UNIVERSITÀ DEGLI STUDI "ROMA TRE"
Facoltà di Scienze Matematiche, Fisiche e Naturali

Tesi di Dottorato di Ricerca in Matematica

Degenerations and applications: polynomial interpolation and secant degree

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To my family
Alla mia famiglia

Acknowledgements

First I express my sincerest gratitude to my supervisor, Professor Ciro Ciliberto, who has supported me thoughout my thesis with his patience and knowledge, allowing me the room to work in my own way.
I am also very grateful to the group of Trento, with a particular thanks to Dr. Roberto Pignatelli and Prof. Marco Andreatta that have advised me in my important choices. In particular I thank them for encouraging me to make this beautiful and illuminating experience in Rome. I am very thankful to Dr. Claudio Fontanari for his moral support and precious suggestions.
Voglio esprimere la mia gratitudine a tutta la mia famiglia e in particolare ai miei genitori per il supporto morale e finanziario durante tutti i miei anni di studio, prima a Trento e poi a Roma: senza il loro aiuto questa tesi non esisterebbe.
A tutti i miei amici trentini per essermi stati vicini nella distanza, durante questi ultimi anni e per lo svago che mi hanno regalato durante le mie "fughe da Roma", Barbara, Fabrizia, Gianna, Ilaria, Maria, Mario, Martina, Michele, Tatiana... grazie ragazzi!
Un grazie di cuore a tutti i miei colleghi di Roma Tre (e non solo) per l'affetto che mi hanno saputo dimostrare, nei momenti gioiosi e in quelli bui del dottorato.
Il ringraziamento più sentito va a Massimo, insostituibile, che con amore e incommensurabile pazienza mi è stato accanto durante questi tre lunghi anni, come sempre regalandomi i momenti più dolci e le parole più rassicuranti.
Finally, I would like to thank the institutions that gave me all the financial supports I needed during these years: the Mathematics Departments of the University of Roma Tre, the INdAM institute and the Opera Universitaria of Trento.

Contents

Introduction

The polynomial interpolation problem is a subject that has been widely studied for centuries. The classical interpolation theory in one variable says that a polynomial f of a given degree d over a field K is uniquely determined by the values it assumes at $d+1$ distinct points of the affine line. The generality of the points is not necessary, the only requirement is that they are distinct. This is nothing else than *Ruffini Theorem* and it is based on the fact that the *Vandermonde* determinant is not zero.
A first generalization is asking not only for values of the polynomial, but also for values of its derivatives. These are linear problems in the vector space of polynomial of degree d. Moreover this can be generalized at the case of more than one variable. A polynomial f of degree at most d in r variables depends on $\binom{r+d}{r}$ parameters. Let $p_1, \ldots, p_n$ be points in the affine r-dimensional space and $m_1, \ldots, m_n$ positive integers. We can impose the vanishing at p_i of the partial derivatives of f up to order $m_i - 1$ or, in other words, that the point p_i has multiplicity at least m_i for the hypersurface $f = 0$. If one chooses integers $r, d, m_1, \ldots, m_n$ such that

$$\sum_{i=1}^{n} \binom{m_i + r - 1}{r} = \binom{r+d}{r},$$

i.e. such that the number of conditions imposed equals the number of parameters on which the polynomials depend, one may ask: is $f \equiv 0$? There is so far no general answer to the question.
The interpolation problem can be reformulated in a different setting. Fix $p_1, \ldots, p_n$ distinct points in $\mathbb{P}^r$ and fix $m_1, \ldots, m_n$ positive integers. Define $\mathscr{L}_{r,d}$ to be the linear system of hypersurfaces of $\mathbb{P}^r$ of degree d and consider

$$\mathscr{L} := \mathscr{L}_{r,d}(-\sum_{i=1}^{n} m_i p_i)$$

the sub-linear system of those divisors of $\mathscr{L}_{r,d}$ having multiplicity at least m_i at p_i, $i = 1, \ldots, n$. A divisor in $\mathscr{L}_{r,d}$ has equation $f = 0$; the assumption of having multiplicity m_i at p_i is translated into the vanishing of all the derivatives of order $m_i - 1$ of f, whose number is $\binom{r+m_i-1}{r}$. Hence the *virtual dimension* of $\mathscr{L}$ is defined to be

$$\text{virt-dim}(\mathscr{L}) := \binom{r+d}{r} - 1 - \sum_{i=1}^{n} \binom{r+m_i-1}{r},$$

i.e. the number of parameters minus the number of conditions. The *expected dimension* is defined to be

$$\text{exp-dim}(\mathscr{L}) := \max\{\text{virt-dim}(\mathscr{L}), -1\}.$$

If the conditions imposed by the assigned points are not linearly independent, the actual dimension of $\mathscr{L}$ is greater than the expected one: in that case we say that $\mathscr{L}$ is *special*. Otherwise, if the actual and the expected dimension coincide, we say that $\mathscr{L}$ is *non-special*.
Let Z be a scheme of lenght $\sum_i \binom{r+m_i-1}{r}$ given by n fat points. We have the following restriction sequence

$$0 \to \mathscr{L} = \mathscr{L}_{r,d}(m_1, \ldots, m_n) \to \mathscr{L}_{r,d} \to \mathscr{L}_{r,d|Z}.$$

In cohomology we get

$$0 \to H^0(\mathbb{P}^r, \mathscr{L}) \to H^0(\mathbb{P}^r, \mathscr{L}_{r,d}) \to H^0(Z, \mathscr{L}_{r,d|Z}) \to H^1(\mathbb{P}^r, \mathscr{L}) \to 0,$$

being $h^1(\mathbb{P}^r, \mathscr{L}_{r,d}) = 0$. Thus $\mathscr{L}$ is non-special if and only if

$$h^0(\mathbb{P}^r, \mathscr{L}) \cdot h^1(\mathbb{P}^r, \mathscr{L}) = 0.$$

The *dimensionality problem* consists in investigating, given a linear system $\mathscr{L}$, if it is non-special. It coincides with the original polynomial interpolation problem and it is open in general.
It is very important to observe that the dimensionality problem depends on the position of the points in $\mathbb{P}^r$. For example consider the linear system of cubics of $\mathbb{P}^2$ with five base points lying on a line L. The expected dimension is four. However, if a cubic curve vanishes at four points, then by *Bezout's Theorem* it must vanish along the whole line L, so the condition imposed by the fifth point is not linearly independent from the conditions imposed by the first four points. For this reason it is convenient to assume that the points $p_1, \ldots, p_n$ are sufficiently general. On the other hand, the dimension of $\mathscr{L}$ is *upper-semicontinuous* in the position of the points $p_1, \ldots, p_n \in \mathbb{P}^r$; it achieves its minimum value when they are in *general position*. In this case we use the following notation for the corresponding linear system:

$$\mathscr{L} = \mathscr{L}_{r,d}(m_1, \ldots, m_n).$$

For $r = 1$, the system $\mathscr{L}_{1,d}(m_1, \ldots, n_n)$ is always non-special. Furthermore, if all the points have multiplicity one, i.e. $m_1 = \cdots = m_n = 1$, the system $\mathscr{L}_{r,d}(1^n)$ is also non-special, see Theorem 1. However, the problem becomes more and more complicated in more variables and higher multiplicities, namely if $r \geq 2$ and $m_1, \ldots, m_n \geq 2$. What is known is essentially concentrated in Theroem 2, a result due to J. Alexander and A. Hirschowitz. They classify the special cases for $r \geq 2$ and $m_1 = \cdots = m_n = 2$. This theorem has an equivalent formulation in terms of higher secant varieties of Veronese embedding of projective spaces (see Theorem 3).
A natural approach to the dimensionality problem of linear systems is via *degenerations*. Degenerations allow us to move the multiple base points of the linear system in special position, arguing with a semicontinuity argument. More precisely, if one finds a specialization of the points, which is good in the sense that the corresponding limit linear system $\mathscr{L}_0$ is non-special, then also the original one is non-special. Computing the limit linear system is in general delicate. A. Hirchowitz in [25] elaborated a degeneration technique, which he called *la méthode d'Horace*, consisting in making iterated specializations of as many points as convenient on a fixed hyperplane and then applying induction on the dimension and on the degree. To be more explicit, let $\mathscr{L} := \mathscr{L}_{r,d}(2^n)$ be the linear system of hypersurfaces of $\mathbb{P}^r$ of degree d singular at a collection of n general points ($m_1 = \cdots = m_n = 2$); the main idea of Hirschowitz was to suppose that h of the n points have support on a fixed hyperplane $\pi \subseteq \mathbb{P}^r$; hence one gets the so called *Castelnuovo exact sequence*:

$$0 \to \mathscr{L}_{r,d-1}(2^{n-h}, 1^h) \to \mathscr{L} \to \mathscr{L}_{r-1,d}(2^h),$$

where the h base points of the kernel system are the residual of the h double points specialized on π. Thus, arguing by induction, if the two external systems are non-special with virtual dimension at least -1, which means that one does not lose any condition in this restriction procedure, i.e. $h^1(\mathbb{P}^r, \mathscr{L}_{r,d-1}(2^{n-h}, 1^h)) = h^1(\pi, \mathscr{L}_{r-1,d}(2^n)) = 0$, then the system $\mathscr{L}$ is non-special too. Unfortunately, this methode does not cover all the possible situations. A refined version, the so called *méthode d'Horace différentielle*, gives a general solution. The original proof, of about a hundred pages proposed by J. Alexander and A. Hirschowitz, is contained in [1]-[3]-[4]-[5] and simplified in [6].

In 2002, K. Chandler presented an easier proof of the Alexander-Hirschowitz Interpolation Theorem (Theorem 2) for $d \geq 4$ in [11]. She still uses the Horace's method, but exploiting subsequent specializations of part of the double base points of the linear system to a hyperplane π. In this way the system degree decreases by one and induction can be applied. In the case with degree three, this method does not work because specializing to hyperplanes one must deal with quadrics that are special. Another problem with cubics is that each of the lines joining pairs of points lies in the base locus of the linear system, hence the standard approach can fail because these lines meet π. K. Chandler transformed the obstruction caused by the presence of lines in the base locus of cubic linear systems on an advantage and completed the proof of Theorem 2, see [12], solving also the case of cubics. The innovation was to specialize some of the points onto a subspace L of codimension 2 and pairs of points on hyperplanes containing L.

A recent improvement of this argument is due to M. C. Brambilla and G. Ottaviani. In a beautiful paper ([8]) they offer a shorter proof of Theorem 2 in the case of $d \geq 4$ and propose a new and simpler degeneration argument in the cubic case. Their argument is similar to that of K. Chandler, but it is more effective. Their main idea is to choose a subspace L of codimension three, instead of two, on which they specialize the points. This choice really simplifies the arithmetic side of the problem.

C. Ciliberto and R. Miranda in [19] and [20] used a different degeneration construction, originally proposed by Z. Ran, see [31], to prove Theorem 2 in the planar case. This approach consists essentially in degenerating the plane to a reducible surface, with two components intersecting in a line, and simultaneously degenerating the linear system $\mathscr{L} = \mathscr{L}_{2,d}(2^n)$ to a linear system $\mathscr{L}_0$ obtained as fibered product of linear systems on the two components over the restricted system on their intersection. The limit linear system $\mathscr{L}_0$ is somewhat easier than the original, in particular this degeneration argument allows to use induction either on the degree or on the number of imposed multiple points. This contruction provides a recursive formula for the dimension of $\mathscr{L}_0$ involving the dimensions of the systems on the two components.

In the first part of this thesis we generalize this approach to the case with $r \geq 3$ and we complete the proof of Theorem 2 with this method, exploiting induction on both d and r. In Section 2.1 we explain our approach which generalizes the one of C. Ciliberto and R. Miranda. It consists in blowing up a point $p \in \mathbb{P}^r$ and twisting by an appropriate negative multiple of the exceptional divisor, obtaining a reducible central fiber which is the union of the exceptional divisor $\mathbb{P}$ and the strict transform $\mathbb{F}$ of the blowing up of $\mathbb{P}^r$ at p in the central fiber of a trivial family $\mathbb{P}^r \times \Delta$ over a disc Δ, with a linear system $\mathscr{L}$ such that $\mathscr{L}$ restricts to $\mathcal{O}_{\mathbb{P}^r}(d)$ on any fiber. The two components intersect along a $(r-1)$-dimensional variety R that is isomorphic to $\mathbb{P}^{r-1}$. Then we specialize some nodes on $\mathbb{F}$ and the remaining on $\mathbb{P}$ and we study the corresponding limit linear systems. This argument does not suffice to cover all the cases, because of an arithmetic obstruction similar to the one that M. C. Brambilla and G.Ottaviani met. Our idea is to perform furher degenerations in order to handle these cases; the interested reader can find the details of the

contructions in Section 2.2 and in Section 2.3.
A tricky point of this approach is the study of the transversality of the restrictions of the systems on the intersection of the two components. In the planar case, C. Ciliberto and R. Miranda proved it using the finitness of the set of inflection points of linear systems on $\mathbb{P}^1$ ([19], Proposition 3.1). In higher dimension transversality is more complicated. In Section 2.1.4, 2.3 and in 3.3 we present our approach to this problem: if at least one of the two restricted systems is a complete linear system, then we are able to compute by hand the dimension of their intersection. Anyhow, this is not sufficient to finish the proof of Theorem 2. For istance, it does not work in the cubic cases. The solution to this obstacle is to blow up a codimension three subspace L of $\mathbb{P}^r$, instead of a point. This approach to the cubic case is not so different from the one of M. C. Brambilla and G. Ottaviani; we propose it in Section 3.1.1.
Also the quartic case must be analysed separately. Indeed, twisting by a negative multiple of the exceptional component $\mathbb{P}$ of the central fiber, we get degree two either in the linear system $\mathscr{L}_{\mathbb{P}}$ on $\mathbb{P}$ or in the kernel system of the restriction map of $\mathscr{L}_{\mathbb{P}}$ to the intersection R. We show Theorem 2 for quartics in Section 3.1.2 by induction on r, with a very geometric argument that exploits the property of cubics of containing all the lines trought two distinct double points.
In Chapter 3 we apply all the techniques described in Chapter 2 and we complete our proof of Theorem 2, for $r \geq 3$. Before considering the higher dimensional case, we analyse in details the linear systems $\mathscr{L}_{3,d}(2^n)$ of surfaces of $\mathbb{P}^3$ with a general collection of double points, in order to make our work as clear as possible to the reader.

Our construction besides its intrinsic intent (on the way we prove nonspeciality of some interesting systems, see Theorem 16) gives hope for further extensions to greater multiplicities.

Let $X \subseteq \mathbb{P}^r$ be a projective, irreducible variety of dimension n. Its *k-secant variety* $\mathrm{Sec}_k(X)$ is defined to be the closure of the union of all the $\mathbb{P}^k$'s in $\mathbb{P}^r$ meeting X in $k+1$ independent points. The general question is the following: if $\mathrm{Sec}_k(X)$ has the expected dimension $(k+1)n+k$, what is the number $\nu_k(X)$ of $(k+1)$-secant $\mathbb{P}^k$'s to X intersecting a general subspace of codimension $(k+1)n+k$ in $\mathbb{P}^r$? Note that

$$\nu_k(X) = \deg(\mathrm{Sec}_k(X)) \cdot \mu_k(X),$$

where $\mu_k(X)$ is defined to be the number of $(k+1)$-secant $\mathbb{P}^k$'s passing through the general point of $\mathrm{Sec}_k(X)$.
There is a long tradition within algebraic geometry that studies the dimension and the degree of k-secant varieties. This is a problem which is unsolved in

general. In the second part of our thesis we describe some partial results obtained with a degeneration approach for projective toric surfaces, in the cases $k = 1$ and $k = 2$. These results can be regarded as the beginning of a similar study for the k-secant varieties of toric surfaces for $k \geq 3$ and, in higher dimension, for $k \geq 1$.

The outline for the second part of this thesis is the following. In Chapter 4 we introduce the objects of our study: toric varieties, toric ideals and toric degenerations. A convex lattice polytope P in $\mathbb{R}^n$ defines a toric variety X_P of dimension n endowed with an ample line bundle and therefore a morphism in $\mathbb{P}^r$, where $r+1$ equals the number of reticular points of P. Some familiar examples are Segre-Veronese embeddings, rational normal scrolls and Del Pezzo surfaces of degree $6, 7, 8$; all of them, and some further examples, have ideals which are generated by quadrics (see Section 5.2); more precisely, the ideal $\mathcal{I}_X$ of X is given by the 2×2 minors of a suitable matrix A. The ideals of the secant variety and of the higher secant varieties of X are strictly related to $\mathcal{I}_X$, in fact the *k-secant ideal* $\mathcal{I}_X^{\{k\}}$ (see section 5.2 for a formal definition) of the variety $\mathrm{Sec}_k(X)$ of the $(k+1)$-secant $\mathbb{P}^k$'s to X, for $k \geq 1$, is generated by the $(k+2) \times (k+2)$ minors of A or, in the scroll case, of another matrix. Nevertheless, in general, there is only a little understanding of these questions. We exploit the knowledge of the defining ideals of these varieties and of their k-secant varieties to approach the computation of the number ν_k for any projective toric surface with $\dim \mathrm{Sec}_k(X) = 3k + 2$, for $k = 1, 2$, with a combinatorial approach via degenerations.
A toric degeneration of X_P is defined by a regular subdivision D of P in sub-polytopes $P_1 \dots P_l$ of dimension n such that $P_i \cap P_j$ is a common face of P_i and P_j (perhaps the empty face), such that

$$\bigcup_{i=1}^{l} P_i = P$$

and, furthermore, such that there exists an integral function F defined over P, which is piecewise linear over the sub-polytopes of D and strictly convex over P. The central fiber X_0 has l irreducible components that are the projective toric varieties defined by the P_i's.
Our approach to the problem of computing the number ν_k is the one of C. Ciliberto, O. Dumitrescu and R. Miranda in [16] that is very close to that of B. Sturmfels and B. Sullivant in [32]. In particular, if $n = 2$, we use *planar* toric degeneration, i.e. we subdivide P in triangles having normalized area equal to one. The ideal $\mathcal{I}_0$ of the central fiber is the monomial initial ideal with respect to a suitable term order $\prec$ which corresponds to the triangulation (see [33]):

$$\mathcal{I}_0 = \mathrm{in}_{\prec}(\mathcal{I}_X)$$

In Chapter 5 we define the k-secant varieties with particular attention to the problem of the computation of the k-secant degree in the toric case; we also introduce the notion of a k-delightful toric degeneration of a toric variety. The basic setup was suggested by B. Sturmfels and B. Sillivant in [32]. In particular they proved that if there exists at least one *skew* $(k+1)$-*sets*, i.e. a subset of $(k+1)$ triangles of D that are pairwise disjoint, then the k-secant variety of X has the expected dimension. Moreover the number of such skew $(k+1)$-sets is a lower bound to the number $\nu_k(X)$:

$$\nu_k(X) \geq \nu_{k+1}(D),$$

see Theorem 34. A planar toric degeneration D for which equality holds is said to be k-*delightful* (according to [32] and [16]).

In Chapter 6 we apply these techniques and we expose our results. We study non-k-delightful cases and we give a partial explanation to the lack of k-delightfulness, improving the lower bound for ν_k given by the number of skew $(k+1)$-sets, for $k=1$ and $k=2$. The main tool is keeping into account the singularities of the configuration D. Our original result is Theorem 35: suppose that p is a reticular point in D such that the union of the triangles having a vertex in p form a convex sub-polytope Q_p of P; for $k \in \{1,2\}$ we exploit the knowledge of the toric surface Z_p defined by Q_p (cfr. Table 6.1 and Table 6.2) to prove that the number $\nu_k(Z_p)$ contributes to $\nu_k(X)$, under the hypothesis that $\dim(\mathrm{Sec}_k(Z_p)) = \dim(\mathrm{Sec}(X)) = 3k+2$. Moreover, if in D there are more than one lattice point, $\{p_i\}_{i\in I}$, such that Q_{p_i} is convex and such that the Q_{p_i}'s are pairwise not overlapping, i.e. $\dim(Q_{p_i} \cap Q_{p_j}) < 2$, then the contributions of such singularities do not interfere to each other and all of them contribute to $\nu_k(X)$.
The *non-overlapping* hypothesis is restrictive, in fact it prevents us from considering all the singularities of D. Indeed the reticular distance between two lattice points p_i and p_j must be at least two, otherwise the corresponding subpolytopes will have intersection of dimension 2. We conjecture that the non-overlapping hypothesis may be removed and that all contributions may sum up to $\nu_{k+1}(D)$ in the computation of $\nu_k(X)$, $k=1,2$ (Section 6.2).

Part I

On the Alexander-Hirschowitz Theorem

Chapter 1

Preliminaries on the interpolation problem

We will work over an algebraically closed field K of characteristic zero. We will denote by $\mathbb{P}^r$ the r-dimensional projective space over K.

1.1 Linear systems of hypersurfaces of $\mathbb{P}^r$

Consider the linear system $\mathscr{L} := \mathscr{L}_{r,d}(m_1^{h_1}, \ldots, m_l^{h_l})$ consisting of hypersurfaces of $\mathbb{P}^r$ of degree d with h_i general points of multiplicity at least m_i, for $i = 1 \ldots, l$. This system is said to be *homogeneous* if all the m_i's are equal. The homogeneous polynomial of degree d in $r+1$ variables form a projective space of dimension

$$\binom{r+d}{r} - 1;$$

$$\binom{r+d}{r} - 1 \tag{1.1}$$

moreover, for a polynomial to have multiplicity at least m_i at a point $p_i \in \mathbb{P}^r$ corresponds to

$$\binom{m_i - 1 + r}{r}$$

linear conditions imposed on the coefficients.

Definition 1. *The* virtual dimension *of $\mathscr{L}$ is defined as*

$$v(\mathscr{L}) = v_{r,d}(m_1^{h_1}, \ldots, m_l^{h_l}) := \binom{r+d}{r} - 1 - \sum_{i=1}^{l} h_i \binom{m_i - 1 + r}{r}.$$

The actual (projective) dimension of the linear system is at least -1, and this is verified when the system is empty.

Definition 2. *The* expected dimension *of $\mathscr{L}$ is defined to be*

$$e(\mathscr{L}) = e_{r,d}(m_1^{h_1}, \dots, m_l^{h_l}) := max\{-1, v_{r,d}(m_1^{h_1}, \dots, m_l^{h_l})\}.$$

As the points vary in $\mathbb{P}^r$, the dimension of the system is upper semicontinuous: there exists a Zariski open set in the parameter space of $(\sum h_i)$-tuples of points where the dimension of the linear system achieves its minimum value; we call it the *(general) dimension* of $\mathscr{L}_{r,d}(m_1^{h_1}, \dots, m_l^{h_l})$ and we denote it by $l_{r,d}(m_1^{h_1}, \dots, m_l^{h_l})$. We have

$$l_{r,d}(m_1^{h_1}, \dots, m_l^{h_l}) \geq e_{r,d}(m_1^{h_1}, \dots, m_l^{h_l}); \tag{1.2}$$

and equality implies the all the conditions imposed by the general points are linearly independent.

Definition 3. *The linear system $\mathscr{L}$ is said to be* non-special *if equality holds in (1.2). Otherwise it is said to be* special.

A linear system $\mathscr{L}$ on $\mathbb{P}^r$ is non-special if and only if

$$h^0(\mathbb{P}^r, \mathscr{L}) = h^1(\mathbb{P}^r, \mathscr{L}) = 0.$$

1.2 The Alexander-Hirschowitz Theorem

The general question is to compute the dimension of linear systems (see the Introduction for a historical remark). In the multiplicity one case, i.e. for the simple interpolation problem, there are no surprises; all such systems have the expected dimension.

Theorem 1 (Multiplicity One Theorem)**.** *If n simple points are in general position in $\mathbb{P}^r$, then the system $\mathscr{L}_{r,d}(1^n)$ is non-special.*

Proof. We prove the claim by induction on the number n of simple points. If $n = 1$, there exists a hypersurface of $\mathbb{P}^r$ of degree d not passing through a general point. For the inductive step we have to prove that an additional general point imposes, on a not empty linear system, a linear condition independent from the previous ones, or, equivalentely, that an additional general point does not lie on every hypersurface of the system. This is surely true being the point in general position. $\square$

In the cases of higher multiplicities, the problem of computing the dimension is still unsolved in general.
For the multiplicity two case, what is known is the following theorem, due to J. Alexander and A. Hirschowitz, that classifies all special systems.

Theorem 2 (Alexander-Hirschowitz). *The linear system $\mathscr{L}_{r,d}(2^n)$ is non-special except in the following cases:*

r	$\forall$	2	3	4	4
d	2	4	4	4	3
n	$\leq r$	5	9	14	7

This theorem has an equivalent reformulation in terms of higher secant varieties of Veronese embeddings. Let $X \subseteq \mathbb{P}^N$ be a projective variety. The *k-secant variety* $\mathrm{Sec}_k(X)$ of X is defined to be the Zariski closure of the union of all the linear spans of $p_1, \ldots, p_{k+1}$ independent points of X (cfr. Section 5.1). We have, counting parameters, that

$$\dim(\mathrm{Sec}_k(X)) \leq \min\{(k+1)n + k, N\} =: \text{exp-dim}(\mathrm{Sec}_k(X)).$$

The variety X is said to be *k-defective* if strict inequality holds; it is said to be *k-non-defective* if equality holds.
The *Veronese variety $V_{r,d}$* is defined to be the image of the Veronese embedding $\nu_{r,d}$ of degree d of $\mathbb{P}^r$ in the projective space of dimension $\binom{r+d}{r} - 1$.

Theorem 3. *The $(n-1)$-secant variety of the Veronese $V_{r,d}$ is non-defective, with the same list of exceptions of Theorem 2.*

A hypersurface of $\mathbb{P}^r$ of degree d corresponds via the Veronese embedding $\nu_{r,d}$ to a hyperplane section of $V_{r,d}$. Moreover, a hyperplane of $\mathbb{P}^r$ has a double point at $p \in \mathbb{P}^r$ if and only if the corresponding hyperplane of $\mathbb{P}^{\binom{r+d}{r}-1}$ is tangent to $V_{r,d}$ at $\nu_{r,d}(p)$. Now, fix $p_1, \ldots, p_n$ general points in $\mathbb{P}^r$ and consider the linear system $\mathscr{L}_{r,d}(2^n)$. It corresponds to the linear system of hyperplanes $\mathbb{P}^{\binom{r+d}{r}-1}$ tangent to $V_{r,d}$ at $\nu_{r,d}(p_1), \ldots, \nu_{r,d}(p_n)$. This linear system has as base locus the general tangent space to $\mathrm{Sec}_{n-1}(V_{r,d})$.

Lemma 4 (Terracini's Lemma). *Let $X \subseteq \mathbb{P}^N$ be an irreducible, non-degenerate, projective variety of dimension r. Let $p_1, \ldots, p_n$ general points of X, with $n \leq N+1$. Then*

$$T_{Sec_{n-1}(X),p} =< T_{X,p_1}, \ldots, T_{X,p_n} >,$$

where $p \in < p_1 \ldots, p_n >$ is a general point in $Sec_{n-1}(X)$.

This proves the equivalence between Theorem 2 and Theorem 3.

Our aim is to propose a proof of Theorem 2, for $r \geq 3$, generalizing the degeneration tecniques introduced in [19] by C. Ciliberto and R. Miranda for the planar case.

1.2.1 The special cases

In this section we briefly describe the special cases of Theorem 2.

Quadrics of $\mathbb{P}^r$

All linear systems of quadric hypersurfaces of $\mathbb{P}^r$ with at most r nodes are in the list of special cases of Theorem 2.
The system $\mathscr{L} = \mathscr{L}_{r,d}(2^2)$ consists of quadric cones with vertex containing the double line through the two double points, so

$$\dim(\mathscr{L}) = \binom{r}{2} - 1 > e(\mathscr{L}) = \max\{-1, \binom{r+2}{r} - 1 - 2(r+1)\},$$

and $h^0(\mathbb{P}^r, \mathscr{L}) = \binom{r}{2}, h^1(\mathbb{P}^r, \mathscr{L}) = 1$.
The system $\mathscr{L} = \mathscr{L}_{r,2}(2^r)$ contains only the double hyperplane of $\mathbb{P}^r$ determined by the r points:

$$\dim(\mathscr{L}) = 0 > e(\mathscr{L}) = -1,$$

and $h^0(\mathbb{P}^r, \mathscr{L}) = 1, h^1(\mathbb{P}^r, \mathscr{L}) = \binom{r}{2}$.
An analogous description is available for n general points, with $2 < n < r$: the linear system

$$\mathscr{L} = \mathscr{L}_{r,2}(2^n)$$

consists of quadric cones with vertex containing the double $(n-1)$-dimensional linear subspace of $\mathbb{P}^r$ determined by the n points: hence

$$\dim(\mathscr{L}) = \binom{r-n+2}{2} - 1 > e(\mathscr{L}),$$

$h^0(\mathbb{P}^r, \mathscr{L}) = \binom{r-n+2}{2}, h^1(\mathbb{P}^r, \mathscr{L}) = \binom{n}{2}$.
The system $\mathscr{L}_{r,2}(2^n)$, with $n \geq r+1$, are empty, and in particular non-special.

Quartics in $\mathbb{P}^r$, with $r = 2, 3, 4$

For $r = 2, 3, 4$, let $n = \binom{r+2}{2} - 1$. The linear system $\mathscr{L}_{r,4}(2^n)$ is expected to be empty. Nevertheless it is special because there is a quartic singular at the given points, i.e. the double quadric through them. Indeed the linear system $\mathscr{L}_{r,2}(1^n)$ is non special by Theorem 1 and it has dimension 0.

Cubics of $\mathbb{P}^4$

Through a general collection of seven points in $\mathbb{P}^4$ there exists a quartic rational normal curve described by the 2×2 minors of

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix},$$

in some system of coordinates. Its secant variety is the cubic surface with equation

$$\det \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix} = 0;$$

it is singular along the whole curve and in particular at the seven points. Thus $\mathscr{L}_{4,3}(2^7)$ is special, having virtual dimension equal to -1.

The important remark that we must do is that for each special system $\mathscr{L}_{r,d}(2^n)$, the general section of it is singular along a positive dimensional variety containing the double base points, as we have described above. This goes back to Terracini (see [35]), but has been more recentely proved by C. Ciliberto and A. Hirschowitz in [17].

Lemma 5. *Let X be a projective variety and $p_1, \ldots, p_n \in X$ general points. Let $\mathscr{L}$ be any a linear system on X. If $\mathscr{L}(-2\sum_{i=1}^n p_i)$ is special, then every section of $\mathscr{L}(-2\sum_{i=1}^n p_i)$ is singular along a positive dimensional variety on which $p_1, \ldots, p_n$ are supported.*

Chapter 2

The degeneration inductive approach

Definition 4. *A* 1-dimensional degeneration *is a morphism $\pi_2 : \mathcal{X} \to \Delta$, where Δ is a complex disc centered at the origin, $\mathcal{X}$ is a smooth $(r+1)$-fold and π_2 is a proper and flat map.*
For every $t \in \Delta$ we will denote the fiber of π_2 over t by $X_t = {\pi_2}^{-1}(t)$.

In a 1-dimensional degeneration of varieties of dimension r all the fibers have dimension r, while the family $\mathcal{X}$ has dimension $r+1$.

The reason to use degeneration is to exploit semicontinuity. If one can prove that a property is satisfied in the central fiber, i.e. the degenerate object, then one can obtain an inequality about the general fiber, i.e. the degenerating object. In our cases, we will study non-speciality of a given linear system $\mathscr{L}_t$ on the general fiber. Semicontinuity will give us the following inequality

$$\dim(\mathscr{L}_0) \geq \dim(\mathscr{L}_t),$$

where $\mathscr{L}_0$ is the limiting system. In this chapter we will see how this successfully gives us informations about the dimension of $\mathscr{L}_t$.

