Geometry and Combinatorics of Toric Arrangements

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Introduction

A *toric arrangement* is a finite family of hypersurfaces in a complex torus T, each hypersurface being the kernel of a character of T.

Although similar arrangements appeared already in the 90s, it is just in the last few years that a systematic theory of toric arrangements and their applications has been developing. Toric arrangements proved to be deeply related with a wide number of topics, including partition functions, integral points in polytopes, zonotopes, and index theory.

Toric arrangements are also closely related with hyperplane arrangements, from different points of view. First of all, every toric arrangement is locally isomorphic to hyperplane arrangements. Secondly, every toric arrangement can be seen as a periodic arrangement of affine subspaces in an affine space. Thirdly, many results known for hyperplane arrangements have an analogue for toric arrangements: for instance, the computation of the cohomology of the complement ([12]), the construction of a *wonderful model* (Chapter 3), and the definition of a polynomial encoding a rich description of the arrangement (Chapter 2).

Furthermore, as explained in a forthcoming book of De Concini and Procesi ([14]), whereas hyperplane arrangements are related with some *differentiable* problems and objects, toric arrangements are related with their *discrete* counterparts: for instance, if the former appear in the computation of volume of polytopes, the latter does while counting the number of their integral points; also, the former are related with *box splines* and *multivariate splines* (functions studied in Approximation Theory), while the latter with *partition functions*; furthermore, the former are associated with *differentiable Dahmen-Micchelli spaces* (spaces of polynomials defined by differential equations), and the latter with *discrete Dahmen-Micchelli spaces* (spaces of quasipolynomials defined by difference equations).

Similarly to what happens for hyperplane arrangements, also for toric arrangements one of the main goal is to understand the topology and the geometry of the complement $\mathcal{R}_X$ of the union of the hypersurfaces. And, if in the theory of hyperplane arrangements a key object is the *intersection poset*, likewise in the theory of toric arrangements a central role is played by the poset $\mathcal{C}(X)$ of the *layers* of the arrangement, i.e. of the connected components of the intersections of the hypersurfaces. For instance, by [12] the cohomology of $\mathcal{R}_X$ is a direct sum of contributions given by the elements of $\mathcal{C}(X)$.

This thesis is composed of three parts, which can be read independently from each other. Although the subject is common, the points of view are rather different: Lie-Theoretic in the first part, Combinatorial in the second, Algebro-Geometrical in the third.

In Chapter 1 we describe the combinatorics of $\mathcal{C}(X)$ for the toric arrangements that are defined by *root systems*. This remarkable class of examples is related to the *Kostant partition function*, which plays an important role in Representation Theory, since it yields efficient computation of weight multiplicities and Littlewood-Richardson coefficients. For these arrangements, by studying the action of the Weyl group, we can provide precise formulae counting the elements of $\mathcal{C}(X)$. Using this formulae we compute the Euler characteristic and the Poincaré polynomial of $\mathcal{R}_X$.

In Chapter 2 we introduce a polynomial $M(x, y)$, which can be considered as the "toric analogue" of the Tutte polynomial. Indeed, the *characteristic polynomial* of $\mathcal{C}(X)$ and the Poincaré polynomial of $\mathcal{R}_X$ are shown to be specializations of $M(x, y)$, as the corresponding polynomials for hyperplane arrangements are specializations of the ordinary Tutte polynomial. We also prove that $M(x, y)$ satisfies a recurrence known as *deletion-restriction*, and that it has positive coefficients. Furthermore, we show that $M(x, 1)$ counts integral points in zonotopes according to the dimension of the minimal face in which they are contained, while $M(1, y)$ is the graded dimension of the related discrete Dahmen-Micchelli space.

In Chapter 3 we build a model $\mathbf{Z}_X$ which contains $\mathcal{R}_X$ as a dense open set, but in which the complement of $\mathcal{R}_X$ is a *normal crossing divisor* $\mathbf{D}$. We call $\mathbf{Z}_X$ the *wonderful model* of the toric arrangement, in analogy with the wonderful model built in [10] for arrangements of subspaces in a vector (or projective) space. Then we develop the "toric analogue" of the combinatorics of *nested sets*, and we use it to define a family of smooth open sets covering the model. In this way we prove the model to be smooth, and we obtain a geometrical and combinatorial description of the irreducible components of $\mathbf{D}$ and of their intersections.

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Table of Contents

Chapter 1

The case of root systems

In this chapter, given the toric arrangement defined by a root system Φ, we describe the poset of its layers and we count its elements. Indeed we show how to reduce to the 0-dimensional layers, and in this case we provide an explicit formula involving the maximal subdiagrams of the affine Dynkin diagram of Φ. Then we compute the Euler characteristic and the Poincaré polynomial of the complement of the arrangement.

1.1 Introduction

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra, $\Phi \subset \mathfrak{h}^*$ and $\Phi^\vee \subset \mathfrak{h}$ respectively the root and coroot systems. The equations $\{\alpha(h) = 0,\ \alpha \in \Phi\}$ define in $\mathfrak{h}$ a family $\mathcal{H}$ of intersecting hyperplanes. Let $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots: the quotient $T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$ is a complex torus of rank n. Each root α takes integer values on $\langle \Phi^\vee \rangle$, hence it induces a map $T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ that we denote by e^α. This is a character of T; let H_α be its kernel:

$$H_\alpha \doteq \{t \in T \mid e^\alpha(t) = 1\}.$$

In this way Φ defines in T a finite family of hypersurfaces

$$\mathcal{T} \doteq \{H_\alpha,\ \alpha \in \Phi^+\}$$

(since clearly $H_\alpha = H_{-\alpha}$). $\mathcal{H}$ and $\mathcal{T}$ are called respectively the *hyperplane arrangement* and the *toric arrangement* defined by Φ (see for instance [12], [14], [36]). We call *spaces* of $\mathcal{H}$ the intersections of elements of $\mathcal{H}$, and *layers* of $\mathcal{T}$ the connected components of the intersections of elements of $\mathcal{T}$. We denote by $\mathcal{L}(\Phi)$ the set of the spaces of $\mathcal{H}$, by $\mathcal{C}(\Phi)$ the set of the layers of $\mathcal{T}$, and by $\mathcal{L}_d(\Phi)$ and $\mathcal{C}_d(\Phi)$ the sets of $d-$dimensional spaces and layers. Clearly if $\Phi = \Phi_1 \times \Phi_2$ then $\mathcal{L}(\Phi) = \mathcal{L}(\Phi_1) \times \mathcal{L}(\Phi_2)$ and $\mathcal{C}(\Phi) = \mathcal{C}(\Phi_1) \times \mathcal{C}(\Phi_2)$, hence from now on we will suppose Φ to be irreducible. Let W be the Weyl group of Φ: since W permutes the roots, its natural action on T induces an action on $\mathcal{C}(\Phi)$.

$\mathcal{H}$ is a classical object, whereas $\mathcal{T}$ has recently been shown ([12]) to provide a geometric way to compute the values of the Kostant partition function. This function counts in how many ways an element of the lattice $\langle\Phi\rangle$ can be written as sum of positive roots, and plays an important role in representation theory, since (by Kostant's and Steinberg's formulae [27], [39]) it yields efficient computation of weight multiplicities and Littlewood-Richardson coefficients, as shown in [7] using results from [1], [4], [11], [40]. The values of Kostant partition function can be computed as a sum of contributions given by the elements of $\mathcal{C}_0(\Phi)$ (see [7, Teor 3.2]).

Furthermore, let $\mathcal{R}_\Phi$ be the complement in T of the union of all elements of $\mathcal{T}$. $\mathcal{R}_\Phi$ is known as the set of the *regular points* of the torus T and has been widely studied (see in particular [12], [28], [29]). The cohomology of $\mathcal{R}_\Phi$

is direct sum of contributions given by the elements of $\mathcal{C}(\Phi)$ (see for instance [12]). Then by describing the action of W on $\mathcal{C}(\Phi)$ we implicitly obtain a $W-$equivariant decomposition of the cohomology of $\mathcal{R}_\Phi$, and by counting and classifying the elements of $\mathcal{C}(\Phi)$ we can compute the Poincaré polynomial of $\mathcal{R}_\Phi$.

We say that a subset Θ of Φ is a *subsystem* if it satisfies the following conditions:

1. $\alpha \in \Theta \Rightarrow -\alpha \in \Theta$
2. $\alpha, \beta \in \Theta$ and $\alpha + \beta \in \Phi \Rightarrow \alpha + \beta \in \Theta$.

For each $t \in T$ let us define the following subsystem of Φ:

$$\Phi(t) \doteq \{\alpha \in \Phi | e^\alpha(t) = 1\}.$$

and denote by $W(t)$ the stabilizer of t.

The aim of Section 2 is to describe $\mathcal{C}_0(\Phi)$, which is the set of points $t \in T$ such that $\Phi(t)$ has rank n. We call its elements the *points* of the arrangement $\mathcal{T}$. Let $\alpha_1, \ldots, \alpha_n$ be simple roots of Φ, α_0 the lowest root (i.e. the opposite of the highest root), and Φ_p the subsystem of Φ generated by $\{\alpha_i\}_{0 \leq i \leq n, i \neq p}$. Let Γ be the affine Dynkin diagram of Φ and $V(\Gamma)$ the set of its vertices (a list of such diagrams can be found for instance in [21] or in [26]). $V(\Gamma)$ is in bijection with $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$, hence we can identify each vertex p with an integer from 0 to n. The diagram Γ_p obtained by removing from Γ the vertex p (and all adjacent edges) is the ordinary Dynkin diagram of Φ_p. Let W_p be the Weyl group of Φ_p, i.e. the subgroup of W generated by all the reflections $s_{\alpha_0}, \ldots, s_{\alpha_n}$ except s_{α_p}. Notice that Γ_0 is the Dynkin diagram of Φ and $W_0 = W$.

Then we prove:

Theorem 1.1.1. *There is a bijection between the $W-$orbits of $\mathcal{C}_0(\Phi)$ and the vertices of Γ, having the property that for every point t in the orbit $\mathcal{O}_p$ corresponding to the vertex p, $\Phi(t)$ is $W-$conjugate to Φ_p and $W(t)$ is $W-$conjugate to W_p.*

As a corollary we get the formula

$$|\mathcal{C}_0(\Phi)| = \sum_{p \in V(\Gamma)} \frac{|W|}{|W_p|}. \tag{1.1.1}$$

In Section 3 we deal with layers of arbitrary dimension. For each layer C of $\mathcal{T}$ we consider the subsystem of Φ

$$\Phi_C \doteq \{\alpha \in \Phi | e^{\alpha}(t) = 1 \; \forall t \in C\}$$

and its *completion* $\overline{\Phi_C} \doteq \langle \Phi_C \rangle_{\mathbb{R}} \cap \Phi$.

Let $\mathcal{K}_d$ be the set of subsystems Θ of Φ of rank $n-d$ that are *complete* (i.e. such that $\Theta = \overline{\Theta}$), and let $\mathcal{C}^{\Phi}_{\Theta}$ be the set of layers C such that $\overline{\Phi_C} = \Theta$. This gives a partition of the layers:

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^{\Phi}_{\Theta}.$$

Notice that the subsystem of roots vanishing on a space of $\mathcal{H}$ is always complete; then $\mathcal{K}_d$ is in bijection with $\mathcal{L}_d$. The elements of $\mathcal{L}_d$ are classified and counted in [34], [36]. Thus the description of the sets $\mathcal{C}^{\Phi}_{\Theta}$ given in Theorem 1.3.1 yields a classification of the layers of $\mathcal{T}$. In particular we show that

$|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1}|\mathcal{C}_0(\Theta)|$, where n_Θ is a natural number depending only on the conjugacy class of Θ, and then

$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta\in\mathcal{K}_d} n_\Theta^{-1}|\mathcal{C}_0(\Theta)|.$$

In Section 4, using results of [12] and [13], we deduce from Theorem 1.1.1 that the Euler characteristic of $\mathcal{R}_\Phi$ is equal to $(-1)^n|W|$. Moreover, Corollary 1.3.2 yields a formula for the Poincaré polynomial of $\mathcal{R}_\Phi$:

$$P_\Phi(q) = \sum_{d=0}^{n}(-1)^d(q+1)^d q^{n-d} \sum_{\Theta\in\mathcal{K}_d} n_\Theta^{-1}|W^\Theta|.$$

By this formula $P_\Phi(q)$ can be explicitly computed.

1.2 Points of the arrangement

1.2.1 Statements

For all facts about Lie algebras and root systems we refer to [23]. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

be the Cartan decomposition of $\mathfrak{g}$, and let us choose nonzero elements

$$X_0, X_1, \ldots, X_n$$

in the one-dimensional subalgebras $\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_n}$: since $[\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'}] = \mathfrak{g}_{\alpha+\alpha'}$ whenever $\alpha, \alpha', \alpha+\alpha' \in \Phi$, we have that $X_0, X_1, \ldots, X_n$ generate $\mathfrak{g}$. Let $a_0 = 1$ and for $p = 1, \ldots, n$ let a_p be the coefficient of α_p in $-\alpha_0$. For each $p = 0, \ldots, n$ we define an automorphism σ_p of $\mathfrak{g}$ by

$$\sigma_p(X_j) \doteq \begin{cases} X_j & \text{if } j \neq p \\ e^{2\pi i a_p^{-1}} X_j & \text{if } j = p \end{cases}$$

Let G be the semisimple and simply connected linear algebraic group having root system Φ; then $\mathfrak{g}$ is the Lie algebra of G, and T is the maximal torus of G corresponding to $\mathfrak{h}$ (see for instance [22]). G acts on itself by conjugacy, and for each $g \in G$ the map $k \mapsto gkg^{-1}$ is an automorphism of G. Its differential $Ad(g)$ is an automorphism of $\mathfrak{g}$.

Remark 1.2.1. For every $t \in \mathcal{C}_0(\Phi)$, let $\mathfrak{g}^{Ad(t)}$ be the subalgebra of the elements fixed by $Ad(t)$. For every $\alpha \in \Phi$ and for every $X_\alpha \in \mathfrak{g}_\alpha$ we have that

$$Ad(t)(X_\alpha) = \; e^{\alpha}(t)X_\alpha$$

and then

$$\mathfrak{g}^{Ad(t)} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi(t)} \mathfrak{g}_\alpha.$$

On the other hand $\mathfrak{g}^{\sigma_p}$ is generated by the subalgebras $\{\mathfrak{g}_{\alpha_i}\}_{0 \leq i \leq n, i \neq p}$. Then $\mathfrak{g}^{Ad(t)}$ and $\mathfrak{g}^{\sigma_p}$ are semisimple algebras having root system respectively $\Phi(t)$ and Φ_p. Our strategy will be to prove that for each $t \in \mathcal{C}_0(\Phi)$, $Ad(t)$ is conjugate to some σ_p. This implies that $\mathfrak{g}^{Ad(t)}$ is conjugate to $\mathfrak{g}^{\sigma_p}$ and then $\Phi(t)$ to Φ_p, as claimed in Theorem 1.1.1.

Then we want to give a bijection between vertices of Γ and $W-$orbits of $\mathcal{C}_0(\Phi)$ showing that, for every t in the orbit $\mathcal{O}_p$, $Ad(t)$ is conjugate to σ_p. However, since some of the σ_p (as well as the corresponding Φ_p) are themselves conjugate, this bijection is not canonical. To make it canonical we should merge the orbits corresponding to conjugate automorphisms: for this we consider the action of a larger group.

Let $\Lambda(\Phi) \subset \mathfrak{h}$ be the lattice of the *coweights* of Φ, i.e.

$$\Lambda(\Phi) \doteq \{h \in \mathfrak{h} | \alpha(h) \in \mathbb{Z} \; \forall \alpha \in \Phi\}.$$

The lattice spanned by the coroots $\langle \Phi^\vee \rangle$ is a sublattice of $\Lambda(\Phi)$; set

$$Z(\Phi) \doteq \frac{\Lambda(\Phi)}{\langle \Phi^\vee \rangle}.$$

This finite subgroup of T coincides with $Z(G)$, the *center* of G. It is well known (see for instance [22, 13.4]) that

$$Ad(g) = id_{\mathfrak{g}} \Leftrightarrow g \in Z(\Phi). \tag{1.2.1}$$

Notice that

$$Z(\Phi) = \{t \in T | \Phi(t) = \Phi\}$$

thus $Z(\Phi) \subseteq \mathcal{C}_0(\Phi)$. Moreover, for each $z \in Z(\Phi), t \in T, \alpha \in \Phi$,

$$e^\alpha(zt) = e^\alpha(z)e^\alpha(t) = e^\alpha(t)$$

and therefore $\Phi(zt) = \Phi(t)$. In particular $Z(\Phi)$ acts by multiplication on $\mathcal{C}_0(\Phi)$. Notice that this action commutes with that of W: indeed, let

$$N \doteq N_G(T)$$

be the normalizer of T in G. We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugacy action of N. The elements of $Z(\Phi) = Z(G)$ commute with the elements of G, hence in particular with the elements of N. Thus we get an action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$.

Let Q be the set of the $Aut(\Gamma)$-orbits of $V(\Gamma)$. If $p, p' \in V(\Gamma)$ are two representatives of $q \in Q$, then $\Gamma_p \simeq \Gamma_{p'}$, thus $W_p \simeq W_{p'}$. Moreover we will see (Corollary 1.2.4(ii)) that σ_p is conjugate to $\sigma_{p'}$. Then we can restate Theorem 1.1.1 as follows.

Theorem 1.2.1. *There is a canonical bijection between Q and the set of $W \times Z(\Phi)-$orbits in $\mathcal{C}_0(\Phi)$, having the property that if $p \in V(\Gamma)$ is a representative of $q \in Q$, then:*

1. *every point t in the corresponding orbit $\mathcal{O}_q$ induces an automorphism conjugate to σ_p;*

2. *the stabilizer of $t \in \mathcal{O}_q$ is isomorphic to $W_p \times Stab_{Aut(\Gamma)}p$.*

This theorem implies immediately the formula:

$$|\mathcal{C}_0(\Phi)| = \sum_{q \in Q} |q| \, \frac{|W|}{|W_p|} \tag{1.2.2}$$

where p is any representative of q. This is clearly equivalent to formula (1.1.1).

Remark 1.2.2. If we view the elements of $\Lambda(\Phi)$ as translations, we can define a group of isometries of $\mathfrak{h}$

$$\widetilde{W} \doteq W \ltimes \Lambda(\Phi).$$

$\widetilde{W}$ is called the *extended affine Weyl group* of Φ and contains the affine Weyl group $\widehat{W} \doteq W \ltimes \langle \Phi^\vee \rangle$ (see for instance [24], [37]).

The action of $W \times Z(\Phi)$ on $\mathcal{C}_0(\Phi)$ is induced by that of $\widetilde{W}$. Indeed $\widetilde{W}$ preserves the lattice $\langle \Phi^\vee \rangle$ of $\mathfrak{h}$, and thus acts on $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ and on $\mathcal{C}_0(\Phi) \subset T$. Since the semidirect factor $\langle \Phi^\vee \rangle$ acts trivially, $\widetilde{W}$ acts as its quotient

$$\frac{\widetilde{W}}{\langle \Phi^\vee \rangle} \simeq W \times Z(\Phi).$$

1.2.2 Examples: the classical root systems

In the following examples we denote by $\mathfrak{S}_n$, $\mathfrak{D}_n$, $\mathfrak{C}_n$ respectively the symmetric, dihedral and cyclic group on n letters.

