\, q\, m^3, \\
\widehat{b}^+ &= 2\, q\, a^3 - q^2\, a^- + a^+ + \eta\, \frac{q}{r}(r\,q - \eta)\, m^- + \eta\, m^+ + 2\,\eta\, q\, m^3.
\end{aligned}
\tag{3.12}
$$

Proof: A straightforward computation based on Eq. (3.8) leads to the following equations:

$$
\begin{aligned}
\widehat{u}(\lambda) &= \frac{(\lambda - \eta + 2\, r\, q)[\, u(\lambda) - q\, v(\lambda)] + r\, w(\lambda)}{\lambda - \eta}, \\
\widehat{v}(\lambda) &= \frac{(\lambda - \eta + 2\, r\, q)^2 v(\lambda) - 2\, r\,(\lambda - \eta + 2\, r\, q) u(\lambda) - r^2\, w(\lambda)}{\lambda - \eta}, \\
\widehat{w}(\lambda) &= \frac{w(\lambda) + 2\, q\, u(\lambda) - q^2\, v(\lambda)}{\lambda - \eta}.
\end{aligned}
$$

Collecting the negative powers of λ, the above formulae give Eqs. (3.12). It is not possible to fix a condition on the parameter η in order to get a real map. Hence the one-point BT (3.12) is a map from $\mathbb{R}^9$ to $\mathbb{C}^9$.

□

The following statement shows how the one-point BT can be written in a symplectic form through a generating function. We restrict our BT $\mathcal{B}_\eta$ to a symplectic leaf of the Lie-Poisson structure by fixing the values of the Casimir functions $C_0^{(3)}, C_1^{(3)}, C_2^{(3)}$:

$$
\mathcal{O} \doteq \left\{ (\mathbf{m}, \mathbf{a}, \mathbf{b}) \in \mathbb{R}^9 |\; C_0^{(3)} \equiv \gamma_1,\; C_1^{(3)} \equiv \gamma_2,\; C_2^{(3)} \equiv 1/2,\, (\gamma_1, \gamma_2) \in \mathbb{R}^2 \right\}. \tag{3.13}
$$

Let us fix the following notation:

$$
\chi^3 \doteq (m^3, a^3, b^3)^T, \qquad \chi^\pm \doteq (m^\pm, a^\pm, b^\pm)^T.
$$

We denote with χ_i^α, $i = 1, 2, 3$, $\alpha = \pm, 3$ the i-th component of the vector χ^α and with ∇^i, $i = 1, 2, 3$, the i-th component of the gradient w.r.t. its subscript.

Proposition 3.2 *The map* $\mathcal{B}_\eta|_{\mathcal{O}}$*, namely the BT (3.12) restricted to the symplectic leaf in Eq. (3.13), admits the following formulation:*

$$
\chi_i^3 = \mathrm{i} \sum_{j=1}^{3} \{\chi_i^3, \chi_j^-\}\, \nabla^j_{\chi^-} F_\eta(\chi^- | \widehat{\chi}^+), \tag{3.14a}
$$

$$
\widehat{\chi}_i^3 = \mathrm{i} \sum_{j=1}^{3} \{\widehat{\chi}_j^+, \widehat{\chi}_i^3\}\, \nabla^j_{\widehat{\chi}^+} F_\eta(\chi^- | \widehat{\chi}^+), \tag{3.14b}
$$

with $i = 1,2,3$, *where* $F_\eta(\chi^-|\widehat{\chi}^+)$ *is the following generating function:*

$$\begin{aligned} F_\eta(\chi^-|\widehat{\chi}^+) \;\doteq\; & \frac{m^-\,\widehat{m}^+}{2\,p} + k\left(\frac{m^-}{b^-} + \frac{\widehat{m}^+}{\widehat{b}^+}\right) - \frac{(1+\eta\,\gamma_2)^2}{4\,k\,\eta^2} + \frac{1}{2}\left(\frac{\gamma_2^2}{4} - \gamma_1\right)\ln\left(\frac{k+1}{k-1}\right) - \\ & -\frac{1}{2\,k}\left[\widehat{b}^+\,a^- + b^-\,\widehat{a}^+ - \eta\,\widehat{a}^+\,a^- + \frac{a^-}{b^-}\left(\frac{a^-}{b^-} + \frac{\eta}{2}a^-\,\widehat{b}^+ - \gamma_2\right) + \right. \\ & \left. + \frac{\widehat{a}^+}{\widehat{b}^+}\left(\frac{\widehat{a}^+}{\widehat{b}^+} + \frac{\eta}{2}\widehat{a}^+\,b^- - \gamma_2\right)\right], \end{aligned} \tag{3.15}$$

with $k^2 \doteq 1 + \eta\, b^-\,\widehat{b}^+$.

Proof: The Casimir functions $C_0^{(3)}, C_1^{(3)}, C_2^{(3)}$ do not change under the map:

$$\mathcal{B}_\eta:\; (C_0^{(3)}, C_1^{(3)}, C_2^{(3)}) \longmapsto (\widehat{C}_0^{(3)}, \widehat{C}_1^{(3)}, \widehat{C}_2^{(3)}) = (C_0^{(3)}, C_1^{(3)}, C_2^{(3)}).$$

Fixing the values of such functions as $C_0^{(3)} \equiv \gamma_1,\; C_1^{(3)} \equiv \gamma_2,\; C_2^{(3)} \equiv 1,\; (\gamma_1,\gamma_2) \in \mathbb{R}^2$, the above invariance allows one to exclude six variables, expressing m^+, a^+, b^+ and $\widehat{m}^-, \widehat{a}^-, \widehat{b}^-$ in term of the components of the vectors χ^3, χ^- and $\widehat{\chi}^3, \widehat{\chi}^+$:

$$\begin{aligned} m^+ &= \frac{1}{b^-}\left[2\,\gamma_1 - 1 - 2\,m^3\,b^3 - \frac{m^-}{b^-}(1-(b^3)^2)\right], \\ a^+ &= \frac{1}{b^-}\left[2\,\gamma_2 - 2\,a^3\,b^3 - \frac{a^-}{b^-}(1-(b^3)^2)\right], \\ b^+ &= \frac{1-(b^3)^2}{b^-}, \\ \widehat{m}^- &= \frac{1}{\widehat{b}^+}\left[2\,\gamma_1 - 1 - 2\,\widehat{m}^3\,\widehat{b}^3 - \frac{\widehat{m}^+}{\widehat{b}^+}(1-(\widehat{b}^3)^2)\right], \\ \widehat{a}^- &= \frac{1}{\widehat{b}^+}\left[2\,\gamma_2 - 2\,\widehat{a}^3\,\widehat{b}^3 - \frac{\widehat{a}^+}{\widehat{b}^+}(1-(\widehat{b}^3)^2)\right], \\ \widehat{b}^- &= \frac{1-(\widehat{b}^3)^2}{\widehat{b}^+}. \end{aligned}$$

Using Eqs. (3.10) we can rewrite Eqs. (3.12) in the following form:

$$
\begin{aligned}
\chi_1^3 &= \frac{m^- m^+}{2p} - \frac{1}{2k^3}\left\{\frac{1}{2}\left[\eta\left(b^-\widehat{a}^+ + a^-\widehat{b}^+\right) - b^-\widehat{b}^+\right]^2 + \right.\\
&+\eta^2\left[(b^-)^2\widehat{b}^+\widehat{m}^+ + (\widehat{b}^+)^2 b^- m^-\right] + \left(1+\frac{1}{2}\eta\,\gamma_2\right)\left(b^-\widehat{a}^+ + a^-\widehat{b}^+\right) -\\
&-(\gamma_2 + 2\,\eta\,\gamma_1)\,b^-\widehat{b}^+ - \eta\left(a^-\widehat{a}^+ + b^-\widehat{m}^+ + \widehat{b}^+ m^-\right) +\\
&\left.+\frac{1}{4}\left(\gamma_2^2 - 4\,\gamma_1\right)\right\},\\
\chi_2^3 &= \frac{1}{2k}\left[\widehat{m}^+ a^- + \eta\left(a^-\widehat{b}^+ + \widehat{a}^+ b^-\right) - \widehat{b}^+ b^- + \gamma_2\right],\\
\chi_3^3 &= \frac{\widehat{m}^+ b^-}{2p} + k,\\
\widehat{\chi}_1^3 &= m^3,\\
\widehat{\chi}_2^3 &= \frac{1}{2k}\left[\widehat{a}^+ m^- + \eta\left(a^-\widehat{b}^+ + \widehat{a}^+ b^-\right) - \widehat{b}^+ b^- + \gamma_2\right],\\
\widehat{\chi}_3^3 &= \frac{\widehat{b}^+ m^-}{2p} + k,
\end{aligned}
$$

where $k^2 \doteq 1 + \eta\, b^- \widehat{b}^+$. It is now easy to check that the above equations are equivalent to Eqs. (3.14a) and (3.14b) with the generating function (3.15).

□

Remark 3.2 *The spectrality property of a BT means that the two coordinates η and μ parametrizing the map are conjugated variables, namely $\mu = -\partial F_\eta/\partial\eta$, where F_η is the generating function of the BT [55, 56, 87].*

In our case, using Eqs. (3.10), (3.11) and (3.15), we obtain

$$\mu = u(\eta) - \frac{\widehat{m}^+}{2p}\, v(\eta) = -\frac{\partial F_\eta(\chi^-|\widehat{\chi}^+)}{\partial\eta},$$

so that the spectrality property holds.

In the next paragraph we shall construct a real BT for our integrable system.

Two-point BTs

According to [41, 53, 87], we now construct a composite map which is a product of the map $\mathcal{B}_{P_1} \doteq \mathcal{B}_{(\eta_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \doteq \mathcal{B}_{(\eta_2,-\mu_2)}$:

$$\mathcal{B}_{P_1,Q_2} \doteq \mathcal{B}_{Q_2}\circ\mathcal{B}_{P_1} :\ \mathcal{L}_3(\lambda) \overset{\mathcal{B}_{P_1}}{\longmapsto} \widehat{\mathcal{L}}_3(\lambda;\eta_1) \overset{\mathcal{B}_{Q_2}}{\longmapsto} \widehat{\widehat{\mathcal{L}}}_3(\lambda;\eta_1,\eta_2).$$

The two maps are inverse to each other when $\eta_1 = \eta_2$ and $\mu_1 = \mu_2$. This two-point BT is defined by the following discrete-time Lax equation:

$$\mathcal{M}_{\eta_1,\eta_2}(\lambda)\,\mathcal{L}_3(\lambda) = \widehat{\widehat{\mathcal{L}}}_3(\lambda;\eta_1,\eta_2)\,\mathcal{M}_{\eta_1,\eta_2}(\lambda), \qquad \forall\,\lambda\in\mathbb{C}, \qquad (\eta_1,\eta_2)\in\mathbb{C}^2, \tag{3.16}$$

where the matrix $\mathcal{M}_{\eta_1,\eta_2}(\lambda)$ is [41, 53, 87]

$$\mathcal{M}_{\eta_1,\eta_2}(\lambda) \doteq \begin{pmatrix} \lambda - \eta_1 + s\,t & t \\ -s^2\,t + (\eta_1-\eta_2)\,s & \lambda - \eta_2 - s\,t \end{pmatrix}, \tag{3.17}$$

with

$$\det \mathcal{M}_{\eta_1,\eta_2}(\lambda) = (\lambda-\eta_1)(\lambda-\eta_2).$$

The spectrality property with respect to two fixed points $(\eta_1, \mu_1) \in \Gamma_3$ and $(\eta_2, \mu_2) \in \Gamma_3$ give

$$s = \frac{u(\eta_1)-\mu_1}{v(\eta_1)} = \frac{\widehat{\widehat{u}}(\eta_2)-\mu_2}{\widehat{\widehat{v}}(\eta_2)} = \frac{\widehat{m}^+}{2\,p}, \tag{3.18}$$

$$\begin{aligned} t &= \frac{(\eta_2-\eta_1)\,v(\eta_1)\,v(\eta_2)}{[\,u(\eta_2)+\mu_2\,]\;v(\eta_1) - [\,u(\eta_1)-\mu_1\,]\;v(\eta_2)} = \\ &= \frac{(\eta_2-\eta_1)\,\widehat{\widehat{v}}(\eta_1)\,\widehat{\widehat{v}}(\eta_2)}{\left[\widehat{\widehat{u}}(\eta_1)+\mu_1\right]\,\widehat{\widehat{v}}(\eta_2) - \left[\widehat{\widehat{u}}(\eta_2)-\mu_2\right]\,\widehat{\widehat{v}}(\eta_1)} = \\ &= \frac{(\eta_1-\eta_2)\;[\,u(\eta_1)+\mu_1\,]\;[\,u(\eta_2)-\mu_2\,]}{[\,u(\eta_1)+\mu_1\,]\;w(\eta_2) - [\,u(\eta_2)-\mu_2\,]\;w(\eta_1)} = \\ &= \frac{(\eta_1-\eta_2)\;\left[\widehat{\widehat{u}}(\eta_1)-\mu_1\right]\;\left[\widehat{\widehat{u}}(\eta_2)+\mu_2\right]}{\left[\widehat{\widehat{u}}(\eta_2)+\mu_2\right]\;\widehat{\widehat{w}}(\eta_1) - \left[\widehat{\widehat{u}}(\eta_1)-\mu_1\right]\;\widehat{\widehat{w}}(\eta_2)} = \\ &= \frac{m^- - \widehat{\widehat{m}}^-}{2\,p}. \end{aligned} \tag{3.19}$$

Now we have two complex Bäcklund parameters η_1, η_2. It is possible to obtain several equivalent formulae [41, 53] for the variables s and t since the points (η_1, μ_1) and (η_2, μ_2) belong to the spectral curve Γ_3, i.e. are bound by the following relations

$$-(2\,\mu_j)^2 = u^2(\eta_j) + v(\eta_j)\,w(\eta_j) = \widehat{\widehat{u}}^2(\eta_j) + \widehat{\widehat{v}}(\eta_j)\,\widehat{\widehat{w}}(\eta_j), \qquad j = 1, 2. \tag{3.20}$$

Together with Eqs. (3.18) and (3.19), the formula (3.16) gives an explicit two-point Poisson integrable map from $\mathcal{L}_3(\lambda)$ to $\widehat{\widehat{\mathcal{L}}}_3(\lambda; \eta_1, \eta_2)$. The map is parametrized by the two points P_1 and Q_2. Obviously, when $\eta_1 = \eta_2$ (i.e. $\mu_1 = \mu_2$) the map turns into an identity map. As we have explicitly shown in [53, 73] the two-point BT can be reduced to a real Poisson integrable map if the following condition holds:

$$\eta_1 = \bar{\eta}_2 \doteq \eta = \mathfrak{Re}(\eta) + \mathrm{i}\,\mathfrak{Im}(\eta) \in \mathbb{C}.$$

Therefore, the two-point map leads to a physical BT $\mathcal{B}_\eta$ with two real parameters. Notice that the physical time step can be taken as $\mathrm{i}\,(\eta_2-\eta_1)/2 = \mathfrak{Im}(\eta)$.

A direct computation based on the similarity transform (3.16) shows that the following statement holds.

Proposition 3.3 *The similarity transform (3.16) defines a real BT $\mathcal{B}_\eta : \mathbb{R}^9 \to \mathbb{R}^9$ that maps the vector $M \doteq (\mathbf{m}, \mathbf{a}, \mathbf{b})^T \in \mathbb{R}^9$ into $\widehat{\widehat{M}} \doteq (\widehat{\widehat{\mathbf{m}}}, \widehat{\widehat{\mathbf{a}}}, \widehat{\widehat{\mathbf{b}}})^T \in \mathbb{R}^9$ according to the following compact formula:*

$$\widehat{\widehat{M}} = \Phi_\eta(s,t)\,M + M_0(s,t;\eta), \tag{3.21}$$

with

$$\Phi_\eta(s,t) \doteq \begin{pmatrix} \mathbb{1}_{3\times3} & \mathbb{O}_{3\times3} & \mathbb{O}_{3\times3} \\ A_\eta(s,t) & \mathbb{1}_{3\times3} & \mathbb{O}_{3\times3} \\ B_\eta(s,t) & A_\eta(s,t) & \mathbb{1}_{3\times3} \end{pmatrix},$$

where $A_\eta(s,t)$ and $B_\eta(s,t)$ are two 3×3 matrices depending on the Bäcklund parameter η and the parameters s, t and $M_0(s,t;\eta)$ is a vector depending on the dynamical variables. The *matrices* $\mathbb{1}_{3\times 3}$ *and* $\mathbb{O}_{3\times 3}$ *are respectively the* 3×3 *identity matrix and the* 3×3 *zero matrix. The entries of* $A_\eta(s,t)$ *and* $B_\eta(s,t)$ *are respectively*

$$
\begin{aligned}
&[\,A_\eta(s,t)\,]_{ii} = 0, \quad i = 1,2,3,\\
&[\,A_\eta(s,t)\,]_{12} = \mathrm{i}\,\alpha_2 = -[\,A_\eta(s,t)\,]_{21},\\
&[\,A_\eta(s,t)\,]_{13} = s\,\alpha_1 - t = -[\,A_\eta(s,t)\,]_{31},\\
&[\,A_\eta(s,t)\,]_{23} = -\mathrm{i}\,(s\,\alpha_1 + t) = -[\,A_\eta(s,t)\,]_{32},
\end{aligned}
$$

$$
\begin{aligned}
[\,B_\eta(s,t)\,]_{11} &= \frac{1}{2}(1-s^2)(\alpha_1^2 - t^2),\\
[\,B_\eta(s,t)\,]_{12} &= -\frac{\mathrm{i}}{2}[\,(t^2 - s^2\,\alpha_1^2) - 2\,\alpha_2\,\mathfrak{Re}(\eta)\,],\\
[\,B_\eta(s,t)\,]_{13} &= \frac{1}{2}s\,\alpha_1\,(\alpha_2 + \mathfrak{Re}(\eta)),\\
[\,B_\eta(s,t)\,]_{21} &= -\frac{\mathrm{i}}{2}[\,(t^2 - s^2\,\alpha_1^2) + 2\,\alpha_2\,\mathfrak{Re}(\eta)\,],\\
[\,B_\eta(s,t)\,]_{22} &= \frac{1}{2}(1+s^2)(\alpha_1^2 - t^2)\\
[\,B_\eta(s,t)\,]_{23} &= -\frac{\mathrm{i}}{2}s\,\alpha_1\,(\alpha_2 + \mathfrak{Re}(\eta)),\\
[\,B_\eta(s,t)\,]_{31} &= \frac{1}{2}\left[\,t\,(\alpha_2 + 2\,\mathfrak{Re}(\eta)) + s\,\alpha_1\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\,\right],\\
[\,B_\eta(s,t)]_{32} &= \frac{1}{2}\left[\,t\,(\alpha_2 + 2\,\mathfrak{Re}(\eta)) - s\,\alpha_1\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\,\right],\\
[\,B_\eta(s,t)]_{33} &= 4\,s\,t\,\alpha_1,
\end{aligned}
$$

where $\alpha_1 \doteq 2\,\mathfrak{Im}(\eta) - s\,t$, $\alpha_2 \doteq \alpha_1 - s\,t$. *The components of the vector* $M_0(s,t;\eta)$ *are*

$$
\begin{aligned}
[\,M_0(s,t;\eta)\,]_1 &= p\,(s\,\alpha_1 - t),\\
[\,M_0(s,t;\eta)\,]_2 &= -p\,(s\,\alpha_1 + t),\\
[\,M_0(s,t;\eta)\,]_3 &= 0,\\
[\,M_0(s,t;\eta)\,]_4 &= p\,\left[\,\frac{s\,\alpha_1}{2}(\alpha_2 + \mathfrak{Re}(\eta)) + t\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\,\right],\\
[\,M_0(s,t;\eta)\,]_5 &= \mathrm{i}\,p\,\left[\,\frac{s\,\alpha_1}{2}(\alpha_2 + \mathfrak{Re}(\eta)) - t\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\,\right],\\
[\,M_0(s,t;\eta)\,]_6 &= 2\,p\,t\,s\,\alpha_1,\\
[\,M_0(s,t;\eta)\,]_7 &= -\frac{\mathrm{i}\,p}{4}\left[\,(\mathfrak{Im}(\eta) + \mathfrak{Re}(\eta))^2(t + \alpha_1) + 8\,s\,t\,\mathfrak{Re}(\eta)(t - \alpha_1)\,\right],\\
[\,M_0(s,t;\eta)\,]_8 &= -\frac{p}{4}\left[\,(\mathfrak{Im}(\eta) + \mathfrak{Re}(\eta))^2(t - \alpha_1) + 8\,s\,t\,\mathfrak{Re}(\eta)(t + \alpha_1)\,\right],\\
[\,M_0(s,t;\eta)\,]_9 &= 2\,[\,M_0(s,t;\eta)\,]_6\,\mathfrak{Re}(\eta).
\end{aligned}
$$

Two-point BTs as discrete-time maps

The two point BT constructed above is a one-parameter (η) time discretization of a family of flows parametrized by the points $P_1 \doteq (\eta_1, \mu_1)$ and $Q_2 \doteq (\lambda_2, -\mu_2)$. Recall that the physical time step can be taken as $\mathrm{i}\,(\eta_2 - \eta_1)/2 = \mathfrak{Im}(\eta)$.

Let us consider the following limit: $\mathfrak{Im}(\eta) \doteq \epsilon \to 0$. Considering Eqs. (3.18) and (3.19) we immediately get

$$
\begin{aligned}
s &= s_0 + O(\epsilon), \qquad s_0 \doteq \frac{u(\eta_2) - \mu_2}{v(\eta_2)}, \\
t &= \epsilon\, t_0 + O(\epsilon^2), \qquad t_0 \doteq -\frac{\mathrm{i}\, v(\eta_2)}{\mu_2}\,.
\end{aligned}
$$

The matrix $\mathcal{M}_{\eta_2}(\lambda)$ given in Eq. (3.17) has the following asymptotics:

$$
\mathcal{M}_{\eta_2}(\lambda) = (\lambda - \eta_2)\left[\mathbb{1}_{2\times 2} + \frac{\mathrm{i}\,\epsilon}{\mu_2\,(\lambda - \eta_2)}\begin{pmatrix} u(\lambda_2) + \mu_2 & v(\lambda_2) \\ w(\lambda_2) & -u(\lambda_2) + \mu_2 \end{pmatrix}\right] + O(\epsilon^2).
$$

If we define the time derivative $\dot{\mathcal{L}}_3(\lambda)$ as $\dot{\mathcal{L}}_3(\lambda) \doteq \lim_{\epsilon\to 0}[\,\widehat{\mathcal{L}_3}(\lambda) - \mathcal{L}_3(\lambda)]/\epsilon$, then in this limit we obtain from Eq. (3.16) the Lax equation for the corresponding continuous flow that our BT discretizes, namely:

$$
\dot{\mathcal{L}}_3(\lambda) = \frac{\mathrm{i}}{\mu_2}\left[\frac{\mathcal{L}_3(\eta_2)}{(\lambda - \eta_2)}, \mathcal{L}_3(\lambda)\right].
$$

that is a Hamiltonian flow with μ_2 given in Eq. (3.20).