2.1 The first degeneration of linear systems

We will generalize to higher dimension the degeneration technique introduced by C. Ciliberto and R. Miranda in [19] for homogeneous planar linear systems, that essentially consists in using a degeneration worked out by Z. Ran in [31]. More precisely, we will degenerate $\mathbb{P}^r$ to a reducible variety and we will study how a linear system on the general fiber degenerates. The limiting system will be easier than the general one, and this will enable us to use induction.
Let Δ be a complex disc with center at the origin and consider the product

$\mathcal{V} = \mathbb{P}^r \times \Delta$ with the natural projections $p_1 : \mathcal{V} \to \mathbb{P}^r$ and $p_2 : \mathcal{V} \to \Delta$; let $V_t = \mathbb{P}^r \times \{t\}$ be the fiber of p_2 over $t \in \Delta$. Take a point $(p, 0)$ in the central fiber V_0 and blow it up to obtain a new $(r+1)$-fold $\mathcal{X}$ with the maps

- $f : \mathcal{X} \to \mathcal{V}$,
- $\pi_1 = p_1 \circ f : \mathcal{X} \to \mathbb{P}^r$ and
- $\pi_2 = p_2 \circ f : \mathcal{X} \to \Delta$.

We have the following commutative diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"'] \arrow[rr, bend left, "\pi_1"] \arrow[dr, "\pi_2"'] & \mathcal{V} \arrow[r, "p_1"'] \arrow[d, "p_2"] & \mathbb{P}^r \\ & \Delta & \end{tikzcd}$$

The so obtained flat morphism $\pi_2 : \mathcal{X} \to \Delta$, with fiber $X_t = \pi_2^{-1}(t)$, $t \in \Delta$, produces a 1-dimensional degeneration of $\mathbb{P}^r$. If $t \neq 0$ then $X_t = V_t$ is a $\mathbb{P}^r$, while for $t = 0$ the fiber X_0 is the union of the strict transform $\mathbb{F}$ of V_0 and the exceptional divisor $\mathbb{P} \cong \mathbb{P}^r$ of the blow-up. The two varieties $\mathbb{P}$ and $\mathbb{F}$ meet transversally along a $(r-1)$-dimensional variety R which is isomorphic to $\mathbb{P}^{r-1}$: it represents a hyperplane on $\mathbb{P}$ and the exceptional divisor on $\mathbb{F}$.
The Picard group of X_0 is the fibered product of $\mathrm{Pic}(\mathbb{P})$ and $\mathrm{Pic}(\mathbb{F})$ over $\mathrm{Pic}(R)$. The Picard group of $\mathbb{P}$ is generated by $\mathcal{O}(1)$, while the Picard group of $\mathbb{F}$ is generated by the hyperplane class H and the class E of the exceptional divisor. A line bundle $\mathcal{N}$ on X_0 corresponds to two line bundles $\mathcal{N}_{\mathbb{F}}$ and $\mathcal{N}_{\mathbb{P}}$, respectively on $\mathbb{F}$ and on $\mathbb{P}$, which agree on the intersection R. i.e. two line bundles of the form

$$\mathcal{N}_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(\sigma), \text{ and } \mathcal{N}_{\mathbb{F}} = \mathcal{O}_{\mathbb{F}}(\tau H - \sigma E),$$

for some σ and τ.
We degenerate the linear system $\mathcal{O}(d)$ on the general fiber of π_2 as follows. Take the line bundle

$$\mathcal{O}_{\mathcal{X}}(d) = \pi_1^*(\mathcal{O}_{\mathbb{P}^r}(d)) :$$

its restriction on the general fiber $X_t \cong \mathbb{P}^r$ is isomorphic to $\mathcal{O}_{\mathbb{P}^r}(d)$, while on the central fiber the restrictions to $\mathbb{P}$ and $\mathbb{F}$ are $\mathcal{O}_{\mathbb{P}}$ and $\mathcal{O}_{\mathbb{F}}(dH)$ respectively. Now let us execute a twist by the bundle $\mathcal{O}_{\mathcal{X}}(-(d-k)\mathbb{P})$: the restriction to X_t is still the same, while the restrictions to $\mathbb{P}$ and $\mathbb{F}$ become

$$\mathcal{O}_{\mathbb{P}}(d-k) \text{ and } \mathcal{O}_{\mathbb{F}}(dH - (d-k)E);$$

the resulting line bundle on X_0 is a flat limit of the bundle $\mathcal{O}_{\mathbb{P}^r}(d)$ on the general fiber. Such a limit is not unique.

We now consider the homogeneus linear system $\mathcal{L}_t := \mathcal{L} = \mathcal{L}_{r,d}(2^n)$ of the hypersurfaces of $\mathbb{P}^r$ of degree d with n assigned general points $p_{1,t}, \ldots, p_{n,t}$ of equal multiplicity $m = 2$. Recall that it has virtual dimension

$$v(\mathcal{L}_t) = \binom{r+d}{r} - 1 - n(r+1).$$

Fix a non-negative integer $b \leq n$ and specialize b points generically on $\mathbb{F}$ and the other $n-b$ points generically on $\mathbb{P}$: i.e. consider a flat family $\{p_{1,t} \ldots, p_{n,t}\}_{t\in\Delta}$ such that $p_{1,0}, \ldots, p_{b,0} \in \mathbb{F}$ and $p_{b+1,0}, \ldots, p_{n,0} \in \mathbb{P}$; we consider these points as limit of n general points in $X_t \cong \mathbb{P}^r$, for $t \to 0$.
The limiting linear system $\mathcal{L}_0$ on X_0 is formed by the divisors in the flat limit of the bundle $\mathcal{O}_{\mathbb{P}^r}(d)$ on the general fiber X_t, having multiplicity 2 at $p_{1,0}, \ldots, p_{n,0}$. This system restricts to $\mathbb{F}$ and to $\mathbb{P}$ to the following systems:

$$\mathcal{L}_{\mathbb{P}} = \mathcal{L}_{r,d-k}(2^{n-b}) \text{ and } \mathcal{L}_{\mathbb{F}} = \mathcal{L}_{r,d}(d-k, 2^b),$$

where the point of multiplicity $d-k$ is the point $p \in V_0 \cong \mathbb{P}^r$ which we blew up to obtain $\mathbb{F}$; we view $\mathbb{F}$ as a $\mathbb{P}^r$ blown up at a point and the corresponding line bundle as a linear system of the same form as the others we are considering.

Definition 5. *We say that the limit linear system $\mathcal{L}_0$ is obtained from $\mathcal{L}$ by a (k, b)-degeneration.*

At the level of vector spaces, the system $\mathcal{L}_0$ is the fibered product of $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ over the restricted system on R which is $\mathcal{O}_R(d-k)$: we have

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W_{\mathbb{P}} \arrow[d] \\ W_{\mathbb{F}} \arrow[r] & H^0(R, \mathcal{O}_R(d-k)) \end{tikzcd}$$

where $W_{\mathbb{P}}$ and $W_{\mathbb{F}}$ are the vector spaces from which one obtains the systems $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ as projectivizations, and

$$W = W_{\mathbb{P}} \times_{H^0(R,\mathcal{O}_R(d-k))} W_{\mathbb{F}}$$

is the fibered product of vector spaces which gives $\mathcal{L}_0$ as its projectivization.
An element of $\mathcal{L}$ consists either of a divisor $S_{\mathbb{P}}$ on $\mathbb{P}$ and a divisor $S_{\mathbb{F}}$ on $\mathbb{F}$, both satisfing the conditions imposed by the multiple points, which restrict to the same divisor on R, or it is a divisor corresponding to a section of the bundle which is identically zero on $\mathbb{P}$ (or on $\mathbb{F}$) and which gives a general divisor in $\mathcal{L}_{\mathbb{F}}$ (or in $\mathcal{L}_{\mathbb{P}}$ respectively) containing R as a component.
If we denote by l_0 the dimension of $\mathcal{L}_0$ on X_0, we have, by upper semicontinuity, that l_0 is at least the dimension of the linear system on the general fiber:

$$l_0 \geq \dim(\mathcal{L}_t) = l_{r,d}(2^n).$$

Lemma 6. *In the above notation, if $l_0 = e_{r,d}(2^n)$, then the linear system $\mathscr{L}$ has the expected dimension, i.e. it is non-special.*

Let us consider the restriction exact sequences to $R \cong \mathbb{P}^{r-1} \subset \mathbb{P}^r$:

$$0 \to \hat{\mathscr{L}}_{\mathbb{P}} \to \mathscr{L}_{\mathbb{P}} \to \mathscr{R}_{\mathbb{P}} \subseteq |\mathcal{O}_{\mathbb{P}^{r-1}}(d-k)|$$

$$0 \to \hat{\mathscr{L}}_{\mathbb{F}} \to \mathscr{L}_{\mathbb{F}} \to \mathscr{R}_{\mathbb{F}} \subseteq |\mathcal{O}_{\mathbb{P}^{r-1}}(d-k)|,$$

where $\mathscr{R}_{\mathbb{P}}$, $\mathscr{R}_{\mathbb{F}}$ denote the restrictions of the systems $\mathscr{L}_{\mathbb{P}}$, $\mathscr{L}_{\mathbb{F}}$ to R and $\hat{\mathscr{L}}_{\mathbb{P}}$, $\hat{\mathscr{L}}_{\mathbb{F}}$ denote the kernel systems:

$$\hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{r,d-k-1}(2^{n-b}) \text{ and } \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{r,d}(d-k+1, 2^b).$$

The kernel $\hat{\mathscr{L}}_{\mathbb{P}}$ consists of those sections of $\mathscr{L}_{\mathbb{P}}$ which vanish identically on R, i.e. the divisors in $\mathscr{L}_{\mathbb{P}}$ containing $R \cong \mathbb{P}^{r-1}$ as component; the same holds for $\hat{\mathscr{L}}_{\mathbb{F}}$.
We denote by $v_{\mathbb{P}}$, $v_{\mathbb{F}}$, $\hat{v}_{\mathbb{P}}$, $\hat{v}_{\mathbb{F}}$ and by $l_{\mathbb{P}}$, $l_{\mathbb{F}}$, $\hat{l}_{\mathbb{P}}$, $\hat{l}_{\mathbb{F}}$ the virtual and the actual dimensions of the various linear systems. We have the following identities:

$$r_{\mathbb{P}} := \dim(\mathscr{R}_{\mathbb{P}}) = l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1,$$

$$r_{\mathbb{F}} := \dim(\mathscr{R}_{\mathbb{F}}) = l_{\mathbb{F}} - \hat{l}_{\mathbb{F}} - 1.$$

We want to compute the dimension l_0 by recursion. The simplest cases occurs when all the divisors in $\mathscr{L}_0$ come from a section which is identically zero on one of the two components: in those cases the matching sections of the other system must lie in the kernel of the restriction map.

Lemma 7. *In the above notation, fixed n, d and b, we have that:*

- *if the system $\mathscr{L}_{\mathbb{P}} = \emptyset$, then $l_0 = \hat{l}_{\mathbb{F}}$;*
- *if the system $\mathscr{L}_{\mathbb{F}} = \emptyset$, then $l_0 = \hat{l}_{\mathbb{P}}$.*

If, on the contrary, the divisors on $\mathscr{L}_0$ consist of a divisor on $\mathbb{P}$ and a divisor on $\mathbb{F}$, both not identically zero, which match on R, then the dimension of $\mathscr{L}_0$ depends on the dimension of the intersection

$$\mathscr{R} := \mathscr{R}_{\mathbb{P}} \cap \mathscr{R}_{\mathbb{F}}$$

of the restricted systems, being $\mathscr{L}_0$ obtained as fibered product. An element of W is a pair $(s_{\mathbb{P}}, s_{\mathbb{F}}) \in W_{\mathbb{P}} \times W_{\mathbb{F}}$ such that the restrictions of $s_{\mathbb{P}}$ and $s_{\mathbb{F}}$ coincide on R:

$$W = \{(s_{\mathbb{P}}, s_{\mathbb{F}}) : s_{\mathbb{P}}|_R = s_{\mathbb{F}}|_R\}.$$

In other words, if W_R is the vector space which corresponds to $\mathscr{R}$, namely $\mathbb{P}(W_R) = \mathscr{R}$, then an element of W is obtained by taking an element s_R of

W_R and choosing pre-images $s_{\mathbb{P}}$ and $s_{\mathbb{F}}$ of such an element in $W_{\mathbb{P}}$ and $W_{\mathbb{F}}$: the choice of s_R depends on $\dim(\mathscr{R})+1$ parameters, and then the choice of $s_{\mathbb{P}}$ and $s_{\mathbb{F}}$ depends on $\hat{l}_{\mathbb{P}}+1$ and $\hat{l}_{\mathbb{F}}+1$ parameters respectively. Thus $\dim(W)=\dim(\mathscr{R})+\hat{l}_{\mathbb{P}}+\hat{l}_{\mathbb{F}}+3$ and, projectively,

$$l_0=\dim(\mathscr{R})+\hat{l}_{\mathbb{P}}+\hat{l}_{\mathbb{F}}+2. \tag{2.1}$$

The crucial point is to compute the dimension of $\mathscr{R}$, from which one obtains l_0. If the systems $\mathscr{R}_{\mathbb{P}},\mathscr{R}_{\mathbb{F}}\subset|\mathcal{O}_{\mathbb{P}^{r-1}}(d-k)|$ are *transversal*, i.e. if they intersect properly inside $|\mathcal{O}_{\mathbb{P}^{r-1}}(d-k)|$, one can simply apply the *Grassmann formula* for the dimension of the intersection $\mathscr{R}$:

$$\dim(\mathscr{R})=\begin{cases}-1 & \text{if } r_{\mathbb{P}}+r_{\mathbb{F}}\leq\binom{d+r-1-k}{r-1}-2\\ r_{\mathbb{P}}+r_{\mathbb{F}}-\binom{d+r-1-k}{r-1}+1 & \text{if } r_{\mathbb{P}}+r_{\mathbb{F}}\geq\binom{d+r-1-k}{r-1}-2\end{cases}$$

Notice that transversality holds if at least one between $\mathscr{L}_{\mathbb{P}}$ and $\mathscr{L}_{\mathbb{F}}$ cuts the complete series on R.

2.1.1 Linear systems with virtual dimension $v\geq-1$

In this section we will see how, under some hypothesis, a $(1,b)$-degeneration can be used to prove non-speciality of a given linear system $\mathscr{L}=\mathscr{L}_{r,d}(2^n)$, $d\geq 4$, with n such that the virtual dimension of $\mathscr{L}$ is at least -1, using the recursive formula (2.1). If the system on the central fiber turns out to be non-special, then, by semicontinuity, also our system $\mathscr{L}$ is non-special.

Proposition 8. *Suppose that there exists an integer b, with $0<b<n$ such that:*

1. *the restricted systems $\mathscr{R}_{\mathbb{F}}$ and $\mathscr{R}_{\mathbb{P}}$ are transversal in $|\mathcal{O}_{\mathbb{P}^{r-1}}(d-1)|$;*
2. *$r_{\mathbb{P}}+r_{\mathbb{F}}\geq\binom{r+d-2}{r-1}-2$ (or, equivalently, $\hat{l}_{\mathbb{P}}+\hat{l}_{\mathbb{F}}\leq v-1$);*
3. *the systems $\mathscr{L}_{\mathbb{P}}$ and $\mathscr{L}_{\mathbb{F}}$ are non special with $v_{\mathbb{P}},v_{\mathbb{F}}\geq-1$.*

Then $\mathscr{L}=\mathscr{L}_{r,d}(2^n)$ is non-special with virtual dimension at least -1.

Proof. If $\mathscr{R}_{\mathbb{P}}$ and $\mathscr{R}_{\mathbb{F}}$ are transversal and if the second condition holds, then the $\mathscr{R}=\mathscr{R}_{\mathbb{P}}\cap\mathscr{R}_{\mathbb{F}}$ has dimension

$$r=r_{\mathbb{P}}+r_{\mathbb{F}}-\binom{r+d-2}{r-1}+1$$

Moreover if the third condition holds, we get

$$\begin{aligned} l_0 &= \left(r_{\mathbb{P}} + r_{\mathbb{F}} - \binom{r+d-2}{r-1} + 1 \right) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 \\ &= l_{\mathbb{P}} + l_{\mathbb{F}} - \binom{r+d-2}{r-1} + 1 \\ &= v_{\mathbb{P}} + v_{\mathbb{F}} - \binom{r+d-2}{r-1} + 1 \\ &= v. \end{aligned}$$

□

2.1.2 Linear systems expected to be empty: $v \leq -1$

In this section we will explain how, performing $(1,b)$-degenerations of $\mathbb{P}^r$ as above, we can prove that $\mathscr{L} = \mathscr{L}_{r,d}(2^n)$, with $d \geq 4$ and virtual dimension $v \leq -1$, is empty.

Proposition 9. *Suppose that there exists an integer b, with $0 < b < n$ such that:*

1. *the kernel systems $\hat{\mathscr{L}}_{\mathbb{P}}$ and $\hat{\mathscr{L}}_{\mathbb{F}}$ are empty;*
2. *the restricted systems $\mathscr{R}_{\mathbb{P}}$ and $\mathscr{R}_{\mathbb{F}}$ do not intersect.*

Then the system $\mathscr{L}$ is empty and therefore non-special.

Proof. We have

$$l_0 = \dim(\mathscr{R}_{\mathbb{P}} \cap \mathscr{R}_{\mathbb{F}}) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 = -1.$$

So $\mathscr{L}$ is empty as expected. □

2.1.3 Some useful lemmas

For what it concerns the analysis of the linear system on $\mathbb{P}$ and the relative kernel system, we can exploit induction on d because they are linear systems of hypersurfaces of lower degree with nodes. Actually this is the reason for performing $(1,b)$-degenerations. However, in general the systems $\mathscr{L}_{\mathbb{F}}$ and $\hat{\mathscr{L}}_{\mathbb{F}}$ are unknown because of the presence of a point of greater multiplicity in their base locus. This section is devoted to the study of such linear systems.

Let us begin with some preliminary results. Consider the linear system of hypersurfaces of $\mathbb{P}^r$ of degree d; each element of this system is described by the vanishing of a homogeneous polynomial $f(x_0,\dots,x_r)$ of degree d in the $x_0,\dots,x_r$'s. We can write:

$$f(x_0,\dots,x_r) = \sum_{i=0}^{d} x_r^i f_{d-i}(x_0,\dots,x_{r-1}),$$

where the f_i's are homogeneous polynomials of degree $d-i$. Let p be the point $p=[0,\ldots,1]\in\mathbb{P}^r$; notice that the hypersurfaces of $\mathbb{P}^r$ of degree d having multiplicity m at p can be written as follows

$$f(x_0,\ldots,x_r)=\sum_{i=0}^{d-m}x_r^i f_{d-i}(x_0,\ldots,x_{r-1}), \tag{2.2}$$

in fact all partial derivatives of f up to order $m-1$ must vanish at p.

Lemma 10. *The linear system $\mathscr{L}_{r,d}(d,2^b)$ is either special of dimension $l_{r-1,d}(2^b)$, or it is empty.*

Proof. A hypersurface of $\mathbb{P}^r$ of degree d having multiplicity d in $p=[0,\ldots,0,1]$ is defined by the vanishing of a homogeneous polynomial of degree d in the $x_0,\ldots,x_{r-1}$'s:

$$f(x_0,\ldots,x_r)=f_d(x_0,\ldots,x_{r-1}).$$

In other words, the linear system $\mathscr{L}_{r,d}(d)$ consists of the cones of degree d with vertex at the point p; as vector space it has dimension

$$h^0(\mathbb{P}^r,\mathscr{L}_{r,d}(d))=\binom{r+d-1}{r-1}=h^0(\mathbb{P}^{r-1},\mathscr{L}_{r-1,d}).$$

This means that, fixed the vertex p, we have to choose hypersurfaces of degree d in a general hyperplane $\{x_r=0\}\cong\mathbb{P}^{r-1}\subseteq\mathbb{P}^r$. Now, let $p_1,\ldots,p_b\neq p$ general points of $\mathbb{P}^r$ and let and $p_1',\ldots,p_b'$ the projections from p to the hyperplane. The conditions imposed on f by b general double points $p_1,\ldots,p_b$ of $\mathbb{P}^r$ are

$$\partial_{x_i}f(p_j),\ i=0,\ldots,r,\ j=1,\ldots,b,$$

i.e.

$$\partial_{x_i}f_d(p_j'),\ i=0,\ldots,r-1,\ j=1,\ldots,b.$$

Thus the number of independent conditions imposed by the nodes $p_1,\ldots,p_b$ on $\mathscr{L}_{r,d}(d)$ is equal to the number of independend conditions imposed on $\mathscr{L}_{r-1,d}$ by $p_1',\ldots,p_b'$, which are general in $\mathbb{P}^{r-1}$. So we get

$$h^1(\mathbb{P}^r,\mathscr{L}_{r,d}(d,2^b))=b+h^1(\mathbb{P}^{r-1},\mathscr{L}_{r-1,d}(2^b))$$
$$h^0(\mathbb{P}^r,\mathscr{L}_{r,d}(d,2^b))=h^0(\mathbb{P}^{r-1},\mathscr{L}_{r-1,d}(2^b)),$$

and this concludes the proof. □

Now, let $p_1,\ldots,p_b\neq p$ be general point of $\mathbb{P}^r$ with homogeneous coordinates $p_j=[p_{j,0},\ldots,p_{j,r-1},p_{j,r}]$. A general divisor of $\mathscr{L}_{\mathbb{F}}$ is given by

$$\begin{aligned} f(x_0,\ldots,x_r) &= f_d(x_0,\ldots,x_{r-1})+x_r f_{d-1}(x_0,\ldots,x_{r-1})\\ &= \textstyle\sum_{i+j+k=d}a_{ijk}x_0^i x_1^j x_2^k+x_r\sum_{l+m+n=d-1}c_{lmn}x_0^l x_1^m x_2^n, \end{aligned} \tag{2.3}$$

using (2.2), such that

$$\begin{cases} \partial_{x_i} f_d(p_{j,0},\dots,p_{j,r-1}) + p_{j,r}\partial_{x_i} f_{d-1}(p_{j,0},\dots,p_{j,r-1}) = 0 \\ f_{d-1}(p_{j,0},\dots,p_{j,r-1}) = 0, \end{cases} \tag{2.4}$$

for $i = 0,\dots,r-1,\ j = 1,\dots,b$. This is a system of $(r+1)b$ linear equations in the coefficients a_{ijk}, c_{lmn}.
Moreover the general divisor of the kernel system $\hat{\mathscr{L}_{\mathbb{F}}}$ on $\{x_r = 0\} \cong \mathbb{P}^{r-1}$ is given by

$$f_d(x_0,\dots,x_{r-1}) = 0$$

such that

$$\partial_{x_i} f_d(p_{j,0},\dots,p_{j,r-1}) = 0,\ i = 0,\dots,r-1,\ j = 1,\dots,b.$$

Lemma 11. *If the system $\mathscr{L}_{r-1,d}(2^b)$ is non special with virtual dimension greater than or equal to -1, then the system $\mathscr{L}_{r,d}(d-1,2^b)$ is non special and nonempty.*

Proof. Each element of the system $\mathscr{L}_{r,d}(d-1)$ of hypersurfaces of $\mathbb{P}^r$ of degree d passing through $p = [0,\dots,0,1]$ with multiplicity $d-1$ is described by a homogeneous polynomial f of degree d, as in (2.3). The conditions for f to be singular at p_j are the (2.4), for $i = 0,\dots,r-1,\ j = 1,\dots,b$; we want to prove that they are linearly independent. Let

$$A = \left(\begin{array}{c} A_0 \\ \hline A_1 \\ \hline \vdots \\ \hline A_{r-1} \end{array}\right)$$

be the $(rb) \times \binom{r+d-1}{r-1}$ matrix defined as follows: the j-th row of A_i is the vector of coefficients of $\partial_{x_i} f_d(p_{j,0},\dots,p_{j,r-1})$, $j = 1,\dots,b$. Similarly define C to be the $(rb) \times \binom{r+d-2}{r-1}$ matrix

$$C = \left(\begin{array}{c} C_0 \\ \hline C_1 \\ \hline \vdots \\ \hline C_{r-1} \end{array}\right)$$

such that the j-th row of C_i is the vector of coefficients, for $j = 1,\dots,b$, of $p_{j,r}\partial_{x_i} f_{d-1}(p_{j,0},\dots,p_{j,r-1})$. Finally define C' to be the $b \times \binom{r+d-2}{r-1}$ matrix having as j-th rows the vector of coefficients of $f_{d-1}(p_{j,0},\dots,p_{j,r-1})$, $j = 1\dots b$.
Notice that the equations in (2.4) are independent if and only if the matrix

$$M = \left(\begin{array}{c|c} A & C \\ \hline 0 & C' \end{array}\right)$$

has maximal rank. A has maximal rank rb, in fact, by the hypothesis, b double points in general position impose independent conditions on the hypersurfaces of $\mathbb{P}^{r-1}$ and

$$rb \leq \binom{d+r-1}{r-1}.$$

Moreover, C' has maximal rank b, in fact b general points impose exactly b linearly independent conditions on the hypersurfaces of $\mathbb{P}^{r-1}$ of degree $d-1$ passing through them, and

$$b \leq \binom{d+r-2}{r-1}.$$

Therefore M has makimal rank $(r+1)b$ and this concludes the proof.

□

2.1.4 Transversality

In this section we will show that, under some hypothesis on the integer b, transversality of the restricted systems of a $(1, b)$-degeneration holds. The reason of the choice $k = 1$ for the twisting parameter sits here: in this way we will be able to describe the restricted system $\mathscr{R}_{\mathbb{F}}$ and in particular its base locus and therefore to compute the dimension of the intersection $\mathscr{R}_{\mathbb{P}} \cap \mathscr{R}_{\mathbb{F}}$ of the restricted systems, that is the crucial point of our proof of Theorem 2.
First of all we describe the linear system $\mathscr{L}_{\mathbb{F}}$ on the strict transform $\mathbb{F}$. Let us study the blow up of $\mathbb{P}^r$ at $p = [0, \dots, 0, 1] \in \mathbb{P}^r$: let $x_0, \dots, x_r$ be homogeneous coordinates for $\mathbb{P}^r$ and let U be the affine open set described by $\{x_r = 1\}$: the affine coordinate are $x_0 \dots, x_{r-1}$. Consider now the blow up of $U \cong \mathbb{A}^r$ at the origin: it is described by

$$\left\{ \operatorname{rank} \begin{pmatrix} x_0, & \dots, & x_{r-1} \\ y_0, & \dots, & y_{r-1} \end{pmatrix} = 1 \right\} \subset \mathbb{A}^r \times \mathbb{P}^{r-1}$$

where $y_0, \dots, y_{r-1}$ are homogeneous coordinates of $\mathbb{P}^{r-1}$. Let $V = \{y_{r-1} = 1\}$ be an affine open set of $\mathbb{P}^{r-1}$: the affine equation of the blow up in $\mathbb{A}^r \times \mathbb{A}^{r-1} \cong \mathbb{A}^{2r-1}$ are

$$x_i = y_i x_{r-1}, \quad i = 0, \dots, r-2. \tag{2.5}$$

The strict transform $\mathbb{F}$ has affine coordinates $y_0, \dots, y_{r-2}, x_{r-1}$ and the exceptional divisor R has equation $x_{r-1} = 0$.
The generic hypersurface of $\mathbb{P}^r$ of degree d with multiplicity $d-1$ at P is described by

$$f_d(x_0, \dots, x_{r-1}) + x_r f_{d-1}(x_0, \dots, x_{r-1}) = 0,$$

using the (2.2), so in affine coordinates

$$f_d(y_0x_{r-1},\dots,y_{r-2}x_{r-1},x_{r-1})+f_{d-1}(y_0x_{r-1},\dots,y_{r-2}x_{r-1},x_{r-1})=0,$$

i.e.

$$F(y_0,\dots,y_{r-1},1):=x_{r-1}f_d(y_0,\dots,y_{r-2},1)+f_{d-1}(y_0,\dots,y_{r-2},1)=0. \quad (2.6)$$

Hence its restriction to R has equations

$$\begin{cases} f_{d-1}(y_0,\dots,y_{r-2},1)=0 \\ x_{r-1}=0 \end{cases} \quad (2.7)$$

Lemma 12 (Transversality Lemma)**.** *Keeping the same notation as above, the restricted systems $\mathscr{R}_{\mathbb{P}}$ and $\mathscr{R}_{\mathbb{F}}$ are transversal in $|\mathcal{O}_{\mathbb{P}^{r-1}}(d-1)|$ if one of the following conditions holds:*

(i.) either the system $\hat{\mathscr{L}}_{\mathbb{P}}$ is non-special and $\hat{v}_{\mathbb{P}}\geq -1$, or

(ii.) the system $\hat{\mathscr{L}}_{\mathbb{F}}\cong\mathscr{L}_{r-1,d}(2^b)$ has dimension $\hat{l}_{\mathbb{F}}=v_{r-1,d}(2^b)\geq -1$.

Proof. (i.) Under the hypothesis, $\mathscr{L}_{\mathbb{P}}$ is non-special and $\mathscr{R}_{\mathbb{P}}$ fills up the whole space $|\mathcal{O}_{\mathbb{P}^{r-1}}(d-1)|$; consequentely $\mathscr{R}=\mathscr{R}_{\mathbb{F}}$ and transversality is trivial.

(ii.) Notice that the kernel of the restriction map $\mathscr{L}_{\mathbb{F}}\to\mathscr{R}_{\mathbb{F}}$ has dimension equal to $l_{r-1,d}(2^b)$, by Lemma 10. Moreover the system $\mathscr{L}_{\mathbb{F}}$ is non-special, by Lemma 11. This allows us the knowledge of the restricted system on R: it is the complete linear system of hypersurfaces of $\mathbb{P}^{r-1}$ of degree $d-1$ containing b simple points $p_1'',\dots,p_b''$, which are the traces on the exceptional divisor R of the b lines through the $(d-1)$-point p, that we blew up, and the b double points specialized on $\mathbb{F}$, i.e.

$$\mathscr{R}_{\mathbb{F}}=\mathscr{L}_{r-1,d-1}(1^b),$$

in fact b points are base points for the hypersurfaces of $\mathbb{P}^{r-1}$ of degree $d-1$. The restriction to R of a general section of $\mathscr{L}_{\mathbb{F}}$ has affine equation given by (2.7). We know that b general points $p_1,\dots,p_b\in\mathbb{F}$ impose independent condition to the polynomial with affine equation of the form (2.6): in other words, the conditions imposed by the vanishing of the partial derivatives at $p_1,\dots,p_b$ are independent, and in particular

$$\partial_{x_{r-1}}F(p_{j,0},\dots,p_{j,r-1})=f_{d-1}(p_{j,0},\dots,p_{j,r-1})=0,\quad j=1,\dots,b.$$

This means that the p_j'''s are base points for the restricted system $\mathscr{R}_{\mathbb{F}}$. Notice also that $\mathscr{R}_{\mathbb{F}}$ has the right dimension:

$$r_{\mathbb{F}}=l_{\mathbb{F}}-\hat{l}_{\mathbb{F}}-1=\binom{r+d-2}{r-1}-1-b.$$

Now, if $\mathscr{L}_{\mathbb{P}}$ is empty, transversality is trivial, being $\mathscr{R} = \emptyset$. Suppose, from now on, that $\mathscr{L}_{\mathbb{P}} \neq \emptyset$:

$$\mathscr{R} = \{S : S \in \mathscr{R}_{\mathbb{P}} \text{ and } p_1, \ldots, p_b \in S\} \subseteq \mathscr{R}_{\mathbb{P}} \subseteq |\mathcal{O}_{\mathbb{P}^{r-1}}(d-1)|.$$

Choosing the b double points generically on $\mathbb{F}$, also $p_1, \ldots, p_b$ are general on R, then they impose b independent condition on $\mathscr{R}$. Therefore we get

$$\dim(\mathscr{R}) = \max\{-1, r_{\mathbb{P}} - b\}$$

hence

$$\dim(\mathscr{R}) = \begin{cases} -1 & \text{if } r_{\mathbb{P}} + r_{\mathbb{F}} \leq \binom{d+1}{2} - 2 \\ r_{\mathbb{P}} + r_{\mathbb{F}} - \binom{d+1}{2} + 1 & \text{if } r_{\mathbb{P}} + r_{\mathbb{F}} \geq \binom{d+1}{2} - 2 \end{cases}$$

i.e. the restricted systems intersect properly.