1. **Case** C_n The roots

$$2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n$$

$(i = 1, \ldots, n)$ take integer values on the points $[\alpha_1^\vee/2], \ldots, [\alpha_n^\vee/2] \in \mathfrak{h}/\langle \Phi^\vee \rangle$, and thus on their sums, for a total of 2^n points of $\mathcal{C}_0(\Phi)$. Indeed, let us introduce the following notation. Fixed a basis $h_1^*, \ldots, h_n^*$ of $\mathfrak{h}^*$, the simple roots of C_n can be written as

$$\alpha_i = h_i^* - h_{i+1}^* \text{ for } i = 1, \ldots, n-1 \text{ , and } \alpha_n = 2h_n^*. \qquad (1.2.3)$$

Then

$$\Phi = \{h_i^* - h_j^*\} \cup \{h_i^* + h_j^*\} \cup \{\pm 2h_i^*\} \ (i, j = 1, \ldots, n \text{ , } i \neq j)$$

and writing t_i for $e^{h_i^*}$, we have that

$$e^\Phi \doteq \{e^\alpha, \alpha \in \Phi\} = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 2}\}.$$

The system of n independent equations

$$\begin{cases} t_1^2 = 1 \\ \ldots \\ t_n^2 = 1 \end{cases}$$

has 2^n solutions: $(\pm 1, \ldots, \pm 1)$, and it is easy to see that all other systems does not have other solutions. The Weyl group $W \simeq \mathfrak{S}_n \ltimes (\mathfrak{C}_2)^n$

acts on $T = (\mathbb{C}^*)^n$ by permuting and inverting its coordinates; the second operation is trivial on $\mathcal{C}_0(\Phi)$. Thus two elements of $\mathcal{C}_0(\Phi)$ are in the same $W-$orbit if and only if they have the same number of negative coordinates. Then we can define the $p-$th $W-$orbit $\mathcal{O}_p$ as the set of points with p negative coordinates. (This choice is not canonical: we may choose the set of points with p positive coordinates as well). Clearly if $t \in \mathcal{O}_p$ then

$$W(t) \simeq (\mathfrak{S}_p \times \mathfrak{S}_{n-p}) \ltimes (\mathfrak{C}_2)^n.$$

Thus $|\mathcal{O}_p| = \binom{n}{p}$ and we get:

$$|\mathcal{C}_0(\Phi)| = \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$

Notice that if $t \in \mathcal{O}_p$ then $-t \in \mathcal{O}_{n-p}$, and $Ad(t) = Ad(-t)$ since $Z(\Phi) = \{\pm(1, \dots, 1)\}$. In fact Γ has a symmetry exchanging the vertices p and $n-p$. Finally notice that $\mathcal{C}_0(\Phi)$ is a subgroup of T isomorphic to $(\mathfrak{C}_2)^n$ and generated by the elements

$$\delta_i \doteq (1, \dots, 1, -1, 1, \dots, 1) \text{ (with the } -1 \text{ at the } i-th \text{ place).}$$

Then we can come back to the original coordinates observing that δ_i is the nontrivial solution of the system ${t_i}^2 = 1$, $t_j = 1 \forall j \neq i$, and using (1.2.3) to get:

$$\delta_i \leftrightarrow \left[\sum_{k=i}^{n} \alpha_k^\vee / 2 \right].$$

2. **Case** D_n We can write $\alpha_n = h_{n-1}^* + h_n^*$ and the others α_i as before; then

$$e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\}.$$

Then each system of n independent equations is $W-$conjugate to one of this form:

$$\begin{cases} t_1 = t_2 \\ \dots \\ t_{p-1} = t_p \\ t_{p-1} = t_p^{-1} \\ t_{p+1}^{\pm 1} = t_{p+2} \\ \dots \\ t_{n-1} = t_n \\ t_{n-1} = t_n^{-1} \end{cases}$$

for some $p \neq 1, n-1$. Then we get the subset of $(\mathfrak{C}_2)^n$ composed by the following $n-$ples:

$$\{(\pm 1, \dots, \pm 1)\} \setminus \{\pm \delta_i,\ i = 1, \dots, n\}$$

which are in number of $2^n - 2n$. However reasoning as before we see that each one represents two points in $\mathfrak{h}/\langle \Phi^\vee \rangle$. Namely, the correspondence is given by:

$$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_{n-1}^\vee - \alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i.$$

From a geometric point of view, the t_is are coordinates of a maximal torus of the orthogonal group, while $T = \mathfrak{h}/\langle \Phi^\vee \rangle$ is a maximal torus of its two-sheets universal covering. Each $W-$orbit corresponding to the four extremal vertices of Γ is a singleton consisting of one of the four points over $\pm(1, \dots, 1)$, all inducing the identity automorphism: indeed $Aut(\Gamma)$ acts transitively on these points. The other orbits are defined as in the case C_n.

3. **Case** B_n This case is very similar to the previous one, but now $\alpha_n = h_n^*$,

$$e^\Phi = \{t_i t_j^{-1}\} \cup \{t_i t_j\} \cup \{t_i^{\pm 1}\}$$

and then we get the points

$$\{(\pm 1, \dots, \pm 1)\} \backslash \{\delta_i\}_{i=1,\dots,n}.$$

In this case the projection is

$$\left\{ \left[\sum_{k=i}^{n-1} \frac{\alpha_k^\vee}{2} \pm \frac{\alpha_n^\vee}{4} \right] \right\} \longrightarrow \delta_i$$

then we have $2^n - n$ pairs of points in $\mathcal{C}_0(\Phi)$.

4. **Case** A_n If we see $\mathfrak{h}^*$ as the subspace of $\langle h_1^*, \dots, h_{n+1}^* \rangle$ of equation $\sum h_i^* = 0$, and T as the subgroup of $(\mathbb{C}^*)^{n+1}$ of equation $\prod t_i = 1$, we can write all the simple roots as $\alpha_i = h_i^* - h_{i+1}^*$; then $e^\Phi = \{t_i t_j^{-1}\}$. In this case Φ has no proper subsystem of its same rank, then all the coordinates must be equal. Therefore

$$\mathcal{C}_0(\Phi) = Z(\Phi) = \left\{ (\zeta, \dots, \zeta) | \zeta^{n+1} = 1 \right\} \simeq \mathfrak{C}_{n+1}.$$

Then $W \simeq \mathfrak{S}_{n+1}$ acts on $\mathcal{C}_0(\Phi)$ trivially and $Z(\Phi)$ transitively, as expected since $Aut(\Gamma) \simeq \mathfrak{D}_{n+1}$ acts transitively on the vertices of Γ. We can write more explicitly $\mathcal{C}_0(\Phi) \subseteq \mathfrak{h}/\langle \Phi^\vee \rangle$ as

$$\mathcal{C}_0(\Phi) = \left\{ \left[\frac{k}{n+1} \sum_{i=1}^{n} i\alpha_i^\vee \right], k = 0, \dots, n \right\}.$$

1.2.3 Proofs

Motivated by Remark 1.2.1, we start to describe the automorphisms of $\mathfrak{g}$ that are induced by the points of $\mathcal{C}_0(\Phi)$.

Lemma 1.2.2. *If $t \in \mathcal{C}_0(\Phi)$, then $Ad(t)$ has finite order.*

Proof. Let $\beta_1, \dots, \beta_n$ linearly independent roots such that $e^{\beta_i}(t) = 1$: then for each root $\alpha \in \Phi$ we have that $m\alpha = \sum c_i \beta_i$ for some m and $c_i \in \mathbb{Z}$, and thus

$$e^{\alpha}(t^m) = e^{m\alpha}(t) = \prod_{i=1}^{n} (e^{\beta_i})^{c_i}(t) = 1.$$

Then $Ad(t^m)$ is the identity on $\mathfrak{g}$, hence by (1.2.1) $t^m \in Z(\Phi)$. $Z(\Phi)$ is a finite group, thus t^m and t have finite order. □

The previous lemma allows us to apply the following

Theorem 1.2.3 (Kač)**.**

1. *Each inner automorphism of $\mathfrak{g}$ of finite order m is conjugate to an automorphism σ of the form*

$$\sigma(X_i) = \zeta^{s_i} X_i$$

with ζ fixed primitive $m-$th root of unity and $(s_0, \dots, s_n)$ nonnegative integers without common factors such that $m = \sum s_i a_i$.

2. *Two such automorphisms are conjugate if and only if there is an automorphism of Γ sending the parameters $(s_0, \dots, s_n)$ of the first in the parameters $(s'_0, \dots, s'_n)$ of the second.*

3. *Let $(i_1, \dots, i_r)$ be all the indices for which $s_{i_1} = \dots = s_{i_r} = 0$. Then $\mathfrak{g}^{\sigma}$ is the direct sum of an $(n-r)-$dimensional center and of a semisimple Lie algebra whose Dynkin diagram is the subdiagram of Γ of vertices $i_1, \dots, i_r$.*

This is a special case of a theorem proved in [25] and more extensively in [21, X.5.15 and 16]. We only need the following

Corollary 1.2.4.

1. *Let σ be an inner automorphism of $\mathfrak{g}$ of finite order m such that $\mathfrak{g}^\sigma$ is semisimple. Then there is $p \in V(\Gamma)$ such that σ is conjugate to σ_p. In particular $m = a_p$ and the Dynkin diagram of $\mathfrak{g}^\sigma$ is Γ_p.*

2. *Two automorphisms σ_p, $\sigma_{p'}$ are conjugate if and only if p, p' are in the same $Aut(\Gamma)-$orbit.*

Proof. If $\mathfrak{g}^\sigma$ is semisimple, then in the third part of Theorem 1.2.3 $n = r$, hence all parameters of σ but one are equal to 0, and the nonzero parameter s_p must be equal to 1, otherwise there would be a common factor, contradicting the first part of the Theorem. Thus we get the first statement. Then the second statement follows from Theorem 1.2.3(ii). $\square$

Let be $t \in \mathcal{C}_0(\Phi)$: by Remark 1.2.1 $\mathfrak{g}^{Ad(t)}$ is semisimple, hence by Corollary 1.2.4(i) $Ad(t)$ is conjugate to some σ_p. Then there is a canonical map

$$\begin{aligned} &\psi : \mathcal{C}_0(\Phi) \to Q \\ &\qquad t \mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \sigma_p \text{ is conjugate to } Ad(t)\}. \end{aligned}$$

Notice that $\psi(t)$ is a well-defined element of Q by Corollary 1.2.4(ii).

We now prove the fundamental

Lemma 1.2.5. *Two points in $\mathcal{C}_0(\Phi)$ induce conjugate automorphisms if and only if they are in the same $W \times Z(\Phi)-$orbit.*

Proof. We recall that $W \simeq N/T$ and the action of W on T is induced by the conjugation action of N; it is also well known that two points of T are $G-$conjugate if and only if they are $W-$conjugate. Then $W-$conjugate points induce conjugate automorphisms. Moreover by (1.2.1)

$$Ad(t) = Ad(s) \Leftrightarrow Ad(ts^{-1}) = id_{\mathfrak{g}} \Leftrightarrow ts^{-1} \in Z(\Phi).$$

Finally suppose that $t, t' \in \mathcal{C}_0(\Phi)$ induce conjugate automorphisms, i.e.

$$\exists g \in G | Ad(t') = Ad(g)Ad(t)Ad(g^{-1}) = Ad(gtg^{-1}).$$

Then $zt' = gtg^{-1}$ for some $z \in Z(\Phi)$. Thus zt' and t are $G-$conjugate elements of T, and hence they are $W-$conjugate, proving the claim.

$\square$

We can now prove the first part of Theorem 1.2.1. Indeed by the previous lemma there is a canonical injective map defined on the set of the orbits of $\mathcal{C}_0(\Phi)$:

$$\overline{\psi} : \frac{\mathcal{C}_0(\Phi)}{W \times Z(\Phi)} \longrightarrow Q.$$

We must show that this map is surjective. The system

$$\alpha_i(h) = 1 \, (\forall i \neq 0, p) \, , \, \alpha_p(h) = a_p^{-1}$$

is composed of n linearly independent equations, then it has a solution $h \in \mathfrak{h}$. Notice that $\alpha_0(h) \in \mathbb{Z}$. Let t be the class of h in T; then

$$e^{\alpha}(t) = 1 \Leftrightarrow \alpha \in \Phi_p.$$

Then by Remark 1.2.1 $Ad(t)$ is conjugate to σ_p and $\Phi(t)$ to Φ_p.

In order to relate the action of $Z(\Phi)$ with that of $Aut(\Gamma)$, we introduce the following subset of W. For each $p \neq 0$ such that $a_p = 1$, set $z_p \doteq w_0^p w_0$, where w_0 is the longest element of W and w_0^p is the longest element of the parabolic subgroup of W generated by all the simple reflections $s_{\alpha_1}, \dots, s_{\alpha_n}$ except s_{α_p}. Then we define

$$W_Z \doteq \{1\} \cup \{z_p\}_{p=1,\dots,n|a_p=1}$$

W_Z has the following properties (see [24, 1.7 and 1.8]):

Theorem 1.2.6 (Iwahori-Matsumoto)**.**

1. *W_Z is a subgroup of W isomorphic to $Z(\Phi)$.*

2. *For each $z_p \in W_Z$, we have that $z_p.\alpha_0 = \alpha_p$, and z_p induces an automorphism of Γ that sends the $0-th$ vertex to the $p-th$ one; this defines an injective morphism $W_Z \hookrightarrow Aut(\Gamma)$.*

3. *The W_Z-orbits of $V(\Gamma)$ coincide with the $Aut(\Gamma)-$orbits.*

Therefore Q is the set of W_Z-orbits of $V(\Gamma)$, and the bijection $\overline{\psi}$ between Q and the set of $Z(\Phi)-$orbits of $\mathcal{C}_0(\Phi)/W$ can be lifted to a noncanonical bijection between $V(\Gamma)$ and $\mathcal{C}_0(\Phi)/W$. Then we just have to consider the action of W on $\mathcal{C}_0(\Phi)$ and prove the

Lemma 1.2.7. *If $t \in \mathcal{O}_p$, then $W(t)$ is conjugate to W_p.*

Proof. Notice that the centralizer $C_N(t)$ of t in N is the normalizer of $T = C_T(t)$ in $C_G(t)$. Then $W(t) = C_N(t)/T$ is the Weyl group of $C_G(t)$. $C_G(t)$ is the subgroup of G of points fixed by the conjugacy by t, then its Lie algebra is

$\mathfrak{g}^{Ad(t)}$, which is conjugate to $\mathfrak{g}^{\sigma_p}$ by the first part of Theorem 1.2.1. Therefore $W(t)$ is conjugate to W_p.

□

This completes the proof of Theorem 1.2.1 and also of Theorem 1.1.1, since by Remark 1.2.1 the map ψ defined in (1.2.4) can also be seen as the map

$$t \mapsto \psi(t) = \{p \in V(\Gamma) \text{ such that } \Phi_p \text{ is conjugate to } \Phi(t)\}.$$

1.3 Layers of the arrangement

1.3.1 From hyperplane arrangements to toric arrangements

Let S be a $d-$dimensional space of $\mathcal{H}$. The set Φ_S of the elements of Φ vanishing on S is a complete subsystem of Φ of rank $n-d$. Then the map $S \to \Phi_S$ gives a bijection between $\mathcal{L}_d$ and $\mathcal{K}_d$, whose inverse is

$$\Theta \to S(\Theta) \doteq \{h \in \mathfrak{h} | \alpha(h) = 0 \ \forall \alpha \in \Theta\}.$$

In [36, 6.4 and C] (following [34] and [6]) the spaces of $\mathcal{H}$ are classified and counted, and the $W-$orbits of $\mathcal{L}_d$ are completely described. This is done case-by-case according to the type of Φ. We now show a case-free way to extend this analysis to the layers of $\mathcal{T}$.

Given a layer C of $\mathcal{T}$ let us consider

$$\Phi_C \doteq \{\alpha \in \Phi | e^\alpha(t) = 1 \ \forall t \in C\}.$$

In contrast with the case of linear arrangements, Φ_C in general is not complete. For each $\Theta \in \mathcal{K}_d$, define $\mathcal{C}^{\Phi}_{\Theta}$ as the set of layers C such that $\overline{\Phi_C} = \Theta$. This is clearly a partition of the set of $d-$dimensional layers of $\mathcal{T}$, i.e.

$$\mathcal{C}_d(\Phi) = \bigsqcup_{\Theta \in \mathcal{K}_d} \mathcal{C}^{\Phi}_{\Theta} \tag{1.3.1}$$

Given any $C \in \mathcal{C}^{\Phi}_{\Theta}$, we call $S(\Theta)$ the *tangent space* at the layer U. Then by [36] the problem of classifying the layers of $\mathcal{T}$ reduces to classify the layers of $\mathcal{T}$ having a given tangent space, i.e. the elements of $\mathcal{C}^{\Phi}_{\Theta}$. In the next section we show that this amounts to classify the points of a smaller toric arrangement, namely that defined by Θ.

1.3.2 Theorems

Let Θ be a complete subsystem of Φ and W^{Θ} its Weyl group. Let $\mathfrak{k}$ and K be respectively the semisimple Lie algebra and the semisimple and simply connected algebraic group of root system Θ, $\mathfrak{d}$ a Cartan subalgebra of $\mathfrak{k}$, $\langle \Theta^{\vee} \rangle$ and $\Lambda(\Theta)$ the coroot and coweight lattices, $Z(\Theta) \doteq \frac{\Lambda(\Theta)}{\langle \Theta^{\vee} \rangle}$ the center of K, D the maximal torus of K defined by $\mathfrak{d}/\langle \Theta^{\vee} \rangle$, $\mathcal{D}$ the toric arrangement defined by Θ on D and $\mathcal{C}_0(\Theta)$ the set of its points.

We also consider the *adjoint group* $K_a \doteq K/Z(\Theta)$ and its maximal torus $D_a \doteq D/Z(\Theta) \simeq \mathfrak{d}/\Lambda(\Theta)$. We recall from [22] that K is the universal covering of K_a, and if D' is an algebraic torus having Lie algebra $\mathfrak{d}$, then $D' \simeq \mathfrak{d}/L$ for some lattice $\Lambda(\Theta) \supseteq L \supseteq \langle \Theta^{\vee} \rangle$; then there are natural covering projections $D \twoheadrightarrow D' \twoheadrightarrow D_a$ with kernels respectively $L/\langle \Theta^{\vee} \rangle$ and $\Lambda(\Theta)/L$. Notice that Θ naturally defines an arrangement on each torus D', and that for $D' = D_a$

the set of its 0-dimensional layers is $\mathcal{C}_0(\Theta)/Z(\Theta)$. Given a point t of some D' we set

$$\Theta(t) \doteq \{\alpha \in \Theta | e^{\alpha}(t) = 1\}.$$

Theorem 1.3.1. *There is a $W^{\Theta}-$equivariant surjective map*

$$\varphi : \mathcal{C}_{\Theta}^{\Phi} \twoheadrightarrow \mathcal{C}_0(\Theta)/Z(\Theta)$$

such that $\ker \varphi \simeq Z(\Phi) \cap Z(\Theta)$ *and* $\Phi_C = \Theta(\varphi(C))$.

Proof. Let $S(\Theta)$ be the subspace of $\mathfrak{h}$ defined in the previous section, and H the corresponding subtorus of T. T/H is a torus with Lie algebra $\mathfrak{h}/S(\Theta) \simeq \mathfrak{d}$, then Θ defines an arrangement $\mathcal{D}'$ on $D' \doteq T/H$. The projection $\pi : T \twoheadrightarrow T/H$ induces a bijection between $\mathcal{C}_{\Theta}^{\Phi}$ and the set of 0-dimensional layers of $\mathcal{D}'$, because $H \in \mathcal{C}_{\Theta}^{\Phi}$ and for each $C \in \mathcal{C}_{\Theta}^{\Phi}$, $\Phi_C = \Theta(\pi(C))$.

Moreover the restriction of the projection $d\pi : \mathfrak{h} \twoheadrightarrow \mathfrak{h}/S(\Theta)$ to $\langle \Phi^{\vee} \rangle$ is simply the map that restricts the coroots of Φ to Θ. Set $R^{\Phi}(\Theta) \doteq d\pi(\langle \Phi^{\vee} \rangle)$; then $\Lambda(\Theta) \supseteq R^{\Phi}(\Theta) \supseteq \langle \Theta^{\vee} \rangle$ and $D' \simeq \mathfrak{d}/R^{\Phi}(\Theta)$. Denote by p the projection $\Lambda(\Phi) \twoheadrightarrow \frac{\Lambda(\Phi)}{\langle \Phi^{\vee} \rangle}$ and embed $\Lambda(\Theta)$ in $\Lambda(\Phi)$ in the natural way. Then the kernel of the covering projection of $D' \twoheadrightarrow D_a$ is isomorphic to

$$\frac{\Lambda(\Theta)}{R^{\Phi}(\Theta)} \simeq p(\Lambda(\Theta)) \simeq Z(\Phi) \cap Z(\Theta).$$

$\square$

We set

$$n_{\Theta} \doteq \frac{|Z(\Theta)|}{|Z(\Phi) \cap Z(\Theta)|}.$$

The following corollary is straightforward from Theorem 1.3.1.

Corollary 1.3.2.
$$|\mathcal{C}_\Theta^\Phi| = n_\Theta^{-1}|\mathcal{C}_0(\Theta)|$$
and then by (1.3.1),
$$|\mathcal{C}_d(\Phi)| = \sum_{\Theta\in\mathcal{K}_d} n_\Theta^{-1}|\mathcal{C}_0(\Theta)|.$$

Notice that two layers C, C' of $\mathcal{T}$ are $W-$conjugate if and only if the two conditions below are satisfied:

1. their tangent spaces are $W-$conjugate , i.e. $\exists w \in W$ such that $\overline{\Phi_C} = w.\overline{\Phi_{C'}}$;

2. C and $w.C'$ are $W^{\overline{\Phi_C}}-$conjugate.

Then the action of W on $\mathcal{C}(\Phi)$ is described by the following remark.

Remark 1.3.1.

1. By Theorem 1.3.1, φ induces a surjective map $\overline{\varphi}$ from the set of the $W^\Theta-$orbits of $\mathcal{C}_\Theta^\Phi$ to the set of the $W^\Theta \times Z(\Theta)-$orbits of $\mathcal{C}_0(\Theta)$, that are described by Theorem 1.2.1.

2. In particular if Θ is irreducible, set Γ^Θ its affine Dynkin diagram, Q^Θ the set of the $Aut(\Gamma)-$orbits of its vertices, Γ_p^Θ the diagram that we obtain from Γ^Θ removing the vertex p, and Θ_p the associated root system. Then there is a surjective map
$$\widehat{\varphi} : \mathcal{C}_\Theta^\Phi \twoheadrightarrow Q^\Theta$$
such that, if $\widehat{\varphi}(C) = q$ and p is a representative of q, then $\Phi_C \simeq \Theta_p$.

1.3.3 Examples

Case F_4. $Z(\Phi) = \{1\}$, thus $n_\Theta = |Z(\Theta)|$. Therefore in this case n_Θ does not depend on the conjugacy class, but only on the isomorphism class of Θ.

