UNIVERSITÀ DEGLI STUDI ROMA TRE
Dottorato di Ricerca in Matematica, XXII Ciclo

Alice Fabbri

Kronecker Function Rings of Domains and Projective Models

THESIS

Advisor
Prof. Bruce Olberding
New Mexico State University

. . . .

Co-Advisor
Prof. Marco Fontana
Università degli Studi Roma Tre

. . . .

AMS classification: 13B22, 13F30, 14A05
Key words: Star operation, Kronecker function ring, Projective model

Contents

Introduction

Star operations were introduced by W. Krull in a series of papers starting from 1936. The purpose was to generalize Kronecker's theory of divisibility, holding in Dedekind domains, to arbitrary integrally closed domains. The main obstacle for this generalization was the restriction on domains enjoying Gauss' Lemma. In fact, Gauss' Lemma holds in an integrally closed domain R if and only if R is a Prüfer domain, i.e., an integral domain in which every nonzero finitely generated ideal is invertible (cf. [22, Corollary 28.5 and Theorem 28.6]).

By means of star operations it is possible to define a special kind of multiplication for ideals, i.e. the $\star$-multiplication, and then to isolate a particular class of star operations, i.e. the *e.a.b.* star operations, enjoying cancellation properties on $\star$-multiplication. In such a situation a more general version of Gauss' Lemma holds in every integrally closed domain and a given integrally closed domain R can be associated to a Kronecker function ring, a Bézout domain (i.e. an integral domain such that every finitely generated ideal is principal) generalizing the classical domain introduced by Kronecker and consisting of the possible gcd's of elements in R. A thorough historical overview on star operations and Kronecker function rings can be found in [17].

It is well-known that the set of valuation overrings of an integral domain may have a large variety of structures. The simplest of these structures is realized in valuation domains, for which valuation overrings form a linearly ordered set.

In the class of integrally closed domains, a way to evaluate how complicated is the set of valuation overrings is to look at the number of Kronecker function rings admitted by the domain. In fact, the more complicated is the set of valuation overrings of an integrally closed domain, the more the possible ways in which the domain can be represented as an intersection of valuation overrings, the more the possible number of distinct Kronecker function rings the domain admits.

Let R be an integral domain with quotient field K. Let $\mathfrak{F}(R)$ denote the set of nonzero fractional ideals of R. A star operation on R is then defined as an application $\star : \mathfrak{F}(R) \to \mathfrak{F}(R)$: $I \mapsto I^\star$, such that for all $I, J \in \mathfrak{F}(R)$ and $x \in K \setminus \{0\}$:

($\star_1$) $R^\star = R$ and $(xI)^\star = xI^\star$;

($\star_2$) $I \subseteq I^\star$, and $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;

($\star_3$) $I^{\star\star} := (I^\star)^\star = I^\star$.

A star operation $\star$ is called *endlich arithmetisch brauchbar* (in brief, *e.a.b.*) if for each finitely generated $I, J, H \in \mathfrak{F}(R)$, $(IJ)^\star \subseteq (IH)^\star$ implies $J^\star \subseteq H^\star$. Given an e.a.b. star operation the *Kronecker function ring of R with respect to $\star$* is defined by:

$$\mathrm{Kr}(R, \star) := \{f/g \mid f, g \in R[X],\ g \neq 0, C(f)^\star \subseteq C(g)^\star\},$$

where $C(f)$ denotes the content of the polynomial $f(X)$ (i.e., $C(f)$ is the ideal of R generated by the coefficients of f). It is known that $\mathrm{Kr}(R, \star)$ is a Bézout domain with quotient field $K(X)$ such that $\mathrm{Kr}(R, \star) \cap K = R$ (see Theorem 1.1.9).

Given any two star operations $\star_1$ and $\star_2$ on R, we say that $\star_1$ and $\star_2$ are *equivalent* (and we write $\star_1 \sim \star_2$) if they agree on finitely generated ideals. If $\star_1$ and $\star_2$ are e.a.b. it is not hard to see that $\star_1 \sim \star_2$ if and only if $\mathrm{Kr}(R, \star_1) = \mathrm{Kr}(R, \star_2)$ (cf. Remark 1.1.14).

Although a celebrated theorem by W. Krull states that an integrally closed domain can be written as an intersection of its valuation overrings, there are in general many possible representations of an integrally closed domain by means of its valuation overrings. To each such representation can be associated an e.a.b. star operation (see Theorem 1.1.8). As a consequence of Krull's intersection theorem, an integral domain admits an e.a.b. star operation if and only if it is integrally closed.

Let us denote by $\mathrm{Zar}(R)$ the set of valuation rings of K containing R. Suppose $\Sigma \subseteq \mathrm{Zar}(R)$ is such that $R = \bigcap_{V \in \Sigma} V$. Then the application $I \mapsto I^{\star_\Sigma} := \bigcap_{V \in \Sigma} IV$ is an e.a.b. star operation (see Theorem 1.1.8). Moreover Theorem 1.1.15 shows that an e.a.b. star operation on R is equivalent to a star operation $\star_\Sigma$ for a suitable $\Sigma \subseteq \mathrm{Zar}(R)$.

Hence, equivalence classes of e.a.b. star operations, and consequently the number of distinct Kronecker function rings, are closely related to representations of R as intersection of its valuation overrings.

Suppose $\Sigma \subseteq \mathrm{Zar}(R)$ is such that $\bigcap_{V \in \Sigma} V = R$, we say that $W \in \Sigma$ is *irredundant* for the representation if $\bigcap \{V \in \Sigma : V \neq W\} \supsetneq R$. If every $W \in \Sigma$ is irredundant then we say $\bigcap_{V \in \Sigma} V (= R)$ is an *irredundant representation* of R (cf. Definition 2.1.2).

In [23, Theorem 1.7], R. Gilmer and W. Heinzer proved that if a Prüfer domain has an irredundant representation, then it is the unique such representation. We recall their result in Theorem 2.1.3.

The valuation overrings of $\mathrm{Kr}(R,\star)$, for an e.a.b. star operation $\star$, are given by Gaussian extensions of valuation overrings of R (for more details on Gaussian extensions see the discussion at page 13). Hence, denoting by V^b the Gaussian extension to $K(X)$ of a valuation ring V of K, we have that an irredundant representation $\bigcap_{V\in\Sigma} V$ of R lifts to an irredundant representation $\bigcap_{V\in\Sigma} V^b$ of $\mathrm{Kr}(R,\star_\Sigma)$ for some e.a.b. star operation depending on Σ (see Proposition 2.1.7). Reciprocally if a maximal Kronecker function ring of R has an irredundant representation, this representation restricts in K to an irredundant representation of R (see Proposition 2.1.8).

So, if an integrally closed domain with a unique Kronecker function ring has an irredundant representation, then it is the unique such representation (see Corollary 2.1.9). More precisely, an integrally closed domain has at least as many distinct Kronecker function rings as the number of irredundant representations of the domain. In general, an integrally closed domain may not have any irredundant representation and either one or more Kronecker function rings. For an example of such situation it is enough to consider a Prüfer domain that does not have an irredundant representation (cf. Examples 2.1.6 (b)).

If I is an ideal of the integral domain R, an element x in the quotient field K of R is *integral* over I, provided there exist $a_i \in I^i$ such that:

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

By [49, Theorem 1, Appendix 4], the set $\overline{I}$ consisting of the elements of K integral over the ideal I, i.e. the *integral closure* of I, coincides with the ideal $\bigcap_{V\in\mathrm{Zar}(R)} IV$, called the *completion* of I.

An integrally closed domain, then, has a unique Kronecker function ring if and only if every e.a.b. star operation is equivalent to the b-operation (or integral closure of ideals, or completion), which coincide, by using the language introduced before, with the star operation $\star_\Sigma$, where $\Sigma = \mathrm{Zar}(R)$. In particular, an integrally closed domain with a unique Kronecker function ring has a rather simple Zariski space, so that it is not possible to generate non equivalent representations of the domain itself.

K. A. Loper called domains having a unique Kronecker function ring, vacant domains, to point out this "lack" in valuation overrings. We will use the same terminology throughout.

Definition. An integrally closed domain is *vacant* if it has a unique Kronecker function ring.

In Chapter 2 we tackle the problem of characterizing vacant domains from different points of view.

We start by studying the property of having a unique Kronecker function ring for distinguished classes of domains of classical ideal theory. We prove that the most celebrated classes (whose definitions are recalled later), i.e. Krull domains, P$\star$MD's, where $\star$ is any star operation, GCD-domains and more in general generalized GCD-domains, have a unique Kronecker function ring if and only if they are Prüfer.

In Section 2.3 we give characterizations of integrally closed domains having a unique Kronecker function ring, by studing their Zariski space and the integral closure of finitely generated ideals (Theorem 2.3.1). We deduce a new characterization of Prüfer domains as integrally closed domains for which whenever I is a finitely generated ideal, I^b is flat.

Motivated by what is to our knowledge the only example in literature of a non-Prüfer vacant domain (namely, [22, Example 12, Section 32]), Section 2.4 is devoted entirely to the study of pseudo-valuation domains (in brief, PVD's; the definition is recalled in Section 1.4). We prove, in Theorem 2.4.9, that having a unique Kronecker function ring for an integrally closed PVD R with maximal ideal M and associated valuation overring V, is equivalent to the requirement that the transcedence degree of the residue field V/M over R/M is 1. For our proof we added the hypothesis that the field V/M is a finite extension of a transcendental extension of R/M.

In order to build new examples of non-Prüfer vacant domains, in Section 2.5, we study how the uniqueness of the Kronecker function ring is preserved in a pullback diagram. We prove in Theorem 2.5.1, that an integrally closed domain R with a divided prime ideal P (i.e. a prime ideal for which $P = PR_P$), such that R_P is a valuation domain, has a unique Kronecker function ring if and only if so has R/P. In particular it will immediately follow that the CPI extension of a domain R with respect to a prime ideal P (i.e., the pullback $\pi^{-1}(R/P)$, where π is the canonical projection of R_P onto R_P/PR_P), for which R_P is a valuation domain has a unique Kronecker function ring if and only if R/P has a unique Kronecker function ring.

By means of the results collected, we give in Section 2.6 new examples of domains having a unique Kronecker function ring. We construct quasi-local domains with this property which are neither valuation domains, nor pseudo-valuation domains. We give next two ways to build semi-quasi-local domains with a unique Kronecker function ring and a pre-assigned number of maximal ideals such that the localization in each of those is not a valuation domain.

In 2001 two generalizations of the concept of Kronecker function rings were proposed: one by F. Halter-Koch [24], and the other one by M. Fontana and K. A. Loper [15, 16]. Halter-Koch's construction starts from an axiomatization of two properties of the classical Kronecker function ring, whilst the Fontana - Loper approach uses semistar operations (see [37, 38]). According to Halter-Koch's axiomatization, if F is a field and X is an indeterminate over F, an *F-function ring* is a domain $H \subseteq F(X)$ such that:

(Ax1) $X, X^{-1} \in H$;

(Ax2) for all $f \in F[X]$, $f(0) \in fH$.

(cf. Definition 1.3.12).

It is easily seen that the Gaussian extension to $F(X)$ of a valuation ring V of F satisfies (Ax1) and (Ax2). Moreover any intersection of F-function rings is still an F-function ring, so that every Kronecker function ring of a domain with quotient field F is an F-function ring (cf. Remark 1.3.13). In general when H is an F-function ring such that $H \cap F$ has quotient field F, then H is a Kronecker function ring of R (in the sense that there exists an e.a.b. star operation $\star$ on R such that $H = \mathrm{Kr}(R, \star)$). However, the notion of F-function ring does not require that $H \cap F$ has quotient field F. Thus $H \cap F$ can even be a field and, in this case, for lack of fractional ideals, it is not possible to build a star operation on $K := H \cap F$ to make H a Kronecker function ring.

In Chapter 3 we show that by shifting focus to projective model we can introduce ideals in this context and build projective star operations to deduce F-function rings in a way very similar to the classical one. These results are a joint work with O. Heubo.

Let F be a field and R a subring (possibly a field) of F. We denote by $\mathrm{Zar}(F/R)$ the Zariski space of F over R and keep the notation $\mathrm{Zar}(R)$ when F is the quotient field of R. A natural topology, called the *Zariski topolgy*, can be introduced on $\mathrm{Zar}(F/R)$ by taking as an open basis the sets of the form:

$$\mathcal{U}(z_1, \dots, z_n) := \{V \in \mathrm{Zar}(F/R) : z_i \in V,\ \forall i = 1, \dots, n\},$$

for all finite subsets $\{z_1, \dots, z_n\} \subseteq F$. The set of prime ideals $\mathrm{Spec}(R)$ of R, is endowed with the Zariski spectral topology. There is a continuous map:

$$\begin{array}{rcl} \varphi : \mathrm{Zar}(F/R) & \longrightarrow & \mathrm{Spec}(R) \\ V & \longmapsto & P := M_V \cap D, \end{array}$$

where M_V is the (unique) maximal ideal of the valuation ring V.

This construction can be generalized, to include also the case in which R is a subfield of F, by introducing the concept of *complete model*. We denote by $L(F/R)$ the set of quasi-local domains S, such that $R \subseteq S \subseteq F$. A basis for a topology on $L(F/R)$ is given by the following collection of open sets:

$$\mathcal{W}(z_1, \ldots, z_n) := \{S \in L(F/R) : z_i \in S\}, \text{ for all finite subsets } \{z_1, \ldots, z_n\} \subseteq F.$$

With this construction, $\mathrm{Zar}(F/R)$ is a topological subspace of $L(F/R)$ and the induced topology coincides with the Zariski topology on $\mathrm{Zar}(F/R)$ (see the discussion at page 59).

For any domain T, with $R \subseteq T \subseteq F$, $V(T) := \{T_P : P \in \mathrm{Spec}(T)\}$ is a subset of $L(F/R)$, and a topological subspace, homeomorphic to $\mathrm{Spec}(T)$.

A *complete model* of F over R is then a collection $M := \bigcup_{i=0}^{n} V(R_i)$, such that:

(M1) $R \subseteq R_i \subseteq F$ and each R_i is finitely generated over R;

(M2) for each $V \in \mathrm{Zar}(F/R)$ there exists a unique $S \in M$ dominated by V.

If the R_i's are such that $R_i = R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$, for some finite set $\{x_0, \ldots, x_n\} \subseteq F$, the model $M = \bigcup_{i=0}^{n} V(R_i)$ is called a *projective model*. (A quick overview on model and ideals in models will be given in Section 3.1. More details can be found in [49, Ch. VI, § 7] and [1, § 6, p.167]).

If $F := K(Y_1, \ldots, Y_n)$ is a purely transcendental extension of K we can associate to $\mathrm{Zar}(F/K)$ a projective model of F over K, such that all the R_i's can be chosen to be integrally closed and have quotient field F (see [49, Lemma 1, Ch. VI, § 17] and [49, pages 119-120]). It is possible indeed to take the underlying domains as $R_i := K[\frac{Y_1}{Y_i}, \ldots, \frac{Y_n}{Y_i}, \frac{Y_1 \cdots Y_n}{Y_i}]$. With a change of variables we have that $R_i = K[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}]$ and the model $M = \bigcup_{i=0}^{n} V(R_i)$ can be endowed with a scheme structure, precisely it can be viewed as the projective scheme $\mathrm{Proj}(S)$ where $S := K[X_0, \ldots, X_n]$.

With this new setting we can consider coherent sheaves of ideals on $\mathrm{Proj}(S)$ and define a projective star operation as an application from the set of coherent sheaves of ideals into itself, satisfying the same properties of classical star operations. For this construction we can reduce to consider homogeneous ideals of S, in view of the bijection between the set of coherent sheaves of ideals of $\mathrm{Proj}(S)$ and homogeneous saturated relevant ideals of S (see Proposition 3.2.7).

Denoting by $\mathcal{H}(S)$ the set of nonzero homogeneous relevant ideals of S, we shall say that an S-submodule J of F is a *homogeneous fractional ideal* if there exists a homogeneous $f \in S$ such that $fJ \in \mathcal{H}(S)$.

A *projective star operation* on S (or equivalently on $\mathrm{Proj}(S)$) is a mapping $\star : \mathfrak{F}(S) \to \mathfrak{F}(S)$; $I \mapsto I^\star$, such that, whenever $I \in \mathcal{H}(S)$ then $I^\star \in \mathcal{H}(S)$, and for every nonzero homogeneous rational function f (i.e., $f = \frac{g}{h}$ with g and $h \neq 0$ homogeneous polynomials in S) in the quotient field of S and every $I, J \in \mathcal{H}(S)$:

(a) $(f)^\star = (f)$, $(fI)^\star = fI^\star$;

(b) $I \subseteq I^\star$ and if $I \subseteq J$ then $I^\star \subseteq J^\star$;

(c) $I^{\star\star} := (I^\star)^\star = I^\star$.

(cf. Definition 3.3.4).

So we will consider projective star operations as maps from the set of homogeneous fractional ideals into itself satisfying (a), (b) and (c) above.

Projective star operations can be sometimes deduced from star operations on S. More precisely if a star operation $\star$ on S preserves homogeneous ideals (namely $I \in \mathcal{H}(S)$ implies $I^\star \in \mathcal{H}(S)$), then it can be restricted to a projective star operation. We compare then the set of star operations of S with the set of projective star operations on $\mathrm{Proj}(S)$, and give examples of star operations on S preserving or not preserving homogeneous ideals: we observe, in Section 3.2, that the identity d-, the saturation *sat*, the b- and the v- operation have the homogeneous preserving property, while for the $v(I)$'s operations (for which $J_{v(I)} := (I : (I : J))$) it is possible, by choosing a suitable I, that for a homogeneous ideal J, $J^\star$ is not homogeneous (Example 3.3.6 (b)).

Because of the fact that we can move ideals of S to each of the R_i's, by using the *dehomogenization* a_i (see the discussion at page 61 and Theorem 3.2.3), and the other way round, by *homogenization* h (see the discussion at page 62 and Theorem 3.2.4), we prove, in Section 3.3, that each projective star operation $\star$ induces a star operation $\star_i$ on each R_i by defining for each ideal I of R_i:

$$I^{\star_i} := {}^{a_i}\left(({}^{h}I)^\star\right).$$

This process can be reversed. If a compatibility condition between star operations on different R_i's is given (cf. Definition 3.3.12), the projective star operation built from those local compatible star operations $\star_0, \ldots, \star_n$ is defined, for each homogeneous ideal J of S, by:

$$J^\star := {}^{sat}\left(\bigcap_{i=0}^{n} {}^{h}\left(({}^{a_i}I)^{\star_i}\right)\right),$$

where, for each ideal J of S:

$${}^{sat}J := \{f \in S : \forall i = 0, \ldots, n \; \exists m_i \geq 0 \text{ such that } X_i^{m_i} f \in J\}.$$

In Section 3.4, we show that the projective b-operation (resp., v-operation) dehomogenizes to the integral closure of ideals (resp., divisorial closure of ideals) in each R_i, and observe that the saturation sat is a projective star operation which dehomogenizes to the identity star operation (Lemma 3.4.1, Lemma 3.4.2 and Proposition 3.4.3 for the b-operation, Proposition 3.4.5 and Remark 3.4.6 for the v-operation). Furthermore given any projective star operation $\star$, the composition $sat \circ \star$ is again a projective star operation and dehomogenizes to the same star operations obtained locally from $\star$. We obtain then a bijection between the $(n+1)$-tuples of compatible star operations, each of those defined on one of the R_i's and projective star operations of the form $sat \circ -$:

$$\left\{ \begin{array}{c} \{\star_0, \ldots, \star_n\} \\ \star_i = \text{ star operation on } R_i, \\ \star_i \text{ compatible with } \star_j,\ \forall i, j \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} sat \circ \star \\ \star = \text{ projective star operation} \\ \text{on } S \end{array} \right\}$$

The e.a.b. property is then defined for projective star operations in the natural way (cf. Definition 3.5.1). We prove that we can associate to such a projective e.a.b. star operation a projective Kronecker function ring, $\mathrm{PKr}(S, sat \circ \star)$, which turns out to be an F-function ring (cf. Corollary 3.5.6), and has a natural interpretation in terms of valuations of F. In particular, the F-function ring $H = \bigcap_{V \in \mathrm{Zar}(F/K)} V^b$ coincides with the projective Kronecker function ring $\mathrm{PKr}(S, sat \circ b)$ (cf. Corollary 3.5.9).

Chapter 1

Background results

This chapter collects the basic background results that will be used throughout Chapter 2 and Chapter 3. To avoid heavy notations and an excess of concepts, some of them are reformulated in a more synthetic version.

For the proofs references are given, except when the used techniques, or results, are of particular interest for our purposes. In this latter case, for the sake of the reader, the complete proof is included.

All rings considered are commutative rings with identity 1. A ring homomorphism φ will always satisfy $\varphi(1) = 1$.

Notation 1.0.1. For a quasi-local domain S, we shall denote by M_S its unique maximal ideal and by k_S its residue field S/M_S.

Given I, J ideals in the integral domain R having quotient field K, we will use the identification:

$$\mathrm{End}(I) = (I : I) = \{x \in K : xI \subseteq I\}.$$

1.1 Star operations and Kronecker function rings

Let R be an integral domain with quotient field K. We shall denote by $\mathfrak{F}(R)$ the set of nonzero fractional ideals of R.

Definition 1.1.1. A *star operation* on R is an application:

$$\begin{array}{rcl} \star : \mathfrak{F}(R) & \longrightarrow & \mathfrak{F}(R) \\ I & \longmapsto & I^\star \end{array}$$

satisfying the following properties for each $I, J \in \mathfrak{F}(R)$ and $x \in K \setminus \{0\}$:

$(\star_1)$ $R^\star = R$ and $(xI)^\star = xI^\star$;

($\star_2$) $I \subseteq I^\star$ and $I \subseteq J \Rightarrow I^\star \subseteq J^\star$;

($\star_3$) $I^{\star\star} := (I^\star)^\star = I^\star$.

Conditions ($\star_1$) and ($\star_2$) imply that if I is an (integral) ideal of R, then $I^\star \subseteq R^\star = R$ is an (integral) ideal too. Moreover if J is a fractional ideal of R, then there exists $x \in R \setminus \{0\}$ such that $xJ = I \subseteq R$. Therefore, by ($\star_1$), $J^\star = (x^{-1}I)^\star = x^{-1}I^\star$, so that the behavior of the star operation $\star$ is completely determined by its action on integral ideals.

Definition 1.1.2. Let $\star$ be a star operation on the integral domain R. A fractional ideal J of R is called a $\star$-*ideal* provided $J^\star = J$.

It is easily seen that, by condition ($\star_3$), J is a $\star$-ideal of R if and only if there exists $H \in \mathfrak{F}(R)$ such that $H^\star = J$.

Definition 1.1.3. Let $\star$ be a star operation on the integral domain R. A $\star$-ideal J of R is called $\star$-*finite* if there exists $H \in \mathfrak{F}(R)$ finitely generated such that $H^\star = J$.

Each finitely generated $\star$-ideal of R is clearly $\star$-finite.

Basic properties of $\star$-ideals are summarized in the following results:

Proposition 1.1.4. ([22, Proposition 32.2]) *Let $\star$ be a star operation on the integral domain R. Then for all $I, J \in \mathfrak{F}(R)$ and for each subset $\{I_\alpha\}$ of $\mathfrak{F}(R)$, we have:*

(a) $(\sum_\alpha I_\alpha)^\star = (\sum_\alpha I_\alpha^\star)^\star$, *if $\sum_\alpha I_\alpha$ is a fractional ideal of R.*

(b) $\bigcap_\alpha I_\alpha^\star = (\bigcap_\alpha I_\alpha^\star)^\star$, *if $\bigcap_\alpha I_\alpha \neq (0)$.*

(c) $(IJ)^\star = (IJ^\star)^\star = (I^\star J^\star)^\star$.

Given a star operation $\star$, the set of $\star$-ideals has in particular the following distinguished properties. Furthermore, each set of ideals $\mathcal{S}$ satisfying those properties yields a star operation $\star$ having $\mathcal{S}$ contained in the set of $\star$-ideals.

Proposition 1.1.5. ([22, (32.3)]) *Let $\star$ be a star operation on the integral domain R. Let $\mathcal{S}$ denote the set of $\star$-ideals of R. The following hold:*

(a) *Each nonzero principal ideal of R can be expressed as an intersection of a set of elements of $\mathcal{S}$.*

(b) *If $J \in \mathcal{S}$ and if $x \in K \setminus \{0\}$, then $xJ \in \mathcal{S}$.*

Proposition 1.1.6. ([22, Proposition 32.4]) *Let $\mathcal{S}$ be a subset of $\mathfrak{F}(R)$ satisfying* (a) *and* (b) *of Proposition 1.1.5. For $J \in \mathfrak{F}(R)$, we define $J^\star$ to be the intersection of the set of elements of $\mathcal{S}$ which contain J. Then $J \to J^\star$ is a star operation on R.*

Proposition 1.1.4 and Proposition 1.1.5 suggest that a special $\star$-multiplication can be defined for $\star$-ideals. In fact, although in general $(IJ)^\star \neq I^\star J^\star$, we have, by (c) of Proposition 1.1.4, that $(I^\star J^\star)^\star = (IJ)^\star$, and we can define $I^\star \times J^\star := (I^\star J^\star)^\star = (IJ)^\star$. Some star operations enjoy good cancellation properties with respect to $\star$-multiplication and form, from our point of view, an important class of star operations.

Definition 1.1.7. Let R be an integral domain. A star operation $\star$ on R is *aritmetisch brauchbar* or, in brief, *a.b.* (resp., *endlich aritmetisch brauchbar*, or, in brief, *e.a.b.*), provided:

$$(IJ)^\star \subseteq (IH)^\star \Longrightarrow J^\star \subseteq H^\star$$

for all $I, J, H \in \mathfrak{F}(R)$, I finitely generated (resp., I, J, H finitely generated).

It follows directly by definition that an a.b. star operation is also e.a.b.. The reverse implication is not true in general. A counterexample has been recently presented by M. Fontana and K. A. Loper in [19].

The following result gives an alternative method to build star operations. This process will be widely used later, in view of the fact that an e.a.b. star operation can be always associated, up to equivalence, to an e.a.b. star operation built as follows.