□

2.2 The approach via collision of fat points

In this section we will construct a degeneration of schemes defined by collections of n nodes to a *fat point*, which is a point with multiplicity, and a collection of $n-c$ general nodes. In other words, we will suppose that c nodes of $\mathbb{P}^r$ collide to a fat point p. To this end, let us perform a $(k, n-c)$-degeneration of $\mathbb{P}^r$ (and of $\mathscr{L}$); we get:

$$\begin{array}{ll} \mathscr{L}_{\mathbb{F}} = \mathscr{L}_{r,d}(d-k, 2^{n-c}) & \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{r,d}(d-k+1, 2^{n-c}) \\ \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{r,d-k}(2^c) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{r,d-k-1}(2^c). \end{array}$$

Notice that if

$$\frac{1}{r+1}\binom{r+d-k-1}{r}$$

is an integer and choosing c equal to that number, then $\hat{v}_{\mathbb{P}} = -1$. Morover, if under this choice $\mathscr{L}_{\mathbb{P}} \neq \emptyset$ is non-special, then the restricted system $\mathscr{R}_{\mathbb{P}} \subseteq |\mathcal{O}_{\mathbb{P}^{r-1}}(d-k)|$ fills up the whole space. Indeed

$$\begin{aligned} r_{\mathbb{P}} &= l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1 \\ &= \binom{r+d-k}{r} - 1 - (r+1)c = \binom{r+d-k-1}{r-1} - 1. \end{aligned}$$

This means that $\mathscr{L}_{\mathbb{P}}$ does not impose matching conditions to $\mathscr{R}_{\mathbb{F}}$ or, equivalently, that the c nodes specialized on the component $\mathbb{P}$ of the central fiber collide to a point of multiplicity $d-k$ on $\mathbb{F}$. The limiting system on $\mathbb{P}$ is the system of surfaces having c general double points, with the minimal degree with respect to the property that such a system is not empty, i.e.

$$\mathscr{L}_{r,d-k}(2^c) \neq \emptyset, \text{ while } \mathscr{L}_{r,d-k-1}(2^c) = \emptyset.$$

The problem of studying non-speciality of $\mathscr{L}$ is now translated into the analysis of non-speciality of $\mathscr{L}_{\mathbb{F}}$, that in some cases will be more easily solved.
We will see how this approach is useful to prove the statement of the Alexander-Hirschowitz Theorem is some cases. In particular we will apply this construction to the linear systems $\mathscr{L}_{3,d}(2^n)$, with $d \equiv 0 \pmod 6$ (Section 3.2).

2.3 The second degeneration

The method consisting of simply specializing the double points some on $\mathbb{P}$ and the others on $\mathbb{F}$ will be not enough to cover all the cases. Trying to prove the non-speciality of a given linear system $\mathscr{L}$, in some cases we are not able to find an integer b such that the limiting system $\mathscr{L}_0$ has dimension equal to $e(\mathscr{L})$. In those cases it is a arithmetic obstruction that prevent us from finding such a b. Then we use another approach in order to overcome the problem. It consistes in degenerating the system $\mathscr{L}_0$ on the central fiber X_0 to a system $\mathscr{L}'_0$ such that some of the points of $\mathbb{P}$ and of $\mathbb{F}$ *approach* R.
Let Δ' be a complex disc around the origin. Consider the trivial family $\mathcal{Z} = Z \times \Delta' \to \Delta'$ with reducible fibers $Z_s = \mathbb{F}_s \cup \mathbb{P}_s \cong X_0$, where $\mathbb{F}_s = \mathbb{F}$ is isomorphic to $\mathbb{P}^r$ blown up at a point, $\mathbb{P}_s = \mathbb{P}$ is isomorphic to $\mathbb{P}^r$ and $\mathbb{F}_s \cap \mathbb{P}_s = R_s \cong \mathbb{P}^{r-1}$, for every $s \in \Delta'$. The Picard group of Z_s is the fibered product of $\mathrm{Pic}(\mathbb{P}_s)$ and $\mathrm{Pic}(\mathbb{F}_s)$ over $\mathrm{Pic}(R_s)$. Consider on Z_s, $s \neq 0$, the linear system $\mathscr{L}'_s := \mathscr{L}_0$, where $\mathscr{L}_0$ is the flat limit of $\mathscr{L} = \mathscr{L}_{r,d}(2^n)$, with respect to the first degeneration. Such a system is given by two linear systems $\mathscr{L}'_{\mathbb{P}_s}$ and $\mathscr{L}'_{\mathbb{F}_s}$ on the two components that agree on the intersection R_s. The system on $\mathbb{P}_s$ (or on $\mathbb{F}_s$) restricts to a system $\mathscr{R}'_{\mathbb{P}_s}$ ($\mathscr{R}'_{\mathbb{F}_s}$ respectively) and the relative kernel, at the level of linear systems, is $\hat{\mathscr{L}}'_{\mathbb{P}_s}$ ($\hat{\mathscr{L}}'_{\mathbb{F}_s}$ respectively). We have the following identities:

$$\hat{\mathscr{L}}'_{\mathbb{F}_s} = \hat{\mathscr{L}}_{\mathbb{F}},\ \hat{\mathscr{L}}'_{\mathbb{P}_s} = \hat{\mathscr{L}}_{\mathbb{P}},\ \mathscr{R}'_{\mathbb{F}_s} = \mathscr{R}'_{\mathbb{F}},\ \mathscr{R}'_{\mathbb{P}_s} = \mathscr{R}'_{\mathbb{P}}, \text{ for } s \neq 0$$

Now, let $\alpha, \beta \in \mathbb{N}$ such that $\beta \leq b$, $\alpha \leq n-b$. Consider on the central fiber the scheme given by

- $n - b - \alpha$ double points in $\mathbb{P}_0 \setminus R_0$,
- $b - \beta$ in $\mathbb{F}_0 \setminus R_0$
- and $\alpha + \beta$ in R_0:

we can consider these nodes as limit of the n general nodes in Z_s ($n-b$ in $\mathbb{F}_s$ and b in $\mathbb{P}_s$). So, on the central fiber Z_0 the systems $\mathscr{L}'_{\mathbb{P}_0}$ and $\mathscr{L}'_{\mathbb{F}_0}$ are still the same, while the kernels are

$$\begin{aligned} \hat{\mathscr{L}}'_{\mathbb{P}_0} &= \mathscr{L}_{r,d-2}(2^{n-b-\alpha}, 1^{\alpha}) \\ \hat{\mathscr{L}}'_{\mathbb{F}_0} &= \mathscr{L}_{r,d}(d, 2^{b-\beta}, 1^{\beta}) \cong \mathscr{L}_{r-1,d}(2^{b-\beta}, 1^{\beta}) \end{aligned}$$

with the following restriction sequences:

$$\begin{array}{ccccccl} 0 & \to & \hat{\mathscr{L}}'_{\mathbb{P}_0} & \to & \mathscr{L}'_{\mathbb{P}_0} & \to & \mathscr{R}'_{\mathbb{P}_0} \subseteq \mathscr{L}_{r-1,d-1}(2^{\alpha}) \\ 0 & \to & \hat{\mathscr{L}}'_{\mathbb{F}_0} & \to & \mathscr{L}'_{\mathbb{F}_0} & \to & \mathscr{R}'_{\mathbb{F}_0} \subseteq \mathscr{L}_{r-1,d-1}(2^{\beta}, 1^{b-\beta}). \end{array}$$

We respectively denote by $\hat{v}'_{\mathbb{P}_0}$, $v'_{\mathbb{P}_0}$, $\hat{v}'_{\mathbb{F}_0}$, $v'_{\mathbb{F}_0}$ and $\hat{l}'_{\mathbb{P}_0}$, $l'_{\mathbb{P}_0}$, $\hat{l}'_{\mathbb{F}_0}$, $l'_{\mathbb{F}_0}$ the virtual and the actual dimensions. As in Section 2.1, we obtain a recursive formula for the dimension of $\mathscr{L}'_0$:

$$l'_0 = \hat{l}'_{\mathbb{P}_0} + \hat{l}'_{\mathbb{F}_0} + \dim(\mathscr{R}'_0) + 2, \tag{2.8}$$

where $\mathscr{R}'_0 := \mathscr{R}'_{\mathbb{P}_0} \cap \mathscr{R}'_{\mathbb{F}_0}$.

Proposition 13. *Keeping the same notations as above, if there are integers b, α, β such that $\alpha \leq n-b, \beta \leq b$ and $l'_0 = e(\mathscr{L})$, then $\mathscr{L}$ is non-special*

Proof. Exploiting upper semicontinuity of the second degeneration, we have $l'_0 \geq l'_s$, $s \neq 0$, $s \in \Delta'$. Moreover, by the first degeneration, we have $l_0 \geq l_t$, $t \neq 0$, $t \in \Delta$. But $l'_s = l_0$ and $l_t = l$ and this concludes the proof. □

Non-speciality of $\mathscr{L}_{\mathbb{F}}$

In this Section we assume that the case of cubics is already solved, i.e. that $\mathscr{L}_{r,3}(2^n)$ is non-special exept if $r=4$ and $n=7$. The proof of this is completely untied from what follows and it will be discussed in Section 3.1.1.

Notice that if we only perform a $(1,b)$-degeneration, we simultaneously get emptyness of $\hat{\mathscr{L}}_{\mathbb{F}}$ and non-speciality of $\mathscr{L}_{\mathbb{F}}$ (under the hypothesys of Lemma 11) if and only if

$$\frac{1}{r}\binom{r+d-1}{r-1}$$

is an integer. In all the remaining cases, if we choose

$$b \geq \left\lfloor \frac{1}{r}\binom{r+d-1}{r-1} \right\rfloor,$$

we preserve emptyness of $\hat{\mathscr{L}}_{\mathbb{F}}$, but we are no longer in the hypothesis of Lemma 11. In that situation, we need to perform a degeneration of the central fiber and of the limit system $\mathscr{L}_0$ as described above. However we need a criterion for the non-speciality of $\mathscr{L}'_{\mathbb{F}_s}$. In this section we will prove that there exists an upper bound on the number k of nodes such that the linear systems of the form

$$\mathscr{L}_{r,d}(d-1, 2^k)$$

are non-special. The proof will be by induction on both r and d. The first two lemmas provide the starting points of the induction: they will describe the cases with $d = 4$ in the first one and $r = 3$ in the second one.

Lemma 14. *Let $r \geq 2$. The linear system $\mathscr{L}_{r,4}(3, 2^k)$, with*

$$k \leq k(r) := \left\lceil \frac{1}{r+1} \binom{r+4}{4} \right\rceil - r - 1,$$

is non special.

Proof. The proof is by induction on r. It suffices to prove the statement for $k(r)$ nodes. For $k < k(r)$, non-speciality of the corresponding linear system is a consequence, being $v(\mathscr{L}_{r,4}(3, 2^{k(r)})) \geq -1$, for $r \geq 2$. The base step is the case $r = 2$: the system $\mathscr{L}_{2,4}(3, 2^2)$ is non-special. Consider now the scheme Z given by the union of the triple point and $k(r-1) < k(r)$ double points. If $\pi \subset \mathbb{P}^r$ is a fixed hyperplane containing the support of Z, then the trace of Z with respect to π is the scheme $Z \cap \pi$, while the residual scheme is given by a point of multiplicity 2 and $k(r-1)$ simple points. Thus we get the following restriction map,

$$\mathscr{L}_{r,4}(3, 2^{k(r)}) \to \mathscr{L}_\pi := \mathscr{L}_{r-1,4}(3, 2^{k(r-1)}),$$

and the kernel is the system

$$\hat{\mathscr{L}} := \mathscr{L}_{r,3}(2^{1+k(r)-k(r-1)}, 1^{k(r-1)}).$$

This gives us the induction on r. The system on the right is non special with virtual dimension at least -1, by the inductive hypothesis; the system on the left is non-special and it has virtual dimension

$$\begin{aligned} \hat{v} &= \binom{r+3}{3} - 1 - (r+1)(k(r)+1) + rk(r-1) \\ &\geq \binom{r+3}{3} - 1 - \binom{r+4}{4} + r^2 + \binom{r+3}{4} - r^2 = -1, \end{aligned}$$

being

$$k(r) \leq \frac{1}{r+1}\binom{r+4}{4} - r - 1 + \frac{r}{r+1} \text{ and } k(r-1) \geq \frac{1}{r}\binom{r+3}{4} - r.$$

for all r. Therefore

$$l_{r,4}(3, 2^{k(r)}) = \dim(\mathscr{L}_\pi) + \dim(\hat{\mathscr{L}}) + 1 = v(\mathscr{L}),$$

and this concludes the proof. $\square$

Lemma 15. *Let $d \geq 4$, and*

$$k_0(d) := \left\lfloor \frac{d^2+2d-3}{4} \right\rfloor .$$

If $k \leq k_0(d)$, then the linear system $\mathscr{L} = \mathscr{L}_{3,d}(d-1, 2^k)$ is non-special.

Proof. We prove the statement by induction on d for a collection of $k_0(d)$ points. The base step is the cae $\mathscr{L}_{3,4}(3, 2^5)$ that is non-special by Lemma 14. For the inductive step, consider the system of surfaces of $\mathbb{P}^3$ of degree d with a point p of multiplicity $d-1$ and $k_0(d)$ double points. Notice that $v(\mathscr{L}_{3,d}(d-1, 2^{k_0(d)})) \geq -1$. Specializing p and

$$h = \left\lfloor \frac{2d+1}{3} \right\rfloor \leq k_0(d)$$

nodes on a general plane π, we obtain the following restriction map:

$$\mathscr{L}_{3,d}(d-1, 2^{k_0(d)}) \to \mathscr{L}_\pi := \mathscr{L}_{2,d}(d-1, 2^h),$$

with kernel

$$\hat{\mathscr{L}} := \mathscr{L}_{3,d-1}(d-2, 2^{k_0(d)-h}, 1^h).$$

Notice that the integer h is the minimal one with respect to the property that $v_\pi \geq -1$. Being

$$h \geq \frac{2d+1}{3} - \frac{2}{3} = \frac{2d-1}{3},$$

then

$$k_0(d) - h \leq \frac{d^2+2d-3}{4} - \frac{2d-1}{3} = \frac{3d^2-2d-5}{12}.$$

Moreover

$$k_0(d-1) = \left\lfloor \frac{d^2-4}{4} \right\rfloor = \left\lfloor \frac{d^2}{4} \right\rfloor - 1 = \begin{cases} d^2/4 - 1 & \text{if d is even} \\ d^2/4 - 1 - 1/4 & \text{if d is odd} \end{cases} .$$

Thus $k_0(d) - h \leq k_0(d-1)$ and non-speciality of $\mathscr{L}_{3,d-1}(d-2, 2^{k_0(d)-h})$ follows from the inductive hypothesis. In particular $\hat{\mathscr{L}}$ is non-special. Not only, it is not empty, in fact it has positive virtual dimension.

$$\begin{aligned} v(\hat{\mathscr{L}}) &= \binom{d+2}{3} - 1 - \binom{d}{3} - 4(k_0(d) - h) - h \\ &\geq \binom{d+2}{3} - 1 - \binom{d}{3} - \frac{3d^2-2d-5}{3} - \frac{2d+1}{3} = \frac{1}{3}. \end{aligned}$$

Finally $l_{3,d}(d-1, 2^{k_0(d)}) = \dim(\hat{\mathscr{L}}) + \dim(\mathscr{L}_\pi) + 1 = e(\mathscr{L})$ and this concludes the proof. $\square$

Now, we prove a result for linear systems of hypersurfaces of degree d of $\mathbb{P}^r$ with a point of multiplicity $d-1$ and k general nodes in its full generality, i.e. for every $r \geq 4$ and $d \geq 4$. To this end, let us define the number

$$k(r,d) := \left\lfloor \frac{1}{r+1}\binom{r+d}{r} - \frac{1}{r+1}\binom{r+d-2}{r} \right\rfloor - (r-2),$$

for every r and d. We want to prove that the linear system $\mathscr{L}_{r,d}(d-1, 2^k)$ is non special, if $k \leq k(r,d)$.

Remark 1. *Notice that $k(3,d)$ is equal to the number $k_0(d)$ defined in Lemma 15, so that result can be employed as base step of the induction on r.*

As in the case with $r = 3$, the trick, given $\mathscr{L}_{r,d}(d-1, 2^{k(r,d)})$, will be to specialize $k(r-1,d)$ nodes on an hyperplane $\pi \cong \mathbb{P}^{r-1}$ containing the support of p as follows:

$$0 \to \hat{\mathscr{L}} \to \mathscr{L}_{r,d}(d-1, 2^{k(r,d)}) \to \mathscr{L}_\pi = \mathscr{L}_{r-1,d}(d-1, 2^{k(r-1,d)}). \tag{2.9}$$

The kernel system is

$$\hat{\mathscr{L}} = \mathscr{L}_{r,d-1}(d-2, 2^{k(r,d)-k(r-1,d)}, 1^{k(r-1,d)}).$$

The $(d-2)$-point is the residual of the $(d-1)$-point and the simple base points are the residual of the nodes specialized on π.
Let us consider first of all the quartic case because it is the starting step of the induction on the degree: the linear system $\mathscr{L}_{r,4}(3, 2^k)$, with $k \leq k(r,4)$, is non-special by Lemma 14; in fact $k(r,4) \leq k(r)$ as it can be easily checked, being

$$\begin{aligned} k(r,4) \leq& \left(\frac{1}{r+1}\binom{r+4}{r} - \frac{1}{r+1}\binom{r+2}{r} - 1\right) - r + 2 \\ =& \frac{1}{r+1}\binom{r+4}{r} - r - \left(\frac{1}{r+1}\binom{r+2}{r} - 1\right) \end{aligned}$$

and

$$k(r) \geq \left(\frac{1}{r+1}\binom{r+4}{r} + 1\right) - r - 1 = \frac{1}{r+1}\binom{r+4}{r} - r.$$

Theorem 16. *The linear system $\mathscr{L}_{r,d}(d-1, 2^k)$, with $k \leq k(r,d)$ and $d \geq 4$, is non-special and it has virtual dimension at least -1.*

Proof. The induction on r is based on the case of linear system in $\mathbb{P}^3$ analysed in Lemma 15; while the base step of the induction on the degree is the case of quartics, already examined in Lemma 14.
Consider the restriction exact sequence in (2.9): $\mathscr{L}_\pi$ is non-special by induction

on r, and $v_\pi \geq -1$. Moreover the system $\hat{\mathscr{L}}$ is non special, applying induction on d, if

$$k(r,d) - k(r-1,d) \leq k(r,d-1). \tag{2.10}$$

Now, being

$$\begin{aligned} k(r,d-1) \geq & \frac{1}{r+1}\binom{r+d-1}{r} - \frac{1}{r+1}\binom{r+d-3}{r} - (r-2) - \frac{r}{r+1}, \\ k(r,d) \leq & \frac{1}{r+1}\binom{r+d}{r} - \frac{1}{r+1}\binom{r+d-2}{r} - (r-2) \\ k(r-1,d) \geq & \frac{1}{r}\binom{r+d-1}{r-1} - \frac{1}{r}\binom{r+d-3}{r-1} - (r-3) - \frac{r-1}{r}; \end{aligned}$$

one can easily check that inequality (2.10) is veriefied if

$$\binom{r+d-1}{r-1} - \binom{r+d-3}{r-1} \geq r^3 - 3r - 1,$$

i.e. for every pair $(r,d) \neq (4,4),(4,5),(4,6),(5,4),(5,5),(6,4),(6,5)$. However, in the exluded cases, it is very easy to check directly by hand that (2.10) holds. Moreover $\hat{\mathscr{L}}$ has positive virtual dimension, for every (r,d) in fact

$$\hat{v} = \binom{r+d-1}{r} - \binom{r+d-3}{r} - (r+1)k(r,d) + rk(r-1,d) \geq r-1.$$

Finally,

$$l_{r,d}(d-1, 2^{k(r,d)}) = \dim(\hat{\mathscr{L}}) + \dim(\mathscr{L}_\pi) + 1 = e(\mathscr{L})$$

and this completes the proof. □

Remark 2. *Lemma 14 provides an upper bound to the number of double points which is bigger than the one we need for the base step of the induction on the degree used in the proof of Theorem 16. Nevertheless $k(r)$ is exactly the number of nodes that we will specialize on the component $\mathbb{F}$ in the proof of Alexander-Hirschowitz Theorem for quartics (see Section 3.1.2).*

Remark 3. *If a linear system of type $\mathscr{L}_{r,d}(d-1,2^k)$ verifies the hypothesis of Lemma 11, it consequentely verifies also the hypothesis of Theorem 16, being*

$$\left\lfloor \frac{1}{r}\binom{r+d-1}{r-1} \right\rfloor \leq k(r,d),$$

for every r and d. The hypothesys of Lemma 11 is too strong. The reason to study those cases separately sits in the fact that also transversality holds in the assumption of Lemma 11 (see Lemma 12). Recall that transversality is the crucial fact in all our computations. So we put stronger assumption, but in that way we get a stronger result concerning the intersection of the restricted systems.

Chapter 3

Proof of the Alexander-Hirschowitz Theorem

The goal of this chapter is to expose a proof of the Alexander-Hirschowitz Theorem (Theorem 2), applying the degeneration techniques described in Chapter 2 for $d \geq 4$ and discussing the case of cubics separately. The proof will be done by induction on both r and d. The induction on the degree will be based on the cases of cubics and quartics, while the induction on the dimension of the ambient space will start from the case of linear systems of surfaces of $\mathbb{P}^3$.
Define

$$n^- = n^-(r,d) := \left\lfloor \frac{1}{r+1}\binom{r+4}{4} \right\rfloor = \frac{1}{r+1}\binom{r+4}{4} - \frac{l^-}{r+1},$$

and

$$n^+ = n^+(r,d) := \left\lceil \frac{1}{r+1}\binom{r+4}{4} \right\rceil = \frac{1}{r+1}\binom{r+4}{4} + \frac{l^+}{r+1},$$

with $l^-, l^+ \in \{0, \dots, r\}$. They are respectively the maximal number of nodes with respect to the property that the linear system of surfaces of degree d with a collection of nodes has virtual dimension at least -1 and the minimal number of nodes such that the corresponding linear system is expected to be empty. Obviously, if $\frac{1}{r+1}\binom{r+d}{r}$ is an integer, then $n^- = n^+$.

Notice that if non-speciality holds for a collection of n^- double points, then it holds, as an easy consequence, for a smaller number of double points. On the other hand, if there are no hypersurfaces of degree d with a given collection of nodes, the same is true adding other nodes; so it suffices to prove emptyness of $\mathscr{L}_{r,d}(2^n)$ for n^+.

3.1 The base steps of the induction on the degree

3.1.1 Cubics

The techniques introduced in the previous sections do not work in the case of cubics, because the limiting system on the exceptional component $\mathbb{P}$ of the central fiber of a $(1,b)$-degeneration is a linear system of quadrics with nodes which is special. We will prove non-speciality of $\mathscr{L}_{r,3}(2^n)$, for $r \geq 3$, $r \neq 4$ by induction on r, with a different degeneration argument.
The starting point is the linear system $\mathscr{L}_{3,3}(2^5)$ of cubic surfaces of $\mathbb{P}^3$, which is empty as expected. Indeed, if we restrict it to a plane π and if we specialize three nodes on it, we get the following sequence:

$$0 \to \mathscr{L}_{3,2}(2^2, 1^3) \to \mathscr{L}_{3,3}(2^5) \to \mathscr{L}_{3,3}(2^5)_{|\pi} \subset \mathscr{L}_{2,3}(2^3).$$

An useful remark is that if a cubic has two double points, then by *Bézout's Theorem*, it must vanish identically on the line joining them. This line meets π at a point, so

$$\mathscr{L}_{3,3}(2^5)_{|\pi} \subseteq \mathscr{L}_{2,3}(2^3, 1) = \emptyset.$$

Moreover the kernel $\mathscr{L}_{3,2}(2^2, 1^3)$ is empty, and this concludes the proof in the case of $\mathbb{P}^3$.
We will study the linear system $\mathscr{L}_{r,3}(2^n)$, for $r \geq 5$. Observe that

$$n^-(r,3) = \begin{cases} \frac{1}{r+1}\binom{r+3}{3} & \text{if } r \equiv 0,1 \pmod 3 \\ \frac{1}{r+1}\binom{r+3}{3} - \frac{1}{3} & \text{if } r \equiv 2 \pmod 3 \end{cases}$$

Let $l \in \mathbb{Z}$ be such that $r = 3l + k$, with $k \in \{0,1,2\}$; define

$$\gamma(r) := \begin{cases} 0 & \text{if } r \equiv 0,1 \pmod 3 \\ l+1 & \text{if } r \equiv 2 \pmod 3). \end{cases}$$

We will prove the following result, due to J. Alexander and A. Hirschowitz (see [5]).

Theorem 17. *Let $r \neq 2 \pmod 3$, $r \neq 4$. Then there are no cubics in $\mathbb{P}^r$ singular at $n^-(r,3)$ general points.*

Let $r = 3k+2$. Then there are no cubics in $\mathbb{P}^r$ singular at $n^-(r,3)$ general points and passing through $\gamma(r)$ additional general points.

The degeneration construction for cubics

Let us consider the trivial family $\mathcal{Y} = \mathbb{P}^r \times \Delta \to \Delta$, where Δ is a complex disc with center at the origin. Let Y_0 be the central fiber. Choose a general linear subspace $L \subset Y_0 = \mathbb{P}^r$ of codimension h:

$$\mathcal{N}_{L|\mathcal{Y}} = \mathcal{O}_L(1)^{\oplus h} \oplus \mathcal{O}_L$$

is the normal sheaf of L in $\mathcal{Y}$. Blowing up L in the family, we obtain a new family $\mathcal{X}$, with maps $\pi_1 : \mathcal{X} \to \mathbb{P}^r$ and $\pi_2 : \mathcal{X} \to \Delta$ and a reducible central fiber X_0 which is the union of the strict transform V of $\mathbb{P}^r$, i.e. $\mathbb{P}^r$ blown up along L, and the exceptional divisor T, which is isomorphic to $\mathbb{P}(\mathcal{N}^*_{L|\mathcal{Y}}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-h}}(1)^{\oplus h} \oplus \mathcal{O}_{\mathbb{P}^{r-h}}(2))$. This variety of dimension r is a $\mathbb{P}^h$-bundle over $L \cong \mathbb{P}^{r-h}$ with the natural map $p : T \to L$. The intersection of the two components of X_0 is a $(r-1)$-dimensional subvariety Q of degree h:

$$Q = \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-h}}(1)^{\oplus h}) \cong \mathbb{P}^{r-h} \times \mathbb{P}^{h-1}.$$

It is the exceptional divisor of the blow up of L in the central fiber. The Picard group of V is generated by the hyperplane class H_V, which corresponds to the line bundle $\mathcal{O}_V(1)$ pull back of $\mathcal{O}_{\mathbb{P}^r}(1)$, and by the divisor Q. The Picard group of T is generated by $\pi := p^*(\mathcal{O}_L(1))$ and by Q; so the $\mathcal{O}(1)$-bundle of $T \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{r-h}}(1)^{\oplus h} \oplus \mathcal{O}_{\mathbb{P}^{r-h}}(2))$ is of the form $H_T = Q + 2\pi$.

Now, consider the line bundle $\mathcal{O}_{\mathcal{X}}(3) = \pi_1^*\mathcal{O}_{\mathbb{P}^r}(3)$. It restricts to $\mathcal{O}_{\mathbb{P}^r}(3)$ on the general fiber; while on the central one we have:

- $|\mathcal{O}_{\mathcal{X}}(3)_{|T}| = |3\pi|$
- $|\mathcal{O}_{\mathcal{X}}(3)_{|V}| = |3H_V|$.

If $r \neq 7$, choose $h = 3$ and twist by $-T$: on the general fiber we do not make any change, while on the special one we get:

- $|3\pi + Q|$ on T,
- $|3H_V - Q|$ on V.

Consider the linear system of cubics on the general fiber

$$\mathscr{L}_t := \mathscr{L} = \mathscr{L}_{r,3}(2^{n^-(r,3)}, 1^{\gamma(r)}),$$

which has virtual dimension -1, for every r. Specializing $n^-(r-3,3)$ nodes and $\gamma(r-3)$ simple points on the component T, we get

$$\begin{array}{rcl} \mathscr{L}_T & = & |3\pi + Q - 2^{n^-(r-3,3)} - 1^{\gamma(r-3)}| \\ \mathscr{L}_V & = & |3H - Q - 2^{r+1} - 1^{\gamma(r)-\gamma(r-3)}|, \end{array}$$

where $\gamma(r) - \gamma(r-3) \in \{0,1\}$.
Notice that, to be precise, instead of $-2^a - 1^b$ we would have to write $-2E_1 - \cdots - 2E_a - E_{a+1} - \cdots - E_b$, where the E_i's are the exceptional divisor of the blow up of $\mathcal{X}$ at these points.
The system $\mathscr{L}_V$ is isomorphic to the linear system of cubic hypersurfaces of $\mathbb{P}^r$ containing a 3-codimensional subspace L, being singular at $r+1$ general

points, and passing through $\gamma(r) - \gamma(r-3)$ general points; we will use the following notation:

$$\mathscr{L}_V \cong \mathscr{L}_{r,3}(L, 2^{r+1}, 1^{\gamma}(r-3)).$$

If we restrict the two linear systems to the intersection Q, we obtain as kernels the following systems:

$$\begin{aligned} \hat{\mathscr{L}_T} &= |3\pi - 2^{n^-(r-3,3)} - 1^{\gamma(r-3)}| \cong \mathscr{L}_{r-3,3}(2^{n^-(r-3,3)}, 1^{\gamma(r-3)}) \\ \hat{\mathscr{L}_V} &= |3H - 2Q - 2^{r+1} - 1^{\gamma(r)-\gamma(r-3)}| \end{aligned}$$

The motivation of this choice is that in this way the kernel $\hat{\mathscr{L}_T}$ is known to be empty, applying induction from $r-3$ to r, if $r \neq 7$.