We say that a space S of $\mathcal{H}$ (respectively a layer C of $\mathcal{T}$) is of a given type if the corresponding subsystem Φ_S (respectively Φ_C) is of that type. Then by [36, Tab. C.9] and Corollary 1.3.2 there are:

1. one space of type "A_0", tangent to one layer of the same type (the whole spaces);
2. 24 spaces of type A_1, each tangent to one layer of the same type;
3. 72 spaces of type $\mathsf{A}_1 \times \mathsf{A}_1$, each tangent to one layer of the same type;
4. 32 spaces of type A_2, each tangent to one layer of the same type;
5. 18 spaces of type B_2, each tangent to one layer of the same type and one layer of type $\mathsf{A}_1 \times \mathsf{A}_1$;
6. 12 spaces of type C_3, each tangent to one layer of the same type and 3 of type $\mathsf{A}_2 \times \mathsf{A}_1$;
7. 12 spaces of type B_3, each tangent to one layer of the same type, one of type A_3 and 3 of type $\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$;
8. 96 spaces of type $\mathsf{A}_1 \times \mathsf{A}_2$, each tangent to one layer of the same type;
9. one space of type F_4 (the origin), tangent to: one layer of the same type, 12 of type $\mathsf{A}_1 \times \mathsf{C}_3$, 32 of type $\mathsf{A}_2 \times \mathsf{A}_2$, 24 of type $\mathsf{A}_3 \times \mathsf{A}_1$, and 3 of type C_4.

Case A_{n-1}. It is easily seen that each subsystem Θ of Φ is complete and is a product of irreducible factors $\Theta_1, \ldots, \Theta_k$, with Θ_i of type $A_{\lambda_i - 1}$ for some positive integers λ_i such that $\lambda_1 + \cdots + \lambda_k = n$ and $n - k$ is the rank of Θ. In other words, as is well known, the $W-$conjugacy classes of spaces of $\mathcal{H}$ are in bijection with the partitions λ of n, and if a space has dimension d then corresponding partition has length $|\lambda| \doteq k$ equal to $d+1$. The number of spaces of partition λ is easily seen to be equal to $n!/b_\lambda$, where b_i is the number of λ_j that are equal to i and $b_\lambda \doteq \prod i!^{b_i} b_i!$ (see [36, 6.72]). Now let g_λ be the greatest common divisor of $\lambda_1, \ldots, \lambda_k$. By Example 4 in Section 1.2.2 we have that

$$|Z(\Theta)| = \lambda_1 \ldots \lambda_k = |\mathcal{C}_0(\Theta)|$$

and $|Z(\Phi) \cap Z(\Theta)| = g_\lambda$. Then by Corollary 1.3.2 $|\mathcal{C}_\Theta^\Phi| = g_\lambda$ and

$$|\mathcal{C}_d(\Phi)| = \sum_{|\lambda|=d+1} \frac{n! g_\lambda}{b_\lambda}.$$

This could also be seen directly as follows. We can view T as the subgroup of $(\mathbb{C}^*)^n$ given by the equation $t_1 \ldots t_n - 1 = 0$. Then Θ imposes the equations

$$\begin{cases} t_1 = \cdots = t_{\lambda_1} \\ \ldots \\ t_{\lambda_1 + \cdots + \lambda_{k-1} + 1} = \cdots = t_n. \end{cases}$$

Thus we have the relation

$$x_1^{\lambda_1} \ldots x_k^{\lambda_k} - 1 = 0.$$

If $g_\lambda = 1$ this polynomial is irreducible, because the vector $(\lambda_1, \ldots, \lambda_k)$ can be completed to a basis of the lattice $\mathbb{Z}^k$. If $g_\lambda > 1$ this polynomial has exactly g_λ irreducible factors over $\mathbb{C}$. Then in every case it defines an affine variety having g_λ irreducible components, which are precisely the elements of $\mathcal{C}^{\Phi}_{\Theta}$.

1.4 Topology of the complement

1.4.1 Theorems

Let $\mathcal{R}_\Phi$ be the complement of the toric arrangement:

$$\mathcal{R}_\Phi \doteq T \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha.$$

In this section we prove that the Euler characteristic of $\mathcal{R}_\Phi$, denoted by E_Φ, is equal to $(-1)^n |W|$. This may also be seen as a consequence of [5, Prop. 5.3]. Furthermore, we give a formula for the Poincaré polynomial of $\mathcal{R}_\Phi$, denoted by $P_\Phi(q)$.

Let $d_1, \ldots, d_n$ be the *degrees* of W, i.e. the degrees of the generators of the ring of $W-$invariant regular functions on $\mathfrak{h}$; it is well known that $d_1 \ldots d_n = |W|$. The numbers $d_1 - 1, \ldots, d_n - 1$ are known as the *exponents* of W; we denote by $\mathcal{P}(\Phi)$ their product:

$$\mathcal{P}(\Phi) \doteq (d_1 - 1) \ldots (d_n - 1).$$

Then we have:

Theorem 1.4.1.

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} \mathcal{P}(\Phi_C)(q+1)^{d(C)} q^{n-d(C)}$$

where $d(C)$ is the dimension of the layer C.

Proof. Let $nbc(\Phi)$ be the number of *no-broken circuit bases* of Φ (whose definition is recalled in Section 2.3.4). By [35], $nbc(\Phi)$ equals the leading coefficient of the Poincaré polynomial of the complement of $\mathcal{H}$ in $\mathfrak{h}$; moreover by [3] this coefficient is equal to $\mathcal{P}(\Phi)$ (these facts can be found also in [14, 10.1]).

Then the claim is a restatement of a known result. Indeed the cohomology of $\mathcal{R}_\Phi$ can be expressed as a direct sum of contributions given by the layers of $\mathcal{T}$ (see for example [12, Theor. 4.2] or [14, 14.1.5]). In terms of Poincaré polynomial this expression is:

$$P_\Phi(q) = \sum_{C \in \mathcal{C}(\Phi)} nbc(\Phi_C)(q+1)^{d(C)} q^{n-d(C)}.$$

$\square$

Now we use the theorem above to compute the Euler characteristic of $\mathcal{R}_\Phi$.

Lemma 1.4.2.

$$E_\Phi = (-1)^n \sum_{p=0}^{n} \frac{|W|}{|W_p|} \mathcal{P}(\Phi_p)$$

Proof. We have

$$E_\Phi = P_\Phi(-1) = (-1)^n \sum_{t \in \mathcal{C}_0(\Phi)} \mathcal{P}(\Phi(t)) \tag{1.4.1}$$

because the contributions of all positive-dimensional layers vanish at -1. Obviously isomorphic subsystems have the same degrees, thus Theorem 1.1.1 yields the statement. $\square$

Theorem 1.4.3.

$$E_\Phi = (-1)^n |W|$$

Proof. By the previous lemma we must prove that

$$\sum_{p=0}^{n} \frac{\mathcal{P}(\Phi_p)}{|W_p|} = 1$$

If we write $d_1^p, \dots, d_n^p$ for the degrees of W_p, the previous identity becomes

$$\sum_{p=0}^{n} \frac{(d_1^p - 1) \dots (d_n^p - 1)}{d_1^p \dots d_n^p} = 1.$$

This identity has been proved in [13], and later with different methods in [18]. $\square$

Notice that W acts on $\mathcal{R}_\Phi$ and then on its cohomology. Then we can consider the *equivariant Euler characteristic* of $\mathcal{R}_\Phi$, that is, for each $w \in W$,

$$\widetilde{E}_\Phi(w) \doteq \sum_{i=0}^{n} (-1)^i \, Tr(w, H^i(\mathcal{R}_\Phi, \mathbb{C})).$$

Let ϱ_W be the character of the regular representation of W. From Theorem 1.4.3 we get the following

Corollary 1.4.4.

$$\widetilde{E}_\Phi = (-1)^n \varrho_W$$

Proof. Since W is finite and acts freely on $\mathcal{R}_\Phi$, it is well known that $\widetilde{E}_\Phi = k\varrho_W$ for some $k \in \mathbb{Z}$. Then to compute k we just have to look at $\widetilde{E}_\Phi(1_W) = E_\Phi$. $\square$

Finally we give a formula for $P_\Phi(q)$ which, together with the mentioned results in [36], allows its explicit computation.

Theorem 1.4.5.

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} n_\Theta^{-1} |W^\Theta|$$

Proof. By formula (1.3.1) we can restate Theorem 1.4.1 as

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{\Theta \in \mathcal{K}_d} \sum_{C \in \mathcal{C}_\Theta^\Phi} \mathcal{P}(\Phi_C)$$

Moreover by Theorem 1.3.1 and Corollary 1.3.2 we get

$$\sum_{C \in \mathcal{C}_\Theta^\Phi} \mathcal{P}(\Phi_C) = n_\Theta^{-1} \sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)).$$

Finally the claim follows by formula (1.4.1) and Theorem 1.4.3 applied to Θ:

$$\sum_{t \in \mathcal{C}_0(\Theta)} \mathcal{P}(\Theta(t)) = (-1)^d \chi_\Theta = |W^\Theta|.$$

$\square$

1.4.2 Examples

Case F_4. In Section 1.3.3 we have given a list of all possible types of complete subsystems, together with their multiplicities. Then we just have to compute the coefficient $n_\Theta^{-1}|W^\Theta|$ for each type. This is equal to:

- 1 for types 1., 2. and 3.
- 2 for types 4. and 8.
- 4 for type 5.
- 24 for types 6. and 7.
- 1152 for type 9.

Thus

$$P_\Phi(q) = 2153q^4 + 1260q^3 + 286q^2 + 28q + 1.$$

Case A_{n-1}. By Section 1.3.3, $n_\Theta^{-1} = \frac{g_\lambda}{\lambda_1 \ldots \lambda_k}$ and $|W^\Theta| = \lambda_1! \ldots \lambda_k!$. Hence by Theorem 1.4.5

$$P_\Phi(q) = \sum_{d=0}^{n} (q+1)^d q^{n-d} \sum_{|\lambda|=d+1} n! b_\lambda^{-1} g_\lambda (\lambda_1 - 1)! \ldots (\lambda_k - 1)!.$$

Chapter 2

A generalized Tutte polynomial

In this chapter we introduce a multiplicity Tutte polynomial $M(x, y)$, which generalizes the ordinary one and has applications to zonotopes, multigraphs and toric arragements. We prove that $M(x, y)$ satisfies a deletion-restriction formula and has positive coefficients. The characteristic polynomial and the Poincaré polynomial of a toric arrangement are shown to be specializations of the associated polynomial $M(x, y)$, as the corresponding polynomials of a hyperplane arrangement are specializations of the ordinary Tutte polynomial. Furthermore $M(1, y)$ computes the graded dimension of the related Dahmen-Micchelli space.

2.1 Introduction

The Tutte polynomial is an invariant naturally associated to a matroid and encoding many of its features, such as the number of the bases and their *internal and external activity* ([41], [8], [14]). If the matroid is defined by a finite list of vectors, it is natural to consider the arrangement obtained by taking the hyperplane orthogonal to each vector. To the poset of the intersections of the hyperplanes one associates its *characteristic polynomial*, which provides a rich combinatorial and topological description of the arrangement ([35], [42]). This polynomial can be obtained as a specialization of the Tutte polynomial.

Given a torus $T = (\mathbb{C}^*)^n$ and a finite list X of characters, i.e. elements of $Hom(T, \mathbb{C}^*)$, we consider the arrangement of hypersurfaces in T obtained by taking the kernel of each element of X. To understand the geometry of this *toric arrangement* one needs to describe the poset $\mathcal{C}(X)$ of the *layers*, i.e. of the connected components of the intersections of the hypersurfaces ([12], [19]). Clearly this poset depends also on the arithmetics of X, and not only on its linear algebra: for instance, the kernel of the identity character λ of $\mathbb{C}^*$ is the point $t = 1$, but the kernel of 2λ has equation $t^2 = 1$, hence is made of two points. Therefore we have no chance to get the characteristic polynomial of $\mathcal{C}(X)$ as a specialization of the ordinary Tutte polynomial $T(x, y)$ of X. In this chapter we define a polynomial $M(x, y)$ that specializes to the characteristic polynomial of $\mathcal{C}(X)$ (Theorem 2.4.6) and to the Poincaré polynomial of the complement $\mathcal{R}_X$ of the toric arrangement (Theorem 2.4.9). In particular $M(1, 0)$ equals the Euler characteristic of $\mathcal{R}_X$, and also the number of connected components of the complement of the arrangement in the *compact torus* $\overline{T} = (\mathbb{S}^1)^n$.

We call $M(x, y)$ the *multiplicity Tutte polynomial* of X, since it satisfies a recursive formula similar to the *deletion-restriction* one that holds for $T(x, y)$. By this recurrence (Theorem 2.3.6) we prove that $M(x, y)$ has positive coefficients (Theorem 2.3.7).

Actually a similar polynomial can be defined more generally for matroids, if we enrich their structure in order to encode some "arithmetic data"; we call such objects *multiplicity matroids*. For instance, we show that every graph with labeled edges defines a multiplicity matroid and hence a multiplicity Tutte polynomial. However in this case the coefficients fail to be positive; then we focus on the case of a list X of vectors in a lattice.

Given such a list, we consider two finite dimensional vector spaces: a space of polynomials $D(X)$, defined by differential equations, and a space of quasipolynomials $DM(X)$, defined by difference equations. These spaces were introduced by Dahmen and Micchelli to study respectively *box splines* and *partition functions*, and are deeply related respectively with the hyperplane arrangement and the toric arrangement defined by X, as explained in the forthcoming book [14]. In particular, $T(1, y)$ is known to be the graded dimension of $D(X)$; then we prove that $M(1, y)$ is the graded dimension of $DM(X)$ (Theorem 2.5.3).

On the other hand, the coefficients of $M(x, 1)$ count integral points in some faces of a convex polytope, the *zonotope* defined by X, which by [14] plays a central role in the picture above (see Theorem 2.2.3). In particular $M(1, 1)$ equals the volume of the zonotope (see Proposition 2.2.1).

2.2 Definitions and examples

2.2.1 Definitions

We start recalling the notions we are going to generalize.

A *matroid* $\mathfrak{M}$ is a pair (X, I), where X is a finite set and I is a family of subsets of X (called the *independent sets*) with the following properties:

1. The empty set is independent;

2. Every subset of an independent set is independent;

3. Let A and B be two independent sets and assume that A has more elements than B. Then there exists an element $a \in A \setminus B$ such that $B \cup \{a\}$ is still independent.

A maximal independent set is called a *basis*. The last axiom implies that all bases have the same cardinality, which is called the *rank* of the matroid. Every $A \subseteq X$ has a natural structure of matroid, defined by considering a subset of A independent if and only if it is in I. Then each $A \subseteq X$ has a rank which we denote by $r(A)$.

The *Tutte polynomial* of the matroid is then defined as

$$T(x, y) \doteq \sum_{A \subseteq X} (x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

From the definition it is clear that $T(1, 1)$ equals the number of bases of the matroid.

In the next sections we will recall the two most important examples of matroid and some properties of their Tutte polynomials.

We now introduce the following definitions.

A *multiplicity matroid* $\mathfrak{M}$ is a triple (X, I, m), where (X, I) is a matroid and m is a function (called *multiplicity*) from the family of all subsets of X to the positive integers.

We say that m is the *trivial multiplicity* if it is identically equal to 1.

We define the *multiplicity Tutte polynomial* of a multiplicity matroid as

$$M(x,y) \doteq \sum_{A\subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

Let us remark that we can endow every matroid with the trivial multiplicity, and then $M(x,y) = T(x,y)$.

Remark 2.2.1. Given any two matroids $\mathfrak{M}_1 = (X_1, I_1)$ and $\mathfrak{M}_2 = (X_2, I_2)$, it is naturally defined a matroid $\mathfrak{M}_1 \oplus \mathfrak{M}_2 = (X, I)$: X is the disjoint union of X_1 and X_2, and $A \in I$ if and only if $A_1 \doteq A \cap X_1 \in I_1$ and $A_2 \doteq A \cap X_2 \in I_2$. Moreover if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have multiplicity functions m_1 and m_2, $m(A) \doteq m_1(A_1) \cdot m_2(A_2)$ defines a multiplicity on $\mathfrak{M}_1 \oplus \mathfrak{M}_2$. We notice that the rank of a subset A is just the sum of the ranks of A_1 and A_2, and so it is easily seen that the (multiplicity) Tutte polynomial of $\mathfrak{M}_1 \oplus \mathfrak{M}_2$ is the product of the (multiplicity) Tutte polynomials of $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

2.2.2 Lists of vectors and zonotopes

Let X be a finite list of vectors spanning a real vector space U, and I be the family of its linearly independent subsets; then (X, I) is a matroid, and the rank of a subset A is just the dimension of the spanned subspace. We denote by $T_X(x,y)$ the associated Tutte polynomial.

We associate to the list X a *zonotope*, that is a convex polytope in U defined as follows:

$$\mathcal{Z}(X) \doteq \left\{ \sum_{x\in X} t_x x, 0 \leq t_x \leq 1 \right\}.$$

Zonotopes play an important role in the theory of hyperplane arrangements, and also in that of *splines*, a class of functions studied in Approximation Theory. (see [14]).

We recall that a *lattice* Λ of rank n is a discrete subgroup of $\mathbb{R}^n$ which spans the real vector space $\mathbb{R}^n$. Every such Λ can be generated from some basis of the vector space by forming all linear combinations with integral coefficients; hence the group Λ is isomorphic to $\mathbb{Z}^n$. We will use the word *lattice* always with this meaning, and not in the combinatorial sense (poset with *join* and *meet*).

Then let X be a finite list of elements in a lattice Λ, and let I and r be as above. We denote by $\langle A \rangle_{\mathbb{Z}}$ and $\langle A \rangle_{\mathbb{R}}$ respectively the sublattice of Λ and the subspace of $\Lambda \otimes \mathbb{R}$ spanned by A. Let us define

$$\Lambda_A \doteq \Lambda \cap \langle A \rangle_{\mathbb{R}} :$$

this is the largest sublattice of Λ in which $\langle A \rangle_{\mathbb{Z}}$ has finite index. Then we define m as this index:

$$m(A) \doteq [\Lambda_A : \langle A \rangle_{\mathbb{Z}}] .$$

This defines a multiplicity matroid and then a multiplicity Tutte polynomial $M_X(x, y)$, which is the main subject of this chapter. We start by showing the relations with the zonotope $\mathcal{Z}(X)$ generated by X in

$$U \doteq \Lambda \otimes \mathbb{R}.$$

We already observed that $T_X(1, 1)$ equals the number of bases that can be extracted from X; on the other hand we have:

Proposition 2.2.1. $M_X(1, 1)$ *equals the volume of the zonotope* $\mathcal{Z}(X)$.

Proof. By [38], $\mathcal{Z}(X)$ is paved by a family of polytopes $\{\Pi_B\}$, where B varies among all the bases extracted from X, and

$$vol(\Pi_B) = |det(B)|.$$

On the other hand, when B is a basis,

$$m(B) = [\Lambda : \langle B \rangle_{\mathbb{Z}}] = |det(B)|. \tag{2.2.1}$$

Since

$$M_X(1,1) = \sum_{B \subset X, B basis} m(B)$$

the claim follows. $\square$

Now, let us assume X to be a basis for U. In this case $M_X(x,y)$ is a polynomial in which only the variable x appears, whose coefficients have a remarkable combinatorial interpretation.

We say that a point of U is *integral* if it is contained in Λ. For every $A \subset X$ the zonotope $\mathcal{Z}(A)$ is a face of $\mathcal{Z}(X)$; we say that a point of $\mathcal{Z}(A)$ is *internal* to such face if it is not contained in any smaller face of $\mathcal{Z}(X)$. We denote by $h(A)$ the number of integral points that are internal to $\mathcal{Z}(A)$.

Lemma 2.2.2. *For every $A \subset X$,*

$$h(A) = \sum_{B \subseteq A} (-1)^{|A|-|B|} m(B).$$

Proof. For every $\varepsilon > 0$, let $\underline{\varepsilon}$ be the point in $\Lambda \otimes \mathbb{R}$ of coordinates

$$\underline{\varepsilon} = \sum_{\lambda \in X} \varepsilon \lambda.$$

Let $\mathcal{Z}(X)-\underline{\varepsilon}$ be the polytope obtained translating $\mathcal{Z}(X)$ by $-\underline{\varepsilon}$. It is intuitive (and proved in [14, Prop 2.50]) that when ε is small enough

$$vol\left(\mathcal{Z}(X)\right) = \left|\left(\mathcal{Z}(X) - \underline{\varepsilon}\right) \cap \Lambda\right|.$$

(More in general, this is true when $\underline{\varepsilon}$ is any point outside the *cut-locus* of $\mathcal{Z}(X)$; the interested reader can refer to [14]).

