# SCATTERING ELETTROMAGNETICO E RADIAZIONE DA OGGETTI INSERITI IN UN MEZZO OSPITE 

Cristina Ponti

Docente guida: Prof. Giuseppe Schettini

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# ELECTROMAGNETIC SCATTERING AND RADIATION BY OBJECTS EMBEDDED IN A HOST MEDIUM 

Cristina Ponti

Advisor: Prof. Giuseppe Schettini

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## Introduzione

L'attività di ricerca svolta nel triennio di dottorato si è articolata principalmente intorno a due temi: diffrazione elettromagnetica da oggetti cilindrici, di sezione arbitraria, sepolti in uno strato dielettrico; diffrazione elettromagnetica da strutture periodiche, con particolare riguardo al progetto di un'antenna con elevata direttività.

Il primo filone di ricerca è stato incentrato sulla risoluzione del problema diretto di scattering di un'onda piana monocromatica da parte di un numero arbitrario di oggetti dielettrici o perfettamente conduttori, a sezione arbitraria e di lunghezza infinita, con assi paralleli, sepolti in un mezzo dielettrico stratificato, lineare, omogeneo ed isotropo. Si è pertanto sviluppato un metodo analitico-numerico, nel dominio spettrale, che consente di trattare qualsiasi stato di polarizzazione coerente dei campi elettromagnetici coinvolti, e che permette di ottenere risultati in zona di campo vicino e di campo lontano.

Il metodo analitico è stato implementato in un codice di calcolo sviluppato in linguaggio Fortran, che è risultato versatile ed efficiente riguardo ai tempi di calcolo e all'accuratezza dei risultati.

Inizialmente è stato risolto il problema dello scattering bidimensionale da cilindri perfettamente conduttori, a sezione circolare, sepolti in uno strato dielettrico, ed in una seconda fase si è affrontata l'estensione a diffusori dielettrici.

Il metodo sviluppato è definito con l'acronimo di CWA (Cylindrical Wave Approach), poiché il campo diffuso dai cilindri è espresso mediante un'espansione modale in onde cilindriche, ossia nel prodotto tra una funzione di Hankel ed un fattore angolare esponenziale. La trattazione del problema tiene conto di tutti i contributi di campo che si originano dall'interazione tra l'onda piana monocromatica ed il sistema costituito dalle interfacce piane ed i diffusori cilindrici. Per quanto riguarda l'interazione dell'eccitazione con le interfacce, sono state calcolate le onde piane riflesse e trasmesse, mediante coefficienti di riflessione e trasmissione relativi ad uno strato dielettrico. I campi diffusi dai cilindri, riflessi e trasmessi attraverso le interfacce, sono stati espressi sfruttando lo spettro di onde piane di
un'onda cilindrica. Tale approccio consente di valutare la riflessione e trasmissione delle onde cilindriche che costituiscono le basi di espansione per i campi diffusi dai cilindri, essendo i coefficienti di trasmissione e riflessione noti solamente per le onde piane. Rispetto a quanto già affrontato in precedenza per il problema di scattering da cilindri immersi in un mezzo semi-infinito, è stato necessario ricorrere alla definizione di opportune funzioni cilindriche riflesse e riflesse-trasmesse da associare alle riflessioni multiple che si verificano all'interno dello strato dielettrico. Una trattazione rigorosa del problema deve, infatti, tener conto, almeno da un punto di vista teorico, di infinite riflessioni subite dal campo diffuso da parte delle interfacce piane. L'implementazione numerica ha introdotto un troncamento su tali riflessioni multiple, sotto criteri che consentano di mantenere un'elevata accuratezza dei risultati.

Il metodo analitico e la relativa implementazione numerica, in linguaggio Fortran, sono stati testati mediante prove di auto-consistenza e confronti con la letteratura, ottenendo un ottimo accordo. Tali test hanno inoltre evidenziato come il metodo possa essere esteso al caso di cilindri sepolti in uno strato che termina su un piano di massa, o contenuti in un mezzo semi-infinito, eventualmente coincidente con il mezzo di provenienza del campo di eccitazione. L'esecuzione del codice ha inoltre dimostrato la possibilità di simulare geometrie di interesse nell'ambito del GPR (Ground Penetrating Radar), ad esempio in riferimento all'individuazione di cavità o di sottoservizi.

Il metodo sviluppato per cilindri sepolti perfettamente conduttori e i risultati numerici sono stati pubblicati in [r2] e presentati a congresso [c8, c10, c11]. L'estensione del CWA al caso di cilindri sepolti dielettrici, con i relativi risultati, saranno pubblicati in [r5].

In seguito, per ottenere una caratterizzazione sempre più accurata degli scenari indagati dai sistemi GPR, il metodo è stato esteso considerando una linea di corrente come eccitazione del problema. Associando un'onda cilindrica al campo irradiato dalla linea di corrente, i campi che derivano dalla sua riflessione e trasmissione da parte delle interfacce possono essere valutati utilizzando il concetto di spettro di onde piane di un'onda cilindrica, coerentemente con quanto già formulato per le funzioni cilindriche relative ai campi diffusi. Questa estensione è stata affrontata per cilindri perfettamente conduttori sepolti all'interno di un semispazio dielettrico.

Il secondo filone di ricerca ha avuto come argomento lo studio della diffrazione di un'onda piana monocromatica da parte di strutture EBG (Electromagnetic BandGap), e l'utilizzo di queste nell'ambito del progetto e della realizzazione di antenne ad elevata direttività.

E' noto che tra le principali proprietà dei materiali EBG vi è la presenza di una banda proibita, ossia un intervallo di frequenze all'interno del quale risulta inibita la trasmissione delle onde elettromagnetiche attraverso la struttura. Tuttavia, un'interruzione della periodicità consente di evidenziare la comparsa di picchi di trasmissione all'interno di tale banda. Ciò suggerisce di pensare ad applicazioni per il filtraggio, selettivo in frequenza e nello spazio, della radiazione elettromagnetica proveniente da una determinata sorgente.

In una prima fase, è stato studiato il woodpile, un particolare EBG tridimensionale, la cui cella elementare è costituita dalla sovrapposizione di quattro strati di barre allineate in modo che le barre appartenenti a due strati consecutivi risultano ortogonali, mentre barre parallele sono traslate di mezzo periodo. Del woodpile è stata considerata una particolare configurazione in cui la periodicità della struttura risulta interrotta, ottenendo, in definitiva, che implementa il Metodo agli Elementi Finiti, è stato simulato il comportamento di cavità woodpile in allumina da impiegare nel range delle microonde. Dopo un'estesa caratterizzazione di tali cavità, è stato possibile pensare ad un'applicazione nell'ambito del progetto di antenne di tipo planare, che utilizzando uno strato della cavità woodpile come superstrato, presentano caratteristiche di maggiore direttività rispetto al radiatore di partenza. Come primo radiatore di base è stata progettata un'antenna a doppia slot, costituita da due identiche fessure rettangolari su piano di massa. In seguito si è considerato come radiatore di base un'antenna a microstriscia. Entrambi i progetti hanno evidenziato un incremento di direttività superiore ai 10 dB , rispetto ad una configurazione in cui non si utilizzi il superstrato woodpile, ed un notevole effetto di riduzione dell'ampiezza del fascio principale.

I risultati relativi all'analisi di cavità woodpile ed al progetto di un'antenna a doppia slot con superstrato woodpile sono pubblicati in [r1] e [r4] e sono stati presentati a congresso [c1, c2, c3, c6, c7]. Il progetto di antenne a patch che utilizzano cavità EBG sarà pubblicato in [r3] ed è stato presentato a congresso [c9]. In [r3, c9], i vantaggi dell'impiego di una cavit woodpile per l'incremento di direttivit di un'antenna a patch sono confrontati con quelli derivanti dall'utilizzo di strutture EBG di più semplice realizzazione, con periodicità unidimensionale.

In una seconda fase, è stata affrontata la realizzazione di un woodpile in allumina e di un'antenna a patch da utilizzare come radiatore di base per un'antenna a cavità con superstrato woodpile. Sono stati costruiti due strati di woodpile identici, ed un opportuno supporto che consente di effettuare misure in trasmissione sugli strati, in configurazione di cavità, e di distanziare uno dei due strati da un'antenna a patch realizzata con un laminato Rogers RT/Duroid 5870. I risultati finora prodotti sulla
cavità hanno confermato quanto evidenziato dalle simulazioni in merito al comportamento selettivo in frequenza, mentre da alcune misure recentemente eseguite sull'antenna, con e senza woodpile, è stato confermato l'effetto di incremento di direttività introdotto dall'EBG.

I risultati preliminari delle misure sono stati recentemente presentati a congresso [c12].

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## Part I

## Scattering by buried cylindrical objects

## Introduction to part I

The two-dimensional (2-D) electromagnetic scattering problem from buried cylindrical objects is a subject with important applications to the remote sensing of the earth internal structure, to the detection of explosive mines, of pipes, conduits and tunnels, to the communication through the earth, or to biomedical imaging. For this reason, it has been widely discussed by many authors in the past, both from a theoretical and a numerical point of view, as it will be reviewed in the following.

Plane-wave scattering by a perfectly-conducting or dielectric circular cylinder, in a homogeneous and isotropic medium, is a classical problem [1]-[3]. Such problem takes a more complex form in the presence of a planar discontinuity for the electromagnetic constants, and several resolving techniques have been developed for the scattering by objects above a dielectric half-space or buried in it. In particular, scattering from a subterranean cylindrical inhomogeneity is solved by Howard [4] with an eigenfunction expansion of a two-dimensional Fredholm integral equation for the scattered field. Ogunade [5], extending a previous work by D'Yakonov [6], obtained numerical data for a current line source above a uniform half-space. Mahmoud et al. [7] faced the problem using a multipole expansion for the scattered field. Green's function approach is developed in [8] and an effective scattering model for a real buried object is proposed. In [9], Butler et al. solved an integral equation for current induced on a conducting cylinder near a planar interface, and various forms of the kernel suitable for an efficient numerical evaluation are discussed. A study of scattering by partially-buried and coupled cylinders is presented in [10]. The planewave scattering by a 2-D cylindrical obstacle in a dielectric half-space is treated by Hongo and Hamamura [11]. In this work, Kobayashi's [42] potential concept is exploited: the asymptotic solutions presented, using the saddle-point method, are valid for obstacle size and distance between the obstacle and the interface much larger than the wavelength. A correct version of paper [11] has been reported in [13]. For a deeply-buried cylinder, results are shown in [14]. More recent contributions have been given by Ahmed and Naqvi [15] and by Altuncu et al. [16].

A more general characterization of scattering by buried objects can be dealt
with by considering a multilayered medium. Such a problem turns out to be a really hard task, since multiple reflections occuring between the scatterers and the interfaces must be taken into account. Different methods are reported in the literature, for a general multilayered medium and for the particular case of objects embedded in a slab between two half-spaces. In [17] a theoretical method was developed for the electromagnetic scattering by perfect conducting objects of arbitrary shape embedded in layered media. In particular, three different Mixed-Potential Integral Equations (MPIEs) were formulated; one of them was solved in [18] with the Method of Moments, and numerical results were presented for scatterers placed between two contiguous half-spaces. A solution to scattering by a cylinder buried in layered media with rough interfaces was given in [19], developing the extended boundary-condition method and the generalized scattering-matrix technique. The analytical properties of the scattering by a conducting circular cylinder buried in a stratified ferrite medium were described by Tsalamengas [20], using a combined Green's function and integral-equation approach.

The case of the two interfaces, i.e. of obstacles buried in a dielectric slab, was discussed by some authors. Naqvi et al. [21] gave an asymptotic solution for the scattered field by a perfectly-conducting cylindrical obstacle in a grounded dielectric layer: numerical results for a circular cylinder were reported in [22], for both normal and oblique incidence of a plane wave. A method to study the scattering by gratings embedded in a dielectric slab was proposed by Jia and Yasumoto [23], which combines a generalized scattering matrix with a lattice-sum matrix and a T-matrix of an isolated cylinder. The effect of a lossy medium was discussed by Paknys [25], describing reflection and transmission properties of reinforced concrete: a two-dimensional model was adopted to represent a wire grid embedded in a lossy dielectric slab, and a Method of Moments/Green's function approach was developed.

The case of buried nonmetallic scatterers is also of great interest. For the particular case of dielectric circular cylinders, several solving techniques have been proposed for the scattering by objects both above a dielectric half-space [26]-[27] and buried in it [28]. Two-dimensional scattering from an inhomogeneous dielectric cylinder embedded in a stratified medium is considered in [29], for modeling microwave and optical devices, for nondestructive testing of composite materials, or for remote sensing of soil. Electric Field Integral Equation over the cylinder cross section and Moment Method are employed, and the problem is solved for the case of TM polarization. The scattering problem by a set of dielectric radially-stratified circular cylinders embedded in a finite slab is solved by Lee [30] with the Hertz potential formalism, in the general case of an off-plane incident wave. Numerical re-
sults are given for three equally spaced, homogeneous and parallel silicon cylinders. In [31] a perturbation technique is applied to the scattering by a dielectric cylinder buried in a grounded dielectric layer. The permittivity of the cylinder has a low dielectric contrast from the host medium and the cylinder is considered as a perturbation in the dielectric layer. Extended boundary condition method and scattering matrix technique are developed by Kuo and Moghaddam [32] to study the scattering by a dielectric cylinder in layered media with rough interfaces. Results are reported in terms of bistatic scattering coefficients, also for the cases of two rough interfaces without any buried object, and for a cylinder buried beneath a single rough surface.

Most of the works referenced above solve scattering by buried objects making a combined use of integral equations and numerical discretization techniques, as the Method of Moments. In other works, integrals are solved through asymptotic techniques. Thus, several hypotheses are made in order to simplify the problem, as deeply buried cylinders or dielectric cylinders with low contrast with respect to the host medium. Otherwise, the scattered field is evaluated only in the far-field region. The case of scattering by an isolated cylinder is mainly dealt with, due to the difficulty in developing multiple-interactions in an analytical model.

An exact analysis for two-dimensional scattering by a conducting circular cylinder below a plane surface and by a set of conducting cylinders buried in a dielectric half-space is developed in [33] and [34], respectively, through the Cylindrical Wave Approach (CWA). The concept of plane-wave spectrum of a cylindrical wave [35] is employed: reflection and transmission of cylindrical waves by the plane interface are treated by introducing suitable reflected and transmitted cylindrical functions. The relevant spectral integrals are numerically solved employing adaptive integration procedures of Gaussian type, together with acceleration techniques, as reported in [36], [37]. The analysis reported in [34] is extended to scattering by dielectric circular scatterers in [38].

In this work, a rigorous solution to the two-dimensional scattering problem of a plane wave by a set of circular cylinders, with infinite length and parallel axes, buried in a dielectric slab, is developed. The CWA employed in [34] and [38], to solve scattering by objects buried in a semi-infinite medium, is here applied to a layered geometry. Multiple-reflections phenomena are experienced by the scattered field inside the slab, and they are taken into account by means of generalized reflected and reflected-transmitted cylindrical functions. Thus, the theoretical approach gives an exact solution and the complicate interaction between the cylinders and the interfaces is properly dealt with. The method has been implemented in a numerical Fortran code, that gives results both in near- and far-field regions.

Numerical results can be employed to get a deeper understanding of scenarios investigated by Groung Penetrating Radar (GPR) [39], for detection of buried objects. For a more realistic characterization, further assumptions may be taken into account in the formulation of the scattering problem. To this purpose, the theoretical analysis is extended to the excitation of buried perfectly-conducting cylinders in a semi-infinite medium by a line source. Due to the geometry of the source, the radiated field can be described by a cylindrical wave. Therefore, reflection and transmission of such a wave through the plane interface can be dealt with by means of the spectrum of a cylindrical wave [35], as already implemented for the basis functions of the scattered fields in the mochromatic plane-wave case.

In chapter 1, the theoretical model for the scattering of a plane wave by perfectlyconducting circular cylinders buried in a dielectric slab is dealt with. In chapter 2, the analysis is extended to the case of dielectric cylinders. Numerical results are given in chapter 3 (perfectly-conducting cylinders) and 4 (dielectric cylinders). Several geometrical layouts are analyzed, and results are validated both through convergency tests and comparisons with the literature. In chapter 5, the theoretical analysis reported in chapter 1 is extended to case of excitation by a line source.

## Chapter 1

## Perfectly-conducting cylinders buried in a dielectric layer

In this chapter, a theoretical method to solve the scattering of a plane wave by perfectly-conducting circular cylinders buried in a dielectric layer is developed.

The involved media are assumed linear, isotropic, homogeneous, dielectric, lossless, and separated by planar interfaces. The scatterers have parallel such states axes and they are parallel to the planar interfaces of separation between the media. The whole structure infinitely extends along the direction of the cylinders axes, and the propagation vector of the incident wave lies in the plane orthogonal to the interfaces and the cylinders axes. Thus, the problem can be considered two dimensional.

Two states of linear polarization for the incident field are dealt with, since any other state of polarization can be expressed by the superimposition of

- E or TM polarization - The electric field is parallel to the cylinders axes. The magnetic field and the propagation vector lie in the plane orthogonal to the cylinders axes. For this reason, the E polarization is also called transversemagnetic (TM), being the magnetic field transverse to the axes of the cylindrical scatterers.
- H or TE polarization - The magnetic field is parallel to the cylinders axes. With the electric field lying in the orthogonal plane, the H polarization is also called transverse-electric (TE).

Each component of the electromagnetic field can be obtained by Maxwell equations, once known the field component parallel to the cylinders axes, which coincides with the electric and magnetic field for E and H polarization, respectively. In the whole analysis, this field component is represented by a scalar function $V$, which in each medium of the structure takes into account all the interactions. A time
dependence $e^{-i \omega t}$, being $\omega$ the angular frequency, is assumed for the fields, and will be omitted.

In each medium, the function $V$ is decomposed in the following terms, which arise from the interaction of the incident wave with the interfaces and the cylinders (see Fig. 1.1):

- $V_{\mathrm{i}}$ : plane-wave incident field;
- $V_{\mathrm{r}}$ : plane-wave reflected field, due to the reflection in medium 0 of the incident plane-wave $V_{\mathrm{i}}$ on the slab;
- $V_{t 1}$ : plane-wave transmitted field, representing a downward-propagating wave in medium 1, as the result of the various waves reflected by the slab;
- $V_{\mathrm{r} 1}$ : plane-wave reflected field, representing an upward-propagating wave in medium 1, as the result of the various waves reflected by the slab;
- $V_{\mathrm{t} 2}$ : plane-wave transmitted field, due to the transmission in medium 2 of the plane-wave $V_{\mathrm{t} 1}$ through the slab;
- $V_{\mathrm{s}}$ : field scattered by the cylinders in medium 1 ;
- $V_{\text {sr }(\mathrm{j})}^{1,0}$ : scattered-reflected fields, due to a first $(\mathrm{j}=1)$ reflection of $V_{\mathrm{s}}$ in medium 1 by the upper interface followed by multiple $(\mathrm{j}=2, \ldots, \infty)$ reflections inside the slab;
- $V_{\mathrm{sr}(\mathrm{j})}^{1,2}$ : scattered-reflected fields, due to a first $(\mathrm{j}=1)$ reflection of $V_{\mathrm{s}}$ in medium 1 by the lower interface followed by multiple $(\mathrm{j}=2, \ldots, \infty)$ reflections inside the slab;
- $V_{\mathrm{st}}^{1,0}$ : scattered-transmitted field, due to the transmission in medium 0 of $V_{\mathrm{s}}$ through the upper interface;
- $V_{\mathrm{srt}(\mathrm{j})}^{1,0}$ : multiple scattered-reflected-transmitted fields, due to multiple ( $\mathrm{j}=$ $1, \ldots, \infty)$ reflections inside the slab, and ultimately transmitted in medium 0 through the upper interface;
- $V_{\mathrm{st}}^{1,2}$ : scattered-transmitted field, due to the transmission in medium 2 of $V_{\mathrm{s}}$ through the lower interface;
- $V_{\text {srt }(\mathrm{j})}^{1,2}$ : multiple scattered-reflected-transmitted fields, due to multiple ( $\mathrm{j}=$ $1, \ldots, \infty)$ reflections inside the slab, and ultimately transmitted in medium 2 through the lower interface.


Figure 1.1: Decomposition of the total field.

For field contributions propagating in medium 1, an expansion into cylindrical functions has been employed, which takes into account the circular geometry of the scatterers cross-section. In particular, for the fields $V_{\mathrm{t} 1}$ and $V_{\mathrm{r} 1}$ the expansion of a plane-wave in terms of Bessel functions $J_{m}$ is used, while the addition theorem of Hankel functions is employed for the scattered field $V_{\mathrm{s}}$. Handling the scatteredreflected fields $V_{\operatorname{sr}(\mathrm{j} \mathrm{j}}^{1,0}, V_{\mathrm{sr}(\mathrm{j})}^{1,2}$, the scattered-transmitted and scattered-reflected transmitted fields, like $V_{\mathrm{st}}^{1,0}$ and $V_{\mathrm{srt}(\mathrm{j})}^{1,0}$, is a more difficult task: an expansion with unknown coefficients into cylindrical wave is the most appropriate for scatterers of circular cross-section, but properties of reflection and transmission from planar interfaces are known only for plane waves. Thus, the proposed technique is to take simultaneously into account the two geometries by the concept of plane-wave spectrum of a cylindrical wave.

Finally, a linear system for the unknown coefficients is derived, by imposition of boundary conditions on the cylinders surfaces.

### 1.1 Geometry of the problem

The geometry of the problem is shown in Fig. 1.2: $N$ perfectly-conducting cylinders with circular cross-section are buried in a dielectric slab (medium 1) comprised between two half-spaces (medium 0, above, and medium 2, below). The three media are linear, isotropic, homogeneous, dielectric and lossless. In particular, an upper airfilled dielectric half-space (medium 0 ) is followed by a layer of permittivity $\varepsilon_{1}=\varepsilon_{0} n_{1}^{2}$ (medium 1) and by a lower half-space with $\varepsilon_{2}=\varepsilon_{0} n_{2}^{2}$ (medium 2 ); $\mu_{2}=\mu_{1}=\mu_{0}$ are the vacuum magnetic permeabilities.

A monochromatic plane wave obliquely impinges on the first interface between


Figure 1.2: Geometry of the scattering problem.
medium 0 and the dielectric slab, forming an angle $\varphi^{i}$ with respect to the positive direction of the $x$-axis, and with wavevector $\mathbf{k}^{\mathbf{i}}$ lying in the $(x, z)$ plane. The cylinders axes are parallel to the $y$-axis, and the whole structure infinitely extends along the $y$-direction, so that the problem is two-dimensional.

In the following, a Main Reference Frame (MRF) ( $\mathrm{O}, \xi, \zeta$ ) is introduced, with normalized coordinates $\xi=k_{0} x$ and $\zeta=k_{0} z$, where $k_{0}=\omega / c$ is the vacuum wavenumber. A second Reference Frame $R F_{\mathrm{p}}$ centred on the $p$-th ( $p=1, \ldots, N$ ) cylinder is considered, both in rectangular $\left(\mathrm{O}_{\mathrm{p}}, \xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)$ and polar coordinates $\left(\mathrm{O}_{\mathrm{p}}, \rho_{\mathrm{p}}, \theta_{\mathrm{p}}\right)$, where $\xi_{\mathrm{p}}=k_{0} x_{\mathrm{p}}, \zeta_{\mathrm{p}}=k_{0} z_{\mathrm{p}}, \rho_{\mathrm{p}}=k_{0} r_{\mathrm{p}}$ and

$$
\left\{\begin{array}{l}
\xi=\xi_{\mathrm{p}}+\chi_{\mathrm{p}}  \tag{1.1}\\
\zeta=\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}
\end{array}\right.
$$

The $p$-th cylinder has circular cross-section with normalized radius $\alpha_{\mathrm{p}}=k_{0} a_{\mathrm{p}}$; its axis is centred in $\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}\right)$ in MRF, and it is buried in a dielectric slab of normalized thickness $\Lambda$. The electromagnetic properties of reflection and transmission of the plane-wave fields $V_{\mathrm{r}}, V_{\mathrm{t} 1}, V_{\mathrm{r} 1}$, and $V_{\mathrm{t} 2}$, by the planar interfaces, are taken into account through reflection and transmission coefficients of a plane wave $\Gamma_{\mathrm{ij}}$ and $T_{\mathrm{ij}}=1+\Gamma_{\mathrm{ij}}$ in a layered medium, being $i=0,1$ and $j=1,2$. The two symbols $i, j$ refer to a wave propagating in medium $i$, which impinges on the planar interface with medium $j$. These coefficients are developed in Appendix A, for the two polarizations.

### 1.2 Decomposition of the total field

In a generic point $P(\xi, \zeta)$, the scalar function $V(\xi, \zeta)$ represents, for the fields components parallel to the cylinders axes, the sum of the all the contributions following the interaction of the incident field with the planar interfaces and the circular scatterers, i.e.

- $V^{(0)}(\xi, \zeta)=V_{\mathrm{i}}(\xi, \zeta)+V_{\mathrm{r}}(\xi, \zeta)+V_{\mathrm{st}}^{1,0}(\xi, \zeta)+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{srt}(\mathrm{j})}^{1,0}$
in medium 0 , as
- $V^{(1)}(\xi, \zeta)=V_{\mathrm{t} 1}(\xi, \zeta)+V_{\mathrm{r} 1}(\xi, \zeta)+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}$
in medium 1, and
- $V^{(2)}(\xi, \zeta)=V_{\mathrm{t} 2}(\xi, \zeta)+V_{\mathrm{st}}^{1,2}(\xi, \zeta)+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{stt}(\mathrm{j})}^{1,2}$
in medium 2. In this Section, every field contribution will be analyzed separately.


### 1.2.1 Incident field

The incident field is a monochromatic plane-wave, of wavelength $\lambda_{0}$ and complex amplitude $V_{0}$, with propagation vector $\mathbf{k}^{\mathrm{i}}=k_{\perp}^{\mathrm{i}} \hat{\xi}+k_{\|}^{\mathrm{i}} \hat{\zeta}$, therefore

$$
\begin{equation*}
V_{\mathrm{i}}(x, z)=V_{0} e^{i\left(k_{\perp}^{\mathrm{i}} x+k_{\|}^{\mathrm{i}} z\right)} \tag{1.2}
\end{equation*}
$$

Being $\varphi^{i}$ the angle formed by the propagation vector with the $\xi$ axis, the orthogonal and parallel components of the wave vector to the planar interfaces are expressed as $k_{\perp}^{\mathrm{i}}=k_{0} \cos \varphi^{\mathrm{i}}$ and $k_{\|}^{\mathrm{i}}=k_{0} \sin \varphi^{\mathrm{i}}$, respectively, with $k_{0}=2 \pi / \lambda_{0}$. In the following, the symbols $\perp$ and $\|$ will always be associated to the orthogonal and parallel components of a generic vector with respect to the planar interfaces. Introducing the normalized unit vector $\mathbf{n}^{\mathrm{i}}$, such that $\mathbf{k}^{\mathrm{i}}=k_{0} \mathbf{n}^{\mathrm{i}}$, with $\mathbf{n}^{\mathrm{i}}=n_{\perp}^{\mathrm{i}} \hat{\xi}+n_{\|}^{\mathrm{i}} \hat{\zeta}$

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{i}}=\cos \varphi^{\mathrm{i}}  \tag{1.3}\\
n_{\|}^{\mathrm{i}}=\sin \varphi^{\mathrm{i}}
\end{array}\right.
$$

the $V_{\mathrm{i}}$ field can be written as a function of adimentional coordinates, as follows

$$
\begin{equation*}
V_{\mathrm{i}}(\xi, \zeta)=V_{0} e^{i\left(n_{\perp}^{\mathrm{i}} \xi+n_{\|}^{\mathrm{i}} \zeta\right)} \tag{1.4}
\end{equation*}
$$

As the plane wave impinges on a layered geometry, two waves are excited [1] (see Fig. 1.3), as the result of multiple and infinite reflections through the two
planar interfaces: an upward-propagating wave, formed by the reflected fields $V_{\mathrm{r}}$ and $V_{\mathrm{r} 1}$, and a downward-propagating wave of the transmitted fields $V_{\mathrm{t} 1}$ and $V_{\mathrm{t} 2}$. The plane-wave fields propagating in medium 0,1 , and 2 , respectively, turn out to be the followings

- $V^{(0)}(\xi, \zeta)=V_{\mathrm{i}}(\xi, \zeta)+V_{\mathrm{r}}(\xi, \zeta)$
- $V^{(1)}(\xi, \zeta)=V_{\mathrm{t} 1}(\xi, \zeta)+V_{\mathrm{r} 1}(\xi, \zeta)$
- $V^{(2)}(\xi, \zeta)=V_{\mathrm{t} 2}(\xi, \zeta)$
and they are evaluated in Sections 1.2.2 and 1.2.3.


Figure 1.3: Sketch of the waves excited by a plane wave impinging on a dielectric layer.