The kernel $\hat{\mathscr{L}_V}$ is empty

The kernel system of the component V is isomorphic to the linear system of cubic hypersurfaces of $\mathbb{P}^r$ that are singular along a 3-codimensional subspace L and at $r+1$ general points, and with $\gamma(r) - \gamma(r-3)$ additional base points:

$$\hat{\mathscr{L}_V} \cong \mathscr{L}_{r,3}(2L, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)}).$$

Notice, as a preliminary step, that the linear system $\mathscr{L}_{r,2}(2L, 2)$ has dimension 2. Indeed, if a quadric hypersurface is singular along $L \cong \mathbb{P}^{r-3}$ and at $p \notin L$, then it is singular along $\tilde{L} =< L, p >\cong \mathbb{P}^{r-2}$. So, if $x_0, \ldots, x_r$ are homogeneous coordinates for $\mathbb{P}^r$ and $\tilde{L} = \{x_0 = x_1 = 0\} \subset \mathbb{P}^r$, a quadratic polynomial in $x_0, \ldots, x_r$ vanishing along $\tilde{L}$ is of the form

$$x_0(a_0 x_0 + \cdots a_r x_r) + x_1(b_1 x_1 + \cdots b_r x_r)$$

Furthermore, imposing the vanishing of the first partial derivatives along $\tilde{L}$, we get

$$\mathscr{L}_{r,2}(2L, 2) = \mathscr{L}_{r,2}(2\tilde{L}) = \{a_0 x_0^2 + a_1 x_0 x_1 + b_1 x_1^2 = 0\}$$

depending on two projective parameters.

Proposition 18. *The system $\mathscr{L}_{r,3}(2L, 2^{r+1})$ is empty, for $r \geq 3$.*

Proof. In the first case ($r = 3$), a sublinear space of codimension 3 is a point, so the corresponding system is $\mathscr{L}_{3,3}(2^5)$ that is empty.
A general hyperplane of $\mathbb{P}^4$ will intersect a general line L at a point; so, restricting the linear system $\mathscr{L}_{4,3}(2L, 2^5)$ to the hyperplane containing the supports of four of the five nodes, we get

$$0 \to \mathscr{L}_{4,2}(2L, 2, 1^4) \to \mathscr{L}_{4,3}(2L, 2^5) \to \mathscr{L}_{3,3}(2, 2^4)$$

The kernel system is empty, so our system is empty too.
Similarly, for $r \geq 5$, the statement follows by induction on r and by the sequence

$$0 \to \mathscr{L}_{r,2}(2L, 2, 1^r) \to \mathscr{L}_{r,3}(2L, 2^{r+1}) \to \mathscr{L}_{r-1,3}(2L', 2^r)$$

where $L' \cong \mathbb{P}^{r-4}$ is the intersection of L with the restricting hyperplane. The restricted system is empty by the inductive assumption; the kernel is empty too. $\square$

From this proposition in particular follows the emptyness of $\hat{\mathscr{L}}_V$, for $r \geq 5$, $r \neq 7$.

Matching systems

Let $p_1, \ldots, p_t$, with $t = n^-(r-3,3)$, be the nodes specialized on T. Each of them lies on a distinct fiber of the ruling of T: say $p_i \in f_i \cong \mathbb{P}^3$. This implies that each of the sections of $\mathscr{L}_T$ must contain $f_1, \ldots, f_t$. Therefore the sections of $\mathscr{L}_{T|Q}$ must contain t distinct planes $\sigma_i = f_{i|Q} \cong \mathbb{P}^2$, each of them imposing 3 linear conditions on it.
The sections of $\mathscr{L}_{V|Q}$ must agree with those of $\mathscr{L}_{T|Q}$. Define $\mathscr{L}_V^m \subseteq \mathscr{L}_V$ and $\hat{\mathscr{L}}_V^m \subseteq \hat{\mathscr{L}}_V$ to be the linear systems on V defined by the matching conditions. Similarly, let $\mathscr{L}_T^m \subseteq \mathscr{L}_T$ and $\hat{\mathscr{L}}_T^m \subseteq \hat{\mathscr{L}}_T$ be the corresponding systems on the exceptional component. We will refer to them as the *matching systems*, according to the notation of [21].
The system $\mathscr{L}_V^m$ is isomorphic to the linear system of cubic hypersurfaces of $\mathbb{P}^r$ which contain a linear subspace L of codimension 3 and which are singular at $n^-(r,3)$ nodes, such that $n^-(r-3,3)$ of them are supported on L and $r+1$ are general in $\mathbb{P}^r \setminus L$ and which pass through $\gamma(r) - \gamma(r-3)$ additional general points:

$$\mathscr{L}_V^m \cong \mathscr{L}_{r,3}(\{L, 2^{n^-(r-3,3)}\}, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)}), \quad r \neq 7.$$

We use the notation $\{L, 2^t\}$ for the scheme given by a subspace L and t general nodes supported on it. It suffices to prove the emptyness of $\mathscr{L}_V^m$, for every $r \geq 5$, $r \neq 7$, to conclude. Indeed, if on the contrary the system $\mathscr{L}_t$ on the general fiber is nonempty, then there exists an integer k such that the limiting system $\mathscr{L}_0$ is given by two systems $\mathscr{L}_{V,k}^m$ and $\mathscr{L}_{T,k}^m$ obtained by twisting by $-kT$,

$$\begin{array}{rcl} \mathscr{L}_{T,k}^m & \subseteq & \mathscr{L}_{T,k} = |3\pi + kQ - 2^{n^-(r-3,3)} - 1^{\gamma(r-3)}| \\ \mathscr{L}_{V,k}^m & \subseteq & \mathscr{L}_{V,k} \cong \mathscr{L}_{r,3}(kL, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)}) \end{array}$$

both nonempty. But, if $k = 0$, we would have that $\mathscr{L}_{T,0}^m \subseteq \mathscr{L}_{T,0} \cong \mathscr{L}_{r-3,3}(2^{n^-(r-3,3)}, 1^{\gamma(r-3)}) = \emptyset$. If $k = 1$, then $\mathscr{L}_{V,1}^m = \mathscr{L}_V^m = \emptyset$ by assumption. Finally, if $k \geq 2$, then $\mathscr{L}_{V,k}^m \subseteq \mathscr{L}_{r,3}(2L, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)}) = \hat{\mathscr{L}}_V = \emptyset$ (see Proposition 18).

We will prove that the matching system $\mathscr{L}_V^m$ is empty, for $r \geq 5$, by induction from $r-3$ to r, starting from the cases $r = 5, 6, 7$. This proof is very similar to the one of M. C. Brambilla and G. Ottaviani in [8], Section 5.
We need two preliminary results.

Proposition 19. *The system $\mathscr{K}_2(r) := \mathscr{L}_{r,3}(\{L_1, 2^3\}, \{L_2, 2^3\}, \{L_3, 2^3\})$, with $L_1, L_2, L_3 \cong \mathbb{P}^{r-3}$ three general subspaces of $\mathbb{P}^r$, is empty for $r \geq 6$.*

Proof. For $r = 6$ it suffices to make the computation explicitly. Indeed if we choose generically three general subspaces of dimension 3, that intersect two by two at a point, and three general points on each of them and if we impose them as nodes for the cubics of $\mathbb{P}^6$, the resulting system is empty. We can check this for example choosing $L_1 = \{x_0 = x_1 = x_2 = 0\}$, $L_2 = \{x_4 = x_5 = x_6 = 0\}$, $L_3 = \{x_3 = x_0 - x_4 = x_2 - x_6\}$ and $p^i_j \in L_i$, $i, j = 1, 2, 3$, as follows:

$$\begin{array}{lll} p^1_1 = [0,0,0,1,1,0,0] & p^1_2 = [0,0,0,1,0,1,1] & p^1_3 = [0,0,0,0,1,0,-1] \\ p^2_1 = [1,1,0,1,0,0,0] & p^2_2 = [0,0,1,1,0,0,0] & p^2_3 = [-1,0,1,1,0,0,0] \\ p^3_1 = [1,1,1,0,1,0,1] & p^3_2 = [1,0,1,0,1,1,1] & p^3_3 = [1,-1,0,0,1,1,0]. \end{array}$$

For $r \geq 7$, we prove the statement by induction from $r-1$ to r. Choose a general hyperplane of $\mathbb{P}^r$: it intersects L_i in a subspace L'_i of dimension $r-4$, for $i = 1, 2, 3$. Moreover specialize the nine nodes on it, three on each L'_i, and consider the following exact sequence

$$0 \to \mathscr{L}_{r,2}(L_1, L_2, L_3) \to \mathscr{K}_2(r) \to \mathscr{K}_2(r-1). \tag{3.1}$$

The kernel system is empty for $r \geq 7$. Indeed in $\mathbb{P}^7$ there are no quadric hypersurfaces vanishing along three general subspaces of dimension four. Similarly, for $r \geq 8$, the kernel of (3.1) is empty. Indeed a subspace of dimension $r-3$ of $\mathbb{P}^r$ imposes $\binom{r-1}{2}$ linear conditions to the system $\mathscr{L}_{r,2}$. If $L_1, L_2, L_3 \subseteq \mathbb{P}^r$ then, $L_i \cap L_j$ is a $\mathbb{P}^{r-6}$, $i \neq j$ and $L_1 \cap L_2 \cap L_3$ is a $\mathbb{P}^{r-9}$ for $r \geq 9$ and it is empty if $r = 8$. Thus the total number of linear conditions imposed to $\mathscr{L}_{r,d}$ is $3\binom{r-1}{2} - 3\binom{r-4}{2} + \binom{r-7}{2}$. Therefore

$$l_{r,2}(L_1, L_2, L_3) = \binom{r+2}{2} - 1 - 3\binom{r-1}{2} + 3\binom{r-4}{2} - \binom{r-7}{2} = -1.$$

□

Proposition 20. *Let $L_1, L_2 \cong \mathbb{P}^{r-3}$ be general subspaces of $\mathbb{P}^r$. The linear system $\mathscr{K}_1(r) = \mathscr{L}_{r,3}(\{L_1, 2^{r-2}\}, \{L_2, 2^{r-2}\}, 2^3)$ is empty for $r \geq 3$, $r \neq 4$.*

Proof. We will prove the statement by induction on r, from $r-3$ to r, starting from the cases $r = 3, 5, 7$. For $r = 3$, one has $\mathscr{K}_1(3) = \mathscr{L}_{3,3}(2^5)$: this system is empty. For $r = 5$ and $r = 7$ it is an explicit computation. For $r = 5$ one can check that there are no cubics in $\mathscr{K}_1(5)$ choosing for example $L_1 = \{x_0 = x_1 = x_2 = 0\}$, $L_2 = \{x_3 = x_4 = x_5 = 0\}$, $p^i_j \in L_i$, for $i = 1, 2,\ j = 1, 2, 3$ and $q_1, q_2, q_3 \in \mathbb{P}^5 \setminus (L_1 \cup L_2)$ as follows:

$$\begin{array}{lll} p^1_1 = [0,0,0,1,0,0] & p^2_1 = [1,0,0,0,0,0] & q_1 = [1,1,0,1,0,1] \\ p^1_2 = [0,0,0,0,1,0] & p^2_2 = [0,1,0,0,0,0] & q_2 = [1,0,1,0,1,1] \\ p^1_3 = [0,0,0,0,0,1] & p^2_3 = [0,0,1,0,0,0] & q_3 = [1,1,1,-1,-1,-1]. \end{array} \tag{3.2}$$

Similarly for $r = 7$, with an explicit computation we conclude, for example choosing $L_1 = \{x_0 = x_1 = x_2 = 0\}$, $L_2 = \{x_5 = x_6 = x_7 = 0\}$, $p^i_j \in L_i$, for $i = 1, 2,\ j = 1, \dots 5$, and $q_1, q_2, q_3 \in \mathbb{P}^7 \setminus (L_1 \cup L_2)$ as follows:

$$\begin{array}{ll} p^1_1 = [0,0,0,0,1,0,0,1] & p^2_1 = [1,0,0,1,0,0,0,0] \\ p^1_2 = [0,0,0,1,0,0,1,0] & p^2_2 = [0,1,0,0,1,0,0,0] \\ p^1_3 = [0,0,0,0,0,1,0,0] & p^2_3 = [0,0,1,1,0,0,0,0] \\ p^1_4 = [0,0,0,0,0,1,1,1] & p^2_4 = [0,1,1,0,0,0,0,0] \\ p^1_5 = [0,0,0,0,-1,1,1,-1] & p^2_5 = [1,1,1,1,0,0,0,0] \\ q_1 = [-1,-1,1,0,0,-1,1,1] & q_2 = [0,1,-1,1,0,1,-1,0] \\ q_3 = [0,1,0,0,-1,1,0,1]. & \end{array}$$

For $r = 6$, $r \geq 8$, we prove the statement exploiting the following restriction exact sequence:

$$0 \to \mathscr{K}_2(r) \to \mathscr{K}_1(r) \to \mathscr{K}_1(r-3).$$

The kernel is empty by Proposition 19 and $\mathscr{K}_1(r-3)$ is empty by induction.

□

Proposition 21. *In the same notation as above, the matching linear system* $\mathscr{L}^m_V = \mathscr{L}_{r,3}(\{L, 2^{n^-(r-3,3)}\}, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)})$, *with* $L \cong \mathbb{P}^{r-3}$, *is empty, for* $r \geq 5$.

Proof. For $r = 5$, the matching system is $\mathscr{L}^m_V = \mathscr{L}_{5,3}(\{L, 2^3\}, 2^6, 1)$. With an explicit computation, one can easily see that there is a unique cubic in $\mathbb{P}^5$ that vanishes along $L = \{x_0 = x_1 = x_2 = 0\}$ and that is singular at the nine points in (3.2) (three of them supported on L). Therefore, there are no cubics passing through one further general point.
For $r = 6$ the matching system is $\mathscr{L}^m_V = \mathscr{L}_{6,3}(\{L, 2^5\}, 2^7)$. Restricting it to a general $L_1 \cong \mathbb{P}^3$ intersecting L in the support p of one of the nodes and specializing on it four general nodes, we get

$$0 \to \mathscr{L}_{6,2}(\{L, 2^4\}, \{L_1, 2^4\}, 2^3) \to \mathscr{L}_{6,3}(\{L, 2^5\}, 2^7) \to \mathscr{L}_{3,3}(\{p, 2\}, 2^4).$$

The kernel is empty by Proposition 20, and the restricted system is $\mathscr{L}_{3,3}(2^5)$ which is empty.
For $r = 7$ the matching system is $\mathscr{L}^m_V = \mathscr{L}_{7,3}(\{L, 2^7\}, 2^8)$, with $L \cong \mathbb{P}^4$. Let

$p_1, \ldots, p_7 \in L$ and $q_1, \ldots, q_8 \in \mathbb{P}^r \setminus L$ be the supports of the fifteen nodes. Let π be a hyperplane such that $L \cap \pi = L' \cong \mathbb{P}^3$ and such that $p_4, \ldots, p_7 \in L'$; moreover specialize on π the points $q_2, \ldots, q_8$:

$$\mathscr{L}_V^m \xrightarrow{\phi} \mathscr{L}_V^m{}_{|\pi} \subseteq \mathscr{L}_{6,3}(\{L', 2^4\}, 2^7).$$

Notice that the line joining q_1 and p_i is contained in all the sections of $\mathscr{L}_V^m$ and it intersects π at a point, for $i = 1, 2, 3$. Therefore

$$\mathscr{L}_V^m{}_{|\pi} \subseteq \mathscr{L}_{6,3}(\{L', 2^4\}, 2^7, 1^3).$$

It is non-special as consequence of the previous point (case $r = 6$), moreover it is empty, having virtual dimension equal to -1. Furthermore the kernel of the restriction map ϕ is

$$\mathscr{L}_{7,2}(\{L, 2^3\}, 2, 1^7).$$

It is easy to check that $l_{7,2}(\{L, 2^3\}, 2) = 6$, choosing for example $L = \{x_0 = x_1 = x_2 = 0\}$ and

$$\begin{array}{ll} p_1 = [0,0,0,0,0,0,0,1] & p_2 = [0,0,0,0,0,0,1,0] \\ p_3 = [0,0,0,0,0,1,0,0] & q_1 = [1,0,0,0,0,0,0,0]. \end{array}$$

Therefore, imposing seven further general base points, the resulting system is empty.
For $r \geq 8$, the statement follows by induction restricting to a general $\mathbb{P}^{r-3}$ and making specializations of the points as follows:

$$\mathscr{L}_{r,3}(\{L, 2^{n^-(r-3,3)}\}, 2^{r+1}, 1^{\gamma(r)-\gamma(r-3)}) \to \mathscr{L}_{r-3,3}(\{L', 2^{n^-(r-6,3)}\}, 2^{r-2}, 1^{\gamma(r)-\gamma(r-3)}),$$

where $L' = L \cap \mathbb{P}^{r-3} \cong \mathbb{P}^{r-6}$: the kernel is $\mathscr{K}_1(r) = \emptyset$ and this concludes the proof. □

Finally, being $\mathscr{L}_V^m = \hat{\mathscr{L}}_T^m = \emptyset$, for $r \geq 5$, $r \neq 7$, then $\mathscr{L} = \mathscr{L}_t$ is empty.

Remark 4. *For $r = 7$, the emptyness of the matching system does not suffice to conclude that the system of cubics of $\mathbb{P}^7$ with fifteen nodes is empty (and in particular non-special), because the kernel system $\hat{\mathscr{L}}_T \cong \mathscr{L}_{4,3}(2^7)$ on the other component is nonempty (see Section 1.2.1). Nevertheless this is crucial because it represents the starting point of the induction from $r-3$ to r, for $r \geq 10$, $r \equiv 1 \pmod 3$; so we will analyse this case separately.*

Cubics in $\mathbb{P}^7$

In the case $r = 7$ this method fails. We would have to blow up a $L \cong \mathbb{P}^4$, but this is not the right thing to do, because cubics with seven nodes are defective there. To avoid the problem, we will reproduce the same argument, but blowing up a subspace L_1 of codimension four, instead of three, in the central fiber of the trivial family $\mathbb{P}^7 \times \Delta$. Let us denote by T the exceptional component of the new special fiber, and with V the strict transform, as above. Let $p : T \to L_1$ the natural map and $\pi := p^*\mathcal{O}_{L_1}(1)$. Twist by $-T$ and consider the limit of the linear system of cubics of $\mathbb{P}^7$:

- $|3\pi + Q|$ on T and
- $|3H_V - Q|$ on V, where H_V is the pull-back of an hyperplane, and Q is the exceptional divisor of the blow up along $L_1 \cong \mathbb{P}^3$.

Consider the system $\mathscr{L}_{7,3}(2^{15})$ on the general fiber. To prove its emptyness, we use the same trick as in the general case: we specialize the points on the two components as follows:

$$\begin{array}{rcl} \mathscr{L}_T & = & |3\pi + Q - 2^5|, \\ \mathscr{L}_V & = & |3H_V - Q - 2^{10}| \cong \mathscr{L}_{7,3}(L_1, 2^{10}) \end{array}$$

The kernels of the restriction to Q are

$$\begin{array}{rcl} \hat{\mathscr{L}}_T & \cong & \mathscr{L}_{3,3}(2^5) = \emptyset, \\ \hat{\mathscr{L}}_V & = & |3H_V - 2Q - 2^{10}| \cong \mathscr{L}_{7,3}(2L_1, 2^{10}). \end{array}$$

Each node specialized on T selects a fiber of the ruling of T. Each fiber cuts a $\mathbb{P}^3$ at the intersection Q, which corresponds to a fiber of the ruling of Q. So, as in the general case, the matching system on V is

$$\mathscr{L}^m_{V,7} \cong \mathscr{L}_{7,3}(\{L_1, 2^5\}, 2^{10})$$

and it has virtual dimension -1. To prove its emptyness, we make two subsequent specialization of the general nodes, five on L_2 and five on L_3, where $L_2, L_3 \cong \mathbb{P}^3$ are general subspaces of $\mathbb{P}^7$, as we did for the general case:

$$0 \to \mathcal{K}_1 \to \mathscr{L}_{7,3}(\{L_1, 2^5\}, 2^{10}) \to \mathscr{L}_{3,3}(2^5) \to 0$$

where $\mathcal{K}_1 := \mathscr{L}_{7,3}(\{L_1, 2^5\}, \{L_2, 2^5\}, 2^5)$ and

$$0 \to \mathcal{K}_2 \to \mathcal{K}_1 \to \mathscr{L}_{3,3}(2^5) \to 0$$

where $\mathcal{K}_2 := \mathscr{L}_{7,3}(\{L_1, 2^5\}, \{L_2, 2^5\}, \{L_3, 2^5\})$. With an explicit computation we prove that $\mathcal{K}_2$ is empty, choosing for example

$$\begin{array}{rcl} L_1 & = & \{x_0 = x_1 = x_2 = x_3 = 0\}, \\ L_2 & = & \{x_4 = x_5 = x_6 = x_7 = 0\}, \\ L_3 & = & \{x_0 - x_4 = x_1 - x_5 = x_2 - x_6 = x_3 - x_7 = 0\} \end{array}$$

and choosing $p^i_j \in L_i$, for $i = 1, 2, 3$ and $j = 1 \ldots , 5$, as follows:

$$\begin{array}{ll}
p^1_1 = [0, 0, 0, 0, 0, 0, -1, 1] & p^1_2 = [0, 0, 0, 0, 0, 0, 1, 0] \\
p^1_3 = [0, 0, 0, 0, 0, 1, 0, 0] & p^1_4 = [0, 0, 0, 0, 1, 0, 1, 0] \\
p^1_5 = [0, 0, 0, 0, 1, 1, 1, 1] & p^2_1 = [1, -1, 0, 0, 0, 0, 0, 0] \\
p^2_2 = [0, 1, 0, 0, 0, 0, 0, 0] & p^2_3 = [0, 0, 1, 0, 0, 0, 0, 0] \\
p^2_4 = [0, 1, 0, 1, 0, 0, 0, 0] & p^2_5 = [1, 1, 1, 1, 0, 0, 0, 0] \\
p^3_1 = [1, 1, 1, 0, 1, 1, 1, 0] & p^3_2 = [0, 1, 1, 1, 0, 1, 1, 1] \\
p^3_3 = [1, 1, 0, 1, 1, 1, 0, 1] & p^3_4 = [1, 0, 1, 1, 1, 0, 1, 1] \\
p^3_5 = [1, -1, 1, -1, 1, -1, 1, -1]. &
\end{array}$$

Therefore also $\mathcal{K}_1$ is empty and, as consequence, $\mathscr{L}^m_{V,7} = \emptyset$. This completes the proof of Theorem 17.

3.1.2 Quartics

Quartics in $\mathbb{P}^3$

If $n > 9$ the system is empty because there exists a unique quartic surface singular at nine points (see Section 1.2.1). For $n = 8$, we prove non-speciality of the corresponding linear system performing a $(1, 4)$-degeneration:

$$\begin{array}{ll}
\mathscr{L}_{\mathbb{F}} = \mathscr{L}_{3,4}(3, 2^4) & \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{3,4}(4, 2^4) \cong \mathscr{L}_{2,4}(2^4) \\
\mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,3}(2^4) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{3,2}(2^4)
\end{array}$$

The system $\mathscr{L}_{\mathbb{P}}$ is non-special as we have seen in the previous section, while the system $\mathscr{L}_{\mathbb{F}}$ is non-special by Lemma 11; $\mathscr{R}_{\mathbb{F}}$ is the complete series $\mathscr{L}_{2,3}(1^4)$ and the restricted systems intersect transversally (see the proof of Lemma 12). Hence, by Proposition 8, we get

$$\begin{aligned}
l_0 &= \dim(\mathscr{R}) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 \\
&= (l_{\mathbb{P}} - 4) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 = v_{3,4}(2^8).
\end{aligned}$$

It follows that also the system of quartic surfaces of $\mathbb{P}^3$ having n nodes, with $n < 8$, is non-special.

Quartics in $\mathbb{P}^4$

The systems of quartics with n nodes in $\mathbb{P}^4$, for $n > 14$, is obviously empty, being $l_{4,4}(2^{14}) = 0$ (see Section 1.2.1).
Performing a $(1, 8)$-degeneration of $\mathbb{P}^4$, we prove that the system $\mathscr{L}_{4,4}(2^{13})$ is non-special, exactly as for the case of $\mathbb{P}^3$. Indeed

$$\begin{array}{ll}
\mathscr{L}_{\mathbb{F}} = \mathscr{L}_{4,4}(3, 2^8) & \hat{\mathscr{L}}_{\mathbb{F}} \cong \mathscr{L}_{3,4}(2^8) \\
\mathscr{L}_{\mathbb{P}} = \mathscr{L}_{4,3}(2^5) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{4,2}(2^5) = \emptyset
\end{array}$$

Both systems $\mathscr{L}_{\mathbb{P}}$ and $\mathscr{L}_{\mathbb{F}}$ are non special with positive dimension, moreover $\mathscr{R}_{\mathbb{F}} = \mathscr{L}_{3,3}(1^8)$ and transversality holds, so

$$\begin{aligned} l_0 &= \dim(\mathscr{R}) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 \\ &= (l_{\mathbb{P}} - 8) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 = v_{4,4}(2^{13}). \end{aligned}$$

As consequence, for a smaller number of double points, the system of quadrics of $\mathbb{P}^4$ results to be non-special.

Quartics in $\mathbb{P}^r$, $r \geq 5$

Let $n^- := n^-(r,4)$ and $n^+ := n^+(r,4)$. We will prove non-speciality of the system of quartic hypersurfaces of $\mathbb{P}^r$, with $r \geq 5$, with a collection of n nodes, with $n^- \leq n \leq n^+$; in all the remaining cases, non-speciality follows. The expected dimension of $\mathscr{L}$ is

$$e(\mathscr{L}) = \begin{cases} -1 + l^- & \text{if } n = n^- \\ -1 & \text{if } n = n^+ \end{cases}.$$

Let us perform a $(1, n-r-1)$-degeneration. We will show that, under this choice, the two kernel system are both empty and the intersection $\mathscr{R}$ of the restricted systems has dimension equal to $e(\mathscr{L})$. On the two components of the central fiber we get the following linear systems:

$$\mathscr{L}_{\mathbb{F}} = \mathscr{L}_{r,4}(3, 2^{n-r-1}) \text{ and } \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{r,3}(2^{r+1}).$$

The system $\mathscr{L}_{\mathbb{P}}$ is non-special by Theorem 17. Furthermore the system $\mathscr{L}_{\mathbb{F}}$ is non-special by Lemma 14. Being $\hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{r,2}(2^{r+1}) = \emptyset$, the restriction map to R

$$\mathscr{L}_{\mathbb{P}} \hookrightarrow \mathscr{R}_{\mathbb{P}} \subseteq |\mathcal{O}_{\mathbb{P}^{r-1}}(3)|$$

is injective. We want to describe the image $\mathscr{R}_{\mathbb{P}}$. We know that a cubic singular at two points must contain the whole line joining them. Similarly, if a cubic has k nodes, then it must contains all the $\binom{k}{2}$ lines joining the points. Consequentely, when we restrict to the hyperplane R, the image of the cubics in $\mathscr{L}_{\mathbb{P}}$ must contain the traces of these lines as base points; so we get

$$\mathscr{R}_{\mathbb{P}} \subseteq \mathscr{L}_{r-1,3}(1^{\binom{r+1}{2}}).$$

Actually, these $\binom{r+1}{2}$ points give linearly independent conditions, and therefore $\mathscr{R}_{\mathbb{P}}$ is the complete series.

Proposition 22. *In the setting of above, the system $\mathscr{R}_{\mathbb{P}}$ is the complete linear system of cubics of R with $\binom{r+1}{2}$ base points and* $\dim(\mathscr{R}_{\mathbb{P}}) = \binom{r+2}{r-1} - 1 - \binom{r}{2}$.

Proof. We have to prove that the $\binom{r+1}{2}$ points on R, traces of the lines joining the $r+1$ nodes $p_1, \ldots, p_{r+1}$ specialized on the component $\mathbb{P}$, impose independent conditions. If we prove that this is true for quadrics, it will be true in higher degree and in particular for cubics.
The proof will be by induction on r. The base step ($r = 3$) is easy: let $p_1, \ldots, p_4$ points in $\mathbb{P} \cong \mathbb{P}^3$: three of them, say p_1, p_2, p_3, span a plane π, which cuts a line π' on $R \cong \mathbb{P}^2$; on this line we will have the three distinct points given as traces of the three lines $< p_i, p_j >$, $i \neq j$, $i, j = 1, 2, 3$. The line π' splits off the system of conics through these three points, thus

$$\mathscr{L}_{2,2}(1^6) = \pi' + \mathscr{L}_{2,1}(1^3),$$

where the three base points of the system on the right are the projection of p_1, p_2, p_3 from p_4 to R and they will not lie on a line, by generality. So our system is empty.
For the inductive step, suppose that $\mathscr{L}_{r-2,2}(1^{\binom{r}{2}}) = \emptyset$ and consider $\mathscr{L}_{r-1,2}(1^{\binom{r+1}{2}})$: we have to show that there are no quartics in $R \cong \mathbb{P}^{r-1}$ through these points. On the hyperplane $\pi \subset \mathbb{P}^r$ spanned by $p_1, \ldots, p_r$, we have the $\binom{r}{2}$ lines joining the r points two by two. Now, π cuts $\pi' \cong \mathbb{P}^{r-2}$ on R and π' contains the $\binom{r}{2}$ traces of the lines on π. These points are in general position by induction. Therefore the system of quadrics of π' containing these points is empty; this means that π' splits off $\mathscr{L}_{r-1,2}(1^{\binom{r+1}{2}})$:

$$\mathscr{L}_{r-1,2}(1^{\binom{r+1}{2}}) = \pi' + \mathscr{L}_{r-1,1}(1^r).$$

Notice finally that the

$$r = \binom{r+1}{2} - \binom{r}{2}$$

points on R correspond to the lines $< p_{r+1}, p_i >, i = 1, \ldots, r$; precisely they are the projections $p'_1, \ldots, p'_r$ of $p_1, \ldots, p_r$ from p_{r+1} to R. Therefore, being $p_1, \ldots, p_r$ in general position in $\mathbb{P}^r$, then also their projections are in general position in R, so

$$\mathscr{L}_{r-1,1}(1^r) = \mathscr{L}_{r-1,1}(p'_1, \ldots, p'_r) = \emptyset.$$

Notice finally that

$$\dim(\mathscr{R}_{\mathbb{P}}) = \binom{r+2}{r-1} - 1 - \binom{r+1}{2} = \binom{r+2}{2} - 1 = \dim(\mathscr{L}_{\mathbb{P}}).$$

□

The system $\hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{r,4}(4, 2^{n-r-1})$ has dimension $l_{r-1,4}(2^{n-r-1})$ and, being

$\mathscr{L}_{r-1,4}(2^{n-r-1})$ non-special and

$$\begin{aligned} v_{r-1,4}(2^{n-r-1}) &= \binom{r+3}{4} - 1 - r(n-r-1) \\ &\leq \binom{r+3}{4} - 1 - \frac{r}{r+1}\binom{r+4}{4} + r(r+1) \\ &= -\frac{r^3 - 3r^2 - 2r}{8} - 1 \leq -1, \end{aligned}$$

it is empty. Finally, observe that

$$\begin{aligned} l_{\mathbb{F}} - \binom{r+1}{2} &= \binom{r+4}{r} - 1 - \binom{r+2}{r} - (r+1)(n-r-1) - \binom{r+1}{2} \\ &= v(\mathscr{L}) \end{aligned}$$

therefore, the intersection $\mathscr{R}$ of the restricted systems has dimension

$$\dim \mathscr{R} = \max\left\{ l_F - \binom{r+1}{2}, -1 \right\} = e(\mathscr{L}).$$

Thus, we conclude applying Formula (2.1).

Remark 5. *This discussion does not apply if $r = 3, 4$. Indeed the kernel on the component $\mathbb{F}$ would be isomorphic to $\mathscr{L}_{2,4}(2^5)$ and $\mathscr{L}_{3,4}(2^9)$ respectively, that are special and in particular nonempty.*

3.2 The proof in $\mathbb{P}^3$

In this section we will apply the techniques introduced in Chapter 2 to the case $r = 3$ and $d \geq 5$. This plays the role of the starting point of the induction on r. We will investigate the non-speciality of the linear system $\mathscr{L}_{3,d}(2^n)$, for $n^-(3,d) \leq n \leq n^+(3,d)$.