Notice that by construction $\mathcal{Z}(X) - \underline{\varepsilon}$ contains all the integral points which are internal to the faces $\mathcal{Z}(A)$, $A \subseteq X$, and none of those which are on the opposite faces; hence

$$\left|\left(\mathcal{Z}(X) - \underline{\varepsilon}\right) \cap \Lambda\right| = \sum_{A \subseteq X} h(A).$$

Moreover by Formula (2.2.1) $m(X)$ equals the volume of $\mathcal{Z}(X)$. Thus we proved:

$$m(X) = \sum_{A \subseteq X} h(A).$$

Then we get the claim by inclusion-exclusion principle, since the intersection of two faces $\mathcal{Z}(A_1)$, $\mathcal{Z}(A_2)$ is the face $\mathcal{Z}(A_1 \cap A_2)$. $\square$

We can now prove that the coefficient of x^k equals the number of integral points of $\mathcal{Z}(X) - \underline{\varepsilon}$ that are internal to some $k-$codimensional face:

Theorem 2.2.3. *Let X be a basis for U. Then*

$$M_X(x, y) = \left(\sum_{A \subseteq X,\, |A| = n-k} h(A) \right) x^k.$$

Proof. By definition

$$M_X(x, y) = \sum_{A \subseteq X} m(A)(x - 1)^{n-|A|}.$$

The coefficient of x^k in this expression is

$$\sum_{A \subseteq X,\, |A| \leq n-k} (-1)^{n-k-|A|} \binom{n-|A|}{k} m(A).$$

By the previous Lemma, or claim amounts to prove that the coefficient of x^k is

$$\sum_{A \subseteq X,\, |A|=n-k} \sum_{B \subseteq A} (-1)^{|A|-|B|} m(B) = \sum_{B \subseteq X,\, |B| \leq n-k} (-1)^{n-k-|B|} \binom{n-|B|}{k} m(B)$$

because every $B \subseteq X$ is contained in exactly

$$\binom{n-|B|}{n-k-|B|} = \binom{n-|B|}{k}$$

sets $A \subseteq X$ of cardinality $n-k$. Then we get the claim. □

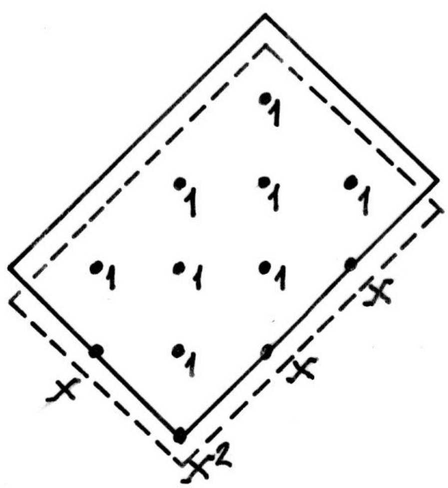
Example 2.2.1. Consider the list in $\mathbb{Z}^2$

$$X = \{(3,3), (-2,2)\}.$$

Then

$$M_X(x,y) = (x-1)^2 + 5(x-1) + 12 = x^2 + 3x + 8.$$

Indeed the picture of the zonotope $\mathcal{Z}(X)$ with its integral points is:



2.2.3 Graphs

Let G be a finite graph and X be the set of its edges. We view each $A \subseteq X$ as a subgraph of G, having the same set of vertices $V(G)$ of G and A as set of edges. We define I as the set of the *forests* in G (i.e, subgraphs whose connected components are simply connected). Then (X, I) is a matroid with rank function

$$r(A) = |V(G)| - c(A)$$

where $c(A)$ is the number of connected components of A.

Remark 2.2.2. If G has no loops nor multiple edges, let us take a vector space $\widetilde{U}$ with basis $e_1, \ldots, e_n$ in bijection with $V(G)$, and associate to the edge connecting two vertices i and j the vector $e_i - e_j$. In this way we get a list X_G of vectors in bijection with X and spanning a hyperplane U in $\widetilde{U}$. Since in this correspondence the rank is preserved and forests correspond to linearly independent sets, G and X_G define the same matroid and have the same Tutte polynomial.

Now let us assume every edge $e \in X$ to have an integer label $m_e > 0$. Then by defining

$$m(A) \doteq \prod_{e \in A} m_e$$

we get a multiplicity matroid and then a multiplicity Tutte polynomial $M_G(x, y)$.

We may view the labels m_e as multiplicities of the edges in the following way. Let us define a new graph G_m with the same vertices of G, but with m_e edges between the two vertices incident to $e \in X$. Then let $S(G_m)$ be the set of *simple* subgraphs of G_m, i.e subgraphs with at most one edge connecting any two vertices, and at most one loop on every vertex. It is then clear that

$$M_G(x, y) \doteq \sum_{A \in S(G_m)} (x-1)^{r(X)-r(A)} (y-1)^{|A|-r(A)}.$$

In particular, $M_G(2, 1)$ equals the number of forests of G_m, and $M_G(1, 1)$ the number of *spanning trees* (i.e., trees connecting all the vertices) of G_m.

2.3 Deletion-restriction formula and positivity

The central idea that inspired Tutte in defining the polynomial $T(x, y)$, was to find the most general invariant satisfying a recurrence known as *deletion-restriction* (or *deletion-contraction*). Such recurrence allows to reduce the computation of the Tutte polynomial to some trivial cases. We will explain this algorithm in the two examples above, i.e. when the matroid is defined by a list of vectors or by a graph. Then we will show that in both cases also the polynomial $M(x, y)$ satisfies a similar recursion.

2.3.1 Graphs

Let G be a finite graph, and $e \in X$ be an edge that is not a loop; then we define two new graphs. G_1 is obtained from G by removing the edge e; G_2 is obtained from G by removing the edge e and identifying the two vertices that were connected by e (hence, if there are other edges between these two vertices, they become loops). Then we have the following

Theorem 2.3.1.

$$T_G(x, y) = T_{G_1}(x, y) + T_{G_2}(x, y)$$

if e is contained in some cycle;

$$T_G(x,y) = xT_{G_2}(x,y)$$

otherwise.

We generalize this theorem as follows. If G is a labeled a graph and $e \in X$ is an edge that is not a loop, we define two labeled graphs as follows. G_1 is obtained from G by replacing by $m_e - 1$ the label m_e of e (or by removing the edge e, if $m_e - 1 = 0$). G_2 is obtained from G by removing the edge e and identifying the two vertices that were connected by e. Let e be en edge contained in some cycle; then we have:

Theorem 2.3.2.

$$M_G(x,y) = M_{G_1}(x,y) + M_{G_2}(x,y)$$

Proof. We denote by $m_1(A)$ the multiplicity of A in G_1 and by $m_2(\overline{A})$ the multiplicity of the image $\overline{A}$ of A in G_2. We distinguish two cases.

If $m_e = 1$, we divide the sum expressing $M_G(x,y)$ into two parts, the first over the sets A not containing e:

$$\sum_{A \subseteq G_1} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} = M_{X_1}(x,y)$$

since clearly $r(X) = r(X_1)$ and $m(A) = m_1(A)$. The second part is over the sets A containing e:

$$|\overline{A}| = |A| - 1,\ r(\overline{A}) = r(A) - 1,\ r(G_2) = r(G) - 1,\ m_2(\overline{A}) = m(A).$$

Therefore

$$\sum_{A \subseteq G, e \in A} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} =$$

$$\sum_{\overline{A}\subseteq G_2} m_2(\overline{A})(x-1)^{r(X_2)-r(\overline{A})}(y-1)^{|\overline{A}|-r(\overline{A})} = M_{G_2}(x,y).$$

If on the other hand $m_e > 1$, for every $A \subset X$ such that $e \notin A$, we set $A_e \doteq A \cup \{e\}$. Then

$$m(A) = m_1(A) \text{ and } m(A_e) = m_1(A_e) + m_2(\overline{A_e})$$

and

$$|\overline{A_e}| = |A_e| - 1,\ r(\overline{A_e}) = r(A_e) - 1,\ r(G_2) = r(G) - 1.$$

Hence

$$M_G(x,y) = \sum_{A\subseteq G, e\notin A} \big(m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} +$$
$$+ m(A_e)(x-1)^{r(X)-r(A_e)}(y-1)^{|A_e|-r(A_e)}\big) =$$
$$= \sum_{A\subseteq G_1} m_1(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} +$$
$$+ \sum_{\overline{A_e}\subseteq G_2} m_2(\overline{A_e})(x-1)^{r(G_2)-r(\overline{A_e})}(y-1)^{|\overline{A_e}|-r(\overline{A_e})} = M_{G_1}(x,y) + M_{G_2}(x,y).$$

□

If on the other hand e is not contained in any cycle, we observe that in the notations above, for every $A \subseteq G, e \notin A$

$$r(A_e) = r(A) + 1 \text{ and } m(A_e) = m_e m(A).$$

Thus it is easily seen that

$$M_G(x,y) = (x - 1 + m_e) M_{G_2}(x,y).$$

By applying recursively this formula and the theorem above we can reduce to the case of graphs with only loops; then by Remark 2.2.1 we can assume them to have only one vertex. If G_0 is such a graph, let h be the number of its loops and $m_1, \ldots, m_h$ be their labels. Then clearly

$$M_{G_0}(x, y) = \prod_{i=1}^{h} \big(m_i(y-1)+1\big).$$

Remark 2.3.1. We can identify an ordinary graph with the labeled graph

$$m_e = 1 \; \forall e \in X.$$

Then the formulae above reduce to Theorem 2.3.1, and $T_{G_0}(x, y) = y^h$.

In this way we see that the coefficients of $T_G(x, y)$ are always positive, while the coefficients of $M_G(x, y)$ are not.

2.3.2 Lists of vectors

Let X be a finite list of elements spanning a vector space U, and let $v \in X$ be a nonzero element. We define two new lists: the list $X_1 \doteq X \setminus \{v\}$ of elements of U and the list X_2 of elements of $U/\langle v\rangle$ obtained by reducing X_1 modulo v. Assume that v is dependent in X, i.e. $v \in \langle X_1\rangle_{\mathbb{R}}$. Then we have the following well-known formula:

Theorem 2.3.3.

$$T_X(x, y) = T_{X_1}(x, y) + T_{X_2}(x, y)$$

It is now clear why we defined X as a list, and not as a set: even if we start with X made of (nonzero) distinct elements, in X_2 some vector may appear many times (and some vector may be zero).

Notice that by applying recursively the above formula, our problem reduces to compute $T_Y(x, y)$ when Y is the union of a list Y_1 of k linearly independent vectors and of a list Y_0 of h zero vectors ($k, h \geq 0$). In this case the Tutte polynomial is easily computed:

Lemma 2.3.4.

$$T_Y(x, y) = x^k y^h.$$

Proof. Given any $v \in Y_1$, since

$$\langle Y \rangle_{\mathbb{R}} = \langle Y \setminus \{v\} \rangle_{\mathbb{R}} \oplus \langle \{v\} \rangle_{\mathbb{R}}$$

by Remark 2.2.1 we have that

$$T_Y(x, y) = x\, T_{Y \setminus \{v\}}(x, y).$$

Hence by induction we get that $T_Y = x^k\, T_{Y_0}$. Finally

$$T_{Y_0}(x, y) = \sum_{j=0}^{h} \binom{h}{j} (y-1)^j = \big((y-1)+1\big)^h = y^h.$$

□

Thus we get:

Theorem 2.3.5. *$T_X(x, y)$ is a polynomial with positive coefficients.*

2.3.3 Lists of elements in finitely generated abelian groups.

We now want to show a similar recursion for the polynomial $M_X(x, y)$. Inspired by [16], we notice that in order to do this, we need to work in a larger category. Indeed, whereas the quotient of a vector space by a subspace is still a vector space, the quotient of a lattice by a sublattice is not a lattice, but a *finitely generated abelian group*. for instance in the 1-dimensional case, the quotient of $\mathbb{Z}$ by $m\mathbb{Z}$ is the cyclic group of order m.

Then let Γ be a finitely generated abelian group. For every subset S of Γ we denote by $\langle S\rangle$ the generated subgroup. We recall that Γ is isomorphic to the direct product of a lattice Λ and of a finite group Γ_t, which is called the *torsion subgroup* of Γ. We denote by π the projection $\pi : \Gamma \to \Lambda$.

Let X be a finite subset of Γ; for every $A \subseteq X$ we set

$$\Lambda_A \doteq \Lambda \cap \big\langle \pi(A) \big\rangle_{\mathbb{R}}$$

and

$$\Gamma_A \doteq \Lambda_A \times \Gamma_t.$$

In other words, Γ_A is the largest subgroup of Γ in which $\langle A\rangle$ has finite index.

Then we define

$$m(A) \doteq \big[\Gamma_A : \langle A\rangle\big].$$

We also define $r(A)$ as the rank of $\pi(A)$. In this way we defined a multiplicity matroid, to which is associated a multiplicity Tutte polynomial:

$$M_X(x,y) \doteq \sum_{A\subseteq X} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)}.$$

It is clear that if Γ is a lattice, these definitions coincide with the ones given in the previous sections.

If on the opposite hand Γ is a finite group, $M(x,y)$ is a polynomial in which only the variable y appears; furthermore this polynomial, evaluated at $y=1$, gives the order of Γ. Indeed the only summand that does not vanish is the contribution of the empty set, which generates the trivial subgroup.

Now let $\lambda \in X$ be a nonzero element such that

$$\pi(\lambda) \in \big\langle \pi\big(X \setminus \{\lambda\}\big)\big\rangle_{\mathbb{R}} \tag{2.3.1}$$

We set

$$X_1 \doteq X \setminus \{\lambda\} \subset \Gamma$$

and we denote by $\overline{A}$ the image of every $A \subseteq X$ under the natural projection

$$\Gamma \longrightarrow \Gamma/\langle\lambda\rangle.$$

Since $\Gamma/\langle\lambda\rangle$ is a finitely generated abelian group and $\overline{A}$ is a subset of it, $m(\overline{A})$ is defined. Notice that

$$m(\overline{A}) \doteq \big[(\Gamma/\langle\lambda\rangle)_{\overline{A}} : \langle\overline{A}\rangle\big] = \big[\Gamma_A/\langle\lambda\rangle : \langle A\rangle/\langle\lambda\rangle\big] = \big[\Gamma_A : \langle A\rangle\big] = m(A).$$

We denote by X_2 the subset $\overline{X_1}$ of $\Gamma/\langle\lambda\rangle$. Then we have the following deletion-restriction formula.

Theorem 2.3.6.

$$M_X(x,y) = M_{X_1}(x,y) + M_{X_2}(x,y).$$

Proof. The sum expressing $M_X(x,y)$ splits into two parts, the first over the sets $A \subseteq X_1$:

$$\sum_{A \subseteq X_1} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} = M_{X_1}(x,y)$$

since clearly $r(X) = r(X_1)$. The second part is over the sets A such that $\lambda \in A$. For such sets we have that:

$$|\overline{A}| = |A| - 1,\ r(\overline{A}) = r(A) - 1,\ r(X_2) = r(X) - 1,\ m(\overline{A}) = m(A).$$

Therefore

$$\sum_{A\subseteq X,\lambda\in A} m(A)(x-1)^{r(X)-r(A)}(y-1)^{|A|-r(A)} =$$
$$\sum_{\overline{A}\subseteq X_2} m(\overline{A})(x-1)^{r(X_2)-r(\overline{A})}(y-1)^{|\overline{A}|-r(\overline{A})} = M_{X_2}(x,y).$$

□

Then we prove:

Theorem 2.3.7. *$M_X(x,y)$ is a polynomial with positive coefficients.*

Proof. By applying recursively the formula above, we can reduce to the case of lists that do not contain any λ satisfying condition (2.3.1). Any such list Y is made of elements of some quotient $\Gamma(Y)$ of Γ, and is the disjoint union of a list Y_0 of h zeros ($h \geq 0$), and of a list Y_1 such that $\pi(Y_1)$ is a basis of $\Lambda(Y)\otimes\mathbb{R}$. (Here we denoted by π the projection $\Gamma(Y)\to\Lambda(Y)$, where $\Gamma(Y)\simeq\Lambda(Y)\times\Gamma(Y)_t$ is the product of the lattice and of the torsion subgroup). Then we first notice that

$$M_{Y_0} = |\Gamma(Y)_t|\sum_{j=0}^{h}\binom{h}{j}(y-1)^j = |\Gamma(Y)_t|\big((y-1)+1\big)^h = |\Gamma(Y)_t|y^h.$$

Furthermore it is easily seen that

$$M_Y(x,y) = M_{Y_0}(x,y)M_{Y_1}(x,y).$$

Finally the positivity of $M_{Y_1}(x,y)$ follows from Theorem 2.2.3. □

2.3.4 Statistics

Usually, polynomials with positive coefficients encode some statistics: in other words, their coefficients *count* something.

For instance, the Tutte polynomial embodies two statistics on the set of the bases, called internal and external activity. Although they can be stated for an abstract matroid (see for example [14, Section 2.2.2]), we give such definitions for a list X of vectors. Let B be a basis extracted from X.

1. We say that $v \in X \setminus B$ is *externally active* for B if v is a linear combination of the elements of B following it (in the total ordering fixed on X);

2. we say that $v \in B$ is *internally active* for B if there is no element w in X preceeding v such that $\{w\} \cup (B \setminus \{v\})$ is a basis.

3. the number $e(B)$ of externally active elements is called the *external activity* of B;

4. the number $i(B)$ of internally active elements is called the *internal activity* of B;

Then in [8] is proved the following result:

Theorem 2.3.8.

$$T_X(x, y) = \sum_{B \subseteq X,\, B basis} x^{i(B)} y^{e(b)}.$$

Hence the coefficients of $T_X(x, y)$ count the number of the basis having a given internal and external activity.

Since also $M_X(x, y)$ has positive coefficients, it is natural to wonder which are the statistics involved. When X is an integer basis of the lattice, we have Theorem 2.2.3; in the general case, we leave this question open:

Problem 2.3.1. *Give a combinatorial interpretation of the coefficients of* $M_X(x, y)$.

We say that a basis B of X is a *no-broken circuit* if $e(B) = 0$. We denote by $nbc(X)$ the number of no-broken circuit bases of X. It is clear from Theorem 2.3.8 that

$$nbc(X) = T_X(1, 0). \tag{2.3.2}$$

We will use this formula in the next section.

2.4 Application to arrangements

In this Section we describe some geometrical objects related to the lists considered in Section 2.2.2, and show that many of their features are encoded in the polynomials $T_X(x, y)$ and $M_X(x, y)$.

2.4.1 Recall on hyperplane arrangements

Let X be a finite list of elements of a vector space U. Then in the dual space $V = U^*$ a *hyperplane arrangement* $\mathcal{H}(X)$ is defined by taking the orthogonal hyperplane of each element of X. Conversely, given an arrangement of hyperplanes in a vector space V, let us choose for each hyperplane a nonzero vector in V^* orthogonal to it; let X be the list of such vectors. Since every element of X is determined up to scalar multiples, the matroid associated to X is well defined; in this way a Tutte polynomial is naturally associated to the hyperplane arrangement.

The importance of the Tutte polynomial in the theory of hyperplane arrangements is well known. Here we just recall some results that we generalize in the next sections.

To every sublist $A \subseteq X$ is associated the subspace $A^\perp$ of V that is the intersection of the corresponding hyperplanes of $\mathcal{H}(X)$; in other words, $A^\perp$ is the subspace of vectors that are orthogonal to every element of A. Let $\mathcal{L}(X)$ be the set of such subspaces, partially ordered by reverse inclusion, and having as minimal element $\mathbf{0}$ the whole space $V = \emptyset^\perp$. $\mathcal{L}(X)$ is called the *intersection poset* of the arrangement, and is "the most important combinatorial object associated to a hyperplane arrangement" (R. Stanley).

We also recall that to every finite poset $\mathcal{P}$ is associated a *Moebius function*

$$\mu : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}$$

recursively defined as follows:

$$\mu(L, M) = \begin{cases} 0 & \text{if } L > M \\ 1 & \text{if } L = M \\ -\sum_{L \leq N < M} \mu(L, N) & \text{if } L < M. \end{cases}$$

Notice that the poset $\mathcal{L}(X)$ is *ranked* by the dimension of the subspaces; then we define *characteristic polynomial* of the poset as

$$\chi(q) \doteq \sum_{L \in \mathcal{L}(X)} \mu(\mathbf{0}, L) q^{dim(L)}.$$

This is an important invariant of $\mathcal{H}(X)$. Indeed, let $\mathcal{M}_X$ be the complement in V of the union of the hyperplanes of $\mathcal{H}(X)$. Let $P(q)$ be *Poincaré polynomial* of $\mathcal{M}_X$, i.e. the polynomial having as coefficient of q^k the $k-$th Betti number of $\mathcal{M}_X$. Then if V is a complex vector space, by [35] we have the following theorem.

Theorem 2.4.1.

$$P(q) = (-q)^n \chi(-1/q).$$

If on the other hand V is a real vector space, by [42] the number $Ch(X)$ of *chambers* (i.e., connected components of $\mathcal{M}_X$) is:

Theorem 2.4.2.

$$Ch(X) = (-1)^n \chi(-1).$$

The Tutte polynomial $T_X(x, y)$ turns out to be a stronger invariant, in the following sense. Assume that $\underline{0} \notin X$; then

Theorem 2.4.3.

$$(-1)^n T_X(1 - q, 0) = \chi(q).$$

The proof of these theorems can be found for instance in [14, Theorems 10.5, 2.34 and 2.33].

2.4.2 Toric arrangements and their generalizations

Let $\Gamma = \Lambda \times \Gamma_t$ be a finitely generated abelian group, and define

$$T_\Gamma \doteq Hom(\Gamma, \mathbb{C}^*).$$

T_Γ has a natural structure of abelian linear algebraic group: indeed it is the direct product of a complex torus T_Λ of the same rank as Λ and of the finite group ${\Gamma_t}^*$ dual to Γ_t (and isomorphic to it).

Moreover Γ is identified with the group of characters of T_Γ: indeed given $\lambda \in \Lambda$ and $t \in T_\Gamma$ we can take any representative $\varphi_t \in Hom(\Gamma, \mathbb{C})$ of t and set

$$\lambda(t) \doteq e^{2\pi i \varphi_t(\lambda)}.$$

When this is not ambiguous we will denote T_Γ by T.

Let $X \subset \Lambda$ be a finite subset spanning a sublattice of Λ of finite index. The kernel of every character $\chi \in X$ is a (non-connected) hypersurface in T:

$$H_\chi \doteq \{t \in T | \chi(t) = 1\}.$$

The collection $\mathcal{T}(X) = \{H_\chi, \chi \in X\}$ is called the *generalized toric arrangement* defined by X on T.

We denote by $\mathcal{R}_X$ the complement of the arrangement:

$$\mathcal{R}_X \doteq T \setminus \bigcup_{\chi \in X} H_\chi$$

and by $\mathcal{C}_X$ the set of all the connected components of all the intersections of the hypersurfaces H_χ, ordered by reverse inclusion and having as minimal elements the connected components of T.