### 1.2.2 Reflected fields

The plane-wave reflected by the first interface has propagation vector $\mathbf{k}^{\mathrm{r}}=k_{\perp}^{\mathrm{r}} \hat{\xi}+k_{\|}^{\mathrm{r}} \hat{\zeta}$; being $\varphi^{\mathrm{r}}$ the angle formed with the positive direction of the $\xi$ axis, it is $k_{\perp}^{\mathrm{r}}=k_{0} \cos \varphi^{\mathrm{r}}$ and $k_{\|}^{\mathrm{r}}=k_{0} \sin \varphi^{\mathrm{r}}$. Otherwise, the propagation vector can be expressed as $\mathbf{k}^{\mathrm{r}}=k_{0} \mathbf{n}^{\mathrm{r}}$, where $\mathbf{n}^{\mathrm{r}}$ is the propagation unit vector, with components $n_{\perp}^{\mathrm{r}}=\cos \varphi^{\mathrm{r}}$, and $n_{\|}^{\mathrm{r}}=$ $\sin \varphi^{\mathrm{r}}$. Thus, the field reflected by the first interface has the following expression

$$
\begin{equation*}
V_{\mathrm{r}}(\xi, \zeta)=V_{r} e^{i\left(n_{\perp}^{\mathrm{r}} \xi+n_{\|}^{\mathrm{r}} \zeta\right)} \tag{1.5}
\end{equation*}
$$

The angle $\varphi^{\mathrm{r}}$ is evaluated by means of the Snell law, leading to $\varphi^{\mathrm{r}}=\pi-\varphi^{\mathrm{i}}$. Therefore, the components of the reflected unit vector are

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{r}}=-n_{\perp}^{\mathrm{i}}  \tag{1.6}\\
n_{\|}^{\mathrm{r}}=n_{\|}^{\mathrm{i}}
\end{array}\right.
$$

and the final expression of the reflected plane-wave in medium 0 is

$$
\begin{equation*}
V_{\mathrm{r}}(\xi, \zeta)=\Gamma_{01}\left(n_{\|}^{\mathrm{i}}\right) V_{0} e^{i\left(-n_{\perp}^{\mathrm{i}} \xi+n_{\|}^{\mathrm{i}} \zeta\right)} \tag{1.7}
\end{equation*}
$$

where $V_{\mathrm{r}}=\Gamma_{01}\left(n_{\|}^{\mathrm{i}}\right) V_{0}$.
In a similar way, the field $V_{\mathrm{r} 1}$ reflected by the second interface can be obtained

$$
\begin{equation*}
V_{\mathrm{r} 1}(\xi, \zeta)=V_{\mathrm{r} 1} e^{i n_{1}\left[n_{\perp}^{\mathrm{r}}(\xi-\Lambda)+n_{\|}^{\mathrm{r}} \zeta\right]} \tag{1.8}
\end{equation*}
$$

where the propagation vector $\mathbf{k}^{\mathrm{r} 1}$, with components $k_{\perp}^{\mathrm{r} 1}=k_{1} \cos \varphi^{\mathrm{r} 1}$ and $k_{\|}^{\mathrm{r} 1}=$ $k_{1} \sin \varphi^{\mathrm{r} 1}$, has been expressed as a function of the unit vector $\mathbf{n}^{\mathrm{r} 1}$, as $\mathbf{k}^{\mathrm{r}}=k_{1} \mathbf{n}^{\mathrm{r} 1}$. The normal and parallel components of $\mathbf{n}^{\mathrm{r} 1}$ are $n_{\perp}^{\mathrm{r} 1}=\cos \varphi^{\mathrm{r} 1}$ and $n_{\|}^{\mathrm{r} 1}=\sin \varphi^{\mathrm{r} 1}$, respectively. According to the Snell law, the reflected angle $\varphi^{\mathrm{r} 1}$ satisfies the following relation: $\varphi^{\mathrm{r} 1}=\pi-\varphi^{\mathrm{t} 1}$. Thus, the components of the unit vector $\mathbf{n}^{\mathrm{r} 1}$ can be expressed as follows

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{r} 1}=-n_{\perp}^{\mathrm{t} 1}  \tag{1.9}\\
n_{\|}^{\mathrm{rr}}=n_{\|}^{\mathrm{t1}}
\end{array}\right.
$$

once known $n_{\perp}^{\mathrm{t} 1}$ and $n_{\|}^{\mathrm{t} 1}$. Being also $V_{\mathrm{r} 1}=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right)$

$$
\begin{equation*}
V_{\mathrm{r} 1}(\xi, \zeta)=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t}}(\xi-\Lambda)+n_{\|}^{\mathrm{t1}} \zeta\right]} \tag{1.10}
\end{equation*}
$$

The boundary conditions can be solved in an easier way, if the reflected field in medium 1 is expressed through a modal expansion which takes into account the cylindrical geometry of the scatterers. Introducing in equation (1.10) the (1.1), we get

$$
\begin{align*}
& V_{\mathrm{r} 1}(\xi, \zeta)=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}}\left(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}\right)\right]} \\
& =V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{i n_{1}\left(n_{\perp}^{\mathrm{r} 1} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{r} 1} \zeta_{\mathrm{p}}\right)} \tag{1.11}
\end{align*}
$$

which represents the reflected field associated to a point of coordinates $(\xi, \zeta)$ of $(M R F)$, as a function of the coordinates $\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)$ in $R F_{\mathrm{p}}$. Employing the expansion of a plane wave into Bessel functions [45]

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{r}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{r}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} e^{-i \ell \varphi^{r 1}} \tag{1.12}
\end{equation*}
$$

the reflected field in medium 1 can be written in the searched form

$$
\begin{align*}
V_{\mathrm{r} 1}(\xi, \zeta)= & V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{t 1}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{t 1} \eta_{\mathrm{p}}\right]} \\
& \times \sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} e^{-i \ell \varphi^{\mathrm{r} 1}} \tag{1.13}
\end{align*}
$$

### 1.2.3 Transmitted fields

The plane wave transmitted through the first interface has propagation vector $\mathbf{k}^{\mathrm{t1}}=$ $k_{\perp}^{\mathrm{t}} \hat{\xi}+k_{\|}^{\mathrm{t1}} \hat{\zeta}$. Being $\varphi^{\mathrm{t1}}$ the angle formed by $\mathbf{k}^{\mathrm{t} 1}$ with the positive direction of the $\xi$ axis, it follows: $k_{\perp}^{t 1}=k_{1} \cos \varphi^{\mathrm{t} 1}$ and $k_{\|}^{\mathrm{t1}}=k_{1} \sin \varphi^{\mathrm{t1}}$, with $k_{1}=n_{1} k_{0}$ wave number in medium 1. The vector $\mathbf{k}^{\text {t1 }}$ can be expressed by means of the unit vector $\mathbf{n}^{\text {t1 }}$ as $\mathbf{k}^{\mathrm{t} 1}=k_{1} \mathbf{n}^{\mathrm{t} 1}$, where $\mathbf{n}^{\mathrm{t} 1}=n_{\perp}^{\mathrm{t} 1} \hat{\xi}+n_{\|}^{\mathrm{t} 1} \hat{\zeta}$, with $n_{\perp}^{\mathrm{t} 1}=\cos \varphi^{\mathrm{t} 1}$ and $n_{\|}^{\mathrm{t} 1}=\sin \varphi^{\mathrm{t} 1}$. The amplitude of the transmitted field is $V_{0} V_{\mathrm{t} 1}=T_{01}\left(n_{\|}^{\mathrm{i}}\right)$, and the field transmitted through the first interface is

$$
\begin{equation*}
V_{\mathrm{t} 1}(\xi, \zeta)=T_{01}\left(n_{\|}^{\mathrm{i}}\right) V_{0} e^{i n_{1}\left(n_{\perp}^{\mathrm{t}} \xi+n_{\|}^{\mathrm{t1}} \zeta\right)} \tag{1.14}
\end{equation*}
$$

From the Snell law, $\varphi^{\mathrm{t1}}=\arcsin \left(k_{0} \sin \varphi^{\mathrm{i}} / k_{1}\right)=\arcsin \left(n_{\|}^{\mathrm{i}} / n_{1}\right)$, which yields

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{t} 1}=\sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}}  \tag{1.15}\\
n_{\|}^{\mathrm{t1}}=n_{\|}^{\mathrm{i}} / n_{1}
\end{array}\right.
$$

As already done in Section 1.2 .2 on equation (1.10), the field in (1.14) can be expressed in a point of coordinates $(\xi, \zeta)$ of $(M R F)$, as a function of the coordinates $\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)$ in $R F_{\mathrm{p}}$. Introducing in (1.14) the (1.1), we get

$$
\begin{equation*}
V_{\mathrm{t} 1}(\xi, \zeta)=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{i n_{1}\left(n_{\perp}^{\mathrm{t1}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{t1}} \zeta_{\mathrm{p}}\right)} \tag{1.16}
\end{equation*}
$$

The expansion of a transmitted plane-wave into Bessel functions is

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{t} 1} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{t} 1} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} e^{-i \ell \varphi^{\mathrm{t1}}} \tag{1.17}
\end{equation*}
$$

which leads to the final expression for the field transmitted in medium 1

$$
\begin{align*}
& V_{\mathrm{t} 1}(\xi, \zeta)=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{t 1}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} \\
& \quad \times \sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} e^{-i \ell \varphi^{\mathrm{t1}}} \tag{1.18}
\end{align*}
$$

The plane wave transmitted in medium 2, through the second interface, has propagation vector $\mathbf{k}^{\mathrm{t} 2}=k_{\perp}^{\mathrm{t} 2} \hat{\xi}+k_{\|}^{\mathrm{t} 2} \hat{\zeta}$. Being $\varphi^{\mathrm{t} 2}$ the angle formed by $\mathbf{k}^{\mathrm{t} 2}$ with the positive direction of the $\xi$ axis, it follows: $k_{\perp}^{\mathrm{t2}}=k_{2} \cos \varphi^{\mathrm{t} 2}$ and $k_{\|}^{\mathrm{t} 2}=k_{2} \sin \varphi^{\mathrm{t} 2}$, with $k_{2}=n_{2} k_{0}$ wavenumber in medium 2. Expressing the vector $\mathbf{k}^{\mathrm{t} 2}$ by means of the unit vector $\mathbf{n}^{\mathrm{t} 2}$, we get $\mathbf{k}^{\mathrm{t} 2}=k_{2} \mathbf{n}^{\mathrm{t} 2}$, where $\mathbf{n}^{\mathrm{t} 2}=n_{\perp}^{\mathrm{t} 2} \hat{\xi}+n_{\|}^{\mathrm{t} 2} \hat{\zeta}$, with $n_{\perp}^{\mathrm{t} 2}=\cos \varphi^{\mathrm{t} 2}$ and $n_{\|}^{\mathrm{t} 2}=\sin \varphi^{\mathrm{t} 2}$. The field transmitted through the second interface is

$$
\begin{equation*}
V_{\mathrm{t} 2}(\xi, \zeta)=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) T_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{2}\left[n_{\perp}^{\mathrm{t} 2}(\xi-\Lambda)+n_{\|}^{\mathrm{t} 2} \zeta\right]} \tag{1.19}
\end{equation*}
$$

with $V_{\mathrm{t} 2}=V_{0} T_{01}\left(n_{\|}^{\mathrm{i}}\right) T_{12}\left(n_{\|}^{\mathrm{i}}\right)$, amplitude of the field.
From the Snell law, $\varphi^{\mathrm{t} 2}=\arcsin \left(k_{1} \sin \varphi^{\mathrm{t1}} / k_{2}\right)=\arcsin \left(n_{\|}^{\mathrm{t1}} n_{1} / n_{2}\right)$, thus

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{t} 2}=\sqrt{1-\left(n_{\| 1}^{\mathrm{tt}} n_{1} / n_{2}\right)^{2}}  \tag{1.20}\\
n_{\|}^{\mathrm{t} 2}=n_{\|}^{\mathrm{t} 1} n_{1} / n_{2}
\end{array}\right.
$$

### 1.2.4 Scattered field

The scattered field $V_{\mathrm{s}}$ is written as the sum of the fields $V_{\mathrm{s}(\mathrm{q})}$, with $\mathrm{q}=1, \ldots, N$, scattered by each cylinder

$$
\begin{equation*}
V_{\mathrm{s}}(\xi, \zeta)=\sum_{\mathrm{q}=1}^{N} V_{\mathrm{s}(\mathrm{q})}(\xi, \zeta) \tag{1.21}
\end{equation*}
$$

The field $V_{\mathrm{s}(\mathrm{q})}$ is, in turn, the sum of infinite cylindrical functions $C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{1} \zeta_{\mathrm{q}}\right)=$ $H_{\mathrm{m}}^{(1)}\left(n_{1} \rho_{\mathrm{q}}\right) e^{i \mathrm{~m} \theta_{\mathrm{q}}}$, with unknown coefficients $c_{\mathrm{qm}}$

$$
\begin{equation*}
V_{\mathrm{s}(\mathrm{q})}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{1} \zeta_{\mathrm{q}}\right) \tag{1.22}
\end{equation*}
$$

where $H_{\mathrm{m}}^{(1)}\left(n_{1} \rho_{\mathrm{q}}\right)$ is the first-kind Hankel function of integer order m [45], and q stands for the $q$-th cylinder. The scattered field (1.21) can be expressed in the coordinates of $M R F$, introducing in (1.22) the (1.1), where $p$ is replaced with $q$

$$
\begin{equation*}
V_{\mathrm{s}}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} C W_{\mathrm{m}}\left[n_{1}\left(\xi-\chi_{\mathrm{q}}\right), n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right] \tag{1.23}
\end{equation*}
$$

Imposition of the boundary condition on the $p$-th cylinder surface is made easier if the scattered field is expressed as a function of the coordinates of $R F_{\mathrm{p}}$. Introducing equation (1.22) in (1.21) and isolating the term for $q=p$

$$
\begin{align*}
& V_{\mathrm{s}}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{pm}} C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{p}}, n_{1} \zeta_{\mathrm{p}}\right)+ \\
& \quad+\sum_{\substack{\mathrm{q}=1 \\
\mathrm{q} \neq \mathrm{p}}}^{N} V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{1} \zeta_{\mathrm{q}}\right) \tag{1.24}
\end{align*}
$$

By means of the addition theorem of Hankel functions [42], the wave emitted by the $q$-th cylinder, with $\mathrm{q} \neq \mathrm{p}$, can be centred on the $R F_{\mathrm{p}}$ system

$$
\begin{equation*}
H_{\mathrm{m}}^{(1)}\left(n_{1} \rho_{\mathrm{q}}\right) e^{i \mathrm{~m} \theta_{\mathrm{q}}}=e^{i \mathrm{~m} \theta_{\mathrm{qp}}} \sum_{\ell=-\infty}^{+\infty}(-1)^{\ell} H_{\mathrm{m}+\ell}(1)\left(n_{1} \rho_{\mathrm{qp}}\right) e^{i \ell \theta_{\mathrm{qp}}} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{-i \ell \theta_{\mathrm{p}}} \tag{1.25}
\end{equation*}
$$

It follows

$$
\begin{gather*}
V_{\mathrm{s}}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{pm}} C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{p}}, n_{1} \zeta_{\mathrm{p}}\right)+ \\
+\sum_{\substack{\mathrm{q}=1 \\
\mathrm{q} \neq \mathrm{p}}}^{N} V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \sum_{\ell=-\infty}^{+\infty} C W_{\mathrm{m}+\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)(-1)^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{-i \ell \theta_{\mathrm{p}}} \tag{1.26}
\end{gather*}
$$

where $\left(\xi_{\mathrm{qp}}, \zeta_{\mathrm{qp}}\right)$ are the coordinates relevant to ( $\rho_{\mathrm{qp}}, \theta_{\mathrm{qp}}$ ) (see Fig. 1.2). Employing the property $(-1)^{\ell} J_{\ell}(\cdot)=J_{-\ell}(\cdot)$, and replacing in the latter sum $\ell$ with $-\ell$, equation (1.26) becomes

$$
\begin{gather*}
V_{\mathrm{s}}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{pm}} C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{p}}, n_{1} \zeta_{\mathrm{p}}\right)+ \\
+\sum_{\substack{\mathrm{q}=1 \\
\mathrm{q} \neq \mathrm{p}}}^{N} V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \sum_{\ell=-\infty}^{+\infty} C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right) J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.27}
\end{gather*}
$$

Finally, equation (1.27) can be written in the following more compact form, representing the scattered field associated to a point of coordinates $(\xi, \zeta)$ in $M R F$, as a function of the coordinates in $R F_{\mathrm{p}}$

$$
\begin{gather*}
V_{\mathrm{s}}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times  \tag{1.28}\\
\times\left[C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}\right]
\end{gather*}
$$

and obtained introducing the Kronecker symbols $\delta_{\mathrm{qp}}$ and $\delta_{\ell \mathrm{m}}$, putting in evidence the term $J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{-i \ell \theta_{\mathrm{p}}}$, and taking into account the definition of the cylindrical functions $C W_{\mathrm{m}}$.

### 1.2.5 Scattered-reflected fields

The scattered-reflected, scattered-transmitted and scattered-reflected-transmitted fields are the results of the interaction of the field $V_{s}$ scattered by the cylinders and the two planar interfaces which limit the dielectric slab. The description of such an interaction is a difficult task, since two reference frames with different symmetry are employed: cylindrical coordinate systems relevant to the cylinders and the Cartesian reference frame relevant to the planar discontinuities. The field scattered by each cylinder has been expressed in terms of cylindrical waves emitted by the cylinder itself, but properties of reflection and transmission through a planar surface are known only for plane waves, by means of reflection and transmission coefficients.

Such a problem can be solved employing the plane-wave spectrum of a cylindrical wave [35]. It means that each cylindrical wave radiated by the $p$-th cylinder can be expressed as a superimposition of spectral plane-waves: reflection and transmission phenomena can be studied for each plane wave of the spectrum. Finally, the reflected, transmitted and reflected-transmitted plane-waves are superimposed to express the scattered-reflected, scattered-transmitted and scattered-reflected-transmitted fields, respectively. As far as fields propagating in medium 1 are concerned, each plane wave can be expanded into Bessel functions, in a similar way to what developed for the fields $V_{\mathrm{r} 1}$ and $V_{\mathrm{t} 1}$. The scattered-reflected, scatteredtransmitted and scattered-reflected-transmitted fields are obtained once the relevant scattered-reflected, scattered-transmitted and scattered-reflected-transmitted fields by each one of the $N$ cylinders have been summed up.

In medium 1, the angular spectrum of the cylindrical wave $C W_{\mathrm{m}}$ emitted by the $p$-th cylinder is given by

$$
\begin{equation*}
C W_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{p}}, n_{1} \zeta_{\mathrm{p}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} F_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{p}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{p}}} d n_{\|}^{\mathrm{s}} \tag{1.29}
\end{equation*}
$$

The evaluation of the function $F_{\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right)$ has been carried out in [35], and it has been further developed in [40] in the following form taking into account of a different behaviour for $\left|n_{\|}^{\mathrm{s}}\right| \geq 1$ and $\left|n_{\|}^{\mathrm{s}}\right| \leq 1$

$$
F_{\mathrm{m}}\left(\xi, n_{\|}\right)=\frac{2 e^{i \xi \sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}}}{\sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}} \times \begin{cases}\left(\sqrt{\left(n_{\|}^{\mathrm{s}}\right)^{2}-1}+n_{\|}^{\mathrm{s}}\right)^{\mathrm{m}} & \text { with } \quad\left|n_{\|}^{\mathrm{s}}\right| \geq 1  \tag{1.30}\\ e^{-i \mathrm{marccos} n_{\|}^{\mathrm{s}}} & \text { with }\left|n_{\|}^{\mathrm{s}}\right| \leq 1\end{cases}
$$

which applies to $\xi>0$.
An expression for $F_{\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right)$ applying to $\xi<0$ can be also obtained from [35]

$$
F_{\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right)=\frac{2 e^{-i \xi \sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}}}{\sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}} \times \begin{cases}\left(\sqrt{\left(n_{\|}^{\mathrm{s}}\right)^{2}-1}+n_{\|}^{\mathrm{s}}\right)^{-\mathrm{m}} & \text { with }  \tag{1.31}\\ e^{i \mathrm{~m} \arccos n_{\|}^{\mathrm{s}}} \mid \geq 1 \\ \text { with } & \left|n_{\|}^{\mathrm{s}}\right| \leq 1\end{cases}
$$

Thus, we have the following property

$$
\begin{equation*}
F_{\mathrm{m}}\left(-\xi, n_{\|}^{\mathrm{s}}\right)=F_{-\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right) \tag{1.32}
\end{equation*}
$$

Equations (1.30) and (1.31) can be written in a more compact way, making use of the properties of function arcosine in a complex domain [41], and considering $n_{\|}^{\mathrm{s}} \in(-\infty,+\infty)$

$$
F_{\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right)=\frac{2}{\sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}} e^{i|\xi| \sqrt{1-\left(n_{\|}^{\mathrm{s}}\right)^{2}}} \begin{cases}e^{-i \mathrm{marccos} n_{\|}^{\mathrm{s}}}, & \xi \geq 0  \tag{1.33}\\ e^{i \mathrm{marccos} n_{\|}^{\mathrm{s}}}, & \xi \leq 0\end{cases}
$$

Equation (1.29) gives the values of cylindrical function $C W_{\mathrm{m}}$ in a plane of abscissa $\xi_{\mathrm{p}}$. The evaluation of scattered-reflected fields has been performed following the approach developed in [34] and [40] for cylinders embedded in a homogeneous medium, i.e. a dielectric half-space, and placed below and above a planar interface, respectively. In these works, scattered-reflected fields are expansions of reflected cylindrical functions, where, in turn, each plane wave of the spectrum is obtained evaluating the reflection of the generic plane-wave defined in (1.33). The approach led to define Reflected Cylindrical Functions, where a single reflection was evaluated, as due to the single interface geometry of the medium hosting the cylinders.

In the present work, to describe multiple reflections established by a geometry with double interface, generalized Reflected Cylindrical Functions are introduced

$$
\begin{equation*}
R W_{\mathrm{m}}^{(1, \cdot)(\mathrm{j})}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} F_{\mathrm{m}}\left(\xi, n_{\|}^{\mathrm{s}}\right) e^{i n_{\|}^{\mathrm{s}} \zeta} d n_{\|}^{\mathrm{s}} \tag{1.34}
\end{equation*}
$$

In (1.34), $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)$ and $\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)$ are the plane-wave reflection coefficients, with respect to the medium $0 /$ medium 1 and medium $1 /$ medium 2 interfaces, respectively, and the exponents $f$ and $g$ are the number of reflections occurred at such interfaces, being $\mathrm{j}=\mathrm{f}+\mathrm{g}$ the total number of reflections. The symbol $(1, \cdot)$ may correspond to $(1,0)$ or $(1,2)$, according to the first reflection occurring at the upper or at the lower interface (Fig. 1.4), respectively, thus giving a different value in the $\xi$ argument of the plane wave $F_{\mathrm{m}}$.


Figure 1.4: Sketch of the reflected cylindrical waves excited inside the slab, with first reflection either at the upper (solid arrows) and lower (dashed arrows) interface.

The cylindrical functions $R W_{\mathrm{m}}^{1,0(\mathrm{j})}(\xi, \zeta)$ and $R W_{\mathrm{m}}^{1,2(\mathrm{j})}(\xi, \zeta)$ represent the basis functions of the fields $V_{\mathrm{sr}(\mathrm{j})}^{1,0}$ and $V_{\mathrm{sr}(\mathrm{j})}^{1,2}$, which are the scattered reflected fields, with
first reflection occurred at the medium $0 /$ medium 1 and medium $1 /$ medium 2 interface, respectively.

## Scattered-reflected field from upper interface

The scattered-reflected fields $V_{\mathrm{sr}(\mathrm{j})}^{1,0}$, relevant to a first reflection on the upper interface, placed in $\xi=0$, are now evaluated.

We start from the scattered-reflected field with 1 order of reflection, $V_{\operatorname{sr}(1)}^{1,0}$. We want to derive the reflected cylindrical wave which is excited when the wave $C W_{\mathrm{m}}(1.29)$ impinges on the plane of abscissa $\xi_{\mathrm{q}}=-\chi_{\mathrm{q}}$. Thus, we evaluate the generic plane-wave $F_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$ of the expansion (1.29), with propagation vector $n_{\perp}^{\mathrm{s}} \hat{\xi}_{\mathrm{q}}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$, on the upper interface, and we get $F_{\mathrm{m}}\left(-n_{1} \chi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$. In the plane $\xi_{\mathrm{q}}$, a reflected plane-wave $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left(-n_{1} \chi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{-i n_{1} n_{\perp}^{\mathrm{s}}\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right)}$ is obtained, with propagation vector $-n_{\perp}^{\mathrm{s}} \hat{\xi}_{q}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$. We can include the second exponential of this expression in $F_{\mathrm{m}}$, making use of definition (1.33), and we get $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{s} \zeta_{\mathrm{q}}}$. This leads to define reflected cylindrical waves $C W_{\mathrm{m}}^{\mathrm{r}}$ [34], relevant to a single reflection $\mathrm{j}=1$

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} . \tag{1.35}
\end{equation*}
$$

According to definition (1.34) of generalized Reflected Cylindrical Functions, the field generated by one reflection of the cylindrical wave (1.29) is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(1)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,0(1)}\left[-n_{1}\left(2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.36}
\end{equation*}
$$

By a superimposition of scattered-reflected fields by the $N$ cylinders, and making use of (1.1) in (1.35), the scattered-reflected field from the upper interface $V_{\operatorname{dr}(1)}^{1,0}$ can be written as

$$
\begin{equation*}
V_{\mathrm{sr}(1)}^{1,0}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell}^{1,0(1)}\left[-n_{1}\left(\xi+\chi_{\mathrm{q}}\right), n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right] \tag{1.37}
\end{equation*}
$$

The reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, for a number of reflections $\mathrm{j}>1$, are defined starting from (1.35). Considering $\mathrm{j}=2$ reflections, the generic plane-wave $F_{\mathrm{m}}\left[-n_{1}\left(2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right]$ of the expansion (1.35) is evaluated at the lower interface, placed in $\xi_{\mathrm{q}}=\Lambda-\chi_{\mathrm{q}}$, giving $F_{\mathrm{m}}\left[-n_{1}\left(\Lambda+\chi_{\mathrm{q}}\right), n_{\|]}^{\mathrm{s}}\right]$. In the plane $\xi_{\mathrm{q}}$, the reflected plane-wave is $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(\Lambda+\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{i n_{1} n_{\perp}^{\mathrm{s}}\left(\xi_{q}+\chi_{\mathrm{q}}-\Lambda\right)}$, with propagation vector $n_{\perp}^{\mathrm{s}} \hat{\xi}_{\mathrm{q}}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$. Including the second exponential in $F_{\mathrm{m}}$, according to definition (1.33), the reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, with $\mathrm{j}=2$, is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(2)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(2 \Lambda-\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} \tag{1.38}
\end{equation*}
$$

and recalling definition (1.34) we get

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(2)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,0(2)}\left[-n_{1}\left(2 \Lambda-\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.39}
\end{equation*}
$$

If we go on with $\mathrm{j}=3$ reflections, the generical plane-wave $F_{\mathrm{m}}\left[-n_{1}\left(2 \Lambda-\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right]$ of the expansion (1.38) has now to be evaluated at the upper interface, in $\xi_{\mathrm{q}}=-\chi_{\mathrm{q}}$, giving $F_{\mathrm{m}}\left[-n_{1}\left(2 \Lambda+\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right]$. In the plane $\xi_{\mathrm{q}}$, the generic reflected plane-wave is $\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{2} \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(2 \Lambda+\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{s} \zeta_{\mathrm{q}}} e^{-i n_{1} n_{\perp}^{\mathrm{s}}\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right)}$, with propagation vector $-n_{\perp}^{\mathrm{s}} \hat{\xi}_{\mathrm{q}}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$. The reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, with $\mathrm{j}=3$, is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(3)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{2} \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[-n_{1}\left(2 \Lambda+2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} \tag{1.40}
\end{equation*}
$$

and

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(3)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,0(3)}\left[-n_{1}\left(2 \Lambda+2 \chi_{\mathrm{q}}+\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.41}
\end{equation*}
$$

As a general rule, the argument of the plane-wave $F_{\mathrm{m}}$ corresponding to j reflections is $\left\{n_{1}\left[-(\mathrm{j}-1) \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right], \mathrm{n}_{\|}^{\mathrm{s}}\right\}$, when the number of reflections j is odd. With an even number of reflections, the argument of $F_{\mathrm{m}}$ is $\left[n_{1}\left(-\mathrm{j} \Lambda+\xi_{\mathrm{q}}\right), \mathrm{n}_{\|}^{\mathrm{s}}\right]$. Thus, generalized Reflected Cylindrical Functions $R W_{\mathrm{m}}^{1,0(\mathrm{j})}$ in MRF are defined as follows

$$
\begin{gather*}
R W_{\mathrm{m}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right\}= \\
\frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} F_{\mathrm{m}}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}}  \tag{1.42}\\
\left\{\begin{array}{c}
\mathrm{h}=\mathrm{j}, \quad \text { if } \mathrm{j}=2,4,6, \ldots \\
\mathrm{~h}=\mathrm{j}-1, \quad \text { if } \mathrm{j}=1,3,5, \ldots
\end{array}\right.
\end{gather*}
$$

where the first reflection occurs at the upper interface; the symbol h takes into account the number of reflections $j$, distinguishing the cases of odd or even reflections. Moreover, it is $\mathrm{f}=\mathrm{g}=\mathrm{j} / 2$ with even j , and $\mathrm{f}=(\mathrm{j}+1) / 2, \mathrm{~g}=(\mathrm{j}-1) / 2$ when j is odd.

In $M R F$, the scattered-reflected field $V_{\text {sr }(\mathrm{j})}^{1,0}$, with the first reflection occurring downwards at the upper interface, is defined by the following expression

$$
\begin{equation*}
V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right\} \tag{1.43}
\end{equation*}
$$

To make easier the imposition of the boundary conditions on the $p$-th cylinder surface, the diffracted-reflected field has to be expressed in the coordinates of $R F_{\mathrm{p}}$.

In (1.43), we introduce the (1.1)

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times  \tag{1.44}\\
\times R W_{\mathrm{m}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}}\left(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}\right)\right], n_{1}\left(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}
\end{gather*}
$$

and employ definition (1.42)

$$
\begin{align*}
& V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} \times  \tag{1.45}\\
\times & F_{\mathrm{m}}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}}\left(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}\right)\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}}
\end{align*}
$$

Making use of (1.33), the latter equation becomes

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} \times  \tag{1.46}\\
\times F_{\mathrm{m}}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1}\left(n_{\perp}^{\mathrm{sr}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{sr}} \zeta_{\mathrm{p}}\right)} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}}
\end{gather*}
$$

The exponential $e^{i n_{1}\left(n_{\perp}^{\mathrm{sr}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{sr}} \zeta_{\mathrm{p}}\right)}$ in (1.46) represents a reflected plane-wave in the reference frame $R F_{\mathrm{p}}$, with $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$, and $n_{\perp}^{\mathrm{sr}}= \pm n_{\perp}^{\mathrm{s}}$, taking the sign ' + ' for a downward-propagating wave, and '-' for an upward-propagating one. Such plane wave can be expanded into a series of Bessel functions

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{ss}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{ss}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \arcsin \left(n_{\|}^{\mathrm{sr}}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.47}
\end{equation*}
$$

where $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)$ is the angle of propagation $\varphi^{\mathrm{sr}}$.
As regards to upward-propagating waves, according to the Snell law $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=$ $\pi-\arcsin \left(n_{\|}^{\mathrm{s}}\right)$. Being $\arcsin \left(n_{\|}^{\mathrm{s}}\right) \geq \pi / 2$, it follows $\arcsin \left(n_{\|}^{\mathrm{s}}\right)=\pi / 2+\arccos \left(n_{\|}^{\mathrm{s}}\right)$, and the expansion (1.47) becomes

$$
\begin{equation*}
e^{i n_{1}\left(-n_{\perp}^{s} \xi_{\mathrm{p}}+n_{\|}^{s} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} e^{i \ell \arccos \left(n_{\|}^{s}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.48}
\end{equation*}
$$

Downward-propagating waves, instead, satisfy the following relation: $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=$ $\arcsin \left(n_{\|}^{\mathrm{s}}\right)$. Therefore, in this case $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=\pi / 2+\arccos \left(n_{\|}^{\mathrm{s}}\right)$, and the expansion (1.47) becomes

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{s}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{s}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} e^{-i \ell \arccos \left(n_{\|}^{s}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.49}
\end{equation*}
$$

The expansions (1.48) and (1.49) can be introduced in (1.46), and the exponential $e^{ \pm i \ell \arccos \left(n_{\|}\right)}$can be included in $F_{\mathrm{m}}$, according to definition (1.33), leading to

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times \\
\times \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} F_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(n_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}} \tag{1.50}
\end{gather*}
$$

The scattered-reflected field $V_{\mathrm{sr}(\mathrm{j})}^{1,0}$ in a point of coordinates $(\xi, \zeta)$, as a function of the coordinates in $R F_{\mathrm{p}}$, is the following

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times \\
\times R W_{\mathrm{m}+\ell(-1) \mathrm{j}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\},  \tag{1.51}\\
\left\{\begin{array}{c}
\mathrm{h}=\mathrm{j}, \quad \text { if } \mathrm{j}=2,4,6, \ldots \\
\mathrm{~h}=\mathrm{j}-1, \quad \text { if } \mathrm{j}=1,3,5, \ldots
\end{array}\right.
\end{gather*}
$$

## Scattered-reflected field from lower interface

The procedure applied to the scattered-reflected fields $V_{\text {sr }(\mathrm{j})}^{1,0}$ can be extended to the scattered-reflected fields $V_{\operatorname{sr}(\mathrm{j})}^{1,2}$, i.e. scattered-reflected fields excited by a first reflection on the lower interface, placed in $\xi=\Lambda$.

We consider the scattered-reflected field with 1 order of reflection, $V_{\mathrm{sr}(1)}^{1,2}$. The expression of the reflected cylindrical wave excited from reflection of the wave $C W_{\mathrm{m}}$ (1.29) on the plane of abscissa $\xi_{\mathrm{q}}=\Lambda-\chi_{\mathrm{q}}$ is looked for. The generic plane-wave $F_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}^{\mathrm{s}}}$ of the expansion (1.29), with propagation vector $n_{\perp}^{\mathrm{s}} \hat{\xi}_{\mathrm{q}}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$, is evaluated on the lower interface, and we get $F_{\mathrm{m}}\left[n_{1}\left(\Lambda-\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$. The corresponding reflected plane-wave has propagation vector $-n_{\perp}^{\mathrm{s}} \hat{\xi}_{q}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{\mathrm{q}}$, and in the plane $\xi_{\mathrm{q}}$ is $\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}\left(\Lambda-\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{-i n_{1} n_{\perp}^{\mathrm{s}}\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}-\Lambda\right)}$. The second exponential of the latter expression can be included in $F_{\mathrm{m}}$, according to definition (1.33). We get $\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$. Thus, downward-propagating reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, relevant to a single reflection $\mathrm{j}=1$, is derived

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(1)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} \tag{1.52}
\end{equation*}
$$

Recalling definition (1.34), the field generated by one reflection of the cylindrical
wave (1.29) is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(1)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,2(1)}\left[n_{1}\left(2 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.53}
\end{equation*}
$$

The scattered-reflected field from the upper interface $V_{\mathrm{dr}(1)}^{1,2}$ is given by a superimposition of the scattered-reflected fields by the $N$ cylinders, and making use of (1.1) in (1.52) we get

$$
\begin{equation*}
V_{\mathrm{sr}(1)}^{1,2}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell}^{1,2(1)}\left[n_{1}\left(2 \Lambda-\chi_{\mathrm{q}}-\xi\right), n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right] \tag{1.54}
\end{equation*}
$$

The reflected cylindrical waves $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, for a number of reflections $\mathrm{j}>1$, are defined starting from definition (1.52). We consider the case of $\mathrm{j}=2$ reflections: the generic plane-wave $F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right)\right.$, $\left.n_{\|}^{\mathrm{s}}\right]$ of the expansion (1.52) is evaluated at the upper interface, placed in $\xi_{\mathrm{q}}=-\chi_{\mathrm{q}}$, giving $F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda-\chi_{\mathrm{q}}\right), n_{\|]}^{\mathrm{s}}\right]$. In the plane $\xi_{\mathrm{q}}$, the reflected plane-wave is $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda-\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{i n_{1} n_{\perp}^{\mathrm{s}}\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right)}$, with propagation vector is $n_{\perp}^{\mathrm{s}} \hat{\xi}_{q}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{q}$. Including the second exponential in $F_{\mathrm{m}}$ by means of definition (1.33), the reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, with $\mathrm{j}=2$ reflections, is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(2)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} \tag{1.55}
\end{equation*}
$$

which satisfies the following equality with Reflected Cylindrical Functions (1.34)

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(2)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,2(2)}\left[n_{1}\left(2 \Lambda+\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.56}
\end{equation*}
$$

With $\mathrm{j}=3$ reflections, the generic plane-wave $F_{\mathrm{m}}\left[n_{1}\left(2 \Lambda+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right]$ of the expansion (1.55) is evaluated at the upper interface, in $\xi_{\mathrm{q}}=\Lambda-\chi_{\mathrm{q}}$, giving $F_{\mathrm{m}}\left[n_{1}(3 \Lambda-\right.$ $\left.\left.\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right]$. In the plane $\xi_{\mathrm{q}}$, the reflected plane-wave is $\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{2} F_{\mathrm{m}}\left[n_{1}(3 \Lambda-\right.$ $\left.\left.\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{-i n_{1} n_{\perp}^{s}\left(\xi_{q}+\chi_{\mathrm{q}}-\Lambda\right)}$, with propagation vector $-n_{\perp}^{\mathrm{s}} \hat{\xi}_{\mathrm{q}}+n_{\|}^{\mathrm{s}} \hat{\zeta}_{q}$. The reflected cylindrical wave $C W_{\mathrm{m}}^{\mathrm{r}(\mathrm{j})}$, with $\mathrm{j}=3$, is

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(3)}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{2} F_{\mathrm{m}}\left[n_{1}\left(4 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}} \tag{1.57}
\end{equation*}
$$

and it follows

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{r}(3)}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=R W_{\mathrm{m}}^{1,2(3)}\left[n_{1}\left(4 \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right), n_{1} \zeta_{\mathrm{q}}\right] \tag{1.58}
\end{equation*}
$$

As a general rule, the argument of the plane-wave $F_{\mathrm{m}}$ corresponding to j reflections is $\left\{n_{1}\left[(j+1) \Lambda-2 \chi_{q}-\xi_{q}\right], n_{\|}^{\mathrm{s}}\right\}$, when the number of reflections j is odd. With an even number of reflections, the argument of $F_{\mathrm{m}}$ is $\left[n_{1}\left(\mathrm{j} \Lambda+\xi_{\mathrm{q}}\right), \mathrm{n}_{\|}^{\mathrm{s}}\right]$.