The cases with $d \not\equiv 0 \pmod 3$

Observe that if $d = 3k+1$, for some k, then

$$\frac{1}{3}\binom{d+2}{2} = \frac{(k+1)(3k+2)}{2} \in \mathbb{Z};$$

if $d = 3k+2$, for some k, then

$$\frac{1}{3}\binom{d+2}{2} = \frac{(k+1)(3k+4)}{2} \in \mathbb{Z};$$

while if d is a multiple of three this number is not an integer. So, when $d \equiv 1, 2 \pmod 3$, define the integer b as follows:

$$b := \frac{1}{3}\binom{d+2}{2}$$

and perform a $(1, b)$-degeneration of $\mathbb{P}^3$ and of the linear system $\mathscr{L}_{3,d}(2^n)$.

Proposition 23. *Let $d \geq 5$, $d \not\equiv 0 \pmod 3$. Let $n^- \leq n \leq n^+$. Assume that $\mathscr{L}_{\mathbb{P}}$ and $\hat{\mathscr{L}}_{\mathbb{P}}$ are non-special. Then the linear system $\mathscr{L}$ is non-special.*

Proof. We prove that $\mathscr{L}$ is non-special if $n = n_0^+$ and $n = n_0^-$.

- Case $n = n_0^+$. Notice that the kernel system $\hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{3,d}(d, 2^b)$ has dimension
$$\hat{l}_{\mathbb{F}} = v_{2,d}(2^b) = \binom{d+2}{2} - 1 - 3b = -1.$$
Consequentely, $\mathscr{L}_{\mathbb{F}}$ is non-special, by Lemma 11; moreover it cuts out the complete series $\mathscr{R}_{\mathbb{F}} = \mathscr{L}_{2,d-1}(1^b)$ on R by Lemma 12. The kernel system $\hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{3,d-2}(2^{n-b})$ is non special by assumption, and it has virtual dimension at most -1, in fact
$$\begin{aligned}\hat{v}_{\mathbb{P}} &= \binom{d+1}{3} - 1 - 4n + 4b \\ &\leq \binom{d+1}{3} - 1 - \binom{d+3}{3} + \frac{4}{3}\binom{d+2}{2} = \frac{1-d^2}{3} - 1 \leq -1,\end{aligned}$$
being
$$n \geq \frac{1}{4}\binom{d+3}{3}.$$
The system $\mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,d-1}(2^{n-b})$ is non-special by assumption, and $r_{\mathbb{P}} = l_{\mathbb{P}}$. Therefore $\mathscr{R}$ is the system of curves of $\mathscr{R}_{\mathbb{P}}$ with b more simple points (the points imposed by $\mathscr{R}_{\mathbb{F}}$):
$$\dim(\mathscr{R}) = \max\{-1, l_{\mathbb{P}} - b\}.$$
We have
$$\begin{aligned} l_{\mathbb{P}} - b &= \binom{d+2}{3} - 1 - 4n + 3b \\ &\leq \binom{d+2}{3} - 1 - \binom{d+3}{3} + \binom{d+2}{2} - 1.\end{aligned}$$
Therefore $\mathscr{L}_{3,d}(2^{n^+})$ is empty, as expected.

- Case $n = n_0^-$. The system $\hat{\mathscr{L}}_{\mathbb{F}} \cong \mathscr{L}_{2,d}(2^b)$ is empty and $l_{2,d}(2^b) = -1$. Moreover, by Lemma 12, transversality holds and, by Lemma 11, $\mathscr{L}_{\mathbb{F}}$ is non-special. The system $\mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,d-1}(2^{n-b})$ is non special by assumption and its virtual dimension is at least -1 being
$$v_{\mathbb{P}} = \binom{d+2}{3} - 1 - 4(n-b) \geq \binom{d+2}{2} - 1.$$

Finally, the system $\hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{3,d-2}(2^{n-b})$ is non-special, and in particular empty, in fact

$$\begin{aligned}\hat{v}_{\mathbb{P}} &= \binom{d+1}{3} - 1 - 4(n-b) \\ &\leq \binom{d+1}{3} - \binom{d+3}{3} + \frac{4}{3}\binom{d+2}{2} + 3 = \frac{1-d^2}{3} + 3 \leq -1.\end{aligned}$$

Applying Proposition 8, we conclude the proof.

□

Corollary 24. *Keeping the same assumptions as in Proposition 23, $\mathscr{L}_{3,d}(2^n)$ is non-special for every n.*

The cases with $d \equiv 0 \pmod 3$

The case of sextics. We study first of all the linear system

$$\mathscr{L} = \mathscr{L}_{3,6}(2^{21}).$$

We perform a $(1,10)$-degeneration, specializing the first ten points, say $p_1, \ldots, p_{10}$, on $\mathbb{F}$ and the remaining ones, say $p_{11} \ldots, p_{21}$, on $\mathbb{P}$:

$$\mathscr{L}_{\mathbb{F}} = \mathscr{L}_{3,6}(5, 2^{10}) \text{ and } \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,5}(2^{11}).$$

We know by induction that the system $\mathscr{L}_{\mathbb{P}}$ is non-special. Indeed, consider for a moment the system $\mathscr{L}_{3,5}(2^{14})$: the hypothesis of Proposition 23 are satisfied, in fact $n-b$ would be equal to seven, and both the quartics and the cubics of $\mathbb{P}^3$ with seven nodes are non-special, as we have already proved in Section 3.1; in particular $\mathscr{L}_{3,5}(2^{11})$ is non-special.
The system $\mathscr{L}_{\mathbb{F}}$ is non-special and it has dimension 8, by Lemma 15. Now, we want to degenerate the collection of nodes in such a way that one of the points on $\mathbb{F}$ and three of the points on $\mathbb{P}$ approach R. To do that, we perform a degeneration of the central fiber (see Section 2.3). Let $q_1, \ldots, q_{10}$ be the limits of the points specialized on $\mathbb{F}$ and $q_{12}, \ldots, q_{21}$ the limits of the points specialized on $\mathbb{P}$ and

- $q_1, q_{11}, q_{12}, q_{13} \in R_0$,
- $q_2, \ldots, q_{10} \in \mathbb{F}_0 \setminus R_0$ and
- $q_{14}, \ldots, q_{21} \in \mathbb{P}_0 \setminus R_0$.

We get the following restriction exact sequence

$$0 \to \hat{\mathscr{L}}'_{\mathbb{F}_0} = \mathscr{L}_{3,6}(6, 2^9, 1) \to \mathscr{L}'_{\mathbb{F}_0} \to \mathscr{R}'_{\mathbb{F}_0}.$$

The nine points, traces on R of the nine lines through $q_2, \ldots, q_{10}$ and the node supported at q_1, are base points of $\mathscr{R}'_{\mathbb{F}_0}$ (see Lemma 12), thus

$$\mathscr{R}'_{\mathbb{F}_0} \subseteq \mathscr{L}_{2,5}(1^9, 2).$$

Furthermore the kernel system $\hat{\mathscr{L}}'_{\mathbb{F}_0}$ has dimension $v_{2,6}(2^9, 1) = -1$, then $\mathscr{R}'_{\mathbb{F}_0}$ is the complete series $\mathscr{L}_{2,5}(1^9, 2)$.
On the other component, we get

$$0 \to \hat{\mathscr{L}}'_{\mathbb{P}_0} = \mathscr{L}_{3,4}(2^8, 1^3) \to \mathscr{L}'_{\mathbb{P}_0} \to \mathscr{R}'_{\mathbb{P}_0} \subseteq \mathscr{L}_{2,5}(2^3).$$

The kernel is empty and it has virtual dimension exactly -1. Thus also in this case the restricted system is complete.
One has

$$\mathscr{R}'_0 = \mathscr{L}_{2,5}(2^4, 1^9) = \emptyset.$$

Finally, using formula (2.8) for the dimension of the system on the central fiber, we get

$$l'_0 = \hat{l}'_{\mathbb{P}_0} + \hat{l}'_{\mathbb{F}_0} + \dim(\mathscr{R}'_0) + 2 = -1$$

so the limiting system of the second degeneration is empty. By upper semicontinuity, the system on the general fiber is empty and therefore, a fortiori, $\mathscr{L}$ is empty, as expected.

The case $d \equiv 3 \pmod 6$. Let $d = 6k + 3$, $k \geq 1$. Observe that

$$\frac{1}{4}\binom{d+3}{3} = \frac{(k+1)(3k+2)(6k+5)}{2} \in \mathbb{Z}.$$

Consider the linear system $\mathscr{L} = \mathscr{L}_{3,d}(2^n)$ of surfaces of degree d with

$$n = \frac{1}{4}\binom{d+3}{3}$$

nodes: $v(\mathscr{L}) = -1$. Performing a $(1, b)$-degeneration, the limit system restricts to the following systems on the two components of the special fiber:

$$\mathscr{L}_{\mathbb{F}} = \mathscr{L}_{3,d}(d-1, 2^b) \qquad \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,d-1}(2^{n-b}),$$

specializing $p_1, \ldots, p_b$ on $\mathbb{F}$ and $p_{b+1}, \ldots, p_n$ on $\mathbb{P}$. As in the sextic case, let us suppose that β points of $\mathbb{F}$ and α points of $\mathbb{P}$ approach the plane R, performing a degeneration of the special fiber. We obtain the following exact sequences on the central fiber of the second degeneration (see Section 2.3):

$$0 \to \hat{\mathscr{L}}'_{\mathbb{F}_0} = \mathscr{L}_{3,d}(d, 2^{b-\beta}, 1^\beta) \to \mathscr{L}'_{\mathbb{F}_0} \to \mathscr{R}'_{\mathbb{F}_0} \subseteq \mathscr{L}_{2,d-1}(1^{b-\beta}, 2^\beta).$$

and

$$0 \to \hat{\mathscr{L}}'_{\mathbb{P}_0} = \mathscr{L}_{3,d-2}(2^{n-b-\alpha}, 1^\alpha) \to \mathscr{L}'_{\mathbb{P}_0} \to \mathscr{R}'_{\mathbb{P}_0} \subseteq \mathscr{L}_{2,d-1}(2^\alpha),$$

with $\mathscr{L}'_{\mathbb{F}_0} = \mathscr{L}_{\mathbb{F}}$ and $\mathscr{L}'_{\mathbb{P}_0} = \mathscr{L}_{\mathbb{P}}$. Notice that

$$\frac{1}{3}\binom{d+2}{2} = 6k^2 + 9k + 3 + \frac{1}{3}.$$

So, choose

$$b = \frac{1}{3}\binom{d+2}{2} + \frac{2}{3} \in \mathbb{Z}$$

and $\beta = 1$.

Proposition 25. *Keeping the same setting as above, assume that $\hat{\mathscr{L}}'_{\mathbb{P}_0}$ and $\mathscr{L}'_{\mathbb{P}_0}$ are non-special. Then the linear system $\mathscr{L}_{r,3}(2^n)$ is empty.*

Proof. The kernel $\hat{\mathscr{L}}'_{\mathbb{F}_0}$ consists of the cones of degree d with vertex at the d-ple point and having $b-1$ nodes and a simple point; the dimension is

$$\hat{l}'_{\mathbb{F}_0} = v_{2,d}(2^{b-1}, 1) = -1.$$

The system $\mathscr{L}'_{\mathbb{F}_0}$ is non-special, as consequence. Accordingly, $\mathscr{R}'_{\mathbb{F}_0}$ is the complete series $\mathscr{L}_{2,d-1}(1^{b-1}, 2)$, in fact, being $h^0(\mathbb{F}_0, \hat{\mathscr{L}}'_{\mathbb{F}_0}) = h^1(\mathbb{F}_0, \hat{\mathscr{L}}'_{\mathbb{F}_0}) = 0$, then $\dim(\mathscr{R}'_{\mathbb{F}_0}) = l'_{\mathbb{F}_0} = l_{2,d-1}(1^{b-1}, 2)$. On the exceptional component $\mathbb{P}$ of the first degeneration we specialized $n-b$ nodes. Observe that there exists an integer α such that $\hat{v}'_{\mathbb{P}_0} = -1$, namely

$$\alpha = \frac{1}{3}\left[4n - 4b - \binom{d+1}{3}\right] = 4k^2 + 4k.$$

The systems $\hat{\mathscr{L}}'_{\mathbb{P}_0}$ and $\mathscr{L}'_{\mathbb{P}_0}$ are non-special by assumption. Furthermore $\mathscr{L}'_{\mathbb{P}_0}$ cuts the complete series on R_0, i.e. $\mathscr{R}'_{\mathbb{P}_0} = \mathscr{L}_{2,d-1}(2^\alpha)$, being

$$\dim(\mathscr{R}'_{\mathbb{P}_0}) = l'_{\mathbb{P}_0} = l_{2,d-1}(2^\alpha).$$

Finally, the intersection of the restricted systems is

$$\mathscr{R}'_0 = \mathscr{L}_{2,d-1}(2^{\alpha+1}, 1^{b-1}) :$$

it is non-special being the points in general position by construction; it has virtual dimension exactly -1 and this concludes the proof.

□

The case $d \equiv 0 \pmod 6$, $d \neq 6$. We will use the technique described in Section 2.2. Let $k \in \mathbb{N} \setminus \{0, 1\}$ be such that $d = 6k$; notice that

$$c = \frac{1}{4}\binom{d}{3} = \frac{k(3k-1)(6k-1)}{2} \in \mathbb{Z}$$

$$a = \frac{1}{3}\binom{d}{2} = k(6k-1) \in \mathbb{Z}$$

Let us first of all perform a $(2, n-c)$-degeneration:

$$\begin{array}{ll} \mathscr{L}_{\mathbb{F}} = \mathscr{L}_{3,d}(d-2, 2^{n-c}) & \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{3,d}(d-1, 2^{n-c}) \\ \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,d-2}(2^c) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{3,d-3}(2^c). \end{array}$$

Assume that $\mathscr{L}_{\mathbb{P}}$ and $\hat{\mathscr{L}}_{\mathbb{P}}$ are non-special. We have $\hat{v}_{\mathbb{P}} = -1$ and $v_{\mathbb{P}} = \binom{d}{2} - 1$. Therefore the restricted systems intersect transversally, in particular we have that $\mathscr{R} = \mathscr{R}_{\mathbb{F}}$, because $\mathscr{R}_{\mathbb{P}}$ fills up the whole space, namely $\mathscr{R}_{\mathbb{P}} = |\mathcal{O}_{\mathbb{P}^2}(d-2)|$. Observe that $v_{3,d}(2^n) = v_{3,d}(d-2, 2^{n-c})$ and non-speciality of the system $\mathscr{L}_{3,2}(d-2, 2^{n-c})$ implies non-speciality of $\mathscr{L}$.

Proposition 26. *If $d \equiv 0 \pmod 6$ and $d \neq 6$, then the linear system $\mathscr{L} = \mathscr{L}_{3,d}(d-2, 2^{n-c})$ is non-special.*

Proof. Let $b = n - c - a$ (a and c as above). Performing a degeneration of $\mathbb{P}^r$ to $X_0 = \mathbb{F} \cup \mathbb{P}$ (in the usual notation) and specializing on $\mathbb{F}$ b nodes and on $\mathbb{P}$ the remaining a nodes and the $(d-2)$-point, we get:

$$\begin{array}{ll} \mathscr{L}_{\mathbb{F}} = \mathscr{L}_{3,d}(d-1, 2^b) & \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{3,d}(d, 2^b) \cong \mathscr{L}_{2,d}(2^b) \\ \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{3,d-1}(d-2, 2^a) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{3,d-2}(d-2, 2^a) \cong \mathscr{L}_{2,d-2}(2^a). \end{array}$$

We have $\hat{l}_{\mathbb{P}} = l_{2,d-2}(2^a) = v_{2,d-2}(2^a) = -1$. Moreover $a \leq k_0(d-1)$ and $b \leq k_0(d)$, being

$$k_0(d-1) - a \geq \frac{d^2+2d-6}{12} \geq 0 \text{ and } k_0(d) - b \geq \frac{3d^2-3d-12}{8} \geq 0$$

for $d \geq 12$; thus $\mathscr{L}_{\mathbb{P}}$ and $\mathscr{L}_{\mathbb{F}}$ are both non-special by Lemma 15. Moreover the kernel system $\hat{\mathscr{L}}_{\mathbb{F}}$ is empty, being

$$v_{2,d}(2^b) = \binom{d+2}{2} - 3(n-c-a) \leq -\frac{d^2+d+22}{8} \leq -1.$$

Now, if one can show that $\mathscr{R}_{\mathbb{P}} = \mathscr{L}_{2,d-1}(1^a)$, one would have transversality, concluding the proof. Indeed the system $\mathscr{R}$ would contain the section of $\mathscr{R}_{\mathbb{F}}$ that vanish at the a base points imposed by $\mathscr{R}_{\mathbb{P}}$; being those points in general position on R, one would get

$$v(\mathscr{R}) = r_{\mathbb{F}} - a = \binom{d+3}{3} - 1 - \binom{d+1}{3} - 4b - a = -1.$$

The series cut out by $\mathscr{L}_{\mathbb{P}}$ is complete, in fact a surface $S \in \mathscr{L}_{\mathbb{P}}$ is described by the vanishing of a homogeneous polynomial of the form

$$f(x_0, \ldots, x_3) = f_{d-1}(x_0, \ldots, x_2) + x_3 f_{d-2}(x_0, \ldots, x_2),$$

with partial derivatives vanishing at $p_1 \ldots, p_a$, supposing, without waste of generality, $p = [0, 0, 0, 1] \in \mathbb{P}^3$. The restriction C of S to R has equation

$$f_{d-1}(x_0, \ldots, x_2) = 0.$$

We have to prove that $f_{d-1}(p'_j) = 0$, for $j = 1, \ldots, a$ (where p'_j are the projection of p_j from p to R). The linear conditions imposed by the node p_j to S are the following

$$\begin{cases} \partial_{x_i} f_{d-1}(p'_j) + p_{j,r}\partial_{x_i} f_{d-2}(p'_j) = 0, i = 0, 1, 2 \\ f_{d-2}(p'_j) = 0. \end{cases}$$

for $j = 1, \ldots a$. Now, using the *Euler formula* for homogeneous polynomials, we get

$$\begin{aligned} 0 = f_{d-2}(p'_j) &= \frac{1}{d-2}\sum_{i=0}^{3} p'_i \partial_{x_i} f_{d-2}(p'j) \\ &= -\frac{1}{d-2}\frac{1}{p_{j,3}}\sum_{i=0}^{3} p'_i \partial_{x_i} f_{d-1}(p'j) \\ &= -\frac{d-1}{d-2}\frac{1}{p_{j,3}} f_{d-1}(p'_j) \end{aligned}$$

and this prove that $p'_1, \ldots, p'_a$ are base points for $\mathscr{R}_{\mathbb{P}}$. Furthermore

$$r_{\mathbb{P}} = l_{\mathbb{P}} - \hat{l}_{\mathbb{P}} - 1 = l_{2,d-1}(1^a),$$

thus $\mathscr{L}_{\mathbb{P}}$ cuts the complete series on R. □

Corollary 27. *In the same notation as above, if $\mathscr{L}_{3,d-2}(2^c)$ and $\mathscr{L}_{3,d-3}(2^c)$ are non-special, then the system $\mathscr{L}_{3,d}(2^n)$ is non-special too.*

Putting all together, we obtain the following

Theorem 28. *Let $d \geq 3$. For a general collection of n double points, the linear system $\mathscr{L}_{3,d}(2^n)$ is non-special, except if $(d, n) = (4, 9)$.*

Proof. The cubic and quartic cases have been analysed separately in Section 3.1. We will apply induction for $d \geq 5$. The first case we meet is $\mathscr{L}_{3,5}(2^n)$: for $n = 14$ the system has virtual dimension equal to -1. Proposition 23 shows that it is non-special. The sextic case has been already analysed above. For $d \geq 7$, we have to study non-speciality of $\mathscr{L}_{r,d}(2^n)$, supposing that the thesis is true for every smaller degree. Consider the following possibilities:

1. Let either $d = 3k + 2$ or $d = 3k + 1$, $k \geq 2$. We know by induction that $\mathscr{L}_{r,d-1}(2^{n-b})$ and $\mathscr{L}_{r,d-2}(2^{n-b})$ are non-special, so we conclude using Proposition 23.

2. If $d = 6k$, $k \geq 2$, then we exploit Corollary 27; while if $d = 6k+3$, $k \geq 2$, we apply Proposition 25.

□

3.3 The proof in higher dimension

This last Section is devoted to the proof of the following

Theorem 29. *Let $r \geq 4$ and $d \geq 5$. Then the system $\mathscr{L}_{r,d}(2^n)$ is non-special.*

We will study linear system of hypersurfaces of $\mathbb{P}^r$, with $r \geq 4$, of degree $d \geq 5$ with a collection of n nodes in general position; in this range we will never deal with special cases. Take n general points, such that $n^-(r,d) \leq n \leq n^+(r,d)$, then the linear system $\mathscr{L} = \mathscr{L}_{r,d}(2^n)$ has expected dimension

$$e(\mathscr{L}) = \begin{cases} -1 + l^- & \text{if } n = n^- \\ -1 & \text{if } n = n^+ \end{cases}$$

We will show that $\mathscr{L}$ has the expected dimension by induction on r and on d, exploiting as base steps the case of cubics and quartics, for the induction on the degree, and the case of surfaces of $\mathbb{P}^3$ for the induction on the dimension. The technique will be the usual one: we will perform $(1,b)$-degenerations, trying to find a good specialization of the n points on the components $\mathbb{F}$ and $\mathbb{P}$ of the special fiber.
Let us define the number

$$b_0 := \frac{1}{r}\binom{r+d-1}{r-1}.$$

If it is an integer, we will specialize $b = b_0$ point on $\mathbb{F}$; if, to the opposite, b_0 is not an integer, then we will construct a further degeneration of the central fiber letting some of the points go to the intersection of the components, as we already did in $\mathbb{P}^3$ for $d = 6$ and $d \equiv 3 \pmod 6$.

Case $b_0 \in \mathbb{Z}$. Specializing $b = b_0$ points on the component $\mathbb{F}$ of the central fiber, and the remaining $n - b$ on $\mathbb{P}$, we get :

$$\begin{array}{ll} \mathscr{L}_{\mathbb{F}} = \mathscr{L}_{r,d}(d-1, 2^b) & \hat{\mathscr{L}}_{\mathbb{F}} = \mathscr{L}_{r,d}(d, 2^b) \cong \mathscr{L}_{r-1,d}(2^b) \\ \mathscr{L}_{\mathbb{P}} = \mathscr{L}_{r,d-1}(2^{n-b}) & \hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{r,d-2}(2^{n-b}) \end{array}$$

Proposition 30. *Keep the same notation of above. Assume that the linear systems $\mathscr{L}_{r-1,d}(2^b)(\cong \hat{\mathscr{L}}_{\mathbb{F}})$, $\mathscr{L}_{\mathbb{P}}$ and $\hat{\mathscr{L}}_{\mathbb{P}}$ are non-special. Then the linear system $\mathscr{L}$ is non-special too.*

Proof. The system $\hat{\mathscr{L}}_{\mathbb{F}}$ is special of dimension $l_{r-1,d}(2^b)$; the system $\mathscr{L}_{r-1,d}(2^b)$ is non-special, by assumption, and moreover $v_{r-1,d}(2^b) = -1$; therefore the kernel system $\hat{\mathscr{L}}_{\mathbb{F}}$ is empty. As a consequence, the system $\mathscr{L}_{\mathbb{F}}$ is non-special, by Lemma 11 and it cuts the complete series

$$\mathscr{R}_{\mathbb{F}} = \mathscr{L}_{r-1,d-1}(1^b)$$

on R, by Lemma 12. Furthermore the restricted systems intersect transversally.
The system $\mathscr{L}_{\mathbb{P}} = \mathscr{L}_{r,d-1}(2^{n-b})$ on the component $\mathbb{P}$ in non special, by assumption and it is nonempty, in fact

$$\begin{aligned} v_{\mathbb{P}} &= \binom{r+d-1}{r} - 1 - (r+1)(n-b) \\ &\geq \binom{r+d-1}{r} - 1 - \binom{r+d}{r} - l^+ + \frac{r+1}{r}\binom{r+d-1}{r-1} \\ &= b - l^+ - 1 > -1 \end{aligned}$$

The kernel system $\hat{\mathscr{L}}_{\mathbb{P}} = \mathscr{L}_{r,d-2}(2^{n-b})$ is non-special and

$$\begin{aligned} \hat{v}_{\mathbb{P}} &= \binom{r+d-2}{r} - 1 - (r+1)(n-b) \\ &\leq -\binom{r+d-2}{r-1} - 1 + b + l^- \leq -1. \end{aligned}$$

Moreover the dimension of the intersection $\mathscr{R}$ of the restricted systems on R is

$$\dim(\mathscr{R}) = \max\{-1, l_{\mathbb{P}} - b\} = \begin{cases} -1 + l^- & \text{if } n = n^- \\ -1 & \text{if } n = n^+ \end{cases}.$$

Now, we can compute the dimension of the limiting system $\mathscr{L}_0$ on the central fiber with our recursive formula (formula (2.8)):

$$l_0 = \dim(\mathscr{R}) + \hat{l}_{\mathbb{P}} + \hat{l}_{\mathbb{F}} + 2 = \dim(\mathscr{R}) = e(\mathscr{L}).$$

Therefore, by upper semicontinuity, the system $\mathscr{L}$ is non-special. □

Case $b_0 \notin \mathbb{Z}$. We want to analyse the cases in which performing a $(1,b)$-degeneration is not enough. Let us first of all explain why the $(1,b)$-degeneration approach does not suffice. For example when $\mathscr{L}$ is expected to be empty, namely if $n = n^+$, we would like to find a specialization of the n nodes on the components of the special fiber, such that both kernel systems and also the intersection of the restricted systems are empty. Looking for an integer b such that $\hat{\mathscr{L}}_{\mathbb{F}}$ is empty, we notice that the minimal one is

$$b = \left\lceil \frac{1}{r}\binom{r+d-1}{r-1} \right\rceil = \frac{1}{r}\binom{r+d-1}{r-1} + \frac{l'}{r};$$

but we would have a problem with the dimension of the intersection of the restricted systems $\mathscr{R}$ on R (which we wish to be empty); indeed

$$\begin{aligned} l_{\mathbb{P}} - b &= \binom{r+d-1}{r} - 1 - (r+1)n + rb \\ &= \binom{r+d-1}{r} - 1 - \binom{r+d}{r} - l^{+} + \binom{r+d-1}{r-1} + l' = -1 - l^{+} + l' \end{aligned}$$

and we are not able to check if $l' \leq l^{+}$. On the other hand, if we choose

$$b = \left\lfloor \frac{1}{r}\binom{r+d-1}{r-1} \right\rfloor,$$

then $\hat{\mathscr{L}}_{\mathbb{F}}$, which has dimension $l_{r-1,d}(2^b)$, is nonempty.
Thus, we degenerate the central fiber X_0 is such a way that some of the points specialized on $\mathbb{F}$ approach the intersection R with the exceptional component, in order to avoid this arithmetical problem. Let us construct the trivial family $\mathcal{Z} = Z \times \Delta$, where Δ is a disc centered at the origin, with reducible fibers $Z_s = \mathbb{F}_s \cup \mathbb{P}_s$, see Section 2.3. Consider the scheme given by a collection of n nodes such that b of them lie on $\mathbb{F}_s$ and the other $n-b$ lie on $\mathbb{P}_s$, for $s \neq 0$. We suppose that such a scheme degenerates in the following way: the limit on the central fiber Z_0 of the b points on $\mathbb{F}_s$ is a scheme given by $\beta \leq b$ general points on the intersection R_0 of $\mathbb{F}_0$ and $\mathbb{P}_0$ and $b-\beta$ general points on $\mathbb{F}_0 \setminus R_0$.

Let $\mathscr{L}'_s$ be the system on the general fiber, which corresponds to the limit system of the first degeneration, i.e. $\mathscr{L}'_s = \mathscr{L}'_{\mathbb{P}_s} \times_{\mathscr{R}'_{\mathbb{P}_s} \cap \mathscr{R}'_{\mathbb{F}_s}} \mathscr{L}'_{\mathbb{F}_s}$. If we prove that the system $\mathscr{L}'_0$, which is the limiting system of the second degeneration, has dimension equal to $e(\mathscr{L})$, we conclude, by upper semicontinuity, that $\mathscr{L}$ is non-special. We have to choose integers b and β such that the system $\mathscr{L}'_0$ has dimension equal to $e(\mathscr{L})$. Let β be defined as follows:

$$\frac{1}{r}\binom{r+d-1}{r-1} = \left\lfloor \frac{1}{r}\binom{r+d-1}{r-1} \right\rfloor + \frac{\beta}{r}, \qquad \beta \in \{0, \ldots, r-1\}.$$

Choose

$$b = \frac{1}{r}\binom{r+d-1}{r-1} - \frac{\beta}{r} + \beta \in \mathbb{Z}.$$

From now on, we assume that the systems $\mathscr{L}_{r-1,d}(2^{b-\beta})$, $\mathscr{L}_{r,d-1}(2^{n-b+\beta})$ and $\mathscr{L}_{r,d-2}(2^{n-b})$ are non-special.
Consider the following exact restriction sequence on the component $\mathbb{F}_0$:

$$0 \to \hat{\mathscr{L}}'_{\mathbb{F}_0} = \mathscr{L}_{r,d}(d, 2^{b-\beta}, 1^{\beta}) \to \mathscr{L}'_{\mathbb{F}_0} \to \mathscr{R}'_{\mathbb{F}_0} = \mathscr{L}_{r-1,d-1}(1^{b-\beta}, 2^{\beta}) \tag{3.3}$$

The kernel system has dimension

$$\begin{aligned} \hat{l}'_{\mathbb{F}_0} &= l_{r-1,d}(2^{b-\beta}, 1^{\beta}) \\ &= v_{r-1,d}(2^{b-\beta}, 1^{\beta}) \\ &= \binom{r+d-1}{r-1} - 1 - r(b-\beta) - \beta = -1 \end{aligned}$$

so it is empty. The system $\mathscr{L}'_{\mathbb{F}_0}$ is non-special by Lemma 16, in fact

$$\begin{aligned} b &\le \frac{1}{r}\binom{r+d-1}{r-1} + r - 1 \\ &\le \frac{1}{r+1}\binom{r+d}{r} - \frac{1}{r+1}\binom{r+d-2}{r} - (r-2) - \frac{r}{r+1} \le k(r,d) \end{aligned}$$

for $d \ge 6$, if $r = 4, 5$ and for $d \ge 5$ if $r \ge 6$ and moreover the points are in general position being $\beta < b$. (Notice that if $r = 4,\ d = 5$ then $b_0 \in \mathbb{Z}$, so this case is already covered in the previous section; while if $r = 5,\ d = 4$ we have $b = 26 \le k(5,5) = 29$, so $\mathscr{L}'_{\mathbb{F}_0}$ is anyhow non-special.)
As a consequence, $\mathscr{L}'_{\mathbb{F}_0}$ cuts the complete series on R_0, namely

$$\mathscr{R}'_{\mathbb{F}_0} = \mathscr{L}_{r-1,d-1}(1^{b-\beta}, 2^{\beta}),$$

in fact the $b-\beta$ simple points (trace on R_0 of the lines through the $b-\beta$ double points and the $(d-1)$-point) are base points (see Lemma 12). Moreover the system $\hat{\mathscr{L}}'_{\mathbb{P}_0} = \mathscr{L}_{r,d-2}(2^{n-b})$ is empty; indeed it is non-special by assumption and

$$\begin{aligned} \hat{v}'_{\mathbb{P}_0} &= \binom{r+d-2}{r} - 1 - (r+1)(n-b) \\ &\le -\binom{r+d-2}{r-1} - 1 - (r+1) + \frac{1}{r}\binom{r+d-1}{r-1} + r^2 < -1 \end{aligned}$$

for $r \ge 4$, $d \ge 5$. It remains only to prove that $\mathscr{R}'_{\mathbb{P}_0}$ and $\mathscr{R}'_{\mathbb{F}_0}$ intersect transversally on R_0. The intersection $\mathscr{R}'_0$ is given by those divisors of $\mathscr{R}'_{\mathbb{P}_0}$ that are singular at β further general points of R_0 and passing through $b-\beta$ points, the ones imposed by $\mathscr{R}'_{\mathbb{F}_0}$. Let us denote by

$$\hat{\mathscr{L}}'^{m}_{\mathbb{P}_0} \subseteq \hat{\mathscr{L}}'_{\mathbb{P}_0},\ \mathscr{L}'^{m}_{\mathbb{P}_0} \subseteq \mathscr{L}'_{\mathbb{P}_0} \text{ and } \mathscr{R}'^{m}_{\mathbb{P}_0} = \mathscr{R}'_{\mathbb{P}_0}(2^{\beta}, 1^{b-\beta}) \subseteq \mathscr{R}'_{\mathbb{P}_0},$$

the systems defined by these matching conditions. It is clear that

$$\dim(\hat{\mathscr{L}}'^{m}_{\mathbb{P}_0}) = \dim(\hat{\mathscr{L}}'_{\mathbb{P}_0}) = -1,$$

therefore

$$\dim(\mathscr{R}'^{m}_{\mathbb{P}_0}) = \dim(\mathscr{L}'^{m}_{\mathbb{P}_0}).$$

It suffices to prove that if we impose our β nodes to $\mathscr{R}'_{\mathbb{P}_0}$, the resulting system

$$\bar{\mathscr{R}}'_{\mathbb{P}_0} := \mathscr{R}'_{\mathbb{P}_0}(2^{\beta})$$

is non-special, i.e. that

$$\dim(\bar{\mathscr{R}}'_{\mathbb{P}_0}) = \dim(\mathscr{R}'_{\mathbb{P}_0}) - r\beta;$$

the $b-\beta$ simple points give $b-\beta$ independent conditions being them in general position. Notice that $\mathscr{R}'^m_{\mathbb{P}_0} \subseteq \bar{\mathscr{R}}'_{\mathbb{P}_0} \subseteq \mathscr{R}'_{\mathbb{P}_0}$, and in particular that $\mathscr{R}'^m_{\mathbb{P}_0} = \bar{\mathscr{R}}'_{\mathbb{P}_0}(1^{b-\beta})$.
Let $\bar{\mathscr{L}}'_{\mathbb{P}_0} := \mathscr{L}_{r,d-1}(2^{n-b+\beta}) \subset \mathscr{L}'_{\mathbb{P}_0}$ be the linear system of hypersurfaces of $\mathbb{P}_0$ with $n-b$ general nodes on $\mathbb{P}_0$ and β general nodes on $R_0 \subseteq \mathbb{P}_0$. Recall that R_0 is a general hyperplane for $\mathbb{P}_0$ and that $\beta < r$: so the $n-b+\beta$ nodes are in general position in $\mathbb{P}_0$.