Since $rank(\Lambda) = dim(T)$, the maximal elements of $\mathcal{C}(X)$ are 0-dimensional, hence (since they are connected) they are points. We denote by $\mathcal{C}_0(X)$ the set of such layers, which we call the *points* of the arrangement.

Given $A \subseteq X$ let us define

$$H_A \doteq \bigcap_{\lambda \in A} H_\lambda.$$

Lemma 2.4.4. *$m(A)$ equals the number of connected components of H_A.*

Proof. It is clear by definition that $m(X) = |\mathcal{C}_0(X)|$. Then for every $A \subseteq X$, we have that

$$|\mathcal{C}(A)^0| = m(A)$$

where $\mathcal{C}(A)^0$ is the set of the points of the arrangement $\mathcal{T}(A)$ defined by A in T_{Γ_A}. Now let ${H_A}^0$ be the connected component of H_A that contains the identity. This is a subtorus of T_Γ, and the quotient map

$$T_\Gamma \twoheadrightarrow T_\Gamma / {H_A}^0 \simeq T_{\Gamma_A}$$

induces a bijection between the connected components of H_A and the points of $\mathcal{T}(A)$. $\square$

In particular, when Γ is a lattice, T is a torus and $\mathcal{T}(X)$ is called the *toric arrangement* defined by X. Such arrangements have been studied for example in [29], [12]; see [14] for a complete reference. In particular, the complement $\mathcal{R}_X$ has been described topologically and geometrically. In this description the poset $\mathcal{C}(X)$ plays a major role, for many aspects analogous to that of the intersection poset for hyperplane arrangements.

We will now explain the importance in this framework of the polynomial $M_X(x, y)$ defined in Section 2.3.3.

2.4.3 Characteristic polynomial

Let μ be the Moebius function of $\mathcal{C}(X)$; notice that we have a natural rank function given by the dimension of the layers. For every $C \in \mathcal{C}(X)$, let T_C be the connected component of T that contains C. Then we define the characteristic polynomial of $\mathcal{C}(X)$:

$$\chi(q) \doteq \sum_{C \in \mathcal{C}(X)} \mu(T_C, C) q^{dim(C)}.$$

In order to relate this polynomial with $M_X(x, y)$, we prove the following fact. Let us assume that X does not contain $\underline{0}$. For every $C \in \mathcal{C}(X)$, set

$$\mathcal{D}(C) \doteq \{A \subseteq X \mid C \text{ is a connected component of } H_A\}.$$

Lemma 2.4.5.

$$\mu(T_C, C) = \sum_{A\in\mathcal{D}(C)} (-1)^{|A|}.$$

Proof. By induction on the codimension of C. If it is 0 or 1, the statement is trivial; otherwise, by the inductive hypothesis

$$\mu(T_C, C) = -\sum_{D\supsetneq C} \mu(T_C, D) = -\sum_{D\supsetneq C} \sum_{A\in\mathcal{D}(D)} (-1)^{|A|}.$$

Proving that this sum is equal to the claimed one amounts to prove that

$$\sum_{D\supseteq C} \sum_{A\in\mathcal{D}(D)} (-1)^{|A|} = 0.$$

Now, let B be the largest (hence minimum with respect to reverse inclusion) element of $\mathcal{D}(C)$. Every $A \in \mathcal{D}(D)$ for $D \supseteq C$ is a subset of B, and conversely every $A \subseteq B$ is in $\mathcal{D}(D)$ for exactly one $D \supseteq C$ (if there were two such layers D, their union would be connected). Therefore

$$\sum_{D\supseteq C} \sum_{A\in\mathcal{D}(D)} (-1)^{|A|} = \sum_{A\subseteq B} (-1)^{|A|} = 0$$

where the last equality is an elementary combinatorial fact, which is checked by looking at the binomial coefficients of $(1-1)^k$. □

Theorem 2.4.6.

$$(-1)^n M_X(1-q, 0) = \chi(q)$$

Proof. By definition we must prove that

$$(-1)^n \sum_{A\subseteq X} m(A)(-q)^{n-r(A)}(-1)^{|A|-r(A)} = \sum_{C\in\mathcal{C}(X)} \mu(T_C, C)q^{dimC}.$$

We remark that

$$dim(C) = n - r(A)\ \forall A \in \mathcal{D}(C)$$

and

$$(n - r(A)) + (|A| - r(A)) + n \equiv |A| \,(mod\, 2).$$

Thus we have to prove that for every $k = 0, \ldots, n$,

$$\sum_{A \subseteq X, n-r(A)=k} m(A)(-1)^{|A|} = \sum_{C \in \mathcal{C}(X), dim(C)=k} \mu(T_C, C). \tag{2.4.1}$$

By Lemma 2.4.4, each A is in $\mathcal{D}(C)$ for exactly $m(A)$ layers C. Then Formula (2.4.1) is a consequence of Lemma 2.4.4, since $(-1)^{|A|}$ appears $m(A)$ times in the sum. □

Example 2.4.1. Take $T = (\mathbb{C}^*)^2$ with coordinates (t, s) and

$$X = \{(2, 0), (0, 2), (1, 1), (1, -1)\}$$

defining equations:

$$t^2 = 1, s^2 = 1, ts = 1, ts^{-1} = 1.$$

The hypersurfaces H_{t^2} and H_{s^2} have two connected components each; H_{ts} and $H_{ts^{-1}}$ are connected (but their intersection is not). The $0-$dimensional layers are

$$C_1 = (1, 1),\ C_2 = (-1, -1),\ C_3 = (1, -1),\ C_4 = (-1, 1).$$

Notice that C_1 and C_2 are contained in 4 layers of dimension 1 each, while each of C_3 and C_4 lies in 2 layers of dimension 1. Then $\mu(T, C) = -1$ for each of the six $1-$dimensional layers C, and

$$\mu(T, C_1) = \mu(T, C_2) = -(1 - 4) = 3$$

$$\mu(T, C_3) = \mu(T, C_4) = -(1 - 2) = 1.$$

Hence

$$\chi(q) = q^2 - 6q + 8.$$

The polynomial $M_X(x, y)$ is composed by the following summands:

- $(x-1)^2$, corresponding to the empty set;
- $6(x-1)$, corresponding to the 4 singletons, each giving contribution $(x-1)$ or $2(x-1)$;
- 14, corresponding to the 6 pairs: indeed, the basis $X = \{(2,0),(0,2)\}$ spans a sublattice of index 4, while the other bases span sublattices of index 2;
- $8(y-1)$, corresponding to the 4 triples, each contributing with $2(y-1)$;
- $2(y-1)^2$, corresponding to the whole set X.

Hence

$$M_X(x, y) = x^2 + 2y^2 + 4x + 4y + 3.$$

Notice that

$$M_X(1-q, 0) = q^2 - 6q + 8 = \chi(q)$$

as claimed in Theorem 2.4.6.

2.4.4 Poincaré polynomial

For every $C \in \mathcal{C}_X$, let us define

$$X_C \doteq \{\chi \in X | H_\chi \supseteq C\}.$$

Remark 2.4.1. The set X_C defines a hyperplane arrangement in the vector space $V_C \doteq V/{X_C}^{\perp}$; let $\mathcal{L}(X_C)$ be its intersection poset. Let $\mathcal{C}(X, C)$ be the poset of the elements of $\mathcal{C}(X)$ that contain C. The map

$$\psi : \mathcal{C}(X, C) \to \mathcal{L}(X_C)$$

$$D \mapsto {X_D}^{\perp}$$

is an order-preserving bijection. Indeed, given $L \in \mathcal{L}(X_C)$, set

$$A(L) \doteq \left\{\lambda \in X, \lambda|_L = 0\right\}.$$

Then $\psi^{-1}(L)$ is the connected component containing C of $H_{A(L)}$.

Lemma 2.4.7.

$$nbc(X_C) = (-1)^{n-dim(C)}\mu(T_C, C).$$

Proof. By the previous remark,

$$\mu(T_C, C) \doteq \mu_{\mathcal{C}(X)}(T_C, C) = \mu_{\mathcal{C}(X,C)}(T_C, C) = \mu_{\mathcal{L}(X_C)}(V_C, {X_C}^{\perp}) = \chi_{\mathcal{L}(X_C)}(0)$$

since ${X_C}^{\perp}$ is the origin in V_C, and hence the only element of rank 0. Thus by Theorem 2.4.3 and Formula (2.3.2),

$$\chi_{\mathcal{L}(X_C)}(0) = (-1)^{n-dim(C)}T_{X_C}(1, 0) = (-1)^{n-dim(C)}nbc(X_C).$$

$\square$

Let $T_1, \ldots, T_h$ be the connected components of T. We denote by $\mathcal{C}(X)_i$ the set of layers that are contained in T_i. This clearly gives a partition of the layers:

$$\mathcal{C}(X) = \bigsqcup_{i=1}^{h} \mathcal{C}(X)_i.$$

We now give some formulae for the Poincaré polynomial $P(q)$ and the Euler characteristic of $\mathcal{R}_X$. We start from a restatement of a result proved in [12, Theor. 4.2] (see also [14, 14.1.5]). In this paper is considered an arrangement of hypersurfaces in a torus, in which every hypersurface is obtained by translating by an element of the torus the kernel of a character. It is clear that the restriction of the arrangement $\mathcal{T}(X)$ on every T_i is an arrangement of this kind. Then the cohomology of $\mathcal{R}_X \cap T_i$ can be expressed as a direct sum of contributions given by the layers of this arrangement, which are the elements of $\mathcal{C}(X)_i$. In terms of the Poincaré polynomial $P_i(q)$ of $\mathcal{R}_X \cap T_i$, this expression is:

$$P_i(q) = \sum_{C \in \mathcal{C}(X)_i} nbc(X_C)(q+1)^{dim(C)} q^{n-dim(C)}.$$

Thus the Poincaré polynomial of

$$\mathcal{R}_X = \bigsqcup_i (\mathcal{R}_X \cap T_i)$$

is just the sum of these polynomials:

Theorem 2.4.8.

$$P(q) = \sum_{C \in \mathcal{C}(X)} nbc(X_C)(q+1)^{dim(C)} q^{n-dim(C)}.$$

Then we prove:

Theorem 2.4.9.

$$P(q) = q^n M_X\left(\frac{2q+1}{q}, 0\right).$$

Proof. By definition, we have that

$$q^n M_X\left(\frac{2q+1}{q}, 0\right) = \sum_{A \subseteq X} m(A)(q+1)^{n-r(A)} q^{r(A)} (-1)^{|A|-r(A)}.$$

We compare this formula with the one in the previous Theorem. We have to prove that for every $k = 0, \dots, n$ the coefficient of $(q+1)^k q^{n-k}$ is the same in the two expressions. In fact by applying Formula (2.4.1) and then Lemma 2.4.7 we get the claim:

$$(-1)^{n-k} \sum_{A \subseteq X, r(A)=n-k} m(A)(-1)^{|A|} = (-1)^{n-k} \sum_{C \in \mathcal{C}(X), dim(C)=k} \mu(T_C, C) =$$

$$= \sum_{C \in \mathcal{C}(X), dim(C)=k} nbc(X_C).$$

□

Therefore, by comparing Theorem 2.4.6 and Theorem 2.4.9, we get the following formula, which relates the combinatorics of $\mathcal{C}(X)$ with the topology of $\mathcal{R}_X$, and is the "toric" analogue of Theorem 4.1.

Corollary 2.4.10.

$$P(q) = (-q)^n \chi\left(-\frac{q+1}{q}\right).$$

We recall that the *Euler characteristic* of a space can be defined as the evaluation at -1 of its Poincaré polynomial. Hence by Theorem 2.4.9 we have:

Corollary 2.4.11. $(-1)^n M_X(1,0)$ *equals the Euler characteristic of* $\mathcal{R}_X$.

Example 2.4.2. In the case described in Example 2.4.1, Theorem 2.4.9 (or Corollary 2.4.10) implies that

$$P(q) = 15q^2 + 8q + 1$$

and hence the Euler characteristic is

$$P(-1) = 8 = M_X(1,0).$$

2.4.5 Number of regions of the compact torus

In this section we consider the compact abelian group dual to Γ

$$\overline{T} \doteq Hom(\Gamma, \mathbb{S}^1).$$

We assume for simplicity Γ to be a lattice; then $\overline{T}$ is a *compact torus*, i.e. it is isomorphic to $(\mathbb{S}^1)^n$, where we set

$$\mathbb{S}^1 \doteq \{z \in \mathbb{C} \mid |z| = 1\} \simeq \mathbb{R}/\mathbb{Z}.$$

Then every $\chi \in X$ defines a hypersurface in $\overline{T}$:

$$\overline{H_\chi} \doteq \left\{t \in \overline{T} | \chi(t) = 1\right\}.$$

We denote by $\overline{\mathcal{T}(X)}$ this arrangement; clearly its poset of layers is the same as for the arrangement $\mathcal{T}(X)$ defined in the complex torus T. We denote by $\overline{\mathcal{R}_X}$ the complement

$$\overline{\mathcal{R}_X} \doteq \overline{T} \setminus \bigcup_{\chi \in X} \overline{H_\chi}.$$

The compact toric arrangement $\overline{\mathcal{T}(X)}$ has been studied in [19]; in particular the number $R(X)$ of *region*s (i.e. of connected components) of $\overline{\mathcal{R}_X}$ is proved to be a specialization of the characteristic polynomial $\chi(q)$:

Theorem 2.4.12.

$$R(X) = (-1)^n \chi(0).$$

By comparing this result with Theorem 2.4.6 we get the following

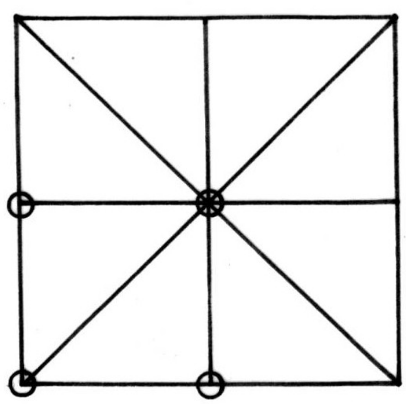
Corollary 2.4.13.

$$R(X) = M_X(1, 0)$$

Example 2.4.3. In the case of Example 2.4.1, we can represent in the real plane with coordinates (x, y) the compact torus $\overline{T}$ as the square $[0,1] \times [0,1]$ with the opposite edges identified. Then the arrangement $\overline{\mathcal{T}(X)}$ is given by the lines

$$x = 0,\ x = 1/2,\ y = 0,\ y = 1/2,\ x = -y,\ x = y.$$

These lines divide the torus in $8 = \chi(0)$ regions:



2.4.6 The case of root systems

We now show a connection with a result proved in Section 1.4. For the convenience of the reader briefly we recall the necessary notations and facts.

Let Φ be a root system, $\langle \Phi^\vee \rangle$ be the lattice spanned by the coroots, and Λ be its dual lattice (which is called the *cocharacters* lattice). Then we define as in Section 2.4.2 a torus $T = T_\Lambda$ having Λ as group of characters. In other words, if $\mathfrak{g}$ is the semisimple complex Lie algebra associated to Φ and $\mathfrak{h}$ is a Cartan subalgebra, T is defined as the quotient $T \doteq \mathfrak{h}/\langle \Phi^\vee \rangle$.

Each root α takes integer values on $\langle \Phi^\vee \rangle$, so it induces a character

$$e^\alpha : T \to \mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*.$$

Let X be the set of this characters; more precisely, since α and $-\alpha$ define the same hypersurface, we set

$$X \doteq \left\{ e^{\alpha},\ \alpha \in \Phi^{+} \right\}.$$

In this way to every root system Φ is associated a toric arrangement. These arrangements are described in Chapter 1; in particular we recall Theorem 1.4.3. Let W be the Weyl group of Φ.

Theorem 2.4.14. *The Euler characteristic of $\mathcal{R}_X$ is equal to $(-1)^n|W|$.*

By comparing this statement with Corollary 2.4.11, we get the following

Corollary 2.4.15.

$$M_X(1,0) = |W|.$$

It would be interesting to have a more direct proof of this fact.

Remark 2.4.2.

1. Let G be the semisimple, simply connected linear algebraic group associated to $\mathfrak{g}$. Then T is the maximal torus of G corresponding to $\mathfrak{h}$, and $\mathcal{R}_X$ is known as the set of *regular points* of T.

2. One may take as Λ the lattice spanned by the roots. But then one obtains as T a maximal torus of the semisimple *adjoint* group G^a, which is the quotient of G by its center.

Example 2.4.4. The toric arrangement described in Example 2.4.1 is that arising from the root system of type C_2. Notice that the order of the Weyl group of type C_2 is

$$8 = P(-1) = M_X(1,0) = R(X).$$

2.5 External activity and Dahmen-Micchelli spaces

Until now we took into account specializations of $T_X(x,y)$ and $M_X(x,y)$ in which the second variable vanishes. However, there is another remarkable specialization of the Tutte polynomial: $T_X(1,y)$, which (by Theorem 2.3.8) is called the *polynomial of the external activity* of X. It is related with the corresponding specialization of $M_X(x,y)$ in a simple way:

Lemma 2.5.1.

$$M_X(1,y) = \sum_{p \in \mathcal{C}_0(X)} T_{X_p}(1,y).$$

Proof. By definition

$$M_X(1,y) = \sum_{A \subseteq X, r(A)=n} m(A)(y-1)^{|A|-n}$$

and

$$T_{X_p}(1,y) = \sum_{A \subseteq X_p, r(A)=n} (y-1)^{|A|-n}.$$

But by Lemma 2.4.4

$$m(A) = |\{p \in \mathcal{C}_0(X) | A \subseteq X_p\}|$$

which is the number of polynomials T_{X_p} in which the summand $(y-1)^{|A|-n}$ appears. □

The previous lemma has an interesting consequence. In [9] to every finite set $X \subset V$ is associated a space $D(X)$ of functions $V \to \mathbb{C}$, and to every finite set $X \subset \Lambda$ is associated a space $DM(X)$ of functions $\Lambda \to \mathbb{C}$. Such spaces are defined as the solutions of a system, respectively of differential equations and of difference equations, in the following way.

For every $v \in V$, let ∂_v be the usual directional derivative

$$\partial_v f(x) \doteq \frac{\partial f}{\partial v}(x)$$

and let ∇_v be the difference operator

$$\nabla_v f(x) \doteq f(x) - f(x - v).$$

Then for every $A \subset X$ we define the differential operator

$$\partial_A \doteq \prod_{v \in A} \partial_v$$

and the difference operator

$$\nabla_A \doteq \prod_{v \in A} \nabla_v.$$

We can now define define the *differentiable Dahmen-Micchelli space*

$$D(X) \doteq \{f : V \to \mathbb{C} \mid \partial_A(f) = 0 \, \forall A \text{ such that } r(X \setminus A) < n\}$$

and the *discrete Dahmen-Micchelli space*

$$DM(X) \doteq \{f : \Lambda \to \mathbb{C} \mid \nabla_A(f) = 0 \, \forall A \text{ such that } r(X \setminus A) < n\}.$$

An explanation of the importance of such spaces would take us too far; the interested reader can find a wide exposition in the book [14]. Let us just mention that the differentiable space $D(X)$ is related with hyperplane arrangements and splines, whereas the discrete space $DM(X)$ is related with toric arrangements and partition functions. Furthermore $DM(X)$ has recently been applied in the index theory of transversally elliptic operators (see [15], [16]).

In order to compare these two spaces, we consider the elements of $D(X)$ as functions $\Lambda \to \mathbb{C}$ by restricting them to the lattice Λ. Since the elements of $DM(X)$ are polynomial functions, they are determined by their restriction. For every $p \in \mathcal{C}_0(X)$, let us define the following map:

$$\varphi_p : \Lambda \to \mathbb{C}$$

$$\lambda \mapsto \lambda(p).$$

(see Section 2.4.2). In [9] (see also [14, Formula 16.1]) is proved the following result.

Theorem 2.5.2.
$$DM(X) = \bigoplus_{p \in \mathcal{C}_0(X)} \varphi_p D(X_p).$$

Since every $D(X_p)$ is a space of polynomials, it is naturally graded; the dimension of the graded parts is known to be given by the coefficients of the polynomial $T_{X_p}(1, y)$ (see [2] or [14, Theorem 11.8]). Then, by the previous theorem, also the space $DM(X)$ is graded, and by Lemma 2.5.1 we have:

Theorem 2.5.3. *$M_X(1, y)$ is the graded dimension of $DM(X)$.*

Chapter 3

Wonderful models

In this chapter we build wonderful models for toric arrangements. We develop the "toric analogue" of the combinatorics of nested sets, which allows to define a family of smooth open sets covering our model. In this way we prove that the model is smooth, and we give a precise geometrical and combinatorial description of the normal crossing divisor.

3.1 Introduction

In this Section we work with a slightly more general notion of toric arrangement, best suited for geometrical applications. This is also the definition used in [12].

Let T be a complex torus and Λ its group of characters.

Let $\widetilde{X}$ be a finite subset of $\Lambda \times \mathbb{C}^*$. For every pair $(\lambda, a) \in \widetilde{X}$ we define the hypersurface of T:

$$H_{\lambda,a} \doteq \{t \in T | \lambda(t) - a = 0\}.$$

The collection

$$\mathcal{T}_{\widetilde{X}} \doteq \left\{ H_{\lambda,a}, (\lambda, a) \in \widetilde{X} \right\}$$

is called the *toric arrangement* defined by $\widetilde{X}$ on T. Such arrangements have been studied for instance in [29], [12]; see [14] for a complete reference.