Generalized Reflected Cylindrical Functions $R W_{\mathrm{m}}^{1,2(\mathrm{j})}$ are finally defined, with a first reflection at the lower interface:

$$
\begin{gather*}
R W_{\mathrm{m}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right\}= \\
\frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} F_{\mathrm{m}}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}},  \tag{1.59}\\
\left\{\begin{array}{c}
\mathrm{h}=\mathrm{j}, \quad \text { if } \mathrm{j}=2,4,6, . . \\
\mathrm{h}=\mathrm{j}+1, \quad \text { if } \mathrm{j}=1,3,5, . .
\end{array}\right.
\end{gather*}
$$

where the symbol $h$ takes into account the number of reflections $j$, distinguishing the cases of odd or even reflections. Moreover, it is $\mathrm{f}=\mathrm{g}=\mathrm{j} / 2$ with even j , and $\mathrm{f}=(\mathrm{j}-1) / 2$ and $\mathrm{g}=(\mathrm{j}+1) / 2$ when j is odd.

In $M R F$, the scattered-reflected fields $V_{\mathrm{sr}(\mathrm{j})}^{1,2}$, relevant to a first reflection occuring upwards at the lower interface, are defined by the following expression

$$
\begin{equation*}
V_{\mathrm{sr}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \xi\right], n_{1}\left(\zeta-\eta_{\mathrm{q}}\right)\right\} \tag{1.60}
\end{equation*}
$$

To make easier the imposition of the boundary conditions on the $p$-th cylinder surface, the diffracted-reflected field is expressed in the coordinates of $R F_{\mathrm{p}}$. Introducing (1.1) in (1.60)

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times  \tag{1.61}\\
\times R W_{\mathrm{m}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}}\left(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}\right)\right], n_{1}\left(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}
\end{gather*}
$$

and employing definition (1.59), the field can be written as

$$
\begin{align*}
& \quad V_{\mathrm{sr}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} \times  \tag{1.62}\\
& \times F_{\mathrm{m}}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}}\left(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}\right)\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}}
\end{align*}
$$

Making use of (1.33), the former equation becomes

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j}}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|)}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} \times  \tag{1.63}\\
\times F_{\mathrm{m}}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1}\left(n_{\perp}^{\mathrm{sr}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{sr}} \zeta_{\mathrm{p}}\right)} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}}
\end{gather*}
$$

The exponential $e^{i n_{1}\left(n_{\perp}^{\mathrm{sr}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{sr}} \zeta_{\mathrm{p}}\right)}$ in (1.63) represents a reflected plane-wave in the
reference frame $R F_{\mathrm{p}}$, with $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$, and $n_{\perp}^{\mathrm{sr}}= \pm n_{\perp}^{\mathrm{s}}$, taking the sign ' + ' for a downward-propagating wave and '-' for an upward-propagating one. Such plane wave can be expanded into a series of Bessel functions

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{sr}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{sr}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \arcsin \left(n_{\|}^{\mathrm{sr}}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.64}
\end{equation*}
$$

being $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)$ the angle of propagation $\varphi^{\mathrm{sr}}$.
According to the Snell law, for the upward-propagating waves it turns out: $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=\pi-\arcsin \left(n_{\|}^{\mathrm{s}}\right)$. Being $\arcsin \left(n_{\|}^{\mathrm{s}}\right) \leq \pi / 2$, it follows $\arcsin \left(n_{\|}^{\mathrm{s}}\right)=\pi / 2-$ $\arccos \left(n_{\|}^{\mathrm{s}}\right)$, and the expansion (1.64) becomes

$$
\begin{equation*}
e^{i n_{1}\left(-n_{\perp}^{s} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{s}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} e^{-i \ell \arccos \left(n_{\|}^{\mathrm{s}}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.65}
\end{equation*}
$$

Downward-propagating waves, instead, satisfy the following relation: $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=$ $\arcsin \left(n_{\|}^{\mathrm{s}}\right)$. Therefore, in this case $\arcsin \left(n_{\|}^{\mathrm{sr}}\right)=\pi / 2-\arccos \left(n_{\|}^{\mathrm{s}}\right)$, and the expansion (1.64) becomes

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{\mathrm{s}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{s}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} e^{i \ell \arccos \left(n_{\|}^{\mathrm{s}}\right)} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{1.66}
\end{equation*}
$$

The expansions (1.65) and (1.66) can be introduced in (1.63), and the exponential $e^{\mp i \ell \arccos \left(n_{\|}\right)}$can be included in $F_{\mathrm{m}}$, according to definition (1.33); this yields to

$$
\begin{gather*}
V_{\mathrm{sr}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times \\
\times \frac{1}{2 \pi} \int_{-\infty}^{+\infty}\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{f}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{g}} F_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}} \tag{1.67}
\end{gather*}
$$

The scattered-reflected fields, associated to a point of coordinated $(\xi, \zeta)$, as a function of the coordinates in $R F_{\mathrm{p}}$, are the following

$$
\begin{align*}
& \quad V_{\mathrm{sr}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} \times \\
& \times R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\},  \tag{1.68}\\
& \left\{\begin{array}{c}
\mathrm{h}=\mathrm{j}, \quad \text { if } \mathrm{j}=2,4,6, . . \\
\mathrm{h}=\mathrm{j}+1, \quad \text { if } \mathrm{j}=1,3,5, . .
\end{array}\right.
\end{align*}
$$

### 1.2.6 Scattered-transmitted fields

To express the diffracted-transmitted fields, the evaluation of the cylindrical wave given in (1.29), through the upper or lower interface, is needed. The procedure applied is shown in detail for the scattered field transmitted from medium 1 to medium 0 .

The followings are the orthogonal and parallel components, respectively, of the unit vector $\mathbf{n}^{\text {st }}$ relevant to the transmitted spectral plane-wave in medium 1

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{st}}=-\sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}}\right)^{2}}  \tag{1.69}\\
n_{\|}^{\mathrm{st}}=n_{1} n_{\|}^{\mathrm{s}}
\end{array}\right.
$$

The transmitted plane-wave corresponds to the spectrum $F_{\mathrm{m}}\left(n_{1} \xi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$ of the expansion (1.29), and in the plane of abscissa $\xi_{\mathrm{q}}$ it is defined as

$$
\begin{equation*}
T_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left(-n_{1} \chi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{-i\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right)} \sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}}\right)^{2}} \tag{1.70}
\end{equation*}
$$

The transmitted cylindrical wave $C W_{\mathrm{m}}^{\mathrm{t}}$, given by the transmission through the upper interface of the wave (1.29), is so expressed as

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{t}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left(-n_{1} \chi_{\mathrm{q}}, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{-i\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right) \sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}}\right)^{2}}} d n_{\|}^{\mathrm{s}} \tag{1.71}
\end{equation*}
$$

Once defined the Transmitted Cylindrical Wave Function [34] of order m $T W_{\mathrm{m}}^{1,0}$ as

$$
\begin{equation*}
T W_{\mathrm{m}}^{1,0}(\xi ; \zeta, \chi)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{10}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left(-n_{1} \chi, n_{\|}^{\mathrm{s}}\right) e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta} e^{-i(\xi+\chi) \sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}}\right)^{2}}} d n_{\|}^{\mathrm{s}} \tag{1.72}
\end{equation*}
$$

we get that the field given by the transmission of the cylindrical wave (1.29) corresponds to

$$
\begin{equation*}
C W_{\mathrm{m}}^{\mathrm{t}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)=T W_{\mathrm{m}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}} ; \chi_{\mathrm{q}}\right) \tag{1.73}
\end{equation*}
$$

The scattered-transmitted field by the $q$-th cylinder is given by a superimposition of the transmitted functions $C W_{\mathrm{m}}^{\mathrm{t}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)$, with unknown coefficients

$$
\begin{equation*}
V_{\mathrm{st}(\mathrm{q})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} C W_{\mathrm{m}}^{\mathrm{t}}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right) \tag{1.74}
\end{equation*}
$$

With a superimposition of the $N$ scattered-transmitted field by each cylinder, and taking into account equations (1.1) where $p$ is replaced with $q$, we get

$$
\begin{equation*}
V_{\mathrm{st}}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} T W_{\mathrm{m}}^{1,0}\left(\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ; \chi_{\mathrm{q}}\right) \tag{1.75}
\end{equation*}
$$

which is the scattered-transmitted field $V_{\mathrm{st}}^{(1,0)}$.
In the same way, from the cylindrical wave (1.29), the Transmitted Cylindrical Functions $T W_{\mathrm{m}}^{1,2}$, for transmission from medium 1 to medium 2, can be derived

$$
\begin{equation*}
T W_{\mathrm{m}}^{1,2}(\xi-\Lambda, \zeta ; \chi)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{12}\left(n_{\|}^{\mathrm{s}}\right) F_{\mathrm{m}}\left[n_{1}(\Lambda-\chi), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta} e^{i(\xi+\chi-\Lambda) \sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}} / n_{2}\right)^{2}}} d n_{\|}^{\mathrm{s}} \tag{1.76}
\end{equation*}
$$

The orthogonal and parallel components, respectively, of the unit vector $\mathbf{n}^{\text {st }}$ relevant to the transmitted spectral plane-wave in medium 2 are

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{st}}=\sqrt{1-\left(n_{1} n_{\|}^{\mathrm{s}} / n_{2}\right)^{2}}  \tag{1.77}\\
n_{\|}^{\text {st }}=n_{1} n_{\|}^{\mathrm{s}} / n_{2}
\end{array}\right.
$$

The scattered field $V_{\mathrm{st}}^{1,2}$, transmitted from medium 1 to medium 2 , is the following

$$
\begin{equation*}
V_{\mathrm{st}}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} T W_{\mathrm{m}}^{1,2}\left(\xi-\Lambda-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ; \Lambda-\chi_{\mathrm{q}}\right) \tag{1.78}
\end{equation*}
$$

### 1.2.7 Scattered reflected-transmitted field

The scattered-reflected-transmitted fields $V_{\mathrm{srt}(\mathrm{j})}^{1,0}$ and $V_{\mathrm{srt}(\mathrm{j})}^{1,2}$ are scattered fields transmitted in medium 0 or medium 2, respectively, after multiple reflections inside the dielectric layer.

In order express the scattered-reflected-transmitted fields, the following ReflectedTransmitted Cylindrical Functions are introduced

$$
\begin{gather*}
R T W_{\mathrm{m}}^{(1, \cdot)(\mathrm{j})}\left(\xi, \zeta ; \chi, \mathrm{h} \Lambda, n_{1}, n\right)= \\
\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{(1 \cdot)}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{p}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{r}} F_{\mathrm{m}}\left(\chi+\mathrm{h} \Lambda, n_{\|}^{\mathrm{s}}\right) e^{i n_{\perp}^{\mathrm{st}}(\xi+\chi)} e^{i n n_{\|}^{\mathrm{s}} \zeta} d n_{\|}^{\mathrm{s}} \tag{1.79}
\end{gather*}
$$

The symbol $(1, \cdot)$ stands for $(1,0)$ when the field is transmitted from medium 1 to medium 0, being $T_{(1 \cdot)}\left(n_{\|}^{\mathrm{s}}\right)$ the plane-wave transmission coefficient $T_{10}\left(n_{\|}^{\mathrm{s}}\right)$. When the transmission occurs from medium 1 to medium $2,(1, \cdot)$ stands for $(1,2)$, and the plane-wave transmission coefficient is $T_{12}\left(n_{\|}^{\mathrm{s}}\right) . \Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right), \Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)$ are the planewave reflection coefficients, with respect to the medium $0 /$ medium 1 and medium $1 /$ medium 2 interfaces, respectively. The exponents $p$ and $r$ stands for the number of reflections occurred at such interfaces, being $j=p+r$ the total number of reflections.

The field $V_{\text {srt }(\mathrm{j})}^{1,0}$ is the scattered-reflected-transmitted field in medium 0, i.e. it undergoes j reflections inside the slab before transmission through the upper interface in medium 0 . The spectral reflected-transmitted plane-wave is obtained by the
generic reflected plane-wave of the spectrum in equation (1.42) and (1.59), evaluated at the $\xi=0$ interface.

The propagation unit vector $\mathbf{n}^{\text {srt }}$ of the reflected-transmitted plane-wave in medium 1 has components

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{srt}}=-\sqrt{1-\left(n_{1} n_{\|}^{\mathrm{sr}}\right)^{2}}  \tag{1.80}\\
n_{\|}^{\mathrm{srt}}=n_{1} n_{\|}^{\mathrm{sr}}
\end{array}\right.
$$

being $\mathbf{n r}^{\mathrm{sr}}$ the propagation unit vector of the plane waves of the spectrum (1.42), with components $n_{\perp}^{\mathrm{sr}}=n_{\perp}^{\mathrm{s}}$ and $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$, or of spectrum (1.59), with $n_{\perp}^{\mathrm{sr}}=-n_{\perp}^{\mathrm{s}}$ and $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$. Finally, we can state that

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{sst}} \equiv n_{\perp}^{\mathrm{st}}  \tag{1.81}\\
n_{\|}^{\mathrm{srt}} \equiv n_{\|}^{\mathrm{st}}
\end{array}\right.
$$

with $n_{\|}^{\text {st }}$ and $n_{\perp}^{\text {st }}$ given by (1.69).
In particular, final transmission in medium 1 depends just on upward-propagating plane-waves (1.42) and (1.59). Transmission spectrum is evaluated from the spectrum of Reflected Cylindrical Functions (1.42), with first reflection on upper interface and an even number of reflections j. Instead, as regards to the spectrum of Reflected Cylindrical Functions (1.59), where the first reflection occurs at the lower interface, such waves are upward-propagating for an odd number of reflections $j$ (Fig. 1.5).


Figure 1.5: Sketch of the reflected-transmitted cylindrical waves propagating in medium 0, excited by a multiple-reflected wave with first reflection either at the upper (solid arrows) and lower (dashed arrows) interface of the slab, corresponding to even and odd reflections, respectively.

We consider now the upward-propagating Reflected Cylindrical Functions defined in $\operatorname{MRF}$ (1.42), corresponding to even values of $\mathrm{h}=\mathrm{j}$. The evaluation of the
generic reflected plane-wave $\left.\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}}{2}} F_{\mathrm{m}}\left[n_{1}\left(-\mathrm{j} \Lambda+\xi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right)\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$ of the expansion (1.42), on the upper interface, leads to $\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}} F_{\mathrm{m}}\left[n_{1}(-\mathrm{j} \Lambda-\right.$ $\left.\left.\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$. The corresponding reflected-transmitted plane-wave in the plane of abscissa $\xi_{\mathrm{q}}$ is $T_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}} F_{\mathrm{m}}\left[n_{1}\left(-\mathrm{j} \Lambda-\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{i n_{\perp}^{\mathrm{st}}\left(\xi_{q}+\chi_{\mathrm{q}}\right)}$. Thus, the reflected-transmitted cylindrical waves in medium 0 , for an even number of reflections j inside the slab, are

$$
\begin{gather*}
C W_{\mathrm{m}}^{r t(\mathrm{j})}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j}{2}} \times \\
\times F_{\mathrm{m}}\left[n_{1}\left(-\mathrm{j} \Lambda-\chi_{\mathrm{q}}\right), n_{\|}^{\mathrm{s}}\right] e^{i n_{\perp}^{\mathrm{st}}\left(\xi_{\mathrm{q}}+\chi_{\mathrm{q}}\right)} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}},  \tag{1.82}\\
\text { with } \quad \mathrm{j}=2,4,6, . .
\end{gather*}
$$

which compared with definition (1.79) give

$$
\begin{equation*}
C W_{\mathrm{m}}^{r t(\mathrm{j})}(\xi, \zeta)=R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left[\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ;-\chi_{\mathrm{q}},-\mathrm{j} \Lambda, n_{1}, n_{0}\right] \tag{1.83}
\end{equation*}
$$

Upward-propagating Reflected Cylindrical Functions defined in MRF (1.59) are, instead, associated to odd values of $\mathrm{h}=\mathrm{j}$. The generic reflected plane-wave $\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}}{2}}$ $\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}+1}{2}} F_{\mathrm{m}}\left\{n_{1}\left[(\mathrm{j}+1) \Lambda-2 \chi_{\mathrm{q}}-\xi_{\mathrm{q}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$ of the expansion (1.59) is evaluated on the upper interface, leading to $\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{j-1}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}+1}{2}} F_{\mathrm{m}}\left\{n_{1}[(\mathrm{j}+1) \Lambda-\right.$ $\left.\left.\chi_{\mathrm{q}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}}$. In the plane $\xi_{\mathrm{q}}$, the reflected-transmitted plane-wave has amplitude $T_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}-1}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}+1}{2}} F_{\mathrm{m}}\left\{n_{1}\left[(\mathrm{j}+1) \Lambda-\chi_{\mathrm{q}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} e^{i n_{\perp}^{\mathrm{st}}\left(\xi_{q}+\chi_{\mathrm{q}}\right)}$. The reflected-transmitted cylindrical waves in medium 0 , for an odd number of reflections j inside the slab, are

$$
\begin{gather*}
C W_{\mathrm{m}}^{r t(\mathrm{j})}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}-1}{2}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\frac{\mathrm{j}+1}{2}} \times \\
\times F_{\mathrm{m}}\left\{n_{1}\left[(\mathrm{j}+1) \Lambda-\chi_{\mathrm{q}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{\perp}^{\mathrm{st}}\left(\xi_{q}+\chi_{\mathrm{q}}\right)} e^{i n_{1} n_{\|}^{\mathrm{s}} \zeta_{\mathrm{q}}} d n_{\|}^{\mathrm{s}},  \tag{1.84}\\
\text { with } \quad \mathrm{j}=1,3,5, . .
\end{gather*}
$$

and recalling definition (1.79) we get

$$
\begin{equation*}
C W_{\mathrm{m}}^{r t(\mathrm{j})}(\xi, \zeta)=R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left[\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ;-\chi_{\mathrm{q}},(\mathrm{j}+1) \Lambda, n_{1}, n_{0}\right] \tag{1.85}
\end{equation*}
$$

From equations (1.82) and (1.84), general expression for Reflected-Transmitted Cylindrical Functions in medium 0 of order $\mathrm{m} R T W_{\mathrm{m}}^{1,0(\mathrm{j})}$ can be derived

$$
\begin{gather*}
R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left[\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ;-\chi_{\mathrm{q}},(-1)^{\mathrm{j}+1} \mathrm{~h} \Lambda, n_{1}, n_{0}\right]= \\
=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{10}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{p}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{r}} F_{\mathrm{m}}\left\{n_{1}\left[(-1)^{\mathrm{j}+1} \mathrm{~h} \Lambda-\chi_{q}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{\perp}^{\mathrm{s}} \xi} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}} \tag{1.86}
\end{gather*}
$$

where the spectrum $F_{\mathrm{m}}$ is defined in (1.33), and it corresponds to the definition for $\xi \geq 0$ and $\xi \leq 0$ when the number of reflections is odd or even, respectively. We remark that $\mathrm{p}+\mathrm{r}=\mathrm{j}$; moreover, it is $\mathrm{p}=\mathrm{r}=\mathrm{j} / 2$ when j is even, and $\mathrm{p}=(\mathrm{j}-1) / 2$, $\mathrm{r}=(\mathrm{j}+1) / 2$ when j is odd. The scattered reflected-transmitted fields by the $q$-th cylinder is given by a superimposition of the transmitted functions $R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left(\xi_{\mathrm{q}}, \zeta_{\mathrm{q}}\right)$, with unknown coefficients

$$
\begin{equation*}
V_{\mathrm{srt}(\mathrm{q})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left[\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ; \chi_{\mathrm{q}},(-1)^{\mathrm{j}+1} \mathrm{~h} \Lambda, n_{1}, n_{0}\right] \tag{1.87}
\end{equation*}
$$

A superimposition of the $N$ scattered-transmitted field by each cylinder yields

$$
\begin{gather*}
V_{\mathrm{srt}(\mathrm{j})}^{1,0}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R T W_{\mathrm{m}}^{1,0(\mathrm{j})}\left[\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ; \chi_{\mathrm{q}},(-1)^{\mathrm{j}+1} \mathrm{~h} \Lambda, n_{1}, n_{0}\right] \\
\left\{\begin{array}{c}
\mathrm{h}=\mathrm{j}, \quad \text { if } \mathrm{j}=2,4,6, . . \\
\mathrm{h}=\mathrm{j}+1, \quad \text { if } \mathrm{j}=1,3,5, . .
\end{array}\right. \tag{1.88}
\end{gather*}
$$

which is the scattered reflected-transmitted field $V_{\text {srt(j) }}^{1,0}$ in a point of coordinates $(\xi, \zeta)$ in MRF.

In a similar way, the field and $V_{\text {srt }(\mathrm{j})}^{1,2}$, transmitted in medium 2, can be evaluated by means of reflected-transmitted cylindrical waves relevant to the propagation from medium 1 to medium 2.

General expression for Reflected-Transmitted Cylindrical Functions in medium 2 is derived from down-propagating waves in the spectra of Reflected Cylindrical Functions (1.42) and (1.59), and it is the following

$$
\begin{gather*}
R T W_{\mathrm{m}}^{1,2(\mathrm{j})}\left[\xi-\Lambda-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ;-\chi_{\mathrm{q}},(-1)^{\mathrm{j}} \mathrm{~h} \Lambda, n_{1}, n_{2}\right]= \\
=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{12}\left(n_{\|}^{\mathrm{s}}\right)\left[\Gamma_{10}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{p}}\left[\Gamma_{12}\left(n_{\|}^{\mathrm{s}}\right)\right]^{\mathrm{r}} F_{\mathrm{m}}\left\{n_{1}\left[(-1)^{\mathrm{j}} \mathrm{~h} \Lambda-\chi_{\mathrm{q}}\right], n_{\|}^{\mathrm{s}}\right\} e^{i n_{\perp}^{\mathrm{s}}(\xi-\Lambda)} e^{i n_{1} n_{\|}^{\mathrm{s}}\left(\zeta-\eta_{\mathrm{q}}\right)} d n_{\|}^{\mathrm{s}} \tag{1.89}
\end{gather*}
$$

The propagation unit vector $\mathbf{n}^{\text {srt }}$ of the reflected-transmitted plane-wave in medium 2 has components

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{srt}}=\sqrt{1-\left(n_{1} n_{\|}^{\mathrm{sr}} / n_{2}\right)^{2}}  \tag{1.90}\\
n_{\|}^{\mathrm{srt}}=n_{1} n_{\|}^{\mathrm{sr}} / n_{2}
\end{array}\right.
$$

being $\mathbf{n r}^{\text {sr }}$ the propagation unit vector of the plane waves of the spectrum (1.42), with components $n_{\perp}^{\mathrm{sr}}=-n_{\perp}^{\mathrm{s}}$ and $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$, or of spectrum (1.59), with $n_{\perp}^{\mathrm{sr}}=n_{\perp}^{\mathrm{s}}$ and $n_{\|}^{\mathrm{sr}}=n_{\|}^{\mathrm{s}}$. We can also state that

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{sst}} \equiv n_{\perp}^{\mathrm{st}}  \tag{1.91}\\
n_{\|}^{\mathrm{srt}} \equiv n_{\|}^{\mathrm{st}}
\end{array}\right.
$$

with $n_{\|}^{\text {st }}$ and $n_{\perp}^{\text {st }}$ given by (1.77).
In (1.89), the spectrum $F_{\mathrm{m}}$ is defined in (1.33), and it corresponds to the definition for $\xi \geq 0$ or $\xi \leq 0$ when the number of reflections is even or odd, respectively. It is $\mathrm{p}+\mathrm{r}=\mathrm{j}$, and $\mathrm{p}=\mathrm{r}=\mathrm{j} / 2$ when j is even, and $\mathrm{p}=(\mathrm{j}+1) / 2, \mathrm{r}=(\mathrm{j}-1) / 2$ when j is odd.

From (1.89), the scattered reflected-transmitted field in medium $2 V_{\operatorname{srt}(\mathrm{j})}^{1,2}$ in a point of coordinates $(\xi, \zeta)$ in $M R F$ is defined as

$$
\begin{gather*}
V_{\mathrm{srt}(\mathrm{j})}^{1,2}(\xi, \zeta)=V_{0} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R T W_{\mathrm{m}}^{1,2(\mathrm{j})}\left[\xi-\Lambda-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}} ;-\chi_{\mathrm{q}},+(-1)^{\mathrm{j}} \mathrm{~h} \Lambda, n_{1}, n_{2}\right], \\
\left\{\begin{array}{l}
\mathrm{h}=\mathrm{j}, \quad \text { if } \quad \mathrm{j}=2,4,6, . . \\
\mathrm{h}=\mathrm{j}-1, \quad \text { if } \quad \mathrm{j}=1,3,5, . .
\end{array}\right. \tag{1.92}
\end{gather*}
$$

### 1.3 Boundary conditions

Once the expressions of the fields have been obtained, the scattering problem is solved imposing the boundary conditions on the cylinders surfaces, being the boundary condition on the planar interfaces already included in the plane-wave reflection and transmission coefficients. We distinguish the two cases of E and H polarization.

### 1.3.1 E polarization

In case of E polarization, the electric field has only the $E_{y}$ component, corresponding to the scalar function $V$. The boundary condition to be imposed of the $p$-th cylinder is the following:

$$
\begin{equation*}
\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=0, \quad \text { with } \mathrm{p}=1, \ldots, N \tag{1.93}
\end{equation*}
$$

Substituting in (1.93) equations (1.16), (1.11), (1.28), (1.51) and (1.68), with $\rho_{p}=$ $\alpha_{p}$, and simplifying the constant $V_{0}$

$$
\begin{gather*}
T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} \sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \varphi_{\mathrm{t} 1}} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}}+ \\
+T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{tr}} \eta_{\mathrm{p}}\right]} \times \sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \varphi^{\mathrm{rt}}} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}}+ \\
+\sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\ell( }\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}\right]+ \\
+\sum_{\mathrm{j}=1}^{+\infty} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}+ \\
+\sum_{\mathrm{j}=1}^{+\infty} \sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}=0 \tag{1.94}
\end{gather*}
$$

Multiplying the (1.94) by $e^{-i \nu \theta_{\mathrm{p}}}$, integrating between 0 and $2 \pi$ in the variable $\theta_{\mathrm{p}}$, and employing the orthogonality property of exponential functions $\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i \ell \theta_{\mathrm{p}}} e^{-i \nu \theta_{\mathrm{p}}} d \theta_{\mathrm{p}}=$ $\delta_{\ell \nu}$, we obtain

$$
\begin{gather*}
T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} i^{\nu} e^{-i \nu \varphi_{\mathrm{t} 1}} J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)+ \\
+T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{t 1} \eta_{\mathrm{p}}\right]} i^{\nu} e^{-i \nu \varphi^{r 1}} J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)+ \\
+J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\nu}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\nu}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\nu \mathrm{m}}\right]+ \\
+\sum_{\mathrm{j}=1}^{+\infty} J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\nu(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}+ \\
+\sum_{\mathrm{j}=1}^{+\infty} J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\nu(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}=0 \tag{1.95}
\end{gather*}
$$

In (1.95) the term $J_{\nu}\left(n_{1} \alpha_{\mathrm{p}}\right)$ can be simplified; isolating the unknown terms on left side, putting $G_{\nu}(\cdot)=J_{\nu}(\cdot) / H_{\nu}^{(1)}(\cdot)$, multiplying both sides by $G_{\nu}\left(n_{1} \alpha_{p}\right)$, and dividing by $i^{\nu}$, we get

$$
\begin{gather*}
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} i^{-\nu} G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)\left\{\left[C W_{\mathrm{m}-\nu}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{qp}} \delta_{\nu \mathrm{m}}}{G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)}+\right.\right. \\
\quad+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\nu(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{j} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)+\right. \\
\left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\nu(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}\right\}=  \tag{1.96}\\
\quad=-T_{01}\left(n_{\|}^{\mathrm{i}}\right) G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i n_{1}\left[n_{\perp}^{t 1}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\| 1}^{\mathrm{t1}} \eta_{\mathrm{p}]}\right]} e^{-i \nu \varphi_{\mathrm{t} 1}}+ \\
\quad-T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}]}\right]} e^{-i \nu \varphi^{\mathrm{r} 1}}
\end{gather*}
$$

We can substitute $\nu$ with $\ell$, and write the system (1.96) in a more compact way

$$
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}} c_{\mathrm{qm}}-B_{\ell}^{\mathrm{p}}=0 \quad\left\{\begin{array}{l}
\ell=0, \pm 1, \ldots, \pm \infty  \tag{1.97}\\
\mathrm{p}=1, \ldots, N
\end{array}\right.
$$

with

$$
\begin{gather*}
A_{\ell \mathrm{m}}^{\mathrm{qp}}=i^{-\ell} G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)\left\{C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}}{G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)}\right. \\
+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}  \tag{1.98}\\
\left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}\right\} \\
B_{\ell}^{\mathrm{p}}=-G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)\left\{T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi^{\mathrm{t1}}}\right. \\
\left.\quad+\Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\| 1}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi^{\mathrm{r1}}}\right\} \tag{1.99}
\end{gather*}
$$

The (1.97) is a linear system of $(\infty \cdot N)$ equations in $(\infty \cdot N)$ unknowns $c_{\mathrm{qm}}$.

### 1.3.2 H Polarization

For TE polarization, from Maxwell equations the condition is the following

$$
\begin{equation*}
\left\{\frac{\partial}{\partial \rho_{\mathrm{p}}}\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr} \mathrm{j}(\mathrm{j}}^{1,2}\right]\right\} \underset{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}{ }=0, \quad \text { with } \mathrm{p}=1, \ldots, N \tag{1.100}
\end{equation*}
$$

Substituting in (1.100) equations (1.16), (1.11), (1.28), (1.51) and (1.68), with $\rho_{p}=$ $\alpha_{p}$, and simplifying the constant $n_{1} V_{0}$

$$
\begin{gathered}
T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}]}\right]} \sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \varphi_{\mathrm{t1}}} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}}+ \\
+T_{01}\left(n_{\|}^{\mathrm{i}}\right) \Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} \times \sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \varphi^{\mathrm{r} 1}} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}}+ \\
+\sum_{\ell=-\infty}^{+\infty} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\ell( }\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\ell}^{\prime(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}\right]+ \\
+\sum_{\mathrm{j}=1}^{+\infty} \sum_{\ell=-\infty}^{+\infty} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell(-1)^{j}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}+ \\
+\sum_{\mathrm{j}=1}^{+\infty} \sum_{\ell=-\infty}^{+\infty} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}=0 \\
\text { with } J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right)=\frac{1}{n_{1}}\left[\frac{\partial J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)}{\partial \rho_{\mathrm{p}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}^{\text {and } H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)=\frac{1}{n_{1}}\left[\frac{\partial H_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{\partial \rho_{\mathrm{s}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} .} .
\end{gathered}
$$

From a comparison of equation (1.101) with (1.95), it turns out that the system for the H polarization is the same for the E polarization, being now $G_{\ell}(\cdot)=$ $J_{\ell}^{\prime}(\cdot) / H_{\ell}^{\prime(1)}(\cdot)$.

## Chapter 2

## Dielectric cylinders buried in a dielectric layer

The analytical approach applied to the solution of scattering of a plane wave by perfectly-conducting cylinders, buried in a dielectric layer, is now generalizeded to the case of dielectric scatterers.

In addition to the field contributions already considered in Section 1.2, a scattered field transmitted inside the $p$-th cylinder of refraction index $n_{\mathrm{cp}}$ is introduced, in Section 2.1. The latter is expressed into a modal expansion with coefficients which represent further unknowns of the problem.

Boundary conditions for dielectric scatterers are expressed imposing the continuity of both the electric and magnetic field tangential to the cylinders surfaces. A system of matched equations is derived in Section 2.2, for coefficients of the scattered fields external to the cylinders and coefficients of the field internal to the cylinder.