Proposition 31 (Transversality on R'_0)**.** *Keep the same construction as above and assume that $\bar{\mathscr{L}}'_{\mathbb{P}_0}$ and $\hat{\mathscr{L}}'_{\mathbb{P}_0}$ are non-special. Then the linear system $\bar{\mathscr{R}}'_{\mathbb{P}_0}$ in non-special.*

Proof. We have

$$\begin{aligned}
\dim(\bar{\mathscr{L}}'_{\mathbb{P}_0}) &= \binom{r+d-1}{r} - 1 - (r+1)(n-b+\beta)\\
&\geq \binom{r+d-1}{r} - 1 - \binom{r+d}{r} - l^+ + (r+1)\left\lfloor \frac{1}{r}\binom{r+d-1}{r-1}\right\rfloor\\
&\geq \binom{r+d-1}{r} - 1 - \binom{r+d}{r} - r + (r+1)[\frac{1}{r}\binom{r+d-1}{r-1} - 1]\\
&= -1 + \frac{1}{r}\binom{r+d-1}{r-1} - (2r+1) > -1
\end{aligned}$$

for $r \geq 4$ and $d \geq 5$, therefore $\bar{\mathscr{L}}'_{\mathbb{P}_0}$ is nonempty. Consider the restriction map of $\bar{\mathscr{L}}'_{\mathbb{P}_0}$ to R_0:

$$\bar{\mathscr{L}}'_{\mathbb{P}_0} \to \bar{\mathscr{R}}'_{\mathbb{P}_0};$$

the kernel is $\hat{\bar{\mathscr{L}}}'_{\mathbb{P}_0} := \mathscr{L}_{r,d-2}(2^{n-b}, 1^\beta) \subseteq \hat{\mathscr{L}}'_{\mathbb{P}_0} = \emptyset$. The following facts hold:

- $h^0(\mathbb{P}_0, \hat{\bar{\mathscr{L}}}'_{\mathbb{P}_0}) = h^0(\mathbb{P}_0, \hat{\mathscr{L}}'_{\mathbb{P}_0}) = 0$;
- $h^1(\mathbb{P}_0, \hat{\mathscr{L}}'_{\mathbb{P}_0}) = \hat{l}'_{\mathbb{P}_0} - \hat{v}'_{\mathbb{P}_0} > 0$,
- $h^1(\mathbb{P}_0, \hat{\bar{\mathscr{L}}}'_{\mathbb{P}_0}) = h^1(\mathbb{P}_0, \hat{\mathscr{L}}'_{\mathbb{P}_0}) + \beta$;
- $h^0(\mathbb{P}_0, \bar{\mathscr{L}}'_{\mathbb{P}_0}) = h^0(\mathbb{P}_0, \mathscr{L}'_{\mathbb{P}_0}) - (r+1)\beta \geq 0$.

We get the following commutative diagram:

$$\begin{tikzcd}
& 0 & 0 & & \\
& K^{(r+1)\beta} \arrow[u] & V \arrow[u] & 0 & \\
0 \arrow[r] & H^0(\mathbb{P}_0,\mathscr{L}'_{\mathbb{P}_0}) \arrow[r] \arrow[u] & H^0(R_0,\mathscr{R}'_{\mathbb{P}_0}) \arrow[r] \arrow[u] & H^1(\mathbb{P}_0,\hat{\mathscr{L}}'_{\mathbb{P}_0}) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H^0(\mathbb{P}_0,\bar{\mathscr{L}}'_{\mathbb{P}_0}) \arrow[r] \arrow[u, hook] & H^0(R_0,\bar{\mathscr{R}}'_{\mathbb{P}_0}) \arrow[r] \arrow[u, hook] & H^1(\mathbb{P}_0,\hat{\bar{\mathscr{L}}}'_{\mathbb{P}_0}) \arrow[r] \arrow[u] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & K^{\beta} \arrow[u, hook] \arrow[uuull] & \\
& & & 0 \arrow[u] &
\end{tikzcd}$$

It follows that

$$\begin{aligned}
\dim(V) &= h^0(R_0,\mathscr{R}'_{\mathbb{F}_0}) - h^0(R_0,\bar{\mathscr{R}}'_{\mathbb{P}_0}) \\
&= h^0(\mathbb{P}_0,\mathscr{L}'_{\mathbb{P}_0}) + h^1(\mathbb{P}_0,\hat{\mathscr{L}}'_{\mathbb{P}_0}) - h^0(\mathbb{P}_0,\bar{\mathscr{L}}'_{\mathbb{P}_0}) - h^1(\mathbb{P}_0,\hat{\bar{\mathscr{L}}}'_{\mathbb{P}_0}) \\
&= \dim(K^{(r+1)\beta}) - \dim(K^{\beta}) = r\beta.
\end{aligned}$$

Hence $H^0(R_0,\bar{\mathscr{R}}'_{\mathbb{P}_0})$ has codimension equal to $\dim(V) = r\beta$ in $H^0(R_0,\mathscr{R}'_{\mathbb{P}_0})$ and, at the level of linear systems, we have the following equivalence:

$$\dim(\bar{\mathscr{R}}'_{\mathbb{P}_0}) = \dim(\mathscr{R}'{}_{\mathbb{P}_0}) - r\beta.$$

This conclude the proof. □

By consequence, the matching system $\mathscr{R}'^{m}_{\mathbb{P}_0}$, that corresponds to the intersection $\mathscr{R}'$ of the two restricted systems, is non-special, being the $b-\beta$ base points in general position. In particular

$$\dim(\mathscr{R}') = \max\{-1, \dim(\bar{\mathscr{R}}'_{\mathbb{P}_0}) - (b-\beta)\} = \begin{cases} -1+l^- & \text{if } n = n^- \\ -1 & \text{if } n = n^+ \end{cases}$$

Proposition 32. *In the above notation, assume that the linear systems* $\mathscr{L}_{r-1,d}(2^{b-\beta}) (\cong \hat{\mathscr{L}}'_{\mathbb{F}_0})$, $\bar{\mathscr{L}}'_{\mathbb{P}_0}$ *and* $\hat{\mathscr{L}}'_{\mathbb{P}_0}$ *are non-special. Then the linear system* $\mathscr{L}_{r,d}(2^n)$ *is non-special.*

Proof. Following the argument of this section we get $l'_0 = e(\mathscr{L})$. Thus we conclude by upper semicontinuity applyed to the two performed degenerations. □

Putting together Propostion 30 and Proposition 32, the proof of Theorem 29 is now completed, and consequentely also the proof of the Alexander-Hirschowitz theorem (Theorem 2).

Part II

On the degree of the k-secant varieties of toric surfaces

Chapter 4

Toric varieties

4.1 Toric varieties

A separated normal variety X of dimension n is a *toric variety* if it contains a torus $(\mathbb{C}^*)^n$ as a dense open subvariety, with an action

$$(\mathbb{C}^*)^n \times X \to X$$

of $(\mathbb{C}^*)^n$ on X that extends the natural action of the torus on itself. In the practise, toric varieties arise from lattices, fans and polytopes.

Let $N \cong \mathbb{Z}^n$ be a lattice of rank n, that is, a finitely generated free abelian group of rank n. We denote by $N_\mathbb{R}$ the associated real vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and we set $M = \mathrm{Hom}(N, \mathbb{Z})$, which is isomorphic to $\mathbb{Z}^n$. Let $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} \cong \mathrm{Hom}(N_\mathbb{R}, \mathbb{R})$ and denote by $\langle \cdot, \cdot \rangle$ the natural pairing $M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$. Let now V be a vector space; an *integral structure* on V is the datum of a lattice N such that $N_\mathbb{R} = V$; we have $M_\mathbb{R} = V^\vee = \mathrm{Hom}(V, \mathbb{R})$.

Definition 6. *A* polyhedral cone *in a vector space V is the positive hull of a finite set of vectors of V, that is*

$$\sigma := \{a_1 v_1 + \cdots a_s v_s : a_i \in \mathbb{R}^+\}.$$

A rational polyhedral cone *in $V = N_\mathbb{R}$ is a polyhedral cone in V which can be generated by elements of N. A rational polyhedral cone is said to be* strongly convex *if it does not contain any linear subspace; we will call such a strongly convex rational polyhedral cone simply a* cone.

The dimension of σ is the dimension of the linear space $\mathbb{R} \cdot \sigma$ it spans.

Definition 7. *The* dual cone $\sigma^\vee$ *of a rational cone σ in $N_\mathbb{R}$ is the set:*

$$\sigma^\vee = \{u \in M_\mathbb{R} : \langle u, v \rangle \geq 0, \forall v \in \sigma\}.$$

In what follows, we will see how to construct an affine toric variety starting from a cone. A cone σ determines a finitely generated semigroup $S_\sigma = \sigma^\vee \cap M$ described by

$$S_\sigma = \{u \in M : \langle u, v\rangle \geq 0, \forall v \in \sigma\} \subseteq M.$$

The property of finite generation of S_σ is known as *Gordon's Lemma*. The algebra $A_\sigma = \mathbb{C}[S_\sigma]$, corresponding to the semigroup S_σ, is a finitely generated commutative $\mathbb{C}$-algebra: a set of generators $\{u_i\}$ for S_σ determines a set of generators $\{\chi^{u_i}\}$ for $\mathbb{C}[S_\sigma]$ as a complex vector space, with the following operation

$$\chi^u \cdot \chi^{u'} = \chi^{u+u'}.$$

The $\mathbb{C}$-algebra A_σ is commutative and finitely generated, so it determines an affine variety

$$U_\sigma = \mathrm{Spec}(A_\sigma)$$

that we call *affine toric variety*.

There is an order-preserving correspondence between cones and affine toric varieties: if τ is a face of σ, then $S_\sigma \subset S_\tau$, A_σ is a subalgebra of A_τ: this inclusion of algebras induces a morphism of affine varieties $U_\tau \to U_\sigma$ which embeds U_τ as a principal open subset of U_σ.
We construct general toric varieties by combining affine ones taking fans which are families of cones instead of single cones.

Definition 8. *A* fan Δ *in* $N_\mathbb{R}$ *is a collection of cones such that:*

1. *if* τ *is a face of a cone* σ*, then* τ *is a cone of* Δ*;*

2. *if* σ_1, σ_2 *are cones of* Δ*, then* $\sigma_1 \cap \sigma_2$ *is a common face of them.*

We define the *toric variety* $X(\Delta)$ as the disjoint union of the affine toric varieties U_σ associated to the cones $\sigma \in \Delta$; two cones σ and σ' with a common face $\tau = \sigma \cap \sigma'$ are glued together along the affine subvarieties U_τ. These identifications are compatible because of the order-preserving correspondence between cones and affine toric varieties; $X(\Delta)$ is separated because the diagonal map $U_\tau \to U_\sigma \times U_{\sigma'}$ is a closed embedding.

In the second part of this section, we describe a combinatorial way to define projective toric varieties endowed with a base point free and ample line bundle.

Definition 9. *A* convex polytope P *in a vector space* V *of finite dimension* n*, is the convex hull of a finite set* $\mathcal{A}$ *of points of* V*, that is a set of the form*

$$P = \left\{ \sum_i \lambda_i u_i : \lambda_i \in \mathbb{R}_{\geq 0}, \sum_i \lambda_i = 1 \right\}.$$

A *face* F of a polytope P is the intersection of P with a supporting affine hyperplane. A *facet* of P is a face of codimension one. We call *vertices* the faces of dimension zero and *edges* the faces of dimension one.
The *normalized Ehrhart polynomial* of a polytope P is the numerical function

$$\begin{array}{rccc} E_P : & \mathbb{N} & \rightarrow & \mathbb{N} \\ & t & \mapsto & \sharp(\mathbb{Z}\mathcal{A} \cap tP). \end{array}$$

It is known that E_P is a polynomial of degree $\dim(P)$:

$$E_P = \sum_{i=0}^{\dim(P)} \frac{c_i}{i!} t^i.$$

The leading coefficient $c_{\dim(P)}$ is denoted by $\mathrm{Vol}(P)$ and it is called the *(normalized) volume* of P. If $\dim(P) = n$, we have

$$\mathrm{Vol}(P) = \frac{V(P)}{n!},$$

where $V(P)$ is the usual Euclidean volume of P (see [33]). If $\dim(P) = n = 2$, then we write $Area(P)$ for the normalized volume of P.

Suppose now that $V = M_{\mathbb{R}}$. We construct from P a fan Δ_P, and then a toric variety $X_P = X(\Delta_P)$, as follows: for each face Q of P, define

$$\sigma_Q = \left\{ v \in N_{\mathbb{R}} : \langle u, v \rangle \leq \left\langle u', v \right\rangle \text{ for all } u \in \mathbb{Q} \text{ and } u' \in P \right\}.$$

Observe that σ_Q is the dual cone of the cone $\sigma_Q^{\vee}$ consisting of all vectors pointing from points of Q to points of P: $\sigma_Q^{\vee}$ is generated by the vectors $u' - u$, where u' and u vary among the vertices of P and Q respectively. The σ_Q's form a fan, as Q varies among the faces of P (for a complete proof see [24]). The degree of the projective toric variety X_P is equal to the normalized volume $\mathrm{Vol}(P)$ (see [33]).
The toric variety X_P is irreducible, reduced, separated and normal.

4.1.1 Toric degenerations

Let P be any polytope in $M_{\mathbb{R}}$.

Definition 10. *A* subdivision D *of* P *is a partition of* P *given by a finite family* $\{Q_i\}_{i \in I}$ *of convex sub-polytopes of maximal dimension such that*

- $\bigcup_{i \in I} Q_i = P$,
- $Q_i \cap Q_j$, *with* $i \neq j$, *is either a common face or it is empty.*

Given a piecewise linear positive function F defined over a polytope P with values in $\mathbb{R}$, define the *graph polytope* $G(F)$ to be the following object:

$$G(F) := \{(x, z) \in P \times \mathbb{R} : 0 \leq z \leq F(x)\} .$$

Definition 11. *A subdivision D is said to be* regular *if there exists a piecewise linear positive function F with values in $\mathbb{R}$ defined over P, verifying the following requests:*

(a.) each Q_i is the orthogonal projection of the n-dimensional faces of $G(F)$ on $z = 0$;

(b.) F is strictly convex.

We will call such an F a *lifting function* (according to [26]).
Consider for example the triangle P with edges of reticular lenght three (and normalized area nine) and a subdivision D of P in triangles of normalized area one, as in Figure 4.1. This subdivision is regular, in fact there exists a lifting function F, see Figure 4.2 for an example.

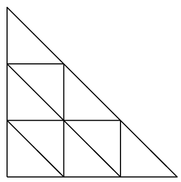


Figure 4.1: A regular subdivision D of P.

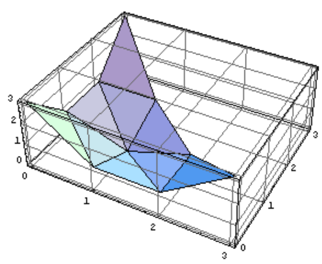


Figure 4.2: A lifting function over P.

Now, let $P_M := P \cap M = \{\underline{m}_0, \ldots, \underline{m}_r\}$ be the set of lattice points of P, where $\underline{m}_i = (m_{i1}, \ldots, m_{in})$, $i = 1 \ldots n$. Given a regular subdivision D of P, we can define the associated morphism as follows:

$$\begin{array}{rlcl} \Phi_D : & (\mathbb{C}^*)^{n+1} & \rightarrow & \mathbb{P}^r \times \mathbb{C} \\ & (\underline{x}, t) & \mapsto & ([t^{F(\underline{m}_0)}\underline{x}^{\underline{m}_0} : \cdots : t^{F(\underline{m}_r)}\underline{x}^{\underline{m}_r}], t). \end{array} \tag{4.1}$$

The closure of $\Phi_D((\mathbb{C}^*)^{n+1})$, for all $t \neq 0$, is a variety X_t projectively equivalent to X_P. Let X_0 be the flat limit of X_t, when t tends to zero: such a variety is the union of the varieties X_{Q_i}, $i \in I$. Indeed, the restriction $F_{|Q_i}$ of F to Q_i has equation $a_1x_1 + \cdots a_nx_n + b$, for some $a_1, \ldots, a_n, b \in \mathbb{R}$; we can always compose Φ_D with the following reparametrization

$$x_1, \ldots, x_n, t \mapsto t^{-a_1}x_1, \ldots, t^{-a_n}x_n, t$$

getting

$$\begin{array}{lcl} (\mathbb{C}^*)^{n+1} & \to & \mathbb{P}^r \times \mathbb{C} \\ (\underline{x}, t) & \mapsto & ([\cdots : t^{F(\underline{m}_i) - F_{Q_i}(\underline{m}_i)}\underline{x}^{\underline{m}_i} : \cdots], t). \end{array}$$

For t tending to zero, we see that X_{Q_i} sits in X_0. The map (4.1) can be extended to a map

$$\begin{array}{lcl} X_P \times \mathbb{C}^* & \to & \mathbb{P}^r \\ (\underline{x}, t) & \mapsto & ([t^{F(\underline{m}_0)}\underline{x}^{\underline{m}_0} : \cdots : t^{F(\underline{m}_r)}\underline{x}^{\underline{m}_r}], t) \end{array}$$

and the flat morphism

$$\pi_D : ([\cdots : t^{F(\underline{m}_i) - F_{Q_i}(\underline{m}_i)}\underline{x}^{\underline{m}_i} : \cdots], t) \mapsto t$$

provides a 1-dimensional embedded[1] degeneration of X_P to X_0.

Definition 12. *The flat morphism π_D is said to be a* toric degeneration *of the toric variety X_P.*

The reducible central fiber X_0 is given by the subdivision D of P: the irreducible components of X_0 are the X_{Q_i}'s. Notice that if $i \neq j$ and Q_i and Q_j have a common face $Q_i \cap Q_j$, then X_{Q_i} and X_{Q_j} intersect along $X_{Q_i \cap Q_j}$. In the example drown in Figure 4.1, the central fiber of the toric degeneration is a reducible surface given by the union of nine planes, each one corresponding to a triangle of the configuration of D; the intersection between the components are easily depicted looking at the figure.

Definition 13. *If $n = 2$ and the reducible central fiber X_0 is a union of planes, i.e. if the subdivision D of the polytope P is a triangulation of it, we say that π_D is a* planar toric degeneration *of X_P.*

[1] A degeneration $\pi : \mathcal{X} \to \Delta$, with Δ a complex disc centered at the origin, is said to be *embedded* if $\mathcal{X} \subseteq \Delta \times \mathbb{P}^r$ and the following diagram commutes.

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d] & \Delta \times \mathbb{P}^r \arrow[dl, "\pi"] \\ \Delta & \end{tikzcd}$$

In this case the family D of sub-polytopes of P is a simplicial complex[2], whose maximal simplices are the Q_i's.
In the next chapters we will deal with toric degenerations of toric surfaces, and we will use the notation

$$X_0 = \lim_D X =$$

to say that X_0 is the flat limit, for t tending to zero, of X; namely X_0 is the central fiber of the toric degeneration and $X_t \cong X$ is the general one.

4.2 Toric ideals

In this section we will define a special class of ideals and their interesting properties.
Let K be any field and let $K[x_0, \dots, x_r]$ be the polynomial ring in $r+1$ indeterminates. Fix a subset $\mathcal{A} = \{\underline{\alpha}_0, \dots, \underline{\alpha}_r\} \subset \mathbb{Z}^{n+1}$ and suppose that all the vectors of $\mathcal{A}$ lye on a hyperplane of $\mathbb{R}^{n+1}$, i.e. suppose that there exists a vector $\omega \in \mathbb{Q}^{n+1}$ such that $\omega^T \cdot \underline{\alpha}_i = 1$, for all i. Identify each vector $\underline{a}_i \in \mathbb{Z}^{n+1}$ with a monomial $\underline{t}^{\underline{a}_i}$ in the Laurent polynomial ring $K[t_1^{\pm 1}, \dots, t_{n+1}^{\pm 1}]$. Consider the semigroups homomorphism

$$\begin{array}{cccc} \pi_{\mathcal{A}}: & \mathbb{N}^{r+1} & \to & \mathbb{Z}^{n+1} \\ & (u_0, \dots, u_r) & \mapsto & u_0\underline{\alpha}_0 + \dots + u_r\underline{\alpha}_r \end{array}$$

and the corresponding semigroup algebras homomorphism

$$\begin{array}{cccc} \hat{\pi}_{\mathcal{A}}: & K[x_0, \dots, x_r] & \to & K[t_1^{\pm 1}, \dots, t_{n+1}^{\pm 1}] \\ & x_i & \mapsto & \prod_{j=1}^{n+1} t_j^{\alpha_{ij}}. \end{array}$$

We denote with $\mathcal{I}_{\mathcal{A}}$ the kernel of the map $\hat{\pi}_{\mathcal{A}}$ and we call it the *homogeneous toric ideal* of $\mathcal{A}$.
Notice that a binomial of the form $x_0^{u_0} \cdots x_r^{u_r} - x_0^{v_0} \cdots x_r^{v_r}$, where $\pi_{\mathcal{A}}(u_0, \dots, u_r) = \pi_{\mathcal{A}}(v_0, \dots, v_r)$, lies in $\mathcal{I}_{\mathcal{A}}$. Moreover each polynomial in $\mathcal{I}_{\mathcal{A}}$ is a K-linear combination of binomials of that form (for a complete proof see [33], chapter 4). Therefore

$$\{x_0^{u_0} \cdots x_r^{u_r} - x_0^{v_0} \cdots x_r^{v_r} : \pi_{\mathcal{A}}(u_0, \dots, u_r) = \pi_{\mathcal{A}}(v_0, \dots, v_r)\}$$

is a generating set for $I_{\mathcal{A}}$. The hypothesis that the $\underline{\alpha}_i$'s lie on a hyperplane ensures that $\mathcal{I}_{\mathcal{A}}$ is a homogeneous ideal, in fact, given $\underline{u}, \underline{v}$ such that $u_0\underline{\alpha}_0 +$

[2] A *simplex* in M is the convex hull of $n+1$ independent points. A *simplicial complex* D is a set of simplices in M that satisfies the following conditions:

- any face of a simplex from D is also in D;
- the intersection of any two simplices $Q_1, Q_2 \in D$ is a face of both Q_1 and Q_2.

$\cdots + u_r\underline{\alpha}_r = v_0\underline{\alpha}_0 + \cdots + v_r\underline{\alpha}_r$, then, multiplying by ω^T on both sides, we get $u_0 + \cdots + u_r = v_0 + \cdots + v_r$, therefore $x_0^{u_0} \cdots x_r^{u_r} - x_0^{v_0} \cdots x_r^{v_r}$ is homogeneous.

One can define in general the projective toric variety associated to any set $\mathcal{A}$; in contrast to the construction of toric varieties via polytopes, such varieties need not be normal.

Example 1. *Consider*

$$\mathcal{A} = \mathcal{A}_{V_3} = \begin{pmatrix} 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \\ 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}.$$

Let the embedding

$$\begin{array}{ccl} \mathbb{P}^2 & \stackrel{\nu_3}{\to} & \mathbb{P}^9 \\ [y_0, y_1, y_2] & \mapsto & [x_{003}, x_{102}, x_{012}, x_{201}, x_{111}, x_{021}, x_{300}, x_{210}, x_{120}, x_{030}] \end{array}$$

be the morphism associated to the map

$$\begin{array}{ccc} K[x_{003}, \ldots, x_{030}] & \to & K[y_0, y_1, y_2] \\ x_{ijk} & \mapsto & y_0^i y_1^j y_2^k \end{array}$$

The polytope corresponding to $\mathcal{A}$ is a triangle with normalized area equal to nine; the projective surface it defines is the 3-ple Veronese embedding $V_3 \subseteq \mathbb{P}^9$ of $\mathbb{P}^2$ (cfr. Figure 4.1 and Figure 4.3). The reticular points of the polytope

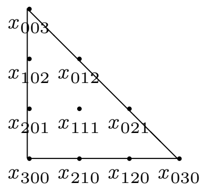


Figure 4.3: The polytope of the Veronese surface $V_3 \subseteq \mathbb{P}^9$.

correspond to the homogeneous coordinates x_{ijk} of $\mathbb{P}^9$. The toric ideal $\mathcal{I}_{V_3}$ of V_3 is generated by the quadratic binomials $x_{i_1,j_1,k_1}x_{i_2,j_2,k_2} - x_{i_3,j_3,k_3}x_{i_4,j_4,k_4}$ such that $(i_1 + i_2, j_1 + j_2, k_1 + k_2) = (i_3 + i_4, j_3 + j_4, k_3 + k_4)$. Its ideal has a nice determinantal presentation, namely it is generated by the 2×2 minors of the following catalecticant matrix*:*

$$A = \begin{pmatrix} x_{300} & x_{210} & x_{201} & x_{120} & x_{111} & x_{102} \\ x_{210} & x_{120} & x_{111} & x_{030} & x_{021} & x_{012} \\ x_{201} & x_{111} & x_{102} & x_{021} & x_{012} & x_{003} \end{pmatrix}$$

Example 2. *Let $\delta_1, \delta_1 \in \{0, 1, 2, \dots\}$, $\delta_1 \leq \delta_2$, and let $\delta = \delta_1 + \delta_2$. The rational normal scroll $S(\delta_1, \delta_2)$ is the toric surface of degree δ in $\mathbb{P}^{\delta+1}$ corresponding to:*

$$\mathcal{A} = \mathcal{A}_{\delta_1,\delta_2} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & \delta_1 & 0 & 1 & \cdots & \delta_2 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

To each vector of $\mathcal{A}$ corresponds a coordinate of $\mathbb{P}^{\delta+1}$. Let

$$K[x_{100}, \dots, x_{1,0,\delta_1}, x_{010}, \dots, x_{0,1,\delta_2}]$$

be the coordinate ring of $\mathbb{P}^{\delta+1}$. The corresponding polytope is the trapezium drawn in Figure 4.4. The ideal $\mathcal{I}_{\delta_1,\delta_2}$ of $S(\delta_1, \delta_2)$ is generated by the binomials

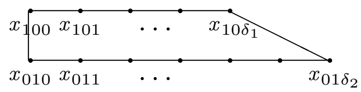


Figure 4.4: The polytope of the rational normal scroll $S(\delta_1, \delta_2) \subseteq \mathbb{P}^{\delta+1}$.

$x_{i_1,j_1,k_1} x_{i_2,j_2,k_2} - x_{i_3,j_3,k_3} x_{i_4,j_4,k_4}$ *such that* $(i_1 + i_2, j_1 + j_2, k_1 + k_2) = (i_3 + i_4, j_3 + j_4, k_3 + k_4)$*, i.e. $\mathcal{I}_{\delta_1,\delta_2}$ is generated by the 2×2 minors of*

$$M = M_{\delta_1,\delta_2} = \begin{pmatrix} x_{100} & \cdots & x_{1,0,\delta_1-1} & x_{010} & \cdots & x_{0,1,\delta_2-1} \\ x_{101} & \cdots & x_{1,0,\delta_1} & x_{011} & \cdots & x_{0,1,\delta_2} \end{pmatrix}$$

For a complete reference see [18, 32].

4.2.1 Initial ideals and regular triangulations

In this section we will briefly recall what the initial ideal of a toric ideal with respect to a term order is (for a complete reference see [33]).
A *term order* $\prec$ on $\mathbb{N}^{r+1}$ is a total order such that the zero vector is the unique minimal element with respect to $\prec$ and such that if $\alpha \prec \beta$ then $\alpha + \gamma \prec \beta + \gamma$, for all $\alpha, \beta, \gamma \in \mathbb{N}^{r+1}$. Given a homogeneous polynomial f and a term order $\prec$, the (unique) *initial monomial* of f with respect to $\prec$ is denoted by $\mathrm{in}_\prec(f)$. If $\mathcal{I} \subseteq K[x_0, \dots, x_r]$ is any ideal, then the corresponding *initial ideal* is the monomial ideal

$$\mathrm{in}_\prec(\mathcal{I}) := \langle \mathrm{in}_\prec(f) : f \in \mathcal{I} \rangle \subseteq K[x_0, \dots, x_r].$$

The passage from $\mathcal{I}$ to its initial ideal is a flat deformation: the zero set of $\mathcal{I}$ is deformed into the zero set of $\mathrm{in}_\prec(\mathcal{I})$ which is a union of linear coordinate subspaces. This operation, when $\mathcal{I}$ defines a projective toric variety X of any dimension n, corresponds to performing a toric degeneration of X to a union of $\mathbb{P}^n$'s. The *initial complex* $\Delta_\prec(\mathcal{I})$ of an ideal $\mathcal{I}$ with respect to $\prec$ is the simplicial complex whose *Stanley-Reisner ideal*, or *non-face ideal*, is the radical of $\mathrm{in}_\prec(\mathcal{I})$, i.e. it is the simplicial complex on the vertex set $\{0, \dots, r\}$ defined by the following rule: a subset $F \subseteq \{0, \dots, r\}$ is a face of the complex if do not exist polynomials $f \in \mathcal{I}$ such that $\mathrm{in}_\prec(\mathcal{I})$ has support on F (see [23], Section 15.8 and [33], chapter 8).