Theorem 1.1.8. ([22, Theorem 32.5]) *Let R be a domain with quotient field K, and assume $\{R_\alpha\}$ is a family of overrings of R such that $R = \bigcap_\alpha R_\alpha$. If $I \in \mathfrak{F}(R)$, we define $I^\star := \bigcap_\alpha IR_\alpha$. Then the mapping $I \to I^\star$ is a star operation on R and $IR_\alpha = I^\star R_\alpha$ for each $I \in \mathfrak{F}(R)$ and each α. If each R_α is a valuation ring, then this star operation is a.b..*

A star operation associated to a family of valuation overrings $\{V_\gamma\}_{\gamma\in\Gamma}$, as in Theorem 1.1.8, was classically called a *w-operation* (cf. [22, Section 32]). Later the name w-operation was used in a different sense to denote a precise star operation, that will be introduced in Section 1.3. Hence, to avoid confusion and also to emphasize the family Γ to which the chosen valuation domains belong to, we will denote by $\star_\Gamma$ the star operation associated to the family Γ, as in Theorem 1.1.8. We will use the name w-operation with the meaning it has nowadays.

Given a domain R and a polynomial $f \in R[X]$, we denote by $C(f)$ the *content* of f, i.e. the ideal of R generated by the coefficients of f.

Each integral domain, endowed with an e.a.b. star operation, can be associated to a Bézout domain, living in a larger quotient field. Nevertheless, the starting domain is strictly related to such a Bézout domain:

Theorem 1.1.9. ([22, Theorem 32.7]) *Let R be an integral domain with quotient field K. Let $\star$ be an e.a.b. star operation on R. Let*

$$\mathrm{Kr}(R,\star) := \{f/g \mid f, 0 \neq g \in R[X],\ C(f)^\star \subseteq C(g)^\star\}.$$

Then:

(a) $\mathrm{Kr}(R,\star)$ *is a domain with identity with quotient field $K(X)$;* $\mathrm{Kr}(R,\star) \cap K = R$.

(b) $\mathrm{Kr}(R,\star)$ *is a Bézout domain.*

(c) *If I is a finitely generated ideal of R, then $I^\star = I\mathrm{Kr}(R,\star) \cap R$.*

If $\star$ is an e.a.b. star operation on the domain R, $\mathrm{Kr}(R,\star)$ is called the *Kronecker function ring of R with respect to the star operation $\star$ and the indeterminate X.*

Since a Kronecker function ring is a Bézout domain, hence an integrally closed domain, by (a) of Theorem 1.1.9 the following corollary is straightforward.

Corollary 1.1.10. ([22, Corollary 32.8]) *If an integral domain R admits an e.a.b. star operation, then R is integrally closed.*

It is easily seen that the converse of Corollary 1.1.10 is also true. For, let R be an integrally closed domain and, as in the Introduction, denote by $\mathrm{Zar}(R)$ the set of valuation overrings of R (the set, or more precisely, the topological space $\mathrm{Zar}(R)$, will be widely investigated in Section 1.2). Consider the mapping from $\mathfrak{F}(R)$ into itself that associates to $I \in \mathfrak{F}(R)$ the fractional ideal $I^b := \bigcap_{V \in \mathrm{Zar}(R)} IV$. Since R is integrally closed $R^b = \bigcap_{V \in \mathrm{Zar}(R)} V = R$ and, by Theorem 1.1.8, b is an e.a.b. star operation on R. This construction corresponds to a well-known star operation, called the b-operation (see Examples 1.1.17 (b)).

It follows that an integral domain R admits an e.a.b. star operation if and only if b is a star operation on R. Hence for each integrally closed domain we can consider the Kronecker function ring $\mathrm{Kr}(R,b)$ and, whenever we assume an integral domain to admit a Kronecker function ring, we need that the domain is integrally closed, by Corollary 1.1.10.

Given a valuation v of a field K with values in an ordered abelian group Γ, if X is an indeterminate for K, the mapping

$$\begin{array}{rcl} w : K[X] & \longrightarrow & \Gamma \cup \{\infty\} \\ f = \sum_{i=0}^{n} a_i X^i & \longmapsto & w(f) := \begin{cases} \infty & \text{if } f = 0; \\ \min_{0 \leq i \leq n}(v(a_i)) & \text{otherwise.} \end{cases} \end{array}$$

induces a valuation, v^b, of $K(X)$ by defining for each $f, g \in K[X] \setminus \{0\}$, $v^b(f/g) := w(f) - w(g)$. The valuation v^b is called the *Gaussian extension*, or the *trivial extension*, of v to $K(X)$ and enjoys the following properties:

(a) v^b extends the valuation v, namely the restriction $v^b \mid_K = v$;

(b) $v^b(X) = 0$;

(c) the class $\overline{X}$ of the residue field of v^b is transcendental over the residue field of v.

For more details on the trivial extension of a given valuation the reader may refer to [11, Section 2] and [22, Section 18].

The following results show how the valuation overrings of an integrally closed domain R are related to the valuation overrings of a Kronecker function ring of R.

Theorem 1.1.11. ([22, Theorem 32.10]) *Let R be an integrally closed domain with quotient field K and let $\star$ be an e.a.b. star operation on R, with Kronecker function ring* $\mathrm{Kr}(R, \star)$*. If W is a valuation overring of* $\mathrm{Kr}(R, \star)$*, then W is the trivial extension of $W \cap K$ to $K(X)$.*

Theorem 1.1.12. ([22, Theorem 32.11]) *Let R be an integrally closed domain with quotient field K and let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a family of valuation overrings of R such that $R = \bigcap_{\lambda \in \Lambda} V_\lambda$. Let $\star_\Lambda : I \mapsto I^{\star_\Lambda}$ be the e.a.b. star operation induced by the family $\{V_\lambda\}_{\lambda \in \Lambda}$ (as in Theorem 1.1.8). Let* $\mathrm{Kr}(R, \star_\Lambda)$ *be the Kronecker function ring of R with respect to $\star_\Lambda$, and, for each $\lambda \in \Lambda$, let V_λ^b denote the trivial extension of V_λ to $K(X)$. Then* $\mathrm{Kr}(R, \star_\lambda) = \bigcap_{\lambda \in \Lambda} V_\lambda^b$.

The problem to determine the number of Kronecker funciton rings admitted by an integrally closed domain cannot be solved just by looking at the number of distinct e.a.b. star operations on the domain. In fact, different e.a.b. star operation may induce the same Kronecker function ring. It is possible anyway to introduce, on the set of star operations on a given domain, an equivalence relation, which, restricted to e.a.b. star operations collects together star operations defining the same Kronecker function ring.

Definition 1.1.13. Let $\star_1$ and $\star_2$ be star operations on the domain R. Then $\star_1$ is *equivalent* to $\star_2$, and we write $\star_1 \sim \star_2$, if for each finitely generated ideal I of R, $I^{\star_1} = I^{\star_2}$.

Remark 1.1.14. ([22, Remark 32.9]) It is easily seen that if $\star_1$ and $\star_2$ are e.a.b. star operations on the (integrally closed) domain R, then $\star_1 \sim \star_2$ if and only if $\mathrm{Kr}(R, \star_1) = \mathrm{Kr}(R, \star_2)$. For suppose $\star_1 \sim \star_2$, and let $f, 0 \neq g \in R[X]$. The content $C(f)$ (resp., $C(g)$) is a finitely generated ideal of R, hence, by assumption, $C(f)^{\star_1} = C(f)^{\star_2}$ (resp., $C(g)^{\star_1} = C(g)^{\star_2}$). Therefore

$$f/g \in \mathrm{Kr}(R, \star_1) \Leftrightarrow C(f)^{\star_1} \subseteq C(g)^{\star_1} \Leftrightarrow C(f)^{\star_2} \subseteq C(g)^{\star_2} \Leftrightarrow f/g \in \mathrm{Kr}(R, \star_2),$$

so that $\mathrm{Kr}(R, \star_1) = \mathrm{Kr}(R, \star_2)$.

For the converse it is enough to apply (c) of Theorem 1.1.9 to conclude that, for each finitely generated ideal I of R, $I^{\star_1} = I\mathrm{Kr}(R, \star_1) \cap K = I\mathrm{Kr}(R, \star_2) \cap K = I^{\star_2}$.

Furthermore, as shown in the following theorem, in each equivalence class of e.a.b. star operations there is a distinguished representative of the form $\star_\Sigma$.

Theorem 1.1.15. ([22, Theorem 32.12]) *Each e.a.b. star operation on an integrally closed domain R is equivalent to a star operation of the form $\star_\Sigma$ for some $\Sigma \subseteq \mathrm{Zar}(R)$.*

We introduce now some of the most commonly used star operations. Other examples require more advanced tools and will be given in Section 1.3. We start with a result that yields a whole family of star operations.

Proposition 1.1.16. ([28, Proposition 3.2]) *Let R be an integral domain and I an ideal of R. If $(I : I) = R$, then the mapping $J \mapsto J^{v(I)} := (I : (I : J))$ is a star operation on R.*

Examples 1.1.17. (a) The identity star operation d, such that, for each $I \in \mathfrak{F}(R)$, $I^d := I$ is trivially a star operation for any domain R.

(b) The *integral closure* b, for which $I^b := \bigcap_{V \in \mathrm{Zar}(R)} IV$, is a star operation on R if and only if R is integrally closed (in fact $R^b = R$ if and only if R is integrally closed). In this case, b is an a.b (hence e.a.b.) star operation by Theorem 1.1.8.

(c) The *divisorial closure* v, for which $I^v := (R : (R : I))$, is a star operation for any domain R (cf. [22, Theorem 34.1 (1)]).

(d) If R is integrally closed, then $(I : I) = R$ for each nonzero finitely generated ideal I of R (cf. [22, Proposition 34.7]). Hence, by Proposition 1.1.16, every finitely generated ideal I of an integrally closed domain defines a star operation of the form $v(I)$. In this setting, the divisorial closure $v = v(R)$.

Given an integrally closed domain R with quotient field K, the collection of Kronecker function rings of R is a poset with respect to inclusion. Moreover if S_1 and S_2 are Kronecker function rings of R, then each T such that $S_1 \subseteq T \subseteq S_2$ is still a Kronecker function ring of R. The set of Kronecker function rings of each integrally closed domain R has a minimum, namely $\mathrm{Kr}(R, b)$, but may not have any maximum element. The following results summarize all these properties.

Theorem 1.1.18. ([22, Theorem 32.15]) *Let R be an integrally closed domain with quotient field K, and let $\mathrm{Kr}(R, b)$ be the Kronecker function ring of R with respect to the b-operation.*

(a) *If S is an integrally closed overring of R, then for each e.a.b. star operation $\star$ on S, the Kronecker function ring $\mathrm{Kr}(S, \star)$ is an overring of $\mathrm{Kr}(R, b)$ such that $\mathrm{Kr}(S, \star) \cap K = S$.*

(b) *If T is an overring of $\mathrm{Kr}(R, b)$, then T is a Kronecker function ring of $T \cap K$.*

By Theorem 1.1.18 (a) has an immediate consequence:

Corollary 1.1.19. ([22, Corollary 32.14]) *Each Kronecker function ring of an integrally closed domain R contains $\mathrm{Kr}(R, b)$, the Kronecker function ring of R with respect to the b-operation.*

It follows from (b) of Theorem 1.1.18 that the set of Kronecker function ring of an integrally closed domain R with quotient field K, is completely determined by those overrings T of $\mathrm{Kr}(R, b)$ such that $T \cap K = R$. Although the set of Kronecker function rings of R has always a minimum and an easy application of Zorn's Lemma shows that maximal elements exist, it is hard in general to determine them. We will give some more details on this problem in Section 2.1, since it is partially related to the representations of the domain R as an intersection of its valuation overrings.

An exception is given by the class of domains we are going to introduce, i.e. the class of *v-domains*. As we will see, such domains have always a Kronecker function ring which is the largest.

Proposition 1.1.20. ([22, Theorem 34.1 (4)]) *Let R be an integral domain with quotient field K. If $\star$ is a star operation on R, then, for each $I \in \mathfrak{F}(R)$, $I^{-1} :=$*

$(R : I)$ is a $\star$-ideal. In particular I^v is a $\star$-ideal and $I^\star \subseteq I^v$. If $I, J \in \mathfrak{F}(R)$, then $I^\star \subseteq J^\star$ implies $I^v \subseteq J^v$. Therefore if the v-operation on R is e.a.b., then the Kronecker function ring of each e.a.b. star operation on R is contained in $\mathrm{Kr}(R, v)$, *the Kronecker function ring of R with respect to the v-operation.*

Definition 1.1.21. An integrally closed domain R such that the v-operation on R is e.a.b. is called a *v-domain*.

Recall that if $R \subseteq S$ is a ring extension, an element $s \in S$ is *almost integral* over R if all powers of s belong to a finite R-submodule of S. The ring R is said *completely integrally closed in S* if every element of S, almost integral over R, is in R. A domain is *completely integrally closed* if it is completely integrally closed in its quotient field.

Examples of v-domains are completely integrally closed domains (cf. [22, Theorem 34.3]) and Prüfer domains (cf. [22, Proposition 32.18]).

1.2 The Zariski space and the space of integrally closed overrings of a domain

Definition 1.2.1. Let R be a domain (possibly a field) contained in a field K. The *Zariski space of R with respect to K* is the set of valuation rings of K containing R:

$$\mathrm{Zar}(K/R) := \{V : V \text{ valuation ring of } K,\ R \subseteq V\}.$$

When K is the quotient field of R, as briefly mentioned in Section 1.1, we will simply write $\mathrm{Zar}(R)$ rather than $\mathrm{Zar}(K/R)$.

Let $V \in \mathrm{Zar}(R)$. It is well-known that $M_V \cap R$ is a prime ideal of R. In general, an application between $\mathrm{Zar}(R)$ and the set of prime ideals of R, $\mathrm{Spec}(R)$, can be considered:

$$\begin{array}{rcl} \varphi : \mathrm{Zar}(R) & \longrightarrow & \mathrm{Spec}(R) \\ V & \longmapsto & M_V \cap R \end{array}$$

The set $\mathrm{Spec}(R)$ is endowed with the Zariski spectral topology, i.e. the topology having as an open basis the complements of the form $D(f) := \mathrm{Spec}(R) \setminus V(f) = \{P \in \mathrm{Spec}(R) : f \notin P\}$, for $f \in R$ (cf. [3, Exercises 15 and 17]).

A natural question is whether there exists a topology on $\mathrm{Zar}(R)$ making φ a continuous map. O. Zariski gave a positive answer to this question introducing the following topology on $\mathrm{Zar}(R)$ (see [49, Ch. IV, § 17]): for each finite subset $\{x_1, \ldots, x_n\}$ of K define

$$\mathcal{U}_R(x_1, \ldots, x_n) := \{V \in \mathrm{Zar}(R) : x_i \in V,\ \forall i = 1, \ldots, n\} = \mathrm{Zar}(R[x_1, \ldots, x_n]).$$

It is easily seen that these sets verify the axioms for an open basis of a topology on $\mathrm{Zar}(R)$.

It is not hard to prove that φ is also surjective and closed (see [22, Theorem 19.6] and [5, Theorem 2.5]). Hence the map φ is close to being an isomorphism, but it is not injective except in rare cases, precisely, in the integrally closed case, if and only if R is a Prüfer domain ([5, Proposition 2.2]).

Although the map φ is not a homeomorphism, $\mathrm{Zar}(R)$ turns out to be a spectral space, meaning that there exists a ring S such that $\mathrm{Zar}(R)$ is homeomorphic to $\mathrm{Spec}(S)$. In [5, Theorem 4.1] it is shown that, for any domain R, $\mathrm{Zar}(R)$ satisfies M. Hochster's necessary and sufficient conditions for a topological space in order to be a spectral space (cf. [30, Proposition 4]). Later, in [7], D. Dobbs and M. Fontana present a homeomorphism between $\mathrm{Zar}(R)$ and $\mathrm{Spec}(\mathrm{Kr}(\overline{R}, b))$, where $\overline{R}$ denotes the integral closure of R. So they find an explicit construction, which works for any integral domain R, of a ring whose prime spectrum is homeomorphic to $\mathrm{Zar}(R)$:

Theorem 1.2.2. ([7, Theorem 2]) *Let R be an integral domain. Then $\mathrm{Zar}(R)$ is homeomorphic to $\mathrm{Spec}(\mathrm{Kr}(\overline{R}, b))$.*

For an integral domain R, the homeomorphism built in [7, Theorem 2], is given by:

$$\begin{array}{rcl} \psi : \mathrm{Zar}(R) & \longrightarrow & \mathrm{Spec}(\mathrm{Kr}(R, b)) \\ V & \longmapsto & M_{V^b} \cap \mathrm{Kr}(R, b) \end{array}$$

It is easily seen that ψ is a bijection. In fact, ψ is obtained as the composition of the following applications:

$$\mathrm{Zar}(R) \xrightarrow{\mathrm{id}} \mathrm{Zar}(\overline{R}) \xrightarrow{\sigma} \mathrm{Zar}(\mathrm{Kr}(\overline{R}, b)) \xrightarrow{\varphi} \mathrm{Spec}(\mathrm{Kr}(\overline{R}, b))$$

As observed earlier, since $\mathrm{Kr}(\overline{R}, b)$ is a Bézout (hence a Prüfer) domain the application φ is a homoeomorphism, in particular, a bijection. Moreover Theorem 1.1.11 and Theorem 1.1.12 imply that if the Kronecker function ring of an integrally closed domain is taken with respect to the b-operation, then the correspondence $\sigma : \mathrm{Zar}(\overline{R}) \to \mathrm{Zar}(\mathrm{Kr}(\overline{R}, b))$, $V \mapsto V^b$ is one-one.

Therefore ψ is a bijection. Furthermore φ and the identity id are homeomorphisms. The fact that σ is a homeomorphism too is guaranteed by the following proposition:

Proposition 1.2.3. ([7, Lemma 1]) *If R is an integrally closed domain with quotient field K, then the bijection $\sigma : \mathrm{Zar}(R) \to \mathrm{Zar}(\mathrm{Kr}(R, b))$ is a homeomorphism.*

Proof. It is easily seen that:

$$\sigma(\mathcal{U}_R(x_1,\dots,x_n)) = \mathcal{U}_{\mathrm{Kr}(R,b)}(x_1,\dots,x_n),$$

so that σ is an open map.

As σ is a bijection, it now suffices to show that σ^{-1} maps the typical subbasic open set $\mathcal{U}_{\mathrm{Kr}(R,b)}(\alpha)$ to an open set. The case $\alpha = 0$ is trivial. If $\alpha \neq 0$, we can write in lowest terms:

$$\alpha = \frac{a_0 + a_1 X + \cdots + a_n X^n}{b_0 + b_1 X + \cdots + b_m X^m}$$

with all the a_i's and b_j's in K. Let v denote the valuation associated to the valuation domain V. Ignoring possible vanishing coefficients, we have:

$$\begin{aligned}
&\sigma^{-1}(\mathcal{U}_{\mathrm{Kr}(R,b)}(\alpha)) = \left\{ V \in \mathrm{Zar}(R) : \alpha \in V^b \right\} = \\
&= \{V \in \mathrm{Zar}(R) : \inf\{v(a_i)\} \geq \inf\{v(b_j)\}\} = \\
&= \bigcup_{i,j} \{V \in \mathrm{Zar}(R) : v(a_i) \leq v(a_\lambda)\ \forall\lambda,\ v(b_j) \leq v(b_\mu)\ \forall\mu,\ v(a_i) \geq v(b_j)\} = \\
&= \bigcup_{i,j} \mathcal{U}_R\left(\left\{ \frac{a_\lambda}{a_i} : 1 \leq \lambda \leq n \right\} \cup \left\{ \frac{b_\mu}{b_j} : 1 \leq \mu \leq m \right\} \cup \left\{ \frac{a_i}{b_j} \right\} \right).
\end{aligned}$$

Thus $\sigma^{-1}(\mathcal{U}_{\mathrm{Kr}(R,b)}(\alpha))$, as a (finite) union of open sets, is open and σ is continuous. □

The correspondence established, for an integrally closed domain R, between $\mathrm{Zar}(R)$ and $\mathrm{Spec}(\mathrm{Kr}(R,b))$, led recently to two different generalizations. On one side O. Heubo proved that the Zariski space $\mathrm{Zar}(K/R)$ is spectral, also in the case in which K is not the quotient field of R. In Heubo's result R may even be a field. This result will be recalled in Section 1.3.

In another direction, B. Olberding considered a natural extension Ψ of ψ to larger spaces, i.e. the space of integrally closed overrings of an integral domain R and the space of prime semigroup ideals of the Kronecker function ring $\mathrm{Kr}(R,b)$. He showed that, given an integrally closed domain R, even though Ψ may not be surjective, the homeomorphism is preserved on the image of Ψ.

Definition 1.2.4. [40, Section 2]. Let R be an integral domain with quotient field K. We define the *space of integrally closed overrings* as the set:

$$\mathrm{Over}(R) := \{S : R \subseteq S \subseteq K,\ S \text{ integrally closed}\},$$

endowed with the *Zariski topology* whose basic open sets are of the form:

$$E_R(x_1, \ldots, x_n) := \{S \in \mathrm{Over}(R) : x_i \in S, \forall i = 1, \ldots, n\},$$

for $\{x_1, \ldots, x_n\}$ that ranges over the finite subsets of K.

In this setting $\mathrm{Zar}(R)$ is a subset of $\mathrm{Over}(R)$ and the Zariski topology of $\mathrm{Over}(R)$ induced on $\mathrm{Zar}(R)$ coincides with the Zariski topology on $\mathrm{Zar}(R)$.

In order to generalize Theorem 1.2.2 we need to consider on the set $\mathrm{Over}(R)$ a topology finer than the Zariski topology.

Definition 1.2.5. Let $\mathrm{Over}(R)$ be the set of integrally closed overrings of an integral domain R with quotient field K. The *b-topology* on $\mathrm{Over}(R)$, is the topology generated by declaring as subbasic open sets those of the form:

$$E_R(I, J) := \left\{S \in \mathrm{Over}(R) : I \subseteq J^{b_S}\right\},$$

where I, J are finitely generated R-submodules of K, and b_S denotes the integral closure of J in S.

We will see in Section 1.3 that such a b_S belongs to a large class of applications that generalize star operations, namely semistar operations.

Proposition 1.2.6. ([40, Proposition 2.5]) *Let R be an integrally closed domain. The mapping*

$$\begin{array}{rcl} h : \mathrm{Over}(R) & \longrightarrow & \mathrm{Over}(\mathrm{Kr}(R, b)) \\ S & \longmapsto & \mathrm{Kr}(S, b) \end{array}$$

is a homeomorphism of $\mathrm{Over}(R)$ *onto its image in* $\mathrm{Over}(\mathrm{Kr}(S, b))$ *with respect to the b-topology.*

Proposition 1.2.7. ([40, Corollary 2.8]) *If R is an integrally closed domain, then the subspace topology on* $\mathrm{Zar}(R)$ *induced by the b-topology on* $\mathrm{Over}(R)$ *is the same as the Zariski topology on* $\mathrm{Zar}(R)$.

Moreover the Zariski topology and the b-topology coincide on $\mathrm{Over}(R)$ when R is a Prüfer domain (cf. [40, (2.2)]).

Definition 1.2.8. Let R be an integral domain. A *semigroup ideal* is a subset J of R such that for all $x \in R$, $xJ \subseteq J$. A semigroup ideal J is *prime* if whenever x, y are elements of R such that $xy \in J$, then $x \in J$ or $y \in J$.

As observed in [40, (2.3)] a nonempty subset P of R is a prime semigroup ideal if and only if $R \setminus P$ is a saturated multiplicatively closed subset of R. Hence, P is a prime semigroup ideal of R if and only if P is a union of prime ideals of R.

Definition 1.2.9. Let R be an integral domain. The *space of prime semigroup ideals of* R is the set $S(R)$ of prime semigroup ideals of R together with the topology obtained by declaring as basic open sets:

$$\mathcal{D}_R(x_1, \ldots, x_n) := \{P \in S(R) : x_i \notin P,\ \forall i = 1, \ldots, n\},$$

where $x_1, \ldots, x_n \in R$.

Given an integrally closed overring S of the integral domain R, it is easily seen that the following set is a prime semigroup ideal of R:

$$P_S := \{x \in R : xS \neq S\}.$$

Recall that an integral domain R is a *QR-domain* if every overring $S \neq K$ of R is a quotient ring of R, namely, $S = R_{P_S}$ (cf. [22, Section 27]). A QR-domain is necessarily Prüfer and Bézout domains are QR-domains (cf. [22, Theorem 27.5]).

Lemma 1.2.10. ([40, Lemma 2.10]) *If R is a QR-domain, then the mapping:*

$$\begin{array}{rcl} g : \mathrm{Over}(R) & \longrightarrow & S(R) \\ S & \longmapsto & P_S \end{array}$$

is a homeomorphism with respect to the b-topology.

Hence, by means of Proposition 1.2.6 and Lemma 1.2.10, we have:

Theorem 1.2.11. ([40, Theorem 2.11]) *If R is an integrally closed domain, then the mapping:*

$$\begin{array}{rcl} \Psi : \mathrm{Over}(R) & \longrightarrow & S(\mathrm{Kr}(R, b)) \\ S & \longmapsto & P_{\mathrm{Kr}(S,b)} \end{array}$$

is a homeomorphism of $\mathrm{Over}(R)$ *onto its image in* $S(\mathrm{Kr}(R, b))$ *with respect to the b-topology on* $\mathrm{Over}(R)$.

The previous result will be used in Section 2.3 (precisely in Proposition 2.3.7) to characterize integrally closed domains having a unique Kronecker function ring and such that each integrally closed overring has a unique Kronecker function ring too (we will call such domains *totally vacant domains*).

1.3 Semistar operations and generalizations of Kronecker function rings

The notions of star operation and Kronecker function ring led to two major developements. On one side A. Okabe and R. Matsuda introduced in [37, 38] the more general concept of a *semistar operation*. This approach was followed afterwards by many authors and, in particular, by M. Fontana and K. A. Loper. In [15, 16, 18] they associated a more general kind of Kronecker function ring to semistar operations and the classical Kronecker function rings are completely included in the class introduced by Fontana and Loper.

On the other hand, F. Halter-Koch axiomatized two main properties of Kronecker function rings and defined the notion of *F-function ring* (cf. [24]). We will give an overview on F. Halter-Koch's generalization, since the concept of F-function ring is crucial for our results in Chapter 3. We will recall also the main definitions and properties of semistar operations. In fact, even if this work is focused basically on star operations, we will apply some of the results on semistar operations to the more restrictive case we consider.

It is worth observing that both of the approaches we just mentioned allow to build a function ring for every domain, hence, not necessarily integrally closed, and the operation associated to the function ring is not anymore required to be e.a.b.. The original purpose that motivated these new constructions, in fact, was precisely to overcome those restrictions on the chosen domain and the star operation.

Let R be an integral domain with quotient field K. Let $\overline{\mathfrak{F}}(R)$ denote the set of nonzero R-modules contained in K.