### 2.1 Field internal to the cylinders

The field $V_{\mathrm{cp}}(\xi, \zeta)$ inside the $p$-th cylinder is now introduced. It can be easily obtained as a modal expansion in terms of first-kind Bessel functions $J_{\ell}$, with expansion coefficients $d_{\mathrm{p} \ell}$ representing further unknowns:

$$
\begin{equation*}
V_{\mathrm{cp}}(\xi, \zeta)=V_{0} \sum_{\ell=-\infty}^{+\infty} d_{\mathrm{p} \ell} J_{\ell}\left(n_{\mathrm{cp}} \rho_{\mathrm{p}}\right) e^{\mathrm{i} \ell \theta_{\mathrm{p}}} \tag{2.1}
\end{equation*}
$$

### 2.2 Boundary conditions

Let us consider Maxwell's equations in a medium of permittivity $\varepsilon$ and permeability $\mu$

$$
\left\{\begin{align*}
\mathbf{E} & =-\frac{1}{i \omega \varepsilon} \nabla \times \mathbf{H}  \tag{2.2}\\
\mathbf{H} & =\frac{1}{i \omega \mu} \nabla \times \mathbf{E}
\end{align*}\right.
$$

Recalling the normalized coordinates defined in Section 1.1, in the (2.2) we can substitute $\nabla=k \tilde{\nabla}$, being $\tilde{\nabla}$ the curl with respect to the normalized coordinates $(\xi, \zeta)$ and $k$ the wave number of the concerned medium; thus we get

$$
\left\{\begin{align*}
\mathbf{E}=\frac{i k}{\omega \varepsilon} \tilde{\nabla} \times \mathbf{H}=i Z \tilde{\nabla} \times \mathbf{H}  \tag{2.3}\\
\mathbf{H}=\frac{k}{i \omega \mu} \tilde{\nabla} \times \mathbf{E}=\frac{1}{i Z} \tilde{\nabla} \times \mathbf{E}
\end{align*}\right.
$$

where $Z$ is the impedance of medium.
Inside the dieletric slab, equations (2.3) can be written in the following way

$$
\left\{\begin{array}{l}
\mathbf{E}=\frac{i Z}{Q} \tilde{\nabla} \times \mathbf{H}  \tag{2.4}\\
\mathbf{H}=\frac{1}{i Z} \tilde{\nabla} \times \mathbf{E}
\end{array}\right.
$$

being the parameter $Q$ defined as

$$
Q= \begin{cases}1 & \text { outside the cylinders }  \tag{2.5}\\ \left(\frac{n_{\mathrm{cp}}}{n_{1}}\right)^{2} & \text { inside the } p \text {-th cylinder }\end{cases}
$$

The curl of the function $V(\xi, \zeta) \hat{y}$ in cylindrical coordinates is

$$
\begin{equation*}
\tilde{\nabla} \times V(\xi, \zeta) \hat{y}=\frac{1}{\rho} \frac{\partial V(\xi, \zeta)}{\partial \theta} \hat{r}-\frac{\partial V(\xi, \zeta)}{\partial \rho} \hat{\theta} \tag{2.6}
\end{equation*}
$$

where $\hat{y}, \hat{r}$ e $\hat{\theta}$ are unit vectors in the cylindrical frame.

### 2.2.1 E Polarization

Once all the field contributions have been defined, the unknown coefficients $c_{\mathrm{qm}}$ and $d_{\mathrm{pm}}$ can be determined by imposing the boundary conditions at the cylinders surfaces

$$
\begin{equation*}
\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=\left[V_{\mathrm{cp}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{2.7}
\end{equation*}
$$

$$
\begin{equation*}
\left\{\frac{\partial}{\partial \rho_{\mathrm{p}}}\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}\right]\right\}_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=\left[V_{\mathrm{cp}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{2.8}
\end{equation*}
$$

We consider equations (2.7), and act as already done on (1.94) in Section 1.3.1: on the left side we get what obtained at left side of (1.97), i.e., $A_{\ell \mathrm{m}}^{\mathrm{qp}}$ and $B_{\ell}^{\mathrm{p}}$. In this case, they are called $A_{\ell \mathrm{m}}^{\mathrm{qp}(1)}$ and $B_{\ell}^{\mathrm{p}(1)}$, defined exactly by (1.98) and (1.99). Now we analyze the same passages on the right side of (2.7): substituting $V_{\text {cp }}$ as given in (2.1), with $\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}$, and simplifying the constant $V_{0}$, it turns out to be

$$
\begin{equation*}
\sum_{\ell=-\infty}^{+\infty} i^{\ell} d_{\mathrm{p} \ell} J_{\ell}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{2.9}
\end{equation*}
$$

Multiplying (2.9) by $e^{-i \nu \theta_{\mathrm{p}}}$, integrating between 0 and $2 \pi$ in the variable $\theta_{\mathrm{p}}$, and exploiting the property of orthogonality of exponential functions, we obtain

$$
\begin{equation*}
i^{\nu} d_{\mathrm{p} \nu} J_{\nu}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right) \tag{2.10}
\end{equation*}
$$

Dividing by $i^{\nu} J_{\nu}\left(n_{1} \alpha_{\mathrm{p}}\right)$, and multiplying by $J_{\nu}\left(n_{1} \alpha_{\mathrm{p}}\right) / H_{\nu}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)$

$$
\begin{equation*}
d_{\mathrm{p} \nu} \frac{J_{\nu}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right)}{H_{\nu}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \tag{2.11}
\end{equation*}
$$

Finally, replacing $\nu$ with $\ell$ in (2.11), from the condition (2.7) we obtain the following expression

$$
\begin{equation*}
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}(1)} c_{\mathrm{qm}}-B_{\ell}^{\mathrm{p}(1)}=L_{\ell}^{\mathrm{p}(1)} d_{\mathrm{p} \ell} \tag{2.12}
\end{equation*}
$$

with $\mathrm{p}=1, \ldots, N, \ell=0, \pm 1, \pm 2, \ldots$. The system coefficients $A_{\ell \mathrm{m}}^{\mathrm{qp}(1)}$ and $B_{\ell}^{\mathrm{p}(1)}$ are equal to the coefficients $A_{\ell \mathrm{m}}^{\mathrm{qp}}$ and $B_{\ell}^{\mathrm{p}}$ defined by (1.98) and (1.99), where $G_{\ell}(\cdot)=$ $G_{\ell}^{(1)}(\cdot)=J_{\ell}(\cdot) / H_{\ell}^{(1)}(\cdot)$, and with the superscripts (1) standing for the boundary conditions (2.7).

The system coefficients are given by

$$
\begin{align*}
& A_{\ell \mathrm{m}}^{\mathrm{qp}(1)}=i^{-\ell} G_{\ell}^{(1)}\left\{C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{q}} \delta_{\ell \mathrm{m}}}{G_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}\right. \\
& +\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}  \tag{2.13}\\
& \left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}\right\} \\
& B_{\ell}^{\mathrm{p}(1)}=-G_{\ell}^{(1)}\left\{T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t} 1}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{t} 1}}\right. \\
& \left.\quad+\Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{r} 1}}\right\} \tag{2.14}
\end{align*}
$$

$$
\begin{equation*}
L_{\ell}^{\mathrm{p}(1)}=\frac{J_{\ell}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right)}{H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \tag{2.15}
\end{equation*}
$$

where $\delta$ is the Kronecker symbol.
Now we consider equation (2.8), and act as already done for (1.101) in Section 1.3.2. On the left side we obtain the same left side of (1.97), with $A_{\ell \mathrm{m}}^{\mathrm{qp}}$ and $B_{\ell}^{\mathrm{p}}$ in this case called $A_{\ell \mathrm{m}}^{\mathrm{qp}(2)}$ and $B_{\ell}^{\mathrm{p}(2)}$, and defined by (1.98) and (1.99). It is now $G_{\ell}=G_{\ell}^{(2)}=J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right) / H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)$. We introduce the same passages on right side of (2.8): substituting $V_{\mathrm{cp}}$ as given in (2.1), with $\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}$, and dividing by $n_{1} V_{0}$, it turns out to be

$$
\begin{equation*}
\sum_{\ell=-\infty}^{+\infty} i^{\ell} d_{\mathrm{p} \ell} \frac{n_{\mathrm{cp}}}{n_{1}} J_{\ell}^{\prime}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{2.16}
\end{equation*}
$$

From a comparison of equation (2.16) with equation (2.9), it can be easily obtained

$$
\begin{equation*}
d_{\mathrm{p} \ell} \frac{n_{\mathrm{cp}}}{n_{1}} \frac{J_{\ell}^{\prime}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right)}{H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \tag{2.17}
\end{equation*}
$$

Finally, from condition (2.8) it turns out

$$
\begin{equation*}
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}(2)} c_{\mathrm{qm}}-B_{\ell}^{\mathrm{p}(2)}=L_{\ell}^{\mathrm{p}(2)} d_{\mathrm{p} \ell} \tag{2.18}
\end{equation*}
$$

being for the system coefficients:

$$
\begin{gather*}
A_{\ell \mathrm{m}}^{\mathrm{qp}(2)}=i^{-\ell} G_{\ell}^{(2)}\left\{C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}}{G_{\ell}^{(2)}\left(n_{1} \rho_{\mathrm{p}}\right)}\right. \\
+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1) \mathrm{j}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}  \tag{2.19}\\
\left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}\right\} \\
B_{\ell}^{\mathrm{p}(2)}=-G_{\ell}^{(2)}\left\{T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{t} 1}}\right.  \tag{2.20}\\
\left.+\Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}]}\right]} e^{-i \ell \varphi_{\mathrm{r} 1}}\right\} \\
L_{\ell}^{\mathrm{p}(2)}=\frac{n_{\mathrm{cp}}}{n_{1}} \frac{J_{\ell}^{\prime}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right)}{H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \tag{2.21}
\end{gather*}
$$

The system established by the equations (2.12) and (2.18) can be solved by eliminating the coefficients $d_{\mathrm{p} \ell}$, thus obtaining a linear system with coefficients $c_{\mathrm{qm}}$ as only unknowns. From system (2.12), it follows

$$
\begin{equation*}
d_{\mathrm{p} \ell}=\frac{\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}(1)} c_{\mathrm{qm}}-B_{\ell}^{\mathrm{p}(1)}}{L_{\ell}^{\mathrm{p}(1)}} \tag{2.22}
\end{equation*}
$$

and from (2.18)

$$
\begin{equation*}
d_{\mathrm{p} \ell}=\frac{\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}(2)} c_{\mathrm{qm}}-B_{\ell}^{\mathrm{p}(2)}}{L_{\ell}^{\mathrm{p}(2)}} \tag{2.23}
\end{equation*}
$$

Equating the right sides of (2.22) and (2.23), the wanted system in the unknowns $c_{\mathrm{qm}}$ is

$$
\begin{equation*}
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} D_{\mathrm{m} \ell}^{\mathrm{qp}} c_{\mathrm{qm}}=M_{\ell}^{\mathrm{p}} \tag{2.24}
\end{equation*}
$$

being $D_{\mathrm{m} \ell}^{\mathrm{qp}}=L_{\ell}^{\mathrm{p}(2)} A_{\ell \mathrm{m}}^{\mathrm{qp}(1)}-L_{\ell}^{\mathrm{p}(1)} A_{\ell \mathrm{m}}^{\mathrm{qp}(2)}$ and $M_{\ell}^{\mathrm{p}}=B_{\ell}^{\mathrm{p}(1)} L_{\ell}^{\mathrm{p}(2)}-B_{\ell}^{\mathrm{p}(2)} L_{\ell}^{\mathrm{p}(1)}$. It can be appreciated how the computational effort needed to solve system (2.18) is the same as the one for perfectly-conducting cylinders.

Once the system (2.24) has been solved, the coefficients $d_{\mathrm{p} \ell}$ can be evaluated by means of equation (2.22) or (2.23). After some algebra, we obtain:

$$
\begin{align*}
& d_{\mathrm{p} \ell}=n_{1} \frac{J_{\ell}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)-J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)}{n_{1} J_{\ell}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)-n_{\mathrm{cp}} J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \\
& \times\left\{\sum _ { \mathrm { q } = 1 } ^ { N } \sum _ { \mathrm { m } = - \infty } ^ { + \infty } i ^ { - \ell } c _ { \mathrm { qm } } \left\{C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)\right.\right. \\
& +\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}  \tag{2.25}\\
& \left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}\right\} \\
& \quad+T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{t} 1}} \\
& \left.\quad+\Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\| 1}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{r} 1}}\right\}
\end{align*}
$$

In the last four terms in curly braces the effect of the planar interfaces is contained, in terms of scattered-reflected fields, and transmitted and reflected plane-wave fields. In absence of the two interfaces, the scattered-reflected fields and the reflected field vanish, while the transmitted field $V_{t 1}$ coincides with the incident field $V_{\mathrm{i}}$, being $T_{01}=1$.

The knowledge of the $c_{\mathrm{qm}}$ and $d_{\mathrm{pm}}$ coefficients gives the total electromagnetic field in any point of space and for both polarizations.

### 2.2.2 H polarization

In this case $\mathbf{H}=V(\xi, \zeta) \hat{y}$. The boundary conditions deriving from the continuity of the tangential components of the magnetic and electric field, to the cylinders surfaces, are:

$$
\begin{gather*}
{\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=\left[V_{\mathrm{cp}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}}  \tag{2.26}\\
\left\{\frac{\partial}{\partial \rho_{\mathrm{p}}}\left[V_{\mathrm{t} 1}+V_{\mathrm{r} 1}+V_{\mathrm{s}}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,0}+\sum_{\mathrm{j}=1}^{+\infty} V_{\mathrm{sr}(\mathrm{j})}^{1,2}\right]\right\}_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=\left(\frac{n_{1}}{n_{\mathrm{cp}}}\right)^{2}\left[V_{\mathrm{cp}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{2.27}
\end{gather*}
$$

From the condition (2.26), equal to (2.7) written for the E polarization, we come to the equation (2.12). Moreover, from the condition (2.27), to be compared with (2.8), an equation equal to (2.18) is obtained, with the only difference that in this case

$$
\begin{equation*}
L_{\ell}^{\mathrm{p}(2)}=\frac{n_{1}}{n_{\mathrm{cp}}} \frac{J_{\ell}^{\prime}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right)}{H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \tag{2.28}
\end{equation*}
$$

Definition of coefficients $d_{\mathrm{p} \ell}$ is

$$
\begin{gather*}
d_{\mathrm{p} \ell}=n_{\mathrm{cp}} \frac{J_{\ell}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)-J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right) H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)}{n_{\mathrm{cp}} J_{\ell}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right) H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)-n_{1} J_{\ell}^{\prime}\left(n_{\mathrm{cp}} \alpha_{\mathrm{p}}\right) H_{\ell}^{(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)} \\
\times\left\{\sum _ { \mathrm { q } = 1 } ^ { N } \sum _ { m = - \infty } ^ { + \infty } i ^ { - \ell } c _ { \mathrm { qm } } \left[C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\right.\right. \\
+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{j}}}^{1,0(\mathrm{j})}\left\{n_{1}\left[-\mathrm{h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\}  \tag{2.29}\\
\quad+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell(-1)^{\mathrm{i}}}^{1,2(\mathrm{j})}\left\{n_{1}\left[\mathrm{~h} \Lambda-\chi_{\mathrm{q}}+(-1)^{\mathrm{j}} \chi_{\mathrm{p}}\right], n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right\} \\
\quad+T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}}\right]} e^{-i \ell \varphi_{\mathrm{t} 1}} \\
\left.\quad+\Gamma_{12}\left(n_{\|}^{\mathrm{i}}\right) T_{01}\left(n_{\|}^{\mathrm{i}}\right) e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}\left(\chi_{\mathrm{p}}-\Lambda\right)+n_{\|}^{\mathrm{t1}} \eta_{\mathrm{p}]}\right]} e^{-i \ell \varphi_{\mathrm{rl}}}\right\}
\end{gather*}
$$

## Chapter 3

## Perfectly-conducting cylinders: results

In this chapter, numerical results from the theoretical analysis on perfectly-conducting cylinders are reported.

Some details on the numerical implementation of the method are given in Section 3.1, in particular for the evaluation of the integrals defining Reflected, Transmitted, and Reflected-Transmitted Cylindrical Functions.

A scattering scenario with an isolated perfectly-conducting cylinder buried in a grounded slab is analyzed in detail in Section 3.2. Truncation criteria for the modal expansions and the multiple reflections are widely dealt with and validated by convergence checks. Furthermore, some self-consistency tests are proposed. The particular geometrical layout is also compared with results given in the literature.

Two scattering problems with cylinders below and above the planar interface between two half-spaces are shown in Section 3.3, highligthing the possibility to apply the method to scattering by cylinders buried in a semi-infinite medium. A study of convergence of numerical results is reported. Finally, results are validated through a comparison with the literature.

In Section 3.4 a practical application of the method to detection of buried utilities is shown, with results for a typical scenario surveyed by $G P R$.

### 3.1 Numerical implementation

The implementation of the method requires the numerical evaluation of the involved cylindrical reflected and transmitted functions. In fact, in all practical cases, the expressions of reflection and transmission coefficients do not allow an analytical evaluation of the integrals in Eqs. (1.42), (1.59), (1.72), (1.86), and (1.89).

The techniques already developed in [36] and [37] for the $R W_{\mathrm{m}}$ functions, and in [33] for the $T W_{\mathrm{m}}$ functions, are generalized to deal with the additional cylindrical functions introduced here, and turn out to be very fast and accurate. The algorithms take into account the infinite extension of the integration domains, and detect where the spectrum of each wave function is oscillating or evanescent.

For what concerns the evanescent spectra, generalized Gaussian method are adopted, consisting of an integration-interval decomposition in subintervals of suitable length on which a fixed low-order Gauss-Legendre rule gives good accuracy.

As far as the homogeneous spectra are regarded, they are decomposed into two terms representing the contribution of partially and totally reflected components. Both terms are evaluated by adaptive generalized Gaussian quadrature rules, based on a decomposition of the integration interval in a suitable number of subdomains; the number of subdomains, and their amplitudes, depend on the oscillatory behavior of the integrand. The oscillations are higher as the expansion order of the cylindrical functions increases. Since the adaptive algorithms are based on the assumption that the local oscillation rate is monotonic, it is necessary to perform a preliminary decomposition of the whole interval in a suitable number of subintervals in which the oscillation rate behaves monotonically.

The implementation of the CWA requires also to establish two truncation criteria.

The first criterion involves the index m in the system (1.97), for which the limitation $M=\left\lfloor 3 n_{1} \alpha\right\rfloor[43\rfloor$ reveals to be a good compromise beetween accuracy and computational heaviness.

The second truncation criterion involves the number of multiple scattered-reflected and scattered-reflected-transmitted fields to be taken into account, i.e. the index j in the equations (1.51), (1.68), (1.75), and (1.88). Such truncation is strictly dependent on the desidered accuracy for the results, once noticed a stable convergence as the number j is increased. In the numerical cases reported in this chapter, a convergence up to the fourth significant figure in the expansion coefficients is looked for. Under such a criterion, it has been obtained that just a limited number of reflections is needed, as it will be pointed out in Sections 3.2.1 and 3.3.2.

### 3.2 Cylinder buried in a grounded slab

Let us consider the geometrical layout of Fig. 3.1, where a perfectly-conducting cylinder is buried in a grounded dielectic slab. In Fig. 3.1 (b), an equivalent (for $\xi<\Lambda$ ) geometry is depicted: according to the image theory, the ground plane is removed while the substrate and the cylinder are doubled and the image source,
represented by a plane wave impinging from a lower half-space, is introduced. For the geometry in Fig. 3.1 (a), it is: $\alpha=\pi, \eta=0, \chi=20 \pi, \Lambda=30 \pi, n_{1}=\sqrt{2}, \varphi^{i}=0$ (normal incidence) and TM polarization.


Figure 3.1: a) Geometrical layout for a cylinder buried in a grounded dielectric slab; b) Equivalent (for $\xi<\Lambda$ ) geometry, according to the image theory.

### 3.2.1 Study of convergence and results

Some tests on convergence for results of geometry in Fig. 3.1 (a) are now discussed. The number of reflections $R$ in Eqs. (1.51) and (1.68), to be taken into account for an accurate representation of the electromagnetic field, can be established by inspecting the convergence of the expansion coefficients $c_{1 \mathrm{~m}}=c_{\mathrm{m}}(\mathrm{m}=-M, \ldots, M$, with $M=$ $\left\lfloor 3 n_{1} \alpha\right\rfloor=13$ ). The magnitude and phase of such coefficients are reported in Tab. 3.1 (in this case $c_{\mathrm{m}}=c_{-\mathrm{m}}$ for symmetry reasons). It can be appreciated that a stable convergence up to the fourth significant figure is reached from the tenth reflection. The convergence is obviously faster if, in place of the perfectly conducting wall, an interface between two dielectric media is present. An estimation of the computer time to obtain the coefficient set, on an Intel Core 2, CPU 6400 @2.13 GHz, RAM 3.25 GB , is reported in the last row of Tab. 3.1.

In Tab. 3.2, the magnitude and phase of the axial current induced on the cylinder of Fig. 3.1 (a) are reported for different choices of $M$. In particular, $M$ is set as the nearest integer lower than $\mu n_{1} \alpha$, with $\mu=1,2, .$. , and the corresponding

| Magnitude |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{m} \backslash R$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 0 | $0.7245 \mathrm{D}+0$ | $0.7278 \mathrm{D}+0$ | $0.7285 \mathrm{D}+0$ | $0.7290 \mathrm{D}+0$ | $0.7291 \mathrm{D}+0$ | $0.7292 \mathrm{D}+0$ | $0.7292 \mathrm{D}+0$ | $0.7292 \mathrm{D}+0$ |
| 1 | $0.7722 \mathrm{D}+0$ | $0.7707 \mathrm{D}+0$ | $0.7704 \mathrm{D}+0$ | $0.7704 \mathrm{D}+0$ | $0.7703 \mathrm{D}+0$ | $0.7703 \mathrm{D}+0$ | $0.7703 \mathrm{D}+0$ | $0.7703 \mathrm{D}+0$ |
| 2 | $0.4829 \mathrm{D}+0$ | $0.4851 \mathrm{D}+0$ | $0.4856 \mathrm{D}+0$ | $0.4859 \mathrm{D}+0$ | $0.4860 \mathrm{D}+0$ | $0.4861 \mathrm{D}+0$ | $0.4861 \mathrm{D}+0$ | $0.4861 \mathrm{D}+0$ |
| 3 | $0.1374 \mathrm{D}+1$ | $0.1372 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ | $0.1371 \mathrm{D}+1$ |
| 4 | $0.5416 \mathrm{D}+0$ | $0.5442 \mathrm{D}+0$ | $0.5448 \mathrm{D}+0$ | $0.5452 \mathrm{D}+0$ | $0.5452 \mathrm{D}+0$ | $0.5453 \mathrm{D}+0$ | $0.5453 \mathrm{D}+0$ | $0.5453 \mathrm{D}+0$ |
| 5 | $0.4058 \mathrm{D}+0$ | $0.4050 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ | $0.4048 \mathrm{D}+0$ |
| 6 | $0.5702 \mathrm{D}-1$ | $0.5732 \mathrm{D}-1$ | $0.5738 \mathrm{D}-1$ | $0.5742 \mathrm{D}-1$ | $0.5743 \mathrm{D}-1$ | $0.5744 \mathrm{D}-1$ | $0.5744 \mathrm{D}-1$ | $0.5744 \mathrm{D}-1$ |
| 7 | $0.1842 \mathrm{D}-1$ | $0.1838 \mathrm{D}-1$ | $0.1837 \mathrm{D}-1$ | $0.1837 \mathrm{D}-1$ | $0.1837 \mathrm{D}-1$ | $0.1837 \mathrm{D}-1$ | 0.1837D-1 | $0.1837 \mathrm{D}-1$ |
| 8 | $0.9758 \mathrm{D}-3$ | 0.9815D-3 | $0.9826 \mathrm{D}-3$ | $0.9834 \mathrm{D}-3$ | $0.9836 \mathrm{D}-3$ | $0.9837 \mathrm{D}-3$ | 0.9838D-3 | $0.9838 \mathrm{D}-3$ |
| 9 | $0.1730 \mathrm{D}-3$ | 0.1726D-3 | $0.1726 \mathrm{D}-3$ | $0.1726 \mathrm{D}-3$ | $0.1726 \mathrm{D}-3$ | $0.1726 \mathrm{D}-3$ | 0.1726D-3 | $0.1726 \mathrm{D}-3$ |
| 10 | $0.4362 \mathrm{D}-5$ | $0.4391 \mathrm{D}-5$ | $0.4397 \mathrm{D}-5$ | $0.4401 \mathrm{D}-5$ | 0.4402D-5 | $0.4403 \mathrm{D}-5$ | 0.4403D-5 | $0.4403 \mathrm{D}-5$ |
| 11 | 0.5603D-6 | $0.5590 \mathrm{D}-6$ | $0.5590 \mathrm{D}-6$ | 0.5588D-6 | $0.5588 \mathrm{D}-6$ | 0.5588D-6 | 0.5588D-6 | $0.5588 \mathrm{D}-6$ |
| 12 | $0.7565 \mathrm{D}-8$ | $0.7623 \mathrm{D}-8$ | $0.7631 \mathrm{D}-8$ | $0.7640 \mathrm{D}-8$ | $0.7642 \mathrm{D}-8$ | $0.7644 \mathrm{D}-8$ | 0.7644D-8 | $0.7644 \mathrm{D}-8$ |
| 13 | $0.8160 \mathrm{D}-9$ | 0.8140D-9 | 0.8139D-9 | 0.8137D-9 | $0.8137 \mathrm{D}-9$ | $0.8137 \mathrm{D}-9$ | $0.8137 \mathrm{D}-9$ | $0.8137 \mathrm{D}-9$ |
| Phase [rad] |  |  |  |  |  |  |  |  |
| $\mathrm{m} \backslash R$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 0 | $0.290512 \mathrm{D}+1$ | $0.2906 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ | $0.2907 \mathrm{D}+1$ |
| 1 | $-0.9824 \mathrm{D}+0$ | $-0.9796 \mathrm{D}+0$ | $-0.9803 \mathrm{D}+0$ | $-0.9798 \mathrm{D}+0$ | $-0.9799 \mathrm{D}+0$ | $-0.9798 \mathrm{D}+0$ | $-0.9798 \mathrm{D}+0$ | $-0.9798 \mathrm{D}+0$ |
| 2 | $0.2462 \mathrm{D}+1$ | $0.2463 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ | $0.2464 \mathrm{D}+1$ |
| 3 | -0.1900D+1 | $-0.1898 \mathrm{D}+1$ | $-0.1898 \mathrm{D}+1$ | $-0.1898 \mathrm{D}+1$ | $-0.1898 \mathrm{D}+1$ | $-0.1898 \mathrm{D}+1$ | -0.1898D+1 | $-0.1898 \mathrm{D}+1$ |
| 4 | -0.2075D+1 | $-0.2074 \mathrm{D}+1$ | $-0.2073 \mathrm{D}+1$ | $-0.2074 \mathrm{D}+1$ | $-0.2074 \mathrm{D}+1$ | $-0.2074 \mathrm{D}+1$ | $-0.2074 \mathrm{D}+1$ | $-0.2074 \mathrm{D}+1$ |
| 5 | $-0.7132 \mathrm{D}+0$ | $-0.7105 \mathrm{D}+0$ | $-0.7112 \mathrm{D}+0$ | $-0.7107 \mathrm{D}+0$ | $-0.7108 \mathrm{D}+0$ | $-0.7107 \mathrm{D}+0$ | $-0.7107 \mathrm{D}+0$ | $-0.7107 \mathrm{D}+0$ |
| 6 | -0.1387D+1 | $-0.1385 \mathrm{D}+1$ | $-0.1384 \mathrm{D}+1$ | $-0.1385 \mathrm{D}+1$ | $-0.1385 \mathrm{D}+1$ | $-0.1385 \mathrm{D}+1$ | -0.1385D+1 | $-0.1385 \mathrm{D}+1$ |
| 7 | $-0.4604 \mathrm{D}+0$ | $-0.4578 \mathrm{D}+0$ | $0-.4585 \mathrm{D}+0$ | $-0.4580 \mathrm{D}+0$ | $-0.4581 \mathrm{D}+0$ | $-0.1380 \mathrm{D}+1$ | -0.1380D+1 | $-0.1380 \mathrm{D}+1$ |
| 8 | $-0.1318 \mathrm{D}+1$ | $-0.1316 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ | $-0.1315 \mathrm{D}+1$ |
| 9 | $-0.4796 \mathrm{D}+0$ | $-0.4772 \mathrm{D}+0$ | $-0.4780 \mathrm{D}+0$ | $-0.4775 \mathrm{D}+0$ | $-0.4776 \mathrm{D}+0$ | $-0.4775 \mathrm{D}+0$ | $-0.4775 \mathrm{D}+0$ | $-0.4775 \mathrm{D}+0$ |
| 10 | -0.1350D+1 | $-0.1348 \mathrm{D}+1$ | $-0.1346 \mathrm{D}+1$ | $-0.1347 \mathrm{D}+1$ | $-0.1346 \mathrm{D}+1$ | -0.1347D+1 | $-0.1347 \mathrm{D}+1$ | $-0.1347 \mathrm{D}+1$ |
| 11 | $-0.5071 \mathrm{D}+0$ | $-0.5049 \mathrm{D}+0$ | $-0.5057 \mathrm{D}+0$ | $-0.5053 \mathrm{D}+0$ | $-0.5054 \mathrm{D}+0$ | $-0.5053 \mathrm{D}+0$ | $-0.5053 \mathrm{D}+0$ | $-0.5053 \mathrm{D}+0$ |
| 12 | -0.1439D+1 | $-0.1436 \mathrm{D}+1$ | $-0.1434 \mathrm{D}+1$ | $-0.1434 \mathrm{D}+1$ | -0.1434D+1 | $-0.1434 \mathrm{D}+1$ | $-0.1434 \mathrm{D}+1$ | $-0.1434 \mathrm{D}+1$ |
| 13 | $-0.5231 \mathrm{D}+0$ | $-0.5211 \mathrm{D}+0$ | $-0.5219 \mathrm{D}+0$ | $-0.5215 \mathrm{D}+0$ | -0.5216D+0 | $-0.5215 \mathrm{D}+0$ | $-0.5215 \mathrm{D}+0$ | $-0.5215 \mathrm{D}+0$ |
| Computer Time [seconds] |  |  |  |  |  |  |  |  |
| $R$ | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|  | 1.766 | 2.625 | 3.625 | 4.891 | 6.219 | 7.844 | 9.546 | 11.375 |

Table 3.1: Magnitude and phase of the expansion coefficients $c_{m}$ for R orders of reflections, and the corresponding computer time expressed in seconds.
matrix size of the system in (1.97) is $M S=(2 M+1)^{2}$. The current is calculated at the points labeled $a$ and $b$ in Fig. 3.1, and it is normalized with respect to the tangential component of the incident magnetic field $H_{i}$, evaluated at the point $a$; $R=10$ reflections are considered. This table is useful to appreciate the efficiency of our algorithm and it can be noted that a stable convergence up to the fourth significant figure is reached from $\mu=3$.

|  |  | a |  | b |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $M$ | $M S$ | magnitude | phase | magnitude | phase |
| $4(\mu=1)$ | 81 | $0.2118 \mathrm{D}+02$ | $0.1498 \mathrm{D}+01$ | $0.1652 \mathrm{D}+02$ | $-.2726 \mathrm{D}+01$ |
| $8(\mu=2)$ | 289 | $0.6134 \mathrm{D}+01$ | $0.1599 \mathrm{D}+01$ | $0.3308 \mathrm{D}+01$ | $-.2524 \mathrm{D}+01$ |
| $13(\mu=3)$ | 729 | $0.5921 \mathrm{D}+01$ | $0.1608 \mathrm{D}+01$ | $0.3141 \mathrm{D}+01$ | $-.2513 \mathrm{D}+01$ |
| $17(\mu=4)$ | 1225 | $0.5921 \mathrm{D}+01$ | $0.1608 \mathrm{D}+01$ | $0.3141 \mathrm{D}+01$ | $-.2513 \mathrm{D}+01$ |

Table 3.2: Magnitude and phase of the normalized current, induced on the cylinder of Fig. 3.1, relevant to the points $a$ and $b$; different values of the truncation index $M$, and of the corresponding matrix size $M S$ of the system in (1.97), are considered.

The convergence of the field with the number of reflections is also explored evaluating the far-field diagram in the air-region, which is obtained from the superposition of the fields (1.75) and (1.88). In Fig. 3.2 the diagram (normalized to its


Figure 3.2: Scattered far-field diagram (normalized to its maximum value) for a perfectly-conducting cylinder in a grounded slab, for different orders of reflections.
maximum value) for the geometry of Fig. 3.1 (a) is reported, being $\theta^{\prime}=\theta-90^{\circ}$ the observation angle, for different reflection orders $R$. It can be noted that the curves for $R=5$ and $R=10$ are superimposed.

In Figs. 3.3 (a)-(b) the absolute error magnitude $\left|\epsilon_{a b s}\right|$ is plotted: it is calculated as the difference between far-field results obtained considering two consecutive orders of reflections. In the plot (a) the error is less than $10^{-2}$, in (b) it is less than $10^{-4}$.

(b)

Figure 3.3: a) Absolute-error magnitude, as difference between far-fields results obtained considering four and five reflections; b) Absolute-error magnitude, as difference between far-fields results obtained considering ten and eleven reflections.

A comparison between the results, relevant to the geometries of Fig. 3.1 (a) and Fig. 3.1 (b), is performed in Fig. 3.4: they are practically coincident. This comparison validates our technique when applied to a two-cylinders geometry, as a particular case of multiple cylinders. Moreover, it points out the accuracy of our numerical analysis in a self-consistent way. A further check on convergence is proposed. In Fig. 3.1 (a), a finite value for the lower half-space refractive index $n_{2}$ has been introduced. In Fig. 3.5, far-field diagrams as a function of the observation angle are reported for increasing values in the lower half-space refractive index $n_{2}$, up to consider the grounded dielectric slab as a limiting case. A gradual convergence


Figure 3.4: Comparison between the results relevant to the geometries of Fig. 3.1 (a) and Fig. 3.1 (b).