Let $\mathcal{A} \subseteq \mathbb{Z}^{n+1}$ be the vector configuration of any toric ideal $\mathcal{I}_\mathcal{A}$. Let γ be a subset of $\mathcal{A}$ and consider the cone spanned by γ, denoting it by $\mathrm{pos}(\gamma)$.

Definition 14. *A* triangulation *of $\mathcal{A}$ is a collection D of subsets of $\mathcal{A}$ such that the set*

$$\{pos(\gamma) : \gamma \in D\}$$

is the set of cones in a simplicial fan whose support is pos($\mathcal{A}$), i.e. the convex hull of the vectors of $\mathcal{A}$.

Notice that $\mathrm{pos}(\mathcal{A})$ is a polytope in $\mathbb{R}^n$ and a triangulation of $\mathcal{A}$ is a toric degeneration of the toric variety $X = \mathcal{V}(\mathcal{I}_\mathcal{A})$. Regular triangulations correspond to regular toric degenerations.
The radical of $\mathrm{in}_\prec(\mathcal{I}_\mathcal{A})$ is a squarefree monomial ideal whose corresponding initial complex $\Delta_\prec(\mathcal{I}_\mathcal{A})$ is a regular triangulation of $\mathcal{A}$. Conversely, every regular triangulation of $\mathcal{A}$ can be interpreted as $\Delta_\prec(\mathcal{I}_\mathcal{A})$, for some $\prec$ ([33], Theorem 8.3).
A triangulation is said to be *full* if every vector of $\mathcal{A}$ is the vertex of some simplex in the triangulation. It is said to be *unimodular* if all the maximal simplices have normalized volume equal to one, i.e. it is a tetrahedron with edges of reticular lenght one (a triangle if $n = 2$). Full unimodular regular triangulations corresponds to toric degenerations to unions of $\mathbb{P}^n$'s.

Chapter 5

Secant varieties of toric surfaces

In this chapter we will introduce the concept of k-secant variety of a variety and in particular, in the toric case, we will define what a k-delightful toric degeneration is.

5.1 k-secant varieties

Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, projective variety of dimension n. Fix an integer $k \geq 1$ and consider the k-th symmetric product $\mathrm{Sym}^k(X)$. We define the *abstract k-th secant variety* of X, $S_X^k \subseteq \mathrm{Sym}^k(X) \times \mathbb{P}^r$, as the Zariski closure of the set

$$\{((x_0, \dots, x_k), z) \in \mathrm{Sym}^k(X) \times \mathbb{P}^r : \dim(\pi) = k \text{ and } z \in \pi\}$$

where $\pi = \langle x_0, \dots, x_k \rangle$. It is irreducible of dimension $(k+1)n + k$. Consider the projection p_X^k on the second factor and define the *k-th secant variety* of X

$$\mathrm{Sec}_k(X) := p_X^k(S_X^k)$$

as the scheme-theoretic image of S_X^k in $\mathbb{P}^r$, i.e.

$$\mathrm{Sec}_k(X) = \overline{\bigcup_{x_i \in X,\ \dim(\langle x_0, \dots, x_k \rangle) = k} \langle x_0, \dots, x_k \rangle} \subseteq \mathbb{P}^r.$$

It is an irreducible algebraic variety of dimension

$$\dim(\mathrm{Sec}_k(X)) \leq \min\{(k+1)n + k, r\} \tag{5.1}$$

Definition 15. *The right hand side of (5.1) is called the* expected dimension *of $Sec_k(X)$. If strict inequality holds, the k-secant variety of X does not have the expected dimension and X is said to be k-*defective*.*

The general fiber of p_X^k is pure of dimension $(k+1)n+k-\dim(\mathrm{Sec}_k(X))$; denote by $\mu_k(X)$ the number of irreducible components of this fiber. If

$$\dim(\mathrm{Sec}_k(X)) = (k+1)n+k \leq r,$$

then p_X^k is generically finite and $\mu_k(X) = \deg(p_X^k)$, i.e. $\mu_k(X)$ is the number of $(k+1)$-secant $\mathbb{P}^k$'s to X passing through the general point of $\mathrm{Sec}_k(X)$. This number is equal to one unless X is k-*weakly defective*; the weakly defective surfaces are classified in [13].

Suppose now that $\mathrm{Sec}_k(X)$ is not k-defective and that $\dim \mathrm{Sec}_k(X) = (k+1)n+k$. Let L be a general linear subspace of $\mathbb{P}^r$ of codimension $\dim(\mathrm{Sec}_k(X))$: X has

$$\nu_k(X) = \mu_k(X) \cdot \deg(\mathrm{Sec}_k(X))$$

$(k+1)$-secant $\mathbb{P}^k$'s meeting L. Let π_L be the projection of X from L to $\mathbb{P}^{(k+1)n+k-1}$: the image X' of X has $\nu_k(X)$ new $(k+1)$-secant $\mathbb{P}^{k-1}$'s that X did not use to have.

Definition 16. *Let $X \subset \mathbb{P}^r$ as above, with $r \geq (k+1)n+k$. The number $\nu_k(X)$ is called the* number of apparent $(k+1)$-secant $\mathbb{P}^{k-1}$*'s to X.*

In particular $\nu_1(X)$ corresponds to the number of double points that X acquires in a general projection to $\mathbb{P}^{2n}$, $\nu_2(X)$ is the number of trisecant lines in a general projection of X to $\mathbb{P}^{3n+1}$ and so on.

Notice that if $\nu_k(X) = 1$, then $\mathrm{Sec}_k(X) = \mathbb{P}^r$ and $\mu_k(X) = 1$ which means that for a general points of $\mathrm{Sec}_k(X)$ there is a unique $(k+1)$-secant $\mathbb{P}^k$.

If one is able to compute the number of apparent $(k+1)$-secant $\mathbb{P}^{k-1}$ of X, one can say something about the degree of its k-secant variety.

We will deal with the surface case. Let X be a smooth surface, the Severi's *double point formula* gives the number of nodes of a general projection of X to $\mathbb{P}^4$:

$$\nu_1(X) = \frac{d(d-5)}{2} - 5g + 6p_a - K^2 + 11, \tag{5.2}$$

where d is the degree, g is the sectional genus, p_a is the arithmetic genus and K is the canonical divisor of X. In particular, if $X = X_P$ is a projective toric surface, then

$$\nu_1(X) = \frac{d^2 - 10d + 5B + 2V - 12}{2}, \tag{5.3}$$

where d is the normalized area of the polytope P, B is the number of lattice points on the boundary and V is the number of vertices of P, see [22].
If X does not contain lines, a formula for $\nu_2(X)$, known as the *LeBarz' trisecant formula for surfaces in* $\mathbb{P}^7$ (see [29, 30]), is

$$\nu_2(X) = \frac{d^3 - 30d^2 + 224d - 3d(5h + K^2 - c_2) + 192h + 56K^2 - 40c_2}{6} \tag{5.4}$$

where H is the hyperplane divisor, c_2 is the second Chern class of X and $h = HK$; if the surface X contains a finite number of lines, the contribution of each line to $\nu_2(X)$ is

$$-\binom{4+a}{3},$$

where $a \in \mathbb{Z}$ is its self-intersection.
For the rational normal surface scrolls there is a formula for the number of trisecant lines of a general projection of X to $\mathbb{P}^7$, due to C. James (see [27]):

$$\nu_2(X) = \binom{d}{3} - 2d^2 + 12d - 3dg + 20g - 20. \tag{5.5}$$

There are similar, but more complicated, formulas for the number $\nu_k(X)$ in the curve case (see [2], chapter VIII), and in the surface case, if X does not contain any line, for $k \leq 5$ (see [29, 30]).

5.2 k-secant ideals

Let $\mathcal{I}$ be an ideal in the polynomial ring $K[x_0, \ldots, x_r]$. The *secant* of $\mathcal{I}$

$$\mathcal{I}^{\{1\}} = \mathcal{I} * \mathcal{I}$$

is an ideal in $K[x_0, \ldots, x_r]$ defined in the following way: take the polynomial ring $K[\underline{x}, \underline{y}] = K[x_0, \ldots, x_r, y_0, \ldots, y_r]$ and consider the map

$$\begin{array}{ccc} K[\underline{x}, \underline{y}] & \rightarrow & K[\underline{x}, \underline{y}] \\ x_i & \mapsto & y_i, \end{array}$$

for $i = 0, \ldots, r$. Denote with $\mathcal{I}'$ the image of $\mathcal{I}$ under that map. Then $\mathcal{I}^{\{1\}}$ is the elimination ideal

$$\mathcal{I}' + \mathcal{I}' + \langle 2y_i - x_i : 0 \leq i \leq r \rangle \cap K[x_0, \ldots, x_r].$$

Similarly, we define the 2-*secant* of $\mathcal{I}$ as

$$\mathcal{I}^{\{2\}} = \mathcal{I} * \mathcal{I} * \mathcal{I}$$

and the k-*secant* of $\mathcal{I}$ as

$$\mathcal{I}^{\{k\}} = \overbrace{\mathcal{I} * \cdots * \mathcal{I}}^{k+1}.$$

For homogeneous prime ideals, the k-secant ideals represent the prime ideals of the k-secant varieties of irreducible projective varieties.

Secants of edge ideals

Let $\mathcal{I} = \mathcal{I}_{\mathcal{A}} \subseteq K[x_0, \ldots, x_r]$ be the homogeneous toric ideal defining a projective toric variety and let $\prec$ be any term order on $K[x_0, \ldots, x_r]$. Set $\mathcal{I}_0 := \mathrm{in}_{\prec}(\mathcal{I})$. Let Δ be the simplicial complex of $\mathcal{I}_0$ (see Section 4.2.1). Moreover define $\Delta^{\{k\}}$ to be the simplicial complex of $\mathcal{I}_0^{\{k\}}$ whose faces are the subset F of $\mathcal{A}$ such that there are no monomials in $\mathcal{I}_0^{\{k\}}$ having support on F. The simplices in $\Delta^{\{k\}}$ are unions of $k+1$ simplices in Δ (see [32], Remark 2.9). In the case of *edge ideals*, we can simplify the study of secant ideals by considering the coloring properties of the graph they reflect. An edge ideal $\mathcal{I}(G)$ is an ideal generated by the squarefree quadratic monomials $x_i x_j$ corresponding to the edges $\{i, j\}$ of a graph G with vertex set $\{0, \ldots, r\}$. The *chromatic number* $\chi(G)$ of a graph is the minimal number of colors which can be used to color the vertices of G in such a way that no adjacent vertices have the same color; it corresponds to the smallest k such that the secant ideal $\mathcal{I}(G)^{\{k-1\}}$ is zero. The ideal $\mathcal{I}(G)^{\{k-1\}}$ has a nice combinatorial description. Given a subset $V \subseteq \{0, \ldots, r\}$ of the vertex set of G, we write G_V for the sub-graph of G which is induced on the set of vertices V; let $m_V = \prod_{i \in V} x_i$ be the monomial corresponding to G.

Theorem 33. *The k-secant ideal $\mathcal{I}(G)^{\{k\}}$ of an edge ideal $\mathcal{I}(G)$ is generated by the squarefree monomials m_V whose corresponding sub-graph G_V has chromatic number strictly greater than $k+1$:*

$$\mathcal{I}(G)^{\{k\}} = \langle m_V : \chi(G_V) > k+1 \rangle .$$

For a proof see [32]. This result is very helpful to compute the degree of the k-secant varieties, via toric degenerations.

Example 3. *For example, consider the triangle in Figure 5.1 (cfr. Table 6.1, sixth row): it describes a toric singular sextic surface $X \subseteq \mathbb{P}^6$. The ideal $\mathcal{I}_X$*

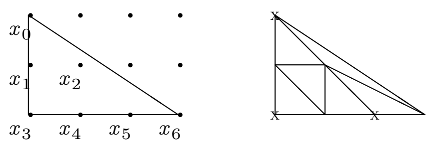


Figure 5.1: A triangle.

of X is generated by the following quadratic binomials:

$$x_0x_3 - x_1^2, x_0x_5 - x_2^2, x_3x_5 - x_4^2, x_4x_6 - x_5^2, x_0x_4 - x_1x_2,$$
$$x_2x_3 - x_1x_4, x_1x_5 - x_2x_4, x_1x_6 - x_2x_5, x_3x_6 - x_4x_5$$

Now, consider the subdivision D of X in Figure 5.1 on the right. The initial ideal $\mathcal{I}_0$ with respect to this planar toric degeneration is an edge ideal: $\mathcal{I}_0 =$

$\mathcal{I}(G)$, where G is the graph with vertex set $\{0,\dots,6\}$ and edge set the set of the non-edges *of D:*

$$\{\{0,3\},\{0,5\},\{3,5\},\{4,6\},\{0,4\},\{2,3\},\{1,5\},\{1,6\},\{3,6\}\},$$

where the vertex i of G corresponds to the coordinate x_i of $\mathbb{P}^6$. Therefore

$$\mathcal{I}_0 = \langle x_0x_3, x_0x_5, x_3x_5, x_4x_6, x_0x_4, x_2x_3, x_1x_5, x_1x_6, x_3x_6 \rangle\,,$$

and, by Theorem 33

$$\mathcal{I}_0^{\{1\}} = \langle x_0x_3x_5 \rangle\,,$$

in fact the only squarefree monomial of degree three such that the x_i's are pairwise disjoint is $x_0x_3x_5$.

Example 4. *Consider the surface X defined by the polytope in Figure 5.2 (cfr. Table 6.1, eleventh row). It is a singular surface of degree eight in $\mathbb{P}^8$ whose*

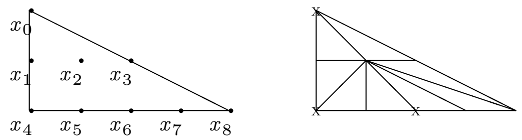


Figure 5.2: A triangle.

ideal is generated by the 2×2 minors of the following matrix:

$$C = \begin{pmatrix} x_4 & x_5 & x_1 & x_6 \\ x_5 & x_6 & x_2 & x_4 \\ x_1 & x_2 & x_0 & x_3 \\ x_6 & x_4 & x_3 & x_8 \end{pmatrix}$$

The initial ideal $\mathcal{I}_0$ with respect to the triangulation in the right hand side of Figure 5.2 is

$$\begin{aligned} \mathcal{I}_0 = \ & \langle x_0x_4, x_0x_5, x_0x_6, x_0x_7, x_0x_8, x_1x_3, x_1x_5, x_1x_6, x_1x_7, x_1x_8, \\ & x_3x_4, x_3x_5, x_3x_6, x_3x_7, x_4x_6, x_4x_7, x_4x_8, x_5x_7, x_5x_8, x_6x_8 \rangle\,. \end{aligned}$$

It is an edge ideal: the corresponding graph G has as edge set the non-edges *of D, i.e. the pairs $\{i,j\}$ such that there is not an edge joining the vertices i and j. Moreover*

$$\begin{aligned} \mathcal{I}_0^{\{1\}} = \ & \langle x_0x_4x_6, x_0x_4x_7, x_0x_4x_8, x_0x_5x_7, x_0x_5x_8, x_0x_6x_8, x_1x_3x_5, \\ & x_1x_3x_6, x_1x_3x_7, x_1x_5x_7, x_1x_5x_8, x_1x_6x_8, x_4x_6x_8 \rangle \end{aligned}$$

and

$$\mathcal{I}_0^{\{2\}} = \langle x_0x_4x_6x_8 \rangle\,.$$

Therefore, using for example the software CoCoA ([15]), one can compute the Hilbert polynomial, and in particular the degree and the dimension, of the algebraic varieties that $\mathcal{I}_0^{\{1\}}$ and $\mathcal{I}_0^{\{2\}}$ respectively define: $\mathcal{V}(\mathcal{I}_0^{\{1\}})$ has dimension five and degree three, while $\mathcal{V}(\mathcal{I}_0^{\{2\}})$ has dimension seven and degree four.

The ideals of some toric surfaces and of their k-secant varieties

In this section we briefly investigate the ideals of the toric surfaces we will deal with in the last chapter.

Consider first of all the surface X defined by the polytope in Figure 5.2, cfr Example 4. The secant variety $\mathrm{Sec}(X)$ of X is generated by the minors of C of order three, so it has dimension five as expected and degree eight; the variety $\mathrm{Sec}_2(X)$ is defined by $\det(C)$, so it is a hypersurface of degree four, and in particular X is 2-defective.

Let now $S_8 \subseteq \mathbb{P}^8$ be the embedding of the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$ via $\mathcal{O}(2,2)$. The defining configuration is

$$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}, \quad \omega = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

A reference is for example [32]. The embedding is the one associated to the map

$$\begin{array}{ccc} K[x_{0000}, \ldots, x_{1111}] & \to & K[y_0, y_1; z_0, z_1] \\ x_{ijkl} & \mapsto & y_i y_j z_k z_l. \end{array}$$

The corresponding polytope is the four sided polygon of edge lenght two with a unique internal lattice point, for example a square as in Figure 5.3 (cfr. Table 6.1, tenth row). The toric ideal $\mathcal{I}_{S_8}$ of S_8 in generated by the 2×2 minors of

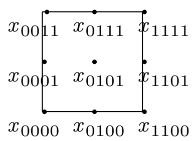


Figure 5.3: A polytope of S_8.

the matrix

$$B = \begin{pmatrix} x_{0000} & x_{0001} & x_{0100} & x_{0101} \\ x_{0001} & x_{0011} & x_{0101} & x_{0111} \\ x_{0100} & x_{0101} & x_{1100} & x_{1101} \\ x_{0101} & x_{0111} & x_{1101} & x_{1111} \end{pmatrix};$$

$\mathcal{I}_{S_8}^{\{1\}}$ is generated by the 3×3 minors of B, so $\deg(\mathrm{Sec}(S_8)) = 10$. The variety $\mathrm{Sec}_2(S_8)$ is a quartic hypersurface defined by $\det(B) = 0$, then S_8 is 2-defective.

Now we consider the 3-ple Veronese embedding V_3 of $\mathbb{P}^2$ in $\mathbb{P}^9$ (described in Example 1, cfr. Table 6.1, last row) and some further toric surfaces which can be obtained as projection of V_3 from a finite number of general points on it. The (toric) ideal of V_3 is generated by the 2×2 minors of A. The ideal $\mathcal{I}_{V_3}^{\{1\}}$ of $\mathrm{Sec}(V_3)$ is generated by the 3×3 minors of A by a result of Kanev (see

[28]). Therefore V_3 is not 1-defective and $\deg \mathrm{Sec}(V_3) = 15$. Moreover $\mathrm{Sec}_2(V_3)$ is a hypersurface, as expected, and it has the quartic *Aronhold* invariant of ternary cubic as equation (see for example [32]).
Consider the Del Pezzo surface X_8 of degree eight in $\mathbb{P}^8$. Its toric configuration corresponds to $\mathcal{A}_{V_3}$ without the vector ${}^T(0,0,3)$ (or ${}^T(0,3,0)$ or ${}^T(3,0,0)$ by simmetry). The associated polytope is the one represented in Figure 5.4 (or in Figure 5.5 respectively).

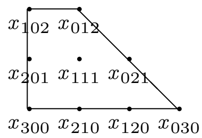


Figure 5.4: A polytope of X_8.

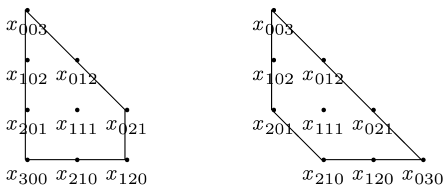


Figure 5.5: Other polytopes of X_8.

Notice that this toric surface is obtained, starting from V_3, by projecting from the point $[0,\dots,0,1] \in V_3 \subseteq \mathbb{P}^9$, as Figure 5.4 suggests. The (toric) ideal $\mathcal{I}_{X_8}$ of X_8 is generated by the 2×2 minors of the matrix A_8 obtained from A by erasing the last column (the one containing the coordinate x_{003}). Indeed X_8 is embedded in $\mathbb{P}^8$ via the linear system of plane cubics passing through $[0,0,1] \in \mathbb{P}^2$, i.e. the cubic curves not containing the monomial y_2^3, where y_0, y_1, y_2 are homogeneous coordinates for $\mathbb{P}^2$, which corresponds to the coordinate x_{003} of $\mathbb{P}^9$. Therefore

$$\mathcal{I}_{X_8} = \mathcal{I}_{V_3} \cap K[\widehat{x_{003}}, \cdots, x_{030}].$$

The secant variety of the projection of a variety equals the projection of the secant variety of that variety. Therefore the ideal $\mathcal{I}_{X_8}^{\{1\}}$ of $\mathrm{Sec}(X_8)$ is generated by the 3×3 minors of A_8, it has the expected dimension and $\deg(\mathrm{Sec}(X_8)) = 10$. Moreover $\mathrm{Sec}_2(X_8)$ fills up $\mathbb{P}^8$ as expected. Indeed, let ν_8 be the embedding of $\mathbb{P}^2$ in $\mathbb{P}^8$, via the linear system of cubic curves through a fixed point $q \in \mathbb{P}^2$ such that $\nu_8(\mathbb{P}^2) = X_8$. Let p_1, p_2, p_3 be general points of $\mathbb{P}^2$; the linear system $\mathscr{L} = \mathscr{L}_{2,3}(2^3, 1)$ of plane cubics singular at p_1, p_2, p_3 and passing through q corresponds to the linear system of hyperplanes of $\mathbb{P}^8$ tangent to X_8 at $\nu_8(p_1), \nu_8(p_2), \nu_8(p_3)$; its base locus is the general tangent space to $\mathrm{Sec}_2(X_8)$.

Now, the non-speciality of $\mathscr{L}$ is equivalent to the 2-non-defectivity of X_8, by *Terracini's Lemma* (see Section 1.2). The same holds for the surfaces described by the polytopes in Figure 5.5.
Let now $X_i \subseteq \mathbb{P}^i$ be the Del Pezzo surface of degree i, for $i = 6, 7$: X_7 is obtained from X_8 by projecting from a general point; X_6 is obtained from X_7 by projecting from a further general point. The corresponding polytopes are drown in the seventh row and in the third row of Table 6.1 respectively. The ideal $\mathcal{I}_{X_i}$ of X_i, for $i = 6, 7$, is generated by the 2×2 minors of the matrix A_i obtained by erasing from A the right columns, in the same way as above. The ideal $\mathcal{I}^{\{1\}}_{X_i}$ is generated by the 3×3 minors of A_i. With an easy computation we get that $\deg(\mathrm{Sec}(X_6)) = 3$, $\mathrm{Sec}_2(X_6) = \mathbb{P}^6$ and that $\deg(\mathrm{Sec}(X_7)) = 6$, $\mathrm{Sec}_2(X_7) = \mathbb{P}^7$. (For a reference see for example [10]).
Now, exploiting the same rule, we complete the description of the toric surfaces in Table 6.1. Let X be the toric surface in $\mathbb{P}^7$ obtained as image of V_3 under the projection of $\mathbb{P}^9$ to $\mathbb{P}^7$ from the line $\{x_{ijk} = 0 : (i, j, k) \neq (0, 0, 3), (1, 0, 2)\} \subseteq \mathbb{P}^9$ or from the line $\{x_{ijk} = 0 : (i, j, k) \neq (0, 0, 3), (0, 1, 2)\} \subseteq \mathbb{P}^9$: the corresponding polytopes are drawn in the eighth row of Table 6.1. As above, the ideal of X and of its secant variety are generated by the minors of order two and three respectively of the matrix obtained by erasing from A the two corresponding columns: $\deg(\mathrm{Sec}(X)) = 6$ and $\mathrm{Sec}_2(X) = \mathbb{P}^7$.
By projecting from a further point to $\mathbb{P}^6$, we get the toric surfaces of degree six whose polytopes are drown in the fourth and the fifth rows: the secant varieties have degree three and the varieties of trisecant planes fill up $\mathbb{P}^6$.
By projecting from a further point to $\mathbb{P}^5$, we get the toric quintic surfaces defined by the polytopes in the first two rows whose secant varieties fill up $\mathbb{P}^5$.

Finally, consider the rational normal surface scrolls $S(\delta_1, \delta_2) \subseteq \mathbb{P}^{\delta+1}$, where $\delta = \delta_1 + \delta_2$ and $\delta_1 \leq \delta_2$ (cfr. Example 2), whose polytopes are the trapezia drown in Table 6.2. All the secant ideals of $S(\delta_1, \delta_2)$ are known to have determinantal presentations, see for example [32] for a recent reference. Define

$$M^{\{1\}}_{\delta_1} = \begin{pmatrix} x_{100} & \dots & x_{1,0,\delta_1-2} \\ x_{101} & \dots & x_{1,0,\delta_1-1} \\ x_{102} & \dots & x_{1,0,\delta_1} \end{pmatrix}, \quad M^{\{1\}}_{\delta_2} = \begin{pmatrix} x_{010} & \dots & x_{0,1,\delta_2-2} \\ x_{011} & \dots & x_{0,1,\delta_2-1} \\ x_{012} & \dots & x_{0,1,\delta_2} \end{pmatrix}.$$

If $\delta_i \in \{0, 1\}$, then $M^{\{1\}}_{\delta_i}$ denotes the empty matrix. Let $\delta \geq 5$. If $\delta_1 \geq 2$, the ideal $\mathcal{I}^{\{1\}}_{\delta_1,\delta_2}$ is generated by the 3×3 minors of the matrix

$$M^{\{1\}} = M^{\{1\}}_{\delta_1,\delta_2} = \left(\begin{array}{c|c} M^{\{1\}}_{\delta_1} & M^{\{1\}}_{\delta_2} \end{array} \right),$$

while if $\delta_1 \in \{0, 1\}$, then $\mathcal{I}^{\{1\}}_{\delta_1,\delta_2}$ is generated by the 3×3 minors of $M^{\{1\}}_{\delta_2}$.

Define

$$M_{\delta_1}^{\{2\}} = \begin{pmatrix} x_{100} & \cdots & x_{1,0,\delta_1-3} \\ x_{101} & \cdots & x_{1,0,\delta_1-2} \\ x_{102} & \cdots & x_{1,0,\delta_1-1} \\ x_{103} & \cdots & x_{1,0,\delta_1} \end{pmatrix}, \quad M_{\delta_2}^{\{2\}} = \begin{pmatrix} x_{010} & \cdots & x_{0,1,\delta_2-3} \\ x_{011} & \cdots & x_{0,1,\delta_2-2} \\ x_{012} & \cdots & x_{0,1,\delta_2-1} \\ x_{013} & \cdots & x_{0,1,\delta_2} \end{pmatrix}.$$

If $\delta_i \in \{0,1,2\}$, $M_{\delta_i}^{\{2\}}$ is the empty matrix. Let $\delta \geq 8$. The ideal $\mathcal{I}_{\delta_1,\delta_2}^{\{2\}}$, if $\delta_1 \geq 3$, is generated by the 4×4 minors of

$$M^{\{2\}} = M_{\delta_1,\delta_2}^{\{2\}} = \left(\begin{array}{c|c} M_{\delta_1}^{\{2\}} & M_{\delta_2}^{\{2\}} \end{array} \right)$$

while if $\delta_1 \in \{0,1,2\}$, $\mathcal{I}_{\delta_1,\delta_2}^{\{2\}}$ is generated by the 4×4 minors of $M_{\delta_2}^{\{2\}}$. The trick is the same for $k \geq 3$.

We are interested in $\mathrm{Sec}_k(S(\delta_1,\delta_2))$, for $k = 1, 2$ and $\delta_1 \leq 2$, as we will appreciate in the next chapter. We can easily compute the dimensions and the degrees.

$\delta_1 = 0$: $S(0,\delta)$ is a rational cone over a rational normal curve $C_\delta \subseteq \mathbb{P}^\delta \subseteq \mathbb{P}^{\delta+1}$ of degree δ.

- $\mathrm{Sec}(S(0,\delta)) = \mathbb{P}^{\delta+1}$ if $\delta \leq 3$, while $S(0,\delta)$ is 1-defective if $\delta \geq 4$;
- $\mathrm{Sec}_2(S(0,\delta)) = \mathbb{P}^{\delta+1}$ if $\delta \leq 5$, while $S(0,\delta)$ is 2-defective if $\delta \geq 6$.

$\delta_1 = 1$:

- $\mathrm{Sec}(S(1,\delta-1)) = \mathbb{P}^{\delta+1}$ if $\delta \leq 4$, while $\mathrm{Sec}(S(1,\delta-1))$ has dimension five and degree $\binom{\delta-2}{2}$ for $\delta \geq 5$, using formula (5.2);
- $\mathrm{Sec}_2(S(1,\delta-1)) = \mathbb{P}^{\delta+1}$ if $\delta \leq 6$, while $S(1,\delta-1)$ is 2-defective if $\delta \geq 7$.

$\delta_1 = 2$:

- For the 1-secant variety of $S(2,\delta-2)$ the same things as for the scroll $S(1,\delta-1)$ hold;
- $\mathrm{Sec}_2(S(2,\delta-2)) = \mathbb{P}^{\delta+1}$ if $\delta \leq 7$, while $\mathrm{Sec}_2(S(2,\delta-2))$ has dimension eight and degree $\binom{\delta}{3} - 2((\delta-3)^2+1)$ for $\delta \geq 8$, applying formula (5.5).

5.3 k-delightfulness and k-secant degree

Let $\prec$ be any term order. The initial ideal of the k-secant ideal $\mathcal{I}^{\{k\}}$ of $\mathcal{I}$ is contained in the k-secant of the initial ideal of $\mathcal{I}$, for $k \geq 1$:

$$\mathrm{in}_\prec(\mathcal{I}^{\{k\}}) \subseteq (\mathrm{in}_\prec(\mathcal{I}))^{\{k\}}. \tag{5.6}$$

For a reference see for example [32], Corollary 4.2.

Definition 17. *If equality holds in (5.6), then $\prec$ is said to be k-*delightful *for the ideal $\mathcal{I}$. It is said to be* delightful *for $\mathcal{I}$ if it is k-delightful for $\mathcal{I}$, for every $k \geq 1$.*

Let us go back to Example 3, Section 5.2. The initial ideal $\mathrm{in}_\prec(\mathcal{I}_X^{\{1\}})$, i.e. the ideal of the flat limit of $\mathrm{Sec}(X)$ with respect to D, must contain $\mathcal{I}_0^{\{1\}}$, by (5.6). Moreover, the dimension (and also the degree) is preserved under flat deformations, so

$$\dim(\mathrm{Sec}(X)) = \dim(\mathcal{V}(\mathrm{in}_\prec(\mathcal{I}^{\{1\}}))) \geq \dim(\mathcal{V}(\mathcal{I}_0^{\{1\}})) = 5,$$

then $\mathrm{Sec}(X)$ is a hypersurface of $\mathbb{P}^6$. Its defining equation is given by a homogeneous cubic polynomial whose initial term, with respect to the term order associated to the triangulation D, is $x_0x_3x_5$. Therefore $\deg(\mathrm{Sec}(X)) = 3$.