Let $\mathcal{R}_{\widetilde{X}}$ be the complement of the arrangement:

$$\mathcal{R}_{\widetilde{X}} \doteq T \setminus \bigcup_{(\lambda,a)\in\widetilde{X}} H_{\lambda,a}.$$

In the present chapter we build a smooth minimal model $\mathbf{Z}_{\widetilde{X}}$ containing $\mathcal{R}_{\widetilde{X}}$ as an open set with complement a normal crossing divisor $\mathbf{D}$, and a proper map $\pi : \mathbf{Z}_{\widetilde{X}} \to T$ extending the identity of $\mathcal{R}_{\widetilde{X}}$. We call $\mathbf{Z}_{\widetilde{X}}$ the *wonderful model* of $\mathcal{T}_{\widetilde{X}}$, in analogy with the wonderful model built by De Concini and Procesi [10] for arrangements of subspaces in a vector (or projective) space. We have been greatly inspired by their work, and also by the general construction [30] of MacPherson and Procesi.

We proceed as follows. In Section 3.2 we give the first definitions, we make some basic remarks and we build the wonderful model. In Section 3.3 we develop the necessary combinatorial tools, i.e. the "toric analogues" of the notions of irreducible set, nested set and adapted basis. In Section 3.4 we define some smooth open sets of the model and we prove that they cover $\mathbf{Z}_{\widetilde{X}}$. In Section 3.5 the open sets are used to prove that the complement of $\mathcal{R}_{\widetilde{X}}$ in $\mathbf{Z}_{\widetilde{X}}$ is a normal crossing divisor, and to describe its irreducible components and their intersections (see Theorem 3.5.3).

3.2 First definitions and remarks

3.2.1 Toric arrangements

Let Λ be a lattice and $U = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ the complex vector space obtained by extending the scalars of Λ.

Let $\widetilde{X}$ be a finite set in $\Lambda \times \mathbb{C}^*$, and set

$$X \doteq \{\lambda | (\lambda, a) \in \widetilde{X}\}.$$

Given $A \subseteq X$, we denote by $\langle A\rangle_{\mathbb{Z}}$ and $\langle A\rangle_{\mathbb{R}}$ respectively the sublattice of Λ and the subspace of U spanned by A. We will always assume the sublattice $\langle X\rangle_{\mathbb{Z}}$ to have finite index in Λ; otherwise we can replace Λ with $\Lambda \cap \langle X\rangle_{\mathbb{C}}$.

Then we define

$$T \doteq \frac{Hom(\Lambda, \mathbb{C})}{Hom(\Lambda, \mathbb{Z})}.$$

The group T is isomorphic to $(\mathbb{C}^*)^n$, and its group of characters $Hom(T, \mathbb{C}^*)$ is identified with Λ: indeed given $\lambda \in \Lambda$ and $t \in T$, we can take any representative $\varphi_t \in Hom(\Lambda, \mathbb{C})$ of t and set

$$\lambda(t) \doteq e^{2\pi i \varphi_t(\lambda)}.$$

For every pair $(\lambda, a) \in \widetilde{X}$ we define:

$$H_{\lambda,a} \doteq \{t \in T | \lambda(t) - a = 0\}.$$

We remark that in general the hypersurfaces $H_{\lambda,a}$ are not connected; and even if they are, their intersections are not (see Remark 3.2.1 and Example 3.2.1 below). Then we consider the set $\mathcal{C}(\widetilde{X})$ of all the connected components of all the intersections of the hypersurfaces $H_{\lambda,a}$. This is a poset (with respect to inclusion) which plays a major role in the study of toric arrangements, for many aspects analogous to that of the intersection poset for hyperplane arrangements. We call the elements of $\mathcal{C}(\widetilde{X})$ the *layers* of the arrangement. Under our assumptions, the minimal elements of $\mathcal{C}(\widetilde{X})$ are 0-dimensional, hence they are points. We denote by $\mathcal{C}_0(\widetilde{X})$ the set of such layers, which we call the *points* of the arrangement.

For every layer C we define

$$\widetilde{X}_C \doteq \left\{(\lambda, a) \in \widetilde{X} | H_{\lambda,a} \supseteq C\right\}.$$

and

$$X_C \doteq \{\lambda | (\lambda, a) \in \widetilde{X}_C\}.$$

The natural surjection $\widetilde{X}_C \longrightarrow X_C$ is indeed a bijection, since the condition $(\lambda, a), (\lambda, b) \in X_C$ implies that λ is identically equal to $a = b$ on C.

3.2.2 Primitive vectors

Fixed a system of coordinates $(t_1, \ldots, t_n)$ on T, for every $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ we have a map

$$\begin{aligned} e(\nu) : T &\to \mathbb{C}^* \\ (t_1, \ldots, t_n) &\mapsto {t_1}^{\nu_1} \cdot \ldots \cdot {t_n}^{\nu_n}. \end{aligned}$$

It is well known that e is an isomorphism between $\mathbb{Z}^n$ and $\Lambda = Hom(T, \mathbb{C}^*)$.

We will assume every $\lambda \in X$ to be *primitive*, i.e. such that

$$\Lambda \cap \langle \lambda \rangle_{\mathbb{C}} = \langle \lambda \rangle_{\mathbb{Z}}.$$

This amounts to require that under the previous isomorphism λ is identified with a vector $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$ such that $GCD(\{\nu_i\}) = 1$.

Remark 3.2.1. This is not a restrictive assumption; indeed, suppose $GCD(\{\nu_i\}) = d > 1$, and write $\nu_i' \doteq \nu_i/d$. Then

$$t_1{}^{\nu_1} \cdot \ldots \cdot t_n{}^{\nu_n} - a = \left(t_1{}^{\nu_1'} \cdot \ldots \cdot t_n{}^{\nu_n'}\right)^d - a = \prod_{i=1}^{d} \left(t_1{}^{\nu_1'} \cdot \ldots \cdot t_n{}^{\nu_n'} - \zeta^i \sqrt[d]{a}\right)$$

where ζ is a primitive $d-$th root of 1. Then there is a primitive element λ' of Λ such that $\lambda = d\lambda'$, and we can write $H_{\lambda,a}$ as the union of its connected components:

$$H_{\lambda,a} = \bigsqcup_{i=1}^{d} H_{\lambda',\zeta^i \sqrt[d]{a}}.$$

Then we can replace every pair $(\lambda, a) \in \widetilde{X}$ with all the pairs $(\lambda', \zeta^i a)$. In this way we get a new set $\widetilde{X}'$ which defines the same toric arrangement as $\widetilde{X}$.

Example 3.2.1. Take $T = (\mathbb{C}^*)^2$ with coordinates (t, s) and

$$\widetilde{X} = \{(t^2, 1), (s^2, 1), (ts, 1), (ts^{-1}, 1)\}.$$

Since $t^2 - 1 = (t+1)(t-1)$, the hypersurfaces H_{t^2} and H_{s^2} have two connected components each; H_{ts} and $H_{ts^{-1}}$ are connected, but their intersection is not.

The points of the arrangement are:

$$p_1 = (1, 1),\ p_2 = (-1, -1),\ p_3 = (1, -1),\ p_4 = (-1, 1).$$

Notice that $\widetilde{X}_{p_1} = \widetilde{X}_{p_2} = \widetilde{X}$, whereas

$$\widetilde{X}_{p_3} = \widetilde{X}_{p_4} = \{(t^2, 1), (s^2, 1)\}.$$

Following Remark 3.2.1, we can replace $\widetilde{X}$ by

$$\widetilde{X}' = \{(t, 1), (t, -1), (s, 1), (s, -1), (ts, 1), (ts^{-1}, 1)\}.$$

3.2.3 Construction of the model

Given a sublattice $\Delta \subset \Lambda$, we define its *completion*

$$\overline{\Delta} \doteq \langle \Delta \rangle_{\mathbb{C}} \cap \Lambda.$$

For every layer $C \in \mathcal{C}(\widetilde{X})$, we consider the lattice $\Lambda_C \doteq \langle X_C \rangle_{\mathbb{Z}}$ and its completion $\overline{\Lambda_C}$.

Remark 3.2.2. The elements of $\overline{\Lambda_C}$ are the characters taking a constant value on C. Indeed, for every $\lambda \in \overline{\Lambda_C}$, we have that $d\lambda \in \Lambda_C$ for some $d > 0$. Then by definition $d\lambda$ takes a constant value a on C; hence

$$\lambda(t)^d = a \ \forall \, t \in C.$$

Since C is connected and the set of dth roots of unity is discrete, the continuous map λ must be constant.

Now let $\lambda_1, \ldots, \lambda_k$ be an *integral basis* of $\overline{\Lambda_C}$ (i.e., a basis spanning over $\mathbb{Z}$ the lattice $\overline{\Lambda_C}$), and let a_i be the constant value assumed by λ_i on C: then the ideal $\mathfrak{I}_C$ of the regular functions on T that vanish on C is generated by

$$\{\lambda_1 - a_1, \ldots, \lambda_k - a_k\}$$

and the normal space to C in T is

$$\mathbf{N}_T(C) \simeq \left(\frac{\mathfrak{I}_C}{\mathfrak{I}_C^2} \right)^* .$$

We denote by $\mathbb{P}_C$ its projectified $\mathbb{P}(\mathbf{N}_T(C))$ and by φ_C the natural map

$$\begin{aligned} \varphi_C : T \setminus C &\to \mathbb{P}_C \\ t &\mapsto [\lambda_1(t) - a_1, \ldots, \lambda_k(t) - a_k]. \end{aligned}$$

Now let us fix a subset $\mathcal{G} \subseteq \mathcal{C}(\widetilde{X})$. By collecting the maps $\{\varphi_C,\, C \in \mathcal{G}\}$ and the inclusion $j : \mathcal{R}_{\widetilde{X}} \hookrightarrow T$, we get a map

$$i_{\mathcal{G}} = j \times \prod_{C \in \mathcal{G}} \varphi_C : \mathcal{R}_{\widetilde{X}} \to T \times \prod_{C \in \mathcal{G}} \mathbb{P}_C$$

We define $\mathbf{Z}_{\widetilde{X},\mathcal{G}}$ as the closure $\overline{i_{\mathcal{G}}(\mathcal{R}_{\widetilde{X}})}$ of the image of $\mathcal{R}_{\widetilde{X}}$.

In the next section we will describe the subsets $\mathcal{G}$ that give arise to models with good geometric properties.

Remark 3.2.3.

1. If we choose another basis $\lambda'_1, \ldots, \lambda'_k$, we get other generators

$$\{\lambda'_1 - a'_1, \ldots, \lambda'_k - a'_k\}$$

of the same ideal $\mathfrak{I}_C$, hence another basis of $\mathfrak{I}_C/\mathfrak{I}_C^2$ and then another system of projective coordinates for $\mathbb{P}_C$; then our construction does not depend on such choice.

2. Since $\prod_{C \in \mathcal{G}} \mathbb{P}_C$ is a projective variety, the restriction $\pi : \mathbf{Z}_{\widetilde{X}} \to T$ of the projection on the first factor T is a projective and thus proper map.

3. Since $i_{\mathcal{G}}$ is injective, we identify $\mathcal{R}_{\widetilde{X}}$ with its image $i_{\mathcal{G}}(\mathcal{R}_{\widetilde{X}})$. Such image is closed in $\mathcal{R}_{\widetilde{X}} \times \prod_{C \in \mathcal{G}} \mathbb{P}_C$, which is open in $T \times \prod_{C \in \mathcal{G}} \mathbb{P}_C$; therefore $\mathbf{Z}_{\widetilde{X}}$ contains $\mathcal{R}_{\widetilde{X}}$ as a dense open set, and the restriction of π to $\mathcal{R}_{\widetilde{X}}$ is j.

3.2.4 Hyperplane arrangements and complete sets

Given a finite set $A \subseteq U$, a *hyperplane arrangement* $\mathcal{H}(A)$ is defined in the dual space $V = U^*$ by taking the orthogonal hyperplane to each element of

A. To every subset $B \subseteq A$ is associated the subspace $B^\perp$ of V that is the intersection of the corresponding hyperplanes of $\mathcal{H}(A)$; in other words, $B^\perp$ is the subspace of vectors that are orthogonal to every element of B. Then we set

$$\mathcal{L}(A) = \{B^\perp, B \subseteq A\}.$$

$\mathcal{L}(A)$ is called the *intersection poset* of $\mathcal{H}(A)$, and its element are called the *spaces* of the arrangement.

Given a subset $B \subset A$, we define its completion

$$\overline{B} \doteq \langle B \rangle_{\mathbb{C}} \cap A.$$

We say that B is *complete* in A if $B = \overline{B}$.

For every $Q \in \mathcal{L}(A)$, let $\alpha(Q)$ be the set of elements of A which are identically equal to 0 on Q; clearly

$$\alpha(Q)^\perp = Q \ \text{ and } \ \alpha(B^\perp) = \overline{B}.$$

Hence we have a bijection between $\mathcal{L}(A)$ and the family of complete subsets of A.

Fix $p \in \mathcal{C}_0(\widetilde{X})$. For every pair $(\lambda, a) \in \widetilde{X_p}$, $\lambda - a \in \mathfrak{I}_p$ defines a vector in $\mathfrak{I}_p/\mathfrak{I}_p^2$ and hence a hyperplane in its dual, which is the normal space to the point, i.e. the tangent space $T(p)$ to p in T. This hyperplane of $T(p)$ is simply the tangent space to the hypersurface $H_{(\lambda,a)}$ in p. In this way X_p defines in $T(p)$ a hyperplane arrangement $\mathcal{H}_p$, which is locally isomorphic (in $\underline{0}$) to our toric arrangement (in p). Then the map

$$C \mapsto (X_C)^\perp$$

is a bijection between layers $C \in \mathcal{C}(\widetilde{X})$ containing p and spaces of $\mathcal{H}_p$.

Remark 3.2.4. In particular we see that, for every layer C containing p, $X_C = \alpha\big((X_C)^{\perp}\big)$ is a complete subset of X_p. Conversely, for every complete subset A of X_p there is a unique layer $C(A)$ such that $X_{C(A)} = A$ and $p \in C(A)$. Namely, $C(A)$ is the connected component containing p of the subvariety of T

$$H_A \doteq \{t \in T \mid \lambda(t) - \lambda(p) = 0 \;\forall\, \lambda \in A\}\,.$$

3.3 Combinatorial notions

3.3.1 Irreducible sets

Let A be a finite subset of Λ. Given a complete subset B, an *integral decomposition* of B is a partition $B = \bigcup_i B_i$ such that

$$\overline{\langle B\rangle_{\mathbb{Z}}} = \bigoplus_i \overline{\langle B_i\rangle_{\mathbb{Z}}}.$$

A *complex decomposition* of B is a partition $B = \bigcup_i B_i$ such that

$$\langle B\rangle_{\mathbb{C}} = \bigoplus_i \langle B_i\rangle_{\mathbb{C}}.$$

Notice that the B_i are necessarily complete.

We say that B is $\mathbb{Z}-$*irreducible* (resp. $\mathbb{C}-$*irreducible*) if it does not have a nontrivial integral (resp. complex) decomposition.

We say that a layer $C \in \mathcal{C}(\widetilde{X})$ is $\mathbb{Z}-$*irreducible* (resp. $\mathbb{C}-$*irreducible*) if X_C is. We denote by $\mathcal{I}$ (resp. by $\mathcal{I}_{\mathbb{C}}$) the set of $\mathbb{Z}-$*irreducible* (resp. $\mathbb{C}-$*irreducible*) layers.

Remark 3.3.1. Clearly every integral decomposition is also a complex decomposition, but not conversely: see the example below. Then in general $\mathcal{I}_{\mathbb{C}} \subsetneq \mathcal{I}$.

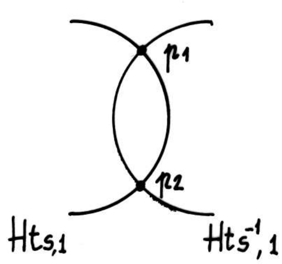
In the language of [30], $\mathcal{C}(\widetilde{X})$ is a *conical stratification* on T, and $\mathcal{I}_{\mathbb{C}}$ is the set of the *irreducible strata*. Then a minimal wonderful model can be obtained by blowing up (in any dimension-increasing order) the elements of $\mathcal{I}_{\mathbb{C}}$. However, in this model the intersections of irreducible components of the normal crossing divisor fail to be connected (see example below). In order to obtain such property (i.e. the last point of Theorem 3.5.3), we will blow up all the elements of $\mathcal{I}$.

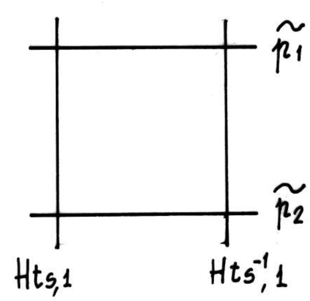
Example 3.3.1. Take $T = (\mathbb{C}^*)^2$ with coordinates (t, s) and

$$\widetilde{X} = \big\{(ts, 1), (ts^{-1}, 1)\big\}.$$

Then X is identified with the subset $\{(1,1),(1,-1)\}$ of $\mathbb{Z}^2$. Thus X is not $\mathbb{C}-$irreducible, but it is $\mathbb{Z}-$irreducible: indeed $\mathbb{Z}(1,1) \oplus \mathbb{Z}(1,-1)$ is a sublattice of index 2 in $\mathbb{Z}^2$.

The hypersurfaces H_{ts} and $H_{ts^{-1}}$ are the irreducible components of a normal crossing divisor; however their intersection consists of two points. By blowing them up we optain a model whose normal crossing divisor has four irreducible components, pairwise intersecting in a single point.





We now prove some properties of integral decompositions, which are known (and easier to prove) for complex decompositions (see for instance [14, Chapter 20.1]).

From now on we will simply call *decompositions* the integral decompositions, and *irreducible* sets (resp. layers) the $\mathbb{Z}-$irreducible sets (resp. layers).

Lemma 3.3.1. *Let $B = B_1 \cup B_2$ be a decomposition and $D \subset B$ be an irreducible subset. Then $D \subseteq B_1$ or $D \subseteq B_2$.*

Proof. Set $D_1 \doteq D \cap B_1$ and $D_2 \doteq D \cap B_2$. We must prove that $D = D_1 \cup D_2$ is a decomposition; then the irreducibility of D implies that D_1 or D_2 is empty. We first notice that

$$\langle D \rangle_{\mathbb{Z}} = \langle D_1 \rangle_{\mathbb{Z}} \oplus \langle D_2 \rangle_{\mathbb{Z}}$$

since

$$\langle D_1 \rangle_{\mathbb{Z}} \cap \langle D_2 \rangle_{\mathbb{Z}} \subseteq \langle B_1 \rangle_{\mathbb{Z}} \cap \langle B_2 \rangle_{\mathbb{Z}} \subseteq \overline{\langle B_1 \rangle_{\mathbb{Z}}} \cap \overline{\langle B_2 \rangle_{\mathbb{Z}}} = \{\underline{0}\}.$$

Then take any $\lambda \in \overline{\langle D \rangle_{\mathbb{Z}}}$. For some positive integer m we have that $m\lambda \in \langle D \rangle_{\mathbb{Z}}$ and then it is written uniquely as $m\lambda = \mu_1 + \mu_2$, with $\mu_1 \in \langle D_1 \rangle_{\mathbb{Z}}$ and $\mu_2 \in \langle D_2 \rangle_{\mathbb{Z}}$. Moreover, since

$$\lambda \in \overline{\langle B \rangle_{\mathbb{Z}}} = \overline{\langle B_1 \rangle_{\mathbb{Z}}} \oplus \overline{\langle B_2 \rangle_{\mathbb{Z}}}$$

λ can be expressed uniquely as $\lambda = \gamma_1 + \gamma_2$, with $\gamma_1 \in \overline{\langle B_1 \rangle_{\mathbb{Z}}}$ and $\gamma_2 \in \overline{\langle B_2 \rangle_{\mathbb{Z}}}$. Then $m\lambda = m\gamma_1 + m\gamma_2 = \mu_1 + \mu_2$ implies $\mu_1 = m\gamma_1$ and $\mu_2 = m\gamma_2$, hence $\gamma_1 \in \overline{\langle D_1 \rangle_{\mathbb{Z}}}$ and $\gamma_2 \in \overline{\langle D_2 \rangle_{\mathbb{Z}}}$. Thus

$$\overline{\langle D \rangle_{\mathbb{Z}}} = \overline{\langle D_1 \rangle_{\mathbb{Z}}} \oplus \overline{\langle D_2 \rangle_{\mathbb{Z}}}.$$

□

Lemma 3.3.2. *Every subset B has a decomposition $B = \bigcup B_i$ into irreducible subsets B_i. This decomposition is unique up to the order.*

Proof. The existence is clear by induction. Now let $B = \bigcup B'_j$ be another decomposition into irreducible subsets. By the previous lemma every B_i is contained in some B'_j and viceversa. Then these factors are the same up to the order. $\square$

3.3.2 Building sets and nested sets of layers

We now recall some general definitions given in [10] and [14, Chapter 20.1], adapting them to our situation.

A family $\mathcal{G}^*$ of subsets of A is a *building set* if every complete subset B of A is decomposed by the maximal elements B_i of $\mathcal{G}^*$ contained in B. Then we say that $B = \bigcup_i B_i$ is the decomposition of B in $\mathcal{G}^*$ or that the B_is are the $\mathcal{G}^*-$factors of B.

A subset $\mathcal{S}^*$ of $\mathcal{G}^*$ is a $\mathcal{G}^*-$*nested set* if given any $B_1, \ldots, B_r \in \mathcal{S}^*$ mutually incomparable,

$$B \doteq B_1 \cup \ldots \cup B_r$$

is a complete set in A with its decomposition in $\mathcal{G}^*$.