Figure 3.5: Scattered far-field diagram (normalized to its maximum value) for a perfectly-conducting circular cylinder buried in a dielectric slab, for different values of refractive index of the lower half-space.
of the diagram shape to the one relevant to the grounded slab is observed, as the refractive index of the half-space is increased, which can be interpreted as a selfconsistency test for our results.

### 3.2.2 Comparison with the literature

A comparison between our results of Fig. 3.4 and the results reported in Fig. 2 of [22] for the same layout is reported in Fig. 3.6: the agreement is quite good.


Figure 3.6: Comparison between our results (solid line) and Fig. 2 of [22] (dashed line).

### 3.3 Cylinder in a semi-infinite medium

Our approach can be used also for the scattering by perfectly conducting cylinders located near the planar interface between two semi-infinite media. This can be shown through a comparison with Fig. 6 of Butler et al. [9], where the magnitude and phase of the axial currents induced on a cylinder with $\alpha=0.35 \pi$ are determined, for TM polarization and normal incidence. The current is normalized with respect to the tangential component of the incident magnetic field $H_{i}$, evaluated at the point labeled $b$ in Fig. 3.7. Two layouts are considered, in Figs. 3.7 (a) and (b), where a cylinder is respectively placed above and below the interface between two
semi-infinite half-spaces. The simulation of the layout in Fig. 3.7(a) is implemented with our method setting $n_{1}=n_{2}$ and an arbitrary $\Lambda$, so as to represent two uniform half-spaces. The cylinder below a planar interface, as in Fig. 3.7 (b), is instead simulated setting $n_{1}=n_{0}=1$.

### 3.3.1 Results and comparisons with the literature

Results for the layout of Fig. 3.7 are reported in Fig. 3.8: the solid and dashed lines refer respectively to the cylinder placed above ( $\chi=0.35 \pi$ ) and below the interface ( $\Lambda-\chi=0.35 \pi$ ). The agreement between our results and those of [9] (circles) is excellent.


Figure 3.7: Geometrical layout for a cylinder near a planar interface: a) Cylinder placed below the interface $n_{1}=n_{2} ;$ b) Cylinder placed above the interface $n_{0}=n_{1}$.

### 3.3.2 Study of convergence

The convergence properties of the layout are investigated for two different choices of the distance between the object and the interface. In Table III, the magnitude and phase of the normalized axial current, induced on the buried cylinder of Fig. 3.7 (a), when $\chi=0.35 \pi$ and $\chi=35 \pi$, are reported. The current is relevant to the points $b$ and $d$ of Fig. 3.7; different values of the truncation index $M$ and of

| $\chi=0.35 \pi$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | b |  | d |  |
| M | $M S$ | magnitude | phase | magnitude | phase |
| $2(\mu=1)$ | 25 | $0.192996 \mathrm{D}+01$ | $0.300452 \mathrm{D}+01$ | $0.392359 \mathrm{D}+00$ | -. $223417 \mathrm{D}+01$ |
| $4(\mu=2)$ | 81 | $0.166856 \mathrm{D}+01$ | $0.292268 \mathrm{D}+01$ | $0.733897 \mathrm{D}+00$ | $0.407724 \mathrm{D}+00$ |
| $6(\mu=3)$ | 169 | $0.173223 \mathrm{D}+01$ | $0.289329 \mathrm{D}+01$ | 0.940199D-01 | $0.156458 \mathrm{D}+01$ |
| $8(\mu=4)$ | 289 | $0.173922 \mathrm{D}+01$ | $0.289573 \mathrm{D}+01$ | 0.969002D-01 | $0.149330 \mathrm{D}+01$ |
| $10(\mu=5)$ | 441 | $0.174020 \mathrm{D}+01$ | $0.289626 \mathrm{D}+01$ | $0.768065 \mathrm{D}+00$ | $0.410202 \mathrm{D}+00$ |
| $13(\mu=6)$ | 729 | $0.174036 \mathrm{D}+01$ | $0.289640 \mathrm{D}+01$ | $0.767990 \mathrm{D}+00$ | 0.410161D+00 |
| $\chi=35 \pi$ |  |  |  |  |  |
|  |  | b |  | d |  |
| M | $M S$ | magnitude | phase | magnitude | phase |
| $2(\mu=1)$ | 25 | $0.234753 \mathrm{D}+01$ | $0.305348 \mathrm{D}+01$ | $0.470936 \mathrm{D}+00$ | $-.228050 \mathrm{D}+01$ |
| $4(\mu=2)$ | 81 | $0.212710 \mathrm{D}+01$ | $0.297919 \mathrm{D}+01$ | $0.178319 \mathrm{D}+00$ | $0.101836 \mathrm{D}+01$ |
| $6(\mu=3)$ | 169 | $0.214622 \mathrm{D}+01$ | $0.298023 \mathrm{D}+01$ | 0.997681D-01 | $0.136332 \mathrm{D}+01$ |
| $8(\mu=4)$ | 289 | $0.214542 \mathrm{D}+01$ | $0.298023 \mathrm{D}+01$ | $0.103147 \mathrm{D}+00$ | $0.134035 \mathrm{D}+01$ |
| $10(\mu=5)$ | 441 | $0.214543 \mathrm{D}+01$ | $0.298023 \mathrm{D}+01$ | $0.103069 \mathrm{D}+00$ | $0.134082 \mathrm{D}+01$ |
| $13(\mu=6)$ | 729 | $0.214543 \mathrm{D}+01$ | $0.298023 \mathrm{D}+01$ | $0.103070 \mathrm{D}+00$ | $0.134082 \mathrm{D}+01$ |

Table 3.3: Magnitude and phase of the normalized current, induced on the cylinder of Fig. 3.7, placed below the interface, for $\chi=0.35 \pi$ and $\chi=35 \pi$, respectively. The current is relevant to the points $b$ and $d$ of Fig. 3.7; different values of $M$, and of the corresponding matrix size $M S$ of the system in (1.97), are considered.
the corresponding matrix size $M S$ of the system in (1.97) are considered. When $\chi=0.35 \pi$, the cylinder touches the interface and the computational convergence of our codes is slightly slower, due to a stronger interaction between the buried object and the interface.


Figure 3.8: Magnitude (a) and phase (b) of the normalized axial current induced on a cylinder of radius $\alpha=0.35 \pi$ placed above ( $\chi=0.35 \pi$, full line) and below ( $\Lambda-\chi=0.35 \pi$, dashed line) the interface between two half-spaces, for normal incidence and TM polarization. A comparison is performed between our results and Fig. 6 of [9] (dots).

### 3.4 Detection of buried utilities

Detection of buried utilities is a common application of subterranean diagnostic by electromagnetic techniques [39]. Buried utilities can be either metallic, i.e. telephone and electricity cables, or dielectric, i.e. plastic pipes for gas or water. The frequency range of operation with a $G P R$ survey depends on the target depth and resolution, and it extends from few MHz up to 2 GHz . The block diagram of this radar system is shown in Fig. 3.9. A few nanoseconds short impulse of electromagnetic energy is launched by a transmitting antenna. The antenna is mounted on a mobile trolley which is moved forward over the soil, at a very close distance from the ground surface. The energy scattered by the target is gathered by the receiving antenna, which is usually identical to the transmitting antenna, and then processed by the receiver, to display the signal in a suitable form for the operator.


Figure 3.9: Block diagram of a GPR system.
Standard depth of buried utilities is around 50 cm , which suggests a frequency range between 1 and 1.5 GHz on which to operate. A metallic cylindrical target with 6 cm diameter is assumed, embedded in ground layer of permittivity $\varepsilon_{1}=4$. In our normalized analysis, these geometrical values correspond to a depth $\chi=5 \pi$ and a radius $\alpha=0.6 \pi$, at a frequency of 1.5 GHz . The layout is sketched in Fig. 3.10. The scattered field is evaluated in the upper medium, at the near-field distance of 5 cm , for a normally-incident plane-wave in TM polarization.

In Fig. 3.11, the scattered field is plotted for a layer of thickness $\mathrm{L}=120 \mathrm{~cm}$ and different permittivities of the lower half-space. Relative permittivities of dry soil, such as dry sand, dry clay and rock, are assumed, with values between 4 and 7 . The assumption of dry materials is made to better meet the approximation of lossless materials of our analysis. Moreover, even in practical surveys, the application of GPR to detection of utilities is limited by attenuation due to wet soil. The effect of
a variation in the thickness of the layer is reported in Fig. 3.12. In both Figs. 3.11 and 3.12, results are compared to the limit case with target buried in a semi-infinite medium, i.e. when $\varepsilon_{1}=\varepsilon_{2}=4$.


Figure 3.10: Geometrical layout of a buried utility.


Figure 3.11: Scattered field along a line parallel to the interface, for the layout of Fig. 3.10, with $\mathrm{L}=120 \mathrm{~cm}$.


Figure 3.12: Scattered field along a line parallel to the interface, for the layout of Fig. 3.10, with $\varepsilon_{2}=7$.

## Chapter 4

## Dielectric cylinders: results

In this chapter, numerical results of the analytical technique developed in chapter 2 are reported.

In Section 4.1, the case of a dielectric cylinder buried in a grounded layer is dealt with. Results are given in terms of scattered field in the air region, and they are compared with the literature. The convergence for the multiple reflections, in terms of expansion coefficients, is also discussed.

Further comparisons with the literature are reported in Section 4.2, for the case of a cylinder embedded in a semi-infinite medium, below or above a planar interface, as possible generalization of the method.

Finally, a layout of scattering by multiple cylinders is examined in Section 4.4.

### 4.1 Cylinder buried in a grounded slab

Let us consider the geometry given in [31], with an isolated cylinder of normalized radius $\alpha=2 \pi$ buried in a grounded dielectric layer (see Fig. 4.1). The cylinder has a low dielectric contrast with the dielectric layer: being $n_{1}=\sqrt{2}$, a perturbation $1+\delta$ from $n_{1}$ is assumed for the refraction index of the cylinder $n_{\mathrm{c} 1}=n_{\mathrm{c}}$. In Fig. 3 of [31], results are reported for a normally-incident plane wave on a dielectric slab with three different thicknesses $\Lambda=12 \pi, 13 \pi, 14 \pi$ and a fixed value $\Lambda-\chi=6 \pi$ of the distance between the cylinder axis and the lower interface. This corresponds to three different depths $\chi$ for the cylinder. The perturbation is $\delta=0.01$ and the polarization is TM. A comparison with our results is reported in Fig. 4.2 in terms of scattered field in the air-filled region, as a function of the scattering angle $\theta^{\prime}=\theta-90^{\circ}$. An excellent agreement can be appreciated. For the given results, a stable convergence up to the fourth significant figure is reached with a number of reflections $R=11$. In Fig. 4.3, the magnitude of the expansion coefficients $c_{1 \mathrm{~m}}=c_{\mathrm{m}}$


Figure 4.1: Geometrical layout for a dielectric cylinder buried in a grounded dielectric slab.
( $\mathrm{m}=-M, \ldots, M$, with $M=\left\lfloor 3 n_{1} \alpha\right\rfloor=23$ ), for a slab thickness $\Lambda=12 \pi$, is shown. A stable convergence can be noticed, as the number of reflections $R$ is increased.


Figure 4.2: Comparison between our results (solid lines) and results in Fig. 3 of [31] (circles).

### 4.2 Cylinder in a semi-infinite medium

Two further comparisons are shown, to both validate the method and stress the possibility of its employment for scattering problems with cylinders located near a planar interface of separation between two homogeneous media.

The first comparison has been performed with [26], where results for a cylinder located above a planar interface of separation between two dielectric half-spaces are


Figure 4.3: Behaviour of the magnitude of the expansion coefficents $c_{m}$ for different numbers of reflections $R$, for the geometrical layout of Fig. 4.1(a) when $\Lambda=12 \pi$.
given. The geometrical layout is sketched in Fig. 4.4(a): a $\mathrm{SiO}_{2}\left(n_{\mathrm{c}}=1.46\right)$ cylinder of $0.35 \mu \mathrm{~m}$ radius is placed on silicon $\left(n_{2}=3.8\right)$ and excited by a plane wave of wavelength $0.6328 \mu \mathrm{~m}$ and in TM polarization, impinging at an angle $\varphi_{i}=-30^{\circ}$. Results are given in Fig. 4.4(b) in terms of far-field scattered intensity I, expressed as the squared modulus of the electric field, as a function of the scattering angle $\theta^{\prime}=\theta-90^{\circ}$. The agreement between our results and the ones of Fig. 2 of [26] is excellent.

To validate the method in the case of a cylinder below the planar interface of separation between two half-spaces, results given in [28] have been considered. A dielectric cylinder of permittivity $n_{c}=1.5$ and radius $\alpha=0.32 \pi$ is located at a depth $\chi=2.6 \pi$, in a homogeneous half-space with relative permittivity 1.2 , assuming $\varepsilon_{r 2}=\varepsilon_{r 1}=1.2$ so as to simulate a semi-infinite medium, as depicted in Fig. 4.5(a). A plane-wave in TE polarization impinges on the structure at an angle $\varphi_{i}=30^{\circ}$. In Fig. 4.5(b) results are reported in terms of far-field radar cross section $\sigma\left(\theta^{\prime}\right)$ : the comparison with the ones of Fig. 2 of [28] shows a good agreement.


Figure 4.4: a) Geometrical layout for a dielectric cylinder on a planar interface; b) Far-field scattered intensity in arbitrary units as a function of the scattering angle $\theta^{\prime}=\theta-90^{\circ}$, with $a=0.35 \mu \mathrm{~m}, n_{\mathrm{c}}=1.46, n_{2}=3.8, \varphi_{i}=-30^{\circ}$, and TM polarization. Comparison between our results (solid line) and Fig. 2 of [26] (circles).


Figure 4.5: a) Geometrical layout for a dielectric cylinder below a planar interface; b) Far-field radar cross section $\sigma$, as a function of the scattering angle $\theta^{\prime}=\theta-90^{\circ}$, for a half-space of permittivity $\varepsilon_{1}=\varepsilon_{2}=1.2$, with $\alpha=0.32 \pi, \chi=2.6 \pi, n_{\mathrm{c}}=1.5$, $\varphi_{i}=30^{\circ}$, and TE polarization. Comparison between our results (solid line) and Fig. 2 of [28] (circles).

### 4.3 Detection of a buried air cavity

In $G P R$ surveys, introduced in Section 3.4, either the detection of concealed objects or the determination of materials internal structure can be concerned. Typical examples of the latter application are archeological investigations, study of geological formations, or inspection of roads, bridges and tunnels. $G P R$ technique has become an established method for non-destructive testing of civil engineering structures. When employed for roads survey, it allows to determine the position of voids and anomalies within the roads structure, as well as to measure the thickness of roads layers. Non metallic targets are investigated, which include air-filled objects, as in the case of voids detection.

Here some results are given for an air-filled cavity in a ground layer. A scheme of the geometrical layout is depicted in Fig. 4.6: a cylinder of variable radius $a$ with permittivity $\varepsilon_{\mathrm{c} 1}=1$ is placed at a fixed distance $h-a=10 \mathrm{~cm}$ from the plane of abscissa $x=0$. The normalized depth and radius are $\chi=k_{0} h$, and $\alpha=k_{0} a$, respectively, evaluated at 1.5 GHz . The slab has permittivity $\varepsilon_{1}=4$, and normalized thickness $\Lambda=k_{0} \mathrm{~L}$, with $\mathrm{L}=80 \mathrm{~cm}$; it is followed by a half-space of permittivity $\varepsilon_{2}=5$ (Fig. 4.7) and $\varepsilon_{2}=7$ (Fig. 4.8). The scattered filed is evaluated in near-field region, along a line of abscissa $x=-5 \mathrm{~cm}$, with a normally-incident plane wave in TM polarization.


Figure 4.6: Geometrical layout of a buried air cavity, placed at a fixed distance $h-a$ from the upper interface, with variable radius $a$.


Figure 4.7: Scattered near-field along a line parallel to the interface, for the geometrical layout of Fig. 4.6, with $h-a=10 \mathrm{~cm}, \mathrm{~L}=80 \mathrm{~cm}, \varepsilon_{2}=4$, and $\varepsilon_{2}=5$.


Figure 4.8: Scattered near-field along a line parallel to the interface, for the geometrical layout of Fig. 4.6, with $h-a=10 \mathrm{~cm}, \mathrm{~L}=80 \mathrm{~cm}, \varepsilon_{2}=4$, and $\varepsilon_{2}=7$.

A geometry with a single air cavity of fixed radius $\mathrm{a}=10 \mathrm{~cm}$, buried at a variable depth $h$ in a layer of permittivity $\varepsilon_{1}=4$ and thickness 100 cm , is sketched in Fig. 4.9. Results are reported in Fig. 4.10, for a lower half-space with $\varepsilon_{2}=5$. Keeping the same layout, results are compared with the cylinder buried at 20 cm from the upper interface, in the cases of semi-infinite medium, and layered medium followed by a half space of permittivity 5 and 7 (Fig. 4.11). It can be noticed how the detection of a cavity, as well as any concealed dishomogeneity, is made easier when the target is buried in a layer followed by a denser medium, because of the reflection from the layer/medium 2 interface.


Figure 4.9: Geometrical layout of a buried air cavity, placed at a varible depth $h$ from the upper interface.


Figure 4.10: Scattered near-field along a line parallel to the interface, for the geometrical layout of Fig. 4.9, with $a=10 \mathrm{~cm}, \mathrm{~L}=100 \mathrm{~cm}, \varepsilon_{2}=4$, and $\varepsilon_{2}=7$.


Figure 4.11: Scattered near-field along a line parallel to the interface, for the geometrical layout of Fig. 4.9, with $a=10 \mathrm{~cm}, d=20 \mathrm{~cm}$, and $\mathrm{L}=100 \mathrm{~cm}$.

### 4.4 Scattering by multiple cylinders

A case of scattering by multiple objects embedded in a dielectric layer is now dealt with, for a finite grating of three equally-spaced cylinders. A normally-incident plane wave of wavelength $\lambda=10.6 \mu \mathrm{~m}$, and in TM polarization, impinges on a $50 \mu \mathrm{~m}$ thick dielectric layer with refraction index $n_{1}=1.333$, followed by a half-space with parametrized refraction index $\left(n_{2}=1,1.333,1.5\right)$. When $n_{2}=1.333$, media 1 and 2 form a homogeneous half-space, in order to perform a comparison with the cases of layered geometry (Fig. 4.12 (a)). Three silicon cylinders ( $n_{\mathrm{c}}=3.4211$ ), with radius of $2.5 \mu \mathrm{~m}$, are located at a depth of $20 \mu \mathrm{~m}$. The scattered field in the upper air-filled region is evaluated in the near-field zone, at a distance $x=-\lambda / 2$, along a line parallel to the upper interface.

In Fig.4.12 (b), results are reported for different permittivities $n_{2}$, when cylinders are spaced of one wavelength $\lambda$. The response is higher when the slab is sorrounded by an air-filled medium $\left(n_{2}=1\right)$, and is instead damped when cylinders are in a homogeneous medium ( $n_{2}=1.333$ ), due to the lack of reflections from the lower interfaces. The lowest response is observed when medium 1 is followed by the denser medium ( $n_{2}=1.5$ ). The effect of cylinder radius $a$ on the near-field responce is reported in Fig. 4.12 (c), where the scattered field is evaluated for the the slab followed by a denser medium ( $n_{2}=1.5$ ).

The scattered field for cylinder radius $a=\lambda$ is compared to the one for a halved radius $a=\lambda / 2$ and a doubled one $a=2 \lambda$. The cylinder-cylinder distance is such that strong mutual-interaction phenomena occur.

Scattered field for radius $a=\lambda$ as a function of the cylinder-cylinder distance $d$ is plotted in Fig. 4.12 (d), when $n_{2}=1.5$ : with cylinders $3 \lambda$ apart, three peaks corresponding to cylinder location appear, but responses are still interacting. With a cylinder-cylinder distance of $6 \lambda$, the peaks are isolated and a precise detection of each cylinder can be performed.


Figure 4.12: a) Geometrical layout for a grid of three equally-spaced silicon cylinders ( $n_{\mathrm{c}}=3.4211$ ), with radius of $2.5 \mu \mathrm{~m}$ and spacing $d=\lambda$, embedded in a layer $50 \mu \mathrm{~m}$ thick at a depth of $20 \mu m$, with refraction index $n_{1}=1.333$. Scattered near-field is evaluated along a line at $x=-\lambda / 2$ for different refraction indexes $n_{2}$ (b), cylinders radius $a$ (c), and cylinder-cylinder distance $d$ (d).

## Chapter 5

## Line source

In this chaper, a different source is dealt with in the problem of scattering by buried objects, i.e. the cylindrical wave radiated from a line current. A two-dimensional scattering problem with two semi-infinite media separated by a planar interface is considered, with the source placed in the upper half-space, and perfecly-conducting cylinders buried in the lower medium. The geometry of the problem is described in Section 5.1.

The theoretical analysis is developed extending the model proposed in chapter 1, for the scattering of a plane-wave, to the cylindrical wave emitted by the line current. The field contributions needed to cope with the new source are given in Section 5.2.

Finally a linear system is derived in Section 5.3, for both E and H polarization.

### 5.1 Geometry of the problem

The geometry of the problem is depicted in Fig. 5.1, with $N$ perfectly-conducting cyrcular cylinders buried in a dielectric half-space. The source is a line of constant current, of infinite-extension and parallel to the planar interfaces and the cylinders axis. The source is placed in air, in vicinity of the planar interface, and its axis has centre in $\left(\chi_{\mathrm{L}}, \eta_{\mathrm{L}}\right)$ in $M R F$. With $\left(\mathrm{O}_{\mathrm{L}}, \xi_{\mathrm{L}}, \zeta_{\mathrm{L}}\right)$ standing for the reference frame $R F_{\mathrm{L}}$ centered on the line source, the following change of coordinates applies

$$
\left\{\begin{array}{l}
\xi=\xi_{\mathrm{L}}+\chi_{\mathrm{L}}  \tag{5.1}\\
\zeta=\zeta_{\mathrm{L}}+\eta_{\mathrm{L}}
\end{array}\right.
$$

where $\chi_{\mathrm{L}}<0$.


Figure 5.1: Geometrical layout for a line source.

### 5.2 Decomposition of the total field

The interaction of the field $V_{\mathrm{i}}(\xi, \zeta)$ with the planar interface, in absence of the cylinders, gives in each medium the following fields

- $V^{(0)}(\xi, \zeta)=V_{\mathrm{i}}(\xi, \zeta)+V_{\mathrm{r}}(\xi, \zeta)$
- $V^{(1)}(\xi, \zeta)=V_{\mathrm{t}}(\xi, \zeta)$
as shown in Fig. 5.2. As far as the diffracted fields are concerned, the fields introduced for the dielectric slab, in Section 5.2, can be applied to the present geometry. In particular, the scattered field $V_{\mathrm{s}}$ does propagate and encounters reflection by the only interface, which is placed in $\xi=0$. Therefore, the scattered-reflected field $V_{\mathrm{sr}(\mathrm{j})}^{1,0}$ is excited, with a single reflection $\mathrm{j}=1$. The scattered-transmitted field $V_{\mathrm{st}}^{1,0}$ propagates in medium 0 , due to transmission of $V_{\mathrm{s}}$ through the interface. The other fields contributions, i.e. scattered-reflected field $V_{\operatorname{sr}(\mathrm{j})}^{1,2}$, scattered-reflected transmitted fields $V_{\operatorname{srt}(\mathrm{j})}^{1,0}$ and $V_{\mathrm{srt}(\mathrm{j})}^{1,2}$, and the scattered-transmitted field $V_{\mathrm{st}}^{1,2}$ are not excited with an unlayered geometry.

Due to the geometry of the source, the incident field $V_{\mathrm{i}}$ is described by a cylindrical wave centered on ( $\chi_{\mathrm{L}}, \eta_{\mathrm{L}}$ ), and the reflected and transmitted fields are determined by means of plane-wave spectrum of a cylindrical wave.


Figure 5.2: Cylindrical waves excited by the line source through the planar interface.

### 5.2.1 Incident field

The line source emits a $y$-directed field proportional to an Hankel funtion of second kind, with argument proportional to the distance from the source to the observation point. If the source is placed as in Fig. 5.1, the field generated in absence of the planar interfaces and the cylinders can be written as

$$
\begin{equation*}
V_{\mathrm{i}}(\xi, \zeta)=V_{0} H_{0}^{(2)}\left[n_{1} \sqrt{\left(\xi-\chi_{\mathrm{L}}\right)^{2}+\left(\zeta-\eta_{\mathrm{L}}\right)^{2}}\right] \tag{5.2}
\end{equation*}
$$

The cylindrical functions $H_{0}^{(1)}$ can be expressed in a spectrum of plane waves

$$
\begin{equation*}
C W_{0}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} F_{0}\left(\xi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\|}^{\mathrm{i}} \zeta_{\mathrm{L}}} d n_{\|}^{\mathrm{i}} \tag{5.3}
\end{equation*}
$$

where the function $F_{0}$ evaluated in $\xi_{L}>0$ is

$$
\begin{equation*}
F_{\mathrm{o}}\left(\xi_{L}, n_{\|}\right)=\frac{2 e^{i \xi_{L} \sqrt{1-n_{\|}^{2}}}}{\sqrt{1-n_{\|}^{2}}} \tag{5.4}
\end{equation*}
$$

### 5.2.2 Reflected field

We want to evaluate the field $V_{\mathrm{r}}$ due to the reflection in medium 0 , at the interface placed in $\xi=0$, of the incident field $V_{\mathrm{i}}$. For the incident cylindrical wave in (5.2), the plane-wave representation given in (5.3) is employed. The generical plane-wave is evaluated in $\xi=0$, leading to $F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\|}^{r}\left(\zeta-\eta_{\mathrm{L}}\right)}$. The reflected plane-wave in $\xi=0$ has amplitude $\Gamma_{01}\left(n_{\|}^{\mathrm{i}}\right) F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\perp}^{\mathrm{r}} \xi} e^{i n_{\|}^{\mathrm{r}}\left(\zeta-\eta_{\mathrm{L}}\right)}$, where the parallel and
perpendicular components of the propagation vector are $n_{\|}^{\mathrm{r} 1}=n_{\|}^{\mathrm{i}}$ and $n_{\perp}^{\mathrm{r} 1}=-n_{\perp}^{\mathrm{i}}$, respectively. The exponential $e^{i n_{\perp}^{r 1} \xi}$ can be included in the spectrum $F_{0}$, according to definition (5.4), and we get the 0 -th order cylindrical wave reflected in medium 0

$$
\begin{equation*}
C W_{0}^{\mathrm{r}}(\xi, \zeta)=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} \Gamma_{01}\left(n_{\|}^{\mathrm{i}}\right) F_{0}\left(\xi-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\|}^{\mathrm{i}} \zeta_{\mathrm{L}}} d n_{\|}^{\mathrm{i}} \tag{5.5}
\end{equation*}
$$

The field $V_{\mathrm{r}}$ as a function of a point of coordinates $(0, \xi, \zeta)$ in $M R F$ is

$$
\begin{equation*}
V_{\mathrm{r}}(\xi, \zeta)=V_{0} C W_{0}^{\mathrm{r}}(\xi, \zeta) \tag{5.6}
\end{equation*}
$$

### 5.2.3 Transmitted field

The cylindrical wave transmitted in medium 1 is evaluated from (5.3). The generical plane wave of the spectrum (5.3) is $F_{0}\left(\xi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\|}^{\mathrm{i}} \zeta_{\mathrm{L}}}$, with propagation vector $n_{\perp}^{i} \hat{\xi_{\mathrm{L}}}+$ $n_{\|}^{i} \hat{\zeta_{\mathrm{L}}}$, and at the air-medium 1 interface in $\xi=0$ it is $F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{\|}^{\mathrm{i}}\left(\zeta-\eta_{\mathrm{L}}\right)}$. In the reference frame ( $\mathrm{O}, \xi, \zeta$ ), the transmitted plane-wave in medium 1 has propagation vector $n_{\perp}^{\mathrm{t1}} \hat{\xi}+n_{\|}^{\mathrm{t} 1} \hat{\zeta}$, being

$$
\left\{\begin{array}{l}
n_{\perp}^{\mathrm{t}}=\sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}}  \tag{5.7}\\
n_{\|}^{\mathrm{t}}=n_{\|}^{\mathrm{i}} / n_{1}
\end{array}\right.
$$

 For the 0 -th order transmitted cylindrical wave in medium 1 we obtain

$$
\begin{gather*}
C W_{0}^{\mathrm{t}}\left(\xi, \zeta ;-\chi_{\mathrm{L}}\right)= \\
=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{01}\left(n_{\|}^{\mathrm{i}}\right) F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{1}} \sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}} \xi  \tag{5.8}\\
e
\end{gather*}
$$

Making use of the change of coordinates (5.1) defined in Section 1.1 of chapter 1, equation (5.8) can be written as follows

$$
\begin{gather*}
C W_{0}^{\mathrm{t}}\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}} ;-\chi_{\mathrm{L}}\right)= \\
=\frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{01}\left(n_{\|}^{\mathrm{i}}\right) F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{1}} \sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}} \chi_{\mathrm{p}}  \tag{5.9}\\
e^{i n_{\|}^{\mathrm{i}}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{L}}\right)} e^{i n_{1}\left(n_{\perp}^{\mathrm{t}} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{t}} \zeta_{\mathrm{p}}\right)} d n_{\|}^{\mathrm{i}}
\end{gather*}
$$

where the exponential $e^{i n_{1}\left(n_{\perp} \xi_{\mathrm{P}}+n_{\|}^{t} \zeta_{\mathrm{P}}\right)}$ stands for a transmitted plane wave in $R F_{\mathrm{p}}$, which can be expanded into Bessel functions

$$
\begin{equation*}
e^{i n_{1}\left(n_{\perp}^{t} \xi_{\mathrm{p}}+n_{\|}^{\mathrm{t}} \zeta_{\mathrm{p}}\right)}=\sum_{\ell=-\infty}^{+\infty} i^{\ell} e^{-i \ell \varphi_{\mathrm{t}}} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \tag{5.10}
\end{equation*}
$$

With the position $e^{-i \ell \varphi_{\mathrm{t}}}=e^{-i \ell \arctan \frac{n_{\|}^{i}}{\sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}}}}$, we obtain for the transmitted field

$$
\begin{gather*}
V_{\mathrm{t}}\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)=V_{0} \sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}} \times} \\
\times \frac{1}{2 \pi} \int_{-\infty}^{+\infty} T_{01}\left(n_{\|}^{\mathrm{i}}\right) F_{0}\left(-\chi_{\mathrm{L}}, n_{\|}^{\mathrm{i}}\right) e^{i n_{1}} \sqrt{1-\left(n_{\|}^{\mathrm{i}} / n_{1}\right)^{2}} \chi_{\mathrm{p}}  \tag{5.11}\\
e^{i n_{\|}^{\mathrm{i}}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{L}}\right)} e^{-i \ell \arctan \frac{n_{\|}^{\mathrm{i}}}{\sqrt{1-\left(n_{\|}^{\left.\mathrm{i} / n_{1}\right)^{2}}\right.}} d n_{\|}^{\mathrm{i}}}
\end{gather*}
$$

The field transmitted in medium 1 , associated to a point of coordinates $(0, \xi, \zeta)$, as a function of the coordinates in $R F_{\mathrm{p}}$, can be written in the following more compact form

$$
\begin{equation*}
V_{\mathrm{t}}\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)=V_{0} \sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} C W_{0 \ell}^{\mathrm{t}}\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}-\eta_{\mathrm{L}} ;-\chi_{\mathrm{L}}\right) \tag{5.12}
\end{equation*}
$$

### 5.3 Boundary conditions

The scattering problem is solved imposing the boundary conditions on the cylinders'surfaces, as already exlplained in Section 1.3. We distinghuish the two cases of TM and TE polarization.