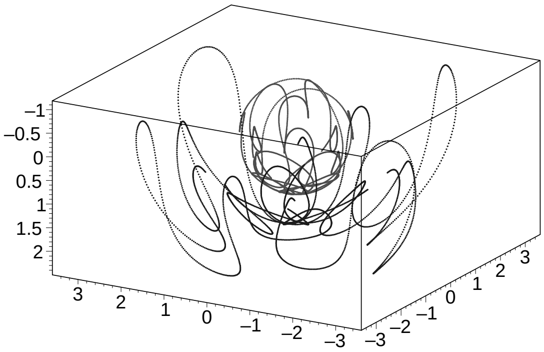
Numerics

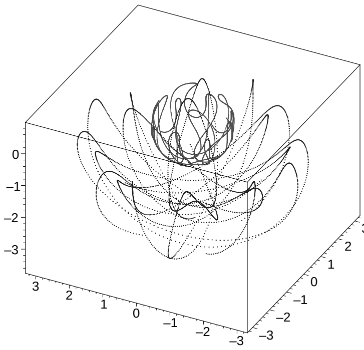
We now present some 3D plots corresponding to the real reduction of the two-point BT (3.21). They are obtained using a MAPLE 8 program that is a slightly different version of the MATLAB program developed by V.B. Kuznetsov in [53].

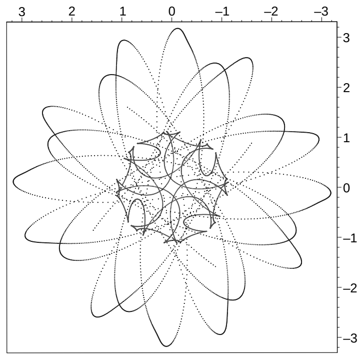
The input parameters are:

- the intensity of the external field, p;
- the Bäcklund parameter $\eta \doteq \mathfrak{Re}(\eta) + \mathrm{i}\,\mathfrak{Im}(\eta)$. Here $\mathfrak{Im}(\eta)$ is the time-step of the discretization;
- the number of iteration of the map, N;
- the initial values of the coordinate functions, $M \doteq (\mathbf{m}, \mathbf{a}, \mathbf{b})$.

The output is a 3D plot of $N + N$ consequent points $(a^1 - b^1, a^2 - b^2, a^3 - b^3)$ and (b^1, b^2, b^3). We remark that the vector $(a^1 - b^1, a^2 - b^2, a^3 - b^3)$ describes the position of the satellite (of unitary mass), as explained in Subsection 2.4.2, and the vector (b^1, b^2, b^3) is the position of the centre of mass of the Lagrange top (of unitary mass). As expected, the points (b^1, b^2, b^3) lie on the sphere $C_2^{(3)} \doteq \langle\, \mathbf{b}, \mathbf{b}\,\rangle$ =constant, of some radius defined by the initial data.







3.1.2 BTs for the rational Lagrange chain

The continuous-time rational Lagrange chain is described in the previuos Chapter, see Subsection 2.5.2. Let us recall the main features of this integrable many-body model.

The Lax matrix is given by, see Eq. (2.91):

$$\mathcal{L}_{M,2}(\lambda) \doteq \mathbf{p} + \sum_{i=1}^{M} \left[\frac{\mathbf{m}_i}{\lambda - \lambda_i} + \frac{\mathbf{a}_i}{(\lambda - \lambda_i)^2} \right], \tag{3.22}$$

where $\mathbf{p} \doteq (0, 0, p)$ as in the Lagrange case, and the $6M$ local coordinates obey to the Lie-Poisson algebra $\oplus^M \mathfrak{e}^*(3)$:

$$\left\{ m_k^\alpha, m_j^\beta \right\} = -\delta_{k,j}\, \varepsilon_{\alpha\beta\gamma}\, m_k^\gamma, \quad \left\{ m_k^\alpha, a_j^\beta \right\} = -\delta_{k,j}\, \varepsilon_{\alpha\beta\gamma}\, a_k^\gamma, \quad \left\{ a_k^\alpha, a_j^\beta \right\} = 0, \tag{3.23}$$

with $1 \le k, j \le M$. According to Proposition 2.14 the Lax matrix (3.22) satisfies a linear r-matrix algebra with r-matrix given by $r(\lambda) = -\Pi/(2\,\lambda)$, where Π is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$.

Moreover, we know from Proposition 2.21 that the spectral curve $\Gamma_{M,2} : \det(\mathcal{L}_{M,2}(\lambda) - \mu\,\mathbb{1}) = 0$ reads

$$\Gamma_{M,2} : \; -\mu^2 = \frac{1}{4}\langle \mathbf{p}, \mathbf{p} \rangle + \frac{1}{2} \sum_{k=1}^{M} \left[\frac{R_k}{\lambda - \lambda_k} + \frac{S_k}{(\lambda - \lambda_k)^2} + \frac{C_k^{(1)}}{(\lambda - \lambda_k)^3} + \frac{C_k^{(2)}}{(\lambda - \lambda_k)^4} \right], \tag{3.24}$$

where

$$R_k \doteq \langle\, \mathbf{p}, \mathbf{m}_k \,\rangle + \sum_{\substack{j=1\\ j\neq k}}^{M} \left[\frac{\langle\, \mathbf{m}_k, \mathbf{m}_j \,\rangle}{\lambda_k - \lambda_j} + \frac{\langle\, \mathbf{m}_k, \mathbf{a}_j \,\rangle - \langle\, \mathbf{m}_j, \mathbf{a}_k \,\rangle}{(\lambda_k - \lambda_j)^2} - 2\,\frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\lambda_k - \lambda_j)^3} \right],$$

$$S_k \doteq \langle\, \mathbf{p}, \mathbf{a}_k \,\rangle + \frac{1}{2}\langle\, \mathbf{m}_k, \mathbf{m}_k \,\rangle + \sum_{\substack{j=1\\ j\neq k}}^{M} \left[\frac{\langle\, \mathbf{a}_k, \mathbf{m}_j \,\rangle}{\lambda_k - \lambda_j} + \frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\lambda_k - \lambda_j)^2} \right],$$

are $2M$ independent involutive Hamiltonians and

$$C_k^{(1)} \doteq \langle\, \mathbf{m}_k, \mathbf{a}_k \,\rangle, \qquad C_k^{(2)} \doteq \frac{1}{2}\langle\, \mathbf{a}_k, \mathbf{a}_k \,\rangle, \qquad 1 \le k \le M, \tag{3.26}$$

are the $2M$ Casimir functions of the Lie-Poisson brackets (3.23). Fixing their values one gets a $2M$-dimensional symplectic leaf

$$\mathcal{O} \doteq \left\{ (\mathbf{m}_k, \mathbf{a}_k) \in \mathbb{R}^6,\ 1 \le k \le M \mid C_k^{(1)} \equiv \ell_i,\ C_k^{(2)} \equiv 1/2 \right\}. \tag{3.27}$$

In the construction of BTs we shall use the complex variables $m_i^{\pm} \doteq m_i^1 \pm \mathrm{i}\, m_i^2$, $a_i^{\pm} \doteq a_i^1 \pm \mathrm{i}\, a_i^2$, $1 \le i \le M$. In terms of such variables the non-trivial Lie-Poisson brackets (3.23) read

$$\{m_i^3, m_j^{\pm}\} = \pm\, \delta_{i,j}\, m_j^{\pm}, \qquad \{m_i^+, m_j^-\} = 2\,\mathrm{i}\, \delta_{i,j}\, m_j^3,$$
$$\{m_i^3, a_j^{\pm}\} = \{a_i^3, m_j^{\pm}\} = \pm\,\mathrm{i}\, \delta_{i,j}\, a_j^{\pm}, \qquad \{m_i^+, a_j^-\} = \{a_i^+, m_j^-\} = 2\,\mathrm{i}\, \delta_{i,j}\, a_j^3.$$

Using the above complex variables we can write the Lax matrix (3.22) in the following form:

$$\mathcal{L}_{M,2}(\lambda) \doteq -\frac{\mathrm{i}}{2} \begin{pmatrix} u(\lambda) & v(\lambda) \\ w(\lambda) & -u(\lambda) \end{pmatrix}, \tag{3.28}$$

with

$$u(\lambda) \doteq p + \sum_{i=1}^{M} \left[\frac{m_i^3}{\lambda - \lambda_i} + \frac{a_i^3}{(\lambda - \lambda_i)^2} \right],$$
$$v(\lambda) \doteq \sum_{i=1}^{M} \left[\frac{m_i^-}{\lambda - \lambda_i} + \frac{a_i^-}{(\lambda - \lambda_i)^2} \right],$$
$$w(\lambda) \doteq \sum_{i=1}^{M} \left[\frac{m_i^+}{\lambda - \lambda_i} + \frac{a_i^+}{(\lambda - \lambda_i)^2} \right].$$

The construction of one-, and two-point BTs is essentially the same that we have performed in the case of the first extension of the Lagrange top in the previous Subsection. As a matter of fact we can use the "universal" ansatzes given in Eqs. (3.9) and (3.17) for the intertwining matrix. Moreover, setting $M = 1$ in the following discretization we recover the discrete-time Lagrange top considered in our paper [53].

One-point BTs

A one-point BT can be constructed performing the following similarity transform:

$$\mathcal{B}_\eta:\ \mathcal{L}_{M,2}(\lambda) \longmapsto \widehat{\mathcal{L}}_{M,2}(\lambda;\eta) \doteq \mathcal{M}_\eta(\lambda)\, \mathcal{L}_{M,2}(\lambda)\, \mathcal{M}_\eta^{-1}(\lambda) = -\frac{\mathrm{i}}{2} \begin{pmatrix} \widehat{u}(\lambda) & \widehat{v}(\lambda) \\ \widehat{w}(\lambda) & -\widehat{u}(\lambda) \end{pmatrix}, \tag{3.30}$$

where we use again the $\widehat{\ }$-notation for the updated variables. Here the intertwining matrix $\mathcal{M}_\eta(\lambda)$ is the one given in Eq. (3.9), where the variables r and q are "a priori" indeterminate dynamical variables.

Comparing the asymptotics in $\lambda \to \infty$ in both sides of Eq. (3.30) we readily get

$$r = \frac{1}{2\,p}\sum_{i=1}^{M} m_i^-, \qquad q = \frac{1}{2\,p}\sum_{i=1}^{M} \widehat{m}_i^+. \tag{3.31}$$

We are looking for an explicit single-valued map from $\mathcal{L}_{M,2}(\lambda)$ to $\widehat{\mathcal{L}}_{M,2}(\lambda;\eta)$. Hence we can proceed as in the previuos Subsection using the spectrality property of the BTs. Eq. (3.30) defines a map $\mathcal{B}_P$ parametrized by the point $P \doteq (\eta,\mu) \in \Gamma_{M,2}$, see Eq. (3.24): there are two points on $\Gamma_{M,2}$, $P \doteq (\eta,\mu)$ and $Q \doteq (\eta,-\mu)$, corresponding to the same η and sitting one above the other because of the hyperelliptic involution:

$$(\eta,\mu) \in \Gamma_{M,2}: \quad \det(\mathcal{L}_{M,2}(\eta) - \mu\,\mathbb{1}) = 0 \;\Leftrightarrow\; \mu^2 + \det(\mathcal{L}_{M,2}(\eta)) = 0.$$

This spectrality property give us the formula

$$q = \frac{u(\eta)-\mu}{v(\eta)} = -\frac{w(\eta)}{u(\eta)+\mu}. \tag{3.32}$$

Now Eq. (3.30) gives an integrable Poisson map from $\mathcal{L}(\lambda)$ to $\widehat{\mathcal{L}}(\lambda;\eta)$.

Let us introduce the following notation:

$$M^\alpha \doteq \{m_i^\alpha\}_{i=1}^M, \qquad A^\alpha \doteq \{m_i^\alpha\}_{i=1}^M, \qquad \alpha = \pm, 3.$$

The following statement shows how the one-point BT obtained in Eq. (3.30) can be written in a symplectic form through a generating function.

Proposition 3.4 *The map $\mathcal{B}_\eta|_{\mathcal{O}}$, namely the BT (3.30) restricted to the symplectic leaf in Eq. (3.27), admits the following formulation:*

$$a_i^3 = a_i^- \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial m_i^-}, \tag{3.33a}$$

$$m_i^3 = a_i^- \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial a_i^-} + m_i^- \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial m_i^-}, \tag{3.33b}$$

$$\widehat{a}_i^3 = \widehat{a}_i^+ \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial \widehat{m}_i^+}, \tag{3.33c}$$

$$\widehat{m}_i^3 = \widehat{a}_i^+ \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial \widehat{a}_i^+} + \widehat{m}_i^+ \,\frac{\partial F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)}{\partial \widehat{m}_i^+}, \tag{3.33d}$$

where $F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+)$ is the following generating function:

$$\begin{aligned} F_\eta(A^-,M^- \mid \widehat{A}^+,\widehat{M}^+) \;\doteq\; & \frac{1}{2\,p}\sum_{i,j=1}^{M} m_i^-\,\widehat{m}_j^+ + \sum_{i=1}^{M} k_i\left(\frac{m_i^-}{a_i^-} + \frac{\widehat{m}_i^+}{\widehat{a}_i^+} - \frac{1}{\eta-\lambda_i}\right) - \\ & - \log\prod_{i=1}^{M}\left(\frac{1+k_i}{1-k_i}\right)^{\ell_i}, \end{aligned} \tag{3.34}$$

with $k_i^2 \doteq 1 + (\eta-\lambda_i)\,a_i^-\,\widehat{a}_i^+$.

Proof: First, because the Casimir functions given in Eq. (3.26) do not change under the map, namely

$$m_i^3\,a_i^3 + \frac{1}{2}\left(m_i^-\,a_i^+ + m_i^+\,a_i^-\right) = \widehat{m}_i^3\,\widehat{a}_i^3 + \frac{1}{2}\left(\widehat{m}_i^-\,\widehat{a}_i^+ + \widehat{m}_i^+\,\widehat{a}_i^-\right) \equiv \ell_i,$$

$$(a_i^3)^2 + a_i^-\,a_i^+ = (\widehat{a}_i^3)^2 + \widehat{a}_i^-\widehat{a}_i^+ \equiv 1,$$

we can exclude the $4M$ variables a_i^+, m_i^+ and $\widehat{a}_i^-, \widehat{m}_i^-$, $1 \le i \le M$, using the following substitutions:

$$a_i^+ = \frac{1-(a_i^3)^2}{a_i^-}, \qquad m_i^+ = \frac{2\,\ell_i}{a_i^-} - \frac{2m_i^3\,a_i^3}{a_i^-} - \frac{m_i^-}{(a_i^-)^2}\,[1-(a_i^3)^2]\,,$$

$$\widehat{a}_i^- = \frac{1-(\widehat{a}_i^3)^2}{\widehat{a}_i^+}, \qquad \widehat{m}_i^+ = \frac{2\,\ell_i}{\widehat{a}_i^+} - \frac{2\widehat{m}_i^3\,\widehat{a}_i^3}{\widehat{a}_i^+} - \frac{\widehat{m}_i^+}{(\widehat{a}_i^+)^2}\,[1-(\widehat{a}_i^3)^2].$$

Now we have $4M + 4M$ (old and new) independent variables: a_i^-, a_i^3, m_i^-, m_i^3 and $\widehat{a}_i^+, \widehat{a}_i^3$, $\widehat{m}_i^+, \widehat{m}_i^3$, $1 \le i \le M$.

The map (3.30) explicitly reads

$$\widehat{u}(\lambda) = \frac{(\lambda-\eta+2\,r\,q)[\,u(\lambda) - q\,v(\lambda)] + r\,w(\lambda)}{\lambda-\eta},$$

$$\widehat{v}(\lambda) = \frac{(\lambda-\eta+2\,r\,q)^2 v(\lambda) - 2\,r\,(\lambda-\eta+2\,r\,q)u(\lambda) - r^2\,w(\lambda)}{\lambda-\eta},$$

$$\widehat{w}(\lambda) = \frac{w(\lambda) + 2\,q\,u(\lambda) - q^2\,v(\lambda)}{\lambda-\eta}.$$

Equating the residues at $\lambda = \lambda_i$ in both sides of the above equations we obtain, after a straightforward computation,

$$a_i^3 = \frac{a_i^-}{2\,p}\sum_{j=1}^M \widehat{m}_j^+ + k_i, \tag{3.35a}$$

$$m_i^3 = \frac{\ell_i}{k_i} + \frac{\eta-\lambda_i}{2\,k_i}\left(\widehat{a}_i^+\,m_i^- + a_i^-\,\widehat{m}_i^+\right) - \frac{a_i^-\,\widehat{a}_i^+}{2\,k_i} + \frac{m_i^-}{2\,p}\sum_{j=1}^M \widehat{m}_j^+\,, \tag{3.35b}$$

$$\widehat{a}_i^3 = \frac{\widehat{a}_i^+}{2\,p}\sum_{j=1}^M m_j^- + k_i, \tag{3.35c}$$

$$\widehat{m}_i^3 = \frac{\ell_i}{k_i} + \frac{\eta-\lambda_i}{2\,k_i}\left(\widehat{a}_i^+\,m_i^- + a_i^-\,\widehat{m}_i^+\right) - \frac{a_i^-\widehat{a}_i^+}{2\,k_i} + \frac{\widehat{m}_i^+}{2\,p}\sum_{j=1}^M m_j^-\,, \tag{3.35d}$$

where $k_i^2 \doteq 1 + (\eta-\lambda_i)\,a_i^-\widehat{a}_i^+$. It is now easy to check that Eqs (3.35a-3.35b-3.35c-3.35d) are equivalent to Eqs. (3.33a-3.33b-3.33c-3.33d) with the generating function (3.34).

□

Remark 3.3 *Let us have a look at the spectrality property of the constructed BT. Using Eqs. (3.31), (3.32) and (3.34) we obtain*

$$\mu = u(\eta) - \left(\frac{1}{2\,p}\sum_{i=1}^M \widehat{m}_i^+\right)\,v(\eta) = -\frac{\partial F_\eta(A^-, M^- \mid \widehat{A}^+, \widehat{M}^+)}{\partial\eta}.$$

The above one-point BT is a complex map. In order to obtain a physical map we shall construct a two-point BT in the next Subsection.

Two-point BTs

According to [41, 53, 87], we construct a composite map which is a product of the map $\mathcal{B}_{P_1} \doteq \mathcal{B}_{(\eta_1,\mu_1)}$ and $\mathcal{B}_{Q_2} \doteq \mathcal{B}_{(\eta_2,-\mu_2)}$:

$$\mathcal{B}_{P_1,Q_2} \doteq \mathcal{B}_{Q_2} \circ \mathcal{B}_{P_1} : \; \mathcal{L}_{M,2}(\lambda) \overset{\mathcal{B}_{P_1}}{\longmapsto} \widehat{\mathcal{L}}_{M,2}(\lambda;\eta_1) \overset{\mathcal{B}_{Q_2}}{\longmapsto} \widehat{\widehat{\mathcal{L}}}_{M,2}(\lambda;\eta_1,\eta_2).$$

The two maps are inverse to each other when $\eta_1 = \eta_2$ and $\mu_1 = \mu_2$. This two-point BT is defined by the following discrete-time Lax equation:

$$\mathcal{M}_{\eta_1,\eta_2}(\lambda)\,\mathcal{L}_{M,2}(\lambda) = \widehat{\widehat{\mathcal{L}}}_{M,2}(\lambda;\eta_1,\eta_2)\,\mathcal{M}_{\eta_1,\eta_2}(\lambda), \qquad \forall\,\lambda \in \mathbb{C}, \qquad (\eta_1,\eta_2) \in \mathbb{C}^2, \tag{3.36}$$

where the matrix $\mathcal{M}_{\eta_1,\eta_2}(\lambda)$ is, see Eq. (3.17),

$$\mathcal{M}_{\eta_1,\eta_2}(\lambda) \doteq \begin{pmatrix} \lambda - \eta_1 + s\,t & t \\ -s^2\,t + (\eta_1 - \eta_2)\,s & \lambda - \eta_2 - s\,t \end{pmatrix},$$

with

$$\det \mathcal{M}_{\eta_1,\eta_2}(\lambda) = (\lambda - \eta_1)(\lambda - \eta_2).$$

The spectrality property with respect to two fixed points $(\eta_1,\mu_1) \in \Gamma_{M,2}$ and $(\eta_2,\mu_2) \in \Gamma_{M,2}$ give

$$s = \frac{u(\eta_1) - \mu_1}{v(\eta_1)} = \frac{\widehat{\widehat{u}}(\eta_2) - \mu_2}{\widehat{\widehat{v}}(\eta_2)} = \frac{1}{2\,p}\sum_{i=1}^{M} \widehat{m}_i^{+}, \tag{3.37}$$

$$\begin{aligned}
t &= \frac{(\eta_2 - \eta_1)\,v(\eta_1)\,v(\eta_2)}{[\,u(\eta_2) + \mu_2\,]\;v(\eta_1) - [\,u(\eta_1) - \mu_1\,]\;v(\eta_2)} = \\
&= \frac{(\eta_2 - \eta_1)\,\widehat{\widehat{v}}(\eta_1)\,\widehat{\widehat{v}}(\eta_2)}{\left[\widehat{\widehat{u}}(\eta_1) + \mu_1\right]\,\widehat{\widehat{v}}(\eta_2) - \left[\widehat{\widehat{u}}(\eta_2) - \mu_2\right]\,\widehat{\widehat{v}}(\eta_1)} = \\
&= \frac{(\eta_1 - \eta_2)\;[\,u(\eta_1) + \mu_1\,]\;[\,u(\eta_2) - \mu_2\,]}{[\,u(\eta_1) + \mu_1\,]\;w(\eta_2) - [\,u(\eta_2) - \mu_2\,]\;w(\eta_1)} = \\
&= \frac{(\eta_1 - \eta_2)\;\left[\widehat{\widehat{u}}(\eta_1) - \mu_1\right]\;\left[\widehat{\widehat{u}}(\eta_2) + \mu_2\right]}{\left[\widehat{\widehat{u}}(\eta_2) + \mu_2\right]\;\widehat{\widehat{w}}(\eta_1) - \left[\widehat{\widehat{u}}(\eta_1) - \mu_1\right]\;\widehat{\widehat{w}}(\eta_2)} = \\
&= \frac{1}{2\,p}\sum_{i=1}^{M}\left(m_i^{-} - \widehat{\widehat{m}}_i^{+}\right).
\end{aligned} \tag{3.38}$$

We have two complex Bäcklund parameters η_1, η_2. Recall that the points (η_1,μ_1) and (η_2,μ_2) belong to the spectral curve $\Gamma_{M,2}$, namely

$$-(2\,\mu_j)^2 = u^2(\eta_j) + v(\eta_j)\,w(\eta_j) = \widehat{\widehat{u}}^2(\eta_j) + \widehat{\widehat{v}}(\eta_j)\,\widehat{\widehat{w}}(\eta_j), \qquad j = 1,2.$$

Together with Eqs. (3.37) and (3.38), the formula (3.36) gives an explicit two-point Poisson integrable map from $\mathcal{L}_{M,2}(\lambda)$ to $\widehat{\widehat{\mathcal{L}}}_{M,2}(\lambda;\eta_1,\eta_2)$. The map is parametrized by the two points P_1 and Q_2. If $\eta_1 = \eta_2$ (i.e. $\mu_1 = \mu_2$) the map turns into an identity map. The constructed two-point BT can be reduced to a real Poisson integrable map if $\eta_1 = \bar{\eta}_2 \doteq \eta = \mathfrak{Re}(\eta) + \mathrm{i}\,\mathfrak{Im}(\eta) \in \mathbb{C}$. Therefore, the two-point map leads to a physical BT $\mathcal{B}_\eta$ with two real parameters. Notice that the physical time step can be taken as $\mathrm{i}\,(\eta_2 - \eta_1)/2 = \mathfrak{Im}(\eta)$.