Definition 1.3.1. A *semistar operation* on R is an application: $* : \overline{\mathfrak{F}}(R) \to \overline{\mathfrak{F}}(R)$; $E \mapsto E^*$ satisfying for all $E, F \in \overline{\mathfrak{F}}(R)$ and $x \in K \setminus \{0\}$ the following conditions:

($*_1$) $(xE)^* = xE^*$;

($*_2$) $E \subseteq E^*$, and $E \subseteq F \Rightarrow E^* \subseteq F^*$;

($*_3$) $E^{**} := (E^*)^* = E^*$.

Remark 1.3.2. A star operation $\star$ can always be lifted to a semistar operation $*$, by defining $E^* := E^\star$ if $E \in \mathfrak{F}(R)$, and $E^* := K$ otherwise. Conversely a semistar operation $*$ such that $R^* = R$ can be restricted to the set of nonzero fractional ideals, i.e. its restriction to $\mathfrak{F}(R)$ is well-defined, and induces a star operation on

R. Because of this property such a semistar operation is often called a *(semi)star* operation.

Definition 1.3.3. Given a star operation $\star$ on R, an ideal I of R such that $I^\star = I$ is called a $\star$-*ideal*. A $\star$-*prime* is a $\star$-ideal which is also prime and a $\star$-*maximal* ideal is a maximal element for the set of $\star$-primes.

To each semistar operation $*$ on R (and in particular to each star operation) it is always possible to associate:

(a) a semistar operation of *finite type*, denoted by $*_f$:

$$J^{*_f} := \bigcup \{F^* : F \subseteq J,\ F \in \mathfrak{F}(R) \text{ finitely generated}\},$$

for all $J \in \overline{\mathfrak{F}}(R)$. If $\star$ is a star operation of finite type, an application of Zorn's lemma shows that the set of $\star$-maximal ideals of R, denoted by $\star - \mathrm{Max}(R)$, is nonempty.

(b) A *stable* semistar operation of finite type, denoted by $\tilde{*}$:

$$J^{\tilde{*}} := \bigcup \{(J : I) : I \subseteq R, I^* = R,\ I \text{ finitely generated}\}$$

for all $J \in \overline{\mathfrak{F}}(R)$ (see [16, Remark 2.8]).

(A semistar operation $*$ is called *stable* if for each pair $I, J \in \overline{\mathfrak{F}}(R)$ we have that $(I \cap J)^* = I^* \cap J^*$).

(c) An e.a.b. semistar operation of finite type, denoted by $*_a$:

$$I^{*_a} = \bigcup \{((IH)^* : H^*) : H \in \mathfrak{F}(R) \text{ finitely generated}\},$$

if I is finitely generated, and then, for each nonzero R-module J contained in K:

$$J^{*_a} := \bigcup \{F^{*_a} : F \subseteq J,\ F \in \mathfrak{F}(R) \text{ finitely generated}\}.$$

(see [16, § 4]).

Definition 1.3.4. A star (resp., semistar) operation $\star$ (resp., $*$) on an integral domain R is *of finite type* if $\star_f = \star$ (resp., $*_f = *$).

It is easily seen that $(\star_f)_f = \star_f$ so that $\star_f$ is always of finite type.

Remark 1.3.5. It is worth observing that starting from a star operation $\star$, $\star_f$ and $\tilde{\star}$ are star operations too, whilst $\star_a$ may not be a star operation, but a proper semistar operation. For an example of such situation, suppose R is an integrally closed domain which is not a v-domain. We claim that v_a is not a star operation. Suppose by way of contradiction that $R^{v_a} = R$. This is equivalent to $(I_v : I_v) = R$, for each I finitely generated (by definition of v_a), and, by [22, Theorem 34.6], this holds if and only if R is a v-domain. Hence $R^{v_a} \supsetneq R$ and v_a is a proper semistar operation.

Furthermore, for each $J \in \mathfrak{F}(R)$, the following equality holds:

$$J^{\tilde{\star}} = \bigcap_{P \in \star_f - \mathrm{Max}(R)} JR_P.$$

(cf. [16, Corollary 2.7 and Remark 2.8]).

Examples 1.3.6. Let R be an integral domain, not necessarily integrally closed. Let K be the quotient field of R.

1. The mapping $d : J \mapsto J$, for every $J \in \overline{\mathfrak{F}}(R)$ is the identity semistar operation.

2. The mapping $b : J \mapsto J^b := \bigcap_{V \in \mathrm{Zar}(R)} JV$, for each $J \in \overline{\mathfrak{F}}(R)$ is a semistar operation, called the *b-operation.* If $\overline{R}$ denotes the integral closure of R, we have $R^b = \overline{R}$. If R is integrally closed the restriction of the semistar operation b to the set $\mathfrak{F}(R)$ coincides with the star operation b considered in Examples 1.1.17 (b). Moreover, it is not hard to prove that $b = d_a$, hence b is e.a.b. and of finite type.

3. The mapping $v : J \mapsto J^v := (R : (R : J))$, for each $J \in \overline{\mathfrak{F}}(R)$, defines a semistar operation, called the *v-operation.* Since $R^v = R$ the semistar operation v restricts naturally to the set $\mathfrak{F}(R)$ and the restriction coincides with the star operation v considered in Examples 1.1.17 (c).

4. The mapping $t : J \mapsto J^t := \bigcup\{F^v : F \subseteq J,\ F \in \mathfrak{F}(R)$ finitely generated$\}$, for each $J \in \overline{\mathfrak{F}}(R)$ is the semistar operation of finite type associated to the semistar operation v, and is called the *t-operation.* According to Remark 1.3.5, the restriction of the semistar operation t to the set of fractional ideals of R is a star operation (of finite type).

5. The mapping $w : J \mapsto J^w := \bigcup\{(J : I) : I \subseteq R, I^v = R,\ I$ finitely generated$\}$, for each $J \in \overline{\mathfrak{F}}(R)$ is a semistar operation called the *w-operation.* the w-operation is the stable semistar operation associated to the semistar operation v. Also in this case, as observed in Remark 1.3.5, the restriction of w

to the set $\mathfrak{F}(R)$ is a (stable) star operation and, for each $J \in \mathfrak{F}(R)$, $J^w = \bigcap_{P \in t-\mathrm{Max}(R)} JR_P$.

It is possible to give to the set of star (resp., semistar) operations on an integral domain a structure of a partially ordered set:

Definition 1.3.7. Given $\star_1$ and $\star_2$ star (resp., semistar) operations on R we say that $\star_1$ is *coarser* than $\star_2$, or, equivalently, that $\star_2$ is *finer* than $\star_1$, (and we write $\star_1 \leq \star_2$) if the following equivalent conditions hold for any nonzero (fractional) ideal I of R (resp. R-submodule of K):

(a) $I^{\star_1} \subseteq I^{\star_2}$.

(b) $(I^{\star_1})^{\star_2} = I^{\star_2}$.

(c) $(I^{\star_2})^{\star_1} = I^{\star_2}$.

In particular, given a star operation $\star$ on the domain R, we have: $\star \leq v$, $\star_f \leq t$ and $\tilde{\star} \leq w$ (cf. Theorem 1.1.20).

Remark 1.3.8. Recall that the equivalence between two star operations $\star_1$ and $\star_2$ on the integral domain R holds if for every $I \in \mathfrak{F}(R)$ finitely generated, $I^{\star_1} = I^{\star_2}$. So if we consider the finite type star operations $(\star_1)_f$ and $(\star_2)_f$, associated to $\star_1$ and $\star_2$ respectively, it is easily seen that $\star_1 \sim \star_2$ if and only if $(\star_1)_f = (\star_2)_f$. Furthermore every star operation $\star$ on R is equivalent to the finite type star operation $\star_f$.

As mentioned earlier, semistar operations were introduced to generalize the theory of star operations holding for integrally closed domains to arbitrary domains. Recall that an integral domain R is integrally closed if and only if $(I : I) = R$ for each nonzero finitely generated ideal I of R (cf. [22, Proposition 34.7]).

Definition 1.3.9. Let R be an integral domain, and $*$ a semistar operation on R. The semistar integral closure of R with respect to $*$ is the domain:

$$R^{[*]} := \bigcup \{(I^* : I^*) \mid I \in \mathfrak{F}(R) \text{ finitely generated}\}.$$

As remarked in [10] for any semistar operation $*$, $R^* \subseteq R^{[*]}$ and $R^{*_a} = R^{[*]}$ (see [15, Proposition 4.5]).

Definition 1.3.10. Let R be an integral domain and $*$ a semistar operation on R.

(a) R is *quasi-$*$-integrally closed* if $R^* = R^{[*]}$.

(b) R is *$*$-integrally closed* if $R = R^{[*]}$.

Proposition 1.3.11. ([10, Lemma 4.13]) *Let R be an integral domain and $*$ a semistar operation on R.*

(a) *If $*$ is e.a.b., then $R^* = R^{[*]}$ (i.e. R is quasi-$*$-integrally closed).*

(b) *R is quasi-$\tilde{*}$-integrally closed if and only if $R^{\tilde{*}}$ is integrally closed.*

More details on semistar integral closures and Kronecker function rings with respect to semistar operations can be found, for instance, in [10, 15, 16, 18].

As pointed out in Section 1.1, a Kronecker function ring is a Bézout domain, with good additional properties, that can be associated to a given integrally closed domain. F. Halter-Koch considered the opposite point of view. More precisely he studied domains having the good properties of Kronecker function rings, by axiomatizing two properties enjoyed by each Kronecker function ring, and he obtained a larger class of domains. This class does not depend on a base domain, but just on a field, that, in the classical case, coincides with the quotient field of the base domain R.

Definition 1.3.12. Let F be a field and X an indeterminate for F. An *F-function ring* is a domain $H \subseteq F(X)$ such that:
(Ax1) X and $X^{-1} \in H$;
(Ax2) for each $f \in F[X]$, $f(0) \in fH$.
If $R = H \cap F$, then H is called a *function ring of R*.

Remark 1.3.13. It is easily seen that if H is an F-function ring then (cf. [24, Remarks 1]):

1. If $F_0 \subseteq F$ is a subfield, then $H \cap F_0(X)$ is an F_0-function ring.

2. Every ring H', with $H \subseteq H' \subseteq F(X)$ is also an F-function ring.

3. The intersection of any family of F-function rings is again an F-function ring. Furthermore, if the chosen family consists of F-function rings of R, the intersection is again an F-function ring of R.

4. Every F-function ring of R is contained in a maximal F-function ring of R (by Zorn's Lemma) and contains a (unique) smallest F-function ring of R (namely, the intersection of all of them).

Conditions (Ax1) and (Ax2) are enough to obtain, for each F-function ring, all the properties enjoyed by Kronecker function rings. Thus Kronecker function rings associated to star operations become a subclass of Halter-Koch's F-function rings.

Theorem 1.3.14. ([24, Theorem 2.2]) *Let $H \subseteq F(X)$ be an F-function ring and $R := H \cap F$.*

(a) *If $f = a_0 + a_1X + \cdots + a_nX^n \in F[X]$, then $fH = a_0H + a_1H + \cdots + a_nH$.*

(b) *H is a Bézout domain with quotient field $F(X)$.*

(c) *R is integrally closed in F. In particular R is an integrally closed domain, and the quotient field of R is relatively algebraically closed in F.*

The same thorough classification made for valuation overrings of a Kronecker function ring (cf. Theorem 1.1.11), can be found for F-function rings.

Proposition 1.3.15. *Let F be a field and $H \subseteq F(X)$ be an F-function ring. If $V \in \mathrm{Zar}(F(X)/H)$ then $V = (V \cap F)^b$.*

Proof. The proof goes exactly as the one for Kronecker function rings.

Let V be a valuation overring of H. Denote by v the valuation associated to V, by w the restriction $v\mid_F$ and by v^b the Gaussian extension of w to $F(X)$. We want to prove that $v = v^b$. It is enough to show that if $\alpha := f_0 + f_1X + \cdots + f_nX^n \in F[X]$, we have $v(\alpha) = v^b(\alpha)\,(= (\inf_{0\leq i\leq n}(w(f_i)))$. Since X and X^{-1} are in $H \subseteq V$, then $v(X) = 0$. Thus, for each $f \in F$, $v(fX^i) = v(f) = w(f)$. Therefore $v(\alpha) \geq v^b(\alpha)$.

For each $i = 0, \ldots, n$, by (Ax2), $f_i/\alpha \in H$. Hence $v(\alpha) \leq v(f_i) = w(f_i)$, for each $i = 0, \ldots, n$. Thus $v(\alpha) \leq v^b(\alpha)$ and equality holds. □

Next we want to point out the greater generality of an F-function ring compared to Kronecker function rings. For this purpose we propose the example that motivated mostly our study on F-function rings (cf. Chapter 3).

Example 1.3.16. Let $K \subseteq F$ be a field extension which is not algebraic. Consider the Zariski space $\mathrm{Zar}(F/K)$. Since $K \subseteq F$ is not algebraic, the algebraic closure of K in F is strictly contained in F. The domain $H := \bigcap_{V\in\mathrm{Zar}(F/K)} V^b$ is an F-function ring. The intersection $H \cap F$ is the algebraic closure $\overline{K}^F$ of K in F. It is straightforward to observe that, in this case, it is not possible to build a star operation $\star$ on $\overline{K}^F$ such that H is a Kronecker function ring of $\overline{K}^F$ in the sense of Section 1.1. In fact, $\overline{K}^F$ being a field, it has just one nonzero ideal, and only a trivial star operation (i.e., the identity).

1.4 Basic properties of domains of classical ideal theory

Part of this work is devoted to a wide ranging study of a new class of domains, namely integrally closed domains having a unique Kronecker function ring. So we thought it was something natural to compare the class we were interested in with distinguished classes of domains, well-known in classical ideal theory. We give here a synthetic overview on these classes and their main properties.

In the literature there can be found a large amount of characterizations for the domains we are going to mention. Hence, in our presentation, we just wish to point out those properties which are relevant for our purpose. For deeper overviews we added some references, which however might not be exhaustive either.

Proposition 1.4.1. ([22, Theorem 22.1]) *For an integral domain R the following conditions are equivalent:*

(a) *Every nonzero finitely generated ideal of R is invertible.*

(b) *For each maximal ideal M of R, R_M is a valuation domain.*

(c) *For each prime ideal P of R, R_P is a valuation domain.*

An integral domain is a *Prüfer domain* if it satisfies one of (equivalently, all) the conditions in Proposition 1.4.1.

Proposition 1.4.2. ([22, Theorem 24.7 and Theorem 26.2]) *For an integrally closed domain R the following are equivalent:*

(a) *R is a Prüfer domain.*

(b) *Each overring S of R is integrally closed.*

(c) *Each overring S of R is a Prüfer domain.*

(d) *Each ideal of R is integrally closed.*

From the point of view of star operations, Prüfer domains have a very simple structure, as shown in the next results.

Proposition 1.4.3. ([22, Proposition 32.18]) *Let R be a Prüfer domain. Each star operation on R is a.b., hence e.a.b., and any two star operations on R are equivalent.*

Proposition 1.4.4. ([22, Proposition 34.12]) *Let R be an integrally closed domain. Then R is a Prüfer domain if and only if $I = I^t$ for each nonzero ideal I of R.*

Therefore Prüfer domains are characterized, amongst integrally closed domains, by the condition that the star operation t is the identity.

Recall that Dedekind domains are integral domains in which each nonzero ideal is invertible. So Dedekind domains are (one-dimensional, Noetherian) Prüfer domains (cf. [22, Theorem 37.1] or [32, Theorem 96]).

More details on Prüfer domains can be found, for instance, in [13, 22, 39].

Since invertibility of ideals has an important role in characterizing domains from an ideal-theoretic point of view, it is natural to ask what may happen when generalizing invertibility: so, what if invertibility with respect to the identity star operation d is replaced by invertibility with respect to a star (resp., semistar) operation? A first answer is given in [22, Section 34] for the special case of the v-operation. Then, in [14] and [20], the case of an arbitrary semistar operation is examined.

Definition 1.4.5. Let R be an integral domain and $\star$ (resp., $*$) a star (resp., semistar) operation on R. A nonzero ideal I of R is called *$\star$-invertible* if $(II^{-1})^\star = R$ (resp., $(II^{-1})^* = R^*$).

It turns out that, by means of $\star$-invertibility it is possible to define a new class of Prüfer-like domains, i.e. Prüfer $\star$-multiplication domains.

Definition 1.4.6. Let $\star$ be a star operation (resp., a semistar operation) on a domain R; R is a *Prüfer $\star$-multiplication domain* (in brief, a *P$\star$MD*) if each finitely generated ideal I of R, is $\star_f$-invertible. (cf. for instance [14] and [20]).

So Definition 1.4.6 includes, in the special case $\star = v$, the well-known class of PvMD's (cf. for instance [22, Section 34]) and, in the case $\star = d$, one has back the definition of Prüfer domain. Furthermore, because of the fact that the set of star (more generally semistar) operations on an integral domain is a poset having v as maximum, if R is a P$\star$MD, then it is also a PvMD (the proof is straightforward, and will be given in more details in the proof of Proposition 2.2.8).

Definition 1.4.7. An integral domain R is a *finite conductor* domain (in brief, an *FC-domain*) if for each $a, b \in R$ the intersection $aR \cap bR$ is finitely generated (cf. [47]).

Definition 1.4.8. An integral domain R is a *generalized GCD-domain* (in brief, a *gGCD-domain*) if the intersection of two integral invertible ideals is invertible, or, equivalently, if the intersection of two principal ideals is invertible (cf. [2]).

It is clear by definition that a gGCD-domain is an FC-domain.

Remark 1.4.9. In [47, Theorem 2] is shown that an integrally closed FC-domain is a PvMD, hence also an integrally closed gGCD-domain is a PvMD. So, dealing with integrally closed domains, we can reduce to the case of a PvMD in each of the three cases above.

Prüfer domains generalize valuation domains by globalization, i.e. a Prüfer domain is locally a valuation domain. A different generalization is given by pseudo-valuation domains. This class of domains shares many properties with valuation domains and each pseudo-valuation domain is associated, in a sense that will be clarified later, to a valuation domain.

Definition 1.4.10. Let R be a domain with quotient field K. A prime ideal P of R is called *strongly prime* if whenever $xy \in P$ for $x, y \in K$ then either $x \in P$ or $y \in P$.

Definition 1.4.11. An integral domain R is called a *pseudo-valuation domain*, (in brief, *PVD*) if every prime ideal of R is strongly prime.

A PVD is quasi-local, more precisely its prime spectrum is linearly ordered (cf. [25, Corollary 1.3]). Furthermore it is easily seen that each valuation domain is a PVD. For suppose V is a valuation domain, P is a prime ideal of V, and $xy \in P$, with x, y in the quotient field of V. Then, if x and y are in V, either x or y is in P, because P is prime. If $x \notin V$, then $x^{-1} \in V$, and $y = (xy)x^{-1} \in P$.

Theorem 1.4.12. ([25, Theorem 2.7]) *For a quasi-local domain (R, M_R) the following conditions are equivalent:*

(a) *(R, M_R) is a PVD;*

(b) *R has a (unique) valuation overring V with maximal ideal M_R;*

(c) *There exists a valuation overring V in which every prime ideal of R is also a prime ideal of V.*

It is not hard to see that, even though, given a PVD R, the valuation domain V associated to R is unique, the converse is not true. So, starting form a valuation domain V, we can possibly have many different PVD's with V as associated valuation domain. In fact, by following the same argument proposed in [25, Example 2.1], if $V = F + M_V$ is a valuation domain and K is a proper subfield of F, the domain $R := K + M_V$ is a PVD and V is its associated valuation domain. Hence, a proper subfield K' of F, $K' \neq K$, yields $K' + M_V$ as another PVD associated to V.

Proposition 1.4.13. ([25, Proposition 2.6]) *Let R be a PVD with maximal ideal M. If P is a nonmaximal prime ideal of R, then R_P is a valuation domain.*

Another way to characterize PVD's is by means of pullback constructions (cf. [12]). In fact, every PVD arises as the pullback of a valuation ring V with a subfield F of the residue field k_V of V over k_V:

$$\begin{tikzcd} R := \pi^{-1}(F) \arrow[r, two heads] \arrow[d, hook] & F \arrow[d, hook] \\ V \arrow[r, two heads, "\pi"] & k_V \end{tikzcd}$$

In fact if R is a PVD with maximal ideal M, then by Theorem 1.4.12, $V = \mathrm{End}(M)$ is the unique valuation overring of R having M as maximal ideal. Then, by choosing $F := R/M$ we have the desired pullback diagram. Conversely if R is the pullback $\pi^{-1}(F)$ then R is quasi-local with maximal ideal M (cf. [12, Proposition 2.1]). So, by Theorem 1.4.12 (b), R is a PVD.

The analogues of Prüfer domains for PVD's are given by those domains which are locally PVD. These domains have been studied by D. Dobbs and M. Fontana in [6].

Definition 1.4.14. An integral domain R is a *locally pseudo-valuation domain* (in brief an *LPVD*) if R_M is a PVD for each maximal ideal M of R.

In [6, Proposition 2.2] is shown that an integral domain R is an LPVD if and only if every prime ideal of R is locally strongly prime, namely, for each $P \in \mathrm{Spec}(R)$, PR_P is strongly prime in R_P.

To conclude we recall that, following H. Matsumura (cf. [35, Section 12]):

Definition 1.4.15. A domain R with quotient field K is a *Krull domain* if $R = \bigcap_{\lambda \in \Lambda} R_\lambda$, where each R_λ is a DVR and the family $\{R_\lambda\}_{\lambda \in \Lambda}$ has finite character (i.e. each nonzero $x \in K$ is invertible in all but finitely many R_λ). The family $\{R_\lambda\}_{\lambda \in \Lambda}$ is called a *defining family* for R.

Remark 1.4.16. It is worth remarking that sometimes the definition of Krull domain includes also the condition that each R_λ is a localization of R at a height-one prime ideal. Actually this additional requirement is not restrictive, in fact, as shown in [35, Theorem 12.3], a Krull domain always admits a defining family consisting of localizations at height-one primes. The reason for including the existence in the definition is because such a defining family is uniquely determined and is a minimal defining family, so that it is sometimes called **the** defining family for the Krull domain R.

A Krull domain, being an intersection of completely integrally closed domains, is itself completely integrally closed. In particular each Krull domain is a v-domain. Moreover the following characterization will be very useful:

Proposition 1.4.17. ([31]) *An integral domain R is a Krull domain if and only if $(II^{-1})^t = R$ for each nonzero ideal I of R.*

Chapter 2

Integral domains having a unique Kronecker function ring

2.1 Representations of an integrally closed domain

Definition 2.1.1. Let R be an integrally closed domain with quotient field K. Let Σ be a subset of $\mathrm{Zar}(R)$, such that:

$$R = \bigcap_{V \in \Sigma} V.$$

Then we say that Σ is a *representation* of R.

For some classes of domains the possible (or, in some cases, effective) representations admitted has been completely characterized, but it is still of interest to know what happens in the general case. Evidently the more complicated is the set of valuation overrings of the domain, the more varied are representations that can be generated. We will focus on irredundant representations, as we mean in the following definition. In general, representations are involved also in different kinds of problems, for instance, to determine the domains in between a given domain and the quotient field, or related to topological properties of the Zariski spaces (cf. [23, 34, 40, 41, 42]).

Definition 2.1.2. Let Σ be a representation of an integrally closed domain R. An element $W \in \Sigma$ such that $\bigcap\{V \in \Sigma : V \neq W\} \supsetneq R$ is called *irredundant* for the representation Σ. If each $W \in \Sigma$ is irredundant, we say that Σ is an *irredundant representation* of R.

It is well-known that a Krull domain R has always an irredundant representation. This representation is obtained by intersecting the members of the defining family of

R (cf. Remark 1.4.16). Even Krull domains, for which the existence of an irredundant representation is guaranteed, may fail the uniqueness of such representation. For instance a 2-dimensional UFD is a Krull domain and has infinitely many irredundant representations (to generate such representations it is enough to replace one of the DVR's W, with a 2-dimensional valuation domain $V \subsetneq W$).

Prüfer domains have a sort of the opposite property. A Prüfer domain may not have any irredundant representation, but if it does, the representation is unique and the valuation overrings it consists of are characterized in the following way.

Let R be a Prüfer domain. Let $\mathcal{F}$ denote the set of the maximal ideals M of R having the following property: there is a finitely generated ideal I of R such that M is the unique maximal ideal of R containing I.

Theorem 2.1.3. ([23, Theorem 1.7]) *Let R be a Prüfer domain, and let $\{P_\alpha\}$ be a collection of prime ideals of R such that $R = \bigcap_\alpha R_{P_\alpha}$. If the representation $R = \bigcap_\alpha R_{P_\alpha}$ is irredundant, then $\{P_\alpha\} = \mathcal{F}$.*

Corollary 2.1.4. ([23, Corollary 1.8]) *A Prüfer domain R has an irredundant representation if and only if $R = \bigcap_{M \in \mathcal{F}} R_M$.*

Corollary 2.1.5. ([23, Corollary 1.9]) *If a Prüfer domain R has an irredundant representation, then it is unique.*

Examples 2.1.6. (a) A very easy example of a Prüfer domain having an irredundant representation is given by the integers $\mathbb{Z} = \bigcap_{p \in \mathbb{Z}} \mathbb{Z}_{(p)}$, where p ranges over the prime numbers. This is in fact also a Krull domain.

(b) Another example of a Prüfer (in fact a Bézout) domain having an irredundant representation is given by the domain E consisting of the entire functions over $\mathbb{C}$ (i.e., complex functions which are analytic in the whole plane). In this case the family $\mathcal{F}$ consists of those maximal ideals associated to the points of $\mathbb{C}$, called *fixed* maximal ideals. Precisely, for each $z \in \mathbb{C}$, $M_z = (Z - z)$ is a maximal ideal having height one. By [13, Proposition 8.1.1], E is a Bézout domain having infinite dimension and $E = \bigcap_{z \in \mathbb{C}} E_{M_z}$. Now it is not hard to see that this representation is irredundant. For, observe that, given any $\alpha \in \mathbb{C}$, the function $f(Z) := 1/(Z - \alpha)$ belongs to all E_{M_z} with $z \neq \alpha$, so that $f \in \left(\bigcap_{\alpha \neq z \in \mathbb{C}} E_{M_z}\right)$. On the other hand f is not analytic in α, hence $f \notin E$. By the generality of the coiche of $\alpha \in \mathbb{C}$, it follows that the representation $E = \bigcap_{z \in \mathbb{C}} E_{M_z}$ is irredundant.