### 5.3.1 E polarization

In case of E polarization the electric field has the only $E_{y}$ component, corresponding to the scalar function $V$. The boundary condition to be imposed of the $p$-th cylinder is the following

$$
\begin{equation*}
\left[V_{\mathrm{t}}+V_{\mathrm{s}}+V_{\mathrm{r}(1)}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=0, \quad \text { with } \mathrm{p}=1, \ldots, N \tag{5.13}
\end{equation*}
$$

which is similar to the condition (1.93), in Section 1.3.1.
Substituting in (5.13) equations (5.12), (1.28), (1.51) evaluated for $(\mathrm{j}=1)$, with $\rho_{p}=\alpha_{p}$, and simplifying the constant $V_{0}$

$$
\begin{gather*}
\sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} C W_{0 \ell}^{\mathrm{t}}\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}-\eta_{\mathrm{L}} ;-\chi_{\mathrm{L}}\right)+ \\
+\sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}\right]+ \\
\quad+\sum_{\ell=-\infty}^{+\infty} J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell}^{1,0(1)}\left[-n_{1}\left(\chi_{\mathrm{q}}+\chi_{\mathrm{p}}\right), n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right]=0 \tag{5.14}
\end{gather*}
$$

Multiplying (5.14) by $e^{-i \nu \theta_{\mathrm{p}}}$, integrating between 0 and $2 \pi$ in the variable $\theta_{\mathrm{p}}$, and employing the orthogonality property of exponential functions $\frac{1}{2 \pi} \int_{0}^{2 \pi} e^{i \ell \theta_{\mathrm{p}}} e^{-i \nu \theta_{\mathrm{p}}} d \theta_{\mathrm{p}}=$ $\delta_{\ell \nu}$, we obtain

$$
\begin{gather*}
i^{\nu} J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) C W_{0 \nu}^{\mathrm{t}}\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}-\eta_{\mathrm{L}} ;-\chi_{\mathrm{L}}\right)+ \\
+J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\nu}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\nu}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\nu \mathrm{m}}\right]+ \\
+J_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\nu}^{1,0(1)}\left[n_{1}\left(\chi_{\mathrm{q}}+\left(\chi_{\mathrm{p}}\right), n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right]=0\right. \tag{5.15}
\end{gather*}
$$

The term $J_{\nu}\left(n_{1} \alpha_{\mathrm{p}}\right)$ can be simplified; moreover, isolating the unknown terms on left side, putting $G_{\nu}(\cdot)=J_{\nu}(\cdot) / H_{\nu}^{(1)}(\cdot)$, multiplying both sides by $G_{\nu}\left(n_{1} \alpha_{p}\right)$, and dividing by $i^{\nu}$

$$
\begin{gather*}
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} i^{-\nu} G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)\left\{C W_{\mathrm{m}-\nu}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{qp}} \delta_{\nu \mathrm{m}}}{G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)}+\right. \\
+  \tag{5.16}\\
\left.+R W_{\mathrm{m}+\nu}^{1,0(1)}\left[-n_{1}\left(\chi_{\mathrm{q}}+\chi_{\mathrm{p}}\right), n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right]\right\}= \\
= \\
=-G_{\nu}\left(n_{1} \rho_{\mathrm{p}}\right)\left(n_{1} \rho_{\mathrm{p}}\right) C W_{0 \nu}^{\mathrm{t}}\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}-\eta_{\mathrm{L}} ;-\chi_{\mathrm{L}}\right)
\end{gather*}
$$

We can substitute $\nu$ with $\ell$, and write the system (5.16) in a more compact way:

$$
\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} A_{\ell \mathrm{m}}^{\mathrm{qp}} c_{\mathrm{qm}}=B_{\ell}^{\mathrm{p}} \quad\left\{\begin{array}{l}
\ell=0, \pm 1, \ldots, \pm \infty  \tag{5.17}\\
\mathrm{p}=1, \ldots, N
\end{array}\right.
$$

with

$$
\begin{gather*}
A_{\ell \mathrm{m}}^{\mathrm{qp}}=i^{-\ell} G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)\left\{C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{\delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}}{G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)}\right. \\
\left.+\sum_{\mathrm{j}=1}^{+\infty} R W_{\mathrm{m}+\ell}^{1,0(\mathrm{j})}\left[-n_{1}\left(\chi_{\mathrm{q}}+\chi_{\mathrm{p}}\right), n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right]\right\}  \tag{5.18}\\
B_{\ell}^{\mathrm{p}}=-G_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right) C W_{0 \ell}^{\mathrm{t}}\left(\chi_{\mathrm{p}}, \eta_{\mathrm{p}}-\eta_{\mathrm{L}} ;-\chi_{\mathrm{L}}\right) \tag{5.19}
\end{gather*}
$$

The (5.18) is a linear system of $(\infty \cdot N)$ equations in $(\infty \cdot N)$ unknowns $c_{\mathrm{qm}}$.

### 5.3.2 H Polarization

For TE polarization, from Maxwell equations, the condition is the following

$$
\begin{equation*}
\left\{\frac{\partial}{\partial \rho_{\mathrm{t}}}\left[V_{\mathrm{t}}+V_{\mathrm{s}}+V_{\mathrm{sr}(1)}^{1,0}\right]\right\}_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}=0, \quad \text { with } \mathrm{p}=1, \ldots, N \tag{5.20}
\end{equation*}
$$

Substituting in (5.20) equations (5.12), (1.28), and (1.51) with $\mathrm{j}=1$, evaluated in $\rho_{p}=\alpha_{p}$, and simplifying the constant $n_{1} V_{0}$

$$
\begin{gathered}
\sum_{\ell=-\infty}^{+\infty} i^{\ell} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} C W_{0 \ell}^{\mathrm{t}}\left(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}}\right)+ \\
+\sum_{\ell=-\infty}^{+\infty} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}}\left[C W_{\mathrm{m}-\ell}\left(n_{1} \xi_{\mathrm{qp}}, n_{1} \zeta_{\mathrm{qp}}\right)\left(1-\delta_{\mathrm{qp}}\right)+\frac{H_{\ell}^{\prime(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right)} \delta_{\mathrm{qp}} \delta_{\ell \mathrm{m}}\right]+ \\
+\sum_{\ell=-\infty}^{+\infty} J_{\ell}^{\prime}\left(n_{1} \rho_{\mathrm{p}}\right) e^{i \ell \theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} c_{\mathrm{qm}} R W_{\mathrm{m}+\ell}^{1,0(1)}\left[n_{1}\left(\chi_{\mathrm{q}}+\chi_{\mathrm{p}}\right), n_{1}\left(\eta_{\mathrm{p}}-\eta_{\mathrm{q}}\right)\right]=0 \\
\text { with } J_{\ell}^{\prime}\left(n_{1} \alpha_{\mathrm{p}}\right)=\frac{1}{n_{1}}\left[\frac{\partial J_{\ell}\left(n_{1} \rho_{\mathrm{p}}\right)}{\partial \rho_{\mathrm{p}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \text { and } H_{\ell}^{\prime(1)}\left(n_{1} \alpha_{\mathrm{p}}\right)=\frac{1}{n_{1}}\left[\frac{\partial H_{\ell}^{(1)}\left(n_{1} \rho_{\mathrm{p}}\right)}{\partial \rho_{\mathrm{s}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}^{(5.21)}
\end{gathered}
$$

From a comparison of equation (5.21) with (5.15), it turns out that the system for the H polarization is the same for the E polarization, being now $G_{\ell}(\cdot)=J_{\ell}^{\prime}(\cdot) / H_{\ell}^{\prime(1)}(\cdot)$.

## Part II

## Scattering and radiation by Electromagnetic Band-Gap materials

## Introduction to part II

Electromagnetic Band-Gap (EBG) materials, also known as photonic crystals (PCs) or photonic bandgap (PBG) materials, are periodic arrangements of dielectric or/and metallic inclusions inside a host medium, with interesting applications from the microwave to optical frequencies [46].

Properly designed EBGs allow to control the propagation of electromagnetic waves, i.e. they display frequency intervals in which propagation of waves is inhibited or allowed only along certain directions. The ability to control the flow of electromagnetic waves is due to their photonic band structure, which is completely analogous to the concept of electronic band structure. Periodic space variations of the permittivity are origin of band gaps, just as electron waves traveling in the periodic potential of a crystal are arranged into energy bands separated by band gaps. Periodicity can be in one, two or three spatial directions, but an omni-directional band gap could be obtained only in the case of three-dimensional (3D) crystals.

The first periodic structure with a full band gap, as proposed by Ho et al. [47], was conceived as a periodic arrangement of dielectric spheres, in a latticelike diamond. However, this structure is very hard to fabricate, especially in the micrometer- and submicrometer-scale. An ingenious structure with diamond lattice symmetry was proposed by Yablonovitch [48]. It was the first experimental structure that proved the existence of a full photonic band gap, in agreement with the theoretical calculations. Another example of a 3D structure with complete band gap is the so-called woodpile, a layer-by-layer structure, where gratings of parallel dielectric rods are stacked toghether. Within each layer, rods are spaced with period $d$, and rods belonging to two neighbouring layers are rotated of $90^{\circ}$. Depending on the dimensions of the rods cross section and on the period, the woodpile exhibit band gaps from few gigaherz up to optical frequencies. This structure was designed by [49]; it was first fabricated [50] by alumina cylinders, and it was demonstrated to have a full 3D band gap from 12 to 14 GHz .

Such structures are very attractive for their frequency and spacial selectivity, which suggests the implementation of innovative microwave and optical devices.

Nevertheless, fabrication and testing may be a very difficult task, especially in the infrared and optical regions.

As regards the woodpile structure, different constructive techniques are proposed in the literature, depending on the frequency range of application. In the microwave frequency regime, the fabrication is really made up layer-by-layer, in the sense that, once the first layer is built, the second one is directly glued on its top at the touching point of the rods. This is the technique employed by Ozbay et al. [50]; for higher frequencies, rods are directly etched from the basic material, as in [51], where a silicon woodpile crystal with a band gap centered on 500 GHz has been fabricated. Mechanical methods with computer-controlled diamond saw, designed for the silicon industry, are applied: parallel grooves are opened up from both sides of a silicon wafer, with a rotation of $90^{\circ}$ from one side to the other. One half of a vertical period of the woodpile crystal is so obtained. Then, automatic alignement for the stacking of the layer-by-layer structure is performed, after having cut the grooved wafers in such a specific way to achieve an offset of half a period between two consecutive layers. To build a woodpile at still higher frequencies, in the nearinfrared wavelengths, GaAs stripes are stacked with a wafer fusion technique, and the stripes alignement is performed by means of laser diffraction, as reported in [52].

Interesting applications of 3D EBGs concern layouts where the periodicity is broken up. In fact, transmission peaks are displayed inside the band gap, which can be employed to filter or localize electromagnetic waves in the desired directions. The interruption of the periodicity may be performed with a spacing between two EBG layers of identical thickness. When the image theory is applied, by removing the lower part of the structure and placing a ground plane in line with the symmetry plane, a cavity delimited by a lower perfect-reflecting wall and an upper partiallyreflecting wall is obtained. A physical understanding of this structure was proposed by Jackson et al. in [53] in terms of leaky-wave propagation modes, for the case of an upper wall made of a dielectric multilayer. The whole structure produced a narrow conical beam, with radiation direction strictly dependent on frequency. An application is proposed in [54], where a multilayer is employed as a cover for a feed of a reflector antenna. When modes excited inside the cavity are, instead, below the cut-off of the guided modes, they can be interpreted as evanescent modes of a leaky-wave cavity [55]. In this case the radiation direction is just normal, with respect to the layers, and is unchanged with frequency.

As far as antenna applications of EBGs are concerned, many interesting results are given in literature, to implement directive and high-efficiency radiators. Surfacewave suppression is accomplished when EBGs are used as substrates of patch an-
tennas [56]. An improvement of directivity is obtained when they are employed as antenna superstrates. In this case, EBGs are employed as cavities, with defect modes that couple with the radiation coming from the antenna. In [57], the ground plane of a patch antenna is used to open a defect mode inside the photonic band gap crystal used as cover. A woodpile resonator antenna based on woodpile EBG material and metallic ground plane is proposed in [58]. In [59], when EBGs are employed as superstrates of a patch antenna, two classes of defects are considered, one introduced by the ground plane of the antenna and the other produced by a row of defect rods with different dielectric constants in the EBG structure. A ceramic woodpile EBG structures for millimeter-wave applications is given in [60]. Moreover, for microstrip patch antennas, a combined EBGs use as superstrate and substrate can be devised [61].

In this part of the thesis, a study carried out on the woodpile structure and its application to the design of a high-directivity antenna is presented.

A woodpile EBG made of alumina, with band gap in the $12-14 \mathrm{GHz}$ frequency range, is studied in chapter 1. Properties of cavities obtained by this woodpile structure are investigated, in terms of rods orientation with respect to the polarization of the field and the permittivity of rods constituent material. A comparison with unidimensional cavities, i.e., the multilayer film, is also performed. Results are simulated with commercial software Ansoft HFSS v11.

In chapter 2, the implementation of the woodpile and multilayer cavities to the design of high-directivity radiators is presented. Two kind of planar radiators are considered: a double-slot antenna and a patch antenna.

After this preliminary study, a woodpile structure has been deviced. This implementation had to face restictions given by the experimental setup which would be employed for measurements on the final structure. In chapter 3, the design of the structure, and details on the fabrication of two woodpile layers, are reported. In the same chapter, experimental data on these layers are shown, and compared to simulated results.

The woodpile layers have been finally employed as superstrates of a patch antenna. In chapter 4 , the most interesting experimental results for this new composite high-gain radiator are reported.

## Chapter 1

## Properties of EBG materials

Electromagnatic Band-Gap materials are one-, two- or three-dimensional periodic structures. The analysis carried on a woodpile structure, i.e., a three-dimensional periodicity EBG material, is reported in Section 1.1. Properties of a woodpile crystal with band gap in the $12-14 \mathrm{GHz}$ frequency range, and the relevant cavities made up with two identycal woodpile layers separated by a spacing, are investigated by means of commercial software Ansoft HFSS v11. Cavities with high-selectivity frequency response in the microwave range are looked for. Thus the effect of the permittivity of the constituent materials and other constructive parameters are taken into account.

Fabrication of a woodpile structure is really a hard task, with many problems related to the tolerances in the rods alignment, gluing of the rods, etc. For this reason, in Section 1.2, properties of simpler 1D EBG geometries showing a similar frequency behavior, and possibly smaller extension, are investigated too. Design resonance-frequency has been chosen such that properties of these structures are directly comparable. The role of the design parameters such as layers thickness, permittivity, and number of layers is studied in order to exhibit a unitary transmission at the desired frequency. A both compact, in terms of spatial extension, and frequency-selective resonator is aimed at.

### 1.1 The woodpile structure

The woodpile is a 3D EBG, made of a stack of rectangular-cross-section rods (Fig. 1.1 (a)). The elementary cell has four layers of rods, and rods belonging to two consecutive layers are orthogonal, while parallel rods have an offset of half a period from each other.

The woodpile considered in the following has square-cross-section rods of side $w=3.2 \mathrm{~mm}$; the periods of the elementary cell in the $x, y$, and $z$-directions are
$d_{x}=d_{y}=11.2 \mathrm{~mm}$ and $d_{z}=12.8 \mathrm{~mm}$, respectively (Fig. 1.1 (b)). Constituent material of the rods is alumina, a ceramic material with permittivity between 8.4 and 10. In the simulated results, alumina has relative permittivity $\varepsilon_{r, \text { al }}=8.4$, with $\tan \delta=0.002$.

A woodpile with infinite extension in the $x y$ plane and two unit-cells superimposed in the $z$-direction is considered; a unit-amplitude monochromatic plane wave with wavelength $\lambda$ impinges on the structure with normal incidence. In Fig. 1.4, the transmission coefficient through the woodpile is plotted as a function of frequency, for the chosen parameters, and it is seen that the crystal exhibits a band gap inside the Ku band, from 12 to 14 GHz .


Figure 1.1: a) The woodpile unit-cell; b) Woodpile EBG.

### 1.1.1 Analysis of woodpile cavities

The woodpile is now used as a cavity (Fig. 1.3), with two identical layers separated by a spacing of width $h$ along the $z$-direction. The electromagnetic behavior depends on the orientation of the woodpile with respect to the polarization of the incident electric field. In our simulations, we consider two layouts: in Fig. 1.3 (a) the most internal alignment of rods is orthogonal to the polarization of the incident electric field, while it is parallel in Fig. 1.3 (b). In both cavities, a spacing $h=23.4 \mathrm{~mm}$ is introduced: it corresponds to about a wavelength, when $\mathrm{f}=12.8 \mathrm{GHz}$. In the first case, two transmission peaks appear, centered at $\mathrm{f}=11.4 \mathrm{GHz}$ and $\mathrm{f}=14.5 \mathrm{GHz}$, as it can be seen from Fig. 1.4, where the magnitude of the transmission coefficient $\eta_{t}$ through the woodpile is plotted as a function of frequency. In the second case, a transmission peak appears inside the band gap, centered at $\mathrm{f}=12.44 \mathrm{GHz}$ (Fig. 1.5), which is very close to the theoretical resonance of 12.8 GHz corresponding to the chosen value of $h$. Anyway, in both cases, the transmission peaks don't appear


Figure 1.2: Magnitude of the transmission coefficient $\eta_{t}$ for one woodpile layer.


Figure 1.3: Woodpile cavities symmetrical with respect to the $x y$ plane, with most internal rods orthogonal (a) and parallel (b) to the polarization of the incident electric field.
at exactly 12.8 GHz . This happens, in fact, just in the case of a simpler cavity with plane parallel interfaces. In the woodpile cavity, instead, a frequency shift in the resonance is like to occur, according to the more complex geometry. As a general rule, reduction in the spacing $h$ shifts the resonce towards higher frequencies, due to a decrease of the wavelength. On the contrary, as $h$ is increased, the resonance is shifted towards lower frequencies. In Fig. 1.6, a parametric variation of $h$ is reported, for the structure of Fig. 1.3 (b). A cavity with alumina rods of permittivity $\epsilon_{r, \mathrm{al}}=9.8$ and without losses is considered in Fig. 1.7. A comparison with results in Fig. 1.6 shows that, on equal values of $h$, the peak is much narrower and shifted
towards lower frequencies with a higher-permittivity of the woodpile constiutuent material. Moreover, peaks reach the unitary transmission efficiency in the higherpermittivity cavity due to the absence of losses, besides they are slightly attenuated in the lower-permittivity cavity becoause of losses.


Figure 1.4: Magnitude of the transmission coefficient $\eta_{t}$, for the structure depicted in Fig. 1.3 (a), as a function of frequency $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 1.5: Magnitude of the transmission coefficient $\eta_{t}$, for the structure depicted in Fig. 1.3 (b), as a function of frequency $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 1.6: Magnitude of the transmission coefficient $\eta_{t}$, for the structure depicted in Fig. 1.3 (b), as a function of frequency, for different spacings $h\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 1.7: Magnitude of the transmission coefficient $\eta_{t}$, for the structure depicted in Fig. 1.3 (b), as a function of frequency, for different spacings $h\left(\varepsilon_{r, \text { al }}=9.8\right)$.

### 1.2 Unidimensional cavities

The simplest possible EBG is the multilayer film, a structure with 1D periodicity along the z -direction, which is realized by alternating layers with different materials: the design parameters are, therefore, the layers thicknesses and the permittivity of the constituent materials. At optical frequencies, this structure is also known as Bragg mirror. As a multilayer, it can be studied in terms of the multiple reflection and refraction phenomena which occur at the interfaces between the different media. As an EBG cavity, instead, the simplest approach is the analysis of its band structure.

Multilayer EBG cavities, with two identical mirrors alternating dielectric alumina layers (thickness $d_{1}$, relative permittivity $\varepsilon_{r 1}=8.4, \tan \delta=0.002$ ) with air layers $\left(d_{2}, \varepsilon_{r 2}\right)$, and separated by a spacing $h$, have been designed (Fig. 1.8). A fixed value for the spacing was chosen as $h=23.4 \mathrm{~mm}$, which should correspond to a wavelength, so that a resonance occurs at about 12.50 GHz . The simple case


Figure 1.8: Geometry of the multilayer cavity.
of an EBG made of a single alumina layer has been examined. A first band gap appears between the normalized frequencies $\omega d / 2 \pi c$ of 0 and 0.2 (Fig. 1.9), being $d$ the period along the orthogonal direction, and $c$ the speed of light. An alumina layer of thickness $d_{1}=0.5 d=2.4 \mathrm{~mm}$, , such that $\omega d / 2 \pi c=0.1$, being $c$ the speed light, has been considered, and employed to design a cavity formed by two alumina layers with a spacing $h$ in between. With the choice $\omega d / 2 \pi c=0.1$, a transmission peak about centered on the band gap is expected; the results show a transmission peak at 12.12 GHz . Anyway, a single layer does not allow a good isolation of the peak, and an increase in the number of layers in each mirror is needed to lower the surrounding band gap. A well-isolated peak is observed as the number of mirror

|  | Number of layers | Maximum $\eta_{i}$ | $Q=f_{0} / \triangle f$ | Frequency [GHz] |
| :---: | :---: | :---: | :---: | :---: |
| 1 | $\varepsilon_{r 2}=8.4, d_{1}=2.4 \mathrm{~mm}$ | 0.99 | 13.57 | 12.49 |
|  | $\varepsilon_{r 2}=8.4, d_{1}=2.85 \mathrm{~mm}$ | 0.99 | 11.01 | 12.11 |
| 3 | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.4 \mathrm{~mm}, d_{2}=2.4 \mathrm{~mm}$ | 0.97 | 80.51 | 12.88 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.4 \mathrm{~mm}, d_{2}=5.4 \mathrm{~mm}$ | 0.95 | 156.25 | 12.50 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.85 \mathrm{~mm}, d_{2}=2.4 \mathrm{~mm}$ | 0.97 | 62.10 | 12.24 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=2.1, d_{1}=2.4 \mathrm{~mm}, d_{2}=5.4 \mathrm{~mm}$ | 0.98 | 43.93 | 12.30 |
| 5 | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.4 \mathrm{~mm}, d_{2}=2.4 \mathrm{~mm}$ | 0.88 | 324.25 | 12.97 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.4 \mathrm{~mm}, d_{2}=5.4 \mathrm{~mm}$ | 0.64 | 624.50 | 12.49 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=1, d_{1}=2.85 \mathrm{~mm}, d_{2}=2.4 \mathrm{~mm}$ | 0.88 | 312.25 | 12.49 |
|  | $\varepsilon_{r 2}=8.4, \varepsilon_{r 2}=2.1, d_{1}=2.4 \mathrm{~mm}, d_{2}=5.4 \mathrm{~mm}$ | 0.92 | 157.50 | 12.32 |

Table 1.1: Transmission properties of some multilayer cavities with mirrors of 1,3 or 5 layers
layers is increased (Fig. 1.10), together with a shift of the resonance towards higher frequencies. Employing lossy materials, just a limited number of layers can be taken into account, due to a gradual reduction in the peak amplitude, from a value of 0.99 with a single alumina layer, up to 0.62 with a seven-layer mirror (four alumina layers alternate with three air layers). Since the goal resonance frequency is 12.50 GHz , the transmission peak of these multilayer mirrors has to be moved to the left. Different choices are possible, as a separate variation in the thicknesses $d_{1}$ and $d_{2}$ of the alumina and air layers, respectively. A considerable increase in the air-layer thickness $d_{2}$ is needed to centre the resonance at about 12.50 GHz ; with a three-layer cavity it is $d_{2}=5.4 \mathrm{~mm}$, that means, with $d_{1}=2.4 \mathrm{~mm}$, a total mirror thickness of 13.2 mm . Instead, a more compact cavity can be obtained when the alumina thickness $d_{1}$ is changed. With the three-layer cavity it is now $d_{1}=2.85 \mathrm{~mm}$, yielding a total mirror thickness of only 8.1 mm , and a spatial reduction of more than $30 \%$ with respect to a variation in the value of $d_{2}$. A further possibility is to substitute the air layers with a material of different permittivity $\varepsilon_{r 2}$. Moreover, this choice allows a practical implementation of the cavity, since the intermediated material between the alumina layers would be a filling material to fix them. Low-permittivity materials can be employed, as Teflon $\left(\varepsilon_{r 2}=2.1\right)$, with a frequency behaviour quite similar. All the results are summed up in Table 1.1, where the maximum transmission efficiency, quality factor $Q=f_{0} / \triangle f$, and resonance frequency are reported.


Figure 1.9: Magnitude of the transmission coefficient $\eta_{t}$ as a function of the normalized frequency, for a single alumina layer.


Figure 1.10: Magnitude of the transmission coefficient $\eta_{t}$ of an alumina/air multilayer cavity, with $d_{1}=d_{2}=2.4 \mathrm{~mm}$.


Figure 1.11: Magnitude of the transmission coefficient $\eta_{t}$ of a multilayer cavity with different number of layers, with $d_{1}=2.85 \mathrm{~mm}$ and $d_{2}=2.4 \mathrm{~mm}$.

## Chapter 2

## Properties of EBG-covered planar antennas

On the basis of the analysis performed in the previous chapter, the EBG cavities given in Sections 1.1 and 1.2 have been used as superstrates of planar antennas. The idea is to employ the effect of angular filtering, carried out by the woodpile and multilayer cavity, to enhance the directivity of a basically low-directive antenna. This can be implemented through the image theory, with an halving with respect to the cavity symmetry-plane, where a ground plane is introduced. By this way, a planar antenna can be mounted on this ground and a final antenna with an EBG superstrate is obtained. The radiation coming from the basic antenna couples with the evanescent modes of the cavity and is filtered through the normal direction: a directive broadside pattern is therefore expected, for frequencies near the resonance of the corresponding cavity. A new illumination field is also noticeable, on an equivalent aperture defined on the EBG upper surface.

Two basic radiators containing a ground plane are here considered: a double-slot antenna (Section 2.1) and a patch antenna (Section 2.2). A comparison between the simple 1D EBG walls and a 3D woodpile EBG is performed in the case of the patch antenna, stressing the main differences and similarities in the trade-off between radiative performances and complexity/compactness of the EBG structure.

### 2.1 Double-slot antenna

The double-slot antenna depicted in Fig. 2.1 (a) has been analyzed. It is made of slots of width $w_{s}=4 \mathrm{~mm}$, length $l_{s}=9.6 \mathrm{~mm}$, and axis distance $d_{s}=9.6 \mathrm{~mm}$. The antenna is fed from below by the $\mathrm{TE}_{10}$ mode of a square cross-section waveguide (19.2 mm $\times 19.2 \mathrm{~mm}$ ), whose transverse electric field is plotted in Fig. 2.1 (b). The


Figure 2.1: a) Front view of the double-slot antenna; b) Excitation electric field.
double-slot antenna presents an acceptable matching at 14.75 GHz , when $\left|\mathrm{S}_{11}\right|=$ -16.8 dB , as it can be seen from Fig. 2.2, where the magnitude of the $\mathrm{S}_{11}$ scattering parameter is reported as a function of frequency. The radiation patterns in the


Figure 2.2: Magnitude of the reflection coefficient $\mathrm{S}_{11}$ of the antenna in Fig. 2.1(a), as a function of frequency.
planes $\varphi=0$ and $\varphi=90^{\circ}$, at the matching frequency, are reported in Fig. 2.3, and the low-directivity behavior of the antenna can be noticed. The radiated field is about ominidirectional, apart from two lateral lobes in the $\varphi=0^{\circ}$-pattern, which are due to the array factor introduced by two slots arranged along the $x$ direction.


Figure 2.3: Directivity of the double-slot antenna as a function of frequency, in the planes $\varphi=0$ (a) and $\varphi=90^{\circ}$ (b), when $\mathrm{f}=14.75 \mathrm{GHz}$.

### 2.1.1 Antenna with woodpile superstrate

Now we considered the compound radiator obtained by the use of the woodpile cavities analyzed in Section 1.1.1 as superstrates of the double-slot antenna. Placing the ground plane of the antenna in the symmetry plane of such cavities (Fig. 2.4), and removing the lower woodpile layer, a new radiator is obtained, which filters radiation coming from the basic antenna according to the frequency behaviour of the relevant cavity (Fig. 2.5). Let us start with a cavity where the most internal


Figure 2.4: Halving of the cavity in the symmetry plane.
rods are orthogonal to the excitation field, as in Fig. 1.4 of Section 1.1, being $\varepsilon_{r, \mathrm{al}}=8.4$ the rods permittivity, with $\tan \delta=0.002$. For the given radiator, the excitation field is the aperture field of the waveguide $\mathrm{TE}_{10}$ mode, as in Fig. 2.1 (b). The spacing between the ground plane and the woodpile is now $h / 2=11.7 \mathrm{~mm}$, which is half the height of the cavity investigated. A woodpile of finite extension $137.6 \mathrm{~mm} \times 137.6 \mathrm{~mm}$ in the $x y$ plane is considered. The antenna shows a poor matching, as noticeable in Fig. 2.7. Anyway, a comparison between the directivity plots, normalized at their maximum value, at 10.8 GHz (Fig. 2.8), and the ones relevant to the antenna without EBG, points out that an interaction between the field radiated by the slot antenna and the woodpile cavity has occurred, even if it is not sufficiently appreciable.

We consider now a $90^{\circ}$-rotation of the woodpile in the $x y$ plane: rods are now parallel to the excitation field (Fig. 2.9). The antenna covered by the woodpile layer is matched at 12.35 GHz , as shown in Fig. 2.10. At this new frequency, a highlydirective behavior occurs. This improvement can be observed in Fig. 2.11, where the directivity plots of the antenna in the planes $\varphi=0$ and $\varphi=90^{\circ}$ are directly compared to the ones of the double-slot antenna without the woodpile cover, and evaluated at the same frequency of 12.35 GHz .


Figure 2.5: Geometry of a woodpile-covered double-slot antenna.

The properties of the antenna with internal rods parallel to the excitation field have been further investigated. In Fig. 2.12, a map of the electric field strength on the upper face of woodpile is reported. The field significantly extends across a region which is much larger than the area of the antenna itself. In particular, assuming that the electric field is reduced of one tenth from the peak value, the active region covers an area of about $98.4 \mathrm{~mm} \times 41 \mathrm{~mm}$, which is approximately $5 \times 2$ times larger than the area covered by the double-slot antenna. As expected, the effective area is increased with respect to the double-slot feeding antenna, since a directivity enhancement must correspond to an increase in antenna aperture.

Let us consider what happens to the directivity when the woodpile dimensions are enlarged in the $x y$ plane, from the starting value of $137.6 \mathrm{~mm} \times 137.6 \mathrm{~mm}$, to $160.0 \mathrm{~mm} \times 160.0 \mathrm{~mm}$, and to $182.4 \mathrm{~mm} \times 182.4 \mathrm{~mm}$. In Fig. 2.13, directivity is reported, in the frequency interval $12-12.6 \mathrm{GHz}$, for the chosen woodpile extensions. In the three cases, directivity keeps larger than 16 dB , and reaches a maximum near the resonant frequency of the cavity ( 12.45 GHz ). Moreover, the important result that a too large woodpile is not needed emerges. Anyway, these remarks are not enough to establish a bandwidth, but an investigation of the Side Lobe Level (SLL) and an overall insight in the radiation patterns are needed, to find a compromise between directivity and SLL. The SLL are reported in Figs. 2.14 and 2.15 for the smallest and largest woodpile. Starting from a comparison between Fig. 2.14 and


Figure 2.6: Antenna covered by a woodpile superstrate with rods orthogonal to the excitation electric field.
the correspondent directivity in Fig. 2.13, a SLL greater than 10 dB but a directivity lower than 20 dB are found, for frequencies in the lower part of the interval; on the contrary, for frequencies higher than 12.5 GHz , the directivity is still high but the SLL becomes unacceptable, due to the presence of grating lobes, and the radiation pattern turns out to be similar to the one without the woodpile superstrate. Indeed, as the frequency is increased, a progressive rising of the lateral lobes takes place: they are included into the main beam, which is finally broadened. The bandwidth can so be located between 12.3 and 12.5 GHz . When a larger woodpile ( 182.4 mm $\times 182.4 \mathrm{~mm}$ ) is employed, the SLL reaches values of more than 25 dB (Fig. 2.15), but the bandwidth is extremely reduced, because for frequencies greater than 12.5 GHz the radiation pattern is already non-directive, with the secondary lobes higher than the main lobe, so that the SLL parameter is not significant.

The antenna radiative behaviour near the resonance and the described gratinglobe phenomenon can be further pointed out by observing the radiation patterns. In the next Figs. 2.16 and 2.17 the directivity is plotted for frequencies greater than the resonance one, for the minimum and maximum woodpile extension, respectively. In both cases, the first lateral lobes are merged into the main lobe at 12.5 GHz , and the central part of the radiation pattern changes from a one-lobe behaviour to a three-lobes one at 12.65 GHz . Anyway, this last attribute is more appreciable in the second case (Fig. 2.17), where at 12.65 GHz the main beam is even split. This phenomenon is also more pronounced in the $\varphi=90^{\circ}$ plane, since in the $\varphi=0$ plane the radiation pattern of the double-slot antenna itself is narrower due to the array


Figure 2.7: Magnitude of the reflection coefficient $S_{11}$ of the antenna in Fig. 2.5, with most internal rods orthogonal to field excitation, as a function of frequency $\left(\varepsilon_{r, \text { al }}=8.4\right)$.
factor along the $x$-direction.
The effect of the woodpile position with respect to the excitation field from the double-slot antenna can also be investigated. As already mentioned, the woodpile is oriented with the most internal rods parallel to the electric excitation field. Anyway, two different symmetries are possible. The shown results have been obtained with the symmetry of Fig. 13 (a), but the symmetry of Fig. 13 (b) is possible, too. A comparison in terms of directivity is reported in Fig. 2.19, for the smallest woodpile, at 12.35 GHz : the main-lobe shape is approximately the same, but some differences can be observed in the lateral lobes. A similar behaviour can be observed in Fig. 12.20 , for the largest woodpile. In this second case the coincidence between the main beam is slightly better, maybe due to the lower edge-effects occurring with a large woodpile cover.

Finally, a woodpile-covered double-slot antenna, with alumina rods of higher permittivity $\varepsilon_{r, \text { al }}=9.8$ has been simulated. A narrow bandwidth response, with a peak centered on $\mathrm{f}=12.27 \mathrm{GHz}$, is observed in Fig. 2.21: these results are in complete agreement with the transmission efficiency of the corresponding cavity in Fig. 1.7 of Section 1.1.1, with $h=23.4 \mathrm{~mm}$. The radiation diagrams at the matching frequency (Fig. 2.22) are more directive than the ones observed in Fig. 2.11, where a lower-permittivity alumina was employed.