Two-point BTs for the Lagrange top

Now we just present the explicit form of the (real) two-point BTs for the Lagrange top. Obviuosly they can be obtained setting $M = 1$ in Eq. (3.36). However, the explicit construction of such integrable Poisson maps is contained in our work [53].

Let us drop the lower index in the coordinates $(\mathbf{m}_1, \mathbf{a}_1)$. The following proposition holds.

Proposition 3.5 *The similarity transform (3.36), with $M = 1$, defines a real BT $\mathcal{B}_\eta : \mathbb{R}^6 \to \mathbb{R}^6$ that maps the vector $M \doteq (\mathbf{m}, \mathbf{a})^T \in \mathbb{R}^6$ into $\widehat{\widehat{M}} \doteq (\widehat{\widehat{\mathbf{m}}}, \widehat{\widehat{\mathbf{a}}})^T \in \mathbb{R}^6$ according to the following compact formula:*

$$\widehat{\widehat{M}} = \Phi_\eta(s,t)\, M + M_0(s,t;\eta), \tag{3.39}$$

with

$$\Phi_\eta(s,t) \doteq \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbb{O}_{3\times 3} \\ A_\eta(s,t) & \mathbb{1}_{3\times 3} \end{pmatrix},$$

where $A_\eta(s,t)$ is a 3×3 matrix depending on the Bäcklund parameter η and the parameters s, t and $M_0(s,t;\eta)$ is a vector depending on the dynamical variables. The matrices $\mathbb{1}_{3\times 3}$ and $\mathbb{O}_{3\times 3}$ are respectively the 3×3 identity matrix and the 3×3 zero matrix. The entries of $A_\eta(s,t)$ are:

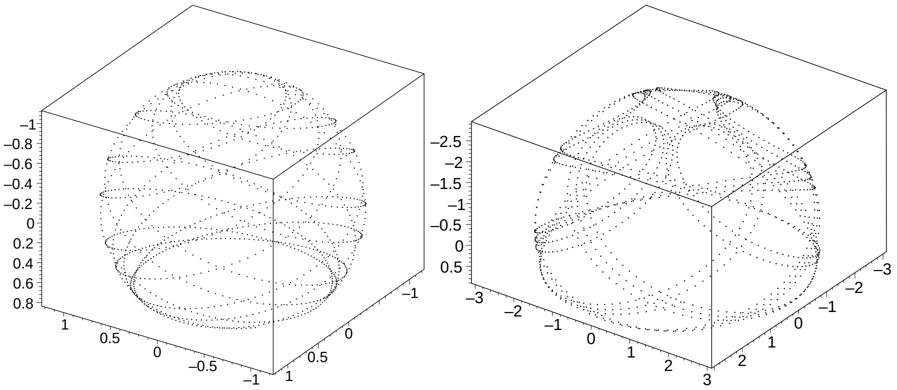
$$\begin{aligned}
&[\,A_\eta(s,t)\,]_{ii} = 0, \quad i = 1,2,3,\\
&[\,A_\eta(s,t)\,]_{12} = \mathrm{i}\,\alpha_2 = -[\,A_\eta(s,t)\,]_{21},\\
&[\,A_\eta(s,t)\,]_{13} = s\,\alpha_1 - t = -[\,A_\eta(s,t)\,]_{31},\\
&[\,A_\eta(s,t)\,]_{23} = -\mathrm{i}\,(s\,\alpha_1 + t) = -[\,A_\eta(s,t)\,]_{32},
\end{aligned}$$

where $\alpha_1 \doteq 2\,\mathfrak{Im}(\eta) - s\,t$, $\alpha_2 \doteq \alpha_1 - s\,t$. The components of the vector $M_0(s,t;\eta)$ are

$$\begin{aligned}
&[\,M_0(s,t;\eta)\,]_1 = p\,(s\,\alpha_1 - t),\\
&[\,M_0(s,t;\eta)\,]_2 = -p\,(s\,\alpha_1 + t),\\
&[\,M_0(s,t;\eta)\,]_3 = 0,\\
&[\,M_0(s,t;\eta)\,]_4 = p\,\left[\frac{s\,\alpha_1}{2}(\alpha_2 + \mathfrak{Re}(\eta)) + t\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\right],\\
&[\,M_0(s,t;\eta)\,]_5 = \mathrm{i}\,p\,\left[\frac{s\,\alpha_1}{2}(\alpha_2 + \mathfrak{Re}(\eta)) - t\,(\alpha_2 - 2\,\mathfrak{Re}(\eta))\right],\\
&[\,M_0(s,t;\eta)\,]_6 = 2\,p\,t\,s\,\alpha_1.
\end{aligned}$$

Remark 3.4 *Notice that the map defined in Eq. (3.39) can be recovered from the two-point BT (3.21) for the first extension of the Lagrange top. The differences between these two maps are: i) the dynamical information contained in the parameters s and t; ii) the Lie-Poisson brackets satisfied by the coordinates. In particular in the map (3.39) we do not have the generator b^α, $\alpha = 1,2,3$.*

Finally we can show two 3D plots (correspondent to different initial data and obtained with a MAPLE 8 program) describing the discrete dynamics given in Eq. (3.39) of the axis of symmetry of the top on the sphere $\langle\, \mathbf{a}, \mathbf{a}\,\rangle$ =constant.



3.2 Alternative approach to integrable discretizations

In this Section we shall construct an integrable Poisson map for the $\mathfrak{su}(2)$ rational Gaudin model. Precisely, we shall present an integrable discrete-time version of the flow generated by the equations of motion, see Eq. (2.38),

$$\dot{\mathbf{y}}_i = \left[\lambda_i\,\mathbf{p} + \textstyle\sum_{j=1}^{N}\mathbf{y}_j\,,\,\mathbf{y}_i\right], \qquad 1 \le i \le N.$$

Then, the contraction procedure described in Chapter 2 enables us to obtain, from the constructed map, integrable discretizations both for the one-body and many-body extended rational $\mathfrak{su}(2)$ hierarchies. In fact, we shall construct an integrable discrete-time version of the equations of motion of the extended Lagrange tops, see Eq. (2.75),

$$\dot{\mathbf{z}}_i = [\,\mathbf{z}_0, \mathbf{z}_i\,] + [\,\mathbf{p}, \mathbf{z}_{i+1}\,], \qquad \mathbf{z}_N \equiv \mathbf{0}, \qquad 0 \le i \le N-1,$$

and of the rational Lagrange chain, see Eq. (2.95),

$$\begin{cases} \dot{\mathbf{m}}_i = [\,\mathbf{p}, \mathbf{a}_i\,] + \left[\mu_i\,\mathbf{p} + \sum_{k=1}^{M}\mathbf{m}_k, \mathbf{m}_i\right], \\[2mm] \dot{\mathbf{a}}_i = \left[\mu_i\,\mathbf{p} + \sum_{k=1}^{M}\mathbf{m}_k, \mathbf{a}_i\right]. \end{cases}$$

Our starting-point consists in the discrete-time Lagrange top obtained in [19]. As a matter of fact our integrable discretization of the extended Lagrange tops can be seen as a generalization of it (from $N = 2$ to an arbitrary N). Let us recall some of the main results concerning such a discretization.

3.2.1 The discrete-time Lagrange top of Suris-Bobenko

In [19] the authors construct an integrable Poisson map for the Lagrange top, that is different from the one we have obtained through Bäcklund transformations, see Subsection 3.1.2. Precisely, they are interested in finding an integrable discrete-time version of the equations of motion (in the rest frame) of the Lagrange top generated by the Hamiltonian, see Eq. (2.79),

$$H_1^{(2)} \doteq \langle\,\mathbf{p}, \mathbf{z}_1\,\rangle + \frac{1}{2}\langle\,\mathbf{z}_0, \mathbf{z}_0\,\rangle. \tag{3.40}$$

They read

$$\begin{cases} \dot{\mathbf{z}}_0 = [\,\mathbf{p}, \mathbf{z}_1\,]\,, \\ \\ \dot{\mathbf{z}}_1 = [\,\mathbf{z}_0, \mathbf{z}_1\,]\,, \end{cases} \tag{3.41}$$

where $\mathbf{z}_0 \in \mathbb{R}^3$ is the vector of kinetic momentum of the body, $\mathbf{z}_1 \in \mathbb{R}^3$ is the vector pointing from the fixed point to the center of mass of the body and $\mathbf{p} \doteq (0, 0, p)$ is the constant vector along the external field. Recall that the Lie-Poisson algebra of the model is $\mathfrak{e}^*(3)$:

$$\left\{ z_0^\alpha, z_0^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_0^\gamma, \qquad \left\{ z_0^\alpha, z_1^\beta \right\} = -\varepsilon_{\alpha\beta\gamma}\, z_1^\gamma, \qquad \left\{ z_1^\alpha, z_1^\beta \right\} = 0.$$

As we have seen in Subsection 2.4.1 the Lax matrix of the Lagrange top can be written as

$$\mathcal{L}_2(\lambda) \doteq \mathbf{p} + \frac{\mathbf{z}_0}{\lambda} + \frac{\mathbf{z}_1}{\lambda^2}.$$

It satisfies the linear r-matrix structure

$$\{\mathcal{L}_2(\lambda) \otimes \mathbb{1}, \mathbb{1} \otimes \mathcal{L}_2(\mu)\} + [\, r(\lambda - \mu), \mathcal{L}_2(\lambda) \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{L}_2(\mu)\,] = 0, \qquad \forall\, (\lambda, \mu) \in \mathbb{C}^2, \tag{3.42}$$

with $r(\lambda) = -\Pi/(2\,\lambda)$. Moreover the equations of motion (3.41) admit the following Lax representation:

$$\dot{\mathcal{L}}_2(\lambda) = [\,\mathcal{L}_2(\lambda), \mathcal{M}_2(\lambda)\,]\,, \qquad \mathcal{M}_2(\lambda) \doteq \frac{\mathbf{z}_1}{\lambda}.$$

In [19] the authors give a complete derivation of Eqs. (3.41) and an explanation of their Hamiltonian nature and integrability. Then they present a discrete map that corresponds to the discrete version of Eqs. (3.41). Such a map takes the following form:

$$\begin{cases} \widehat{\mathbf{z}}_0 - \mathbf{z}_0 = \varepsilon\, [\,\mathbf{p}, \widehat{\mathbf{z}}_1\,]\,, \\ \\ \widehat{\mathbf{z}}_1 - \mathbf{z}_1 = \dfrac{\varepsilon}{2}\, [\,\mathbf{z}_0, \widehat{\mathbf{z}}_1 + \mathbf{z}_1\,]\,, \end{cases} \tag{3.43}$$

where the $\widehat{\ }$-notation is used for the updated variables and $\varepsilon \in \mathbb{R} \setminus \{0\}$ is a discrete-time step. It is easy to see [2] that the second equation in (3.43) can be uniquely solved for $\widehat{\mathbf{z}}_1$:

$$\widehat{\mathbf{z}}_1 = (\mathbf{1} + \varepsilon\, \mathbf{z}_0)\, \mathbf{z}_1\, (\mathbf{1} + \varepsilon\, \mathbf{z}_0)^{-1},$$

so that Eqs. (3.43) define a map $(\mathbf{z}_0, \mathbf{z}_1) \mapsto (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1)$ approximating, for small ε, the time ε shift along the trajectories of Eqs. (3.41). This distinguish the situation from the map in [63], where Lagrangian equations led to correspondences rather than to maps.

Remark 3.5 *Notice that using our notation, that is different w.r.t. the one used in [19], the* explicit *map (3.43) can be written in the following compact form:*

$$\widehat{\mathbf{z}}_i = (\mathbf{1} + \varepsilon\, \mathbf{z}_0)\, \mathbf{z}_i\, (\mathbf{1} + \varepsilon\, \mathbf{z}_0)^{-1} + \varepsilon\, [\,\mathbf{p}, \widehat{\mathbf{z}}_{i+1}\,]\,, \qquad i = 0, 1,$$

with $\mathbf{z}_2 \equiv \mathbf{0}$. *The above equations are reminescent of the continuous-time equations of motion* $\dot{\mathbf{z}}_i = [\,\mathbf{z}_0, \mathbf{z}_i\,] + [\,\mathbf{p}, \mathbf{z}_{i+1}\,]$, $0 \le i \le N-1$, $\mathbf{z}_N \equiv \mathbf{0}$ *for the extended Lagrange tops of order* N.

In [19] it is proven that the map (3.43) is Poisson with respect to the Lie-Poisson brackets on $\mathfrak{e}^*(3)$, so that the Casimir functions $C_0^{(2)} \doteq \langle\, \mathbf{z}_0, \mathbf{z}_1\,\rangle$, $C_1^{(2)} \doteq \langle\, \mathbf{z}_1, \mathbf{z}_1\,\rangle/2$, are integrals of

[2] Recall that for any $\xi, \eta \in \mathfrak{su}(2)$ we have $\xi\,\eta = -\frac{1}{4}\langle\, \xi, \eta\,\rangle + \frac{1}{2}[\,\xi, \eta\,]$.

motion. It is also obvious that $H_0^{(2)} \doteq \langle\, \mathbf{p}, \mathbf{z}_0 \,\rangle$ is an integral of motion. Most remarkably, this map has another integral of motion - a deformed version of the Hamiltonian (3.40) - given by

$$H_1^{(2)}(\varepsilon) \doteq \frac{1}{2}\langle\, \mathbf{z}_0, \mathbf{z}_0 \,\rangle + \langle\, \mathbf{p}, \mathbf{z}_1 \,\rangle + \frac{\varepsilon}{2}\langle\, \mathbf{p}, [\, \mathbf{z}_0, \mathbf{z}_1 \,]\,\rangle. \tag{3.44}$$

The function (3.44) is in involution with $H_0^{(2)}$, which renders the map (3.43) completely integrable.

Remark 3.6 *Note that the map (3.43) corresponds to the Hamiltonian flow direcly generated by a "deformed" version of the physical Hamiltonian (3.40). This situation does not occur in the case of integrable discretizations through Bäcklund transformations, where the constructed integrable Poisson maps discretize a family of flows of the integrable system (and not a particular one).*

A remarkable feature of the map (3.43) is that it admits a Lax representation and an r-matrix formulation. Obviuosly, since the discrete integrals are deformed with respect to the continuous ones, also the discrete Lax matrix will be deformed. The following statements holds; see [19, 91] for further details.

Proposition 3.6 *The map (3.43) has the following Lax representation:*

$$\widehat{\mathcal{L}}_2(\lambda;\varepsilon) = \mathcal{U}_2^{-1}(\lambda;\varepsilon)\,\mathcal{L}_2(\lambda;\varepsilon)\,\mathcal{U}_2(\lambda;\varepsilon),$$

with the matrices

$$\mathcal{L}_2(\lambda;\varepsilon) \doteq \mathbf{p} + \frac{\mathbf{z}_0}{\lambda} + \frac{\mathbf{z}_1 + \frac{\varepsilon}{2}\,[\,\mathbf{z}_0, \mathbf{z}_1\,] + \frac{\varepsilon^2}{2}\,C_1^{(2)}\,\mathbf{p}}{\lambda^2}, \tag{3.45a}$$

$$\mathcal{U}_2(\lambda;\varepsilon) \doteq \mathbf{1} + \varepsilon\,\frac{\widehat{\mathbf{z}}_1}{\lambda}. \tag{3.45b}$$

Notice that the Lax matrix given in Eq. (3.45a) has a rational dependence on the spectral parameter λ. The r-matrix formulation of the map (3.43) can be given in terms of an alternative Lax matrix, that is obtained from (3.45a) by a straightforward computation. It reads

$$\mathcal{L}_2^p(\lambda;\varepsilon) \doteq \lambda^{-1}\,\mathbf{p} + \sum_{k\geq 0}\lambda^{2k}\,\mathbf{x}_{2k} + \sum_{k\geq 0}\lambda^{2k+1}\,\mathbf{x}_{2k+1}, \tag{3.46}$$

where

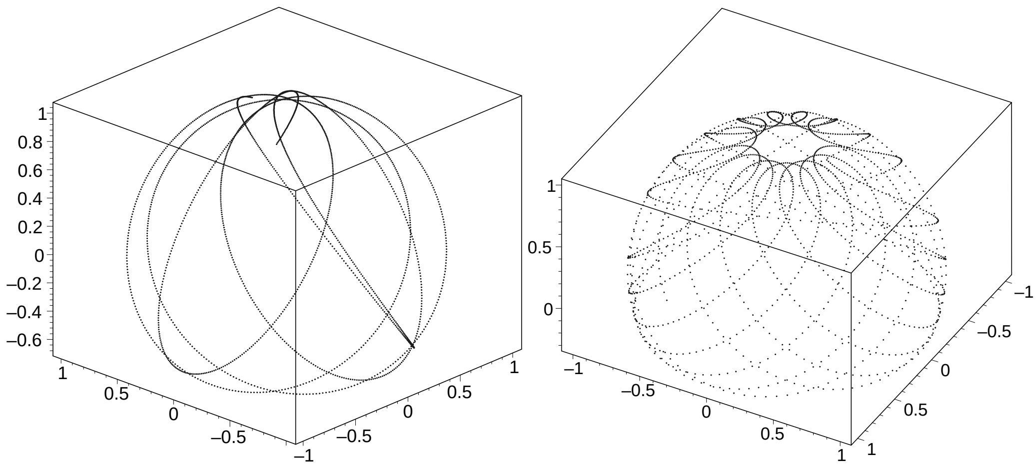
$$\mathbf{x}_{2k} \doteq \left(-\frac{\varepsilon^2}{2}\,C_1^{(2)}\right)^k \mathbf{z}_0,$$

$$\mathbf{x}_{2k+1} \doteq \left(-\frac{\varepsilon^2}{2}\,C_1^{(2)}\right)^k \left(\mathbf{z}_1 + \frac{\varepsilon}{2}\,[\,\mathbf{z}_0, \mathbf{z}_1\,]\right).$$

Obviuosly the integrals $H_0^{(2)}, H_1^{(2)}(\varepsilon), C_0^{(2)}, C_1^{(2)}$ can be obtained as spectral invariants of the Lax matrices (3.45a) or (3.46). Their involutivity is ensured thanks to the following statement.

Proposition 3.7 *The Lax matrix (3.46) satisfies the linear r-matrix structure (3.42) with the same r-matrix (up to a redefinition of the spectral parameter). In particular:*

$$\left\{x_i^\alpha, x_j^\beta\right\} = -\varepsilon_{\alpha\beta\gamma}\,x_{i+j}^\gamma, \qquad i,j \geq 0.$$



Numerics

The integrable Poisson map given in Eq. (3.43) can be easily iterated. We present here some 3D plots, obtained using a MAPLE 8 program, corresponding to such a map.

The input parameters are:

- the intensity of the external field, p;
- the discretization parameter, ε;
- the number of iteration of the map, N;
- the initial values of the coordinate functions, $(\mathbf{z}_0, \mathbf{z}_1)$.

The output is a 3D plot of N consequent points (z_1^1, z_1^2, z_1^3), describing the evolution of the axis of symmetry of the top on the surface $\langle\, \mathbf{z}_1, \mathbf{z}_1 \,\rangle$=constant. These plots show the typical (discrete-time) precession of the axis. Compare these with the classical continuous-time pictures in [6, 48].

3.2.2 Integrable discretizations of extended Lagrange tops: the strategy

In this paragraph we would like to present the strategy used to construct integrable discretizations for the extended Lagrange tops. First of all, recall that this integrable hierarchy is governed by the Lax matrix, see Eq. (2.68),

$$\mathcal{L}_N(\lambda) \doteq \mathbf{p} + \sum_{i=0}^{N-1} \frac{\mathbf{z}_i}{\lambda^{i+1}}, \tag{3.48}$$

where the $3N$ coordinate functions z_i^α's obey to the Lie-Poisson brackets

$$\left\{ z_i^\alpha, z_j^\beta \right\} = \begin{cases} -\varepsilon_{\alpha\beta\gamma}\, z_{i+j}^\gamma & i+j < N, \\ 0 & i+j \geq N. \end{cases} \tag{3.49}$$

The Lax matrix given in Eq. (3.48) satisfies a linear r-matrix algebra with $r(\lambda) = -\Pi/(2\,\lambda)$ and the N involutive Hamiltonians are given by, see Eq. (2.73a),

$$H_k^{(N)} \doteq \langle\, \mathbf{p}, \mathbf{z}_k \,\rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle\, \mathbf{z}_i, \mathbf{z}_{k-i-1} \,\rangle, \qquad 0 \leq k \leq N-1. \tag{3.50}$$

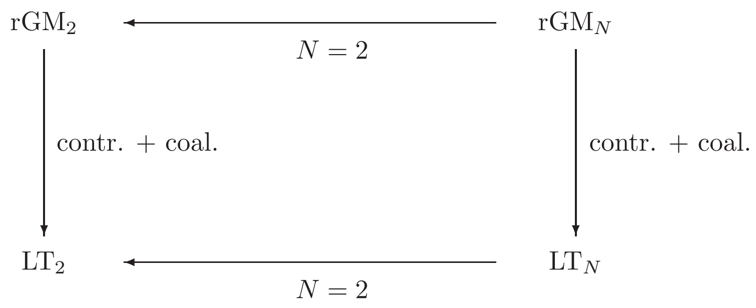
Our aim, as in the Lagrange case, is to find the discrete flow generated by the discrete version of the Hamiltonian $H_1^{(N)}$. We recall that, in the continuous-time setting, the equations of motion generated by $H_1^{(N)}$ are given by

$$\dot{\mathbf{z}}_i = [\,\mathbf{z}_0, \mathbf{z}_i\,] + [\,\mathbf{p}, \mathbf{z}_{i+1}\,], \qquad 0 \le i \le N-1, \quad \mathbf{z}_N \equiv \mathbf{0}. \tag{3.51}$$

Problem: *To construct an integrable discrete-time version of Eqs. (3.51). In other words, to find an integrable Poisson map and its involutive integrals for the hierarchy of the extended Lagrange tops.*

This problem has been solved by means of the contraction procedure described in the previous Chapter.