By [30], an equivalent definition is the following. A *flag* $\mathcal{F}^*$ is a sequence $B_1 \subset \cdots \subset B_k$ of subsets of A. A set $\mathcal{S}^* = \{B_1, \ldots, B_s\}$ is *nested* if there is a flag $\mathcal{F}^*$ such that all the elements of $\mathcal{S}^*$ are $\mathcal{G}^*-$factors of elements of $\mathcal{F}^*$.

The family $\mathcal{I}^*$ of all irreducible subsets of A is clearly a building set. In particular, we call nested sets the $\mathcal{I}^*-$nested sets. Then a *nested set* is a family $\mathcal{S}^*$ of irreducible subsets such that for every $B_1, \ldots, B_r \in \mathcal{S}^*$ mutually incomparable,

$$B \doteq B_1 \cup \ldots \cup B_r$$

is a complete set in A with its decomposition into irreducible subsets.

Now let $p \in \mathcal{C}_0(\widetilde{X})$ be a point of the arrangement, and let C be any layer containing p. Let $\mathcal{G}^*$ be a building set in X_p, and let $X_C = \bigcup_i X_i$ be the decomposition of X_C in $\mathcal{G}^*$. We recall that X_C is in bijection with $\widetilde{X}_C$; then let $\widetilde{X_i}$ be the subset of $\widetilde{X_C}$ corresponding to X_i. Set

$$H_i \doteq \bigcap_{(\lambda,a)\in\widetilde{X_i}} H_{(\lambda,a)}$$

and let C_i be the connected component of H_i containing C. Following Remark 3.2.4, $C_i = C(X_i)$ is the only layer containing C and such that $X_{C_i} = X_i$. We call the C_is the $\mathcal{G}-$*factors* of C; clearly $C = \cap C_i$.

Then we can associate to every building set $\mathcal{G}^*$ a *building set of layers* $\mathcal{G}$ defined as the set of all the $\mathcal{G}-$factors of all the elements of $\mathcal{C}(\widetilde{X})$. In particular for $\mathcal{G}^* = \mathcal{I}^*$ we get that the set $\mathcal{I}$ of all irreducible layers is a building set.

A *flag* $\mathcal{F}$ of layers is a sequence $C_1 \subset \cdots \subset C_k$. A set of layers

$$\mathcal{S} = \{C_1, \ldots, C_s\}$$

is $\mathcal{G}-$*nested* if there is a flag $\mathcal{F}$ such that all the elements of $\mathcal{S}$ are $\mathcal{G}-$factors of elements of $\mathcal{F}$. We say that $\mathcal{S}$ is a *nested set of layers* if it is $\mathcal{I}-$nested, i.e. if there is a flag $\mathcal{F}$ such that all the elements of $\mathcal{S}$ are irreducible factors of elements of $\mathcal{F}$.

Remark 3.3.2. From now on we will assume for simplicity $\mathcal{G} = \mathcal{I}$, and then we will focus on the model $\mathbf{Z}_{\widetilde{X}} \doteq \mathbf{Z}_{\widetilde{X},\mathcal{I}}$ defined as the closure of the image of the map

$$i_{\mathcal{I}} = j \times \prod_{C \in \mathcal{I}} \varphi_C : \mathcal{R}_{\widetilde{X}} \to T \times \prod_{C \in \mathcal{I}} \mathbb{P}_C.$$

However, all the results below may be extended to the case of an arbitrary building set $\mathcal{G}$.

We call the minimum element of the flag the *center* of $\mathcal{S}$. This is a well defined layer by the following Lemma:

Lemma 3.3.3. *Let $\mathcal{S}$ be a nested set. Then*

$$C(\mathcal{S}) \doteq \bigcap_{C \in \mathcal{S}} C$$

is connected (and then is a layer).

Proof. Let $M(\mathcal{S})$ be the set of minimal elements of $\mathcal{S}$; clearly

$$C(\mathcal{S}) = \bigcap_{C \in M(\mathcal{S})} C.$$

The elements of $M(\mathcal{S})$ are pairwise incomparable, hence

$$\overline{\Lambda_{C(\mathcal{S})}} = \sum_{C \in S} \overline{\Lambda_C} = \bigoplus_{C \in M(\mathcal{S})} \overline{\Lambda_C}.$$

Let us choose an integral basis $\underline{b}_C$ for each of the lattices $\overline{\Lambda_C}$, $C \in M(\mathcal{S})$. Then

$$\underline{b} = \bigcup_{C \in M(\mathcal{S})} \underline{b}_C$$

is an integral basis for $\overline{\Lambda_{M(\mathcal{S})}}$. For any $\lambda \in \overline{\Lambda_C}$, λ takes a constant value a_λ on C by Remark 3.2.2. It follows that the elements $\lambda - a_\lambda$, $\lambda \in \underline{b}$ generate the ideal of definition of $C(\mathcal{S})$, which is clearly irreducible since $\underline{b}$ is a basis of a split direct summand in Λ. $\square$

Remark 3.3.3. Notice that our proof clearly implies that the intersection $C(\mathcal{S}) = \cap_{C\in M(\mathcal{S})}$ is transversal.

A nested set of layers is *maximal* if it is not contained in a larger one; this happens if and only if $\mathcal{S}$ contains all the irreducible factors of a maximal flag. In this case the center of $\mathcal{S}$ is a point $p = p(\mathcal{S})$. We denote by $\mathfrak{M}$ the set of all maximal nested set of layers of $\mathcal{C}(\widetilde{X})$ and by $\mathfrak{M}_p$ the set of those having center p. Then we have the partition

$$\mathfrak{M} = \bigsqcup_{p\in\mathcal{C}_0(\widetilde{X})} \mathfrak{M}_p.$$

The following fact is clear from the definitions (and from Remark 3.2.4):

Lemma 3.3.4. *If $\mathcal{S} = \{C_1, \ldots, C_s\} \in \mathfrak{M}_p$ is a maximal nested set of layers of center p, then*

$$\mathcal{S}^* \doteq \{X_{C_1}, \ldots, X_{C_s}\}$$

is a maximal nested set in X_p.

Conversely, given a maximal $\mathcal{G}^*-$nested set $\widehat{\mathcal{S}}$ in X_p, there is a unique $\mathcal{S} \in \mathfrak{M}_p$ such that $\mathcal{S}^* = \widehat{\mathcal{S}}$; namely

$$\mathcal{S} \doteq \left\{C(A_i),\ A_i \in \widehat{\mathcal{S}}\right\}.$$

In particular $|\mathcal{S}| = |\mathcal{S}^*| = n$, the rank of X (see [14, Theor 20.9]).

Finally we prove an elementary result that we will use frequently in the next sections. Take $\mathcal{S} \in \mathfrak{M}_p$.

Lemma 3.3.5.

1. *Let $C \in \mathcal{I}$ and $p \in C$. Then there is an element $\overline{C} \in \mathcal{S}$ which is the maximum among all the elements of $\mathcal{S}$ contained in C; we call it the* $\mathcal{S}-$core *of C.*

2. *Let C be an element of $\mathcal{S}$ which is not minimal in it. Then there is an element $s(C) \in \mathcal{S}$ which is the maximum among all the elements of $\mathcal{S}$ properly contained in C; we call it the* successor *of C.*

Proof. The proof is the same for both statements. Let C' and C'' be two elements of $\mathcal{S}$ which are contained (or, for the second statement, properly contained) in C. Then $X_C \subset X_{C'} \cap X_{C''}$; hence $X_{C'} \cup X_{C''}$ is not a decomposition. Since $X_{C'}$ and $X_{C''}$ are in the nested set $\mathcal{S}^*$, they must be comparable; then also C' and C'' are. □

3.3.3 Adapted bases

Given a nested set $\mathcal{S}$, we say that an integral basis $\underline{b} \doteq \lambda_1 \dots, \lambda_n$ for the lattice Λ is *adapted to* $\mathcal{S}$ if for every $C \in \mathcal{S}$, $\underline{b} \cap \overline{\Lambda_C}$ is an integral basis for $\overline{\Lambda_C}$.

Lemma 3.3.6. *There exists an integral basis $\underline{b}^{\mathcal{S}}$ for Λ adapted to $\mathcal{S}$.*

Proof. Let us define

$$\Lambda_{\mathcal{S}} \doteq \sum_{D \in \mathcal{S}} \overline{\Lambda_D}.$$

Notice that

$$\Lambda_{\mathcal{S}} = \bigoplus_{C \in M(\mathcal{S})} \overline{\Lambda_D}$$

where $M(\mathcal{S})$ is the set of minimal (and hence pairwise incomparable) elements of $\mathcal{S}$. then by definition $\Lambda_{\mathcal{S}} = \overline{\Lambda_{\mathcal{S}}}$. We will prove, by induction on the cardinality of $\mathcal{S}$, that there is a basis of $\Lambda_{\mathcal{S}}$ adapted to $\mathcal{S}$. Then our claim follows: indeed, since the lattice $\Lambda_{\mathcal{S}}$ either coincide with Λ or is a split direct summand of it, the basis of $\Lambda_{\mathcal{S}}$ can be completed to a basis of Λ.

If $\mathcal{S}$ contains only one element C, the statement is trivial since $\Lambda_{\mathcal{S}} = \overline{\Lambda_C}$ and every basis of this lattice is adapted to $\mathcal{S}$.

Otherwise, take a minimal $C \in \mathcal{S}$, and set $\mathcal{S}' = \mathcal{S} \setminus \{C\}$. Since $\mathcal{S}'$ is nested, by inductive hypothesis the lattice

$$\Lambda_{\mathcal{S}'} = \sum_{D \in \mathcal{S}'} \overline{\Lambda_D}$$

has an integral basis adapted to $\mathcal{S}'$. Since $\Lambda_{\mathcal{S}'} = \overline{\Lambda_{\mathcal{S}'}}$ we can complete the chosen basis of $\Lambda_{\mathcal{S}'}$ to an integral basis $\underline{b}$ of $\Lambda_{\mathcal{S}}$ using elements of $\overline{\Lambda_C}$. We claim that this basis is adapted to $\mathcal{S}$. Let us take D in $\mathcal{S}$. If $D \neq C$ there is nothing to prove. Then assume $D = C$. In this case we know that

$$\Lambda_{\mathcal{S}} = \overline{\Lambda_C} \oplus \bigoplus_{D \in M(\mathcal{S}) \setminus \{C\}} \overline{\Lambda_D}.$$

By construction, every element in $\underline{b}$ either lies in $\overline{\Lambda_C}$ or in $\bigoplus_{D \in M(\mathcal{S}) \setminus \{C\}} \overline{\Lambda_D}$. Then every $\lambda \in \overline{\Lambda_C}$ is in the span of $\underline{b} \cap \overline{\Lambda_C}$, proving our claim. $\square$

To every maximal set of layers $\mathcal{S} \in \mathfrak{M}_p$ we associate a function

$$p_{\mathcal{S}} : \Lambda \longrightarrow \mathcal{S}$$

in the following way. For every $\lambda \in \Lambda$ we set $a \doteq \lambda(p)$, and we define $p_{\mathcal{S}}(\lambda)$ as the maximum element of $\mathcal{S}$ on which λ is identically equal to a. This is well defined by Lemma 3.3.5: indeed $p_{\mathcal{S}}(\lambda) = \overline{H_{(\lambda,a)}}$. This function has the following properties:

Lemma 3.3.7.

1. *For every $C \in \mathcal{I}$ there exists $\lambda \in X_C$ such that $p_{\mathcal{S}}(\lambda) = \overline{C}$.*

2. *The restriction of $p_{\mathcal{S}}$ to an adapted basis $\underline{b}$ is a bijection.*

Proof. For every $C \in \mathcal{I}$, let $M(C)$ be the (possibly empty) set of the elements of $\mathcal{S}$ properly containing C and minimal with this property. Such elements are pairwise incomparable, hence $\bigcup_{D\in M(C)} X_D$ is a decomposition. Since $X_C \supset X_D$ for every $D \in M(C)$,

$$X_C \supset \bigcup_{D\in M(C)} X_D$$

and this inclusion is proper, because X_C is irreducible. Then there exists

$$\lambda \in X_C \setminus \bigcup_{D\in M(C)} X_D.$$

By definition $p_{\mathcal{S}}(\lambda) = \overline{C}$, then the first statement is proved.

Now assume $C \in \mathcal{S}$, and let $\underline{b}$ be an adapted basis to $\mathcal{S}$: then by definition $\underline{b} \cap \overline{\Lambda_C}$ is a basis for $\overline{\Lambda_C}$ and

$$\bigsqcup_{D\in M(C)} \left(\underline{b} \cap \overline{\Lambda_D}\right) \text{ is a basis for } \bigoplus_{D\in M(C)} \overline{\Lambda_D}.$$

Since C is irreducible

$$\overline{\Lambda_C} \supsetneq \bigoplus_{D\in M(C)} \overline{\Lambda_D}.$$

Then there exists

$$\lambda \in \left(\underline{b} \cap \overline{\Lambda_C}\right) \setminus \bigsqcup_{D \in M(C)} \left(\underline{b} \cap \overline{\Lambda_D}\right).$$

Clearly $p_{\mathcal{S}}(\lambda) = C$. Then we proved that the restriction of $p_{\mathcal{S}}$ to $\underline{b}$ is surjective; therefore it is bijective, since $|\underline{b}| = n = |\mathcal{S}|$. $\square$

3.4 Open sets and smoothness

3.4.1 Definition of the open sets

To every $\mathcal{S} \in \mathfrak{M}_p$ we associate a nonlinear change of coordinates $f_{\mathcal{S}}$ and an open set $\mathcal{V}_{\mathcal{S}}$ defined as follows.

Let us take a basis of Λ adapted to $\mathcal{S}$, and denote it by

$$\underline{b}^{\mathcal{S}} = (\lambda_C)_{C \in \mathcal{S}}$$

where $\lambda_C \doteq p_{\mathcal{S}}^{-1}(C)$. Set $a_C \doteq \lambda_C(p)$. Since $\underline{b}^{\mathcal{S}}$ is integral, $(\lambda_C - a_C)_{C \in \mathcal{S}}$ is a system of coordinates on T.

Consider $\mathbb{C}^n$ with coordinates $\underline{z}^{\mathcal{S}} = (z_C)_{C \in \mathcal{S}}$, and its open set

$$\widetilde{U_{\mathcal{S}}} \doteq \left\{ (z_C) \in \mathbb{C}^n | \prod_{D \subseteq C} z_D \neq -a_C \; \forall C \in \mathcal{S} \right\}.$$

Define a map $f_{\mathcal{S}} : \widetilde{U_{\mathcal{S}}} \to T$ in the given coordinates as

$$\lambda_C \left(f_{\mathcal{S}}(\underline{z}^{\mathcal{S}})\right) = \left(\prod_{D \subseteq C} z_D \right) + a_C$$

or equivalently as the nonlinear change of coordinates

$$\lambda_C - a_C = \prod_{D \subseteq C} z_D. \tag{3.4.1}$$

Then $f_{\mathcal{S}}(\underline{0}) = p$.

Notice that on the open set of T where $\lambda_C - a_C \neq 0 \forall C \in \mathcal{S}$, the map $f_{\mathcal{S}}$ can be inverted by the following formula:

$$z_C = \begin{cases} \lambda_C - a_C & \text{, if } C \text{ is minimal in } \mathcal{S} \\ \frac{\lambda_C - a_C}{\lambda_{s(C)} - a_{s(C)}} & \text{, otherwise} \end{cases} \tag{3.4.2}$$

where $s(C)$ is the successor defined in Lemma 3.3.5.

Let us define the open set of T

$$T_p \doteq T \setminus \bigcup_{p \notin C} C$$

and set $U_{\mathcal{S}} \doteq {f_{\mathcal{S}}}^{-1}(T_p)$. We denote again by $f_{\mathcal{S}}$ the restriction $U_{\mathcal{S}} \to T_p$.

Now take any $\lambda \in \Lambda$; set $a \doteq \lambda(p)$ and $C \doteq p_{\mathcal{S}}(\lambda)$.

Since $\underline{b}^{\mathcal{S}}$ is adapted to $\mathcal{S}$, an integral basis for $\overline{\Lambda_C}$ is given by

$$\underline{b}^{\mathcal{S}} \cap \overline{\Lambda_C} = \{\lambda_D, D \supseteq C\}.$$

In particular λ can be expressed in this basis, and since $p_{\mathcal{S}}(\lambda) = C$, λ does not lie in the span of $\{\lambda_D, D \supsetneq C\}$: then

$$\lambda = m_C \lambda_C + \sum_{D \supsetneq C} m_D \lambda_D$$

for some integers m_D and a nonzero integer m_C. The previous identity, considered as an equality of regular functions on T, can be written as

$$\lambda = \lambda_C^{m_C} \prod_{D \supsetneq C} \lambda_D^{m_D}.$$

Then we have:

$$\lambda - a = \left(\lambda_C^{m_C} \prod_{D \supsetneq C} \lambda_D^{m_D} - a_C^{m_C} \prod_{D \supsetneq C} \lambda_D^{m_D} \right) + \left(a_C^{m_C} \prod_{D \supsetneq C} \lambda_D^{m_D} - a \right) \tag{3.4.3}$$

and we can write the first summand as

$$\prod_{D \supsetneq C} \lambda_D^{m_D} \left(\lambda_C^{m_C} - a_C^{m_C}\right) = \beta_C(\lambda_C - \alpha_C)$$

where

$$\beta_C \doteq \prod_{D \supsetneq C} \lambda_D^{m_D} \prod_{\zeta^{m_C}=1, \zeta \neq 1} (\lambda_C - \zeta a_C)$$

is a regular function on T which is invertible on C. Working in the same way on the second summand of Formula (3.4.3) we see that, for some regular functions $\{\beta_D, D \in \mathcal{S}\}$,

$$\lambda - a = \beta_C \left(\lambda_C - a_C\right) + \sum_{D \supsetneq C} \beta_D(\lambda_D - a_D).$$

By operating the change of coordinates (3.4.1), we get:

$$\lambda - a = \left(\beta_C \prod_{E \subseteq C} z_E + \sum_{D \supsetneq C} \beta_D \prod_{E \subseteq D} z_E\right) = \left(\prod_{E \subseteq C} z_E\right) \cdot p_\lambda(\underline{z}^{\mathcal{S}}) \tag{3.4.4}$$

where we set

$$p_\lambda(\underline{z}^{\mathcal{S}}) \doteq \beta_C + \sum_{D \supsetneq C} \beta_D \prod_{D \supseteq E \supsetneq C} z_E.$$

We define $\mathcal{V}_{\mathcal{S}}$ as the open set of $U_{\mathcal{S}}$ where

$$\prod_{\lambda \in X_p} p_\lambda(\underline{z}^{\mathcal{S}}) \neq 0.$$

Let us remark that $\underline{0} \in \mathcal{V}_{\mathcal{S}}$, since for every $\lambda \in X_p$ we have that $p_\lambda(\underline{0}) = \beta_C(p) \neq 0$. Furthermore in $\mathcal{V}_{\mathcal{S}}$, for every $\lambda \in X_p$, we have the equality of regular functions

$$\prod_{E \subseteq p_{\mathcal{S}}(\lambda)} z_E = \frac{\lambda - a}{p_\lambda(\underline{z}^{\mathcal{S}})}. \tag{3.4.5}$$

3.4.2 Properties of the open sets

Let us define the open set of $\mathcal{V}_{\mathcal{S}}$

$$\mathcal{V}_{\mathcal{S}}{}^0 \doteq \{\underline{z} \in \mathcal{V}_{\mathcal{S}} \mid z_C \neq 0 \forall C \in \mathcal{S}\}.$$

We denote by $A_{\mathcal{S}}$ the open set of T $f_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}}) \cap \mathcal{R}_{\widetilde{X}}$. We remark that by Formula (3.4.5) $f_{\mathcal{S}}^{-1}(A_{\mathcal{S}}) = \mathcal{V}_{\mathcal{S}}{}^0$ and the restriction of $f_{\mathcal{S}}$ to $\mathcal{V}_{\mathcal{S}}{}^0$ maps it into $A_{\mathcal{S}}$. By composing this map with the inclusion $A_{\mathcal{S}} \hookrightarrow \mathcal{R}_{\widetilde{X}}$ and with the application $\phi_C : \mathcal{R}_{\widetilde{X}} \to \mathbb{P}_C$ defined in Section 3.2.3, we get a map

$$\psi_C : \mathcal{V}_{\mathcal{S}}{}^0 \longrightarrow \mathbb{P}_C.$$

Lemma 3.4.1. *For every $C \in \mathcal{I}$ and $\mathcal{S} \in \mathfrak{M}_p$, the map ψ_C extends uniquely to a map*

$$\widetilde{\psi_C} : \mathcal{V}_{\mathcal{S}} \to \mathbb{P}_C.$$

Proof. Let p be the center of $\mathcal{S}$. If C does not contain p the statement is clear: indeed since $\mathcal{V}_{\mathcal{S}} \subset U_{\mathcal{S}}$, for every $u \in \mathcal{V}_{\mathcal{S}}$ we have that $t \doteq f_{\mathcal{S}}(u) \notin C$ so that for at least one index j, $\lambda_j(t) \neq a_j$. Then the projective coordinate $\lambda_j(t) - a_j$ of $\mathbb{P}_C$ is nonzero.