In Fig. 2.23 the directivity is reported as a function of frequency, considerering three different woodpile dimensions in the $x y$ plane. Differently from the one in Fig. 2.13, now the frequency interval extends from 11.9 to 12.4 GHz : this shift in the response is strictly related to the position of the resonant frequency of the cavity, which is lower with respect to the other case. Moreover, the directivity is reported in frequency intervals progressively reduced as the woodpile is enlarged in the $x y$ plane: for higher frequencies the grating lobes are too high, and the antenna behaviour is not efficient. A high-permittivity woodpile is extremely selective in both space and frequency. The spatial selectivity turns into lower SLL (Fig. 2.24), but the frequency makes the bandwidth more critical.


Figure 2.8: Directivity as a function of the angle $\theta$ for the woodpile-covered slot antenna with most internal rods orthogonal to the excitation field (solid line), and for the antenna without superstrate (dashed line): a) $\varphi=0$; b) $\varphi=90^{\circ}$, when $\mathrm{f}=$ $10.80 \mathrm{GHz}\left(\varepsilon_{r, \mathrm{al}}=8.4\right)$.


Figure 2.9: Antenna covered by a woodpile superstrate with rods parallel to the excitation electric field.


Figure 2.10: Magnitude of the reflection coefficient $\mathrm{S}_{11}$ of the antenna in Fig. 2.5, with most internal rods parallel to field excitation, as a function of frequency $\left(\varepsilon_{r, \text { al }}=\right.$ 8.4).


Figure 2.11: Directivity as a function of the angle $\theta$ for the woodpile-covered slot antenna with most internal rods parallel to the excitation field (solid line), and for the antenna without superstrate (dashed line): a) $\varphi=0$; b) $\varphi=90^{\circ}$, when $\mathrm{f}=$ $12.35 \mathrm{GHz}\left(\varepsilon_{r, \mathrm{al}}=8.4\right)$.

## E field [V/m]




Figure 2.12: Electric field strength on the upper surface of woodpile, for the radiating device of Fig. 2.9


Figure 2.13: Directivity as a function of frequency, for different woodpile extensions in the $x y$ plane $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.14: SLL, as a function of frequency, for the minimum woodpile extension $(137.6 \mathrm{~mm} \times 137.6 \mathrm{~mm})\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.15: SLL, as a function of frequency, for the maximum woodpile extension $(182.4 \mathrm{~mm} \times 182.4 \mathrm{~mm})$. $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.16: Directivity as a function of the angle $\theta$ and of frequency, for the smallest woodpile ( $137.6 \mathrm{~mm} \times 137.6 \mathrm{~mm}$ ): a) $\varphi=0 ;$ b) $\varphi=90^{\circ}$ plane $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.17: Directivity as a function of the angle $\theta$ and of frequency, for the largest woodpile ( $182.4 \mathrm{~mm} \times 182.4 \mathrm{~mm}$ ): a) $\varphi=0 ;$ b) $\varphi=90^{\circ}$ plane ( $\varepsilon_{r, \text { al }}=8.4$ ).


Figure 2.18: : Possible symmetries for the woodpile, with most internal rods parallel to the excitation electric field: a) Symmetry A; b) Symmetry B..


Figure 2.19: Directivity as a function of the angle $\theta$ for symmetry A (solid line) and B (dashed line) of Fig. 13, and with the smallest woodpile ( $137.6 \times 137.6 \mathrm{~mm}$ ): a) $\varphi=0 ;$ b) $\varphi=90^{\circ}$ plane $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.20: Directivity as a function of the angle for symmetry A (solid line) and B (dashed line) of Fig. 13, and with the largest woodpile ( $182.4 \mathrm{~mm} \times 182.4 \mathrm{~mm}$ ):
a) $\varphi=0 ;$ b) $\varphi=90^{\circ}$ plane ( $\varepsilon_{r, \text { al }}=8.4$ ).


Figure 2.21: Magnitude of the reflection coefficient $\mathrm{S}_{11}$ of the antenna in Fig. 2.5, with most internal rods parallel to field excitation, as a function of frequency $\left(\varepsilon_{r, \text { al }}=\right.$ 9.8).


Figure 2.22: Directivity as a function of the angle $\theta$ of the woodpile-covered slot antenna with most internal rods parallel to the excitation field (solid line), and of the antenna without superstrate (dashed line): a) $\varphi=0$; b) $\varphi=90^{\circ}$, when $\mathrm{f}=$ $12.27 \mathrm{GHz}\left(\varepsilon_{r, \mathrm{al}}=9.8\right)$.


Figure 2.23: Directivity as a function of frequency, for different woodpile extensions in the $x y$ plane $\left(\varepsilon_{r, \text { al }}=9.8\right)$.


Figure 2.24: SLL, as a function of frequency, for the minimum woodpile extension $(137.6 \mathrm{~mm} \times 137.6 \mathrm{~mm})$. $\left(\varepsilon_{r, \text { al }}=9.8\right)$.

### 2.2 Patch antenna

In this Section, a patch antenna is assumed as the reference source. It is made of a substrate of Rogers RT/duroid 5880 of relative permittivity 2.2 and thickness 0.787 mm , with a square patch of side 7.6 mm , fed by a coaxial probe (Fig. 2.25). The


Figure 2.25: Geometry of a patch antenna.
ground-plane and EBG-superstrate extension in the $x y$ plane is $a_{x}=a_{y}=92.6 \mathrm{~mm}$, which corresponds to almost four wavelengths. As a typical patch antenna, a broad radiation diagram with a maximum gain lower than 8 dB is observed; this behaviour can be related to a small extension of the illumination area, which is just limited to the patch.

### 2.2.1 Antenna with multilayer superstrate

Let us consider a single alumina layer of thickness $d_{1}=2.4 \mathrm{~mm}$, corresponding to the cavity presented in Section 1.2, as a superstrate placed at a distance $h / 2=11.7 \mathrm{~mm}$ from the patch ground-plane. In Fig. 2.26, the magnitude of the E-field on the upper face of the dielectric layer is reported with respect to the $x$ - and $y$-directions. Considering a field-amplitude reduction of a tenth with respect to its maximum value, the illumination field covers an area of ( $29.6 \mathrm{~mm} \times 35.4 \mathrm{~mm}$ ). A maximum gain of 14.77 dB at the matching frequency of the antenna ( 12.12 GHz ), and of 15.42 dB at the resonant frequency of the corresponding cavity ( 12.49 GHz ), presented in

Section 1.2, have been evaluated. At both frequencies, the effect of the dielectric cover on the final directivity of the antenna can be appreciated through a direct comparison with the radiation diagram of the patch antenna alone (Fig. 2.27).


Figure 2.26: Normalized magnitude of the E-field on the equivalent aperture evaluated at the resonant frequency of the cavity $(12.29 \mathrm{GHz})$, for a patch antenna covered by an alumina layer with $d_{1}=2.4 \mathrm{~mm}$.

An improvement in the antenna performances is expected when a multilayer EBG is employed, since a more pronounced transmission efficiency is observed (Table 1, Section 1.2). The cavity with alumina layers of 2.85 mm thickness alternating with air layers of 2.4 mm is now adopted. The magnitude of the E-field on the equivalent aperture and the radiation diagrams in the planes $\varphi=0$ and $\varphi=90^{\circ}$ are reported in Fig. 2.29-2.28, for an EBG with 1, 3, and 5 layers, and are evaluated at the resonant frequencies of the corresponding cavities. With a single alumina layer the illumination area is ( 26.5 mm 21.6 mm ), and the antenna behaviour is similar to the one describe above, with a maximum gain of 15.95 dB at the cavity resonance. A rise to a number of two alumina layers produces an enlargement of the equivalent illumination area up to ( $31 \mathrm{~mm} \times 29 \mathrm{~mm}$ ), with a maximum gain of 18 dB . In both cases, a non-uniform illumination gives good results in terms of gain. Moreover, we remark that when the superstrate is employed, a new matching for the antenna is needed by varying the feed position along the patch, with the result that the matching frequency never corresponds to the resonant frequency. It means that these gains, which are evaluated at the input of the antenna transmission line, are really noticeable values, considering that a significant return loss is introduced at the resonance due to the mismatching.


Figure 2.27: Directivity plots as a function of $\theta$, for a patch antenna covered by an alumina layer with $d_{1}=2.85 \mathrm{~mm}$. The plots are evaluated at the resonant frequency of the cavity $(\mathrm{f}=12.49 \mathrm{GHz})$ and at the matching frequency ( $\mathrm{f}=12.12 \mathrm{GHz}$ ), and compared with the radiation diagram of the patch antenna without EBG superstrate:
a) $\varphi=0$; b) $\varphi=90^{\circ}$.


Figure 2.28: 2D maps for the E-field strength on the equivalent aperture of a patch antenna with a multilayer superstrate ( $d_{1}=2.85 \mathrm{~mm}$ (alumina) and $d_{2}=2.4 \mathrm{~mm}$ (air)), evaluated at the resonant frequency of the multilayer cavity: a) 1 layer (12.11 $\mathrm{GHz})$; b) 3 layers ( 12.42 GHz ); c) 5 layers ( 12.49 GHz ).

With the employment of a five-layer superstrate, a more uniform illumination is produced, and the equivalent area covers the whole EBG extension. Anyway, a worse radiative behaviour is observed, with a larger main-beam width and a maximum gain of 8.96 dB , since the losses introduced by a thicker mirror deteriorate the cavity efficiency. The results for the radiation diagrams with these geometries are reported in Fig. 2.30. In terms of bandwidth, the response reflects the frequency behaviour of the corresponding cavity acting as a superstrate, as put in evidence by the magnitude of the reflection coefficient $\mathrm{S}_{11}$ (Fig. 2.31). A larger bandwidth is observed for a single layer, up to a peaked matching response for the five-layer EBG. With three and five layers, multiple peaks appear in the $\left|\mathrm{S}_{11}\right|$; anyway the enhanced-directivity effect due to the interaction with the EBG is displayed just in correspondence of the first one, which is close to the resonant frequency.


Figure 2.29: Normalized magnitude of the E-field on the equivalent aperture of a patch antenna with a multilayer superstrate ( $d_{1}=2.85 \mathrm{~mm}$ (alumina) and $d_{2}=$ 2.4 mm (air)), evaluated at the resonant frequency of a 1 -, 3 -, 5 -layers cavity ( 12.11 $\mathrm{GHz}, 12.42 \mathrm{GHz}, 12.49 \mathrm{GHz}):$ a) $\varphi=0$; b) $\varphi=90^{\circ}$.


Figure 2.30: Directivity plots as a function of, for a patch antenna with a multilayer superstrate ( $d_{1}=2.85 \mathrm{~mm}$ (alumina) and $d_{2}=2.4 \mathrm{~mm}$ (air)), evaluated at the resonant frequency of a 1-, 3 -, 5 -layers cavity ( $12.11 \mathrm{GHz}, 12.42 \mathrm{GHz}, 12.49 \mathrm{GHz}$ ):
a) $\varphi=0 ;$ b) $\varphi=90^{\circ}$.


Figure 2.31: Magnitude of the scattering parameter $S_{11}$, as a function of frequency, of a patch antenna with a 1-, 3 -, 5 -layers superstrate ( $d_{1}=2.85 \mathrm{~mm}$ (alumina) and $d_{2}=2.4 \mathrm{~mm}$ (air)).

### 2.2.2 Antenna with woodpile superstrate

A comparison with a woodpile-covered patch antenna has been performed. The single woodpile layer has a total thickness of 12.8 mm in the $z$-direction, and it is $(92.4 \mathrm{~mm} \times 92 \mathrm{~mm})$ large in the $x y$ plane. Along the $x$-direction, the amplitude of the illumination field, evaluated on the upper side of the EBG, is gradually reduced with the distance from the centre, but its value is never negligible, so that it covers the structure in its whole extensions. In the $y$-directed illumination field, a step profile can be observed, with local maxima in line with the rods (Figs. 2.32-2.33). The illumination field extends over the entire structure, as in the five-layer 1D EBG, but in this case a maximum peak gain of 18.89 dB is evaluated at the cavity resonant frequency of 12.44 Ghz , which is a really significant value, notwithstanding a strong return loss of -5.53 dB (Fig. 2.34). It means that best performances can be obtained only with a 3D cavity, notwithstanding the high values of return loss. In Fig. 2.11, the directivity plots of the antenna in the planes $\varphi=0$ and $\varphi=90^{\circ}$ are directly compared to the ones of the double-slot antenna without the woodpile cover, and evaluated at the same frequency of 12.44 GHz .


Figure 2.32: Normalized magnitude of the E-field along the $x$ - and $y$-directions, on the equivalent aperture of a woodpile-covered patch antenna, evaluated at the resonant frequency of the woodpile cavity $(12.44 \mathrm{GHz})$.


Figure 2.33: 2D E-field map on the aperture of a woodpile-covered patch antenna, evaluated at the resonant frequency of the woodpile cavity ( 12.44 GHz ).


Figure 2.34: Magnitude of the reflection coefficient $S_{11}$ of the woodpile-covered patch antenna, with most internal rods parallel to the excitation field, as a function of frequency $\left(\varepsilon_{r, \text { al }}=8.4\right)$.


Figure 2.35: Directivity as a function of the angle $\theta$ for the woodpile-covered patch antenna with most internal rods parallel to the excitation field (solid line), and for the antenna without superstrate (dashed line): a) $\varphi=0$; b) $\varphi=90^{\circ}$, when $\mathrm{f}=$ $12.44 \mathrm{GHz}\left(\varepsilon_{r, \mathrm{al}}=8.4\right)$.

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## Chapter 3

## Design, fabrication and measurements of a woodpile

The results presented in chapters 1 and 2 are preliminary studies to the implementation of a planar antenna with woodpile superstrate. Once we resolved to fabricate a woodpile structure, the practical setup available for measurements had to be taken into account. For this reason, a new woodpile unit cell has been designed, consistent with the anechoic chamber and the spectrum analyzer which would be employed in the experimental analysis.

In Section 3.1, the design of a woodpile unit cell satisfying the constraint af a transmission peak centered on about 11 GHz is described.

Next, two woodpile prototypes in alumina have been built. Details concerning the fabrication are given in Section 3.2.

A Teflon support has been devised, with frames that keep parallel and at the desired distance the woodpile layers, in order to form a cavity. Transmission measurements through the layers have been performed in a shielded anechoic chamber, in two different layouts of the experimental setup. Several sets of measurements have been carried out, and all the experimental results are presented and discussed in Section 3.3.

### 3.1 Design of the woodpile unit cell

In Section 1.1 the elementary cell of a woodpile structure has been introduced. Considering square cross-section rods, the design parameters of cell are the side $w$ and the periods $d=d_{x}=d_{y}$ and $d_{z}=4 w$ (see Fig. 1.1 (a)). The new woodpile structure has been designed in view to be employed as superstrate of a patch antenna. Thus, simulations have been run on a cavity made up of two woodpile layers of


Figure 3.1: Transmission efficiency of a cavity with spacing $h=27.3 \mathrm{~mm}$ and period $d=11.2 \mathrm{~mm}$.
infinite extension, and separated by a spacing $h$. For reasons of compatibility with the experimental setup, available for measurements on the woodpile cavity, and on the woodpile-covered antenna, the constraint is a transmission peak at a frequency around 11 GHz . The commercial rods chosen to fabricate the woodpile structure have a square cross-section of side $w=3.18 \mathrm{~mm}$ and are made of alumina. Being unknown the exact permittivity value of such rods, a relative permittivity $\varepsilon_{r}=9$ is assumed in the simulations. It represents a mean value, being for the real part of alumina permittivity $8.4 \leq \varepsilon \leq 10$. Once fixed side $w$ and $\varepsilon_{r}$, the free parameters are the period $d$ of the cell along the $x$ - and $y$-directions and the spacing $h$. A cavity satisfying specifications of good transmission efficiency and period as highest as possible is looked for. The latter is to meet requirements of reasonable fabrication costs, since a higher numbers of rods is needed when the period is reduced.

A cavity with spacing $h=12 \mathrm{~mm}$, corresponding to a wavelength $\lambda$ for the frequency of 11 GHZ , has been simulated, with electric field polarized as parallel to the most internal rods of the cavity. As starting value, a period $d=11.2 \mathrm{~mm}$ has been assumed, corresponding to the the one of the woodpile cell described in Section 1.1. For the chosen parameters, the transmission occurs at a frequency higher than 11 GHz (Fig. 3.1).

The transmission peak may be shifted to the design frequency through a reduc-


Figure 3.2: Transmission efficiency of a cavity with spacing $h=27.3 \mathrm{~mm}$ and different periods $d$.
tion in the period $d$. In Fig. 3.2 the transmission efficiency is reported for decreasing values of $d$. When $d=7 \mathrm{~mm}$ the peak is centered on 11 GHz . Anyway, a larger period would be needed to employ a lower number of alumina rods, i.e. to reduce the fabrication costs.

To increase the period while keeping the transmission peak at 11 GHz , an increase in the cavity spacing $h$ is needed. Couples of parameters satisfying the specification of a peak centered on 11 GHz are reported in Fig. 3.3. With high values of $d$ and $h$, the band-gap at the left of the transmission peak is less and less pronounced. A cavity with period $d=8 \mathrm{~mm}$ has been chosen, as a good compromise in the trade off cavity efficiency/costs.

### 3.2 Fabrication of the woodpile structure

As already stressed, the practical implementation of the woodpile cavity has to deal with the available experimental setup. In particular, rods length, i.e. the minimum woodpile dimensions along $x$ - and $y$-directions, has been chosen such as to minimize edge-diffraction effects.

For transmission measurements on the cavity, the setup employs two iden-


Figure 3.3: Transmission efficiency for a cavity with different couples $h, d$ of spacing and period.
tical horn antennas, as depicted in Fig. 3.4 (a). The minimum far-field distance is $r_{\text {min }}=2 D^{2} / \lambda$, being $\mathrm{D}=9 \mathrm{~cm}$ the diagonal length of our horn aperture. At $11 \mathrm{GHz}, \mathrm{r}_{\text {min }}=54 \mathrm{~cm}$. Moreover, the horn has a Half Power Beam Width $\mathrm{HPBW}=18^{\circ}=2 \alpha$. From the simple scheme in Fig. 3.4 (b), the minimum woodpile dimension needed by the transmission horn to see a periodic structure of infinite extension is $R_{\text {min }}=2 r_{\text {min }} \tan \alpha=17.10 \mathrm{~cm}$. In order to meet this requirement, a length of $25 d=20 \mathrm{~cm}$, corresponding to about 7 wavelengths, has been assigned to the alumina rods. We expect that a $(20 \mathrm{~cm} \times 20 \mathrm{~cm})$ woodpile should provide a good directivity enhancement when placed at a suitable distance from a planar antenna.

The alignment of the rods has been performed by means of a metallic structure made up of equally-spaced steel pins mounted on the four external sides of an aluminum plate (Fig. 3.6). The design of such structure is sketched in Fig. 3.5, where all the geometrical constraints, as the rods cross-section and length, and the period $d$ are taken into account.

Each layer of the woodpile structure has been formed placing the rods through two opposite rows of pins, with a distance corresponding to the period $d$. Once a layer has been arranged, the neighbouring one has been obtained with an alignment


Figure 3.4: a) Sketch of the experimental setup for the woodpile-cavity measurement; b) Scheme for the evaluation of the woodpile dimensions.
through the two remaining rows of pins, thus achieving a rotation of $90^{\circ}$ between rods of two consecutive layers. Moreover, the new rods have been glued on the top of the rods of the previous layer, at the touching points. Thus, two identical woodpile prototypes have been fabricated; a detail of the final structure is reported in Fig. 3.7.

### 3.3 Measurements on the woodpile structure

The woodpile structure and the cavities have been measured in a shielded anechoic chamber of external dimensions ( $3.20 \mathrm{~m} \times 3.20 \mathrm{~m} \times 2.70 \mathrm{~m}$ ), placed in Department of Electronic Engineering at Sapienza University of Rome. Experimental data have been acquired by a vectorial network analyzer (HP 8510w) and next processed.

In the following sections, the results of four sets of measurements are reported. The magnitude of the transmission efficiency through the structure, as function of frequency, has been evaluated, in the $8-12 \mathrm{GHz}$ range. With measurements on a single layer, or on two consecutive layers, the band-gap of the woodpile crystal has been determined. Then, woodpile cavities have been formed introducing a spacing $h$ between the layers; depending on the layers orientation, both asymmetric and symmetric cavities have been assembled, considering several values of $h$. As far as symmetric cavities are concerned, which are the most interesting in view to their implementation into a resonating antenna, two different alignements of the most


Figure 3.5: Sketch of the metallic support employed to align rods in a woodpile configuration.
internal rods with respect to the polarization of the electric field have been arranged, i.e. orthogonal and parallel to the field.

Transmission measurements have been carried out in two different layouts. A first experimental setup has included two identical horn antennas, with diagonal of 9 cm , as receiving and transmitting antennas. Results for three sets of measurements are shown in the following, for three different distances between the antennas. The first set has been carried out at a 1 m distance, which turned out to be too high, since strong edge-diffraction due to the finite woodpile dimensions was observed. Thus, shorter distances of 44 cm , i.e. Fresnel distance, and 54 cm , corresponding to minimum far-field distance, have been considered. A second experimental setup has employed a parabolic reflector with a 40 cm diameter as transmitting antenna, and the horn as receiving antenna. In this case, being the minimum far-field distance of 10 m too long for the available anechoic chamber, the receiving antenna has been


Figure 3.6: Metallic support employed to align rods in a woodpile configuration.
placed in Fresnel zone of the parabolic reflector.
The experimental results are compared to simulations run with Ansoft HFSS v11. Woodpile has been simulated for three values of alumina permittivity (8.4, 9, and 9.8), being unknown the relative permittivity of the employed alumina in the $8-12 \mathrm{GHz}$ frequency range. The agreement between HFSS simulations and measurements is good; possible differences could be ascribed to the fact that, in the simulated results, the excitation is a monochromatic plane-wave, the involved materials are supposed to be lossless, and the woodpile extension is infinite in the $x$ and $y$ directions, so that diffraction effects at the edges of the prototypes are neglected.

### 3.3.1 Description of the experimental setup

The experimental part of this work has needed the working out of a suitable workbench, which could support the woodpile layers between the two antennas in a vertical position and space the layers at the desired distance. Moreover, a support stable enough to bear the considerable weight of the alumina-made woodpile was called for. The support would be employed for three different classes of measure-


Figure 3.7: Detail of the woodpile layer.
ments. First, the band-gap of the woodpile structure would be measured, both with a single layer and two attached woodpile layers. This would ask a structure with two frames where one or two attached layers could be mounted on. Second, the same structure would be employed to perfom cavity measurements, i.e. on the two woodpile layers, separated by an arbitrary spacing $h$. Thus, at least one of the two frames of the structure should be movable, with a view to removing one of the two layers away from the other one. Third, as it will be clearer in the next chapter, the idea was to employ the same structure to build a woodpile-covered patch antenna. Thus, a third frame, where to place a patch radiator, would be needed. The design of the final structure is Fig. 3.8: it is made of a Teflon support where three PVC frames are mounted on. On the first frame the patch antenna is mounted, while the other two frames are for the woodpile layers. The external frames are movable, while the central one is fixed. Low permittivity materials, as Teflon and PVC, have been chosen, in order to minimize the impact with the objects under measurement. The structure in Fig. 3.8 is finally placed on a PVC table, with such a height that the structure under measurement is aligned to beam emitted by the antennas.


Figure 3.8: Support for the experimental measuremnts on the woodpile layers.

### 3.3.2 Experimental results: first setup

The first experimental setup has employed two identical horn antennas in Transmission and Reception, with characteristics given in Table 3.1). With the present arrangement, experimental data have been collected in three different sets of measurements, relevant to three different distances between the antennas, of $1 \mathrm{~m}, 44 \mathrm{~cm}$ and 54 cm , respectively.

|  | Transmitting and Receiving Antenna (TX, RX) |
| :---: | :---: |
| Type | Horn |
| Dimension | $d=9 \mathrm{~cm}$ |
| Gain | $16 \mathrm{~dB} @ 9.5 \mathrm{GHz}$ |

Table 3.1: Characteristics of the transmititng and receiving antenna employed in the first set of measurements ( $d=$ diagonal).

First set: $d_{\text {ant }}=1 \mathrm{~m}$
The first set of measurements has been carried out with the two horn antennas at a 1 m distance. The fixed woodpile layer is at half-way between the antennas, while the layer placed in the movable frame is closer to the receiving antenna, at a distance $d_{\text {ant }} / 2-h$, i.e. $50 \mathrm{~cm}-\mathrm{h}$, being $h$ the cavity spacing, as it gets gradually moved away from the fixed layer in order to enlarge the cavity.

The calibration measurements on the anechoic chamber point out that the present experimental setup is inappropriate for the structure under measurement. The distance between the two antennas and the woodpile is probably too high, since a positive transmission efficiency, comprised between 0 and 3 dB , is recorded as the experimental support is introduced inside the anechoic chamber, as evident in Fig. 3.9. The effect of a PEC surface introduced at half-way between the antennas is shown in Fig. 3.10: the surface has the same dimension of the woodpile layers, and its screening effect is not very strong.

Calibration results are confirmed by the measured transmission efficiency $\eta_{t}$ through the woodpile structure. In the plots of Figs. ??-3.15, strong diffractions effects at the edges of the layers can be appreciated, which leads to measured values of $\eta_{\mathrm{t}}$ greater than 0 dB .

Transmission efficiency has been measured on two consecutive woodpile layers (3.11), i.e. the band gap of the structure has been examined for a normally-incident radiation.


Figure 3.9: Calibration measurements on the anechoic chamber.


Figure 3.10: Calibration measurements on the anechoic chamber: effect of a PEC surface.

Two kinds of cavities have been measured, both asymmetric (Figs. 3.12 and 3.13) and symmetric with internal rods orthogonal to the polarization of the incident electric field (Figs. 3.15 and 3.15), with $h=36,37 \mathrm{~mm}$. Transmission peaks emerging inside the band gap can be observed, in agreement with prediction of simulated results by HFSS.


Figure 3.11: Transmission efficiency as a function of frequency for two woodpile layers.


Figure 3.12: Asymmetrical cavity with $h=36 \mathrm{~mm}$.


Figure 3.13: Asymmetrical cavity with $h=37 \mathrm{~mm}$.


Figure 3.14: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=36 \mathrm{~mm}$.


Figure 3.15: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=37 \mathrm{~mm}$.

Second set: $d_{\text {ant }}=44 \mathrm{~cm}$
The second set of measurements has employed the two horn antennas at a 44 cm distance, in the Fresnel zone. A shorter distance between the antennas has been chosen in the effort to reduce edge diffraction due to the finite dimensions of the layers.

The fixed woodpile layer is at half-way between the antennas, and the layer placed in the movable frame is closer to the receiving antenna, at a distance $d_{\text {ant }} / 2-$ $h$, i.e. $22 \mathrm{~cm}-\mathrm{h}$, as it gets gradually moved away from the fixed layer in order to enlarge the cavity.

With a closer distance $d_{\text {ant }}$, the calibration measurements on the chamber point out that a lower diffraction is introduced by the plastic support placed between the antennas. As it can be appreciated in Fig. 3.16, the noise due to the frames is negligible with respect to the one of Fig. 3.9. Edge-diffraction is still observed, with a slightly-positive transmission efficiency for frequency between 10 and 12 GHz . Neverthelss, a narrower distance between the two antennas turns out to be more appropriate. This is confirmed by the experiment with the PEC layer, that, as placed at half-way between the antennas, exhibits a more screening effect, with reflections lower than -25 dB (Fig. 3.17).

Measurement for the band gap on a single woodpile layer is too noisy to appreciate the comparison with HFSS results. The agreement is good when band gap is evaluated in case of transmission through two consecutive layers (Fig. 3.18).

Cavities have been measured, both asymmetric (Figs. 3.19 and 3.20), and symmetric with internal rods orthogonal (Fig. 3.21) and parallel (Figs. 3.22 and 3.23) to the polarization of the incident electric field. For this second set of measurements, transmission efficiency is always lower than 0 dB , and the agreement with simulated results is good, even if the wave-front emitted by the antenna impinging on the woodpile layers is quite far from the plane-wave approximation assumed in the simulated results. Some mismatches between experimental/simulated curves, in terms of position and amplitude of the peaks, especially in the higher part of the frequency interval, can be observed in some plots. This may be due to the uncertainty introduced by a manual placing of the layers at the wanted distance $h$, with respect to an external graduated scale.


Figure 3.16: Calibration measurements on the anechoic chamber.


Figure 3.17: Calibration measurements on the anechoic chamber: effect of a PEC surface.


Figure 3.18: Transmission efficiency as a function of frequency for two woodpile layers.


Figure 3.19: Asymmetrical cavity with $h=36 \mathrm{~mm}$.


Figure 3.20: Asymmetrical cavity with $h=37 \mathrm{~mm}$.


Figure 3.21: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=32 \mathrm{~mm}$.


Figure 3.22: Symmetrical cavity with most internal rods parallel to the electric field, and $h=36 \mathrm{~mm}$.


Figure 3.23: Symmetrical cavity with most internal rods parallel to the electric field, and $h=40 \mathrm{~mm}$.

Third set: $d_{\text {ant }}=54 \mathrm{~cm}$
The third set of measurements has been carried out with the antennas at a 54 cm distance, i.e. the minium far-field distance between the antennas. The woodpile cavity is placed at 12 cm from the receiving antenna and is placed in the near field radiated from the transmitting antenna.

Calibration measurements have been carried out for empty anechoic-chamber, chamber with the wood table, and with table, Teflon support and PVC frames. Results are compared in Fig. 3.24.


Figure 3.24: Calibration measurements on the anechoic chamber.
This set of measurements benefits from the introduction of a graduated scale fixed to the Teflon support; anyway, small uncertainties on the exact value of $h$, i.e. on the exact peaks position, are still possible.

Transmission though symmetrical cavities has been measured and compared to numerical results with HFSS, and the agreement is quite good. Cavities with most internal rods orthogonal to the electric field have been measured from the starting value $h=36 \mathrm{~mm}$ up to $h=42 \mathrm{~mm}$, with step 1 mm (see Figs. 3.25-3.31). In the case case of rods parallel to the electric field, spacings from $h=40 \mathrm{~mm}$ up to $h=48 \mathrm{~mm}$ with step 1 mm have been considered (Figs. 3.32-3.40). A good agreemnt with the HFSS data can be appreciated.


Figure 3.25: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=36 \mathrm{~mm}$.


Figure 3.26: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=37 \mathrm{~mm}$.


Figure 3.27: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=38 \mathrm{~mm}$.


Figure 3.28: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=39 \mathrm{~mm}$.


Figure 3.29: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=40 \mathrm{~mm}$.


Figure 3.30: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=41 \mathrm{~mm}$.


Figure 3.31: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=42 \mathrm{~mm}$.


Figure 3.32: Symmetrical cavity with most internal rods parallel to the electric field, and $h=40 \mathrm{~mm}$.


Figure 3.33: Symmetrical cavity with most internal rods parallel to the electric field, and $h=41 \mathrm{~mm}$.


Figure 3.34: Symmetrical cavity with most internal rods parallel to the electric field, and $h=42 \mathrm{~mm}$.


Figure 3.35: Symmetrical cavity with most internal rods parallel to the electric field, and $h=43 \mathrm{~mm}$.


Figure 3.36: Symmetrical cavity with most internal rods parallel to the electric field, and $h=44 \mathrm{~mm}$.


Figure 3.37: Symmetrical cavity with most internal rods parallel to the electric field, and $h=45 \mathrm{~mm}$.


Figure 3.38: Symmetrical cavity with most internal rods parallel to the electric field, and $h=46 \mathrm{~mm}$.


Figure 3.39: Symmetrical cavity with most internal rods parallel to the electric field, and $h=47 \mathrm{~mm}$.


Figure 3.40: Symmetrical cavity with most internal rods parallel to the electric field, and $h=48 \mathrm{~mm}$.

### 3.3.3 Experimental results: second setup

The second setup has employed a parabolic reflector antenna in Transmission and an horn antenna in Reception. As already mentioned, the minimum far-field distance for the employed reflector antenna would be 10 m , of course too large for our in-door measurements. A distance $d_{\mathrm{ant}}=125 \mathrm{~cm}$ between the antenna has been arranged, which is the largest allowed by the anechoic chamber. The cavity has been placed at 12 cm from the receiving antenna.

|  | Transmitting Antenna (TX) | Receiving Antenna (RX) |
| :---: | :---: | :---: |
| Type | Parabolic reflector | Horn |
| Dimension | $D=36 \mathrm{~cm}, f / D=0.42$ | $d=9 \mathrm{~cm}$ |
| Gain | $25 \mathrm{~dB} @ 9.5 \mathrm{GHz}$ | $16 \mathrm{~dB} @ 9.5 \mathrm{GHz}$ |

Table 3.2: Characteristics of the transmititng and receiving antenna employed in the first set of measurements ( $D=$ diameter, $f=$ focal length, $d=$ diagonal).

Calibration measurements (Fig. 3.41) show low edge diffraction, confirmed by the behaviour with a PEC surface between the antennas (Fig. 3.42), with reflections lower than -20 dB almost over the whole frequency interval $8-12 \mathrm{GHz}$.

Measurements of the band gap for one (Fig. 3.43) and two (Fig. 3.44) layers is really good. Differently from the experimental setup with two horn antennas, the plot are more noisy, with very fast oscillations, but the envelope follows very well the simulated results. This is noticeable also for the plots of transmission through the cavities (Figs. 3.45-3.53), where the measured peaks perfectly match the one given by the simulations.