Let us look now at Example 4 (see Section 5.2). We have that

$$\deg(\mathrm{Sec}(X)) = 6 > \deg \mathcal{I}_0^{\{1\}} = 3,$$

then the term order associated to the degeneration D is not 1-delightful for the ideal of X. Furthermore, one can check that the monomial $x_0x_4x_6x_8$ defining $\mathcal{I}_0^{\{2\}}$ is the initial term of the quadratic polynomial $\det(C)$ defining $\mathrm{Sec}_2(X)$, thus X is 2-delightful.

k-delightful toric degenerations

Let D be a toric degeneration of a toric variety X of dimension n. Any subset of D of m pairwise disjoint planes, i.e. $m(n+1)$ vertices of D such that they form the vertices of m disjoint tetrahedra of D, will span a linear subspace of $\mathbb{P}^r$ of dimension $m(n+1)-1$. A subset of this type is said to be a *skew m-set*; we denote by $N_m(D)$ the set of such skew m-sets and by $\nu_m(D)$ its cardinality, see [16, 32]. Consider the following result, due to Sturmfels and Sullivant ([32], Theorem 5.4), which gives a lower bound to the number $\nu_k(X)$ for toric varieties.

Theorem 34. *If there exists a toric degeneration D of X to a union of $\mathbb{P}^n$'s with at least one skew $(k+1)$-set, then $Sec_k(X)$ has the expected dimension and $\nu_k(X)$ is bounded below by the number of skew $(k+1)$-sets:*

$$\nu_k(X) \geq \nu_{k+1}(D). \tag{5.7}$$

Proof. Notice first of all that it must be $(k+1)n + k \leq r$. Let $\mathcal{I}$ be the ideal of X and let $\mathcal{I}_0 = \mathrm{in}_\prec(\mathcal{I})$ be the ideal of the central fiber X_0. The simplicial complex of X_0 is D; let $D^{\{k\}}$ be the simplicial complex of $\mathcal{I}_0^{\{k\}}$: the simplices in $D^{\{k\}}$ are the unions of $k+1$ simplices in D, (see [32], Remark 2.9). Notice that

the simplices of $D^{\{k\}}$ of maximal dimension are the skew $(k+1)$-sets and the subspaces they span sit in the flat limit of $\mathrm{Sec}_k(X)$. Therefore, if there exists at least one skew $(k+1)$-set in D, then $\mathrm{Sec}_k(X)$ has the expected dimension $(k+1)n+k$, having at least an irreducible linear component of dimension $(k+1)n+k$.
Moreover, the toric variety described by $D^{\{k\}}$ is the reduced union of the coordinate subspaces in $\mathbb{P}^r$ given by the skew $(k+1)$-sets. Notice that different skew $(k+1)$-sets could span the same subspace π of $\mathbb{P}^r$ and that for the general point of π there is a unique subspace of dimension k meeting the $k+1$ planes each at a point, for each skew $(k+1)$-set spanning π. Furthermore, the limit of the k-secant variety of X contains the variety defined by the k-secant of $\mathcal{I}_0$, by the (5.6), thus inequality (5.7) is proven. □

Sturmfels and Sullivant in [32] conjectured that if the lower bound in (5.7) holds with equality, then D is k-delightful. We will call such degenerations k-delightful, according to Ciliberto, Dumitrescu e Miranda in [16].

Now, consider the two following examples. Let V_3 be the Veronese surface in $\mathbb{P}^9$. We have $\deg(\mathrm{Sec}(V_3)) = 15$ and $\mu_1(V_3) = 1$, therefore $\nu_1(V_3) = 15$ is the number of nodes that X aquires in a general projection to $\mathbb{P}^4$. Let D be the triangulation shown in Figure 5.6: $\nu_3(D) = 12$: twelve nodes of the image of

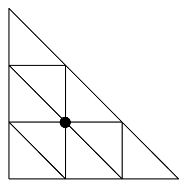


Figure 5.6: A planar degeneration of V_3.

X in $\mathbb{P}^4$ correspond to the pairs of planes that are disjoint in X_0, but whose projections meet in $\mathbb{P}^4$; so D is not 1-delightful because strict inequality holds in (5.7).
As a second example consider the Del Pezzo surface X_6 of degree 6 in $\mathbb{P}^6$: $\nu_1(X_6) = \deg(\mathrm{Sec}(X_6)) = 3$. Consider the triangulation D' of the hexagon in Figure 5.7. We get $\nu_2(D') = 0$.
In both cases the sextuple central point, marked in the figures, causes an obstruction to the presence of skew 2-sets in D and in D': in both examples it counts for three more nodes in a general projection to $\mathbb{P}^4$.

How do the singularities of the configuration influence the lack of delightfulness? This question was asked in [16] by Ciliberto, Dumitrescu and Miranda. Our aim is to give an explanation to this phenomenon. In the next chapter we

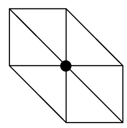


Figure 5.7: A planar degeneration of X_6

will expose some partial results in this direction, for $n = 2$ and $k = 1, 2$.

Chapter 6

Some speculations on the lack of k-delightfulness

Let $P \subseteq M_{\mathbb{R}}$ be the defining polytope of a projective toric surface X and let D be a planar toric degeneration of X to a union of planes X_0. Let $p \in P_M$ be a lattice point of P and let $Q^1, \dots, Q^\delta \in D$ be the triangles in D having a vertex in p: $Q^1 \cap \cdots \cap Q^\delta = \{p\}$. Suppose that the union of the Q^i's is a convex planar figure, namely a sub-polytope Q_p of P, of (normalized) area

$$\mathrm{Area}(Q_p) = \delta.$$

The configurations in Figure 6.1 are admitted, while the ones in Figure 6.2 are not admitted. Let $Z = Z_p$ be the projective toric sub-variety of degree δ of

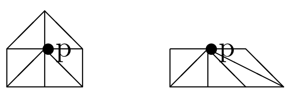


Figure 6.1: Admitted configurations.

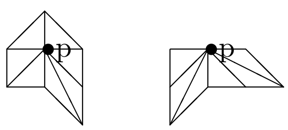


Figure 6.2: Not admitted configurations.

X defined by Q_p and let Z_0 be the union of δ planes defined by the Q^i's.

If p is an internal lattice point, i.e. $p \in P^\circ \cap M$, we will call it an *elliptic singularity* for D because Z_0 is a reduced cycle of planes intersecting at a point (corresponding to p): it has sectional genus one, being the general hyperplane

section a cycle of lines. If p is a boundary point, i.e. $p \in \partial P \cap M$, we will say that p is a *rational singularity* for D because the general hyperplane section of Z_0 is a chain of lines.
Let now p_1, p_2 be two singularities for D with the properties described above. If $\dim(Q_{p_1} \cap Q_{p_2}) < 2$, i.e. if Q_{p_1} and Q_{p_2} intersect in a common proper face (perhaps the empty face), then we will say that Q_{p_1} and Q_{p_2} are *non-overlapping*.

6.1 An improved lower bound for ν_k, $k = 1, 2$

This section is devoted to the proof of the following result that improves the lower bound for ν_k of Theorem 34 for the cases $k = 1$ and $k = 2$.

Theorem 35. *Let $k \in \{1, 2\}$. Let $X = X_P$ be a projective toric surface such that* $\dim Sec_k(X) = 3k+2$. *Let D be any triangulation of P; let $\{p_i\}_{i \in I} \subseteq P_M$, $\{Q_{p_i}\}_{i \in I}$ and $\{Z_{p_i}\}_{i \in I}$ be as above. Assume that*

1. $\dim Sec_k(Z_{p_i}) = 3k + 2$, *for $i \in I$,*
2. *there exists a regular subdivision D_i^1 of P containing Q_{p_i} and such that the polytopes of D_i^1 are unions of polytopes of D, for every $i \in I$ and*
3. *the polytopes $\{Q_{p_i}\}_{i \in I}$ are pairwise non-overlapping.*

Then D is not k-delightful. Moreover

$$\nu_k(X) \geq \nu_{k+1}(D) + \sum_{i \in I} \nu_k(Z_{p_i}). \tag{6.1}$$

6.1.1 The case $k = 1$

Let $X = X_P$ be a projective toric surface such that $\dim \mathrm{Sec}(X) = 5$. Let D be a planar toric degeneration of X and let p be an elliptic or rational singularity for D. Let $Q = Q_p = \cup_{i=1}^{\delta} Q^i$ be the sub-polytope of P corresponding to p and let $Z = Z_p$ be the projective toric surface of degree δ defined by Q: $Z \subseteq \mathbb{P}^{\delta'} \subseteq \mathbb{P}^r$, where

$$\delta' = \begin{cases} \delta & \text{if } p \text{ is elliptic} \\ \delta + 1 & \text{if } p \text{ is rational} \end{cases}$$

We will prove that the flat limit of the secant variety of Z sits in the flat limit of the secant variety of X and in particular that it is a component of degree $\nu_1(Z)$ of it. For this reason, we will assume that $\delta' \geq 5$. If, on the contrary, $\delta' < 5$, then the secant variety of Z has dimension less than five, so it does not

contribute to the degree computation.
Our approach consists in considering a toric degeneration D^1 of X, if it exists, such that the variety Z is an irreducible component of the central fiber. To this end, assume that an intermediate regular partition D^1 of P given by Q and the by the families $\{T_k\}_{k\in I_1}$ and $\{S_j\}_{j\in I_2}$, where the T_k's are triangles of D and S_j's are convex unions of triangles of D, exists; the reader can see some examples in Figure 6.3, Figure 6.4 and Figure 6.5.

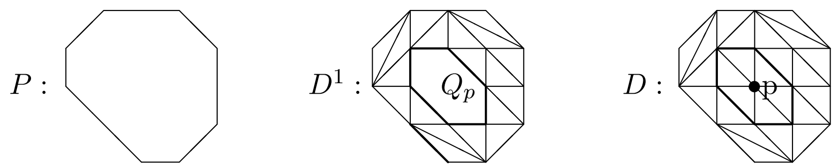


Figure 6.3: An example of decomposed degeneration, $Z_p = X_6$.

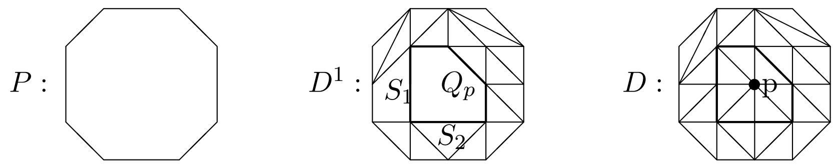


Figure 6.4: An example of decomposed degeneration, $Z_p = X_7$.

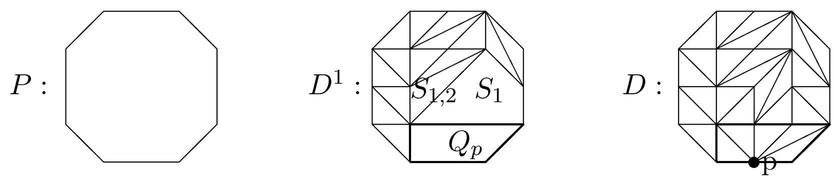


Figure 6.5: An example of decomposed degeneration, $Z_p = S(2, 3)$.

By decomposing the degeneration D of X to X_0 into two subsequent degenerations and by exploting the fact that the degree is preserved under flat deformations, we are able to improve the lower bound for the number $\nu_1(X)$ of Theorem 34.

We need the following definition that generalizes to arbitrary irreducible varieties the concept of *join* of linear spaces.

Definition 18. *Let $X, Y \subseteq \mathbb{P}^r$ be irreducible varieties. Let $J_{X,Y} \subseteq X \times Y \times \mathbb{P}^r$ be the* abstract join *of X and Y defined as the Zariski closure of the set*

$$\{((x, y), z) : x \neq y, z \in \langle x, y\rangle\} \subseteq X \times Y \times \mathbb{P}^r.$$

It is irreducible of dimension $\dim(X) + \dim(Y) + 1$*. Consider the projection* $p_{X,Y}$ *on the second factor and define the* join *of* X *and* Y

$$J(X,Y) := p_{X,Y}(J_{X,Y}),$$

to be the scheme-theoretic image of $J_{X,Y}$ *in* $\mathbb{P}^r$*. It is an irreducible variety of dimension*

$$\dim J(X,Y) \leq \min\{\dim X + \dim Y + 1, r\}.$$

Proposition 36. *Keeping the same setting as above, if there exists in* D *a singularity* p *as in Table 6.1 or in Table 6.2 and if there exists a regular subdivision* D^1 *of* P *as above, then*

$$\nu_1(X) \geq \nu_2(D) + \nu_1(Z). \tag{6.2}$$

Proof. 1. Let us consider first of all the degeneration D^1 of X. Let X_t^1 be the fiber of D^1: $X_t^1 \cong X$, for $t \neq 0$, while X_0^1 is the reduced union of the toric surfaces given by D^1 that are: Z, $\{X_{T_k}\}_{k \in I_1}$ and $\{X_{S_j}\}_{j \in I_2}$. We have that

- the secant variety of Z,
- the secant variety of S_j, for $j \in I_2$ and
- all the joins $J(Z, X_{S_j})$, $J(Z, X_{T_k})$, $J(X_{S_{j_1}}, X_{S_{j_2}})$, $J(X_{T_{k_1}}, X_{T_{k_2}})$, $J(X_{S_j}, X_{T_k})$, for $j, j_1, j_2 \in I_2, j_1 \neq j_2$ and $k, k_1, k_2 \in I_1, k_1 \neq k_2$,

sit in the flat limit $\lim_{D^1} \mathrm{Sec}(X)$ of the secant variety of X, with respect to D^1. Notice that

$$\{J(X_{T_{k_1}}, X_{T_{k_2}}) : k_1 \neq k_2 \text{ and } \dim J(X_{T_{k_1}}, X_{T_{k_2}}) = 5\} = N_2(D^1).$$

2. We consider now the second degeneration D^2 which has as general fiber $X_s^2 \cong X_0^1$, $s \neq 0$, and as central fiber the reduced union of planes $X_0^2 \cong X_0$. The flat limit, with respect to D^2, of $\lim_{D^1} \mathrm{Sec}(X)$, that is nothing but $\lim_D \mathrm{Sec}(X)$, contains as component the flat limits, with respect to D^1 of all the components of $\lim_{D^1} \mathrm{Sec}(X)$, namely the following:

   - $\lim_{D^2} \mathrm{Sec}(Z)$,
   - $\lim_{D^2} \mathrm{Sec}(X_{S_j})$ and
   - the flat limit, with respect to D^2, of all the joins between components of X_s^2, $s \neq 0$.

   In particular the components of maximal dimension, i.e. of dimension equal to $\dim(\mathrm{Sec}(X)) = 5$, contribute to the computation of the degree of

$\lim_D \mathrm{Sec}(X)$, namely of $\nu_1(X)$. Unfortunately, we are not able to determine how because we do not know the degree of all of them. Nevertheless we can give at least a partial explanation.
First of all, the $\nu_1(D^1)$ skew 2-sets of D^1 are skew 2-sets also for D^2: $N_2(D^1) \subseteq N_2(D^2)$. Moreover, in D^2 there are further pairs of disjoint triangles, and the $\mathbb{P}^5$'s they define are 5-dimensional components of the flat limits, with respect to D^2, of the joins of components of X_0^1. Thus, the whole set $N_2(D^2) = N_2(D)$ sits certainly in the flat limit and it corresponds to $\nu_2(D)$ linear irreducible distinct components of $\lim_D \mathrm{Sec}(X)$. Moreover $\lim_{D^2} \mathrm{Sec}(Z)$ is a component of the flat limit of $\mathrm{Sec}(X)$ of degree

$$\deg\left(\lim_{D^2}(\mathrm{Sec}(Z))\right) = \deg(\mathrm{Sec}(Z)) = \nu_1(Z).$$

All these contributions do not interfere to each other, because they come from different components of the limit of $\mathrm{Sec}(X)$ with respect to D^1. Hence the number $\nu_2(D) + \nu_1(Z)$ provides a lower bound for $\nu_1(X)$.

□

What does it happen if in D there are more than one singularity? If there are singularities $\{p_i\}_{i \in I}$ in D satisfying the hypotheses of Proposition 36 and the non-overlapping property, then the contributions given by the degrees of the $\mathrm{Sec}(Z_{p_1})$'s do not interfere to each other. To see this, let us decompose the degeneration D taking subdivisions D_i^1 and D_i^2, for each i. The flat limit of the secant variety of Z_{p_i} with respect to D_i^2 sits in the flat limit of the secant variety of X with respect to D, for every i, by Proposition 36. Moreover, the non-overlapping assumption assures that $\lim_{D_i^2} \mathrm{Sec}(Z_{p_i})$ and $\lim_{D_j^2} \mathrm{Sec}(Z_{p_j})$ are two different components of $\lim_D \mathrm{Sec}(X)$, for all $i, j \in I$, $i \neq j$; hence the respective degrees sum up to $\nu_2(D)$. This proves Theorem 35 for the case $k = 1$.

6.1.2 The case $k = 2$

In this section we will make the same analysis for the varieties of trisecant planes of toric surfaces.
Let $X = X_P$ be a toric surface such that $\dim \mathrm{Sec}_2(X) = 8$. Let D be any triangulation of P.
There are only two types of elliptic singularities we are interested in, namely the ones such that Z_p is either the Veronese surface V_3 in $\mathbb{P}^9$ or the del Pezzo surface X_8 of degree eight in $\mathbb{P}^8$. Indeed in all the remaining cases (see Table 6.1, third column) the 2-secant variety has dimension less than eight.
On the other hand, the only toric surface with sectional genus zero such that its 2-secant variety has dimension eight and such that there exists a toric

degeneration of it to a union of planes all of them intersecting at a single point is the rational normal scroll $S(2, \delta - 2) \subseteq \mathbb{P}^{\delta+1}$, with $\delta \geq 7$, (see Table 6.2, third column).
Using the same contruction and making the same remarks as we did in the previous section for the case $k = 1$, we obtain the following result.

Proposition 37. *Let $X = X_P$ be a toric surface such that $\dim Sec_2(X) = 8$ and let D be a triangulation of P. Let $p \in P_M$ be a multiple point such that Z_p is either V_3, or X_8, or $S(2, \delta - 2)$, with $\delta \geq 7$. Assume furthermore that there exists an intermediate regular subdivision D^1 of P given by Q_p and either triangles of D or unions of triangles of D. Then*

$$\nu_2(X) \geq \nu_3(D) + \nu_2(Z_p). \tag{6.3}$$

Proof.

1. Let D^1 be a toric degeneration of X as above. The following varieties are distinct components of the flat limit $\lim_{D^1} \mathrm{Sec}_2(X)$:
   - $\mathrm{Sec}_2(Z)$;
   - $J(X_{S_{j_1}}, J(X_{S_{j_2}}, X_{S_{j_3}}))$, for every $j_1, j_2, j_3 \in I_2$; notice that if $j_1 = j_2 = j_3 = j$, then it is the 2-secant variety of X_{S_j};
   - $J(X_{T_{k_1}}, J(X_{T_{k_2}}, X_{T_{k_3}}))$, for every $k_1, k_2, k_3 \in I_1$, that is a skew 3-sets of D^1 if $T_{k_1} \cap T_{k_2} \cap T_{k_3} = \emptyset$;
   - $J(X_{S_j}, J(X_{T_{k_1}}, X_{T_{k_2}}))$, $J(J(X_{S_{j_1}}, X_{S_{j_2}}), X_{T_k})$, for $j, j_1, j_2 \in I_2$ and $k, k_1, k_2 \in I_1$, and
   - $J(Z, J(X_{S_{j_1}}, X_{S_{j_2}}))$, $J(Z, J(X_{S_j}, X_{T_k}))$, $J(Z, J(X_{T_{k_1}}, X_{T_{k_2}}))$ for every $j, j_1, j_2 \in I_2$ and every $k, k_1, k_2 \in I_1$;
   - $J(\mathrm{Sec}Z, X_{S_j})$, $J(\mathrm{Sec}Z, X_{T_k})$, $j \in I_2$, $k \in I_1$;
2. Then, looking at the second degeneration D^2, we see that the skew 3-sets of D^2 (that are the skew 3-sets of D) and the limit $\lim_{D^2} \mathrm{Sec}_2(Z)$ sit in the flat limit of $\mathrm{Sec}_2(X)$ with respect to D, with the same argument as in Proposition 36.

□

If there are more than one singularity in D, $\{p_i\}_{i \in I}$, satisfying the hypotheses of Proposition 37, and such that the polytopes $\{Q_{p_1}\}_{i \in I}$ are pairwise non-overlapping, then

$$\nu_2(X) \geq \nu_3(D) + \sum_{i \in I} \nu_2(Z_{p_i}).$$

From this follows Theorem 35 for the case $k = 2$.

To conclude this section we explore, given a regular subdivision D, the existence of an intermediate regular subdivision D^1.
Assume first of all that either the edges of Q have reticular lenght equal to one or they lye on the boundary of P (under this assumption p must be an elliptic singularity, cfr. Table 6.1). The family of sub-polytopes of P given by Q and by the Area$(P)-\delta$ triangles of D not having a vertex in p form a partition of P (see Figure 6.3). Such a subdivision is regular. Indeed, given a lifting function F_D over D, one can always find a lifting function F_{D^1} over D^1, exploiting the fact that strict convexity is a local property: it is enough to flatten F_D over Q. More precisely, one can always assume that $F_D(\underline{m}) \gg 2$, for $\underline{m} \notin Q$ and that

$$F_D(\underline{m}) = \begin{cases} 1-\epsilon & \text{if } \underline{m} = p \\ 1 & \text{if } \underline{m} \in Q_M \setminus \{p\} \end{cases},$$

with $0 < \epsilon \ll 1$. Hence, a lifting function F_{D^1} for D^1 is the following:

$$F_{D^1}(\underline{m}) := \begin{cases} 1 & \text{if } \underline{m} = p \\ F_D(\underline{m}) & \text{if } \underline{m} \in P_M \setminus \{p\} \end{cases}$$

Suppose now that Q has edges $L_1 \ldots, L_m$ of lenght respectively $l_1, \ldots, l_m > 1$. Let us contruct a partition of P containing Q, triangles and unions of triangles of D, using the following algorithm.
INPUT: a triangulation D of P.
OUTPUT: a subdivision D^1 of P containing Q.

- Let S_i be the minimal convex union of triangles of D such that $S_i \cap Q = L_i$, for $i = 1, \ldots, m$. If all the S_i has edges either of lenght one or lying on ∂P we stop.
- Otherwise, for each $i = 1, \ldots, m$, let $L_{i,1}, \ldots, L_{i,m_i}$ be the edges of S_i of lenght respectively $l_{i,1}, \ldots, l_{i,m_i} > 1$, for $i \in \{1, \ldots, m\}$. Let $S_{i,j}$ be the minimal convex union of triangles of D such that $S_{i,j} \cap S_i = L_{i,j}$, $i = 1, \ldots, m$, $j = 1, \ldots, m_i$. If all the $S_{i,j}$'s have edges either of lenght one or contained in ∂P, then we stop.
- Otherwise we go on as above, until all the polytopes have edges either of lenght one, or contained in ∂P.

Notice that this process is finite. The output is a complex D^1 whose maximal polyhedra are Q, the S_i's, the $S_{i,j}$'s, etc. and the remaining triangles T_k's of D. See for example Figure 6.5. If one is able to flatten the lifting function F_D over Q, the S_i's, the $S_{i,j}$'s, etc. and to rescale it over the T_k's, in such a way that the resulting piecewise linear function is strictly convex over P, one

has found a lifting function F_{D^1} for D^1 to be regular.
At this point it is not difficult to define D^2: it is sufficient to take triangulations D_Q of Q, D_{S_i} of S_i, $D_{S_{i,j}}$ of $S_{i,j}$, etc., such that, combining them, one gives rise to the full regular triangulation D of P.

6.2 A conjecture

In the previous sections we showed that the presence in D of a reticular point, which is the common vertex of a sufficiently large number of triangles, induces an obstruction to the k-delightfulness of D, for $k = 1, 2$. Moreover the contributions of the singularities do not interfere, provided that the corresponding sub-polytopes do not overlap. We conjecture that the non-overlapping hypothesis may be removed.

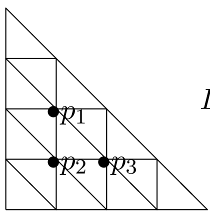
Conjecture 38. *Let $k \in \{1, 2\}$. Let D be a planar toric degeneration of a toric surface $X = X_P$ with $\dim(Sec(X)) = 3k + 2$. Let $\{p_i\}_{i \in I} \subseteq P_M$ be the set of all lattice points such that conditions* 1. *and* 2. *of Theorem 35 are satisfied. Then*

$$\nu_k(X) \geq \nu_{k+1}(D) + \sum_{i \in I} \nu_k(Z_{p_i}), \; k = 1, 2. \tag{6.4}$$

Let us show some examples in the case $k = 1$.

Example 5. *Consider the* 4*-ple Veronese embedding V_4 of $\mathbb{P}^2$ in $\mathbb{P}^{14}$ and the regular subdivisions D and D' of the triangle of edge lenght four (and normalized area sixteen) shown in Figure 6.6. We know that $\nu_1(V_4) = \deg(Sec(V_4)) =$*

D :



D' :

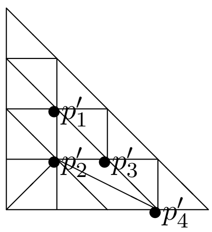


Figure 6.6: Planar toric degenerations of $V_4 \subseteq \mathbb{P}^{14}$

75, using Formula (5.2). One can easily check that $\nu_2(D) = 66$. Moreover $\nu_1(Z_{p_1}) = \nu_1(Z_{p_2}) = \nu_1(Z_{p_3}) = 3$. Therefore the four contributions sum up to restore the secant degree:

$$\nu_2(D) + \nu_1(X_{p_1}) + \nu_1(X_{p_2}) + \nu_1(X_{p_3}) = 75,$$

and (6.4) *holds with equality. Similarly, looking at the figure on the right, one can check that*

$$\nu_2(D') + \nu_1(X_{p'_1}) + \nu_1(X_{p'_2}) + \nu_1(X_{p'_3}) + \nu_1(X_{p'_4}) = 60 + 3 + 10 + 1 + 1 = 75.$$

Example 6. *Let X be the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{11}$ via $\mathcal{O}(2,3)$: one has $\nu_1(X) = \deg(Sec(X)) = 35$. Consider the three planar degenerations of X shown in Figure 6.7. In the first two cases, the sum of the number of skew*

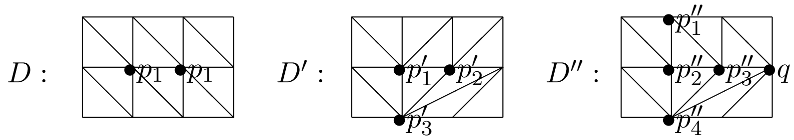


Figure 6.7: Planar toric degenerations of $X \subseteq \mathbb{P}^{11}$

2*-sets and of the contributions of the singularities restores the secant degree:*

$$\upsilon_2(D) + \nu_1(X_{p_1}) + \nu_1(X_{p_2}) = 29 + 3 + 3 = 35$$

and

$$\upsilon_2(D') + \nu_1(X_{p_1'}) + \nu_1(X_{p_2'}) + \nu_1(X_{p_3'}) = 28 + 3 + 1 + 3 = 35.$$

In the third configuration, something different happens. One can see that:

$$\begin{aligned} \upsilon_2(D'') + \nu_1(X_{p_1''}) + \nu_1(X_{p_2''}) + \nu_1(X_{p_3''}) + \nu_1(X_{p_4''}) &= 29 + 1 + 1 + 1 + 1 \\ &= 33 < 35 \end{aligned}$$

In D'' there is a lattice point q which is the common vertex of five triangles: certainly it causes an obstruction to the presence of skew 2-sets, but we are not able, so far, to check how, because the polygon given by the triangles around it is not convex, so the above description does not make sense.

An intention for the future is to fully understand the lack of k-delightfulness in order to give an answer to Conjecture 38 and to have a complete explanation of this phenomenon. Moreover it would be interesting to prove something similar for the cases $k \geq 3$ and in higher dimension.
Notice that in the surface case, the expected dimension of $\mathrm{Sec}_3(X)$ is $\min\{11, r\}$. No one of the singularities in Table 6.1 is interesting in this case, because $\dim \mathrm{Sec}_3(Z_p) < 11$. Also in the rational case (Table 6.2) there are no examples of rational normal scrolls in $\mathbb{P}^\delta$, with $\delta \geq 11$ that could contribute to ν_3, indeed

- if $\delta_1 \leq 2$, then $S(\delta_1, \delta_2)$ is 3-defective, while
- if $\delta_1 \geq 3$ there are no triangulations of the defining polytope such that all the triangles have a common vertex p; indeed p would lye on one of the two horizontal edges of the trapezium, but they are both too long.

Therefore for $k = 3$, thus also for $k \geq 4$, the description done for $k = 1, 2$ does not work; the causes to the lack of k-delightfulness must be hidden elsewhere.

6.3 Tables

In the following tables, we summarize the singularities that influence the lack of k-delightfulness of a planar toric degeneration, for $k = 1, 2$. We already described their ideals and k-secant ideals, for $k = 1, 2$, in Section 5.2.
In the first column we drow the subdivision of the poltope $Q = Q_p$, in the remaining columns we write the degree, the 1-secant degree and the 2-secant degree of the surface $Z = Z_p$ defined by Q.
In all cases, $\deg(\mathrm{Sec}_k(Z)) = \nu_k(Z)$, i.e. $\mu_k(X) = 1$, for $k = 1, 2$ (see [13], where the surfaces with $\mu_k > 1$ are classified).

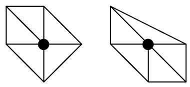
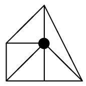
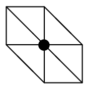
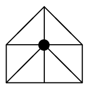
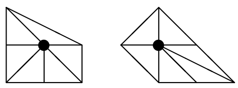
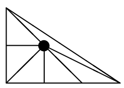
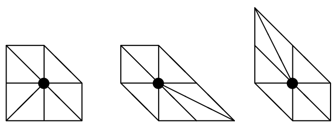
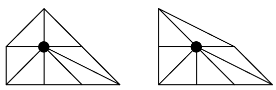
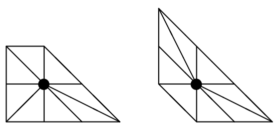
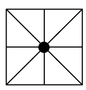
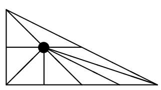
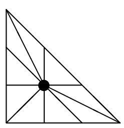
	triangulation of Q	$\deg(Z)$	$\nu_1(Z)$	$\nu_2(Z)$
1.		5	1	/
2.		5	1	/
3.		6	3	/
4.		6	3	/
5.		6	3	/
6.		6	3	/
7.		7	6	/
8.		7	6	/
9.		8	10	1
10.		8	10	/
11.		8	10	/
12.		9	15	4

Table 6.1: The elliptic case

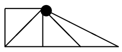
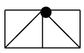
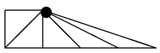
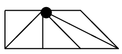
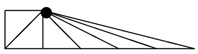
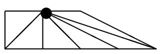
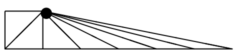
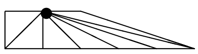
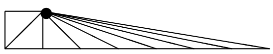
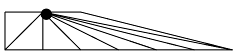
	triangulation of Q	$\deg(Z)$	$\nu_1(Z)$	$\nu_2(Z)$
1.		4	1	/
2.		4	1	/
3.		5	3	/
4.		5	3	/
5.		6	6	/
6.		6	6	/
7.		7	10	/
8.		7	10	1
9.		8	15	/
10.		8	15	4
11.	$S(1, \delta - 1)$	$\delta \geq 9$	$\binom{\delta-2}{2}$	/
12.	$S(2, \delta - 2)$	$\delta \geq 9$	$\binom{\delta-2}{2}$	$\binom{\delta}{3} - 2((\delta - 3)^2 + 1)$

Table 6.2: The rational case

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