Let us schematically recall our results, in the continuous-time setting, with the following diagram:



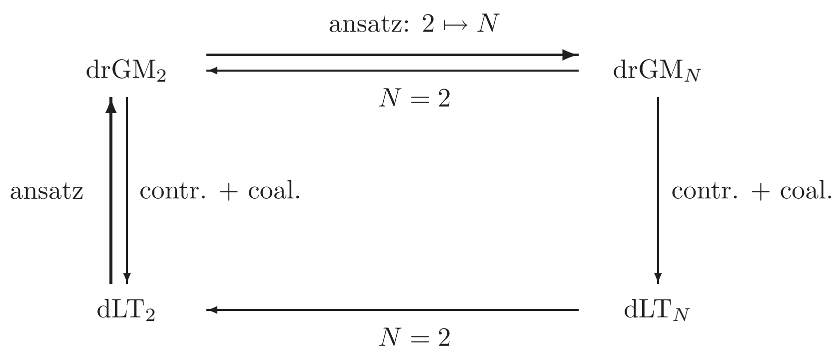
We have proven that by means of a generalized Inönü-Wigner contraction on the Lie-Poisson algebra associated with the N-site $\mathfrak{su}(2)$ rational Gaudin model (rGM$_N$), and a proper pole coalescence performed on the parameters λ_i's of the Lax matrix, we are able to obtain a one-body hierarchy of integrable systems, the *extended Lagrange tops of order N* (LT$_N$). This situation can be seen in the following table:

	rGM$_N$	LT$_N$
LP brackets	$\{y_i^\alpha, y_j^\beta\} = -\delta_{i,j}\,\varepsilon_{\alpha\beta\gamma}\,y_i^\gamma$	$\{z_i^\alpha, z_j^\beta\} = -\theta(N-i-j)\,\varepsilon_{\alpha\beta\gamma}\,z_{i+j}^\gamma$
Lax matrix	$\mathcal{L}_\mathcal{G}(\lambda) \doteq \mathbf{p} + \sum_{i=1}^N \frac{\mathbf{y}_i}{\lambda-\lambda_i}$	$\mathcal{L}_N(\lambda) \doteq \mathbf{p} + \sum_{i=0}^{N-1} \frac{\mathbf{z}_i}{\lambda^{i+1}}$
N Integrals	$H_i \doteq \langle\,\mathbf{p}, \mathbf{y}_i\,\rangle + \sum_{j\neq i}^N \frac{\langle\,\mathbf{y}_i, \mathbf{y}_j\,\rangle}{\lambda_i-\lambda_j}$	$H_i^{(N)} \doteq \langle\,\mathbf{p}, \mathbf{z}_i\,\rangle + \frac{1}{2}\sum_{j=0}^{i-1}\langle\,\mathbf{z}_j, \mathbf{z}_{i-j-1}\,\rangle$
Selected Ham.	$\mathcal{H}_\mathcal{G} \doteq \sum_{i=1}^N \lambda_i\,H_i$	$H_1^{(N)}$
E.o.m.	$\dot{\mathbf{y}}_i = [\,\lambda_i\,\mathbf{p} + \sum_{j=1}^N \mathbf{y}_j\,,\,\mathbf{y}_i]$	$\dot{\mathbf{z}}_i = [\,\mathbf{z}_0, \mathbf{z}_i\,] + [\,\mathbf{p}, \mathbf{z}_{i+1}\,], \quad \mathbf{z}_N \equiv \mathbf{0}$

This algebraic procedure preserves the linear r-matrix formulation of the ancestor model. Obviously, fixing $N=2$, we can obtain the standard Lagrange top (LT$_2$) from the two-body $\mathfrak{su}(2)$ rational Gaudin model (rGM$_2$).

Our aim is now to repeat the above procedure at the discrete-time level. In this case we do not have a proper discrete N-site $\mathfrak{su}(2)$ rational Gaudin model (drGM_N) to start the procedure. Hence our starting point is now the discrete Lagrange top (dLT_2), namely the integrable discretization described in Subsection 3.2.1.

Roughly speaking we have to overcome the following problems (see the second diagram):



1. To make an ansatz for a discrete version of the two-body $\mathfrak{su}(2)$ rational Gaudin model (drGM_2) in order to recover from it, by means of the contraction procedure, the dLT_2.

2. If the ansatz works for $N = 2$ we have to generalize it to an arbitrary N, in order to get a discrete version of the N-body $\mathfrak{su}(2)$ rational Gaudin model (drGM_N).

3. To perform a contraction procedure and a pole coalescence on the drGM_N. The resulting system should be the discrete hierarchy of the extended Lagrange tops (dLT_N).

Remark 3.7 *We notice that the most delicate step in the above list is the point 1. Obviously this "inverse" procedure is not unique. As a matter of fact we are able to construct a discrete-time N-body $\mathfrak{su}(2)$ rational Gaudin model, namely an integrable discrete version of the equations of motion $\dot{\mathbf{y}}_i = [\,\lambda_i\,\mathbf{p}+\sum_{j=1}^{N}\mathbf{y}_j\,,\,\mathbf{y}_i]$, $1 \le i \le N$, and their complete family of discrete involutive integrals. Moreover we shall also construct the contracted versions of them, obtaining an integrable discrete version of the equations of motion (3.51). Nevertheless we are not able, up to now, to construct the Lax representation for these maps. For instance we still have not found the generalization, to an arbitrary N, of the discrete Lax matrix (3.46). Therefore the integrability and the Poisson property of the constructed maps will be proven without using the Lax technique.*

The discrete-time two-body $\mathfrak{su}(2)$ rational Gaudin model

We now present a discrete version of the equations of motions for the two-body $\mathfrak{su}(2)$ rational Gaudin model. We shall prove that a contraction procedure and a pole coalescence on them provide the equations of motion (3.43) of the discrete-time Lagrange top of Suris-Bobenko.

Fixing $N = 2$ in Eqs. (2.38) we get:

$$\begin{cases} \dot{\mathbf{y}}_1 = \lambda_1 \, [\, \mathbf{p}, \mathbf{y}_1 \,] + [\, \mathbf{y}_2, \mathbf{y}_1 \,] \,, \\ \\ \dot{\mathbf{y}}_2 = \lambda_2 \, [\, \mathbf{p}, \mathbf{y}_2 \,] + [\, \mathbf{y}_1, \mathbf{y}_2 \,] \,. \end{cases} \tag{3.52}$$

We know that Eqs. (3.52) are Hamiltonian equations generated by the Hamiltonian

$$\mathcal{H}_{\mathcal{G}} \doteq \lambda_1 \, H_1 + \lambda_2 \, H_2 = \langle \, \mathbf{p}, \lambda_1 \, \mathbf{y}_1 + \lambda_2 \, \mathbf{y}_2 \, \rangle + \langle \, \mathbf{y}_1, \mathbf{y}_2 \, \rangle, \tag{3.53}$$

where the two independent integrals H_1, H_2 are given by

$$H_1 \doteq \langle \, \mathbf{p}, \mathbf{y}_1 \, \rangle + \frac{\langle \, \mathbf{y}_1, \mathbf{y}_2 \, \rangle}{\lambda_1 - \lambda_2}, \qquad H_2 \doteq \langle \, \mathbf{p}, \mathbf{y}_2 \, \rangle + \frac{\langle \, \mathbf{y}_1, \mathbf{y}_2 \, \rangle}{\lambda_2 - \lambda_1}. \tag{3.54}$$

A quite natural ansatz for the discrete version of Eqs. (3.52) is given by:

$$\begin{cases} \widehat{\mathbf{y}}_1 = (\mathbf{1} + \varepsilon \, \lambda_1 \, \mathbf{p}) \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_1 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1} \, (\mathbf{1} + \varepsilon \, \lambda_1 \, \mathbf{p})^{-1}, \\ \\ \widehat{\mathbf{y}}_2 = (\mathbf{1} + \varepsilon \, \lambda_2 \, \mathbf{p}) \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_2 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1} \, (\mathbf{1} + \varepsilon \, \lambda_2 \, \mathbf{p})^{-1}, \end{cases} \tag{3.55}$$

where $\varepsilon \in \mathbb{R} \setminus \{0\}$ is a discrete-time step. Note that Eqs. (3.55) give at order ε the continuous-time equations of motion (3.52). Moreover the *explicit* map (3.55) is the composition of two non-commuting conjugations: hence its Poisson property is straightforward. But we prefer to prove the integrability and the Poissonicity of such a map in the next Subsection, in the case of an arbitrary N.

Let us perform the contration described in the previous Chapter, see Eq. (2.43), and the pole coalescence $\lambda_1 \equiv \vartheta \, \nu_1$, $\lambda_2 \equiv \vartheta \, \nu_2$, on the map (3.55). An easy computation leads to the following equations:

$$\begin{aligned} \widehat{\mathbf{y}}_1 \quad &= \quad (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_1 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1} + \\ &\quad + \varepsilon \, [\, \mathbf{p}, \vartheta \, \nu_1 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_1 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1}] + O(\vartheta^2), \qquad (3.56) \\ \widehat{\mathbf{y}}_2 \quad &= \quad (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_2 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1} + \\ &\quad + \varepsilon \, [\, \mathbf{p}, \vartheta \, \nu_1 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2) \, \mathbf{y}_2 \, (\mathbf{1} + \varepsilon \, \mathbf{y}_1 + \varepsilon \, \mathbf{y}_2)^{-1}] + O(\vartheta^2). \qquad (3.57) \end{aligned}$$

Taking into account Eq. (2.43), i.e. $\mathbf{z}_0 \doteq \mathbf{y}_1 + \mathbf{y}_2$, $\mathbf{z}_1 \doteq \vartheta \, (\nu_1 \, \mathbf{y}_1 + \nu_2 \, \mathbf{y}_2)$ and performing the contraction limit $\vartheta \to 0$, we get from Eqs. (3.56-3.57) the following equations:

$$\begin{cases} \widehat{\mathbf{z}}_0 = \mathbf{z}_0 + \varepsilon \, [\, \mathbf{p}, \widehat{\mathbf{z}}_1 \,] \,, \\ \\ \widehat{\mathbf{z}}_1 = (\mathbf{1} + \varepsilon \, \mathbf{z}_0) \, \mathbf{z}_1 \, (\mathbf{1} + \varepsilon \, \mathbf{z}_0)^{-1}. \end{cases} \tag{3.58}$$

The above map coincides with the one given in Eq. (3.43), describing the dynamics of the discrete-time Lagrange top.

In the remaining part of this paragraph we present the two involutive integrals of the map (3.55) - found by pure inspection - proving that their contracted versions coincide with the involutive integrals $H_0^{(2)}, H_1^{(2)}(\varepsilon)$ of the discrete-time Lagrange top.

We claim that the discrete version of the integrals in Eq. (3.54) is given by

$$H_1(\varepsilon) \doteq \langle \, \mathbf{p}, \mathbf{y}_1 \, \rangle + \frac{\langle \, \mathbf{y}_1, \mathbf{y}_2 \, \rangle}{\lambda_1 - \lambda_2} \left(1 + \frac{\varepsilon^2}{4} \, \lambda_1 \, \lambda_2 \, \langle \, \mathbf{p}, \mathbf{p} \, \rangle \right) - \frac{\varepsilon}{2} \, \langle \, \mathbf{p}, [\, \mathbf{y}_1, \mathbf{y}_2] \, \rangle, \tag{3.59a}$$

$$H_2(\varepsilon) \doteq \langle \, \mathbf{p}, \mathbf{y}_2 \, \rangle + \frac{\langle \, \mathbf{y}_1, \mathbf{y}_2 \, \rangle}{\lambda_2 - \lambda_1} \left(1 + \frac{\varepsilon^2}{4} \, \lambda_1 \, \lambda_2 \, \langle \, \mathbf{p}, \mathbf{p} \, \rangle \right) - \frac{\varepsilon}{2} \, \langle \, \mathbf{p}, [\, \mathbf{y}_2, \mathbf{y}_1] \, \rangle, \tag{3.59b}$$

so that the discrete version of the Gaudin Hamiltonian (3.53) reads

$$\begin{aligned}\mathcal{H}_{\mathcal{G}}(\varepsilon) &\doteq \lambda_1 H_1(\varepsilon) + \lambda_2 H_2(\varepsilon) = \\ &= \langle \mathbf{p}, \lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 \rangle + \langle \mathbf{y}_1, \mathbf{y}_2 \rangle \left(1 + \frac{\varepsilon^2}{4} \lambda_1 \lambda_2 \langle \mathbf{p}, \mathbf{p} \rangle \right) - \\ &\quad - \frac{\varepsilon}{2}(\lambda_1 - \lambda_2)\langle \mathbf{p}, [\mathbf{y}_1, \mathbf{y}_2] \rangle. \end{aligned} \tag{3.60}$$

It is possible to prove by direct verification that the integrals $H_1(\varepsilon), H_2(\varepsilon)$ and the Casimir functions $C_1 \doteq \langle \mathbf{y}_1, \mathbf{y}_1 \rangle/2$, $C_2 \doteq \langle \mathbf{y}_2, \mathbf{y}_2 \rangle/2$ are preserved by the map (3.55). Moreover $H_1(\varepsilon), H_2(\varepsilon)$ are in involutions with respect to the Lie-Poisson brackets on $\mathfrak{su}^*(2) \oplus \mathfrak{su}^*(2)$.

Let us perform the contraction procedure, and the pole coalescence $\lambda_1 \equiv \vartheta\, \nu_1$, $\lambda_2 \equiv \vartheta\, \nu_2$, on the integral (3.60):

$$\begin{aligned}\mathcal{H}_{\mathcal{G}}(\varepsilon, \vartheta) &= \langle \mathbf{p}, \vartheta\, \nu_1 \mathbf{y}_1 + \vartheta\, \nu_1 \mathbf{y}_2 \rangle + \frac{1}{2}\langle \mathbf{y}_1 + \mathbf{y}_2, \mathbf{y}_1 + \mathbf{y}_2 \rangle \left(1 + \frac{\varepsilon^2}{4} \vartheta^2 \nu_1 \nu_2 \langle \mathbf{p}, \mathbf{p} \rangle \right) - \\ & -(C_1 + C_2) \left(1 + \frac{\varepsilon^2}{4} \vartheta^2 \nu_1 \nu_2 \langle \mathbf{p}, \mathbf{p} \rangle \right) + \frac{\varepsilon}{2}\langle \mathbf{p}, [\mathbf{y}_1 + \mathbf{y}_2, \vartheta\, \nu_1 \mathbf{y}_1 + \vartheta\, \nu_2 \mathbf{y}_2] \rangle. \end{aligned} \tag{3.61}$$

Hence:

$$\mathcal{H}_{\mathcal{G}}(\varepsilon, \vartheta) \xrightarrow{\vartheta \to 0} \frac{1}{2}\langle \mathbf{z}_0, \mathbf{z}_0 \rangle + \langle \mathbf{p}, \mathbf{z}_0 \rangle + \frac{\varepsilon}{2}\langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_1] \rangle,$$

that is the integrals $H_1^{(2)}(\varepsilon)$ given in Eq. (3.44) of the discrete-time Lagrange top. Moreover we have

$$H_1(\varepsilon) + H_2(\varepsilon) = \langle \mathbf{p}, \mathbf{y}_1 + \mathbf{y}_2 \rangle = \langle \mathbf{p}, \mathbf{z}_0 \rangle = H_0^{(2)}.$$

Remark 3.8 *Notice that the matrix*

$$\mathcal{L}_{\mathcal{G}}(\lambda, \varepsilon) \doteq \mathbf{p} + \frac{\mathbf{j}_1(\varepsilon)}{\lambda - \lambda_1} + \frac{\mathbf{j}_2(\varepsilon)}{\lambda - \lambda_2}, \tag{3.62}$$

with

$$\begin{aligned}\mathbf{j}_1(\varepsilon) &\doteq \mathbf{y}_1 - \frac{\varepsilon}{2}[\mathbf{y}_1, \mathbf{y}_2] + \frac{\varepsilon^2}{4}\langle \mathbf{y}_1 - \mathbf{y}_2, \lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 \rangle, \\ \mathbf{j}_2(\varepsilon) &\doteq \mathbf{y}_2 - \frac{\varepsilon}{2}[\mathbf{y}_2, \mathbf{y}_1] + \frac{\varepsilon^2}{4}\langle \mathbf{y}_2 - \mathbf{y}_1, \lambda_1 \mathbf{y}_1 + \lambda_2 \mathbf{y}_2 \rangle,\end{aligned}$$

has the following properties: i) for $\varepsilon = 0$ it coincides with the Lax matrix for the continuous-time two-body $\mathfrak{su}(2)$ rational Gaudin model; ii) its contracted version coincides with the Lax matrix (3.45a) for the discrete-time Lagrange top. Obviously the matrix (3.62) is not the only one which satisfies such requirements. As a matter of fact it is not the Lax matrix for the map (3.55) and we still have not found its correct version.

3.2.3 The discrete-time $\mathfrak{su}(2)$ rational Gaudin model

Proposition 3.8 *The map*

$$\mathcal{D}_\varepsilon^N : \mathbf{y}_i \mapsto \widehat{\mathbf{y}}_i \doteq (\mathbf{1} + \varepsilon\, \lambda_i\, \mathbf{p}) \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^N \mathbf{y}_j\right) \mathbf{y}_i \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^N \mathbf{y}_j\right)^{-1} (\mathbf{1} + \varepsilon\, \lambda_i\, \mathbf{p})^{-1}, \tag{3.63}$$

with $1 \le i \le N$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$, is Poisson with respect to the brackets (2.26) on $\oplus^N \mathfrak{su}^*(2)$ and has N independent and involutive integrals of motion assuring its complete integrability:

$$H_k(\varepsilon) \doteq \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1 \\ j \neq k}}^N \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \lambda_j \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1 \\ j \neq k}}^N \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle, \tag{3.64}$$

with $1 \le k \le N$.

Proof: First of all we immediately notice that the map (3.63) reproduces at order ε the continuous-time equations of motion $\dot{\mathbf{y}}_i = \left[\lambda_i \, \mathbf{p} + \sum_{j=1}^N \mathbf{y}_j \, , \, \mathbf{y}_i \right]$, $1 \le i \le N$. Moreover it can be written as the composition of two non-commuting conjugations, namely

$$\mathcal{D}_\varepsilon^N = (\mathcal{D}_\varepsilon^N)_2 \circ (\mathcal{D}_\varepsilon^N)_1,$$

where

$$(\mathcal{D}_\varepsilon^N)_1 : \; \mathbf{y}_i \mapsto \mathbf{y}_i^* \doteq \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right) \, \mathbf{y}_i \, \left(\mathbf{1} + \varepsilon \sum_{j=1}^N \mathbf{y}_j \right)^{-1}, \tag{3.65a}$$

$$(\mathcal{D}_\varepsilon^N)_2 : \; \mathbf{y}_i^* \mapsto \widehat{\mathbf{y}}_i \doteq (\mathbf{1} + \varepsilon \, \lambda_i \, \mathbf{p}) \, \mathbf{y}_i^* \, (\mathbf{1} + \varepsilon \, \lambda_i \, \mathbf{p})^{-1}, \tag{3.65b}$$

with $1 \le i \le N$. Notice that $(\mathcal{D}_\varepsilon^N)_1 \circ (\mathcal{D}_\varepsilon^N)_2 \neq (\mathcal{D}_\varepsilon^N)_2 \circ (\mathcal{D}_\varepsilon^N)_1$.

As we told in Remark 3.7 we do not have a Lax representation and an r-matrix formulation for the map (3.63). Hence we have to prove by a direct computation that:

1. the map (3.63) is Poisson with respect to the Lie-Poisson brackets (2.26) on $\oplus^N \mathfrak{su}^*(2)$;
2. the map (3.63) preserves the functions $\{H_k(\varepsilon)\}_{k=1}^N$ given in Eq. (3.64);
3. the functions $\{H_k(\varepsilon)\}_{k=1}^N$ are in involution with respect to the Lie-Poisson brackets (2.26) (their independence is obvious).

We may start our proof:

1. The Poisson property of the map $\mathcal{D}_\varepsilon^N$ is a consequence of the Poisson property of the maps $(\mathcal{D}_\varepsilon^N)_1$ and $(\mathcal{D}_\varepsilon^N)_2$. In fact $(\mathcal{D}_\varepsilon^N)_1$ is a Hamiltonian flow on $\oplus^N \mathfrak{su}^*(2)$ with respect to the Hamiltonian
$$I_1 \doteq \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^N \langle \, \mathbf{y}_j , \mathbf{y}_k \, \rangle.$$
On the other hand $(\mathcal{D}_\varepsilon^N)_2$ is a Hamiltonian flow on $\oplus^N \mathfrak{su}^*(2)$ with respect to the Hamiltonian
$$I_2 \doteq \sum_{k=1}^N \langle \, \mathbf{p}, \lambda_k \, \mathbf{y}_k^* \, \rangle.$$
2. Notice that the maps (3.65a-3.65b) imply respectively the following relations:

$$\langle \, \mathbf{y}_i^*, \mathbf{y}_j^* \, \rangle = \langle \, \mathbf{y}_i, \mathbf{y}_j \, \rangle, \qquad \mathbf{y}_i^* + \frac{\varepsilon}{2} \sum_{j=1}^N [\, \mathbf{y}_i^*, \mathbf{y}_j \,] = \mathbf{y}_i + \frac{\varepsilon}{2} \sum_{j=1}^N [\, \mathbf{y}_j, \mathbf{y}_i \,], \tag{3.66a}$$

$$\langle \, \mathbf{p}, \widehat{\mathbf{y}}_j \, \rangle = \langle \, \mathbf{p}_i, \mathbf{y}_j^* \, \rangle, \qquad \widehat{\mathbf{y}}_i + \frac{\varepsilon}{2} \lambda_i \, [\, \widehat{\mathbf{y}}_i, \mathbf{p} \,] = \mathbf{y}_i^* + \frac{\varepsilon}{2} \, \lambda_i \, [\, \mathbf{p}, \mathbf{y}_i^* \,], \tag{3.66b}$$

with $1 \le i, j \le N$. We have:

$$
\begin{aligned}
\widehat{H}_k(\varepsilon) &= \langle \mathbf{p}, \widehat{\mathbf{y}}_k \rangle + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\langle \widehat{\mathbf{y}}_k, \widehat{\mathbf{y}}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \, \lambda_j \, \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1\\ j\neq k}}^{N} \langle \mathbf{p}, [\widehat{\mathbf{y}}_k, \widehat{\mathbf{y}}_j] \rangle = \\
&= \langle \mathbf{p}, \mathbf{y}^*_k \rangle + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\langle \mathbf{y}^*_k, \mathbf{y}^*_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \, \lambda_j \, \langle \mathbf{p}, \mathbf{p} \rangle \right) + \frac{\varepsilon}{2} \sum_{\substack{j=1\\ j\neq k}}^{N} \langle \mathbf{p}, [\mathbf{y}^*_k, \mathbf{y}^*_j] \rangle = \\
&= \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j} \left(1 + \frac{\varepsilon^2}{4} \lambda_k \, \lambda_j \, \langle \mathbf{p}, \mathbf{p} \rangle \right) - \frac{\varepsilon}{2} \sum_{\substack{j=1\\ j\neq k}}^{N} \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle = H_k(\varepsilon),
\end{aligned}
$$

with $1 \le k \le N$. Here we have used Eqs. (3.66b) in the first step and Eqs. (3.66a) in the second one.