(c) Let F be a field which is not algebraically closed, and let $F[X_1, \ldots, X_n]$ be the polynomial ring in $n \geq 2$ indeterminates over F. Let Σ be the subset of

$\mathrm{Zar}(F[X_1, \dots, X_n])$ consisting of all the valuation domains having residue field F. Then, by [33, Theorem 1.2], the domain $R := \bigcap_{V \in \Sigma} V$ is a Prüfer domain with quotient field $F(X_1, \dots, X_n)$.

In [41, Example 6.4], B. Olberding proves that such a Prüfer domain has no irredundant representatives. Precisely, each nonzero proper finitely generated ideal I of R is contained in at least two distinct maximal ideals of R, so that the family $\mathcal{F}$ is empty. The assumption that the field F is not algebraically closed is essential.

The Prüfer case has strong consequences also in the case of arbitrary integrally closed domains. In fact, each representation Σ of an integrally closed domain R, can be lifted to a representation of a Prüfer domain, namely the Kronecker function ring $\mathrm{Kr}(R, \star_\Sigma)$.

Proposition 2.1.7. ([23, Proposition 2.1]) *Let R be an integrally closed domain. If R has an irredundant representation Σ, then R has a Kronecker function ring having an irredundant representation, namely* $\mathrm{Kr}(R, \star_\Sigma)$.

Proposition 2.1.8. ([23, Proposition 2.3]) *Let S be a maximal Kronecker function ring of the integrally closed domain R. If S has an irredundant representation then R has an irredundant representation.*

Although each representation Σ of an integrally closed domain R is associated to a Kronecker function ring of R by means of the star operation $\star_\Sigma$, in general two different representations of the same domain may or may not induce the same Kronecker function ring. An interesting case is given by assuming that the integrally closed domain R has a unique Kronecker function ring. In fact, in this case, each representation of R lifts to a representation of the unique Kronecker function ring $\mathrm{Kr}(R, b)$ of R, which is, of course, a Prüfer domain.

Therefore, an integrally closed domain having a unique Kronecker function ring behaves just like a Prüfer domain from the point of view of representations. More precisely:

Corollary 2.1.9. ([23, p. 310]) *Let R be an integrally closed domain having a unique Kronecker function ring. Then R has an irredundant representation if and only if* $\mathrm{Kr}(R, b)$ *has an irredundant representation. Such a representation, if it exists, is unique.*

Remark 2.1.10. More generally, an integrally closed domain R has at least as many Kronecker function rings as the number of its irredundant representations as intersection of valuation overrings. Thus the number of irredundant representations of an integrally closed domain gives a lower bound for the number of Kronecker function rings the domain has. For, if R does not have any irredundant representation, it has anyway $\mathrm{Kr}(R, b)$ as a Kronecker function ring. So that the number of Kronecker function rings of R is always bigger than or equal to 1.

Suppose R has n irredundant representations, with n not necessarily finite. Then, by Proposition 2.1.7, each such representation of R corresponds to an irredundant representation of a Kronecker function ring of R. Now it is enough to observe that since a Kronecker function ring is a Bézout domain, if it has an irredundant representation, then it is unique. Hence each irredundant representation of R is associated to one and only one Kronecker function ring of R. Therefore, the number of Kronecker function rings of R is bigger than or equal to the number of irredundant representations of the domain R.

We will need to refer to the property of having a unique Kronecker function ring repeatedly throughout this work. Hence we thought it was useful to have a name for the property. Recently, K. A. Loper referred to domains having a unique Kronecker function ring as *vacant domains*.

Definition 2.1.11. Let R be an integrally closed domain. Then R is a *vacant domain* if it has a unique Kronecker function ring.

A reason to justify the term *vacant* is that the Zariski space of a domain having a unique function ring is rather simple. So, unlike the large variety of valuation overrings that can be found in the Zariski space of an arbitrary integrally closed domain, domains having a unique Kronecker function ring have a lack from this point of view.

2.2 Vacant domains and domains of classical ideal theory

First of all, we wish to compare the class of vacant domains with other classical classes of integral domains. It is clear that Dedekind domains (and, more in general, generalized Dedekind domains), Bézout domains and Prüfer domains are vacant. We shall see that many distinguished classes of domains, not included in the class of Prüfer domains (e.g., UFD's, Krull domains, GCD-domains, PvMD's and more generally P$\star$MD's), when provided with the additional property of being vacant, fall into the class of Prüfer domains.

We start by remarking the well-known fact that Prüfer domains are vacant.

Remark 2.2.1. Suppose $\star$ is a star operation on an integral domain R and I is an invertible ideal of R. Then, by [22, Lemma 32.17], for each $J \in \mathfrak{F}(R)$ $(IJ)^\star = IJ^\star$. Therefore, in a Prüfer domain, since every finitely generated ideal is invertible, each star operation is e.a.b. and there is a unique equivalence class of star operations, including the identity star operation d (see [22, Proposition 32.18]). So, Prüfer domains have a unique Kronecker function ring.

Since their introduction, star operations first, and semistar operations later, were widely used to characterize whole classes of domains. For instance, as we saw earlier, Prüfer domains can be characterized, amongst integrally closed domains, by the fact that for each nonzero ideal I, $I^t = I$ (cf. Proposition 1.4.4).

Other examples are given by those domains in which the v-operation coincide with the identity, called *divisorial domains* and studied by W. Heinzer in [26]. Or by domains in which the w-operation is the indentity called *DW-domains* (see [36, 44]).

A vacant domain has a unique Kronecker function ring, hence, for such a domain, there is a unique equivalence class of star operation that can be represented by the star operation b. This condition on equivalence classes is not as restrictive as it looks like. The main obstacle is that we cannot say, for an arbitrary vacant domain, how many star operations the equivalence class consists of. Even worse we cannot say which star operations, other than b, are in the class. Nevertheless we can use the technique of comparing star operations to study whether or not given classes of domains, characterized by special behaviors of their ideals under distinguished star operations, are vacant.

However, in the very general case, it is hard to characterize vacant domains by means of star operations.

Proposition 2.2.2. *Let R be an integrally closed domain. If R is vacant then R is a DW-domain.*

Proof. By Proposition 1.3.11 (b) we have that $R = R^w$ is integrally closed if and only if R is quasi-w-integrally closed. Hence $R = R^w = R^{[w]} = R^{w_a}$ so we have that $R^{w_a} = R$ and $(w_a)_{|\mathfrak{F}(R)}$ is an e.a.b. star operation on R. With a little abuse of notation, we will still write w_a instead of its restriction. Since R is vacant necessarily $w_a \sim b$ and, being star operations of finite type, $w_a = b$. By [16, Corollary 4.5], $w \leq \widetilde{w_a} = \widetilde{(w_a)_f}$. Hence $w = \tilde{b} = d$. □

By Remark 2.2.1 and Proposition 2.2.2, it follows that, if R is an integrally closed domain, the following implications hold:

$$R \text{ is a Prüfer domain } \Longrightarrow R \text{ is a vacant domain } \Longrightarrow R \text{ is a DW-domain.}$$

None of the implications above can be reversed. An example of a vacant domain which is not Prüfer is [22, Example 12, p. 409], whilst a DW-domain which is not vacant will be presented at the end of this chapter (Example 2.6.8).

Corollary 2.2.3. *Let R be an integrally closed domain, X an indeterminate for R. Then $R[X]$ is vacant if and only if R is a field.*

Proof. If R is a field the assertion is trivial. Conversely if R is not a field then $R[X]$ is not a DW-domain (see [36, Proposition 2.12]), so by Proposition 2.2.2, $R[X]$ is not vacant. □

Remark 2.2.4. In [44, Theorem 3.7], G. Picozza and F. Tartarone proved that an integrally closed DW-domain is a Prüfer domain if and only if it is an FC-domain. This follows from the fact that an integrally closed finite conductor domain is a PvMD, according to [47, Theorem 2]. Thus, by Proposition 2.2.2, a finite conductor vacant domain is a Prüfer domain, and a vacant PvMD is a Prüfer domain.

Remark 2.2.5. It is worth observing that a domain R for which the v-operation is e.a.b., i.e. a v-domain, is vacant if and only if $b = t$. First of all, note that a v-domain is necessarily integrally closed, having v as an e.a.b. star operation (cf. Corollary 1.1.10).

Suppose R is vacant, then the b-operation is the unique e.a.b. star operation of finite type and t has to be equal to b. Conversely if $b = t$ and $\star$ is any e.a.b. star operation on R, then $b \leq \star_f \leq t$, hence $b = \star_f = t$ and $b \sim \star$. Therefore R is vacant.

Proposition 2.2.6. *If R is a vacant Krull domain (in particular a vacant UFD, or a vacant integrally closed Noetherian domain), then R is a Dedekind domain.*

Proof. By Proposition 1.4.17, R is a Krull domain if and only if $(II^{-1})^t = R$ for each nonzero ideal I of R, and the t-operation is e.a.b.. So if R is vacant $b = t$ and we have that $R = (II^{-1})^t = (II^{-1})^b$. Therefore $R = II^{-1}$ for any nonzero ideal I of R. In fact, suppose by way of contradiction $II^{-1} \subsetneq R$, then $II^{-1} \subseteq M$ for some maximal ideal M of R. So that $(II^{-1})^b \subseteq M^b = M \subsetneq R$, a contradiction. Thus every nonzero ideal of R is invertible and R is a Dedekind domain. □

Proposition 2.2.7. *If R is a vacant generalized GCD-domain (in particular, a vacant GCD-domain), then R is a Prüfer domain.*

Proof. A generalized GCD-domain is finite conductor, then R is a finite conductor vacant domain, which is Prüfer by the discussion in Remark 2.2.4. $\square$

Proposition 2.2.8. *If R is a vacant P$\star$MD, for some star operation $\star$ on R, then R is a Prüfer domain.*

Proof. It is easily seen that if R is a P$\star$MD, for some star operation $\star$ on R, then R is also a PvMD. (For suppose I is a finitely generated ideal of R such that $(II^{-1})^{\star_f} = R$, then $R = (II^{-1})^{\star_f} \subseteq (II^{-1})^t \subseteq R$, so that $(II^{-1})^t = R$ and R is a PvMD.) Therefore, by Remark 2.2.4, a vacant P$\star$MD is a Prüfer domain. $\square$

2.3 Characterizations of vacant domains

Theorem 2.3.1. *Let R be an integrally closed domain with quotient field K. The following are equivalent:*

(a) *R is vacant.*

(b) *Whenever $R = R_1 \cap \cdots \cap R_n$, with R_i integrally closed overrings of R, then $\mathrm{Zar}(R) = \bigcup_{i=1}^n \mathrm{Zar}(R_i)$.*

(c) *Whenever $R = R_1 \cap \cdots \cap R_n$, with R_i integrally closed overrings of R, then $\mathrm{Kr}(R, b) = \mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b)$ (where the b-operation is taken with respect to the appropriate domain).*

(d) *Whenever $R = R_1 \cap \cdots \cap R_n$, with R_i integrally closed overrings of R, then $I^b = (IR_1)^b \cap \cdots \cap (IR_n)^b$ for all finitely generated ideals I of R (where the b-operation is taken with respect to the appropriate domain).*

We will prove that (a)⇒(c)⇒(d)⇒(c)⇒(b)⇒(a).

The equivalence of (a) and (b) is an unpublished result of B. Olberding.

Proof. (a)⇒(c). Observe that since $\mathrm{Kr}(R_i, b) \cap K = R_i$, then $\mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b) \cap K = R_1 \cap \cdots \cap R_n = R$. Hence $\mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b)$ is a Kronecker function ring of R. Since R is vacant by hypothesis, $\mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b) = \mathrm{Kr}(R, b)$.

(c)⇒(d). Suppose whenever $R_1 \cap \cdots \cap R_n = R$ for integrally closed overrings $R_1, \ldots, R_n$ of R, then $\mathrm{Kr}(R, b) = \mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b)$. If I is a finitely generated ideal of R then $I^b = I\mathrm{Kr}(R, b) \cap K$. Furthermore, since $\mathrm{Kr}(R, b)$ is a Bézout domain, $I\mathrm{Kr}(R, b)$ is a principal ideal $f\mathrm{Kr}(R, b)$, for some $f \in I\mathrm{Kr}(R, b)$. Hence:

$$\begin{aligned} I^b &= I\mathrm{Kr}(R, b) \cap K = f\mathrm{Kr}(R, b) \cap K = \\ &= f\left(\mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b)\right) \cap K = \\ &= f\mathrm{Kr}(R_1, b) \cap \cdots \cap f\mathrm{Kr}(R_n, b) \cap K = \\ &= (I\mathrm{Kr}(R_1, b) \cap K) \cap \cdots \cap (I\mathrm{Kr}(R_n, b) \cap K) = \\ &= (IR_1)^b \cap \cdots \cap (IR_n)^b. \end{aligned}$$

Therefore $I^b = (IR_1)^b \cap \cdots \cap (IR_n)^b$.

(d)$\Rightarrow$(c). It is enough to apply (d) to the ideal $I = R$.

(c)$\Rightarrow$(b). Suppose that $\mathrm{Kr}(R, b) = \mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b)$, whenever $R = R_1 \cap \cdots \cap R_n$, with R_i integrally closed overrings of R. Let $V \in \mathrm{Zar}(R)$, then $V^b = (\mathrm{Kr}(R_1, b) \cap \cdots \cap \mathrm{Kr}(R_n, b))_P$ for some prime ideal P of $\mathrm{Kr}(R, b)$. Therefore, denoting by $S := \mathrm{Kr}(R, b) \setminus P$:

$$V^b = \mathrm{Kr}(R_1, b)_S \cap \cdots \cap \mathrm{Kr}(R_n, b)_S \Rightarrow V^b = \mathrm{Kr}(R_i, b)_S, \text{ for some } i,$$

then $V = V^b \cap K \supseteq \mathrm{Kr}(R_i, b) \cap K = R_i$ and $V \in \bigcup_{i=1}^n \mathrm{Zar}(R_i)$.

(b)$\Rightarrow$(a). Suppose by way of contradiction that R is not vacant. Then there exists an e.a.b. star operation $\star$ on R, such that $\mathrm{Kr}(R, \star)$ is a Kronecker function ring of R distinct from $\mathrm{Kr}(R, b)$. Let $\alpha \in \mathrm{Kr}(R, \star) \setminus \mathrm{Kr}(R, b)$. Therefore $\mathrm{Kr}(R, b)[\alpha] \cap K = R$. Let $\mathcal{U}^b(\alpha)$ be the open subset of $\mathrm{Zar}(\mathrm{Kr}(R, b))$ consisting of the valuation overrings of $\mathrm{Kr}(R, b)[\alpha]$. By Proposition 1.2.3, the preimage $\sigma^{-1}(\mathcal{U}^b(\alpha))$ is a finite union of open sets of the form $\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_n)$ in $\mathrm{Zar}(R)$ and

$$\mathcal{U}^b(\alpha) = \sigma(\mathcal{U}(A_1) \cup \cdots \cup \mathcal{U}(A_n)) = \mathcal{U}^b(A_1) \cup \cdots \cup \mathcal{U}^b(A_n).$$

Denoting by $S_i := \overline{R[A_i]}$, it is easily seen that, for each i, $S_i \subseteq \bigcap_{V \in \mathcal{U}(A_i)} V$, so that:

$$S_1 \cap \cdots \cap S_n \subseteq \mathrm{Kr}(R, b)[\alpha] \cap K = R \Rightarrow S_1 \cap \cdots \cap S_n = R.$$

By assumption $\mathrm{Zar}(R) = \bigcup_{i=1}^n \mathrm{Zar}(S_i) = \bigcup_{i=1}^n \mathcal{U}(A_i)$. Therefore, by using the fact that σ is a homeomorphism:

$$\mathrm{Zar}(\mathrm{Kr}(R, b)) = \sigma(\mathrm{Zar}(R)) = \bigcup_{i=1}^n \mathcal{U}^b(A_i),$$

so that $\alpha \in \mathrm{Kr}(R, b)$, which is a contradiction. $\square$

A natural question arising from Theorem 2.3.1 is whether is possible to handle the number of overrings intersecting to R in such characterization. As we are about to see, this is closely related to a more general question, namely: given a vacant domain R, is every integrally closed overring of R vacant too?

Definition 2.3.2. An integrally closed domain R is *m-vacant* if whenever there exist m integrally closed overrings $R_1, \ldots, R_m$ such that $R = R_1 \cap \cdots \cap R_m$, then the Zariski space $\mathrm{Zar}(R) = \bigcup_{i=1}^{m} \mathrm{Zar}(R_i)$.

As an easy Corollary of Theorem 2.3.1 we have that:

Corollary 2.3.3. *An integrally closed domain R is vacant if and only if it is m-vacant for each $m \geq 2$.*

Definition 2.3.4. An integrally closed domain R is *totally vacant* if every integrally closed overring of R is vacant.

Remark 2.3.5. It is easily seen that for an integrally closed domain R n-vacant always implies $(n-1)$-vacant. For if $R = S_1 \cap \cdots \cap S_{n-1}$ with S_i integrally closed overrings of R, let V be any valuation overring of S_1, then $R = S_1 \cap \cdots \cap S_{n-1} \cap V$. By assumption R is n-vacant so if $W \in \mathrm{Zar}(R)$ is not a valuation overring of S_i for all $i = 1, \ldots, n-1$, necessarily $W \supseteq V$. Hence $W \supseteq V \supseteq S_1$, and $\mathrm{Zar}(R) = \bigcup_{i=1}^{n-1} \mathrm{Zar}(S_i)$.

Although we were not able to prove the converse of the previous statement, we will see that it reverses when we consider the whole space of integrally closed overrings of a given integrally closed domain R. We will also give a topological characterization of totally vacant domains. Recall that $\mathrm{Over}(R)$ denotes the set of integrally closed overrings of R. We have the following:

Proposition 2.3.6. *Let R be an integrally closed domain. Then R is totally vacant if and only if every $S \in \mathrm{Over}(R)$ is 2-vacant.*

Proof. $(\Rightarrow)$. This is clear since every integrally closed overring of R is vacant and vacant implies 2-vacant.

$(\Leftarrow)$. Denote by (P_n) the property: "every element in $\mathrm{Over}(R)$ is n-vacant". We will prove then the Proposition by induction on n. The basis (P_2) of the induction is given by the hypothesis. Suppose (P_n) is true, so that every element of $\mathrm{Over}(R)$ is n-vacant. We will show that then every element in $\mathrm{Over}(R)$ is $(n+1)$-vacant. For, let $S \in \mathrm{Over}(R)$ and suppose $S = T_1 \cap \cdots \cap T_{n+1}$. If V is an element of $\mathrm{Zar}(S)$ we need to show that $V \in \mathrm{Zar}(T_i)$ for some $i \in \{1, \ldots, n+1\}$. If $V \in \mathrm{Zar}(T_1)$

we are done, so suppose $V \notin \mathrm{Zar}(T_1)$. By (P_2) it follows that $V \in \mathrm{Zar}(T)$ where $T := T_2 \cap \cdots \cap T_{n+1}$. Since we assumed (P_n) to be true we have $V \in \mathrm{Zar}(T_i)$ for some $i \in \{2, \ldots, n+1\}$, which proves the proposition. $\square$

Let Ψ be the mapping considered in Theorem 1.2.11. A topological characterization of totally vacant domains follows.

Proposition 2.3.7. *Let R be an integrally closed domain. Then R is totally vacant if and only if Ψ is a homeomorphism with respect to the b-topology.*

Proof. The mapping Ψ is obtained as the composition $g \circ h$, where:

$$\begin{array}{rlcrl} h : \mathrm{Over}(R) & \longrightarrow \mathrm{Over}(\mathrm{Kr}(R,b)) & \qquad & g : \mathrm{Over}(\mathrm{Kr}(R,b)) & \longrightarrow S(\mathrm{Kr}(R,b)) \\ S & \longmapsto \mathrm{Kr}(S,b) & & T & \longmapsto P_T \end{array}$$

Since $\mathrm{Kr}(R,b)$ is a Bézout domain, hence a QR-domain, g is a homeomorphism (see Lemma 1.2.10). On the other hand, according to Proposition 1.2.6, h is a homeomorphism of $\mathrm{Over}(R)$, with respect to the b-topology, onto its image in $\mathrm{Over}(\mathrm{Kr}(R,b))$. We need to show that h is surjective if and only if R is totally vacant.

Suppose R is totally vacant and let T be an overring of $\mathrm{Kr}(R,b)$. Note that T is necessarily integrally closed since $\mathrm{Kr}(R,b)$ is a Bézout domain. The integrally closed overring $S := T \cap K$ of R is vacant since R is totally vacant. Hence $T = \mathrm{Kr}(S,b) = h(S)$.

If h is surjective the conclusion is straightforward. $\square$

To conclude this section, we can give the following characterization of Prüfer domains.

Proposition 2.3.8. *The following are equivalent for an integral domain R:*

(a) *R is a Prüfer domain.*

(b) *For each finitely generated ideal I of R, I^b is flat (as an R-module).*

Proof. (a)⇒(b). This is clear because in a Prüfer domain each ideal is flat.

(b)⇒(a). In [48, Corollary 6] Zafrullah characterized generalized GCD-domains with the following property: for each finitely generated ideal I there exists a star operation $\star$, such that $I^\star$ is flat. Hence, if (b) holds, R is a gGCD-domain, which is finite conductor. We prove that the property stated in (b) implies vacant, so that R is a finite conductor vacant domain, and then a Prüfer domain by Remark 2.2.4.

In a recent paper by G. Picozza and F. Tartarone, [43, Theorem 1.4], it is shown that every flat ideal of an integral domain R is a t-ideal. It follows directly that, assuming (b), then for every finitely generated ideal I of R, $I^b = (I^b)^t = I^t$, so that R is vacant, according to Remark 2.2.5. Thus R is a vacant FC-domain, which is a Prüfer domain. $\square$

2.4 Vacant pseudo-valuation domains

The only example in the literature of a vacant domain which is not a Prüfer domain is given in [22, Example 12, Section 32] and is a pseudo-valuation domain. So we wish to characterize pseudo-valuation domains with respect to the property of being vacant.

Lemma 2.4.1. *Let $k \subseteq F$ be a field extension and let $\alpha, \beta, \gamma \in F$ be transcendental over k. If γ is algebraic over both $k(\alpha)$ and $k(\beta)$, then α and β are algebraically dependent.*

Proof. α, β and γ are supposed to be transcendental over k, then γ is algebraic over $k(\alpha)$ (resp. $k(\beta)$) if and only if α (resp. β) is algebraic over $k(\gamma)$. The field extension $k(\alpha) \subseteq k(\alpha, \gamma) \subseteq k(\alpha, \gamma, \beta)$ is algebraic, being a composition of two algebraic extensions. Hence $k(\alpha) \subseteq k(\alpha, \beta)$ is algebraic too, indeed it is contained in the algebraic extension $k(\alpha) \subseteq k(\alpha, \gamma, \beta)$. $\square$

Lemma 2.4.2. *Let R be an integrally closed domain and $P \in \mathrm{Spec}(R)$. Suppose $V := \mathrm{End}(P)$ is a valuation ring with maximal ideal $M_V = P$. Then for every $W \in \mathrm{Zar}(R)$ either $W \subseteq V$ or $V \subseteq W$.*

Proof. Consider the two cases: either $PW = W$ or $PW \neq W$. If $PW = W$ then $V = \mathrm{End}(P) \subseteq \mathrm{End}(PW) = W$.

Let $PW \neq W$ and suppose $W \not\subseteq V$. Then there exists $x \in W \setminus V$. So $x^{-1} \in M_V = P$. But $x, x^{-1} \in W$ and so $1 = x \cdot x^{-1} \in PW \neq W$, a contradiction. Hence $W \subseteq V$. $\square$

Remark 2.4.3. By Lemma 2.4.2 if $\mathrm{End}(P)$ is a valuation overring of R with maximal ideal P and S is an integrally closed overring of R which is not a valuation domain, necessarily $S \subset V := \mathrm{End}(P)$. For if $\mathrm{Zar}(S)$ is the Zariski space of S, then clearly $\mathrm{Zar}(S) \subseteq \mathrm{Zar}(R)$, hence every valuation overring of S is comparable to V. So only two cases are possible: every valuation overring of S contains V, or there exists $W \in \mathrm{Zar}(S)$ such that $W \subseteq V$. In the first case S is an overring of V, and then is a valuation domain. In the second case $S \subseteq W \subseteq V$ as required.

Notation 2.4.4. Let F and R be domains (possibly fields) with $R \subseteq F$. We shall denote by $\overline{R}^F$ the integral closure of R in F. If R and F are fields, with the same notation we mean the algebraic closure of R in F.

Proposition 2.4.5. *Let $R = (R, M_R)$ be a quasi-local integrally closed domain and let $k_R := R/M_R$. Suppose $D := \mathrm{End}(M_R)/M_R$ is an integral domain and $\mathrm{End}(M_R) \neq R$. If there exist $X, Y \in D$ transcendental and algebraically independent over k_R, then R is not vacant.*

Proof. R is quasi-local with maximal ideal M_R, then we can write R as the pullback of the following diagram:

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d, hook] & \frac{R}{M_R} = k \arrow[d, hook] \\ \mathrm{End}(M_R) \arrow[r, two heads] & \frac{\mathrm{End}(M_R)}{M_R} = D \end{tikzcd}$$

with X and Y in $\mathrm{End}(M_R)/M_R$ algebraically independent over k_R. Since R is integrally closed, k_R must be integrally closed in D and $k_R = \overline{k_R(X)}^D \cap \overline{k_R(Y)}^D$ (it follows combining Lemma 2.4.1 and the fact that if an element γ is integer over k_R, then it is, obviously, algebraic over k_R).

It follows that $k_R = \overline{k_R[X]}^D \cap \overline{k_R[Y]}^D$ this intersection being between the two members of the previous equality. By pulling $\overline{k_R[X]}^D$ and $\overline{k_R[Y]}^D$ back we obtain two integrally closed overrings of R, namely $\overline{R[x]}$ and $\overline{R[y]}$ where x and y belong to $V \setminus R$ and map onto X and Y respectively. Moreover $R = \overline{R[x]} \cap \overline{R[y]} = R[x] \cap R[y]$. To prove that $\overline{R[x]}$ (resp. $\overline{R[y]}$) is the pullback of $\overline{k_R[X]}^D$ (resp. $\overline{k_R[Y]}^D$), let x be an element of V that maps onto X, so that $X = x + M_R$, then it is clear that $R[x]$ is contained in the pullback of $k_R[X]$. For the reverse containment it is enough to prove that the preimage of X is all contained in $R[x]$. Suppose $t \in \mathrm{End}(M_R)$ is another element that maps onto X, then $t + M_R = x + M_R$ and $t - x = m \in M_R$. Now we can conlcude that t is in $R[x]$ because $m \in M_R \subseteq R \subseteq R[x]$ and $x \in R[x]$. Then the integral closure of $R[x]$ is the pullback of the integral closure of $k_R[X]$ in K.