Then assume $p \in C$, and let $\overline{C}$ be its $\mathcal{S}-$core (see Lemma 3.3.5). By the first part of Lemma 3.3.7, there exists $\lambda_1 \in X_C$ such that $p_{\mathcal{S}}(\lambda_1) = \overline{C}$. Since we assumed (Remark 3.2.1) every element of X_C to be primitive, we can complete $\{\lambda_1\}$ to an integral basis $\{\lambda_1, \dots, \lambda_k\}$ of $\overline{\Lambda_C}$. Then if we set $a_i \doteq \lambda_i(p)$, we have that

$$[\lambda_1 - a_1, \dots, \lambda_k - a_k]$$

is a system of projective coordinates for $\mathbb{P}_C$.

Since $\underline{b}^{\mathcal{S}}$ is adapted to $\mathcal{S}$, an integral basis for $\overline{\Lambda_{\overline{C}}}$ is given by

$$\underline{b}^{\mathcal{S}} \cap \overline{\Lambda_{\overline{C}}} = \left\{\lambda_D, D \supseteq \overline{C}\right\}.$$

In particular every $\lambda_i \in \overline{\Lambda_C} \subseteq \overline{\Lambda_{\overline{C}}}$ can be expressed in this basis, and since $p_{\mathcal{S}}(\lambda_1) = \overline{C}$, λ_1 does not lie in the span of $\left\{\lambda_D, D \supsetneq \overline{C}\right\}$.

After making the nonlinear change of coordinates (3.4.1) as in Formula (3.4.4), we can divide every projective coordinate by $\prod_{E \subseteq \overline{C}} z_E$; in this way we get that the map $\psi_C : {\mathcal{V}_{\mathcal{S}}}^0 \longrightarrow \mathbb{P}_C$ is given by

$$\underline{z} \mapsto \left[p_{\lambda_1}(\underline{z}),\ p_{\lambda_2}(\underline{z}) \prod_{\overline{C} \subsetneq E \subseteq D_2} z_E,\ \ldots,\ p_{\lambda_k}(\underline{z}) \prod_{\overline{C} \subsetneq E \subseteq D_k} z_E \right]$$

where we set $D_i \doteq p_{\mathcal{S}}(\lambda_i)$. Since by definition $p_{\lambda_1}(\underline{z}) \neq 0$ for $\underline{z} \in \mathcal{V}_{\mathcal{S}}$, this map extends to $\mathcal{V}_{\mathcal{S}}$. Moreover its image is contained in an affine open set of $\mathbb{P}_C$.

Finally the uniqueness of the extension is clear since by its very definition ${\mathcal{V}_{\mathcal{S}}}^0$ is dense in $\mathcal{V}_{\mathcal{S}}$. □

By applying the lemma above to all the layers $C \in \mathcal{I}$, we get that for every $\mathcal{S} \in \mathfrak{M}_p$ the inclusion ${\mathcal{V}_{\mathcal{S}}}^0 \hookrightarrow \mathbf{Z}_{\widetilde{X}}$ extends uniquely to a map

$$j_{\mathcal{S}} : \mathcal{V}_{\mathcal{S}} \to \mathbf{Z}_{\widetilde{X}}.$$

Lemma 3.4.2. *The map $j_{\mathcal{S}}$ is an embedding into a smooth open set.*

Proof. In order to prove that $j_{\mathcal{S}}$ is an embedding, it suffices to see that every coordinate z_C on $\mathcal{V}_{\mathcal{S}}$ can be written as the composition of $j_{\mathcal{S}}$ and a function on $j_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}})$. Then take $C \in \mathcal{S}$. If C is not minimal, let $D = s(C)$ be the successor of C. Since $\underline{b}^{\mathcal{S}}$ is adapted to $\mathcal{S}$, on $\mathbb{P}_D$ we have the projective coordinates

$$[\lambda_E - a_E]_{E\in\mathcal{S},E\supseteq D}$$

and by the proof of the previous lemma $\mathcal{V}_{\mathcal{S}}$ maps into the affine subset where $\lambda_D - a_D \neq 0$. Then we can read the coordinate z_C in $\mathbb{P}_D$ by Formula (3.4.2):

$$z_C = \frac{\lambda_C - a_C}{\lambda_D - a_D}.$$

If on the other hand C is minimal in $\mathcal{S}$, then $z_C = \lambda_C - a_C$.

In this way all the coordinates z_C can be recovered by the projection of $j_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}}) \subset \mathbf{Z}_{\widetilde{X}}$ on T or on some $\mathbb{P}_D$; hence our map is an embedding. Moreover, since $(z_C)_{C\in\mathcal{S}}$ is a system of coordinates on $j_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}})$, in every point the differential of $j_{\mathcal{S}}$ has rank $|\mathcal{S}| = n$. Then $j_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}})$ is smooth. □

Remark 3.4.1. By abuse of notation, from now on we will write $\mathcal{V}_{\mathcal{S}}$ for $j_{\mathcal{S}}(\mathcal{V}_{\mathcal{S}})$, identifying this set with its isomorphic image in $\mathbf{Z}_{\widetilde{X}}$.

3.4.3 Smoothness of the model

Let us define

$$\mathbf{Y}_{\widetilde{X}} \doteq \bigcup_{\mathcal{S}\in\mathfrak{M}} \mathcal{V}_{\mathcal{S}}.$$

In this section we prove that $\mathbf{Y}_{\widetilde{X}} = \mathbf{Z}_{\widetilde{X}}$, and hence $\mathbf{Z}_{\widetilde{X}}$ is smooth. The main step is the following lemma, which tells that every curve in $\mathcal{R}_{\widetilde{X}}$ that "has limit" in T, "has limit" in $\mathbf{Y}_{\widetilde{X}}$. Let $D_\varepsilon \doteq \{s \in \mathbb{C} \mid |s| < \varepsilon\}$.

Lemma 3.4.3. *Let $f : D_\varepsilon \to T$ be a curve such that $f(D_\varepsilon \setminus \{0\}) \subseteq \mathcal{R}_{\widetilde{X}}$.*

Then f lifts to a curve in $\mathbf{Y}_{\widetilde{X}}$.

Proof. Given such a f, let $C_f \in \mathcal{C}(\widetilde{X})$ be the smallest layer containing $f(0)$, and let $p \in \mathcal{C}_0(\widetilde{X})$ be a point contained in C_f. For every $\lambda \in X_p$, we have that locally, near $s = 0$, we can write

$$\lambda(f(s)) - a = s^{n_\lambda} q_\lambda(s)$$

with $a = \lambda(p)$, $n_\lambda \geq 0$ and $q_\lambda(0) \neq 0$.

For every integer $h \geq 0$, let us define

$$A_h \doteq \{\lambda \in X_p | n_\lambda \geq h\}.$$

Notice that $A_0 = X_p$ and $A_{h+1} \subseteq A_h$; by taking all the irreducible factors of the elements of this flag we get a nested set in X_p. Let us complete it to a maximal nested set $\mathcal{S}^*$; by Lemma 3.3.4, to $\mathcal{S}^*$ is naturally associated a maximal nested set of layers $\mathcal{S} \in \mathfrak{M}_p$.

We claim that for a such $\mathcal{S}$, the curve $f : D_\varepsilon \setminus \{0\} \to \mathcal{R}_{\widetilde{X}}$ extends to a map $f : D_\varepsilon \to \mathcal{V}_\mathcal{S}$.

First notice that $f(0) \in T_p$: indeed for every layer D containing $f(0)$ we have that $C_f \subseteq D$ by minimality and then $p \in D$. Then we have to prove that:

1. $z_C\big(f(s)\big)$ is defined in 0 for every $C \in \mathcal{S}$;

2. $p_\lambda\big(f(0)\big) \neq 0$ for every $\lambda \in X_p$.

Take $C \in \mathcal{S}$; if C is minimal in $\mathcal{S}$ then $z_C(f(s)) = \lambda_C(f(s)) - a_C$ and there is nothing to prove. Otherwise, let $D = s(C)$ be the successor of C. Then by 3.4.2

$$z_C(f(s)) = \frac{\lambda_C(f(s)) - a_C}{\lambda_D(f(s)) - a_D} = s^{n_{\lambda_C} - n_{\lambda_D}} \frac{q_{\lambda_C}(s)}{q_{\lambda_D}(s)}$$

and $n_{\lambda_C} \geq n_{\lambda_D}$ by the definition of $\mathcal{S}$, so z_C is well defined in 0.

As for the second claim, given any $\lambda \in X_p$ set $C \doteq p_{\mathcal{S}}(\lambda)$ and take the vector λ_C of the adapted basis $\underline{b}^{\mathcal{S}}$.

Then by definition of $\mathcal{S}$, $n_\lambda = n_{\lambda_C}$, and by Formulae (3.4.1) and (3.4.4) we have

$$p_\lambda = \frac{\lambda - a}{\lambda_C - a_C}.$$

Therefore

$$p_\lambda(f(0)) = \frac{\lambda\big(f(0)\big) - a}{\lambda_C\big(f(0)\big) - a_C} = \frac{q_\lambda(0)}{q_{\lambda_C}(0)} \neq 0.$$

□

Theorem 3.4.4. $\mathbf{Y}_{\widetilde{X}} = \mathbf{Z}_{\widetilde{X}}$. *In particular* $\mathbf{Z}_{\widetilde{X}}$ *is smooth.*

Proof. By the well known *valuative criterion for properness* (see for instance [20]), the previous lemma amounts to say that the map

$$\pi|_{\mathbf{Y}_{\widetilde{X}}} : \mathbf{Y}_{\widetilde{X}} \to T$$

is proper. Since also the projection

$$T \times \prod_{C \in \mathcal{I}} \mathbb{P}_C \to T$$

is proper, the embedding

$$\mathbf{Y}_{\widetilde{X}} \to T \times \prod_{C \in \mathcal{I}} \mathbb{P}_C$$

is proper as well; therefore its image is closed, and thus it coincides with $\mathbf{Z}_{\widetilde{X}}$.

Therefore $\mathbf{Z}_{\widetilde{X}}$ is smooth, since it is union of smooth open sets. □

3.5 The normal crossing divisor

3.5.1 Technical lemmas

For every $C \in \mathcal{I}$, let us define a divisor $\mathbf{D}_C \subset \mathbf{Z}_{\widetilde{X}}$ as follows. Take a $\mathcal{S} \in \mathfrak{M}$ such that $C \in \mathcal{S}$. In the open set $\mathcal{V}_\mathcal{S}$ take the divisor of equation $z_C = 0$; let $\mathbf{D}_C$ be the closure of this divisor in $\mathbf{Z}_{\widetilde{X}}$. The following lemma implies that $\mathbf{D}_C$ does not depend on the choice of $\mathcal{S}$, and yields the theorem below, which describes the geometry of $\mathbf{Z}_{\widetilde{X}} \setminus \mathcal{R}_{\widetilde{X}}$.

Lemma 3.5.1. *Take any two maximal nested sets of layers $\mathcal{S} \in \mathfrak{M}_p$ and $\mathcal{Q} \in \mathfrak{M}_q$. Let $\{z_C^\mathcal{S}, C \in \mathcal{S}\}$ and $\{z_C^\mathcal{Q}, C \in \mathcal{Q}\}$ be the corresponding sets of coordinates on $\mathcal{V}_\mathcal{S}$ and $\mathcal{V}_\mathcal{Q}$.*

Then for every $C \in \mathcal{S}$:

1. *if $C \in \mathcal{S} \setminus \mathcal{Q}$, $z_C^\mathcal{S}$ is invertible as a function on $\mathcal{V}_\mathcal{S} \cap \mathcal{V}_\mathcal{Q}$;*

2. *if $C \in \mathcal{S} \cap \mathcal{Q}$, $z_C^\mathcal{S}/z_C^\mathcal{Q}$ is regular and invertible as a function on $\mathcal{V}_\mathcal{S} \cap \mathcal{V}_\mathcal{Q}$.*

Proof. If $q \notin C$, then $C \in \mathcal{S} \setminus \mathcal{Q}$, and the (first) statement is proved as follows. Take $x \in \mathbf{Z}_{\widetilde{X}}$ such that $z_C^\mathcal{S}(x) = 0$: then by Formula (3.4.1) $\pi(x) \in C$, where $\pi : \mathbf{Z}_{\widetilde{X}} \to T$ is the projection defined in Remark 3.2.3. Therefore $\pi(x) \notin T_q$, hence $x \notin \mathcal{V}_\mathcal{Q}$, proving the claim.

Therefore we can assume $q \in C$ and proceed by induction as in the proof of [14, Lemma 20.39].

- First let us assume C to be a minimal element in $\mathcal{I}$; then necessarily $C \in \mathcal{S} \cap \mathcal{Q}$. We recall that $z_C^\mathcal{S} = \lambda_C^\mathcal{S} - a_C^\mathcal{S}$; set

$$D \doteq p_\mathcal{Q}(\lambda_C^\mathcal{S}) \supseteq C.$$

Then for some function a

$$z_C^{\mathcal{S}} = a \prod_{E \in \mathcal{Q}, D \supseteq E} z_E^{\mathcal{Q}} = a z_C^{\mathcal{Q}} \prod_{E \in \mathcal{Q}, D \supseteq E, E \neq C} z_E^{\mathcal{Q}}.$$

In the same way $z_C^{\mathcal{Q}} = \lambda_C^{\mathcal{Q}} - a_C^{\mathcal{Q}}$, and if we set

$$D' \doteq p_{\mathcal{S}}(\lambda_C^{\mathcal{Q}}) \supseteq C$$

we get

$$z_C^{\mathcal{Q}} = a' \prod_{F \in \mathcal{S}, D' \supseteq F} z_F^{\mathcal{Q}} = a' z_C^{\mathcal{S}} \prod_{F \in \mathcal{S}, D' \supseteq F, F \neq C} z_F^{\mathcal{S}}.$$

for some function a'. Since both D and D' contain C, by substituting we get:

$$z_C^{\mathcal{S}} = z_C^{\mathcal{S}}\, a\, a' \prod_{E \in \mathcal{Q}, D \supseteq E, E \neq C} z_E^{\mathcal{Q}} \prod_{F \in \mathcal{S}, D' \supseteq F, F \neq C} z_F^{\mathcal{S}}.$$

Therefore

$$aa' \prod_{E \in \mathcal{Q}, D \supseteq E, E \neq C} z_E^{\mathcal{Q}} \prod_{F \in \mathcal{S}, D' \supseteq F, F \neq C} z_F^{\mathcal{S}} = 1$$

and hence

$$\frac{z_C^{\mathcal{S}}}{z_C^{\mathcal{Q}}} = a \prod_{E \in \mathcal{Q}, D \supseteq E, E \neq C} z_E^{\mathcal{Q}}$$

is invertible, as claimed.

- Now let us take any $C \in \mathcal{S}$. By induction, we can assume that our claims are true for every $D \subsetneq C$, $D \in \mathcal{S} \cup \mathcal{Q}$ (if $D \in \mathcal{Q} \setminus \mathcal{S}$, by symmetry $z_D^{\mathcal{Q}}$ is assumed to be invertible on $\mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{Q}}$).

  Let $D = \overline{C} \in \mathcal{Q}$ be the $\mathcal{Q}-$core of C. Take $\lambda \in X_C$ such that $p_{\mathcal{Q}}(\lambda) = D$, and set $G \doteq p_{\mathcal{S}}(\lambda)$. Then $G \supseteq C$ and λ takes on D and on G the same constant value $a \doteq \lambda(p)$. Notice that D is the $\mathcal{Q}-$core of G.

Then for some invertible b, b'

$$\lambda - a = b \prod_{E \in \mathcal{Q}, D \supseteq E} z_E^{\mathcal{Q}} = b' \prod_{F \in \mathcal{S}, G \supseteq F} z_F^{\mathcal{S}}.$$

Hence

$$1 = b^{-1} b' \prod_{F \in \mathcal{S} \setminus \mathcal{Q}, G \supseteq F} z_F^{\mathcal{S}} \prod_{E \in \mathcal{Q} \setminus \mathcal{S}, D \supseteq E} z_E^{\mathcal{Q}^{-1}} \prod_{F \in \mathcal{S} \cap \mathcal{Q}, D \supseteq F} z_F^{\mathcal{S}} z_F^{\mathcal{Q}^{-1}}. \quad (3.5.1)$$

We can now prove the first claim. If $C \notin \mathcal{Q}$ then $D \subsetneq C$. Then all the factors in equation (3.5.1) are regular: those of type $z_F^{\mathcal{S}}, F \in \mathcal{S} \setminus \mathcal{Q}, G \supseteq F$ obviously, the others by inductive assumption, since they involve elements properly contained in C. Since $z_C^{\mathcal{S}}$ appears as one of the factors in (3.5.1) it is invertible.

In the same way if $C \in \mathcal{Q}$, and then $D = C$, all the factors in (3.5.1) but (eventually) $z_C^{\mathcal{S}} z_C^{\mathcal{Q}^{-1}}$ are regular; then also $z_C^{\mathcal{S}} z_C^{\mathcal{Q}^{-1}}$ must be regular and invertible.

□

Lemma 3.5.2. *Let be $C \in \mathcal{I}$.*

1. *The divisor $\mathbf{D}_C$ is well defined.*

2. *If $C \notin \mathcal{S}$, then $\mathbf{D}_C \cap \mathcal{V}_{\mathcal{S}} = \emptyset$.*

Proof. 1. Let $\mathcal{S}, \mathcal{Q}$ be two maximal nested set of layers containing C. Then by the second point of Lemma 3.5.1, $z_C^{\mathcal{S}}$ and $z_C^{\mathcal{Q}}$ have the same zeros in $\mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{Q}}$, which is an open dense set in $\mathcal{V}_{\mathcal{S}}$ and in $\mathcal{V}_{\mathcal{Q}}$. Then the closures of the two divisors coincide.

2. Let $\mathcal{Q}$ be a maximal nested set of layers containing C. Then by the first point of Lemma 3.5.1, $z_C^{\mathcal{Q}}$ is invertible as a function on $\mathcal{V}_{\mathcal{S}} \cap \mathcal{V}_{\mathcal{Q}}$. Therefore the divisor of $\mathcal{V}_{\mathcal{Q}}$ defined by $z_C^{\mathcal{Q}} = 0$ is contained in $\mathbf{Z}_{\widetilde{X}} \setminus \mathcal{V}_{\mathcal{S}}$. Since this set is closed, it also contains $\mathbf{D}_C$ which is the closure of the divisor.

□

3.5.2 The main theorem

Now let us define

$$\mathbf{D} = \bigcup_{C \in \mathcal{I}} \mathbf{D}_C.$$

The geometry of the divisor $\mathbf{D}$ is described by the following theorem.

Theorem 3.5.3.

1. $\mathbf{Z}_{\widetilde{X}} \setminus \mathbf{D} = \mathcal{R}_{\widetilde{X}}$.

2. $\mathbf{D}$ *is a normal crossing divisor whose irreducible components are the divisors* $\mathbf{D}_C, C \in \mathcal{I}$.

3. *Let be* $\mathcal{N} \subseteq \mathcal{I}$, *and*

$$\mathbf{D}_{\mathcal{N}} \doteq \bigcap_{C \in \mathcal{N}} \mathbf{D}_C.$$

Then $\mathbf{D}_{\mathcal{N}} \neq \emptyset$ *if and only if* $\mathcal{N}$ *is nested.*

4. *If* $\mathcal{N}$ *is nested,* $\mathbf{D}_{\mathcal{N}}$ *is smooth and irreducible.*

Proof. By Theorem 3.4.4, we can check each statement on every open set $\mathcal{V}_{\mathcal{S}}, \mathcal{S} \in \mathfrak{M}$.

Then the first claim, by the second part of Lemma 3.5.2, amounts to note that

$$\big(\mathbf{Z}_{\widetilde{X}} \setminus \mathbf{D}\big) \cap \mathcal{V}_{\mathcal{S}} = \mathcal{V}_{\mathcal{S}} \setminus \bigcup_{C \in \mathcal{S}} (\mathbf{D}_C \cap \mathcal{V}_{\mathcal{S}}) = {\mathcal{V}_{\mathcal{S}}}^0 = \mathcal{R}_{\widetilde{X}} \cap \mathcal{V}_{\mathcal{S}}.$$

(for the definition of ${\mathcal{V}_{\mathcal{S}}}^0$ see the beginning of Section 3.4.2).

The second statement is obvious since

$$\mathbf{D} \cap \mathcal{V}_{\mathcal{S}} = \bigcup_{C \in \mathcal{S}} (\mathbf{D}_C \cap \mathcal{V}_{\mathcal{S}}) = \{\underline{z} \in \mathcal{V}_{\mathcal{S}} | z_C = 0 \text{ for some } C \in \mathcal{S}\}$$

is by definition a normal crossing divisor in $\mathcal{V}_{\mathcal{S}}$.

For the third statement, note that if $\mathcal{N}$ is not nested it is not contained in any maximal nested set of layers; then for every $\mathcal{S} \in \mathfrak{M}$, $\mathbf{D}_{\mathcal{N}} \cap \mathcal{V}_{\mathcal{S}} = \emptyset$ by the second part of Lemma 3.5.2. On the other hand, if $\mathcal{N}$ is nested it can be completed to some $\mathcal{S} \in \mathfrak{M}$, and

$$\mathbf{D}_{\mathcal{N}} \cap \mathcal{V}_{\mathcal{S}} = \{\underline{z} \in \mathcal{V}_{\mathcal{S}} | z_C = 0 \forall C \in \mathcal{N}\}$$

which is clearly nonempty, smooth and irreducible. Since

$$\mathbf{D}_{\mathcal{N}} = \bigcup_{\mathcal{S} \supseteq \mathcal{N}} (\mathbf{D}_{\mathcal{N}} \cap \mathcal{V}_{\mathcal{S}})$$

also the last statement follows.

□

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