3. Let us write the functions $\{H_k(\varepsilon)\}_{k=1}^N$ in the following way:

$$
H_k(\varepsilon) \doteq h_k^0 - \frac{\varepsilon}{2} \, h_k^1 + \frac{\varepsilon^2}{4} \, \langle \mathbf{p}, \mathbf{p} \rangle \, h_k^2, \qquad 1 \le k \le N,
$$

where

$$
h_k^0 \doteq \langle \mathbf{p}, \mathbf{y}_k \rangle + \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\langle \mathbf{y}_k, \mathbf{y}_j \rangle}{\lambda_k - \lambda_j}, \tag{3.67a}
$$

$$
h_k^1 \doteq \sum_{\substack{j=1\\ j\neq k}}^{N} \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle, \tag{3.67b}
$$

$$
h_k^2 \doteq \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k \, \lambda_j}{\lambda_k - \lambda_j} \, \langle \mathbf{y}_k, \mathbf{y}_j \rangle. \tag{3.67c}
$$

Therefore we have:

$$
\begin{aligned}
\{H_k(\varepsilon), H_i(\varepsilon)\} &= \left\{h_k^0, h_i^0\right\} - \frac{\varepsilon}{2} \left(\left\{h_k^0, h_i^1\right\} + \left\{h_k^1, h_i^0\right\} \right) + \\
&\quad + \frac{\varepsilon^2}{4} \left[\langle \mathbf{p}, \mathbf{p} \rangle \left(\left\{h_k^0, h_i^2\right\} + \left\{h_k^2, h_i^0\right\} \right) + \left\{h_k^1, h_i^1\right\} \right] - \\
&\quad - \frac{\varepsilon^3}{8} \left(\left\{h_k^1, h_i^2\right\} + \left\{h_k^2, h_i^1\right\} \right) + \frac{\varepsilon^4}{16} \langle \mathbf{p}, \mathbf{p} \rangle^2 \left\{h_k^2, h_i^2\right\}.
\end{aligned} \tag{3.68}
$$

We know that $\left\{h_k^0, h_i^0\right\} = 0$, $1 \le k, i \le N$, since the integrals $\left\{h_k^0\right\}_{k=1}^N$ are the ones of the continuous-time $\mathfrak{su}(2)$ rational Gaudin model. Let us explicitly compute the other brackets in Eq. (3.68) using Eqs. (3.67a-3.67b-3.67c) and the Lie-Poisson brackets $\{y_i^\alpha, y_k^\beta\} = -\delta_{i,k} \, \varepsilon_{\alpha\beta\gamma} \, y_i^\gamma$, $1 \le i, k \le N$. We shall obviously assume $k \neq i$ and the summation over the repeated greek indices (running from 1 to 3) is used. Moreover, in the brackets $\left\{h_k^0, h_i^1\right\} + \left\{h_k^1, h_i^0\right\}$ and $\left\{h_k^0, h_i^2\right\} + \left\{h_k^2, h_i^0\right\}$ we shall explicitly write the order of $|\mathbf{p}|$ appearing in the computation.

At order ε we have:

$$
\begin{aligned}
&\left[\,\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}\,\right]_{O(|\mathbf{p}|)} = \\
&= p^\beta\,\varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{\substack{l=1\\ l\neq i}}^{N} \left[\frac{1}{\lambda_k - \lambda_j}\,\{y_k^\alpha\, y_j^\alpha, y_i^\rho\, y_l^\sigma\} + \frac{1}{\lambda_i - \lambda_l}\,\{y_k^\rho\, y_j^\sigma, y_i^\alpha\, y_l^\alpha\} \right] = \\
&= -p^\beta\,\varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\varepsilon_{\alpha\sigma\gamma}}{\lambda_k - \lambda_j} (y_k^\gamma\, y_j^\alpha\, y_i^\rho + y_j^\gamma\, y_i^\rho\, y_k^\alpha) - \\
&\quad -p^\beta\,\varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\varepsilon_{\rho\alpha\gamma}}{\lambda_i - \lambda_k} (y_k^\gamma\, y_j^\sigma\, y_i^\alpha + y_i^\gamma\, y_j^\sigma\, y_k^\alpha) - \\
&\quad -p^\beta\,\varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\varepsilon_{\sigma\alpha\gamma}}{\lambda_i - \lambda_j} (y_k^\rho\, y_j^\gamma\, y_i^\alpha + y_i^\gamma\, y_j^\alpha\, y_k^\rho).
\end{aligned}
$$

The above expression vanishes if we swap the indices α and γ in each second term in the three brackets. Then we have:

$$
\begin{aligned}
&\left[\,\{h_k^0, h_i^1\} + \{h_k^1, h_i^0\}\,\right]_{O(|\mathbf{p}|^2)} = \\
&= p^\alpha\, p^\beta\, \varepsilon_{\beta\rho\sigma} \sum_{l=1}^{N} [\,\{y_k^\alpha, y_i^\rho\, y_l^\sigma\} + \{y_k^\rho\, y_l^\sigma, y_i^\alpha\}\,] = \\
&= p^\alpha\, p^\beta\, (\varepsilon_{\beta\rho\sigma}\, \varepsilon_{\alpha\sigma\gamma} + \varepsilon_{\beta\gamma\sigma}\, \varepsilon_{\sigma\alpha\rho})\, y_k^\gamma\, y_i^\rho,
\end{aligned}
$$

that vanishes using the properties of the tensor $\varepsilon_{\alpha\beta\gamma}$.

At order ε^2 we get:

$$
\begin{aligned}
&\left[\,\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}\,\right]_{O(|\mathbf{p}|)} = \\
&= p^\alpha \sum_{\substack{l=1\\ l\neq i}}^{N} \frac{\lambda_i\,\lambda_l}{\lambda_i - \lambda_l} \left\{ y_k^\alpha, y_i^\beta y_l^\beta \right\} - p^\alpha \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k\,\lambda_j}{\lambda_k - \lambda_j} \left\{ y_i^\alpha, y_k^\beta y_j^\beta \right\} = \\
&= -p^\alpha \varepsilon_{\alpha\beta\gamma} \left(\frac{\lambda_i\,\lambda_k}{\lambda_i - \lambda_k} y_i^\beta y_k^\gamma + \frac{\lambda_i\,\lambda_k}{\lambda_i - \lambda_k} y_i^\gamma y_k^\beta \right),
\end{aligned}
$$

that vanishes swapping the indices γ and β in the second term. Moreover,

$$
\begin{aligned}
&\left[\,\{h_k^0, h_i^2\} + \{h_k^2, h_i^0\}\,\right]_{O(|\mathbf{p}|^0)} = \\
&= \sum_{\substack{j=1\\ j\neq k}}^{N} \sum_{\substack{l=1\\ l\neq i}}^{N} \frac{\lambda_i\,\lambda_l + \lambda_k\,\lambda_j}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_l)} \left\{ y_k^\alpha\, y_j^\alpha, y_i^\beta\, y_l^\beta \right\} = \\
&= -\varepsilon_{\alpha\beta\gamma} \sum_{\substack{j=1\\ j\neq k}}^{N} \frac{\lambda_k(\lambda_i^2 - \lambda_j^2) - \lambda_i(\lambda_k^2 - \lambda_j^2) - \lambda_j(\lambda_i^2 - \lambda_k^2)}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_k)(\lambda_i - \lambda_j)} y_k^\gamma\, y_j^\alpha\, y_i^\beta = \\
&= -\varepsilon_{\alpha\beta\gamma} \sum_{\substack{j=1\\ j\neq k}}^{N} y_k^\gamma\, y_j^\alpha\, y_i^\beta.
\end{aligned}
$$

On the other hand:

$$\left\{h_k^1, h_i^1\right\} = p^\alpha\, p^\sigma\, \varepsilon_{\alpha\beta\gamma}\varepsilon_{\sigma\rho\mu} \sum_{\substack{j=1\\ j\neq k}}^N \sum_{\substack{l=1\\ l\neq i}}^N \left\{y_k^\alpha\, y_j^\gamma, y_i^\rho\, y_l^\mu\right\} =$$

$$= p^\sigma\, p^\sigma\, \varepsilon_{\alpha\beta\gamma} \sum_{\substack{j=1\\ j\neq k}}^N y_k^\gamma\, y_j^\alpha\, y_i^\beta,$$

where we have used the properties of the tensor $\varepsilon_{\alpha\beta\gamma}$. Hence we get:

$$\langle\, \mathbf{p}, \mathbf{p}\,\rangle\, \left(\left\{h_k^0, h_i^2\right\} + \left\{h_k^2, h_i^0\right\}\right) + \left\{h_k^1, h_i^1\right\} = 0.$$

At order ε^3 we have:

$$\left\{h_k^1, h_i^2\right\} + \left\{h_k^2, h_i^1\right\} =$$

$$= -p^\beta\, \varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^N \sum_{\substack{l=1\\ l\neq i}}^N \left[\frac{\lambda_k\, \lambda_j}{\lambda_k - \lambda_j} \left\{y_k^\alpha\, y_j^\alpha, y_i^\rho\, y_l^\sigma\right\} + \frac{\lambda_i\, \lambda_l}{\lambda_i - \lambda_l} \left\{y_k^\rho\, y_j^\sigma, y_i^\alpha\, y_l^\alpha\right\}\right] =$$

$$= p^\beta\, \varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^N \frac{\varepsilon_{\alpha\sigma\gamma}\, \lambda_k\, \lambda_j}{\lambda_k - \lambda_j} (y_k^\gamma\, y_j^\alpha\, y_i^\rho + y_j^\gamma\, y_i^\rho\, y_k^\alpha) -$$

$$- p^\beta\, \varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^N \frac{\varepsilon_{\rho\alpha\gamma}\, \lambda_k\, \lambda_i}{\lambda_i - \lambda_k} (y_k^\gamma\, y_j^\sigma\, y_i^\alpha + y_i^\gamma\, y_j^\sigma\, y_k^\alpha) -$$

$$- p^\beta\, \varepsilon_{\beta\rho\sigma} \sum_{\substack{j=1\\ j\neq k}}^N \frac{\varepsilon_{\sigma\alpha\gamma}\, \lambda_i\, \lambda_j}{\lambda_i - \lambda_j} (y_k^\rho\, y_j^\gamma\, y_i^\alpha + y_i^\gamma\, y_j^\alpha\, y_k^\rho).$$

The above expession vanishes if we swap the indices α and γ in each second term in the three brackets.

Finally, at order ε^4, we get:

$$\left\{h_k^2, h_i^2\right\} = \sum_{\substack{j=1\\ j\neq k}}^N \sum_{\substack{l=1\\ l\neq i}}^N \frac{\lambda_k\, \lambda_j\, \lambda_i\, \lambda_l}{(\lambda_k - \lambda_j)(\lambda_i - \lambda_l)} \left\{y_k^\alpha\, y_j^\alpha, y_i^\beta\, y_l^\beta\right\} =$$

$$= -\varepsilon_{\alpha\beta\gamma} \sum_{\substack{j=1\\ j\neq k}}^N y_j^\alpha\, y_i^\beta\, y_k^\gamma \left[\frac{\lambda_k^2\, \lambda_j\, \lambda_i}{(\lambda_i - \lambda_k)(\lambda_k - \lambda_j)} + \frac{\lambda_k\, \lambda_j\, \lambda_i^2}{(\lambda_i - \lambda_j)(\lambda_k - \lambda_i)} - \frac{\lambda_k\, \lambda_j^2\, \lambda_i}{(\lambda_i - \lambda_j)(\lambda_k - \lambda_j)}\right].$$

A direct verification allows one to check that the expression in the square brackets vanishes.

□

Using the integrals of motion given in Eq. (3.64) we can compute the discrete-time version of the Gaudin Hamiltonian (2.37). It reads:

$$\begin{aligned}\mathcal{H}_{\mathcal{G}}(\varepsilon) \doteq \sum_{k=1}^{N} \lambda_k\, H_k(\varepsilon) \;&=\; \sum_{k=1}^{N} \langle\, \mathbf{p}, \lambda_k\, \mathbf{y}_k \,\rangle + \frac{1}{2} \sum_{\substack{j,k=1\\ j\neq k}}^{N} \langle\, \mathbf{y}_k, \mathbf{y}_j \,\rangle \left(1 + \frac{\varepsilon^2}{4}\, \lambda_k\, \lambda_j\, \langle\, \mathbf{p}, \mathbf{p} \,\rangle\right) - \\ &\quad - \frac{\varepsilon}{4} \sum_{\substack{j,k=1\\ j\neq k}}^{N} (\lambda_k - \lambda_j)\, \langle\, \mathbf{p}, [\, \mathbf{y}_k, \mathbf{y}_j \,] \,\rangle. \end{aligned} \tag{3.69}$$

Moreover we still have a linear integral given by $\sum_{k=1}^{N} H_k(\varepsilon) = \sum_{k=1}^{N} \langle\, \mathbf{p}, \mathbf{y}_k \,\rangle$, as in the continuous-time case.

3.2.4 Discrete-time extended Lagrange tops

We know from Subsection 3.2.2 that fixing $N = 2$ in Proposition 3.8 we can recover, by the contraction procedure and the pole coalescence, the discrete-time Lagrange top considered in [19].

Our aim is now to perform the contraction procedure on the discrete-time N-site $\mathfrak{su}(2)$ rational Gaudin model, in order to construct an integrable discretization for the whole hierarchy of extended Lagrange tops. In particular we want to find the integrable discrete-time version of the equations of motion $\dot{\mathbf{z}}_i = [\, \mathbf{z}_0, \mathbf{z}_i \,] + [\, \mathbf{p}, \mathbf{z}_{i+1} \,]$, $\mathbf{z}_N \equiv \mathbf{0}$, with $0 \le i \le N-1$.

Proposition 3.9 *The map*

$$\widetilde{\mathcal{D}}_{\varepsilon}^{N} \,:\; \mathbf{z}_i \mapsto \widehat{\mathbf{z}}_i \doteq (\mathbf{1} + \varepsilon\, \mathbf{z}_0)\, \mathbf{z}_i\, (\mathbf{1} + \varepsilon\, \mathbf{z}_0)^{-1} - 2 \sum_{j=1}^{N-i-1} \left(-\frac{\varepsilon}{2}\right)^{j} \mathrm{ad}_{\mathbf{p}}^{j}\, \widehat{\mathbf{z}}_{j+i}, \tag{3.70}$$

with $0 \le i \le N-1$ and $\varepsilon \in \mathbb{R} \setminus \{0\}$, is Poisson with respect to the brackets (3.49) and has N independent and involutive integrals of motion assuring its complete integrability:

$$H_k^{(N)}(\varepsilon) \doteq \langle\, \mathbf{p}, \mathbf{z}_k \,\rangle + \frac{1}{2} \sum_{i=0}^{k-1} \langle\, \mathbf{z}_i, \mathbf{z}_{k-i-1} \,\rangle + \frac{\varepsilon}{2} \langle\, \mathbf{p}, [\, \mathbf{z}_0, \mathbf{z}_k] \,\rangle + \frac{\varepsilon^2}{8} \langle\, \mathbf{p}, \mathbf{p} \,\rangle \sum_{i=0}^{k-1} \langle\, \mathbf{z}_{i+1}, \mathbf{z}_{k-i} \,\rangle, \tag{3.71}$$

with $0 \le k \le N-1$.

Remark 3.9 *We immediately notice that fixing $N = 2$ in Eqs. (3.70) and (3.71) we obtain respectively the map (3.43) and the integrals $H_0^{(2)}, H_1^{(2)}(\varepsilon)$ of the discrete-time Lagrange top, see Subsection 3.2.1.*

Moreover Eq. (3.70) defines an explicit *map. For instance, let us fix $N = 3$ in Eq. (3.70). We get:*

$$\begin{cases} \widehat{\mathbf{z}}_0 = \mathbf{z}_0 + \varepsilon\, [\, \mathbf{p}, \widehat{\mathbf{z}}_1 \,] - \dfrac{\varepsilon^2}{2}\, [\, \mathbf{p}, [\, \mathbf{p}, \widehat{\mathbf{z}}_2 \,]], \\[2ex] \widehat{\mathbf{z}}_1 = (\mathbf{1} + \varepsilon\, \mathbf{z}_0)\, \mathbf{z}_1\, (\mathbf{1} + \varepsilon\, \mathbf{z}_0)^{-1} + \varepsilon\, [\, \mathbf{p}, \widehat{\mathbf{z}}_2 \,], \\[2ex] \widehat{\mathbf{z}}_2 = (\mathbf{1} + \varepsilon\, \mathbf{z}_0)\, \mathbf{z}_2\, (\mathbf{1} + \varepsilon\, \mathbf{z}_0)^{-1}. \end{cases} \tag{3.72}$$

The map $(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2) \mapsto (\widehat{\mathbf{z}}_0, \widehat{\mathbf{z}}_1, \widehat{\mathbf{z}}_2)$ approximates, for small ε, the time ε shift along the trajectories of the continuous-time equations of motion of the first Leibniz extension of the Lagrange *top:*

$$\begin{cases} \dot{\mathbf{z}}_0 = [\,\mathbf{p}, \mathbf{z}_1\,], \\ \dot{\mathbf{z}}_1 = [\,\mathbf{z}_0, \mathbf{z}_1] + [\,\mathbf{p}, \mathbf{z}_2\,], \\ \dot{\mathbf{z}}_2 = [\,\mathbf{z}_0, \mathbf{z}_2\,]. \end{cases}$$

Proof: Let us construct the map (3.70) through the usual contraction procedure and the pole coalescence $\lambda_i \equiv \vartheta\,\nu_i$, $1 \le i \le N$, performed on the map (3.63).

Consider the map $(\mathcal{D}_\varepsilon^N)_1$ in Eq. (3.65a). Using Eq. (2.43) we immediately get:

$$\begin{aligned} \mathbf{z}_i^* \doteq \sum_{k=1}^N \vartheta^i\,\nu_k^i\,\mathbf{y}_k^* &= \sum_{k=1}^N \vartheta^i\,\nu_k^i\,\Big(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^N \mathbf{y}_j\Big)\,\mathbf{y}_k\,\Big(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^N \mathbf{y}_j\Big)^{-1} = \\ &= (\mathbf{1} + \varepsilon\,\mathbf{z}_0)\,\mathbf{z}_i\,(\mathbf{1} + \varepsilon\,\mathbf{z}_0)^{-1}, \end{aligned}$$

with $0 \le i \le N-1$. Hence the contracted version of $(\mathcal{D}_\varepsilon^N)_1$ is given by

$$(\widetilde{\mathcal{D}}_\varepsilon^N)_1 :\ \mathbf{z}_i \mapsto \mathbf{z}_i^* = (\mathbf{1} + \varepsilon\,\mathbf{z}_0)\,\mathbf{z}_i\,(\mathbf{1} + \varepsilon\,\mathbf{z}_0)^{-1}, \qquad 0 \le i \le N-1. \tag{3.73}$$

On the other hand, considering the map $(\mathcal{D}_\varepsilon^N)_2$ in Eq. (3.65b) and the map in Eq. (2.43), a straightforward computation leads to:

$$\begin{aligned} \sum_{k=1}^N \vartheta^i\,\nu_k^i\,\widehat{\mathbf{y}}_k &= \sum_{k=1}^N \vartheta^i\,\nu_k^i\,(\mathbf{1} + \varepsilon\,\vartheta\,\nu_k\,\mathbf{p})\,\mathbf{y}_k^*\,(\mathbf{1} + \varepsilon\,\vartheta\,\nu_k\,\mathbf{p})^{-1} = \\ &= \sum_{k=1}^N \sum_{j\ge 0} \vartheta^{i+j}\,\nu_k^{i+j}\,(-\varepsilon)^j\,(\mathbf{1} + \varepsilon\,\vartheta\,\nu_k\,\mathbf{p})\,\mathbf{y}_k^*\,\mathbf{p}^j = \\ &= \mathbf{z}_i^* + 2\sum_{j=1}^{N-i-1} \left(\frac{\varepsilon}{2}\right)^j \mathrm{ad}_{\mathbf{p}}^j\,\mathbf{z}_{j+i}^* + O(\vartheta^N), \end{aligned}$$

with $0 \le i \le N-1$. Performing the limit $\vartheta \to 0$ we obtain the contracted version of the map $(\mathcal{D}_\varepsilon^N)_2$. It reads:

$$(\widetilde{\mathcal{D}}_\varepsilon^N)_2 :\ \mathbf{z}_i^* \mapsto \widehat{\mathbf{z}}_i = \mathbf{z}_i^* + 2\sum_{j=1}^{N-i-1} \left(\frac{\varepsilon}{2}\right)^j \mathrm{ad}_{\mathbf{p}}^j\,\mathbf{z}_{j+i}^*, \qquad 0 \le i \le N-1. \tag{3.74}$$

Therefore, the contraction of the map $\mathcal{D}_\varepsilon^N$ given in Eq. (3.63) is given by

$$\widetilde{\mathcal{D}}_\varepsilon^N = (\widetilde{\mathcal{D}}_\varepsilon^N)_2 \circ (\widetilde{\mathcal{D}}_\varepsilon^N)_1\ :\ \mathbf{z}_i \mapsto \widehat{\mathbf{z}}_i = (\mathbf{1} + \varepsilon\,\mathbf{z}_0)\,\mathbf{z}_i\,(\mathbf{1} + \varepsilon\,\mathbf{z}_0)^{-1} - 2\sum_{j=1}^{N-i-1} \left(-\frac{\varepsilon}{2}\right)^j \mathrm{ad}_{\mathbf{p}}^j\,\widehat{\mathbf{z}}_{j+i},$$

with $0 \le i \le N-1$. The above map is the one given in Eq. (3.70).

The Poisson property of the map $\widetilde{\mathcal{D}}_\varepsilon^N$ is a consequence of the Poisson property of the map $\mathcal{D}_\varepsilon^N$ in Eq. (3.63). In fact $\widetilde{\mathcal{D}}_\varepsilon^N$ is the composition of two non-commuting Poisson maps: $(\widetilde{\mathcal{D}}_\varepsilon^N)_1$ is a Hamiltonian flow on the contraction of $\oplus^N \mathfrak{su}^*(2)$ with respect to the Hamiltonian $\langle\,\mathbf{z}_0, \mathbf{z}_0\,\rangle$; $(\widetilde{\mathcal{D}}_\varepsilon^N)_2$ is a Hamiltonian flow on the contraction of $\oplus^N \mathfrak{su}^*(2)$ with respect to the Hamiltonian $\langle\,\mathbf{p}, \mathbf{z}_1^*\,\rangle$.

We now construct, by contraction of the functions (3.64), the integrals (3.71) of the Poisson map (3.70). We know that fixing $\varepsilon = 0$ in Eq. (3.64) we recover the integrals of motion of the continuous-time $\mathfrak{su}(2)$ rational Gaudin model. Their contraction gives the integrals of motion of the continuous-time extended Lagrange tops, see Remark 2.11. Therefore it is enough to perform the contraction procedure just on the two ε-dependent terms of the integrals (3.64). We get:

$$\begin{aligned}\sum_{k=1}^{N} \vartheta^i \nu_k^i H_k(\varepsilon) &= \langle \mathbf{p}, \mathbf{z}_i \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \langle \mathbf{z}_m, \mathbf{z}_{i-m-1} \rangle - \\ &\quad - \frac{\varepsilon}{4} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} (\vartheta^i \nu_k^i - \vartheta^i \nu_j^i) \langle \mathbf{p}, [\mathbf{y}_k, \mathbf{y}_j] \rangle + \\ &\quad + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{\substack{j,k=1 \\ j \neq k}}^{N} \vartheta^{i+1} \frac{\nu_k^{i+1} \nu_j - \nu_j^{i+1} \nu_k}{\nu_k - \nu_j} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = \\ &= \langle \mathbf{p}, \mathbf{z}_i \rangle + \frac{1}{2} \sum_{m=0}^{i-1} \langle \mathbf{z}_m, \mathbf{z}_{i-m-1} \rangle + \frac{\varepsilon}{2} \langle \mathbf{p}, [\mathbf{z}_0, \mathbf{z}_i] \rangle + \\ &\quad + \frac{\varepsilon^2}{8} \langle \mathbf{p}, \mathbf{p} \rangle \sum_{m=0}^{i-1} \sum_{\substack{j,k=1 \\ j \neq k}}^{N} (\vartheta \nu_k)^{m+1} (\vartheta \nu_j)^{i-m} \langle \mathbf{y}_k, \mathbf{y}_j \rangle = H_i^{(N)}(\varepsilon),\end{aligned}$$

with $0 \le i \le N-1$. In the above computation we have taken into account the polynomial identity

$$\nu_k^{i+1} \nu_j - \nu_j^{i+1} \nu_k = (\nu_k - \nu_j) \sum_{m=0}^{i-1} \nu_k^{m+1} \nu_j^{i-m}.$$

The involutivity of the integrals $\{H_k^{(N)}(\varepsilon)\}_{k=0}^{N-1}$ is ensured thanks to Proposition 2.10.