Define $R' := R[x^{-1}, y^{-1}]$, then $R \subsetneq R' \subsetneq V$ since $k_R \subsetneq k_R[X^{-1}, Y^{-1}] \subsetneq D$. Observe that since R is quasi-local we can choose exactly x^{-1} (resp. y^{-1}) as the element of $\mathrm{End}(M_R)$ that maps onto X^{-1} (resp. Y^{-1}). For suppose x' and y' are such that $x' \mapsto X^{-1}$ and $y' \mapsto Y^{-1}$, then $xx' \mapsto XX^{-1} = 1$, hence $xx' = 1 + m$, $m \in M_R$ and M_R coincides with the Jacobson radical $\mathrm{Jac}(R)$. Thus $xx' = u$ is invertible in R. Then $R[x'] = R[ux^{-1}] = R[x^{-1}]$, and the same holds for y. Also $x^{-1}, y^{-1} \in \mathrm{End}(M_R)$ since $M_R \subseteq \mathrm{Jac}(\mathrm{End}(M_R))$. For suppose that $x \in \mathrm{End}(M_R)$ and $\mathrm{End}(M_R) = x\mathrm{End}(M_R) + M_R$. Then there exists $y \in \mathrm{End}(M_R)$ and $m \in M_R$ such that $1 = xy + m$. But then $xy = 1 - m$ is invertible in R (since $m \in M_R$), so x is invertible in $\mathrm{End}(M_R)$.

Let P be the preimage of the ideal (X^{-1}, Y^{-1}) of $k[X^{-1}, Y^{-1}]$. P is a prime ideal of R' since its image is a prime ideal of $k_R[X^{-1}, Y^{-1}]$. Then there exists a valuation overring W of R' centered on P, that is such that $x^{-1}, y^{-1} \in M_W$. Hence by taking the representation of $R = \overline{R[x]} \cap \overline{R[y]}$, $W \supseteq R$ (since $W \supseteq R' \supseteq R$), but $W \not\supseteq \overline{R[x]}$ and $W \not\supseteq \overline{R[y]}$. Thus R is not vacant by Theorem 2.3.1. $\square$

As a direct consequence of Proposition 2.4.5, we have the following:

Corollary 2.4.6. *Let $R = (R, M_R)$ be an integrally closed PVD which is not a valuation domain. Let $V := \mathrm{End}(M_R)$. If $\mathrm{trdeg}(V/M_R, R/M_R) \geq 2$ then R is not vacant.*

Definition 2.4.7. Let R, S be integrally closed domains with quotient field K, we denote by $\mathrm{Zar}_S(R) := \{W \in \mathrm{Zar}(R) \,|\, S \not\subseteq W\}$.

We proceed now with a characterization of PVD's with respect to the property of being vacant. The following lemma will be crucial for the proof of main theorem.

Lemma 2.4.8. ([41, Lemma 4.1]) *Let K be a field, and let F be a finitely generated field extension of K of transcendence degree 1. Let A be a proper K-subalgebra of F having quotient field F, and let Σ be a collection of valuation rings containing K and having quotient field F. Suppose that there is a valuation ring U containing K and having quotient field F such that $\left(\bigcap_{V\in\Sigma} V\right) \cap A \subseteq U$. Then $U \in \Sigma$ or $A \subseteq U$.*

Theorem 2.4.9. *Let R be an integrally closed PVD, not a valuation domain, with maximal ideal M and let $V := \mathrm{End}(M)$. Suppose $\mathrm{End}(M)/M$ is finite over a transcendental extension of R/M, then the following are equivalent:*

(a) *R is not vacant;*

(b) $\mathrm{trdeg}(V/M, R/M) \geq 2$*;*

(c) *R has uncountably many Kronecker function rings.*

Proof. Since R is a PVD, then $V = \mathrm{End}(M)$ is a valuation overring of R, and R is the pullback of the following diagram:

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d, hook] & \frac{R}{M} = k \arrow[d, hook] \\ V \arrow[r, two heads, "\pi"] & \frac{V}{M} = k_V \end{tikzcd}$$

In such a diagram R is integrally closed if and only if k is algebraically closed in k_V, more generally every intermedate ring T between R and V is integrally closed if and only if $\pi(T)$ is integrally closed in $k_V = \pi(V)$ (see [21, Theorem 1.2] and observe that since V is a valuation ring the integral closure of R (resp. T) in V coincides with the integral closure of R (resp. T) in its quotient field).

(a)$\Rightarrow$(b). Observe first that when applying Lemma 2.3.1 in this situation, by Remark 2.4.3 it follows that if some $W \in \mathrm{Zar}(R) \setminus (\bigcup_{i=1}^{n} \mathrm{Zar}(S_i))$, then $W \subseteq V$. Without loss of generality we can in fact reduce to the case $S_i \subseteq V$, for each $i = 1, \dots, n$. For suppose that R is not vacant, hence, by Theorem 2.3.1, $R = S_1 \cap \cdots \cap S_n$ for integrally closed overrings S_i of R.

By Remark 2.4.3 each S_i is comparable to V. In particular it is not possible that $V \subseteq S_i$ for each i, because in this case the intersection of the S_i's would contain V. Furthermore if some S_j contains V it can be cancelled from the intersection. Then we can reduce to an intersection $R = S_{i_1} \cap \cdots \cap S_{i_k}$, with each $S_{i_j} \subset V$.

Assume, by way of contradiciton, that $\mathrm{trdeg}(k_V/k) = 1$. Our strategy will be to show that every $W \subset V$ is necessary for any representation of R. So that, whenever $R = S_1 \cap \cdots \cap S_n$, once we have reduced, as explained above, to $S_i \subseteq V$ for each i, necessarily $\mathrm{Zar}(R) = \bigcup_{i=1}^{n} \mathrm{Zar}(S_i)$, and the conclusion will follow by Theorem 2.3.1. Hence, we will focus on just the elements of $\mathrm{Zar}_V(R)$, which correspond, by [12, Theorem 2.4], to the valuation rings of k_V containing k.

Suppose then $k_V = k(X, \alpha_1, \dots, \alpha_n)$, where X is transcendental over k and, for each $i = 1, \dots, n$, α_i is algebraic over $k(X)$. In particular if $U \in \mathrm{Zar}(k_V/k)$, then U extends some element $U' \in \mathrm{Zar}(k(X)/k)$, and U and U' have the same rank (see [22, Theorem 19.16]) so that U is rank one (discrete). Hence $\pi^{-1}(U)$ has dimension $1 + \dim(V)$, and given any two distinct $W_1, W_2 \in \mathrm{Zar}_V(R)$ they are incomparable because they have the same dimension. In fact, by [12, Theorem 2.4], there exist $U_1, U_2 \in \mathrm{Zar}(k(X)/k)$ such that $W_i = \pi^{-1}(U_i)$ for $i = 1, 2$, and $\dim(U_1) = \dim(U_2) = 1$, so that $\dim(W_1) = 1 + \dim(V) = \dim(W_2)$.

Thanks to the bijection established in [12, Theorem 2.4], between $\mathrm{Zar}(k_V/k)$ and $\mathrm{Zar}_V(R)$, it is enough to show that the representation of $k = \overline{k}^{k_V}$ given by all the valuation rings of K is irredundant. Suppose by way of contradiction that there exists $U \in \mathrm{Zar}(k_V/k)$ which is redundant, then $U \supseteq \bigcap_{V \in \mathrm{Zar}(k_V/k)\setminus\{U\}} V$. Let W be any element of $\mathrm{Zar}(k_V/k) \setminus \{U\}$ and denote by $\Sigma_W := \mathrm{Zar}(k_V/k) \setminus \{U, W\}$. Then by Lemma 2.4.8 we have that:

$$U \supseteq \left(\bigcap_{V \in \Sigma_W} V \right) \cap W \Longrightarrow U \supseteq W \quad \text{or} \quad U \in \Sigma_W.$$

Since U and W have the same dimension the containment cannot hold, but by assumption $U \notin \Sigma_W$, hence we have a contradiction. We can conclude that R is vacant, since the unique possible representation of R is given by intersecting all the elements of $\mathrm{Zar}(k_V/k)$ and it is possible to generate just one Kronecker function ring.

(b)$\Rightarrow$(c). Suppose now the transcendence degree of $\mathrm{End}(M)/M$ over R/M is strictly greater than 1.

Denote by $\mathbf{X} := \{X_i\}_{i\in I}$ a transcendence basis of $\mathrm{End}(M)/M = k_V$ over $R/M = k$, hence $k_V = k(\mathbf{X}, \alpha_1, \ldots, \alpha_n)$ where α_1 is algebraic over $k(\mathbf{X})$, and α_j is algebraic over $k_V(\mathbf{X}, \alpha_1, \ldots, \alpha_{j-1})$ for each $j = 2, \ldots, n$.

To prove this implication we will build uncountably many irredundant representations of k of the form $k[\mathbf{X}] \cap V_\gamma$, where γ is a positive irrational real number. This is possible because the transcendence degree of the field extension $V/M \supseteq R/M$ is strictly greater than 1, in fact, as we are about to see, the construction needs at least two distinct fixed variables.

Let γ be a positive irrational real number. Let $G := \bigoplus_{i\in I} \mathbb{R}$, ordered lexicographically. Let X_1, X_2 be fixed elements in $\mathbf{X}$. We define v_γ as a valuation such that:

$$\begin{aligned}
&v_\gamma(X_1) := (-1, 0, \ldots, 0, \ldots)\\
&v_\gamma(X_2) := (-\gamma, 0 \ldots, 0, \ldots)\\
&v_\gamma(X_i) := (0, \ldots, 0, \overset{i-\text{th}}{-\gamma}, 0, \ldots), \quad \text{for each } i \in I,\ i \neq 1, 2.
\end{aligned}$$

Note that, since γ is irrational, the value of v_γ can be easily determined for each $f \in k[\mathbf{X}]$ (and, more generally $f \in k(\mathbf{X})$), because, for each pair $n_1, n_2 \in \mathbb{Z}$ there is always a strict inequality between n_1 and γn_2. Hence:

$$v_\gamma(f) = \left(\min\left\{-\mathrm{ord}_{X_1}(f), -\gamma \cdot \mathrm{ord}_{X_2}(f)\right\}, 0, -\gamma \cdot \mathrm{ord}_{X_3}(f), \ldots\right).$$

We have that $k = k[\mathbf{X}] \cap V_\gamma$, where $k[\mathbf{X}] = \bigcap k[\mathbf{X}]_{(f)}$ and f ranges over the irreducible polynomials of $k[\mathbf{X}]$. It is well-known that the above representation of

$k[\mathbf{X}]$ is irredundant (recall that $k[\mathbf{X}]$ is a UFD) and that V_γ is irredundant in that representation of k. For instance, for each $i \in I$, $X_i \in k[\mathbf{X}]$, but $X_i \notin V_\gamma$. We need to prove that for each f irreducible polynomial of $k[\mathbf{X}]$, the valuation ring $k[\mathbf{X}]_{(f)}$ is irredundant.

Suppose by way of contradiction that for some $f \in k[\mathbf{X}]$:

$$k[\mathbf{X}]_{(f)} \supseteq \bigcap_{g \neq f} k[\mathbf{X}]_{(g)} \cap V_\gamma = k.$$

In this case we have that $1/f \in k[\mathbf{X}]_{(g)}$ for each $g \neq f$, as both f and g are irreducible. Since f is a polynomial only a finite number of indeterminates appear in f, hence without loss of generality we can assume that the only indeterminates appearing in f are $X_1, \dots, X_m$, for some $m \geq 1$. Denoting by $n_i = \mathrm{ord}_{X_i}(f)$, we have:

$$v_\gamma(f) = \left(\min\left\{-n_1, -\gamma n_2\right\}, 0, -\gamma n_3, \dots, -\gamma n_m, 0 \dots\right),$$

Therefore $f \notin V_\gamma$ and $f^{-1} \in V_\gamma$, that is $f^{-1} \in k$ which is a contradiction.

We can now extend the previous representation of k to a representation in valuation rings of k_V. Clearly we have $k = \overline{k[\mathbf{X}]}^{k_V} \cap V_\gamma^{(1)} \cap \dots \cap V_\gamma^{(t)}$ where $V_\gamma^{(j)}$ are all the possible extensions of V_γ to k_V. All the possible extensions of V_γ to k_V are finitely many because k_V is a finite extension of $k(\mathbf{X})$, hence by [11, Corollary 3.2.3] t is less than or equal to the degree of the extension $k_V \supseteq k(\mathbf{X})$.

By [22, Theorem 43.13] we have that $\overline{k[\mathbf{X}]}^{k_V}$ is a Krull domain. Moreover its defining family is given by the extensions of the members in the defining family of $k[\mathbf{X}]$. Therefore, no elements in the extended defining family are redundant, so we could possibly have that some of the $V_\gamma^{(j)}$ is redundant. In any case, we cannot omit all of the $V_\gamma^{(j)}$ to represent k, otherwise when contracting in $k(\mathbf{X})$ we would have $k = k[\mathbf{X}]$ which is clearly not possible. Hence, we have the irredundant representation required.

When γ ranges over the possible positive irrational real numbers, we get uncountably many irredundant representations and uncountably many Kronecker function rings. For, if γ and α are two distinct positive irrational numbers, there exists a rational number $q = \frac{r}{s}$ such that $\gamma < q < \alpha$, because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then

$$v_\gamma\left(\frac{X_1^r}{X_2^s}\right) = \gamma \cdot s - r < 0, \quad v_\alpha\left(\frac{X_1^r}{X_2^s}\right) = \alpha \cdot s - r > 0.$$

Thus $\frac{X_1^r}{X_2^s} \in V_\alpha \setminus V_\gamma$ and $\frac{X_2^s}{X_1^r} \in V_\gamma \setminus V_\alpha$.

(c)$\Rightarrow$(a). If R has uncountably many Kronecker function rings, clearly R has more than one Kronecker function ring and it is not vacant. $\square$

2.5 Pullbacks of vacant domains

As a matter of fact pullback constructions have always been very useful to build examples. So we aim to characterize how the property of being vacant is preserved in some kind of pullback diagrams in order to provide new examples of vacant domains.

Recall that if V is a valuation domain with maximal ideal M_V, then $V = \mathrm{End}(M_V)$. Hence if P is a divided prime ideal of R and R_P is a valuation domain, then, by Remark 2.4.3, every integrally closed overring of R is comparable to $R_P = \mathrm{End}(P)$. In particular every integrally closed overring of R which is not a valuation domain is contained in R_P. So, we have the following:

Theorem 2.5.1. *Let R be an integrally closed domain such that R_P is a valuation domain for some nonmaximal divided prime ideal P of R. Then R is n-vacant if and only if R/P is n-vacant. In particular, R is vacant if and only if R/P is vacant.*

Proof. By hypothesis it follows that $V := R_P = \mathrm{End}(P)$ and the domain R is the pullback of the following diagram:

$$\begin{tikzcd} R \arrow[r, two heads] \arrow[d, hook] & R/P \arrow[d, hook] \\ V := \mathrm{End}(P) \arrow[r, two heads, "\pi"] & k_V \end{tikzcd}$$

where we set, as usual, $k_V := R_P/P$.

By Lemma 2.4.2 every valuation overring of R is comparable to V. As observed in Remark 2.4.3 this means that every integrally closed overring of R is comparable to V.

($\Rightarrow$). Suppose, by way of contradiction, that R/P is not n-vacant. Then, by Theorem 2.3.1, there exist integrally closed overrings $T_1, \ldots, T_n$ of R/P and $W \in \mathrm{Zar}(R/P)$ such that $R/P = T_1 \cap \cdots \cap T_n$ and $W \not\supseteq T_i$, for each $i = 1, \ldots, n$. By [21, Theorem 1.2] we have that $S_i := \pi^{-1}(T_i)$, $i = 1, \ldots, n$ are integrally closed overrings of R and $S_1 \cap \cdots \cap S_n = R$. Since $V = \mathrm{End}(P)$ is a valuation domain and the quotient field of W is k_V, then $W' := \pi^{-1}(W)$ is a valuation domain too (see [12, Theorem 2.4]). Then $W = \pi(\pi^{-1}(W)) = \pi(W') \supseteq \pi(S_i) = \pi(\pi^{-1}(T_i)) = T_i$, a contradiction. Therefore R is not n-vacant.

($\Leftarrow$). Suppose, by way of contradiction, that R is not n-vacant. Then, by Theorem 2.3.1, there exist $S_1, \ldots, S_n \subseteq V$, integrally closed overrings of R, and $W \in \mathrm{Zar}(R)$ such that $R = S_1 \cap \cdots \cap S_n$ and $W \not\supseteq S_i$, for all $i = 1, \ldots, n$. It follows by Lemma 2.4.2 that $W \subseteq V$ since V contains each S_i. Hence P is a prime ideal of W and

W/P is a valuation overring of R/P contained in k_V; that is, W/P is a valuation ring of k_V.

Clearly, for each i, P is a prime ideal of S_i, as $S_i \subseteq \mathrm{End}(P)$, and since S_i is integrally closed so is S_i/P (by [21, Theorem 1.2]). In particular $R/P = (S_1/P) \cap \cdots \cap (S_i/P)$ and $W/P \not\supseteq S_i/P$, for each $i = 1, \ldots, n$. For suppose $x \in S_1 \setminus W$, then $\pi(x) \in (S_1/P) \setminus (W/P)$. Since $x \notin W$ then $x^{-1} \in M_W$ and $\pi(x^{-1}) \in M_W/P$ which is the maximal ideal of the valuation ring W/P. Since $\pi(x^{-1}) = \pi(x)^{-1}$ then $\pi(x) \notin W/P$.

With the same argument, we can prove that for $x_i \in S_i \setminus W$, $\pi(x_i) \in (S_i/P) \setminus (W/P)$. Thus there exist integrally closed overrings $S_1/P, \ldots, S_n/P$ of R/P and a valuation overring W/P of R/P such that $R/P = (S_1/P) \cap \cdots \cap (S_n/P)$ and $W/P \not\supseteq S_i/P$, for each i, and R/P is not n-vacant. $\square$

Corollary 2.5.2. *Let R be an integral domain, P a nonmaximal divided prime ideal of R such that R_P is a valuation domain. Then every integrally closed overring of R is vacant if and only if every integrally closed overring of R/P is vacant.*

Proof. For each integrally closed overring T of R/P, $T = \pi(S) = S/P$ for some integrally closed overring S of R. By applying Theorem 2.5.1 to the diagram:

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[d, hook] & S/P \arrow[d, hook] \\ \mathrm{End}(P) \arrow[r, two heads, "\pi"] & \mathrm{End}(P)/P \end{tikzcd}$$

S is vacant if and only if S/P is vacant. It is straightforward that the overrings of R which are also overrings of $\mathrm{End}(P)$ are vacant, as they are valuation domains. $\square$

Though we could not generalize Theorem 2.5.1 to the case in which R_P is not a valuation domain, we give the following description of the Zariski space of an integrally closed domain R having a divided prime ideal P. We shall see that the correspondence for Zariski spaces is not as good as in the case of prime spectra.

Let R and S be integrally closed domains, with S an overring of R. The Zariski space of R can always be split into the disjoint union $\mathrm{Zar}(S) \cup \mathrm{Zar}_S(R)$. Given an integrally closed domain R with a nonmaximal divided prime ideal P, we will describe $\mathrm{Zar}_{R_P}(R)$ in terms of $\mathrm{Zar}(R/P)$. We will see that, in general, there is not a bijection, unlike the case of prime spectra (see [4, 12] for details), between $\mathrm{Zar}(R/P)$ and $\mathrm{Zar}_{R_P}(R)$.

Lemma 2.5.3. *Let R be an integrally closed domain and P a divided prime ideal of R, let $V \in \mathrm{Zar}(R)$. $V \in \mathrm{Zar}(R_P)$ if and only if one (and only one) of the following holds:*

(a) *$PV = V$;*

(b) *$M_V \cap R = P$.*

Proof. Let $V \in \mathrm{Zar}(R)$. If $PV = V$ then $R_P \subseteq \mathrm{End}(P) \subseteq \mathrm{End}(PV) = V$. Suppose $PV \neq V$, then PV is an ideal of V so that $PV \subseteq M_V$ and $M_V \cap R \supseteq P$. Thus $M_V \cap R = M_V \cap R_P \cap R \subseteq P \cap R = P$.

For the converse, it is enough to observe that if $V \in \mathrm{Zar}(R_P)$ then $M_V \cap R = M_V \cap R_P \cap R \subseteq P$. If $M_V \cap R = P$ there is nothing to prove, so suppose $M_V \cap R = Q \subsetneq P$. Then $M_V \cap R \supseteq PV \cap R \supseteq P$ which is a contradiction, hence $PV = V$. Therefore if $V \in \mathrm{Zar}(R_P)$ then either (a) or (b) holds. $\square$

Proposition 2.5.4. *Let R be an integrally closed domain and P a divided prime ideal of R. Then the following statements hold:*

(a) *if $V \in \mathrm{Zar}_{R_P}(R)$ then $\frac{V \cap R_P}{P} \in \mathrm{Zar}(R/P)$;*

(b) *if $W \in \mathrm{Zar}(R/P)$ there exists $V \in \mathrm{Zar}_{R_P}(R)$ such that $W = \frac{V \cap R_P}{P}$.*

In particular $\mathrm{Zar}(R/P) = \left\{ \frac{V \cap R_P}{P} \mid V \in \mathrm{Zar}_{R_P}(R) \right\}$.

Proof. (a). Suppose $V \in \mathrm{Zar}_{R_P}(R)$, then $R \subseteq V \cap R_P \subsetneq R_P$ and P is a prime ideal of $V \cap R_P$. Let $W := \frac{V \cap R_P}{P}$ and $x \in k_P \setminus W$, we claim that then $x^{-1} \in W$, so that $W \in \mathrm{Zar}(R/P)$. It is clear that k_P is the quotient field of W, since k_P is the quotient field of R/P. Observe that $x \in k_P \setminus W$ if and only if $\pi^{-1}(x) \not\subseteq V \cap R_P$. Hence, by construction, $\pi^{-1}(x) \subseteq R_P$, so that if $t \in \pi^{-1}(x)$ then $t \notin V$. Therefore $t^{-1} \in V$ and $t^{-1} \in R_P$ because $t \in \pi^{-1}(x)$ with $x \neq 0$, hence t is a unit in R_P. Now it is enough to observe that $\pi(t^{-1}) = \pi(t)^{-1} = x^{-1}$.
(b). Let $S := \pi^{-1}(W)$. Then S is quasi-local (integrally closed) and $R \subset S \subset R_P$. Since $W \neq k_P$, then $S \neq R_P$. Let V be a valuation overring of S centered on M_S. We claim that $V \in \mathrm{Zar}_{R_P}(R)$. For suppose by way of contradiction that $V \in \mathrm{Zar}(R_P)$, then $M_V \cap S = M_V \cap R_P \cap S \subseteq P \subset M_S$, a contradiction.

Now $S \subseteq V \cap R_P \subseteq R_P$. We prove that $S = V \cap R_P$ so that $W = \frac{V \cap R_P}{P}$ for some $V \in \mathrm{Zar}_{R_P}(R)$. Suppose by way of contradiction that there exists some $t \in (V \cap R_P) \setminus S$. Then $\pi(t) \notin W$ implies $\pi(t)^{-1} \in M_W$ so that $\pi^{-1}\left(\pi(t)^{-1}\right) \in M_S$. As observed in the proof of part (a) t^{-1} is in particular in $\pi^{-1}\left(\pi(t)^{-1}\right)$ so that

$t^{-1} \in M_S = M_V \cap S$ and $t \in V$, which is a contradiction. Thus $S = V \cap R_P$ and $W = \frac{V \cap R_P}{P}$. $\square$

We give now an explicit example that shows how the correspondence for Zariski spaces is quite far from being a bijection. We propose an integrally closed domain R having a divided nonmaximal prime ideal P, such that:

(1) $R_P = \mathrm{End}(P)$ but is not a valuation domain.

(2) Each $W \in \mathrm{Zar}(R/P)$ admits (at least) two valuation overrings of R, W_1 and W_2 such that $W = \frac{W_i \cap R_P}{P}$, for $i = 1, 2$.

Example 2.5.5. Let R be the pullback of the following diagram:

$$\begin{tikzcd} R := \pi^{-1}(\mathbb{Z}) \arrow[r, two heads] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook] \\ \mathbb{Q}[X,Y]_{(X,Y)} \arrow[r, two heads, "\pi"] & \mathbb{Q} \end{tikzcd}$$

Since $\mathbb{Q}[X, Y]$ is a Krull domain, then so is $R_P := \mathbb{Q}[X, Y]_{(X,Y)}$. Hence $\mathbb{Q}[X, Y]_{(X,Y)}$ is completely integrally closed and $\mathrm{End}(I) = \mathbb{Q}[X, Y]_{(X,Y)}$ for each nonzero ideal I of $\mathbb{Q}[X, Y]_{(X,Y)}$ (see [22, Theorem 34.3]). By construction $PR_P := (X, Y)\mathbb{Q}[X, Y]_{(X,Y)}$ is a divided nonmaximal prime ideal of R.

Consider the two following valuation overrings of $\mathbb{Q}[X, Y]$ and their respective maximal ideals:

$$V_1 := \mathbb{Q}[Y]_{(Y)} + X\mathbb{Q}(Y)[X]_{(X)}, \quad M_{V_1} = Y\mathbb{Q}[Y]_{(Y)} + X\mathbb{Q}(Y)[X]_{(X)}$$
$$V_2 := \mathbb{Q}[X]_{(X)} + Y\mathbb{Q}(X)[Y]_{(Y)}, \quad M_{V_2} = X\mathbb{Q}[X]_{(X)} + Y\mathbb{Q}(X)[Y]_{(Y)}$$

It is easily seen that V_1 and V_2 are not comparable to each other by inclusion. For instance $X/Y \in V_1 \setminus V_2$ and $Y/X \in V_2 \setminus V_1$. Furthermore they are centered on the maximal ideal $(X, Y)\mathbb{Q}[X, Y]$ of $\mathbb{Q}[X, Y]$, so that $V_i \in \mathrm{Zar}(\mathbb{Q}[X, Y]_{(X,Y)})$ and $M_{V_i} \cap \mathbb{Q}[X, Y]_{(X,Y)} = (X, Y)\mathbb{Q}[X, Y]_{(X,Y)}$.

Let $p \in \mathbb{Z}$ be a prime number. Let S be the quasi-local integrally closed overring of R obtained as the pullback $\pi^{-1}(\mathbb{Z}_{(p)})$. Then $R \subset S \subset \mathbb{Q}[X, Y]_{(X,Y)}$.