Results for an asymmetric cavity are reported in Fig. 3.45, with $h=36 \mathrm{~mm}$. Symmetric cavities with most internal rods both orthogonal and parallel to the electric field have been measured, with $h=36,39,42,48 \mathrm{~mm}$. Results are given in Figs. 3.46-3.49 and in Figs. 3.50-3.53, for orthogonal and parallel rods, respectively.


Figure 3.41: Calibration measurements on the anechoic chamber.


Figure 3.42: Calibration measurements on the anechoic chamber: effect of a PEC surface.


Figure 3.43: Transmission efficiency as a function of frequency for a woodpile layer.


Figure 3.44: Transmission efficiency as a function of frequency for two woodpile layers.


Figure 3.45: Asymmetrical cavity with $h=36 \mathrm{~mm}$.


Figure 3.46: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=36 \mathrm{~mm}$.


Figure 3.47: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=39 \mathrm{~mm}$.


Figure 3.48: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=42 \mathrm{~mm}$.


Figure 3.49: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=48 \mathrm{~mm}$.


Figure 3.50: Symmetrical cavity with most internal rods parallel to the electric field, and $h=36 \mathrm{~mm}$.


Figure 3.51: Symmetrical cavity with most internal rods parallel to the electric field, and $h=39 \mathrm{~mm}$.


Figure 3.52: Symmetrical cavity with most internal rods parallel to the electric field, and $h=42 \mathrm{~mm}$.


Figure 3.53: Symmetrical cavity with most internal rods parallel to the electric field, and $h=48 \mathrm{~mm}$.

## Chapter 4

## A woodpile-covered patch antenna

In this chapter, experimental results for woodpile cavities used for gain-enhancement of a patch antenna are presented.

A patch antenna has been fabricated and measured. Its characteristics are presented in Section 4.1. Measurement of its reflection coefficient has given a matching at 10.3 GHz . Thus, symmetric woodpile cavities with a transmission peak at 10.3 GHz have been looked for, in order to maximize the antenna gain enhancement when employed as superstrates. The distance between the woodpile prototypes has been progressively varied up to define two sets of cavities with transmission peak at the searched frequency, one set with most internal rods orthogonal to the electric field, the other one with rods parallel to the electric field. Then, reflection coefficient and gain of these two sets of cavities, employed as superstrates of the patch antenna, have been measured. It has been experimentally found that the gain of the woodpile-covered patch can be enhanced up to 10 dB in comparison with the gain of the patch alone. Radiation pattern of two particular layouts has been measured, with good features of Half-Power Beam Width and Side Lobe Level. A full report of the experimental analysis carried out on the new radiator is given in Section 4.2.

### 4.1 Design and fabrication of a patch antenna

A patch antenna has been realized with a layer of 0.76 mm thick Rogers/RT Duroid 5870 with $\varepsilon_{r}=2.33$, printed on both sides with 1 OZ copper. With a PC-controlled microforge, on one side of the layer a rectangular patch of ( $8 \mathrm{~mm} \times 8.4 \mathrm{~mm}$ ) has been cut. The patch antenna is fed from below by a coaxial probe, centered on the $x$-direction, and placed at a distance of 1.2 mm from the centre with respect to the $y$-direction.

The antenna is shown in Fig. 4.2. A measure of the input scattering parameter
has been performed using a network analyzer; the result is reported in Fig. 4.3. The antenna resonates at 10.3 GHz ; at this frequency, its gain is 6.7 dB and the magnitude of its return loss is $\left|\mathrm{S}_{11}\right|=-11.69 \mathrm{~dB}$.


Figure 4.1: Square-patch antenna of side 8 mm on a Rogers/RT Duroid 5870 substrate with $\varepsilon_{r}=2.33$ and thickness 0.76 mm .


Figure 4.2: Feeding of the patch antenna through coaxial probe.


Figure 4.3: Measured return loss (magnitude) for the antenna in Fig. 4.2.

### 4.2 Measurements on a woodpile-covered patch antenna

According to the image theory, an equivalent configuration for the cavity described in Section 4.1 has been considered, consisting in a halving with respect to the symmetry plane, where a perfect ground plane is introduced. The ground plane of the patch antenna itself has been employed, thus obtaining a new compound radiator with a woodpile superstrate spaced of $h / 2$. This layout has been implemented mounting the patch antenna and the woodpile layer on the experimental support. A photo of the experimental setup is in Fig. 4.5.

### 4.2.1 Characterization of the cavities

It has been observed that the patch antenna resonates at 10.3 GHz . Thus, cavities realized by the two woodpile prototypes at a distance $h$ and which resonate at 10.3 GHz have been looked for. Measurements have been performed on symmetric cavities, with spacings $h$ which are multiples of the wavelength $\lambda$.


Figure 4.4: Measured and simulated radiation patterns for the patch antenna of Fig.
4.2.

### 4.2. MEASUREMENTS ON A WOODPILE-COVERED PATCH ANTENNA161

## Cavities with most internal rods orthogonal to the electric field

Here the plots of transmission efficiency for symmetric cavities with most internal rods orthogonal to the electric field are given. It has been observed that resonance occurs at 10.3 GHz if $h \cong(15 N-5 \pm 1) \mathrm{mm}$, with $N$ positive integer.


Figure 4.5: Experimental setup for the measurements on the antenna.


Figure 4.6: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=11 \mathrm{~mm}$.


Figure 4.7: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=26 \mathrm{~mm}$.


Figure 4.8: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=40 \mathrm{~mm}$.


Figure 4.9: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=55 \mathrm{~mm}$.


Figure 4.10: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=70 \mathrm{~mm}$.


Figure 4.11: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=84 \mathrm{~mm}$.


Figure 4.12: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=99 \mathrm{~mm}$.


Figure 4.13: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=113 \mathrm{~mm}$.


Figure 4.14: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=128 \mathrm{~mm}$.


Figure 4.15: Symmetrical cavity with most internal rods orthogonal to the electric field, and $h=142 \mathrm{~mm}$.

## Cavities with most internal rods parallel to the electric field

In Figs. 4.16-4.25 are reported the plots of transmission efficiency of symmetric cavities with internal rods parallel to the electric field, showing a resonance at 10.3 GHz. This structure resonates when the equivalent length is an integer multiple of $\lambda / 2$, being $\lambda \cong 29.12 \mathrm{~mm}$ the wavelength of the electromagnetic field in air. It has been observed that $h \cong(15 N \pm 1) \mathrm{mm}$, with $N$ positive integer.


Figure 4.16: Symmetrical cavity with most internal rods parallel to the electric field, and $h=16 \mathrm{~mm}$.


Figure 4.17: Symmetrical cavity with most internal rods parallel to the electric field, and $h=31 \mathrm{~mm}$.


Figure 4.18: Symmetrical cavity with most internal rods parallel to the electric field, and $h=45 \mathrm{~mm}$.


Figure 4.19: Symmetrical cavity with most internal rods parallel to the electric field, and $h=60 \mathrm{~mm}$.


Figure 4.20: Symmetrical cavity with most internal rods parallel to the electric field, and $h=75 \mathrm{~mm}$.


Figure 4.21: Symmetrical cavity with most internal rods parallel to the electric field, and $h=90 \mathrm{~mm}$.


Figure 4.22: Symmetrical cavity with most internal rods parallel to the electric field, and $h=104 \mathrm{~mm}$.


Figure 4.23: Symmetrical cavity with most internal rods parallel to the electric field, and $h=119 \mathrm{~mm}$.


Figure 4.24: Symmetrical cavity with most internal rods parallel to the electric field, and $h=133 \mathrm{~mm}$.


Figure 4.25: Symmetrical cavity with most internal rods parallel to the electric field, and $h=148 \mathrm{~mm}$.

### 4.2.2 Measurements of the input return loss

In Section 4.2.1 cavities with a transmission peak at 10.3 GHz , which is equal to the matching frequency measured on the patch antenna (Fig. 4.3), have been found. Thus, they have been employed as superstrates of the patch antenna, placing one layer of the cavity at a distance of $h / 2$ from the radiator ground-plane. The magnitude of the $S_{11}$ parameters are here reported, being the antenna covered by the same cavities given in Section 4.2.1, with rods closest to the patch both orthogonal (Figs. 4.46-4.55) and parallel (Figs. 4.56-4.65) to the electric field.

Results show a particular behaviour of the return loss. Some $h / 2$ distances give a regular frequency plot for the $\left|S_{11}\right|$ parameter, with a peaked matching tipically lower than -12 dB . They alternate to $h / 2$ distances showing an uneven and large band-width responce, with values of return loss higher to the one of the patch antenna without cover.

## Cavities with most internal rods orthogonal to the electric field



Figure 4.26: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=11 \mathrm{~mm}$.


Figure 4.27: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=26 \mathrm{~mm}$.


Figure 4.28: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=40 \mathrm{~mm}$.


Figure 4.29: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=55 \mathrm{~mm}$.


Figure 4.30: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=70 \mathrm{~mm}$.


Figure 4.31: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=84 \mathrm{~mm}$.


Figure 4.32: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=99 \mathrm{~mm}$.


Figure 4.33: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=113 \mathrm{~mm}$.


Figure 4.34: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=128 \mathrm{~mm}$.


Figure 4.35: Measured return loss of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=142 \mathrm{~mm}$.

Cavities with most internal rods parallel to the electric field


Figure 4.36: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=16 \mathrm{~mm}$.


Figure 4.37: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=31 \mathrm{~mm}$.


Figure 4.38: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=45 \mathrm{~mm}$.


Figure 4.39: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=60 \mathrm{~mm}$.


Figure 4.40: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=75 \mathrm{~mm}$.


Figure 4.41: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=90 \mathrm{~mm}$.


Figure 4.42: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=104 \mathrm{~mm}$.


Figure 4.43: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=119 \mathrm{~mm}$.


Figure 4.44: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=133 \mathrm{~mm}$.


Figure 4.45: Measured return loss of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=148 \mathrm{~mm}$.

### 4.2.3 Measurements of gain

Gain has been measured on the same woodpile-covered antennas considered in Section 4.2.2. The best antenna layouts, with lowest as possible reflection coefficient and highest as possible gain, are looked for.

The anechoic chamber has been calibrated with the patch antenna in transmission and the horn in reception. The gain of the woodpile-covered patch antenna, $G_{p w}$, normalized to the gain of the patch alone, $G_{p w}$, is plotted for the same layouts given in Section 4.2.2,

Employing cavities with rods closest to the patch which are parallel to the excitation electric field, a considerable gain-enhancement due to the woodpile is observed at 10.3 GHz , when the equivalent distance between the woodpile and the patch is roughly equal to an even integer multiple of $\lambda / 2$. Instead, when the abovementioned equivalent distance is an odd integer multiple of $\lambda / 2$, at 10.3 GHz the effect of the woodpile is a strong reduction of the gain. In fact, only the modes which are zero at the location of the ground plane exhist when the woodpile is employed as antenna superstrate, and they correspond to integer even multiples of $\lambda / 2$, i.e., integer multiples of the wavelength $\lambda$. Modes which have maxima at the ground
plane, i.e. odd integer multiples of $\lambda / 2$, cannot propagate, because the condition of tangential electric field vanishing on the perfect conductor is not satisfied. Similar results are collected for the configurations with the rods nearest to the patch parallel or orthogonal to the electric field. Moreover, it can be noticed that the highest gain ehnancements are with larger cavities. For example, cavities with $h / 2 \cong 45 \mathrm{~mm}$ give satisfactory results: in the $h / 2=45 \mathrm{~mm}$ case (rods nearest to the patch parallel to the electric field), the gain enhancement is 10.17 dB ; for the $h / 2=42 \mathrm{~mm}$ radiator rods nearest to the patch orthogonal to the electric field), a gain enhancement of 9.44 dB has been measured.

## Cavities with most internal rods orthogonal to the electric field



Figure 4.46: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=11 \mathrm{~mm}$.


Figure 4.47: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=26 \mathrm{~mm}$.


Figure 4.48: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=40 \mathrm{~mm}$.


Figure 4.49: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=55 \mathrm{~mm}$.


Figure 4.50: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=70 \mathrm{~mm}$.


Figure 4.51: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=84 \mathrm{~mm}$.


Figure 4.52: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=99 \mathrm{~mm}$.


Figure 4.53: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=113 \mathrm{~mm}$.


Figure 4.54: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=128 \mathrm{~mm}$.


Figure 4.55: Gain of the patch antenna with woodpile cavity having most internal rods orthogonal to the electric field, and $h=142 \mathrm{~mm}$.

Cavities with most internal rods parallel to the electric field


Figure 4.56: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=16 \mathrm{~mm}$.


Figure 4.57: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=31 \mathrm{~mm}$.


Figure 4.58: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=45 \mathrm{~mm}$.


Figure 4.59: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=60 \mathrm{~mm}$.


Figure 4.60: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=75 \mathrm{~mm}$.


Figure 4.61: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=90 \mathrm{~mm}$.


Figure 4.62: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=104 \mathrm{~mm}$.


Figure 4.63: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=119 \mathrm{~mm}$.


Figure 4.64: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=133 \mathrm{~mm}$.


Figure 4.65: Gain of the patch antenna with woodpile cavity having most internal rods parallel to the electric field, and $h=148 \mathrm{~mm}$.

### 4.2.4 Radiation patterns

According to the results discussed in the previous Sections 4.2.2 and 4.2.3, two layouts have been chosen for measurements on radiation pattern. Measurements have been carried out in the anechoic chamber, with the woodpile-covered patch antenna placed on a rotating support. The radiation pattern has been measured in a range of $180^{\circ}$, with a step of $2^{\circ}$, in two orthogonal planes.

The layout with rods nearest to the patch orthogonal to the electric field has been measured in the case $h / 2=42 \mathrm{~mm}$. Radiation patterns of the woodpile-covered patch antenna in two orthogonal planes, normalized to gain maximum-value, are reported in Fig. 4.66, and compared the radiation pattern measured on the patch antenna without woodpile. In the same plots, simulated results with HFSS, for three values of alumina permittivity, i.e. 8.4, 9, and 9.8, are given. The effect of the cavity is strongly evident, with a gain enhancement of 9.44 dB . The measured half-power beam-width (HPBW) is $18^{\circ}$ in the E-plane and $14^{\circ}$ in the H-plane; the SLL is -7.55 dB in the E-plane, -13.19 dB in the H-plane. The pattern is more directive in the H-plane, with the main lobe much more pronounced than lateral lobes. In the E-plane, the pattern is strongly asymmetric, with high lateral lobes.

The same effect of directivity enhancement can be appreciated in the layout with rods nearest to the patch parallel to the electric field, and $h / 2=45 \mathrm{~mm}$. Radiation patterns are given in Fig. 4.67, being now the gain enhancement of 10.17 dB . The HPBW is $12^{\circ}$ in the E-plane, $14^{\circ}$ in the H-plane; the side-lobe level (SLL) is - 9.84 dB in the E-plane, -13.15 dB in the H-plane. In the H-plane, the radiation diagram is almost symmetric, with a main lobe much more pronounced than lateral lobes. In the E-plane, the pattern is still directive, but with higher lateral lobes.


Figure 4.66: Measured and simulated results for a patch antenna covered by a woodpile cavity, with $h / 2=42 \mathrm{~mm}$.


Figure 4.67: Measured and simulated results for a patch antenna covered by a woodpile cavity, with $h / 2=45 \mathrm{~mm}$.

## Appendix A

## Reflection and transmission in a dielectric layer

The expressions of reflections and transmission coefficients employed for fields (1.7), (1.11), (1.16), (1.19) are given.

## A. 1 TM polarization

In TM polarization, the scalar function $V(\xi, \zeta)$ corresponds to the field $\mathrm{E}_{\mathrm{y}}$. We recall the total field $V(\xi, \zeta)$, relevant to reflection and transmission of the incident plane-wave, in each medium of the single-layer structure

- Medium $0 \quad V_{0} e^{i\left(n_{\perp}^{\mathrm{i}} \xi+n_{\|}^{\mathrm{i}} \zeta\right)}+V_{\mathrm{r}} e^{i\left(-n_{\perp}^{\mathrm{i}} \xi+n_{\|}^{\mathrm{i}} \zeta\right)}$
- Medium $1 \quad V_{\mathrm{t} 1} e^{i n_{1}\left[n_{\perp}^{\mathrm{t} 1}(\xi-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]}+V_{\mathrm{r} 1} e^{i n_{1}\left[-n_{\perp}^{\mathrm{t}}(\xi-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]}$
- Medium $2 \quad V_{\mathrm{t} 2} e^{i n_{2}\left[n_{\perp}^{\mathrm{t}} 2(\xi-\Lambda)+n_{\|}^{\mathrm{t} 2} \zeta\right]}$

The H-field is derived by the E-field by the normalized curl

$$
\begin{equation*}
\mathbf{H}=\frac{k}{i \omega \mu} \tilde{\nabla} \times \mathbf{E}=\frac{1}{i Z} \tilde{\nabla} \times \mathbf{E} \tag{A.2}
\end{equation*}
$$

For the E-field, we impose boundary conditions on the planar interface $\xi=0$

$$
\begin{equation*}
V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}+V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}=V_{\mathrm{t} 1} e^{i n_{1}\left[n_{\perp}^{\mathrm{t} 1}(-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]}+V_{\mathrm{r} 1} e^{i n_{1}\left[-n_{\perp}^{\mathrm{t} 1}(-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]} \tag{A.3}
\end{equation*}
$$

and $\xi=\Lambda$

$$
\begin{equation*}
V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t1}} \zeta}+V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}=V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{2} \zeta} \tag{A.4}
\end{equation*}
$$

From (A.2), we have the following boundary conditions for H -field on the planar interface $\xi=0$

$$
\begin{equation*}
i n_{\perp}^{\mathrm{i}}\left(V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}-V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}\right)=i n_{1} n_{\perp}^{\mathrm{t} 1}\left\{V_{\mathrm{t} 1} e^{i n_{1}\left[n_{\perp}^{\mathrm{t1}}(-\Lambda)+n_{\|}^{\mathrm{t1}} \zeta\right]}-V_{\mathrm{r}} e^{i n_{1}\left[-n_{\perp}^{\mathrm{t1}}(-\Lambda)+n_{\|}^{\mathrm{t1}} \zeta\right]}\right\} \tag{A.5}
\end{equation*}
$$

and $\xi=\Lambda$

$$
\begin{equation*}
i n_{1} n_{\perp}^{\mathrm{t} 1}\left(V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}-V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t1}} \zeta}\right)=i n_{2} n_{\perp}^{\mathrm{t} 2}\left(V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{\mathrm{t} 2} \zeta}\right) \tag{A.6}
\end{equation*}
$$

Boundary conditions (A.5) and (A.6) can be written as

$$
\begin{gather*}
V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}-V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}=\frac{n_{1} n_{\perp}^{\mathrm{t} 1}}{n_{\perp}^{\mathrm{i}}}\left[V_{\mathrm{t} 1} e^{i n_{1}\left(-n_{\perp}^{\mathrm{t} 1} \Lambda+n_{\|}^{\mathrm{t} 1} \zeta\right)}-V_{\mathrm{r} 1} e^{i n_{1}\left(n_{\perp}^{\mathrm{t} 1} \Lambda+n_{\|}^{\mathrm{t1}} \zeta\right)}\right]  \tag{A.7}\\
V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}-V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}=\frac{n_{2} n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t}}} V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{\mathrm{t} 2} \zeta} \tag{A.8}
\end{gather*}
$$

We consider the linear system made by equations (A.12) and (A.7), i.e. by the boundary conditions for both E - and H -field at the interface on $\xi=\Lambda$, removing the exponential in $\zeta$

$$
\left\{\begin{array}{l}
V_{\mathrm{t} 1}+V_{\mathrm{r} 1}=V_{\mathrm{t} 2}  \tag{A.9}\\
V_{\mathrm{t} 1}-V_{\mathrm{r} 1}=\frac{n_{2} n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t} 1}} V_{\mathrm{t} 2}
\end{array}\right.
$$

Solving linear system (A.9) we get the reflection coefficient

$$
\begin{equation*}
\Gamma_{21}=\frac{V_{\mathrm{r} 1}}{V_{\mathrm{t} 1}}=\frac{n_{1} n_{1}^{\mathrm{t} 1}-n_{2} n_{ \pm}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t1}}+n_{2} n_{\perp}^{2}} \tag{A.10}
\end{equation*}
$$

and the transmission coefficient

$$
\begin{equation*}
\mathrm{T}_{21}=\frac{V_{\mathrm{t} 2}}{V_{\mathrm{t} 1}}=\frac{2 n_{1} n_{\perp}^{\mathrm{t} 1}}{n_{1} n_{\perp}^{\mathrm{t1}}+n_{2} n_{\perp}^{\mathrm{t} 2}} \tag{A.11}
\end{equation*}
$$

at the planar interface in $\xi=\Lambda$. Once known the coefficients (A.10) and (A.11), in $\xi=\Lambda$, the linear system formed by equations (A.12) and (A.7) can be solved

$$
\left\{\begin{array}{l}
V_{\mathrm{i}}+V_{\mathrm{r}}=V_{\mathrm{t} 1} e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}-V_{\mathrm{r} 1} e^{i n_{1} n_{\perp}^{\mathrm{t1}} \Lambda}  \tag{A.12}\\
V_{\mathrm{i}}+V_{\mathrm{r}}=\frac{n_{1} n_{\perp}^{\mathrm{t} 1}}{n_{\perp}^{\mathrm{i}}}\left(V_{\mathrm{t} 1} e^{-i n_{1} n_{\perp}^{\mathrm{t1}} \Lambda}-V_{\mathrm{r} 1} e^{i n_{1} n_{\perp}^{\mathrm{t1}} \Lambda}\right)
\end{array}\right.
$$

where the exponential in $\zeta$ has been removed.
We get the reflection coefficient

$$
\begin{equation*}
\Gamma_{01}=\frac{V_{r}}{V_{\mathrm{i}}^{\mathrm{i}}}=\frac{e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1+\frac{n_{2} n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t1}}}\right)\left(1-\frac{n_{1} n_{\perp}^{\mathrm{t}}}{n_{\perp}^{\mathrm{i}}}\right)-e^{i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1-\frac{n_{2} n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{t1}}}\right)\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t} 1}}{n_{\perp}^{\mathrm{i}}}\right)}{e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1+\frac{n_{2} n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{t1}}}\right)\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t}}}{n_{\perp}^{\mathrm{i}}}\right)+e^{i n_{1} n_{\perp}^{\mathrm{t}} \Lambda}\left(1-\frac{n_{2} n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{t1}}}\right)\left(1-\frac{n_{1} n_{\perp}^{\mathrm{t}}}{n_{\perp}^{\mathrm{i}}}\right)} \tag{А.13}
\end{equation*}
$$

and the transmission coefficient

$$
\begin{equation*}
\mathrm{T}_{01}=\frac{V_{\mathrm{t}}}{V_{\mathrm{i}}}=\frac{2\left(1+\frac{n_{2} n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t}}}\right)}{e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1+\frac{n_{2} n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{t1}}}\right)\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t1}}}{n_{\perp}^{\mathrm{i}}}\right)+e^{i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1-\frac{n_{2} n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{t}}}\right)\left(1-\frac{n_{1} n_{\perp}^{\mathrm{t1}}}{n_{\perp}^{\mathrm{i}}}\right)} \tag{А.14}
\end{equation*}
$$

at the planar interface in $\xi=0$.

## A. 2 TE polarization

In TE polarization, the scalar function $V(\xi, \zeta)$ corresponds to the field $\mathrm{H}_{\mathrm{y}}$, given by A.1.

The E-field is derived by the H -field through the normalized curl

$$
\begin{equation*}
\mathbf{E}=\frac{i k}{\omega \varepsilon} \tilde{\nabla} \times \mathbf{H}=i Z \tilde{\nabla} \times \mathbf{H} \tag{A.15}
\end{equation*}
$$

For the H -field, boundary condition on the planar interface $\xi=0$ is

$$
\begin{equation*}
V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}+V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}=V_{\mathrm{t} 1} e^{i n_{1}\left[n_{\perp}^{\mathrm{t} 1}(-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]}+V_{\mathrm{r} 1} e^{i n_{1}\left[-n_{\perp}^{\mathrm{t} 1}(-\Lambda)+n_{\|}^{\mathrm{t} 1} \zeta\right]} \tag{A.16}
\end{equation*}
$$

and in $\xi=\Lambda$

$$
\begin{equation*}
V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}+V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t}} \zeta}=V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{\mathrm{t} 2} \zeta} \tag{A.17}
\end{equation*}
$$

From (A.1), we have the following boundary conditions for H -field on the planar interface $\xi=0$

$$
\begin{equation*}
\frac{i k_{0}}{\omega \epsilon_{0}}\left[i n_{\perp}^{\mathrm{i}}\left(V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}-V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}\right)\right]=\frac{i k_{0}}{\omega \epsilon_{1}}\left[i n_{1} n_{\perp}^{\mathrm{t} 1}\left[V_{\mathrm{t} 1} e^{i n_{1}\left(-n_{\perp}^{\mathrm{t1}} \Lambda+n_{\|}^{\mathrm{t} 1} \zeta\right)}-V_{\mathrm{r} 1} e^{i n_{1}\left(n_{\perp}^{\mathrm{t1}} \Lambda+n_{\|}^{\mathrm{t1}} \zeta\right)}\right]\right. \tag{A.18}
\end{equation*}
$$

and $\xi=\Lambda$

$$
\begin{equation*}
\frac{i k_{0}}{\omega \epsilon_{0}}\left[i n_{1} n_{\perp}^{\mathrm{t} 1}\left(V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}-V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}\right)\right]=\frac{i k_{0}}{\omega \epsilon_{1}}\left[i n_{2} n_{\perp}^{\mathrm{t} 2}\left(V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{\mathrm{t} 2} \zeta}\right)\right] \tag{A.19}
\end{equation*}
$$

Boundary conditions (A.18) and (A.19) can be written as

$$
\begin{gather*}
V_{0} e^{i n_{\|}^{\mathrm{i}} \zeta}-V_{\mathrm{r}} e^{i n_{\|}^{\mathrm{i}} \zeta}=\frac{n_{\perp}^{\mathrm{t} 1}}{n_{1} n_{\perp}^{\mathrm{i}}}\left[V_{\mathrm{t} 1} e^{i n_{1}\left(-n_{\perp}^{\mathrm{t} 1} \Lambda+n_{\|}^{\mathrm{t1} 1} \zeta\right)}-V_{\mathrm{r} 1} e^{i n_{1}\left(n_{\perp}^{\mathrm{t1}} \Lambda+n_{\|}^{\mathrm{t} 1} \zeta\right)}\right]  \tag{A.20}\\
V_{\mathrm{t} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}-V_{\mathrm{r} 1} e^{i n_{1} n_{\|}^{\mathrm{t} 1} \zeta}=\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t} 1}} V_{\mathrm{t} 2} e^{i n_{2} n_{\|}^{\mathrm{t} 2} \zeta} \tag{A.21}
\end{gather*}
$$

We consider the linear system made by equations (A.25) and (A.20), i.e. by the boundary conditions for both E - and H -field at the interface on $\xi=\Lambda$; removing the exponential in $\zeta$

$$
\left\{\begin{array}{l}
V_{\mathrm{t} 1}+V_{\mathrm{r} 1}=V_{\mathrm{t} 2}  \tag{A.22}\\
V_{\mathrm{t} 1}-V_{\mathrm{r} 1}=\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{+1}} V_{\mathrm{t} 2}
\end{array}\right.
$$

Solving linear system (A.22) we get the reflection coefficient

$$
\begin{equation*}
\Gamma_{21}=\frac{V_{\mathrm{r} 1}}{V_{\mathrm{t} 1}}=\frac{n_{2} n_{\perp}^{\mathrm{t} 1}-n_{1} n_{ \pm}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t} 1}+n_{1} n_{\perp}{ }^{2}} \tag{A.23}
\end{equation*}
$$

and the transmission coefficient

$$
\begin{equation*}
\mathrm{T}_{21}=\frac{V_{\mathrm{t} 2}}{V_{\mathrm{t} 1}}=\frac{2 n_{2} n_{\perp}^{\mathrm{t} 1}}{n_{2} n_{\perp}^{\mathrm{t1}}+n_{1} n_{\perp}^{\mathrm{t} 2}} \tag{A.24}
\end{equation*}
$$

at the planar interface in $\xi=\Lambda$. Once known coefficients (A.23) and (A.24), in $\xi=\Lambda$, the linear system formed by equations (A.25) and (A.20) can be solved

$$
\left\{\begin{array}{l}
V_{\mathrm{i}}+V_{\mathrm{r}}=V_{\mathrm{t} 1} e^{-i n_{1} n_{\perp}^{\mathrm{t1}} \Lambda}-V_{\mathrm{r} 1} e^{i i_{1} n_{\perp}^{\mathrm{t1}} \Lambda}  \tag{A.25}\\
V_{\mathrm{i}}+V_{\mathrm{r}}=\frac{n_{\perp}^{\mathrm{t} 1}}{n_{1} n_{\perp}^{\mathrm{i}}}\left(V_{\mathrm{t} 1} e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}-V_{\mathrm{r} 1} e^{i n_{1} n_{\perp}^{\mathrm{t1} \Lambda} \Lambda}\right)
\end{array}\right.
$$

where the exponential in $\zeta$ has been removed.
We get the reflection coefficient

$$
\begin{equation*}
\Gamma_{01}=\frac{V_{\mathrm{r}}}{V_{\mathrm{i}}}=\frac{e^{-i n_{1} n_{\perp}^{\mathrm{t}} \Lambda}\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t1}}}\right)\left(1-\frac{n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)-e^{i n_{1} n_{\perp}^{\mathrm{t}} \Lambda}\left(1-\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t1}}}\right)\left(1+\frac{n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)}{e^{-i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t2}}}{n_{2} n_{\perp}^{\mathrm{t1}}}\right)\left(1+\frac{n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)+e^{i n_{1} n_{\perp}^{\mathrm{t} 1} \Lambda}\left(1-\frac{n_{1} n_{\perp}^{\mathrm{t2}}}{n_{2} n_{\perp}^{\mathrm{t1}}}\right)\left(1-\frac{n_{\perp}^{\mathrm{t}}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)} \tag{А̄.26}
\end{equation*}
$$

and the transmission coefficient

$$
\begin{equation*}
\mathrm{T}_{01}=\frac{V_{\mathrm{t} 1}}{V_{\mathrm{i}}}=\frac{2\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t}}}\right)}{e^{-i n_{1} n_{\perp}^{\mathrm{t1}} \Lambda}\left(1+\frac{n_{1} n_{\perp}^{\mathrm{t} 2}}{n_{2} n_{\perp}^{\mathrm{t}}}\right)\left(1-\frac{n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)+e^{i n_{1} n_{\perp}^{\mathrm{t2}} \Lambda}\left(1-\frac{n_{2} n_{\perp}^{\mathrm{t1}}}{n_{1} n_{\perp}^{\mathrm{t}}}\right)\left(1+\frac{n_{\perp}^{\mathrm{t} 2}}{n_{1} n_{\perp}^{\mathrm{i}}}\right)} \tag{A}
\end{equation*}
$$

at the planar interface in $\xi=0$.

## Conclusions

In the first part of this thesis, the two-dimensional plane-wave scattering problem by a set of perfectly-conducting circular cylinders buried in a dielectric slab has been solved, through the CWA. Reflected-transmitted and multiple-reflected cylindrical functions have been introduced, to consider the cylinder-cylinder interaction and the multiple reflections between cylinders and interfaces. The theoretical approach has been also extended to the cases of dielectric cylindrical scatterers.

A numerical implementation in a Fortran code has been carried out, which can deal with both TM and TE polarization cases and yields results in both the nearand far-field zones. Moreover, it may be applied to any value of the cylinder size and of the distance between obstacles and planar interfaces.

The method has been tested by means of comparisons with the literature and self-consistency tests; moreover, convergence checks have been also performed.

In the reported results, the application of the method to characterization of some scenarios of practical interest has been examined. In the case of perfectlyconducting cylinders, typical geometries with buried utilities have been shown. As regards dielectric targets, buried air-filled cavity has been simulated, which is a typical application of civil engineering.

The theoretical analysis has been extended to the cylindrical wave emitted by a line current as excitation source of the problem. In this case, solution has been given in case of perfectly-conducting cylinders buried in a semi-infinite medium. This extension has been performed in view to achieving a more accurate characterization of the scenarios investigated by GPR.

Future work may regard the study of scattering by a line source with cylinders buried in a layered medium. Moreover, an extension of the method to scattering of a plane wave by objects buried beneath a rough surface is presently considered. Future works may also regard the generalization to scatterers of arbitrary cross-section, and to the presence of lossy media.

In the second part of this thesis, the study of EBGs and their application to
directivity-enhancement of planar antennas have been dealt with.
In particular, a woodpile structure, i.e., an EBG with three dimensional periodicity, has been considered. Properties of cavities made up of two identical woodpile layers have been investigated and implemented to form woodpile-covered antennas. The interaction between woodpile cavities and antennas has been studied in the cases of a double-slot antenna and a microstrip patch.

After this preliminary study, two woodpile prototypes have been fabricated. Measurements have been performed to characterize the transmission properties through the woodpile layers when assembled to form a cavity. Next, they have been employed as superstrates of a patch antenna. Gain measurements on the new radiator have shown an actual directivity enhancement up to 10 dB , in comparison with the patch antenna.

As future developments of this work, further measurements will be performed on the antenna in order to have a full understanding of its interaction with the woodpile cavity. Moreover, simpler EBG structures will be considered in order to have easiness of fabrication and lower costs.

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