□

Remark 3.10 *There is an alternative way to write the map $(\widetilde{\mathcal{D}}_\varepsilon^N)_2$ in Eq. (3.74). Explicitly it reads:*

$$\begin{cases} \widehat{\mathbf{z}}_0 = \mathbf{z}_0^* + \varepsilon \, \mathrm{ad}_{\mathbf{p}} \, \mathbf{z}_1^* + \dfrac{\varepsilon^2}{2} \mathrm{ad}_{\mathbf{p}}^2 \, \mathbf{z}_2^* + ... + \dfrac{\varepsilon^{N-1}}{2^{N-2}} \mathrm{ad}_{\mathbf{p}}^{N-1} \, \mathbf{z}_{N-1}^*, \\ \widehat{\mathbf{z}}_1 = \mathbf{z}_1^* + \varepsilon \, \mathrm{ad}_{\mathbf{p}} \, \mathbf{z}_2^* + \dfrac{\varepsilon^2}{2} \mathrm{ad}_{\mathbf{p}}^2 \, \mathbf{z}_3^* + ... + \dfrac{\varepsilon^{N-2}}{2^{N-3}} \mathrm{ad}_{\mathbf{p}}^{N-2} \, \mathbf{z}_{N-1}^*, \\ ... \\ \widehat{\mathbf{z}}_{N-1} = \mathbf{z}_{N-1}^*. \end{cases} \tag{3.75}$$

Let us define the vectors $\widehat{\mathbf{Z}} \doteq (\widehat{\mathbf{z}}_0, ..., \widehat{\mathbf{z}}_{N-1})^T$ and $\mathbf{Z}^* \doteq (\mathbf{z}_0^*, ..., \mathbf{z}_{N-1}^*)^T$. It is easy to see that always exists a $N \times N$ upper triangular matrix, say $A \doteq (A)_{i,j}$, $1 \le i, j \le N$, such that

$$\widehat{\mathbf{Z}} = e^A \, \mathbf{Z}^*.$$

The explicit form of A is the following one:

$$
\begin{aligned}
&(A)_{i,j} = 0, \qquad i \geq j, \\
&(A)_{i,i+2k+1} = \frac{\varepsilon^{2k+1}\,\mathrm{ad}_{\mathbf{p}}^{2k+1}}{2^{2k}\,(2k+1)}, \qquad 0 \leq k \leq \frac{N-1-i}{2}, \\
&(A)_{i,i+2k+2} = 0, \qquad 0 \leq k \leq \frac{N-2-i}{2}.
\end{aligned}
$$

For instance, fixing $N = 6$, we get:

$$
A = \begin{pmatrix}
0 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & 0 & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{12} & 0 & \dfrac{\varepsilon^5\,\mathrm{ad}_{\mathbf{p}}^5}{80} \\
0 & 0 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & 0 & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{12} & 0 \\
0 & 0 & 0 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & 0 & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{12} \\
0 & 0 & 0 & 0 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & 0 \\
0 & 0 & 0 & 0 & 0 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

so that

$$
e^A = \begin{pmatrix}
1 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & \dfrac{\varepsilon^2\,\mathrm{ad}_{\mathbf{p}}^2}{2} & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{4} & \dfrac{\varepsilon^4\,\mathrm{ad}_{\mathbf{p}}^4}{8} & \dfrac{\varepsilon^5\,\mathrm{ad}_{\mathbf{p}}^5}{16} \\
0 & 1 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & \dfrac{\varepsilon^2\,\mathrm{ad}_{\mathbf{p}}^2}{2} & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{4} & \dfrac{\varepsilon^4\,\mathrm{ad}_{\mathbf{p}}^4}{8} \\
0 & 0 & 1 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & \dfrac{\varepsilon^2\,\mathrm{ad}_{\mathbf{p}}^2}{2} & \dfrac{\varepsilon^3\,\mathrm{ad}_{\mathbf{p}}^3}{4} \\
0 & 0 & 0 & 1 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} & \dfrac{\varepsilon^2\,\mathrm{ad}_{\mathbf{p}}^2}{2} \\
0 & 0 & 0 & 0 & 1 & \varepsilon\,\mathrm{ad}_{\mathbf{p}} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

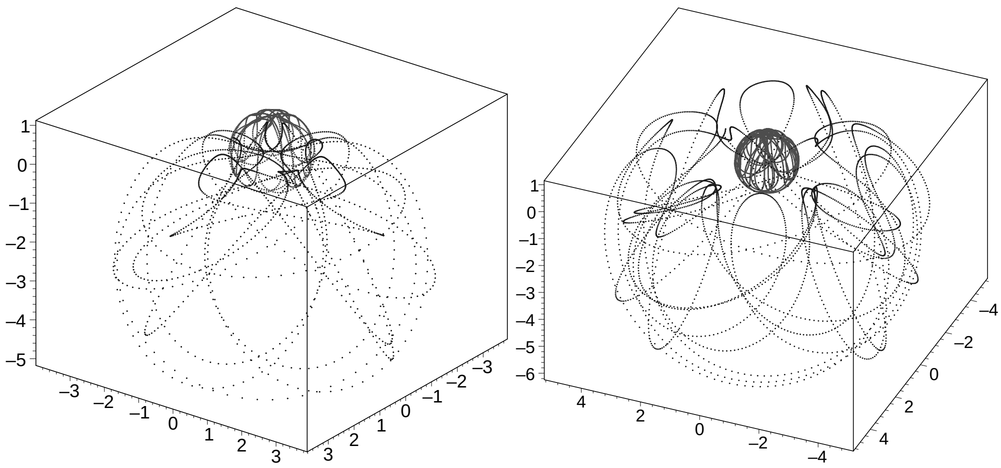
Numerics

Obviously, also the integrable Poisson map given in Eq. (3.70) can be easily iterated, allowing us to obtain some 3D visualizations of the discrete-time dynamics of the extended Lagrange tops. We present here some pictures, obtained using a MAPLE 8 program, corresponding to the discrete-time first extension of the Lagrange top, see Eqs. (3.72).

The input parameters are:

- the intensity of the external field, p;
- the discretization parameter, ε;
- the number of iteration of the map, N;
- the initial values of the coordinate functions, $(\mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2)$.

The output is a 3D plot of N consequent points (z_2^1, z_2^2, z_2^3), describing the evolution of the axis of symmetry of the top on the surface $\langle \mathbf{z}_2, \mathbf{z}_2 \rangle$=constant and N consequent points $(z_1^1 - z_2^1, z_1^2 - z_2^2, z_1^3 - z_2^3)$ describing the evolution of the satellite.



3.2.5 The discrete-time rational Lagrange chain

Using the contraction technique presented in Subsection 2.5.2 we can now construct an integrable discrete-time version of the equations of motion (2.95) of the M-site rational Lagrange chain. They are:

$$\begin{cases} \dot{\mathbf{m}}_i = [\,\mathbf{p}, \mathbf{a}_i\,] + \Big[\,\mu_i\,\mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{m}_i\,\Big]\ , \\[2ex] \dot{\mathbf{a}}_i = \Big[\,\mu_i\,\mathbf{p} + \sum_{k=1}^M \mathbf{m}_k, \mathbf{a}_i\,\Big]\ , \end{cases} \tag{3.77}$$

with $1 \le i \le M$. We recall that they are Hamiltonian equations with respect the Lie-Poisson brackets on $\oplus^M \mathfrak{e}^*(3)$,

$$\Big\{m_i^\alpha, m_j^\beta\Big\} = -\delta_{i,j}\,\varepsilon_{\alpha\beta\gamma}\,m_i^\gamma, \quad \Big\{m_i^\alpha, a_j^\beta\Big\} = -\delta_{i,j}\,\varepsilon_{\alpha\beta\gamma}\,a_i^\gamma, \quad \Big\{a_i^\alpha, a_j^\beta\Big\} = 0, \tag{3.78}$$

with $1 \le i, j \le M$, with the Hamiltonian function given by

$$\mathcal{H}^r_{M,2} \doteq \sum_{k=1}^M \langle\,\mathbf{p}, \mu_k\,\mathbf{m}_k + \mathbf{a}_k\,\rangle + \frac{1}{2}\sum_{i,k=1}^M \langle\,\mathbf{m}_i, \mathbf{m}_k\,\rangle. \tag{3.79}$$

If $M = 1$ the Hamiltonian (3.79) gives the sum of the two integrals of motion of the Lagrange top. Recall that the functions

$$C_k^{(1)} \doteq \langle\,\mathbf{m}_k, \mathbf{a}_k\,\rangle, \qquad C_k^{(2)} \doteq \frac{1}{2}\langle\,\mathbf{a}_k, \mathbf{a}_k\,\rangle, \qquad 1 \le k \le M, \tag{3.80}$$

are Casimirs for the brackets (3.78).

The following propositions holds.

Proposition 3.10 *The map* $\mathcal{D}_\varepsilon^{M,2}$ *defined by*

$$\begin{cases} \widehat{\mathbf{m}}_i \doteq (\mathbf{1} + \varepsilon\,\mu_i\,\mathbf{p})\Big(\mathbf{1} + \varepsilon\sum_{j=1}^M \mathbf{m}_j\Big)\ \mathbf{m}_i\ \Big(\mathbf{1} + \varepsilon\sum_{j=1}^M \mathbf{m}_j\Big)^{-1}(\mathbf{1} + \varepsilon\,\mu_i\,\mathbf{p})^{-1} + \varepsilon\ [\,\mathbf{p}, \widehat{\mathbf{a}}_i\ ]\,, \\[2ex] \widehat{\mathbf{a}}_i \doteq (\mathbf{1} + \varepsilon\,\mu_i\,\mathbf{p})\Big(\mathbf{1} + \varepsilon\sum_{j=1}^M \mathbf{m}_j\Big)\ \mathbf{a}_i\ \Big(\mathbf{1} + \varepsilon\sum_{j=1}^M \mathbf{m}_j\Big)^{-1}(\mathbf{1} + \varepsilon\,\mu_i\,\mathbf{p})^{-1}\,, \end{cases} \tag{3.81}$$

with $1 \le i \le M$ *and* $\varepsilon \in \mathbb{R} \setminus \{0\}$, *is Poisson with respect to the brackets (3.78) and has* $2M$ *independent and involutive integrals of motion assuring its complete integrability:*

$$
\begin{aligned}
R_k(\varepsilon) &\doteq \langle\, \mathbf{p}, \mathbf{m}_k \,\rangle - \frac{\varepsilon}{2} \langle\, \mathbf{p}, \left[\, \mathbf{m}_k, \textstyle\sum_{j=1}^{M} \mathbf{m}_j \,\right] \rangle + \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{M} \Bigg[\left(\frac{\langle\, \mathbf{m}_k, \mathbf{m}_j \,\rangle}{\mu_k - \mu_j} - 2\, \frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\mu_k - \mu_j)^3} \right) \left(1 + \frac{\varepsilon^2}{4}\, \mu_k\, \mu_j \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) + \\
&+ \frac{\langle\, \mathbf{m}_k, \mathbf{a}_j \,\rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4}\, \mu_k^2 \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) - \frac{\langle\, \mathbf{m}_j, \mathbf{a}_k \,\rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4}\, \mu_j^2 \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) \Bigg], \qquad (3.82a)
\end{aligned}
$$

$$
\begin{aligned}
S_k(\varepsilon) &\doteq \langle\, \mathbf{p}, \mathbf{a}_k \,\rangle + \frac{1}{2} \langle\, \mathbf{m}_k, \mathbf{m}_k \,\rangle \left(1 + \frac{\varepsilon^2}{4}\, \mu_k^2 \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) - \frac{\varepsilon}{2} \langle\, \mathbf{p}, \left[\, \mathbf{a}_k, \textstyle\sum_{j=1}^{M} \mathbf{m}_j \,\right] \rangle + \\
&+ \sum_{\substack{j=1\\ j\neq k}}^{M} \left[\frac{\langle\, \mathbf{a}_k, \mathbf{m}_j \,\rangle}{\mu_k - \mu_j} \left(1 + \frac{\varepsilon^2}{4}\, \mu_k\, \mu_j \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) + \frac{\langle\, \mathbf{a}_k, \mathbf{a}_j \,\rangle}{(\mu_k - \mu_j)^2} \left(1 + \frac{\varepsilon^2}{4}\, \mu_k^2 \,\langle\, \mathbf{p}, \mathbf{p} \,\rangle \right) \right], \qquad (3.82b)
\end{aligned}
$$

with $1 \le k \le M$.

Proof: We can use the contraction technique performed in Subsection 2.5.2. Namely we have to consider a discrete-time $\mathfrak{su}(2)$ rational Gaudin model, described by equations of motion (3.63), with $2M$ sites, and to apply, in the contraction limit $\vartheta \to 0$, the map in Eqs. (2.97a-2.97b):

$$
\mathbf{m}_i \doteq \mathbf{y}_{2i} + \mathbf{y}_{2i-1}, \qquad \mathbf{a}_i \doteq \vartheta\, (\nu_{2i}\, \mathbf{y}_{2i} + \nu_{2i-1}\, \mathbf{y}_{2i-1}), \qquad 1 \le i \le M, \qquad (3.83)
$$

where the ν_i's are $2M$ distinct parameters. Moreover the pole coalescence is given by, see Eq. (2.98):

$$
\lambda_{2i} \equiv \vartheta\, \nu_{2i} + \mu_i, \qquad \lambda_{2i-1} \equiv \vartheta\, \nu_{2i-1} + \mu_i, \qquad 1 \le i \le M, \qquad (3.84)
$$

where the λ_i's are the $2M$ parameters of the Gaudin model and the μ_i's are the M parameters of the rational Lagrange chain.

Using Eq. (3.83) and the map (3.65a) with $N = 2M$ we immediately obtain the contracted version of $(\mathcal{D}_{\varepsilon}^{2M})_1$. It reads

$$
\mathbf{m}_i^* = \mathbf{y}_{2i}^* + \mathbf{y}_{2i-1}^* = \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^{M} \mathbf{m}_j \right)\, \mathbf{m}_i\, \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^{M} \mathbf{m}_j \right)^{-1}, \qquad (3.85a)
$$

$$
\mathbf{a}_i^* = \vartheta\, (\nu_{2i}\, \mathbf{y}_{2i}^* + \nu_{2i-1}\, \mathbf{y}_{2i-1}^*) = \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^{M} \mathbf{m}_j \right)\, \mathbf{a}_i\, \left(\mathbf{1} + \varepsilon \textstyle\sum_{j=1}^{M} \mathbf{m}_j \right)^{-1}, \qquad (3.85b)
$$

with $1 \le i \le M$. Using Eqs. (3.83-3.84) and the map (3.65b) with $N = 2M$ we get the contracted version of $(\mathcal{D}_{\varepsilon}^{2M})_2$. It reads

$$
\begin{aligned}
\widehat{\mathbf{m}}_i &= \widehat{\mathbf{y}}_{2i} + \widehat{\mathbf{y}}_{2i-1} = \\
&= (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})\; \mathbf{m}_i^*\; (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})^{-1} + \varepsilon\, \left[\mathbf{p}, (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})\; \mathbf{a}_i^*\; (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})^{-1} \right] + O(\vartheta^2), \quad (3.86a) \\
\widehat{\mathbf{a}}_i &= \vartheta\, (\nu_{2i}\, \widehat{\mathbf{y}}_{2i} + \nu_{2i-1}\, \widehat{\mathbf{y}}_{2i-1}) = (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})\; \mathbf{a}_i^*\; (\mathbf{1} + \varepsilon\, \mu_i\, \mathbf{p})^{-1} + O(\vartheta^2). \qquad (3.86b)
\end{aligned}
$$

Performing the limit $\vartheta \to 0$ in Eqs. (3.86a-3.86b) and combining the resulting equations with the maps in Eqs. (3.85a-3.85b) we obtain the map in Eq. (3.81). Its Poisson property is ensured thanks to the Poisson property of the map (3.63).

Let us construct the integrals (3.82a-3.82b). We can make a computation similar to the one done in the proof of Proposition 2.23. We have:

$$R_i(\varepsilon) = \lim_{\vartheta\to 0} [\, H_{2i}(\varepsilon) + H_{2i-1}(\varepsilon)\,],$$
$$S_i(\varepsilon) = \lim_{\vartheta\to 0} [\, \vartheta\,(\nu_{2i}\, H_{2i}(\varepsilon) + \nu_{2i-1}\, H_{2i-1}(\varepsilon))\,],$$

being $\{H_i(\varepsilon)\}_{i=1}^{2M}$ the set of integrals in Eq. (3.64). We can perform the computation just for the ε-dependent terms, thanks to Proposition 2.23. We get (denoting with $\{R_i\}_{i=1}^{M}$ and $\{S_i\}_{i=1}^{M}$ the sets of continuous-time integrals given in Eqs. (2.93a-2.93b)):

$$\begin{aligned}
H_{2i}(\varepsilon) + H_{2i-1}(\varepsilon) &= R_i - \frac{\varepsilon}{2}\sum_{\substack{j=1\\ j\neq i}}^{M}\langle\, \mathbf{p}, [\, \mathbf{y}_{2i}+\mathbf{y}_{2i-1}, \mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,]\,\rangle + \frac{\varepsilon^2}{4}\,\langle\, \mathbf{p},\mathbf{p}\,\rangle\,\times\\
&\times\sum_{\substack{j=1\\ j\neq i}}^{M}\Bigg\{\frac{(\vartheta\,\nu_{2i}+\mu_i)\,(\vartheta\,\nu_{2j-1}+\mu_j)\,\langle\, \mathbf{y}_{2i}, \mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1 - \frac{\vartheta\,(\nu_{2i}-\nu_{2j-1})}{\mu_i-\mu_j} + \frac{\vartheta^2\,(\nu_{2i}-\nu_{2j-1})^2}{(\mu_i-\mu_j)^2}\right] +\\
&+\frac{(\vartheta\,\nu_{2i}+\mu_i)\,(\vartheta\,\nu_{2j}+\mu_j)\,\langle\, \mathbf{y}_{2i}, \mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1 - \frac{\vartheta\,(\nu_{2i}-\nu_{2j})}{\mu_i-\mu_j} + \frac{\vartheta^2\,(\nu_{2i}-\nu_{2j-1})^2}{(\mu_i-\mu_j)^2}\right] +\\
&+\frac{(\vartheta\,\nu_{2i-1}+\mu_i)\,(\vartheta\,\nu_{2j-1}+\mu_j)\,\langle\, \mathbf{y}_{2i-1}, \mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1 - \frac{\vartheta\,(\nu_{2i-1}-\nu_{2j-1})}{\mu_i-\mu_j} + \frac{\vartheta^2\,(\nu_{2i}-\nu_{2j-1})^2}{(\mu_i-\mu_j)^2}\right] +\\
&+\frac{(\vartheta\,\nu_{2i-1}+\mu_i)\,(\vartheta\,\nu_{2j}+\mu_j)\,\langle\, \mathbf{y}_{2i-1}, \mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1 - \frac{\vartheta\,(\nu_{2i-1}-\nu_{2j})}{\mu_i-\mu_j} + \frac{\vartheta^2\,(\nu_{2i}-\nu_{2j-1})^2}{(\mu_i-\mu_j)^2}\right] + O(\vartheta^5)\Bigg\} =\\
&= R_i - \frac{\varepsilon}{2}\sum_{\substack{j=1\\ j\neq i}}^{M}\langle\, \mathbf{p}, [\, \mathbf{y}_{2i}+\mathbf{y}_{2i-1}, \mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,]\,\rangle +\\
&+\frac{\varepsilon^2}{4}\,\langle\, \mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_i\,\mu_j\,\frac{\langle\, \mathbf{y}_{2i}+\mathbf{y}_{2i-1}, \mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j} +\\
&+\frac{\varepsilon^2}{4}\,\langle\, \mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_i^2\,\frac{\langle\, \mathbf{y}_{2i}+\mathbf{y}_{2i-1}, \vartheta\,(\nu_{2j}\,\mathbf{y}_{2j}+\nu_{2j-1}\,\mathbf{y}_{2j-1})\,\rangle}{(\mu_i-\mu_j)^2} +\\
&+\frac{\varepsilon^2}{4}\,\langle\, \mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_j^2\,\frac{\langle\, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}), \mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,\rangle}{(\mu_i-\mu_j)^2} +\\
&+\frac{\varepsilon^2}{2}\,\langle\, \mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_i\,\mu_j\,\frac{\langle\, \vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}), \vartheta\,(\nu_{2j}\,\mathbf{y}_{2j}+\nu_{2j-1}\,\mathbf{y}_{2j-1})\,\rangle}{(\mu_i-\mu_j)^3} + O(\vartheta^3).
\end{aligned}$$

Using the map (3.83) and performing the limit $\vartheta\to 0$ we obtain the integrals $\{R_i(\varepsilon)\}_{i=1}^{M}$ given in Eq. (3.82a). For the integrals $\{S_i(\varepsilon)\}_{i=1}^M$ we obtain:

$$
\begin{aligned}
&\vartheta\,(\nu_{2i}\,H_{2i}(\varepsilon)+\nu_{2i-1}\,H_{2i-1}(\varepsilon)) = S_i - \frac{\varepsilon}{2}\sum_{\substack{j=1\\ j\neq i}}^{M}\langle\,\mathbf{p},[\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}),\mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,]\,\rangle +\\
&+\frac{\varepsilon^2}{4}\,\mu_i^2\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\langle\,\mathbf{y}_{2i},\mathbf{y}_{2i-1}\,\rangle+\frac{\varepsilon^2}{4}\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\times\\
&\times\sum_{\substack{j=1\\ j\neq i}}^{M}\Bigg\{\frac{\vartheta\,\nu_{2i}\,(\vartheta\,\nu_{2i}+\mu_i)\,(\vartheta\,\nu_{2j-1}+\mu_j)\,\langle\,\mathbf{y}_{2i},\mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i}-\nu_{2j-1})}{\mu_i-\mu_j}\right]+\\
&+\frac{\vartheta\,\nu_{2i}\,(\vartheta\,\nu_{2i}+\mu_i)\,(\vartheta\,\nu_{2j}+\mu_j)\,\langle\,\mathbf{y}_{2i},\mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i}-\nu_{2j})}{\mu_i-\mu_j}\right]+\\
&+\frac{\vartheta\,\nu_{2i-1}\,(\vartheta\,\nu_{2i-1}+\mu_i)\,(\vartheta\,\nu_{2j-1}+\mu_j)\,\langle\,\mathbf{y}_{2i-1},\mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i-1}-\nu_{2j-1})}{\mu_i-\mu_j}\right]+\\
&+\frac{\vartheta\,\nu_{2i-1}\,(\vartheta\,\nu_{2i-1}+\mu_i)\,(\vartheta\,\nu_{2j}+\mu_j)\,\langle\,\mathbf{y}_{2i-1},\mathbf{y}_{2j}\,\rangle}{\mu_i-\mu_j}\left[1-\frac{\vartheta\,(\nu_{2i-1}-\nu_{2j})}{\mu_i-\mu_j}\right]+O(\vartheta^5)\Bigg\}=\\
&=S_i-\frac{\varepsilon}{2}\sum_{\substack{j=1\\ j\neq i}}^{M}\langle\,\mathbf{p},[\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}),\mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,]\,\rangle+\frac{\varepsilon^2}{4}\,\mu_i^2\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\langle\,\mathbf{y}_{2i},\mathbf{y}_{2i-1}\,\rangle+\\
&+\frac{\varepsilon^2}{4}\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_i\,\mu_j\,\frac{\langle\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}),\mathbf{y}_{2j}+\mathbf{y}_{2j-1}\,\rangle}{\mu_i-\mu_j}+\\
&+\frac{\varepsilon^2}{4}\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j=1\\ j\neq i}}^{M}\mu_i^2\,\frac{\langle\,\vartheta\,(\nu_{2i}\,\mathbf{y}_{2i}+\nu_{2i-1}\,\mathbf{y}_{2i-1}),\vartheta\,(\nu_{2j}\,\mathbf{y}_{2j}+\nu_{2j-1}\,\mathbf{y}_{2j-1})\,\rangle}{(\mu_i-\mu_j)^2}+O(\vartheta^3).
\end{aligned}
$$

Notice that

$$\langle\,\mathbf{y}_{2i},\mathbf{y}_{2i-1}\,\rangle=\frac{1}{2}\,\langle\,\mathbf{y}_{2i}+\mathbf{y}_{2i-1},\mathbf{y}_{2i}+\mathbf{y}_{2i-1}\,\rangle-C_{2i}-C_{2i-1},$$

where the functions $C_i\doteq\langle\,\mathbf{y}_i,\mathbf{y}_i\,\rangle$, $1\le i\le 2M$, are Casimirs for $\oplus^{2M}\mathfrak{su}^*(2)$.