Consider, for $i = 1, 2$ the valuation domains:

$$\begin{tikzcd} W_i^{(p)} := \pi_1^{-1}(\mathbb{Z}_{(p)}) \arrow[r, two heads] \arrow[d, hook] & \mathbb{Z}_{(p)} \arrow[d, hook] \\ V_i \arrow[r, two heads, "\pi_1"] & \mathbb{Q} \end{tikzcd}$$

More precisely $W_i^{(p)} = \mathbb{Z}_{(p)} + M_{V_i}$, and $M_{W_i^{(p)}} = p\mathbb{Z}_{(p)} + M_{V_i}$.

Note that, for $i = 1, 2$, $W_i^{(p)}$ is a valuation overring of S. According to what we proved in Proposition 2.5.4 (b), any valuation overring U of S centered on M_S is such that $U \cap R_P = S$. Hence if $W_i^{(p)}$ is also centered on the maximal ideal M_S of S, we can conclude that Proposition 2.5.4 (b) holds. But this is clear since $M_{W_i^{(p)}} = p\mathbb{Z}_{(p)} + M_{V_i}$, and $M_S = p\mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[X,Y]_{(X,Y)} = p\mathbb{Z}_{(p)} + (M_{V_i} \cap S)$.

This process can be done for each prime $p \in \mathbb{Z}$, hence for each valuation domain in $\mathrm{Zar}(\mathbb{Z}) = \mathrm{Zar}(R/P)$ there exist at least two valuation overrings $W_1^{(p)}$ and $W_2^{(p)}$ in $\mathrm{Zar}_{R_P}(R)$ such that $\frac{W_i^{(p)} \cap R_P}{P} = \mathbb{Z}_{(p)}$.

Observe that another example of the same kind can be built by choosing $R_P := k[X,Y,Z]_{(X,Y)}$, $R/P := k[Z]$, so that $R = k[Z] + (X,Y)k[X,Y,Z]_{(X,Y)}$. If k is algebraically closed, the construction is then analogous to the one given in Example 2.5.5. It is enough to replace $\mathbb{Q}$ with $k(Z)$ and p with $Z - a$, for $a \in k$.

2.6 Examples

In this section we build some examples of vacant domains. In particular, we give an example of a quasi-local vacant domain which is neither a valuation domain nor a PVD. Then we produce for any $n \geq 1$ a semi-quasi-local vacant domain having exactly n maximal ideals. Moreover the localization of the resulting domain at any of its maximal ideal is not a valuation domain.

We start with some preliminary results which are needed for the subsequent constructions.

Lemma 2.6.1. *Let R be an integrally closed domain. If R_M is vacant for every $M \in \mathrm{Max}(R)$ then R is vacant.*

Proof. Suppose R is not vacant. Then, by Theorem 2.3.1, there exist integrally closed overrings S_i, $i = 1, \dots, n$ of R and $V \in \mathrm{Zar}(R)$ such that $R = S_1 \cap \dots \cap S_n$ and $V \not\supseteq S_i$, for each $i = 1, \dots, n$. Let $P = M_V \cap R$. Then $V \supseteq R_P$ and, hence, $V \supseteq R_M$ for every maximal ideal containing P. So let M be any maximal ideal of R which contains P. We have $V \in \mathrm{Zar}(R_M)$ and $R_M = (S_1 \cap \dots \cap S_n)_{R \setminus M} = (S_1)_{R \setminus M} \cap \dots \cap (S_n)_{R \setminus M}$ with $(S_i)_{R \setminus M}$ integrally closed overrings of R_M for $i = 1, \dots, n$. Clearly $V \not\supseteq (S_i)_{R \setminus M}$ since $V \not\supseteq S_i$. But then R_M is not vacant, which is a contradiction. $\square$

As a direct consequence of the Lemma above we have:

Proposition 2.6.2. *Let R be an integrally closed domain. The following are equivalent:*

(a) *R is locally vacant;*

(b) *every flat overring of R is vacant.*

Proof. By [45, Theorem 2], T is a flat overring of R if and only of for each maximal ideal M of T, $T_M = R_{M \cap R}$. Hence, if R is integrally closed, so is every flat overring T of R.
(a)$\Rightarrow$(b). Suppose R_P is vacant for any $P \in \mathrm{Spec}(R)$. Let T be a flat overring of R. Then $T_M = R_{M \cap R}$ is vacant and, by Lemma 2.6.1, T is vacant.
(b)$\Rightarrow$(a). It is enough to observe that R_P is a flat overring of R for each prime ideal P of R. $\square$

Example 2.6.3 (Quasi-local vacant domain)**.** Let V be a valuation domain with maximal ideal M_V such that its residue field k_V has a valuation ring W of the form $F(X) + M_W$, where F is a subfield of k_V and X is transcendental over F. Such hypothesis are realized for instance if $V = F[X, Y, Z]_{(Z)}$ so that $k_V = F(X, Y)$. Then, as shown in Theorem 2.4.9, the domain $F+M$ is integrally closed and vacant. In fact $F + M$ is a PVD with purely transcendental residue fields extension having transcendence degree 1. Moreover $F + M$ has the same quotient field as W, namely k_V. Let $R := \pi^{-1}(F + M)$:

$$\begin{tikzcd}
R \arrow[r, two heads] \arrow[dd, hook] & F+M \arrow[r, two heads] \arrow[d, hook] & F \arrow[d, hook] \\
& W := F(X)+M \arrow[r, two heads] \arrow[d, hook] & F(X) \\
V \arrow[r, two heads, "\pi"] & k_V &
\end{tikzcd}$$

It is easily seen that $M_V := \mathrm{Ker}(\pi)$ is a prime ideal of R, and $V = \mathrm{End}(M_V)$. Hence we can apply Theorem 2.5.1 to R and conclude that R is a quasi-local vacant domain which is not a PVD. Furthermore R has a (necessarily unique) irredundant representation as intersection of valuation overrings. This representation is the pullback of the unique irredundant representation of $F+M$. Moreover every integrally closed overring of R is vacant too, since every such overring of R is a Prüfer domain. In fact, if S is an integrally closed overring of R either S is a valuation domain, or, by Remark 2.4.3, S is the pullback of an integrally closed overring of $F + M$. Again by Remark 2.4.3, any such overring of $F+M$ is the pullback of an integrally closed domain in between F and $F(X)$, which is necessarily a Prüfer domain. Now, it is enough to apply [12, Theorem 2.4 (3)], to conclude that S is a Prüfer domain too.

So far, we characterized and gave examples of some class of vacant quasi-local domains. In the following examples we build a semi-local vacant domain. We start by constructing, in (a), a 1-dimensional semi-local vacant domain, then, in (b), we use the same process of Example 2.6.3 to obtain a semi-local vacant domain of any finite dimension.

It will be also pointed out that a vacant domain may have integrally closed overrings which are not Prüfer domains. The example we propose first is a vacant domain R having 2 maximal ideals M_1 and M_2. The localization of R at M_i, $i = 1, 2$, is a PVD, which is not a valuation domain.

Recall that a domain T is a *G-domain* if the quotient field K of T is a finitely generated ring extension of T, or, equivalently, if the nonzero prime ideals of T have nonzero intersection ([32, p. 11-12]).

Let D and T be integral domains with quotient field K.

Proposition 2.6.4. [27, Proposition 1.19]. If D has nonzero Jacobson radical J, and T is a G-domain such that T is contained in only a finite number of rank one valuation rings of K, say $V_1, \ldots, V_n$, and if moreover $D \not\subset V_i$ for each i, then D is a localization of $R := D \cap T$.

Proposition 2.6.5. [27, Proposition 1.15]. If D has nonzero Jacobson radical J and T is 1-dimensional quasi-local, $R := D \cap T$ is an irredundant intersection, and D is a localization of R, then T is centered on a maximal ideal M of R and $R_M = T$.

Example 2.6.6 (Semi-quasi-local vacant domain)**.** (a) Let $F \subseteq K$ be a transcendental field extension, and let X be an element of K that is transcendental over F. Suppose that K has two distinct rank one valuation domains of the form $V_1 = F(X) + M_1$ and $V_2 = F(X) + M_2$. Let $R_1 = F + M_1$ and $R_2 = F + M_2$. Observe that R_1 and R_2 are integrally closed PVD. By Theorem 2.4.9, R_1 and R_2 are vacant.

Since V_i has rank one for $i = 1, 2$, by Lemma 2.4.2 we have that V_i is the only rank one valuation ring of K containing R_i, $i = 1, 2$. Moreover the Jacobson radical of R_i is M_i which is, of course, different from zero. M_i is the only nonzero prime ideal of R_i, hence R_i is a G-domain. We can then apply Proposition 2.6.4 to both R_1 and R_2 and conclude that each of them is a localization of $R := R_1 \cap R_2$.

The intersection $R = R_1 \cap R_2$ is irredundant since there are no containments between V_1 and V_2, hence between R_1 and R_2. For suppose by way of contradiction $R_1 \subseteq R_2$, then $V_2 \in \mathrm{Zar}(R_1)$. So that, by Lemma 2.4.2, V_2 is comparable to V_1, a contradiction.

Since R_1 and R_2 were chosen 1-dimensional we are in the hypothesis of Proposition 2.6.5 for both R_1 and R_2 playing the role of T so that there exist two maximal ideals of R, P_1 and P_2, such that $R_{P_1} = R_1$ and $R_{P_2} = R_2$.

The last thing which remains to be shown is that P_1 and P_2 are the only maximal ideals of R, but that is clear because if $d \in R \backslash (P_1 \cup P_2)$ then $d \notin M_1$ and $d \notin M_2$ so that $d^{-1} \in R_{P_1} \cap R_{P_2} = R_1 \cap R_2 = R$. Thus every element of R not in P_1 or P_2 is a unit, and hence P_1 and P_2 are the only maximal ideals of R.

(b) Combining the previous example with Example 2.6.3 we can increase the dimension of the semi-quasi-local domain just obtained. More precisely if R is the semi-quasi-local domain built in (a) and V is a valuation domain with residue field $K = \mathrm{Qf}(R)$ we can consider the following pullback diagram:

$$\begin{tikzcd} R' := \pi^{-1}(R) \arrow[r, two heads] \arrow[d, hook] & R \arrow[d, hook] \\ V \arrow[r, two heads, "\pi"] & k_V = K \end{tikzcd}$$

then R' is vacant (by Theorem 2.5.1), semi-quasi-local and $\dim(R') = \dim(V) + \dim(R) > 1$, where the equality for dimensions holds by [12, Proposition 2.1].

We give next an example showing that, for each $n \geq 2$, it is possible to build a vacant domain R with exactly n maximal ideals, such that the localization of R at any of its maximal ideals is not a valuation domain.

Example 2.6.7. Let K be a field and $X_1, \ldots, X_n, Z$ algebraically independent variables over K. Let $F = K(X_1, \ldots, X_n)$. Let $f_1, \ldots, f_n$ be distinct irreducible elements of $F[Z]$. Then $V_i := F[Z]_{(f_i)}$ is a valuation domain, in fact a DVR, with quotient field $F(Z)$. Moreover if M_i denotes the maximal ideal of V_i, V_i/M_i is isomorphic to F, that is V_i is of the form $F + M_i$ for each i.

Let $F_i := K(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$ and R_i the PVD obtained as the pullback of the following diagram:

$$\begin{tikzcd} R_i := \pi^{-1}(F_i) \arrow[r, two heads] \arrow[d, hook] & F_i \arrow[d, hook] \\ V_i \arrow[r, two heads, "\pi"] & F \end{tikzcd}$$

Each R_i is integrally closed because the field extension $F_i \subseteq F$ is purely transcendental, and hence F_i is algebraically closed in F.

For $i \neq j$ the valuation domains V_i and V_j are incomparable to each other (they are both DVR's); furthermore the residue field of each V_i is F and $F_i \neq F_j$, so $\{F_i\}_{i=1}^n$ is a collection of distinct proper subfields of F. Then [6, Example 2.5] shows that $R := \bigcap_{i=1}^n R_i$ is an integrally closed LPVD with exactly n maximal ideals, say $Q_1, \dots, Q_n$. Also $R_{Q_i} = R_i$ for each $i = 1, \dots, n$, which is vacant by Theorem 2.4.9. Thus by Lemma 2.6.1 follows that R is a vacant domain having exactly n maximal ideals.

To conclude we give the announced example showing that a DW-domain may not be vacant. Recall that the inverse implication holds, thanks to Proposition 2.2.2

Example 2.6.8. Let X, Y and Z be algebraically independent variables over a field k. The integrally closed domain:

$$\begin{tikzcd} R := \pi^{-1}(k) \arrow[r, two heads] \arrow[d, hook] & k \arrow[d, hook] \\ k(X,Y)[Z]_{(Z)} \arrow[r, two heads, "\pi"] & F := k(X,Y) \end{tikzcd}$$

is not vacant by Theorem 2.4.9 but is a DW-domain as it is one-dimensional (see [44, Proposition 2.9]).

2.7 Discussion and questions

There are still many open problems about vacant domains. Most of them are general problems about intrinsic properties of being vacant. We start by asking what happens in general to localizations and finite integral extensions of a vacant domain.

Question 2.7.1. Let R be a vacant domain and P a prime ideal of R. Is R_P vacant?

Question 2.7.2. Let R be a vacant domain with quotient field K. Let F be a finite field extension of K. Is the integral closure $\overline{R}^F$ of R in F vacant?

It is well-known that for a domain R the following characterization holds: R is a Prüfer domain if and only if each overring of R is integrally closed, if and only if each overring of R is a Prüfer domain ([22, Theorem 26.2]).

Hence we ask:

Question 2.7.3. Is every integrally closed overring of a vacant domain R vacant too?

According to Proposition 2.3.7, the question above can be restated as in the following conjecture.

Conjecture 2.7.4. Let R be an integrally closed domain. Then, for each n, R is n-vacant if and only if R is 2-vacant.

Chapter 3

Projective star operations and graded rings

3.1 Projective models

We observed in Section 1.2 that the Zariski space $\mathrm{Zar}(R)$ of an integral domain R is a spectral space. In a recent paper O. Heubo showed that each Zariski space $\mathrm{Zar}(F/R)$, with R a subdomain of a field F, is spectral.

Proposition 3.1.1. ([29, Proposition 2.7]) *Let F be a field and R a domain (possibly a field) contained in F. Let $H := \bigcap_{V \in \mathrm{Zar}(F/R)} V^b$. Then the mapping:*

$$\begin{array}{rcl} \phi : \mathrm{Zar}(F/R) & \longrightarrow & \mathrm{Zar}(H) \\ V & \longmapsto & V^b \end{array}$$

is a homeomorphism with respect to the Zariski topology.

By Theorem 1.3.14 an F-function ring is a Bézout, hence a Prüfer, domain, so that $\mathrm{Zar}(H)$ is homeomorphic to $\mathrm{Spec}(H)$. Therefore $\mathrm{Zar}(F/R)$ is a spectral space too. In other words, the Zariski spaces $\mathrm{Zar}(F/K)$, for each choice of a field F and a subdomain K of F, from a topological point of view, are just like the prime spectrum of a ring.

We are going to see that, despite the fact that $\mathrm{Zar}(F/K)$ is spectral, we can associate this topological space to a scheme, which is not in general affine. More precisely, in the case we deal with, such a scheme is projective.

In the years around 1940, O. Zariski introduced a precursor to schemes theory from an algebraic point of view. Afterwards his student S. Abhyankar used Zariski's setting to achieve results on resolution of singularities (cf. [1]).

We give here a brief overview on models and ideals in models, which are the algebraic objects corresponding to schemes and coherent sheaves of ideals respectively in algebraic geometry.

Let F be a field and R a subdomain of F. For a quasi-local domain S, we denote by M_S its unique maximal ideal and by k_S the residue field S/M_S.

Consider the set $L(F/R) := \{S : R \subseteq S \subseteq F, S \text{ quasi-local}\}$. The set of quasi-local rings between R and F can be partially ordered by *domination*, as explained in the following definition.

Definition 3.1.2. Let S_1 and S_2 be in $L(F/R)$. We say that S_1 *dominates* S_2 if $S_1 \supseteq S_2$ and $M_{S_1} = M_{S_2} \cap S_1$.

By using the relation of domination we can single out two important kinds of subsets of $L(F/R)$:

Definition 3.1.3. Let X be a subset of $L(F/R)$. We say that X is *irredundant* if for each $V \in \mathrm{Zar}(F/R)$, there exists at most one element $S \in X$ dominated by V.

Reciprocally, we say X to be *complete* if for each $V \in \mathrm{Zar}(F/R)$, there exists at least one element $S \in X$ dominated by V.

Given X and X' subsets of $L(F/R)$ we say that X *dominates* X' if every element of X dominates at least one element of X'.

It is clear that $\mathrm{Zar}(F/R)$ is a subset of $L(F/R)$. We introduce now a topology on $L(F/R)$. We will keep denoting the basic open sets of this topology by $\mathcal{U}$, because, as it will be evident soon, the Zariski topology on $\mathrm{Zar}(F/R)$ coincides with the induced topology. The Zariski topology on $L(F/R)$ is given by declaring as an open basis the following sets:

$$\mathcal{U}(x_1, \ldots, x_n) := \{S \in L(F/R) : x_i \in S, \ \forall i = 1, \ldots, n\}.$$

Notation 3.1.4. Given a domain D, such that $R \subseteq D \subseteq F$, we denote by $V(D) := \{D_P : P \in \mathrm{Spec}(D)\}$. As topological spaces we consider $\mathrm{Spec}(D)$ with the spectral topology and the induced Zariski topology on $V(D)$.

Proposition 3.1.5. ([49, Lemma 1, Ch. VI, § 17]) *Let R be a domain contained in a field F. Let D be a domain in between R and F. The mapping:*

$$\begin{aligned} f : L(F/D) &\longrightarrow \mathrm{Spec}(D) \\ S &\longmapsto M_S \cap D \end{aligned}$$

is continuous. The restriction of f to $V(D)$ is a topological homeomorphism.

We can define now the concept of a model.

Definition 3.1.6. Let F be a field and R a domain contained in F. An *affine model* over R is a set of the form $V(R[x_1,\ldots,x_n])$ with $x_i \in F$, for each $i = 1,\ldots,n$.

A *model* M is defined as an irredundant subset of L which is a finite union of affine models, i.e. $M = \bigcup_{i=1}^n V(R_i)$, with each $R_i \subseteq F$ and finitely generated as an R-algebra.

Definition 3.1.7. Let F be a field and R a domain contained in F. Let $x_0,\ldots,x_n$ be nonzero elements of F. A *projective model* over R is a model $M = \bigcup_{i=0}^n V(R_i)$ whose underlying domains are $R_i := R[\frac{x_0}{x_i},\ldots,\frac{x_n}{x_i}]$.

By [49, Lemma 5, Ch. VI, § 17] a projective model is complete. Therefore a projective model over R is an irredundant and complete subset of $L(F/R)$.

Consider now the following special case: let $S := K[X_0,\ldots,X_n]$, with $X_0,\ldots,X_n$ algebraically independent indeterminates over K. Let the field $F := K(\frac{X_0}{X_1},\ldots,\frac{X_n}{X_1}) = K(\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i}) = F(y_1,\ldots,y_n)$. Denoting by $R_i := K[\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i}]$, a projective model of F over K is then:

$$M := \bigcup_{i=0}^n V(R_i).$$

Furthermore the R_i's are integrally closed domains having quotient field F. According to Proposition 3.1.5 the model M just defined is homeomorphic, as a topological space to:

$$\bigcup_{i=0}^n \mathrm{Spec}(R_i).$$

It is immediately seen that this is the same topological space underlying the projective scheme $\mathrm{Proj}(S) = \bigcup_{i=0}^n D_+(X_i)$, where $D_+(X_i)$ is the set of homogeneous prime ideals of S not containing X_i (for details cf. [9, III.2.1]).

Hence the model M defined above inherits, in a natural way, a structure of a projective scheme, the same one defined for $\mathrm{Proj}(S)$. So, by identifying the model M with the projective scheme $\mathrm{Proj}(S)$ we gain a very useful new setting. In particular, over a scheme, we can consider sheaves of ideals, as we will see in the next section.

3.2 Homogenization, dehomogenization and saturation of ideals

Let K be a field. Let $S := K[X_0,\ldots,X_n]$, the ring in $n+1$ indeterminates over K. For i that ranges from 0 to n, we keep denoting by R_i the ring $K[\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i}]$.

Thus S is graded with the standard grading obtained by giving to each variable degree 1. Hence:

$$S = \bigoplus_{k=0}^{\infty} S_k, \quad \text{where} \quad S_0 = K \quad \text{and} \quad S_m S_n \subseteq S_{n+m}.$$

Definition 3.2.1. An element $f \in S$ is said to be *homogeneous (of degree d)* if $f \in S_d$ for some d.

An ideal I of S is *homogeneous* if I is generated by homogeneous elements, or, equivalently, if for each $f = f_0 + \cdots + f_d \in I$, with f_i homogeneous of degree i, we have $f_i \in I$ for all $i = 0, \ldots, d$.

An ideal P of S is a *homogeneous prime* ideal if P is a homogeneous ideal which is also prime.

Let $f \in S$. The *dehomogenization* of f in R_i is the element

$${}^{a_i}f := f(\frac{X_0}{X_i}, \ldots, 1, \ldots, \frac{X_n}{X_i})$$

of R_i. The application ^{a_i} is a ring homomorphism for each $i = 0, \ldots, n$. Conversely given an element g in R_i, its *homogenization* in S is the homogeneous element

$${}^{h}g := X_i^{n_i} g(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i})$$

of S, where n_i is the degree of g.

Before starting with all the properties of homogeneous and non-homogeneous ideals from the algebraic point of view, we wish to give some details about the algebraic geometry behind our construction. In fact, the spirit we will use for definitions and properties which seemed to be natural to us, when considered in the context of scheme theory.

Keeping the notation above:

Proposition 3.2.2. ([9, Exercise III-13]) *Let I be a homogeneous ideal of S. For each open set $U_i := \mathrm{Spec}(R_i)$, let $\tilde{I}(U_i)$ be the ideal $I \cdot S[X_i^{-1}] \cap R_i$. This definition can be extended in a unique way to other open sets in such a way that $\tilde{I}$ becomes a coherent sheaf of ideals.*

So, a homogeneous ideal of S induces a coherent sheaf of ideals over $\mathrm{Proj}(S)$. The process can be reversed to get a homogeneous ideal of S from a coherent sheaf of ideals, as in [9, Exercise III-14]. The correspondence is not one-one in general. We will get over this restriction by using the *saturation* of a homogeneous ideal of S.

Given a homogeneous ideal I of S, the *dehomogenization* of I:

$$^{a_i}I := \{^{a_i}f \mid f \text{ is homogeneous in } I\}$$

is an ideal of R_i. Observe that the dehomogenization in R_i of an ideal I of S coincides with the ideal $\tilde{I}(U_i)$ considered in Proposition 3.2.2 (cf. also [49, Ch. VII, § 5, p. 185]).

We note that using the fact that the operation ^{a_i} is a ring homomorphism for each $i = 0, \ldots, n$ and the fact that I is generated by homogeneous elements, it is clear that $^{a_i}I = \{^{a_i}f : f \in I\}$. So, for a homogeneous ideal I of S, we have $x \in {}^{a_i}I$ if and only if $x = {}^{a_i}f$ for some $f \in I$.

For each $i = 0, \ldots, n$, the operation $I \mapsto {}^{a_i}I$ is onto the set of all ideals of R_i and preserves inclusion and the usual ideal-theoretic operations:

Theorem 3.2.3. ([49, Theorem 18, Ch. VII, § 5]) *The operation $I \to {}^{a_i}I$ maps the set of all homogeneous ideals in S onto the set of all ideals of R_i. It preserves inclusions and the usual ideal-theoretic operations, i.e., it has the following properties:*

(a) $I \subseteq J$ *implies* $^{a_i}I \subseteq {}^{a_i}J$.

(b) $^{a_i}(I + J) = {}^{a_i}I + {}^{a_i}J$.

(c) $^{a_i}(IJ) = {}^{a_i}I{}^{a_i}J$.

(d) $^{a_i}(I \cap J) = {}^{a_i}I \cap {}^{a_i}J$.

(e) $^{a_i}(I : J) = ({}^{a_i}I : {}^{a_i}J)$.

(f) $^{a_i}(\sqrt{I}) = \sqrt{{}^{a_i}I}$.

The converse of dehomogenization for ideals is given by *homogenization.* Given an ideal I of R_i, let $\mathfrak{C}$ be the set of all the homogeneous polynomials $X_i^{m\,h}f$ ($m \geq 0$, $f \in I$). We denote by ^hI the homogeneous ideal in S which is generated by the elements of $\mathfrak{C}$.

The operation $I \mapsto {}^hI$ is one-one and preserves inclusion and the usual ideal-theoretic operations:

Theorem 3.2.4. ([49, Theorem 17, Ch. VII, § 5]) *The operation $J \mapsto {}^hJ$ maps distinct ideals of R_i into distinct ideals in S; it preserves inclusion and the usual ideal-theoretic operations, i.e., it has the following properties:*

(a) $I \subseteq J$ *implies* $^hI \subseteq {}^hJ$.

(b) ${}^{h}(I+J) = {}^{h}I + {}^{h}J$.

(c) ${}^{h}(IJ) = {}^{h}I{}^{h}J$.

(d) ${}^{h}(I \cap J) = {}^{h}I \cap {}^{h}J$.

(e) ${}^{h}(I : J) = ({}^{h}I : {}^{h}J)$.

(f) ${}^{h}(\sqrt{I}) = \sqrt{{}^{h}I}$.

We note the properties of the composite operations ${}^{a_i h}$ and ${}^{h a_i}$, for each $i = 0, \dots, n$ (see [49, Ch. VII, § 5, p. 182]):

(H1) ${}^{a_i}({}^{h}I) = I$, for any ideal I in R_i;

(H2) ${}^{h}({}^{a_i}I) \supseteq I$, for any ideal I in S;

(H3) $X_i^m({}^{h}({}^{a_i}I)) \subseteq I$, for some integer $m \geq 1$.

A useful characterization of homogeneous prime ideals follows:

Proposition 3.2.5. ([8, Exercise 2.15 (c)]) *Let S be a $\mathbb{Z}$-graded ring. A homogeneous ideal P of S is prime if and only if whenever $fg \in P$ for homogeneous polynomials $f, g \in S$ then $f \in P$ or $g \in P$.*

Proof. ($\Rightarrow$). It follows by definition of a prime ideal.
($\Leftarrow$). Let f, g be elements of S such that $fg \in P$. We assume without loss of generality that S is graded positively. Then we can write $f = F_0 + \dots + F_n$ and $g = G_0 + \dots + G_m$ where all F_i's and G_j's are homogeneous elements of S. By assumption the product $fg \in P$. Then every homogeneous component of $fg \in P$ as P is homogeneous. Observe that if for all $i < k$ both F_i and G_i are in P, then $F_k G_k \in P$. In fact,

$$(fg)_{2k} = F_0G_{2k} + F_1G_{2k-1} + \dots + F_kG_k + \dots + G_1F_{2k-1} + G_0F_{2k}.$$

Thus $F_kG_k = (fg)_{2k} - (F_0G_{2k} + F_1G_{2k-1} + \dots + G_1F_{2k-1} + G_0F_{2k}) \in P$.