Using the map (3.83) and performing the limit $\vartheta\to 0$ we obtain the integrals $\{S_i(\varepsilon)\}_{i=1}^M$ given in Eq. (3.82b).

The Hamiltonians (3.82a-3.82b) are in involution w.r.t. the Lie-Poisson brackets (3.78) thanks to Proposition 2.16.

□

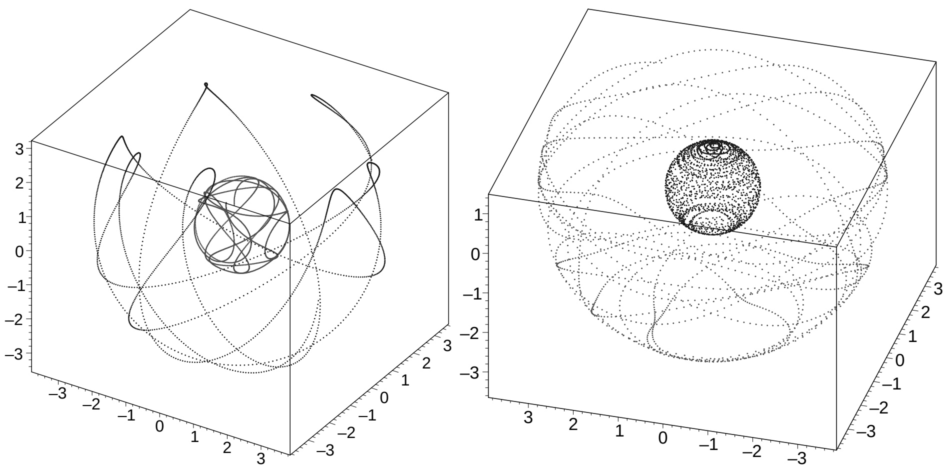
The discrete-time version of the Hamiltonian (2.94) is given by

$$\begin{aligned}\mathcal{H}_{M,2}(\varepsilon) \doteq\ & \sum_{k=1}^{M}[\,\mu_k\,R_k(\varepsilon)+S_k(\varepsilon)\,] = \\ =\ & \sum_{k=1}^{M}\langle\,\mathbf{p},\mu_k\,\mathbf{m}_k+\mathbf{a}_k\,\rangle+\frac{1}{2}\sum_{j,k=1}^{M}\langle\,\mathbf{m}_j,\mathbf{m}_k\,\rangle\left(1+\frac{\varepsilon^2}{4}\,\mu_j\,\mu_k\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\right)- \\ & -\frac{\varepsilon}{4}\sum_{\substack{j,k=1\\ j\neq k}}^{M}(\mu_k-\mu_j)\,\langle\,\mathbf{p},[\,\mathbf{m}_k,\mathbf{m}_j\,]\,\rangle-\frac{\varepsilon}{2}\,\langle\,\mathbf{p},\left[\,\textstyle\sum_{k=1}^{M}\mathbf{a}_k,\sum_{j=1}^{M}\mathbf{m}_j\,\right]\rangle+ \\ & +\frac{\varepsilon^2}{4}\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j,k=1\\ j\neq k}}^{M}\mu_k\langle\,\mathbf{m}_k,\mathbf{a}_j\,\rangle+\frac{\varepsilon^2}{8}\,\langle\,\mathbf{p},\mathbf{p}\,\rangle\sum_{\substack{j,k=1\\ j\neq k}}^{M}\langle\,\mathbf{a}_k,\mathbf{a}_j\,\rangle. \qquad (3.87)\end{aligned}$$

Notice that we still have the linear integral $\sum_{k=1}^{M}R_k(\varepsilon)=\sum_{k=1}^{M}\langle\,\mathbf{p},\mathbf{m}_k\,\rangle$. Moreover, fixing $M=1$ (and $\mu_1=0$) in the Hamiltonian (3.87) we get the discrete-time Hamiltonian (3.44) of the Lagrange top.

Numerics

We present here a visualization, for $M=2$, of the integrable discrete-time evolution of the axes of symmetry of the tops given by the map (3.81).



The input parameters are:

- the intensity of the external field, p;
- the values of the parameters μ_1 and μ_2;
- the discretization parameter, ε;
- the number of iteration of the map, N;
- the initial values of the coordinate functions, $(\mathbf{m}_1,\mathbf{a}_1)$ and $(\mathbf{m}_2,\mathbf{a}_2)$.

The output is a 3D plot of $N+N$ consequent points (a_1^1, a_1^2, a_1^3) and (a_2^1, a_2^2, a_2^3) describing the evolution of the axes of symmetry of the tops respectively on the surfaces $\langle\, \mathbf{a}_1, \mathbf{a}_1 \,\rangle$=constant and $\langle\, \mathbf{a}_2, \mathbf{a}_2 \,\rangle$=constant.

4

Conclusions and open perspectives

We list here the results obtained in the present Thesis together with the related open problems.

- We have presented a general and systematic reduction, based on Inönü-Wigner contractions, of classical Gaudin models. Suitable algebraic and pole coalescence procedures performed on the N-pole Gaudin Lax matrices, enabled us to construct one-body and many-body hierarchies of integrable models sharing the same (linear) r-matrix structure of the ancestor models. Moreover, this technique can be applied for any simple Lie algebra $\mathfrak{g}$ and whatever be the dependence (rational, trigonometric, elliptic) on the spectral parameter.

  Fixing $\mathfrak{g} \equiv \mathfrak{su}(2)$, we have constructed the so called $\mathfrak{su}(2)$ hierarchies. These families of integrable systems are the Leibniz extensions of $\mathfrak{su}(2)$ Gaudin models. For instance, assuming $N = 2$ and a rational dependence on the spectral parameter, we have obtained the standard Lagrange top associated with $\mathfrak{e}^*(3)$, in the one-body case, and a homogeneous long-range integrable chain of interacting Lagrange tops, in the many-body one. For an arbitrary order N of the Leibniz extension, the one-body hierarchy consists of a family of generalized Lagrange tops. They provide an interesting example of integrable rigid body dynamics described by a Lagrange top with $N-2$ interacting heavy satellites.

- Using two different approaches to the integrable discretization problem, we have obtained several integrable Poisson maps for the $\mathfrak{su}(2)$ hierarchies. The method of Bäcklund transformations enabled us to construct families of integrable maps for the first (rational) extension of the standard Lagrange top and for the rational Lagrange chain. On the other hand, we have performed a "guesswork procedure" on the integrable map for the Lagrange top obtained in [19] in order to obtain an integrable discretization for the $\mathfrak{su}(2)$ rational Gaudin model. Therefore, a suitable contraction on such a map, enables us - at least in principle - to construct discrete-time versions of all $\mathfrak{su}(2)$ hierarchies. Nevertheless, up to now, we still have not a Lax representation and an r-matrix formulation for such maps (actually they are known just for the discrete-time Lagrange top considered in [19]). Hence, their integrability and Poisson property is proven by direct verification.

  Indeed, a fundamental task is the construction of a Lax representation for the discrete-time $\mathfrak{su}(2)$ rational Gaudin model. In principle, its knowledge enables us to find the Lax pairs for all the contracted systems. The work is in progress, with the collaboration of Yu.B. Suris.

- A natural extension of our discretizations could be the study of a suitable approach for those models with a trigonometric or elliptic dependence on the spectral parameter instead of a rational one. To the best of our knowledge there are not results in this direction in literature.

- It is well-known that the continuous-time Lagrange top admits a tri-Hamiltonian formulation in terms of the so called Reyman-Semenov-Tian-Shansky tensors. More precisely, the whole family of extended Lagrange tops admits a multi-Hamiltonian formulation in terms of such tensors. For our purposes the multi-Hamiltonian approach should be an alternative tool to prove the integrability property of our discrete-time maps. Our first aim is to construct a Poisson pencil for the discrete-time Lagrange top considered in [19]. Secondly, we have to generalize this construction to an arbitrary (discrete-time) extension of the Lagrange top. The work is in progress, with the collaboration of Yu.B. Suris.

- We are studying a Lagrangian formulation for the rational Lagrange chain, both in the continuous-time and in the discrete-time settings. Our aim is to generalize the results contained in [19] for the Lagrange top. At the moment, we are able to overcome the main difficulty, which is the abscence of the Lagrangian function in the strict sense, due to the degeneracy of the Legendre transform. We have discovered a regularization procedure for this system, which leads to a well-defined Lagrangian formulation of certain nearby systems (with a limit singular on the level of Lagrangians, but regular on the level of Hamiltonians and equations of motions). The work is in progress, with the collaboration of Yu.B. Suris.

- A natural question is whether the the contraction procedure presented in Chapter 2 could be generalized to other Inönü-Wigner contractions. To carry out this program we have first to identify a suitable class of Inönü-Wigner contractions. For our purposes, a good choice is to define Inönü-Wigner contractions through continuous graded contractions for $\mathbb{Z}_N$ graded algebras [62]. It is well-known that they split in two classes: discrete and continuous; in [103] it is proven that all continuous graded contractions can be realized by generalized Inönü-Wigner contractions.

  We have found a linear transformation on the coordinates functions on $\mathfrak{G}_N^* \doteq \oplus^N \mathfrak{g}^*$ that maps the Lie-Poisson tensor $P_N^{\mathfrak{g}}$ into a $\mathbb{Z}_N$ graded one thus allowing us to use the machinery of graded contractions. Remarkably, this linear transformation can be defined for any choice of the Lie algebra $\mathfrak{g}$ and the contraction equations are independent on this choice. Henceforth the solutions of the contraction equations yield a graded contraction on $\mathfrak{G}_N \doteq \oplus^N \mathfrak{g}$ for any choice of the Lie algebra $\mathfrak{g}$. Accordingly, we call these graded contractions "universal". The problem of classifying non-isomorphic universal graded contractions, at least for low values of N, is an interesting and challenging side-problem.

  Once we have fixed the class of Inönü–Wigner contractions we will work with, we have to define the contraction procedure on the Hamiltonians. Preliminary results show that the pole coalescence procedure on the *rational* Gaudin Lax matrices works for some contractions while leads to divergences for others. Such unpleasant feature can be avoided defining the Gaudin Hamiltonians through a bi-Hamiltonian pencil instead of a Lax matrix. Bi-Hamiltonian pencils for the rational Gaudin model can be easily derived through the theory of intertwining operators [30, 79]. We plan to proceed as follows: given a continuous graded contraction on one of the Poisson tensors of the bi-Hamiltonian pencil, we search for a contraction on the second Poisson tensor that preserves the compatibility. Then, we derive a family of involutive Hamiltonians constructing Gel'fand–Zakharevich chains for the Poisson pencil. We expect that this procedure will lead to new integrable systems. This work is in progress with the collaboration of F. Musso.

- An interesting perspective is the quantization of the obtained extended Gaudin models. The first natural candidate to this target could be the rational Lagrange chain. We recall that a standard procedure can be applied to quantize the underlying algebraic structures in order to get a well-known quantum linear r-matrix algebra. An interesting problem concerns the construction of explicit solutions to the spectral problem. As a matter of fact, the simple case of the Lagrange top requires the introduction of Heun functions as eigenfunctions of the Hamiltonian operator (see Appendix A).

Appendix A

Some notes on the quantum Lagrange top

The aim of this Appendix is to briefly show some preliminary results on the study of the quantum Lagrange top. Actually, the original motivation of this topic was the interest on the quantization of Bäcklund transformations for the classical Lagrange top [53]. A collaboration with V.B. Kuznetsov has been initiated, but he tragically died in December 2005.

Let us recall that when one searches for the simplest BT of an integrable system, then one finds a one-dimensional family $\{\mathcal{B}_\eta \,|\, \eta \in \mathbb{C}\}$ of them. The Bäcklund parameter η is canonically conjugate to μ, i.e. $\mu = -\partial F_\eta / \partial \eta$ with F_η generating function of $\{\mathcal{B}_\eta \,|\, \eta \in \mathbb{C}\}$. Here μ is bound to η by the equation of an algebraic curve (dependent on the integrals), which is exactly the characteristic curve that appears in the linearization of the integrable system:

$$W(\eta, \mu; \{H_i\}) \doteq \det(L(\eta) - \mu\, \mathbb{1}) = 0. \tag{A.1}$$

This property is called *spectrality* of the BT [55, 56]. The meaning of equation (A.1) becomes clear if we turn to the quantum case. In the pioneering paper by Pasquier and Gaudin [72], a remarkable connection has been established between the classical BT $\mathcal{B}_\eta$ for the Toda lattice and the famous Baxter's $\mathbb{Q}$-operator [13]. They have constructed certain integral operators $\mathbb{Q}_\eta$, whose properties parallel those of the classical BTs. In particular they commute with the conserved quantities H_i. In the quantum case the canonical transformations are replaced by suitable similarity transformations. The correspondence between the kernel Q_η of $\mathbb{Q}_\eta$ and the generating function F_η of $\mathcal{B}_\eta$ is given by the semiclassical relation $Q_\eta \sim \exp\left(-\mathrm{i}F_\eta/\hbar\right), \, \hbar \to 0$ [55]. Moreover, an interesting property of $\mathbb{Q}_\eta$ is that its eigenvalues $\phi_\nu(\eta)$ on the joint eigenvectors Ψ_ν of H_i and $\mathbb{Q}_\eta$ labelled with the quantum numbers ν, $\mathbb{Q}_\eta \Psi_\nu = \phi_\nu(\eta)\Psi_\nu$, satisfy a certain differential or difference equation, containing the eigenvalues h_i of H_i, which in the classical limit goes into the spectrality equation (A.1).

In the last decade the $\mathbb{Q}$-Baxter operators have been constructed for quantum integrable systems associated with quadratic r-matrix structures (see for instance [25, 54]). To the best of our knowledge, the explicit construction of Baxter operators for $\mathfrak{su}(2)$ rational Gaudin models (associated with a linear r-matrix structure) is still to be done, even if some preliminary (and not published) results have been obtained by V.B. Kuznetsov [52]. Actually the kernel of $\mathbb{Q}_\eta$ can be computed performing a suitable limit procedure on the kernel of $\mathbb{Q}_\eta$ for the XXX $\mathfrak{su}(2)$ Heisenberg model. An interesting task could be to recover the generating function of classical BTs for the $\mathfrak{su}(2)$ Gaudin magnet [41] as a semiclassical limit $Q_\eta \sim \exp\left(-\mathrm{i}F_\eta/\hbar\right), \, \hbar \to 0$.

The first natural reduction of $\mathbb{Q}$-Baxter operators for the $\mathfrak{su}(2)$ rational Gaudin model should allow the construction of $\mathbb{Q}$-Baxter operators for the quantum Lagrange top. In the following we shall give some preliminary results on the study of the spectral problem of the quantum Lagrange top. They are a necessary tool to study the problem of quantization of classical BTs. We remark here that the study of quantum tops is an active research activity, where one can see the intimate connection between quantum integrable systems and special functions; see for instance [57].

The physical model

The quantum Lagrange top (QLT) is a quantum axially symmetric rigid rotator in a constant homogeneous field $\mathbf{p} \doteq (0, 0, p) \in \mathbb{R}^3$. Let us use a different and more convenient notation for the generators of the Lie algebra $\mathfrak{e}(3)$. We denote with J_i, $i = 1, 2, 3$, the components of the angular momentum and with P_i, $i = 1, 2, 3$, those of the vector pointing from the fixed point. They satisfy the following commutation relations:

$$[\, J_\alpha, J_\beta \,] = \mathrm{i}\, \hbar\, \varepsilon_{\alpha\beta\gamma}\, J_\gamma, \qquad [\, J_\alpha, P_\beta \,] = \mathrm{i}\, \hbar\, \varepsilon_{\alpha\beta\gamma}\, P_\gamma, \tag{A.2a}$$

$$[\, P_\alpha, P_\beta \,] = 0, \qquad \alpha, \beta, \gamma = 1, 2, 3, \tag{A.2b}$$

where $\varepsilon_{\alpha\beta\gamma}$ is the full antisymmetric tensor with $\varepsilon_{123} \equiv 1$. Hereafter we set $\hbar \equiv 1$. The center of the universal enveloping algebra of $\mathfrak{e}(3)$ is generated by two Casimir elements,

$$C_1 \doteq \sum_{\alpha=1}^{3} P_\alpha\, J_\alpha = \sum_{\alpha=1}^{3} J_\alpha\, P_\alpha, \qquad C_2 \doteq \sum_{\alpha=1}^{3} P_\alpha\, P_\alpha. \tag{A.3}$$

In parallel with the notation J_1, J_2 and P_1, P_2 we will be using their equivalent complex version $J_\pm \doteq J_1 \pm \mathrm{i}\, J_2$ and $P_\pm \doteq P_1 \pm \mathrm{i}\, P_2$, with the $\mathfrak{e}(3)$ non trivial commutation relations (A.2a-A.2b) replaced by

$$[\, J_3, J_\pm \,] = \pm J_\pm, \qquad [\, J_+, J_- \,] = 2 J_3, \qquad [\, J_3, P_\pm \,] = [\, P_3, J_\pm \,] = \pm P_\pm, \tag{A.4a}$$

$$[\, J_+, P_- \,] = [\, P_+, J_- \,] = 2\, P_3. \tag{A.4b}$$

In the basis $(J_3, J_\pm, P_3, P_\pm)$ the Casimir operators (A.3) read

$$C_1 \doteq \frac{1}{2}\,(J_- P_+ + J_+ X_-) + J_3 P_3, \qquad C_2 \doteq P_3^2 + P_+ P_-. \tag{A.5}$$

The Hamiltonian operator of the QLT in the rest-frame - namely the quantum version of the Hamiltonian (2.79) - can be written as

$$\mathcal{H} \doteq \frac{1}{2}(J_1^2 + J_2^2 + J_3^2) + p\, P_3 = \frac{\{J_+, J_-\}}{4} + \frac{J_3^2}{2} + p\, P_3. \tag{A.6}$$

Here $\{\cdot, \cdot\}$ denotes the anticommutator. We remark that the operator (A.6) can be seen as as the Hamiltonian describing the rotational motion of a rigid symmetric-top molecule in a constant electric field $(0, 0, p)$ [98]. Moreover we are assuming that the top has a unit mass or a unit permanent electric dipole moment.

The six quantum Euler-Poisson equations of motion for the components J_1, J_2, J_3 and P_1, P_2, P_3 can be derived computing their commutators with the Hamiltonian operator (A.6):

$$\begin{aligned}
\dot{P}_1 &= \frac{1}{2}\{J_2, P_3\} - \frac{1}{2}\{J_3, P_2\}, & \dot{J}_1 &= -p\, P_2, \\
\dot{P}_2 &= \frac{1}{2}\{J_3, P_1\} - \frac{1}{2}\{J_1, P_3\}, & \dot{J}_2 &= p\, P_1, \\
\dot{P}_3 &= \frac{1}{2}\{J_1, P_2\} - \frac{1}{2}\{J_2, P_1\}, & \dot{J}_3 &= 0.
\end{aligned}$$

The above equations are the quantum version of Eqs. (2.77) and they go over into the classical ones if we do not follow the ordering of the generators.

The integrability of the QLT requires the existence of a fourth constant of motion. We can immediately notice that $[\, \mathcal{H}, J_3 \,] = 0$. The conservation of J_3 is a direct consequence of the invariance under rotation about the direction of the external field.

For the sake of completeness we briefly present here an r-matrix formulation of the QLT. A straightforward computation leads to the following proposition.

Proposition A.1 *The Lax matrix of the QLT reads*

$$\mathcal{L}(\lambda) \doteq \frac{\mathrm{i}}{\lambda}\begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix} + \frac{\mathrm{i}}{\lambda^2}\begin{pmatrix} P_3 & P_- \\ P_+ & -P_3 \end{pmatrix} + \mathrm{i}\begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix}, \tag{A.8}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. The Lax matrix given in Eq. (A.8) satisfies the linear r-matrix algebra

$$[\,\mathcal{L}(\lambda)\otimes \mathbb{1}, \mathbb{1}\otimes\mathcal{L}(\mu)\,] + [\,r(\lambda-\mu), \mathcal{L}(\lambda)\otimes\mathbb{1} + \mathbb{1}\otimes\mathcal{L}(\mu)\,] = 0,$$

where

$$r(\lambda) \doteq \frac{1}{\lambda}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that the generating function of the integrals of motion of the QLT is given by

$$\mathcal{S}(\lambda) \doteq -\frac{1}{2}\mathrm{tr}\left[\mathcal{L}^2(\lambda)\right] = \frac{C_2}{\lambda^4} + \frac{2\,C_1}{\lambda^3} + \frac{2\,\mathcal{H}}{\lambda^2} + \frac{2\,J_3}{\lambda} + p^2.$$

The spectral problem

Proposition A.2 *[61] Let $\rho > 0$ and consider the manifold of all three-dimensional vectors $\mathbf{P} \doteq (\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta)$, $(\theta,\varphi) \in [\,0,\pi\,]\times[\,0,2\pi\,)$. Let us define the Hilbert space*

$$\mathbb{H} \doteq \left\{\Psi : [\,0,\pi\,]\times[\,0,2\pi\,) \to \mathbb{C}\,|\, \int_0^\pi\int_0^{2\pi} d\theta\, d\varphi\, \sin\theta\,|\,\Psi(\theta,\varphi)\,|^2 < \infty\right\}.$$

The operators $(J_3, J_\pm, P_3, P_\pm)$ defined on $\mathbb{H}$ admit the following irreducible representation

$$J_\pm = e^{\pm\mathrm{i}\varphi}\left(\pm\frac{\partial}{\partial\theta} + \frac{\ell}{\sin\theta} + \mathrm{i}\,\cot\theta\,\frac{\partial}{\partial\varphi}\right), \qquad J_3 = -\mathrm{i}\,\frac{\partial}{\partial\varphi}, \tag{A.9a}$$

$$P_\pm = \rho\, e^{\pm\mathrm{i}\varphi}\sin\theta, \qquad P_3 = \rho\,\cos\theta. \tag{A.9b}$$

The operators (A.9a-A.9b) satisfy the commutation relations given in (A.4a-A.4b). If we refer to the operators given in Eqs. (A.9a-A.9b) we obtain the following formulae for the Casimir operators (A.5):

$$(C_1\,\Psi)(\theta,\varphi) = \rho\,\ell\,\Psi(\theta,\varphi), \qquad (C_2\,\Psi)(\theta,\varphi) = \rho^2\,\Psi(\theta,\varphi), \qquad \forall\,\Psi\in\mathbb{H},$$

where $\ell \in \mathbb{Z}$. The Hamiltonian operator (A.6) becomes the following second-order linear differential operator:

$$\mathcal{H} = -\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{2\sin^2\theta}\left(\ell^2 - \frac{\partial^2}{\partial\varphi^2} + 2\,\mathrm{i}\,\ell\,\cos\theta\,\frac{\partial}{\partial\varphi}\right) + p\,\rho\,\cos\theta. \tag{A.10}$$

Hereafter we set $\rho \equiv 1$. We shall refer also to the Hamiltonian operator describing the free QLT, i.e. $p = 0$, namely

$$\mathcal{H}_0 = -\frac{1}{2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{2\sin^2\theta}\left(\ell^2 - \frac{\partial^2}{\partial\varphi^2} + 2\,\mathrm{i}\,\ell\,\cos\theta\,\frac{\partial}{\partial\varphi}\right). \tag{A.11}$$

Proposition A.3 *Let us consider the spectral problem of the free QLT, namely*

$$(\mathcal{H}_0 - \lambda_0)\Psi_{\lambda_0,m;\ell}(\theta,\varphi) = 0, \tag{A.12a}$$
$$(J_3 - m)\Psi_{\lambda_0,m;\ell}(\theta,\varphi) = 0. \tag{A.12b}$$

The eigenfunctions $\Psi_{\lambda_0,m;\ell}$ *are given by*

$$\Psi_{\lambda_0,m;\ell}(\theta,\varphi) = C_{n,m;\ell}\, e^{\mathrm{i}m\varphi}\, (1-\cos\theta)^{\frac{\alpha}{2}}\, (1+\cos\theta)^{\frac{\beta}{2}}\, P_n^{(\alpha,\beta)}(\cos\theta), \tag{A.13}$$

where $\alpha \doteq |\ell - m|$, $\beta \doteq |\ell + m|$, $(m,\ell) \in \mathbb{Z}\times\mathbb{Z}$, $P_n^{(\alpha,\beta)}(\cos\theta)$ *is a Jacobi polynomial and the normalization constant is*

$$|C_{n,m;\ell}|^2 = \frac{(2\,n+\alpha+\beta+1)\,n!}{2^{\alpha+\beta+2}\,\pi}\frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}. \tag{A.14}$$

The integer number n *is defined through the eigenvalue* λ_0 *of* $\mathcal{H}_0$ *by*

$$\lambda_0 = \frac{1}{2}\,j\,(j+1), \tag{A.15}$$

with $j \in \mathbb{N}$ *defined as*

$$j \doteq n + \frac{\beta+\alpha}{2} = n + \max\left(|\ell|,|m|\right), \qquad j \geq |\ell|,\ j \geq |m|.$$

Proof: Looking at the representation of J_3, see (A.9a), we can immediately factorize the eigenfunctions $\Psi_{\lambda_0,m;\ell}(\theta,\varphi)$, namely

$$\Psi_{\lambda_0,m;\ell}(\theta,\varphi) = \Phi_m(\varphi)\,\Theta_{\lambda_0,m;\ell}(\theta), \qquad \Phi_m(\varphi) = e^{\mathrm{i}m\varphi}, \quad m \in \mathbb{Z}. \tag{A.16}$$

Using the factorization (A.16) we obtain from Eqs. (A.11-A.12a) the differential equation for the function $\Theta_{\lambda_0,m;\ell}(\theta)$. It reads

$$-\frac{1}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\Theta_{\lambda_0,m;\ell}(\theta)}{d\theta}\right) + \left(\frac{m^2 - 2\,\ell\,m\,\cos\theta + \ell^2}{2\,\sin^2\theta} - \lambda_0\right)\Theta_{\lambda_0,m;\ell}(\theta) = 0. \tag{A.17}$$

Let us consider the following mapping on the dependent variable:

$$\Theta_{\lambda_0,m;\ell}(z) \doteq z^{\frac{\alpha}{2}}\,(1-z)^{\frac{\beta}{2}}\,Y_{\lambda_0,m;\ell}(z), \tag{A.18}$$

where

$$z \doteq \frac{1}{2}(1-\cos\theta), \qquad \alpha \doteq |\ell - m|\,, \qquad \beta \doteq |\ell + m|\,, \tag{A.19}$$

with $0 \leq z \leq 1$. We remark that the introduction of the function $Y_{\lambda_0,m;\ell}(z)$ allows one to focus the behaviours of the solutions $\Theta_{\lambda_0,m;\ell}(\theta)$ of Eq. (A.17) in the neighbourhoods of the singularities $z=0$ (i.e. $\theta = 0$) and $z = 1$ (i.e. $\theta = \pi$).

Using the transformation (A.18) we obtain the following Fuchsian differential equation for the function $Y_{\lambda_0,m;\ell}(z)$:

$$\frac{d^2Y_{\lambda_0,m;\ell}(z)}{dz^2} + \left(\frac{\alpha+1}{z} + \frac{\beta+1}{z-1}\right)\frac{dY_{\lambda_0,m;\ell}(z)}{dz} - \frac{q_0}{z\,(z-1)}Y_{\lambda_0,m;\ell}(z) = 0, \tag{A.20}$$

where

$$q_0 \doteq 2\,\lambda_0 - \frac{(\alpha+\beta)(\alpha+\beta+2)}{4}. \tag{A.21}$$

Eq. (A.20) is a Gauss hypergeometric equation. It can be rewritten in the following form:

$$z\,(z-1)\frac{d^2Y_{\lambda_0,m;\ell}(z)}{dz^2}+[\,(\xi+\eta+1)\,z-(\alpha+1)\,]\frac{dY_{\lambda_0,m;\ell}(z)}{dz}+\xi\,\eta\,Y_{\lambda_0,m;\ell}(z)=0, \qquad \text{(A.22)}$$

where

$$\xi \doteq \frac{\alpha+\beta}{2}+\frac{1}{2}\left(1\mp\sqrt{1+8\,\lambda_0}\right), \qquad \eta \doteq \frac{\alpha+\beta}{2}+\frac{1}{2}\left(1\pm\sqrt{1+8\,\lambda_0}\right).$$

The general solution of Eq. (A.22) is given by

$$Y_{\lambda_0,m;\ell}(z)=A\;{}_2F_1\,(\xi,\eta;\gamma\,|\,z)+\;B\;z^{-\alpha}\;{}_2F_1\,(\eta-\gamma+1,\xi-\gamma+1;2-\gamma\,|\,z)\,,$$

where A and B are two normalization constants and ${}_2F_1\,(\xi,\eta;\gamma\,|\,z)$ is the hypergeometric function. Since $-\alpha<0$ we set $B\equiv 0$. Hence the physical eigenfunctions $\Theta_{\lambda_0,m;\ell}(z)$ read

$$\Theta_{\lambda_0,m;\ell}(z)=A\;z^{\frac{\alpha}{2}}\,(1-z)^{\frac{\beta}{2}}\;{}_2F_1\,(\xi,\eta;\gamma\,|\,z)\,. \qquad \text{(A.23)}$$

The solutions (A.23) are well defined in $z=0$. Analyizing the behaviour of this solution in $z=1$ we find that the series appearing in (A.23) must terminate. The resulting solutions are expressed in terms of Jacobi polynomials. Without loss of generality we may choose $\eta>\xi$. Requiring $\xi=-n$, $n\in\mathbb{N}$ and defining $j\doteq n+(\alpha+\beta)/2\in\mathbb{N}$ we find that

$$\lambda_0=\frac{1}{2}\left(n+\frac{\alpha+\beta+1}{2}\right)^2-\frac{1}{8}=\frac{1}{2}\,j\,(j+1),$$

with $j\geq|\ell|$, $j\geq|m|$.

Let us now recall that Jacobi polynomials $P_k^{(s,t)}(x)$, with $-1\leq x\leq 1$ and $k\in\mathbb{N}$ can be defined through the hypergeometric function by the formula [2]

$$P_k^{(s,t)}(x)=\frac{\Gamma(s+1+k)}{\Gamma(s+1)\,k!}\;{}_2F_1\left(-k,k+s+t+1;s+1\,|\,\frac{1}{2}(1-x)\right), \qquad \text{(A.24)}$$

and their orthogonality relation read

$$\int_{-1}^{1}dx\,(1-x)^s\,(1+x)^t\,P_k^{(s,t)}(x)\,P_l^{(s,t)}(x)=\delta_{k,l}\,\frac{2^{s+t+1}\,\Gamma(k+s+1)\,\Gamma(k+t+1)}{(2k+s+t+1)\,\Gamma(k+s+t+1)\,k!}, \qquad \text{(A.25)}$$

with $s>-1$, $t>-1$. Here $\delta_{k,l}$ denotes the usual Kronecker symbol.

From solutions (A.23) we obtain the eigenfunctions $\Psi_{\lambda_0,m;\ell}(\theta,\varphi)$ of $\mathcal{H}_0$ in the form given in Eq. (A.13), where the normalization constant $C_{n,m;\ell}$ (A.14) is obtained requiring

$$\int_0^{\pi}\int_0^{2\pi}d\theta\,d\varphi\,\sin\theta\mid\Psi_{\lambda_0,m;\ell}(\theta,\varphi)\mid^2=1,$$

and using the orthogonality relation (A.25).

□

Remark A.1 *The Hamiltonian $\mathcal{H}_0$ (A.11) can be rewritten as*

$$\mathcal{H}_0=-\frac{1}{2}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta}\right)+\frac{\alpha^2}{8\sin^2\frac{\theta}{2}}+\frac{\beta^2}{8\cos^2\frac{\theta}{2}}, \qquad \text{(A.26)}$$

where α *and* β *are given in Eq. (A.19). The differential operator (A.26) acts on the eigenfunctions* $\Theta_{\lambda_0,m;\ell}(\theta)$*. We now map* $\mathcal{H}_0$ *into an operator on the Hilbert space* $\mathcal{L}^2([0,\pi],d\theta)$ *by writing eigenfunctions of* $\mathcal{H}_0$ *as* $\Theta_{\lambda_0,m;\ell}(\theta) = \sin^{-1/2}(\theta)F_{\lambda_0,m;\ell}(\theta)$*. Hence the eigenvalue equation* $(\mathcal{H}_0 - \lambda_0)\Theta_{\lambda_0,m;\ell}(\theta) = 0$ *is equivalent to* $(\widetilde{\mathcal{H}}_0 - \lambda_0)F_{\lambda_0,m;\ell}(\theta) = 0$*, where*

$$\widetilde{\mathcal{H}}_0 = -\frac{1}{2}\frac{d^2}{d^2\theta} + \frac{\alpha^2 - 1/4}{8\sin^2\frac{\theta}{2}} + \frac{\beta^2 - 1/4}{8\cos^2\frac{\theta}{2}} - \frac{1}{8}. \tag{A.27}$$

The operator (A.27) is exactly a Pöschl-Teller Hamiltonian [33]. The corresponding eigenfunctions and eigenvalues can be found in [33] and they are equivalent to respectively our solutions (A.13) and (A.15).

Remark A.2 *The differential equation (A.20) is equivalent to the following singular Sturm-Liouville problem:*

$$\mathcal{L}_z^{(0)}\, Y_{\lambda_0,m;\ell}(z) = n\,(n+\alpha+\beta+1)\, Y_{\lambda_0,m;\ell}(z), \qquad n \in \mathbb{N}, \quad 0 \le z \le 1, \tag{A.28}$$

with

$$\mathcal{L}_z^{(0)} \doteq z\,(z-1)\frac{d^2}{dz^2} + [\,(\alpha+\beta+2)z - (\alpha+1)\,]\frac{d}{dz}, \tag{A.29}$$

and boundary conditions $\mid Y_{\lambda_0,m;\ell}(0) \mid < \infty$*,* $\mid Y_{\lambda_0,m;\ell}(1) \mid < \infty$*.*

Let us consider the spectral problem of the QLT.

Proposition A.4 *The spectral problem of the QLT is given by the following equations:*

$$(\mathcal{H} - \lambda)\Psi_{\lambda,m;\ell}(\theta,\varphi) = 0, \tag{A.30a}$$

$$(J_3 - m)\Psi_{\lambda,m;\ell}(\theta,\varphi) = 0. \tag{A.30b}$$

The eigenfunctions $\Psi_{\lambda,m;\ell}(\theta,\varphi)$ *are given by*

$$\Psi_{\lambda,m;\ell}(\theta,\varphi) = D_{\lambda,m;\ell}\, e^{\mathrm{i}m\varphi}(1-\cos\theta)^{\frac{\alpha}{2}}(1+\cos\theta)^{\frac{\beta}{2}} \sum_{n=0}^{\infty} c_n\, P_n^{(\alpha,\beta)}(\cos\theta), \tag{A.31}$$

where $\alpha \doteq |\ell - m|$*,* $\beta \doteq |\ell + m|$*,* $(m,\ell) \in \mathbb{Z}\times\mathbb{Z}$*. The coefficients* $\{c_n\}_{n=0}^{\infty}$ *satisfy the three-term recurrence relation*

$$K_n\, c_{n-1} + (L_n - \lambda)\, c_n + M_n\, c_{n+1} = 0, \qquad n \ge 1, \tag{A.32a}$$

$$(L_0 - \lambda)\, c_0 + M_0\, c_1 = 0, \tag{A.32b}$$

with

$$\begin{aligned} K_n &= \frac{2\,p\,(n+\alpha+\beta)(n+\alpha)}{(2\,n+\alpha+\beta-1)(2\,n+\alpha+\beta)}, \\ L_n &= \frac{1}{8}(2\,n+\alpha+\beta+2)(2\,n+\alpha+\beta) - \frac{p\,(\alpha^2-\beta^2)}{(2\,n+\alpha+\beta+2)(2\,n+\alpha+\beta)}, \\ M_n &= \frac{2\,p\,(n+1)(n+\beta+1)}{(2\,n+\alpha+\beta+3)(2\,n+\alpha+\beta+2)}. \end{aligned}$$

The normalization constant $D_{\lambda,m;\ell}$ *is given by*

$$|D_{\lambda,m;\ell}|^2 = \frac{1}{2^{\alpha+\beta+2}\,\pi}\left[\sum_{n=0}^{\infty} c_n^2\, \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2\,n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)\,n!}\right]^{-1}. \tag{A.34}$$

Proof: We can always consider the factorization (A.16), namely

$$\Psi_{\lambda,m;\ell}(\theta,\varphi)=\Phi_m(\varphi)\,\Theta_{\lambda,m;\ell}(\theta),\qquad \Phi_m(\varphi)=e^{\mathrm{i}m\varphi},\quad m\in\mathbb{Z}.$$

Considering the transformation (A.18) with the definitons (A.19) we get the following Fuchsian differential equation:

$$\frac{d^2Y_{\lambda,m;\ell}(z)}{dz^2}+\left(\frac{\alpha+1}{z}+\frac{\beta+1}{z-1}\right)\frac{dY_{\lambda,m;\ell}(z)}{dz}-\frac{4\,p\,z+q}{z\,(z-1)}Y_{\lambda,m;\ell}(z)=0, \tag{A.35}$$

where

$$q\doteq 2\,(\lambda-p)-\frac{(\alpha+\beta)(\alpha+\beta+2)}{4}.$$

Eq. (A.35) is the canonical form of the reduced confluent Heun equation [7].

Let us remark that Eq. (A.35) can be rewritten as the following Sturm-Liouville problem:

$$(\mathcal{L}_z^{(0)}+\mathcal{L}_z^{(1)})\,Y_{\lambda,m;\ell}(z)=q\,Y_{\lambda,m;\ell}(z), \tag{A.36}$$

where $\mathcal{L}_z^{(0)}$ is defined in Eq. (A.29) and $\mathcal{L}_z^{(1)}\doteq -4\,b\,z$. We assume the boundary conditions $\mid Y_{\lambda,m;\ell}(0)\mid<\infty$, $\mid Y_{\lambda,m;\ell}(1)\mid<\infty$.

Solutions to Eq. (A.35) which are analytic in some domain including the singularities at $z=0$ (i.e. $\theta=0$) and $z=1$ (i.e. $\theta=\pi$) are called reduced confluent Heun functions. We explain a formal construction of such functions relative to the points $z=0$ and $z=1$ by means of a series of Jacobi polynomials.

Hence we consider a solution of Eq. (A.35) in the following form [7]:

$$Y_{\lambda,m;\ell}=\sum_{n=0}^{\infty}c_n\,y_{n,m;\ell}, \tag{A.37}$$

with

$$y_{n,m;\ell}\doteq {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1\mid z)=\frac{\Gamma(\alpha+1)\,n!}{\Gamma(\alpha+1+n)}P_n^{(\alpha,\beta)}(1-2z),$$

where $\{c_n\}_{n=0}^{\infty}$ are suitable coefficients to be determined and ${}_2F_1(-n,n+\alpha+\beta+1;\alpha+1\mid z)$ is a local solution of a Gauss hypergeometric Eq. (A.22) which matches Heun's solution at the singularities $z=0$ and $z=1$.

From Eq. (A.36) we obtain

$$(\mathcal{L}_z^{(0)}+\mathcal{L}_z^{(1)})\,Y_{\lambda,m;\ell}(z)=n\,(n+\alpha+\beta+1)\,Y_{\lambda,m;\ell}(z)-4\,p\,z\,Y_{\lambda,m;\ell}(z)=q\,Y_{\lambda,m;\ell}(z). \tag{A.38}$$

Let us consider the following recurrence relation for the functions $y_{n,m;\ell}$ [7]:

$$z\,y_{n,m;\ell}(z)=A_n\,y_{n+1,m;\ell}(z)+B_n\,y_{n,m;\ell}(z)+C_n y_{n-1,m;\ell}(z), \tag{A.39}$$

with

$$A_n=-\frac{(n+\alpha+\beta+1)(n+\alpha+1)}{(2\,n+\alpha+\beta+1)(2\,n+\alpha+\beta+2)},$$
$$B_n=\frac{2\,n\,(n+\alpha+\beta+1)+(\alpha+1)(\alpha+\beta)}{(2\,n+\alpha+\beta+2)(2\,n+\alpha+\beta)},$$
$$C_n=-\frac{n\,(n+\beta)}{(2\,n+\alpha+\beta+1)(2\,n+\alpha+\beta)}.$$

Inserting the expansion (A.37) in Eq. (A.38) and considering the relation (A.39) we obtain the three-term recurrence relation given in Eq. (A.32a).

From solutions (A.37) we obtain the eigenfunctions $\Psi_{\lambda,m;\ell}(\theta,\varphi)$ of $\mathcal{H}$ in the form given in Eq. (A.31), where the normalization constant $D_{n,m;\ell}$ (A.34) is obtained requiring

$$\int_0^{\pi}\int_0^{2\pi} d\theta\, d\varphi\, \sin\theta \mid \Psi_{\lambda,m;\ell}(\theta,\varphi) \mid^2 = 1,$$

and using the orthogonality relation (A.25).

□

To complete the study of the spectral problem we have to compute the eigenvalues λ of $\mathcal{H}$. Actually they can be obtained solving a certain infinite continued fraction, or equivalently, diagonalizing a proper Jacobi matrix. The work is now in progress, with the collaboration of E. Langmann.

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  Refereed

1. D. Levi, M. Petrera, *Discrete reductive perturbation technique*, Jour. Math. Phys. **47**, (2006), 543–563.
2. F. Musso, M. Petrera, O. Ragnisco, G. Satta, *Bäcklund transformations for the rational Lagrange chain*, Jour. Nonlin. Math. Phys. **12**, Suppl. 2 (2005), 240–252.
3. F. Musso, M. Petrera, O. Ragnisco, G. Satta, *A rigid body dynamics derived from a class of extended Gaudin models: an integrable discretization*, Reg. Chaot. Dyn. **10**, Issue 4 (2005), 363–380.
4. F. Musso, M. Petrera, O. Ragnisco, G. Satta, *Gaudin models with* $\mathcal{U}_q(osp(1|2))$ *symmetry*, Nucl. Phys. B, **716** (2005), 543–555.
5. F. Musso, M. Petrera, O. Ragnisco, *Algebraic extensions of Gaudin models*, Jour. Nonlin. Math. Phys. **12**, Suppl. 1 (2005), 482–498.
6. V.B. Kuznetsov, M. Petrera, O. Ragnisco, *Separation of variables and Bäcklund transformations for the symmetric Lagrange top*, Jour. Phys. A **37** (2004), 8495–8512.

Preprints

1. Xiaoda Ji, D. Levi, M. Petrera, *Multiscale expansion of the lattice sine-Gordon equation*, to appear in the Proceedings of "International Conference on Difference Equations, Special Functions and Applications", July 25–30, 2005, Munich, Germany; www.arxiv.org/math-ph/0511006.

- **Oral Communications**

  1. *Discrete integrable mappings for rational Gaudin models*, "Workshop on Geometry and Symmetries of Differential Equations", July 6–12 2006, S. Marinella (Rome), Italy.
  2. *Integrable mappings for extended Lagrange tops*, May 15 2006, Department of Mathematics, Technical University of Munich, Munich, Germany.
  3. *Extended Lagrange tops and rational Lagrange chains: the problem of integrable discretization*, March 31, 2006, Laboratoire de physique théorique et modélisation, Université de Cergy-Pointoise, Paris, France.
  4. *Integrable mappings for extended Lagrange tops*, February 2–4 2006, "Riunione Nazionale PRIN Sintesi", Vietri (SA), Italy.
  5. *Algebraic techniques in the theory of finite-dimensional integrable systems*, September 12–24, 2005, Summer school "Scuola estiva di Fisica Matematica", organized by Istituto Nazionale di Alta Matematica (INdAM), Gruppo Nazionale di Fisica Matematica (GNFM), Ravello (SA), Villa Rufolo, Italy.
  6. *Algebraic extensions of Gaudin models*, March 14, 2005, Département de Mathematique, Université de Cergy-Pointoise, Paris, France.
  7. *Algebraic contractions of Gaudin models*, January 27–29, 2005, "Riunione Nazionale PRIN Sintesi", Vietri (SA), Italy.
  8. *Bäcklund transformations for the classical Lagrange top*, October 31–November 2 2003, "Giornate di Studio sui Sistemi Discreti Integrabili", Lecce, Italy.

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  1. "Geometric Aspects of Integrable Systems", July 16–20 2006, Department of Mathematics, University of Coimbra, Portugal.
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  4. "Solitons, boomerons and isochronous solutions to nonlinear systems", Workshop for Francesco Calogero 70th birthday, February 4–6 2005, Department of Physics, Università degli Studi di Roma Tre and Università La Sapienza, Roma, Italy.
  5. "Riunione Nazionale PRIN Sintesi", January 27–29 2005, Vietri (SA), Italy.
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  9. "Giornate di Studio sui Sistemi Discreti Integrabili", October 31–November 2 2003, Lecce, Italy.
  10. "Workshop on Integrable Systems", March 7–8 2003, Department of Mathematics, University of Leeds, UK.

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