Suppose d is the minimum such that $F_d \in P$ and $G_d \notin P$, and d' is the minimum for which $F_{d'} \notin P$ and $G_{d'} \in P$. If d and d' do not exist we have that the element with smaller degree between f and g is in P. So we can assume that the minimum d does exist, and we have that:

$$(fg)_{2d+1} = F_0G_{2d+1} + \dots + F_{d+1}G_d + G_{d+1}F_d + \dots + F_{2d+1}G_0.$$

Then $F_{d+1}G_d \in P$ and $G_d \notin P$ implies $F_{d+1} \in P$. With the same argument it follows that $F_{d+k} \in P$ for all $k = 1, \dots (n-d)$, so that $f \in P$. $\square$

Definition 3.2.6. Let I be an ideal of S, the *saturation* of I is the ideal:

$$^{sat}I := \{y \in S \,\big|\, \forall i = 0, \ldots, n, \ \exists t_i \geq 0, \ yX_i^{t_i} \in I\}.$$

An ideal I of S is *saturated* if $^{sat}I = I$.

Proposition 3.2.7. ([9, Exercises III-14 and III-16]) *There is a bijective correspondence between homogeneous saturated ideals of S and coherent sheaves of ideals over* $\mathrm{Proj}(S)$.

In view of this correspondence, we will work with homogeneous saturated ideals of S, but all our results can be applied to coherent sheaves of ideals over $\mathrm{Proj}(S)$ and hence on ideals over the projective model $M = \bigcup_{i=0}^{n} V(R_i)$. It is worth remarking that ideals can be defined for models also in the more general case (cf. [1, Chapter 2, § 6, (6.4)]).

Remark 3.2.8. Let I be an ideal of S. Then

$$\begin{aligned} y \in {}^{sat}I &\Longleftrightarrow y \in IS[\frac{1}{X_i}] \cap S \text{ for all } i = 0, \ldots, n. \\ &\Longleftrightarrow y \in \left(IS\left[\frac{1}{X_0}\right] \cap S\right) \cap \cdots \cap \left(IS\left[\frac{1}{X_n}\right] \cap S\right) \end{aligned}$$

Thus

$$^{sat}I = \left(IS\left[\frac{1}{X_0}\right]\right) \cap \cdots \cap \left(IS\left[\frac{1}{X_n}\right]\right).$$

Proposition 3.2.9. *Let I be a homogeneous ideal of S. Then $^{sat}I = \bigcap_{i=0}^{n} {}^{ha_i}I$.*

Proof. Let $f \in {}^{sat}I$. Then for each $i = 0, \ldots, n$, there is a nonnegative integer n such that $X_i^n f \in I$. Set $g := X_i^n f$. Then we have $g = {}^h({}^{a_i}g)X_i^m$, where m is the degree of X_i in the polynomial g. Clearly as $f = X_i^{-n}g \in S$, we have $m \geq n$. So $f = {}^h({}^{a_i}g)X_i^{m-n} \in {}^h({}^{a_i}I)$ for each $i = 0, \ldots, n$. Therefore $^{sat}I \subseteq \bigcap_{i=0}^{n} {}^{ha_i}I$.

Now let $f \in \bigcap_{i=0}^{n} {}^{ha_i}I$ with f a homogeneous polynomial. Then for each $i = 0, \ldots, n$, we can assume without loss of generality that $f = X_i^m({}^hg_i)$ with m_i nonnegative integer and $g_i \in {}^{a_i}I$ (i.e., $g_i = {}^{a_i}\varphi$, $\varphi \in I$). Thus $f = X_i^{m_i h}({}^{a_i}\varphi) = X_i^{m_i}X_i^{-m_{0i}}\varphi$, where m_{0i} is the highest power of X_i that divides φ. Therefore choose a nonnegative integer s such that $s \geq m_{0i} - m_i$ to have $X_i^s f = X_i^{s+m_i-m_{0i}}\varphi \in I$, as $\varphi \in I$. Hence $f \in {}^{sat}I$. Since $\bigcap_{i=0}^{n} {}^{ha_i}I$ is homogeneous, we have $^{sat}I = \bigcap_{i=0}^{n} {}^{ha_i}I$. $\square$

Proposition 3.2.10. *Given I, J homogeneous ideals of S the following properties hold:*

(a) *for each* $i = 0, \dots, n$, ${}^{a_i}I = {}^{a_i\,sat}I$*;*

(b) ${}^{sat}I \subseteq {}^{sat}J \Longleftrightarrow {}^{a_i}I \subseteq {}^{a_i}J$ *for all* $i = 0, \dots, n$*;*

(c) ${}^{sat}I$ *is homogeneous*

(d) *for each (homogeneous) polynomial* $f \in S$, ${}^{sat}(fI) = f^{sat}I$*;*

(e) $I \subseteq {}^{sat}I$ *and if* $I \subseteq J$ *then* ${}^{sat}I \subseteq {}^{sat}J$*;*

(f) ${}^{sat}(I \cap J) = {}^{sat}I \cap {}^{sat}J$.

Proof. (a) The inclusion ${}^{a_i}I \subseteq {}^{a_i\,sat}I$ is trivial since dehomogenization preserves inclusions. For the converse, since ^{a_i} commutes with intersections:

$$ {}^{a_i\,sat}I = {}^{a_i}\left(\bigcap_{j=0}^{n} {}^{ha_j}I\right) = \bigcap_{j=0}^{n} {}^{a_i ha_j}I \subseteq {}^{a_i}I. $$

(b) Suppose that ${}^{sat}I \subseteq {}^{sat}J$. We have $I \subseteq {}^{sat}I \subseteq {}^{sat}J$. Thus for each $i = 0, \dots, n$: ${}^{a_i}I \subseteq {}^{a_i sat}J = {}^{a_i}J$, by (a). Conversely, suppose ${}^{a_i}I \subseteq {}^{a_i}J$ for all $i = 0, \dots, n$. Since the operation ^{h} preserves inclusion, we can conclude using Proposition 3.2.9 that ${}^{sat}I \subseteq {}^{sat}J$.

(c) It is straightforward by Proposition 3.2.9 and the fact that a finite intersection of homogeneous ideals is homogeneous.

(d) Let f be a polynomial in S.

$$ \begin{aligned} {}^{sat}(fI) &= \left((fI)\,S\left[\frac{1}{X_0}\right]\right) \cap \cdots \cap \left((fI)\,S\left[\frac{1}{X_n}\right]\right) \\ &= f\left(\left(IS\left[\frac{1}{X_0}\right]\right) \cap \cdots \cap \left(IS\left[\frac{1}{X_n}\right]\right)\right) \\ &= f^{sat}I. \end{aligned} $$

(e) This is clear by the definition of saturation or by Proposition 3.2.9.

(f) It is clear by combining Proposition 3.2.9 and the fact that the operations ^{a_i} and ^{h} preserve intersection.

By definition of star operations, i.e., Definition 1.1.1, it is easily seen that (d) and (e) imply that *sat* is a star operation on S. Furthermore *sat* is a stable star operation, as shown in (f). $\square$

3.3 Star operations on homogeneous and non-homogeneous ideals

Star operations work in general on nonzero fractional ideals. In our context we want to apply the operations ^{a_i} and ^{h} to the ideals transformed by a star operation. Hence, we extend the concept of being homogeneous to fractional ideals and build an application enjoying the same properties of star operations on such ideals.

Definition 3.3.1. A fractional ideal J of S is *homogeneous* (resp. *saturated*) if there exists a homogeneous polynomial $f \in S$ such that fJ is a homogeneous (resp. saturated) ideal of S.

Definition 3.3.2. If J is a homogeneous fractional ideal of S, then the *dehomogenization* of J is ${}^{a_i}J := \frac{1}{{}^{a_i}f}{}^{a_i}(fJ)$, where f is a homogeneous element of S such that fJ is a homogeneous ideal of S.

For each $i = 0, \ldots, n$, if J is a fractional ideal of R_i i.e., J is an R_i-module in F and there is a nonzero $f \in R_i$ such that fJ is an ideal of R_i, then the *homogenization* of J is ${}^{h}J := \frac{1}{{}^{h}f}{}^{h}(fJ)$.

Remark 3.3.3. For each $i = 0, \ldots, n$ the operation ^{a_i} is well-defined for homogeneous fractional ideals of S: let J be an homogeneous fractional ideal of S. Suppose that there are homogeneous polynomials f and g such that fJ and gJ are homogeneous ideals of S. Then:

$$\frac{1}{{}^{a_i}f}{}^{a_i}(fJ) = \frac{1}{{}^{a_i}g}{}^{a_i}(gJ) \iff {}^{a_i}g^{a_i}(fJ) = {}^{a_i}f^{a_i}(gJ) \iff^{a_i} (gfJ) = {}^{a_i}(fgJ).$$

It is also clear by a similar argument that the operation ^{h} is well-defined for fractional ideals of R_i, for all $i = 0, \ldots, n$.

Observe that if J is a homogeneous fractional ideal of S, then ${}^{a_i}J$ is a fractional ideal of R_i. Conversely given a fractional ideal I of R_i, ${}^{h}I$ is a homogeneous fractional ideal of S. In fact, if $f \in R_i$ is such that $fI \subseteq R_i$, then ${}^{h}f^{h}I \subseteq {}^{h}R_i = S$.

Definition 3.3.4. Let $\overline{\mathcal{H}}(S)$ denote the set of nonzero homogeneous fractional ideals of S. A *projective star operation* on S is a mapping:

$$\begin{array}{rcl} \star : \overline{\mathcal{H}}(S) & \longrightarrow & \overline{\mathcal{H}}(S) \\ I & \longmapsto & I^{\star} \end{array}$$

such that for every nonzero homogeneous rational function f (i.e., $f = \frac{g}{h}$ with $0 \neq h$ and g homogeneous polynomials in S) in the quotient field of S and every $I, J \in \overline{\mathcal{H}}(S)$ the following conditions are satisfied:

(a) $(f)^\star = (f)$, $(fI)^\star = fI^\star$;

(b) $I \subseteq I^\star$ and if $I \subseteq J$ then $I^\star \subseteq J^\star$;

(c) $I^{\star\star} := (I^\star)^\star = I^\star$.

Remark 3.3.5. If $I \mapsto I^\star$ is a projective star operation on S, it is clear that $S = (1) = (1)^\star = S^\star$, and if I is a homogeneous ideal of S, then $I \subseteq I^\star \subseteq S^\star = S$. Hence each projective star operation on S induces a map $I \mapsto I^\star$ from $\mathcal{H}(S)$, the set of homogeneous ideals of S, into $\mathcal{H}(S)$ satisfying conditions (a), (b) and (c). Moreover, for each operation $\star$ from $\mathcal{H}(S)$ onto $\mathcal{H}(S)$ satisfying conditions (a), (b) and (c), if $J \in \overline{\mathcal{H}}(S)$, then there is a homogeneous element $f \in S$ such that $fJ =: I$ is a homogeneous ideal of S. Set $J^\star = \frac{1}{f}I^\star$. It is clear that $\star$ is well-defined and is a projective star operation on S. Therefore, from now on, we will consider a projective star operation on S as a map from $\mathcal{H}(S)$ into $\mathcal{H}(S)$ satisfying conditions (a), (b) and (c) (in condition (a), take f to be homogeneous element of S).

It is clear that a star operation on S which preserves homogeneous ideals is a projective star operation, but, as expected, not every star operation on S is homogeneous preserving. We provide next some examples of projective star operations and an example of a star operation on S that is not a projective star operation.

Example 3.3.6. (a) The identity is clearly by definition a projective star operation. The saturation *sat* is also a projective star operation (Proposition 3.2.10 (c), (d), (e) and (f), together with the fact that ${}^{sat}S = S$). We will see later that the b-operation and the v-operation are projective star operations on S as well.

(b) Let I be an ideal of S. Since S is a Noetherian integrally closed domain, S is completely integrally closed, and so $S = (I : I)$ for each nonzero ideal I of S (see [22, Theorem 34.3]). Therefore, by Proposition 1.1.16, the application $v(I) : \mathfrak{F}(S) \to \mathfrak{F}(S)$, $J \mapsto (I : (I : J))$ is a star operation on S for each ideal I.

Consider the maximal ideal $M := (X_0 - 1, X_1, \ldots, X_n)$ of S and the homogeneous ideal $I := (X_1, \ldots, X_n)$. We shall prove that $I_{v(M)} := (M : (M : I))$ is not homogeneous, and hence $v(M)$ cannot be restricted to a projective star operation. By [28, Lemma 3.1], $I_{v(M)} = \bigcap_{I \subseteq qM} qM$ with q in the quotient field of S. First of all, we observe that $I_{v(M)} \supsetneq I$. Suppose by way of contradiction that $I = I_{v(M)}$. Then, since S is Noetherian, the ideal $(M : I) = (r_1, \ldots, r_n)S$ for some finite set $\{r_1, \ldots, r_n\}$ of the quotient field of S, and $(M : (M : I)) = (M : (r_1, \dots, r_n)S) = \bigcap_{i=1}^{n} r_i^{-1} M$. By setting $q_i := r_i^{-1}$:

$$I = \bigcap_{I \subseteq qM} qM = \bigcap_{i=1}^{n} q_i M = \bigcap_{i=1}^{n} q_i M \cap R \subseteq \bigcap_{i=1}^{n} q_i R \cap R = R,$$

where the last equality holds because I is a prime ideal of height greater than 1 in an integrally closed Noetherian domain. Hence $R = I_v$ by [22, Corollary 44.8], so that

$$R = I_v \subseteq \bigcap_{i=1}^{n} q_i R \cap R \subseteq R.$$

Then, for each i, $q_i R \cap R = R$ and $r_i := q_i^{-1} \in R$. Therefore $I = \frac{1}{r_1} M \cap \cdots \cap \frac{1}{r_n} M \cap R$. We can assume without loss of generality that for all i, $r_i \in R \setminus M$. For if $r_i \in M$ for some i, $\frac{1}{r_i} M = R$ then there is no contribution in the intersection. We have then that $(r_1 \cdots r_n) I = (r_2 \cdots r_n) M \cap \cdots \cap (r_1 \cdots r_{n-1}) M \cap (r_1 \cdots r_n) R$. Thus $I_M = MR_M$ ($\forall i$, $r_i \notin M$), which is a contradiction because I is a prime ideal properly contained in M. So $I \subsetneq I_{v(M)} \subseteq M_{v(M)} = M$.

We prove now that I is maximal among the homogeneous ideals of S contained in M. Since S is Noetherian the ACC on ideals holds and each chain in the set:

$$\mathcal{F} := \{J : J \text{ is homogeneous and } I \subseteq J \subsetneq M\},$$

has a maximal element P. Suppose P is not prime. Then by Proposition 3.2.5 there exist $f, g \in S \setminus P$ homogeneous such that $fg \in P$. We can suppose f is in M because M is prime, so we have $P \subsetneq (P, f) \subsetneq M$, because M is not homogeneous, and this contradicts the maximality of P in $\mathcal{F}$. Hence P is prime and

$$(X_0) \subsetneq (X_0, X_1) \subsetneq \cdots \subsetneq (X_0, \dots, X_{n-1}) \subsetneq P \subsetneq M$$

is a chain of distinct primes of length $n + 2 > \dim(S) = n + 1$, which is impossible. Therefore I is maximal in $\mathcal{F}$ and, since $I \subsetneq I_{v(M)} \subseteq M$, $I_{v(M)}$ is not homogeneous.

We next turn our attention to the "dehomogenization" of a projective star operation. In other words, we prove that a projective star operation $\star$ on S, induces a star operation $\star_i$ on R_i for each $i = 0, \dots, n$.

Proposition 3.3.7. *Let $\star$ be a projective star operation on S. Then the mapping $\star_i : \mathfrak{F}(R_i) \longrightarrow \mathfrak{F}(R_i)$, $I \longmapsto I^{\star_i} := {}^{a_i}(({}^{h}I)^{\star})$, is a star operation on R_i for each $i = 0, \dots, n$.*

Proof. It is enough to prove that conditions $(\star_1)$, $(\star_2)$ and $(\star_3)$ of Definition 1.1.1 hold on (integral) ideals of R_i. Let $g \in R_i$ and I an ideal of R_i;

$$(gI)^{\star_i} = {}^{a_i}({}^{h}(gI))^{\star} = {}^{a_i}({}^{h}g{}^{h}I)^{\star} = g^{a_i}({}^{h}I)^{\star} = gI^{\star_i}.$$

Since ${}^{h}R_i = S$ for each i, condition $(\star_1)$ holds. Condition $(\star_2)$ is straightforward. The fact that $(I^{\star_i})^{\star_i} \supseteq I^{\star_i}$ follows from $(\star_2)$ and we prove that the reverse inclusion holds too. By (H3) we have $X_i^{m\,ha_i}\left(({}^{h}I)^{\star}\right) \subseteq ({}^{h}I)^{\star}$ for some $m \geq 1$. Since $\star$ is a projective star operation,

$$X_i^m\left[{}^{ha_i}\left(({}^{h}I)^{\star}\right)\right]^{\star} = \left[X_i^{m\,ha_i}\left(({}^{h}I)^{\star}\right)\right]^{\star} \subseteq ({}^{h}I)^{\star\star} = ({}^{h}I)^{\star}.$$

Now as ^{a_i} preserves inclusion and ${}^{a_i}(X_i^m) = 1$, we have

$$(I^{\star_i})^{\star_i} = {}^{a_i}\left(\left[{}^{ha_i}\left(({}^{h}I)^{\star}\right)\right]^{\star}\right) = {}^{a_i}\left(X_i^m\left[{}^{ha_i}\left(({}^{h}I)^{\star}\right)\right]^{\star}\right) \subseteq {}^{a_i}({}^{h}I)^{\star} = I^{\star_i}.$$

Then $\star_i$ is a star operation on R_i for any $i = 0, \dots, n$. □

Proposition 3.3.8. *Let $\star$ be a projective star operation on S and $\{\star_0, \dots, \star_n\}$ the set of star operations obtained as in Proposition 3.3.7. Then ${}^{a_i}(I^{\star}) = ({}^{a_i}I)^{\star_i}$ for each homogeneous ideal I of S and each $i = 0, \dots, n$.*

Proof. For each $i = 0, \dots, n$, $\left({}^{ha_i}I\right)^{\star} \supseteq I^{\star} \Longrightarrow ({}^{a_i}I)^{\star_i} = {}^{a_i}\left({}^{ha_i}I\right)^{\star} \supseteq {}^{a_i}(I^{\star})$.
Conversely, by (H3) there exists $m \geq 1$ such that:

$$\begin{aligned} I^{\star} \supseteq X_i^m\left({}^{ha_i}I\right)^{\star} \Longrightarrow {}^{a_i}(I^{\star}) \supseteq {}^{a_i}\left(X_i^m\left({}^{ha_i}I\right)^{\star}\right) = ({}^{a_i}X_i^m)\,{}^{a_i}\left({}^{ha_i}I\right)^{\star} = \\ = ({}^{a_i}I)^{\star_i} \text{ for each } i = 0, \dots, n. \end{aligned}$$

Where the last equality holds by the definition of $\star_i$ given in Proposition 3.3.7. □

Corollary 3.3.9. *Same hypothesis as Proposition 3.3.8. Then*

$${}^{sat}(I^{\star}) = {}^{h}\left(({}^{a_0}I)^{\star_0}\right) \cap \cdots \cap {}^{h}\left(({}^{a_n}I)^{\star_n}\right).$$

Proof. By Proposition 3.2.9,

$$\begin{aligned} {}^{sat}(I^{\star}) &= {}^{ha_0}(I^{\star}) \cap \cdots \cap {}^{ha_n}(I^{\star}) \\ &= {}^{h}\left(({}^{a_0}I)^{\star_0}\right) \cap \cdots \cap {}^{h}\left(({}^{a_n}I)^{\star_n}\right). \end{aligned}$$

The last equality is by Proposition 3.3.8. □

Proposition 3.3.10. *Let $\star$ be a projective star operation on S and I be a homogeneous ideal of S. Then ${}^{sat}I = {}^{sat}(I^{\star})$ if and only if $({}^{a_i}I)^{\star_i} = {}^{a_i}I$ for every $i = 0, \dots, n$ (where $\star_i$ are those built in Proposition 3.3.7).*

Proof. By Proposition 3.3.8 and Proposition 3.2.10, we have

$$\begin{aligned}({}^{a_i}I)^{\star_i} = {}^{a_i}I &\iff {}^{a_i}(I^\star) = {}^{a_i}I \text{ for all } i = 0, \dots, n\\ &\iff {}^{sat}(I^\star) = {}^{sat}I\end{aligned}$$

□

From a projective star operation $\star$ on S, we built star operations $\star_i$ on R_i, $i = 0, \dots, n$. We want to consider the reverse process: how do we get a projective star operation on S from the star operations on R_i, $i = 0, \dots, n$? We start by observing how the star operations $\star_i$, built from a given projective star operation, interact with one another. In fact, we do not expect to be able to build a projective star operation by choosing arbitrarily the star operations $\star_i$.

Moreover, we want to reverse the process of dehomogenization, and then compose homogenization and dehomogenization to go back and forth between the set of projective star operations and $(n+1)$-tuples of star operations, each of them defined on one of the R_i's.

Lemma 3.3.11. *Let $\star$ be a projective star operation on S and $\{\star_0, \dots, \star_n\}$ the set of star operations on $R_0, \dots, R_n$ obtained as in Proposition 3.3.7. Then for each homogeneous ideal I of S,*

(i) $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^h(({}^{a_i}I)^{\star_i})]$ *for all $j = 0, \dots, n$ and $i = 0, \dots, n$.*

(ii) *For each $i = 0, \dots, n$ there exists a nonnegative integer m_i such that* $X_i^{m_i}{}^{a_j}[{}^h(({}^{a_i}I)^{\star_i})] \subseteq ({}^{a_j}I)^{\star_j}$ *for all $j = 0, \dots, n$.*

Proof. For (i), For each $i = 0, \dots, n$, $\left({}^{ha_i}I\right)^\star \supseteq I^\star$, i.e., for each $j = 0, \dots, n$, For each $i = 0, \dots, n$, ${}^{a_j}\left[\left({}^{ha_i}I\right)^\star\right] \supseteq {}^a_j(I^\star)$. By Proposition 3.3.8, $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^h(({}^{a_i}I)^{\star_i})]$ for all $j = 0, \dots, n$ and $i = 0, \dots, n$.

For (ii), a similar argument as for (i) works by using the inclusion $X_i^{m_i}{}^{ha_i}(I^\star) \subseteq I^\star$ for some $m_i \geq 1$ for each $i = 0, \dots, n$. □

By homogenization and dehomogenization we can "move" an ideal of R_i to each of the other R_j's. Condition (i) in Lemma 3.3.11 suggests that the behavior of an ideal of R_i under the star operation $\star_i$ reflects the behavior of that same ideal moved into R_j under $\star_j$. Since a homogeneous ideal of S collects together the behaviors of its dehomogenized components, if we want to glue together a collection of star operations on different R_i's, we define two star operations to be compatible if we can move ideals from R_i to R_j, through S, preserving the behavior of the given star operations. This compatibility has to be satisfied by each pair of star operations that we want to "glue" together into a projective star operation.

In particular it will not be possible to glue together star operations of very different kinds.

Definition 3.3.12. Let $\star_0, \ldots, \star_n$ be star operations on $R_0, \ldots, R_n$ respectively. We say that $\star_0, \ldots, \star_n$ are *pairwise compatible* if $({}^{a_j}I)^{\star_j} \subseteq {}^{a_j}[{}^{h}(({}^{a_i}I)^{\star_i})]$ for all $i, j = 0, \ldots, n$, and all homogeneous ideals I of S.

Proposition 3.3.13. *Let $\{\star_0, \ldots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \ldots, R_n$. Then the mapping:*

$$\begin{array}{rcl} \star : \mathcal{H}(S) & \longrightarrow & \mathcal{H}(S) \\ I & \longmapsto & I^\star := {}^{sat}[\bigcap_{i=0}^{n} {}^{h}\,(({}^{a_i}I)^{\star_i})] \end{array}$$

is a projective star operation on S. Moreover if $\{\star_0, \ldots, \star_n\}$ are built from a projective star operation $\star'$ on S as in Proposition 3.3.7, then $\star = sat \circ \star'$.

Proof. We need to prove that $\star$ satisfies the conditions (a), (b) and (c) of Definition 1.1.1. It is easily seen that $S^\star = S$. Moreover saturation, homogenization and dehomogenization preserve inclusions. So $I \subseteq I^\star$ and $I \subseteq J \Rightarrow I^\star \subseteq J^\star$ are straightforward.

Suppose now that f is a homogeneous element in S. We claim that $(fI)^\star = fI^\star$. Now

$$\begin{aligned} {}^{a_j}[(fI)^\star] &\subseteq {}^{a_j sat}\left({}^{h}[({}^{a_j}fI)^{\star_j}]\right), \text{ (by definition of } \star) \\ &= {}^{a_j h}[({}^{a_j}fI)^{\star_j}], \text{ (by Proposition 3.2.10, (a))} \\ &= {}^{a_j}(fI)^{\star_j} \\ &= {}^{a_j}f({}^{a_j}I)^{\star_j}, \text{ (since } \star_j \text{ is a star operation)} \\ &\subseteq {}^{a_j}f^{a_j}[{}^{h}(({}^{a_i}I)^{\star_i})], \text{ (by the compatibility of the } \star_i\text{'s).} \\ &= {}^{a_j}[f^{h}(({}^{a_i}I)^{\star_i})]. \end{aligned}$$

Hence by Proposition 3.2.10 (b), we have ${}^{sat}[(fI)^\star] \subseteq f^{sat}[{}^{h}(({}^{a_i}I)^{\star_i})]$ for all $i = 0, \ldots, n$. Thus

$$(fI)^\star \subseteq {}^{sat}[(fI)^\star] \subseteq f^{sat}[\bigcap_{i=0}^{n} {}^{h}(({}^{a_i}I)^{\star_i})] = fI^\star.$$

For the other inclusion we have

$${}^{a_j}(fI^\star) = {}^{a_j}f^{a_j}(I^\star) \subseteq {}^{a_j}f^{a_j h}[({}^{a_j}I)^{\star_j}] = {}^{a_j}f({}^{a_j}I)^{\star_j} = [{}^{a_j}(fI)]^{\star_j}.$$

By compatibility of the $\star_i$'s,

$$^{a_j}(fI^\star) \subseteq {}^{a_j}[^h((^{a_i}fI)^{\star_i})], \text{ for all } i = 0, \dots, n,$$

i.e., $^{sat}(fI^\star) \subseteq {}^{sat}\left(^h((^{a_i}fI)^{\star_i})\right)$ for all $i = 0, \dots, n$, by Proposition 3.2.10 (b). Hence

$$fI^\star = {}^{sat}(fI^\star) \subseteq \bigcap_{i=0}^{n} {}^{sat}\left(^h((^{a_i}fI)^{\star_i})\right) = (fI)^\star.$$

So $(fI)^\star = fI^\star$.

For the last condition (c) left, it is clear that $I^\star \subseteq I^{\star\star}$ on one hand. On the other hand,

$$(I^\star)^\star = {}^{sat}\bigcap_{j=0}^{n} {}^h\left((^{a_j}(I^\star)^{\star_j})\right) \subseteq {}^{sat}\bigcap_{j=0}^{n} {}^h\left((^{a_j}I)^{\star_j\star_j}\right) = {}^{sat}\bigcap_{j=0}^{n} {}^h\left((^{a_j}I)^{\star_j}\right) = I^\star.$$

The last part of this proposition follows directly from Corollary 3.3.9. □

To keep a standard notation and avoid confusion between star operations defined on different domains, we shall denote on S the identity, the integral closure of ideals and the divisorial closure of ideals by d, b and v respectively. The same star operations referred to R_i will be denoted by d_i, b_i and v_i respectively (cf. Examples 1.1.17).

Example 3.3.14.

1. The identities d_i's on the R_i's satisfy the compatibility conditions. In fact, for an arbitrary homogeneous ideal I of S, $I \subseteq {}^{ha_i}I$ for each $i = 0, \dots, n$, i.e., $^{a_j}I \subseteq {}^{a_j}(^{ha_i}I)$. Then the projective star operation on S built from the identities on the R_i's is, by Proposition 3.2.9, the saturation ^{sat}.

2. Later (Proposition 3.4.3), we will see that the b_i-operations on the R_i's are pairwise compatible and that the projective star operation $\star$ on S built from those is the composition of the saturation and the b-operation on S, i.e., $\star = sat \circ b$.

3. Also, we will see in Proposition 3.4.5 that the v_i-operations on the R_i's are pairwise compatible and that the projective star operation on S built from those is the v-operation on S.

Proposition 3.3.15. *Let $\{\star_0, \dots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \dots, R_n$ and let $\star$ be the projective star operation on S built in Proposition 3.3.13. Then for each $i = 0, \dots, n$ and for each ideal I of R_i, we have $I^{\star_i} = {}^{a_i}[(^hI)^\star]$.*

Proof. For each $i = 0, \ldots, n$, and each ideal I of R_i, we do have

$$^{a_i}[(^hI)^\star] = {}^{a_i sat}[\bigcap_{k=0}^{n} {}^h(^{a_k h}I)^{\star_k}] = \bigcap_{k=0}^{n} {}^{a_i h}(^{a_k h}I)^{\star_k} = I^{\star_i} \cap \bigcap_{k \neq i, k=0}^{n} {}^{a_i h}(^{a_k h}I)^{\star_k}.$$

Hence

$$I^{\star_i} \supseteq I^{\star_i} \cap \bigcap_{k \neq i, k=0}^{n} {}^{a_i h}(^{a_k h}I)^{\star_k} \supseteq I^{\star_i},$$

by compatibility of the $\star_i$'s. So $I^{\star_i} = {}^{a_i}[(^hI)^\star]$. $\square$

Remark 3.3.16. Given a projective star operation $\star$ on S, we can build star operations $\star_i$ on R_i for each $i = 0, \ldots, n$. These star operations $\star_i$'s are pairwise compatible. Therefore by applying Proposition 3.3.13 to the $n+1$ star operations obtained we get "almost" back the original projective star operation. Indeed we get the projective star operation $sat \circ \star$ on S.

Reciprocally, if we start with a set $\{\star_0, \ldots, \star_n\}$ of pairwise compatible star operations on $R_0, \ldots, R_n$, then we can build a projective star operation $\star$ on S as in Proposition 3.3.13. Then the star operations $\star_i'$ built from $\star$ coincides with $\star_i$ for each $i = 0, \ldots, n$, by Proposition 3.3.15.

Furthermore we have that given any projective star operation $\star$ in S, the composition $sat \circ \star$ is a projective star operation. In particular the projective star operation sat dehomogenizes to the identity on the domain R_i for all $i = 0, \ldots, n$.

We next prove that the same property as in Proposition 3.3.8 holds when we start with a set of pairwise compatible star operations.

Proposition 3.3.17. *Let $\{\star_0, \ldots, \star_n\}$ be a set of pairwise compatible star operations on $R_0, \ldots, R_n$ and $\star$ the projective star operation on S built in Proposition 3.3.13. Then for any homogeneous ideal I of S, we have $(^{a_i}I)^{\star_i} = {}^{a_i}(I^\star)$ for all $i = 0, \ldots, n$.*

Proof. Let $i \in \{1, \ldots, n\}$, by Proposition 3.3.15, we have $(^{a_i}I)^{\star_i} = {}^{a_i}[(^{ha_i}I)^\star]$. On the other hand, $(^{ha_i}I)^\star \supseteq I^\star$, which implies $(^{a_i}I)^{\star_i} = {}^{a_i}\left(^{ha_i}I\right)^\star \supseteq {}^{a_i}\left(I^\star\right)$.

Conversely

$$I^\star \supseteq \left((X_i^m)^{ha_i}I\right)^\star = X_i^m \left(^{ha_i}I\right)^\star \Rightarrow {}^{a_i}\left(I^\star\right) \supseteq {}^{a_i}\left(X_i^m \left(^{ha_i}I\right)^\star\right) = {}^{a_i}[(^{ha_i}I)^\star],$$

where m is any integer for which (H3) is satisfied. $\square$

3.4 The projective b- and v-operation

We first note that if I is a homogeneous ideal of S, then I^b is a homogeneous ideal of S [46, Corollary 5.2.3]. So the b-operation is a projective star operation on S. A natural question is which star operations do we get locally from the b-operation and vice-versa?

Lemma 3.4.1. *Let I be an homogeneous ideal of S. Then ${}^{a_i}(I^b)$ is an integrally closed ideal of R_i for all $i = 0, \ldots, n$.*

Proof. For each i, we have $R_i = S_{M_i} \cap K_i$ where M_i is the multiplicatively closed subset of S consisting of powers of X_i and K_i is the quotient field of R_i. Now for I a homogeneous ideal of S, ${}^{a_i}(I^b) = I^b S_{M_i} \cap K_i = (IS_{M_i})^b \cap K_i$ ([49, VII. § 5.(10')] and [46, Proposition 1.1.4]). Since $(IS_{M_i})^b$ is an integrally closed ideal in S_{M_i}, $(IS_{M_i})^b \cap K_i$ is integrally closed in $S_{M_i} \cap K_i = R_i$. So ${}^{a_i}(I^b)$ is integrally closed in R_i. Hence ${}^{a_i}(I^b)$ is integrally closed in R_i for each $i = 0, \ldots, n$. □

Lemma 3.4.2. *Let I be a homogeneous ideal of S. Then ${}^{a_i}(I^b) = ({}^{a_i}I)^{b_i}$ for all $i = 0, \ldots, n$.*

Proof. Let I be a homogeneous ideal of S. We have $I \subseteq I^b$. Since ^{a_i} preserves inclusion for each i, we have ${}^{a_i}I \subseteq {}^{a_i}(I^b)$. But by Lemma 3.4.1, ${}^{a_i}(I^b)$ is integrally closed. Therefore $({}^{a_i}I)^{b_i} \subseteq {}^{a_i}(I^b)$.

For the reverse inclusion, let $x \in {}^{a_i}(I^b)$. Then we can write $x = {}^{a_i}r$ for some element $r \in I^b$. Thus r satisfies an equation of integral independence of r over I of the form $r^s + c_1 r^{s-1} + \ldots + c_{s-1}r + c_s = 0$ for some positive integer s and $c_j \in I^j$ for each $j = 1, \ldots, s$. Since ^{a_i} is a homomorphism, we have $({}^{a_i}r)^s + {}^{a_i}c_1({}^{a_i}r)^{s-1} + \ldots + {}^{a_i}c_{s-1}{}^{a_i}r + {}^{a_i}c_s = 0$ with ${}^{a_i}c_j \in ({}^{a_i}I)^j$ for each $j = 1, \ldots, s$. Thus $x = {}^{a_i}r \in ({}^{a_i}I)^{b_i}$. Hence ${}^{a_i}(I^b) \subseteq ({}^{a_i}I)^{b_i}$. So ${}^{a_i}(I^b) = ({}^{a_i}I)^{b_i}$. □

The precedent Lemma says that if we start with the b-operation on S, then the star operations $\star_i$ obtained from it as in Proposition 3.3.7 are exactly the b_i-operations on the R_i's.

Proposition 3.4.3. *The b_i-operations satisfy the compatibility condition, hence $({}^{a_j}I)^{b_j} \subseteq {}^{a_j}[{}^{h}(({}^{a_i}I)^{b_i})]$ for any homogeneous ideal I of S and for all $i, j = 0, \ldots, n$.*

Proof. By Lemma 3.4.2, we have ${}^{a_j}[{}^{h}(({}^{a_i}I)^{b_i})] = {}^{a_j}[{}^{ha_i}(I^b)] \supseteq {}^{a_j}(I^b) = ({}^{a_j}I)^{b_j}$. □

Remark 3.4.4. By Lemma 3.4.2, it is clear that if we start with the b-operation (as a projective star operation) on S and construct the star operations $\star_i$'s on R_i's as before, then the $\star_i$'s are exactly the b_i-operations on R_i's. Conversely, if we start with the b_i-operations on the R_i's, since those operations satisfy the compatibility conditions, they produce a projective star operation on S, which is $sat \circ b$, by Proposition 3.3.13.

Suppose I is a homogeneous ideal of S. Then $I_v := (S : (S : I))$ is homogeneous too (see [49, VII. § 2. Theorem 8]), so if we restrict the divisorial closure in S to $\mathcal{H}(S)$, we get a projective star operation on S. Let us denote by v this projective star operation. Recall that the v-operation is bigger than or equal to each star operation on S, in the sense of Definition 1.3.7. Hence we have that every star operation on S, that can be restricted to $\mathcal{H}(S)$, is less than or equal to v.

Proposition 3.4.5. *Let v be the v-operation on S. Then, for every $i = 0, \ldots, n$ the star operation $\star_i$ induced on R_i by v, as in Proposition 3.3.7, is the divisorial closure v_i. In particular v_i and v_j are pairwise compatible for each $i, j = 0, \ldots, n$. Furthermore the projective star operation $sat \circ v = v$.*

Proof. For each $i = 0, \ldots, n$, let J be an ideal of R_i, and let $J^{\star_i} := {}^{a_i}(({}^hJ)_v)$. We will prove that $\star_i = v_i$.

$$\begin{aligned} J^{\star_i} &= {}^{a_i}(({}^hJ)_v) \text{ (by definition of } \star_i) \\ &= {}^{a_i}\left(S : (S : {}^hJ)\right) \text{ (by definition of } v) \\ &= \left({}^{a_i}S : ({}^{a_i}S : {}^{a_ih}J)\right) \ (a_i \text{ commutes with colon}) \\ &= (R_i : (R_i : J)) \text{ (since } {}^{a_i}S = R_i) \\ &= J_{v_i} \text{ (by definition of } v_i). \end{aligned}$$

Hence $\star_i = v_i$ and $\{v_0, \ldots, v_n\}$ are pairwise compatible by Lemma 3.3.11 (i).
If $\star$ is the projective star operation obtained from $\{v_0, \ldots, v_n\}$ as in Proposition 3.3.13 we have that $\star = sat \circ v$, but as discussed before sat is less than or equal to v. So for every $I \in \mathcal{H}(S)$, $({}^{sat}I)_v = {}^{sat}(I_v) = I_v$, and $\star = v$. □

Remark 3.4.6. From Proposition 3.4.5, it follows that v is a maximum also in the set of projective star operations. For suppose $\star$ is a projective star operation such that $v \leq \star$, then $sat \leq v \leq \star$ and $sat \circ \star = \star$. But for each $i = 0, \ldots, n$, $v_i \leq \star_i$, since $v \leq \star$. As v_i is the biggest star operation on R_i it follows that $v_i = \star_i$ for each $i = 0, \ldots, n$. Thus $\star = sat \circ \star = sat \circ v = v$. In particular every divisorial ideal of S is saturated because $(I_v)^{sat} = I_v$.

3.5 Projective Kronecker function rings

To introduce the concept of a Kronecker function ring associated to a projective star operation, we need, first of all, to define the analogue of the e.a.b. property (cf. Definition 1.1.7), for such operations. So we define an e.a.b. projective star operation on S to be a projective star operation that yields e.a.b. star operations on each R_i. Next we show that an e.a.b. projective star operation $\star$ with $sat \circ \star = \star$ satisfies the cancellation property of the classical e.a.b. star operation.

Definition 3.5.1. A projective star operation $\star$ on S is *e.a.b.* if the star operations $\star_i$ on R_i, $i = 0, \ldots, n$, built in Proposition 3.3.7 from $\star$ are all e.a.b. star operations.

From the definition above, we have in particular that $sat \circ b$ is an e.a.b. projective star operation on S.

Lemma 3.5.2. *Let $\star$ be an e.a.b. projective star operation on S such that $\star = sat \circ \star$. Then for any finitely generated homogenous ideals I, J and N of S,*

$$(IN)^\star \subseteq (JN)^\star \Longrightarrow I^\star \subseteq J^\star.$$

Proof. Let I, J, N be finitely generated ideals of S and suppose that $(IN)^{sat\circ\star} \subseteq (JN)^{sat\circ\star}$. Then we have by Proposition 3.2.10 (b): for each $i = 0, \ldots, n$,

$$\begin{aligned} {}^{a_i}\left((IN)^\star\right) \subseteq {}^{a_i}\left((JN)^\star\right) &\Longleftrightarrow ({}^{a_i}I^{a_i}N)^{\star_i} \subseteq ({}^{a_i}J^{a_i}N)^{\star_i} \\ &\Longrightarrow ({}^{a_i}I)^{\star_i} \subseteq ({}^{a_i}J)^{\star_i} \\ &\Longrightarrow {}^{a_i}(I^\star) \subseteq {}^{a_i}(J^\star). \end{aligned}$$

Thus by Proposition 3.2.10 (b), $I^{sat\circ\star} \subseteq J^{sat\circ\star}$. The result follows since $\star = sat \circ \star$. □

Now, we want to build a ring (called later *projective Kronecker function ring*) from an e.a.b. projective star operation $\star$ on S that will behave like the classical Kronecker function ring of an e.a.b. star operation and that will have a relationship with the "affine" Kronecker function rings of the R_i's with respect to the e.a.b. star operations $\star_i$. We start by investigating some properties of the notion of the content ideal of a homogeneous polynomial of $S[T]$, where T is a variable over S.

If $f = f_0 + f_1T + \ldots + f_sT^s \in S[T]$, then the *content* of f, denoted $C(f)$, is the ideal of S generated by the coefficients $f_0, f_1, \ldots, f_s$ of f. Note that if f is a homogenous element of $S[T] = K[X_0, \ldots, X_n, T]$ in $n + 2$ variables, then its coefficients are homogeneous elements of S. Thus $C(f)$ is a homogeneous ideal of S.

Remark 3.5.3. Let $f = f_0 + f_1T + \ldots + f_sT^s$ be a homogeneous element of degree m in $K[X_0, \ldots, X_n, T] = S[T]$. Then

$$\frac{f}{X_i^m} = \frac{f_0}{X_i^m} + \frac{f_1}{X_i^{m-1}}\frac{T}{X_i} + \ldots + \frac{f_s}{X_i^{m-s}}(\frac{T}{X_i})^s \in R_i[\frac{T}{X_i}],$$

for each $i = 0, \ldots, n$.

We also have, for each $i = 0, \ldots, n$:

$$^{a_i}C_S(f) = (^{a_i}f_0, {}^{a_i}f_1, \ldots, {}^{a_i}f_s) = C_{R_i}(\frac{f}{X_i^m}) \qquad (1)$$

Now set

$$L := \left\{ \frac{f}{g} : f, g \text{ homogeneous of same degree in } S[T] \text{ and } g \neq 0 \right\}$$

It is clear that L is a field and it is not hard to see that L is in fact the field $K(\frac{X_0}{T}, \ldots, \frac{X_n}{T})$. Let $\star$ be an e.a.b. projective star operation on S such that $sat \circ \star = \star$. Let

$$\begin{aligned} \mathrm{PKr}(S, \star) :&= \left\{ \frac{f}{g} : f, 0 \neq g \in \mathcal{H}(S[T]), \deg(f) = \deg(g), \text{ and } C(f)^\star \subseteq C(g)^\star \right\} \\ &= \left\{ \frac{f}{g} \in L : C(f)^\star \subseteq C(g)^\star \right\}. \end{aligned}$$

We can immediately note by Lemma 3.5.2 that the set $\mathrm{PKr}(S, \star)$ is well-defined using the fact that for all $f, g \in S[T] \setminus \{0\}$, $C(fg)^\star = (C(f)C(g))^\star$ ([22, Lemma 32.6]). We also note that $\mathrm{PKr}(S, \star)$ quite "looks" like the classical Kronecker function ring, but contrary to the classical one $S \not\subseteq \mathrm{PKr}(S, \star)$. In fact, X_i is not in $\mathrm{PKr}(S, \star)$ for any $i = 0, \ldots, n$. A natural question is whether $\mathrm{PKr}(S, \star)$ is a ring. We give an answer in the next proposition:

Proposition 3.5.4. *Let $\star$ be an e.a.b. projective star operation on S such that $sat \circ \star = \star$. Then $\mathrm{PKr}(S, \star)$ is a domain with quotient field $L = K(\frac{X_0}{T}, \ldots, \frac{X_n}{T})$ (We do have, $K[\frac{X_0}{T}, \ldots, \frac{X_n}{T}] \subseteq \mathrm{PKr}(S, \star) \subseteq K(\frac{X_0}{T}, \ldots, \frac{X_n}{T})$).*

Proof. The fact that $\mathrm{PKr}(S, \star)$ is a domain is proved using the same argument as the one of the classical Kronecker function ring (i.e., [22, Proof of Theorem 32.7 (a)]). It is clear that for each $i = 0, \ldots, n$, $\frac{T}{X_i} \in \mathrm{PKr}(S, \star) \subseteq K(\frac{X_0}{T}, \ldots, \frac{X_n}{T})$. Since $K \subseteq \mathrm{PKr}(S, \star)$, the result follows. □

We next make a connection between the ring $\mathrm{PKr}(S, \star)$ and the classical Kronecker function rings $\mathrm{Kr}(R_i, \star_i)$, $0 \leq i \leq n$, when the $\star_i$'s are pairwise compatible e.a.b. star operations on R_i's and $\star$ is built from the $\star_i$'s as in Proposition 3.3.13. Note in this case that $\star$ is an e.a.b. projective star operation on S and $sat \circ \star = \star$ (Proposition 3.3.13).

Theorem 3.5.5. *Let $\star_0, \ldots, \star_n$ be $n+1$ pairwise compatible e.a.b star operations on $R_0, \ldots, R_n$ respectively. Let $\star$ be the projective star operation on S built from the $\star_i$'s in the sense of Proposition 3.3.13. Then* $\mathrm{PKr}(S, \star) = \bigcap_{i=0}^{n} \mathrm{Kr}(R_i, \star_i)$.

Proof. First we note that the quotient field of $\mathrm{PKr}(S,\star)$ is $L = K(\frac{X_0}{T}, \ldots, \frac{X_n}{T})$ and the quotient field of each $\mathrm{Kr}(R_i, \star_i)$, $0 \leq i \leq n$, is $K(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}, \frac{T}{X_i}) = K(\frac{X_0}{T}, \ldots, \frac{X_n}{T}) = L$. We recall that for any homogeneous element $f \in S[T]$ of degree m, ${}^{a_i}C_S(f) = C_{R_i}(\frac{f}{X_i^m})$, $0 \leq i \leq n$ (see relation (1) in Remark 3.5.3). Hence we have:

$$
\begin{aligned}
X \in \mathrm{PKr}(S,\star) &\Longleftrightarrow X = \frac{f}{g}, f, g \in L, C_S(f)^\star \subseteq C_S(g)^\star \\
&\Longleftrightarrow X = \frac{f}{g}, f, g \in L, {}^{a_i}[C_S(f)^\star] \subseteq {}^{a_i}[C_S(g)^\star] \\
&\Longleftrightarrow X = \frac{f}{g}, f, g \in L, [{}^{a_i}C_S(f)]^{\star_i} \subseteq [{}^{a_i}C_S(g)]^{\star_i}, \ \forall i = 0, \ldots, n \\
&\Longleftrightarrow X = \frac{f}{g}, f, g \in L, C_{R_i}(\frac{f}{X_i^m})^{\star_i} \subseteq C_{R_i}(\frac{g}{X_i^m})^{\star_i}, \ \forall i = 0, \ldots, n \\
&\Longleftrightarrow X = \frac{f}{g} \in \bigcap_{i=0}^{n} \mathrm{Kr}(R_i, \star_i)
\end{aligned}
$$

$\square$

Our next goal is to prove that the projective Kronecker function ring of S with respect to a projective star operation $\star$ on S, which is built from $n+1$ pairwise compatible e.a.b. star operations $\star_0, \ldots, \star_n$ on $R_0, \ldots, R_n$ respectively is an F-function ring in the sense of Halter-Koch (cf. Definition 1.3.12).

Recall that for each $i = 0, \ldots, n$ $R_i = K[\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i}]$ with quotient field $F := K(\frac{X_0}{X_i}, \ldots, \frac{X_n}{X_i})$ (the notation F for the quotient field of R_i is due to the fact that it doesn't depend on i). Since $\star_i$ is an e.a.b. star operation on R_i, we can represent the Kronecker function ring $\mathrm{Kr}(R_i, \star_i)$ of R_i in terms of valuations: $\mathrm{Kr}(R_i, \star_i) = \bigcap_{V \in \Sigma_i} V^b$, where Σ_i is a suitable subset of $\mathrm{Zar}(R_i)$ (cf. Theorem 1.1.12 and Theorem 1.1.15).

By Theorem 3.5.5, we have:

Corollary 3.5.6. *Under the hypothesis of Theorem 3.5.5,* $\mathrm{PKr}(S,\star)$ *is an* F*-function ring.*

Definition 3.5.7. If $\star$ is a projective star operation on S built from compatible e.a.b. star operations $\star_0,\ldots,\star_n$ on $R_0,\ldots,R_n$ respectively, then $\mathrm{PKr}(S,\star)$ is called the *projective Kronecker function ring* of S with respect to $\star$.

As we already observed, the rings of the form $\bigcap_{V\in \mathrm{Zar}(F/K)} V^b$ where $K\subseteq F$ is a field extension, are F-function rings which cannot be always derived from the classical star operations. When we consider the case in which F is a function field of K, we will see that we can deduce such an F-function ring by using the notion of projective star operation we introduced.

Recall that $S=K[X_0,\ldots,X_n]$ and, for each $i=0,\ldots,n$, $R_i=K[\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i}]$ with quotient field $F:=K(\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i})$. Set

$$H:=\bigcap_{V\in \mathrm{Zar}(F/K)} V^b.$$

We want to prove that H is a projective Kronecker function ring $\mathrm{PKr}(S,\star)$ of S, where $\star$ is an appropriate projective star operation on S.

Proposition 3.5.8. *The* F*-function ring* $H=\bigcap_{i=0}^n \mathrm{Kr}(R_i,b_i)$.

Proof. We remark first that $\mathrm{Kr}(R_i,b_i)=\bigcap_{V\in \mathrm{Zar}(F/R_i)} V^b$. Let $V\in \mathrm{Zar}(F/R_i)$. Then $K\subseteq R_i\subseteq V\subseteq F$. Hence $V\in \mathrm{Zar}(F/K)$. Thus $H\subseteq \mathrm{Kr}(R_i,b_i)$ for all $i=0,\ldots,n$. Hence $H\subseteq \bigcap_{i=0}^n \mathrm{Kr}(R_i,b)$.

Now let $V\in \mathrm{Zar}(F/K)$. We have $F\subseteq L:=K(X_0,\ldots,X_n)$. Let w be a valuation that extends the valuation v to L. Pick j such that $w(X_j)=\min\{w(X_i): 0\leq i\leq n\}$. Then $w(\frac{X_i}{X_j})\geq 0$ for all $i=0,\ldots,n$. Hence $R_j\subseteq W\cap F=V\subseteq F$. So $V\in \mathrm{Zar}(F/R_j)$ for some j. Thus $\bigcap_{V\in \mathrm{Zar}(F/R_i)} V^b\subseteq \mathrm{Kr}(R_j,b)\subseteq V^b$. Hence $\bigcap_{i=0}^n Kr(R_i,b)\subseteq H$. □

The same result can be proven by using models:

Proof. Recall that the model $M=\bigcup_{i=0}^n V(R_i)$ is complete, as a projective model. Hence, each $V\in \mathrm{Zar}(F/K)$ dominates at least one domain of the form $(R_i)_{P_i}$ for some i. Therefore $V\in \mathrm{Zar}(F/R_i)=\mathrm{Zar}(R_i)$ because F is the quotient field of the R_i's. So $V^b\in \mathrm{Zar}(\mathrm{Kr}(R_i,b_i))$ and $H\supseteq \bigcap_{i=0}\mathrm{Kr}(R_i,b_i)$. The other inclusion is clear since, for each $i=0,\ldots,n$, a valuation overring W of $\mathrm{Kr}(R_i,b_i)$ is such that $W\cap F\in \mathrm{Zar}(R_i)$ and, in particular, $W\cap F\in \mathrm{Zar}(F/K)$, so that $W=(W\cap F)^b\supseteq H$. □

By Proposition 3.4.3, the b_i's are pairwise compatible e.a.b. star operations on the R_i's and the projective star operation built from those is $sat \circ b$, where b is the b-operation on S. By Theorem 3.5.5, $\mathrm{PKr}(S, sat \circ b) = \bigcap_{i=0}^{n} \mathrm{Kr}(R_i, b_i) = H$. Thus:

Corollary 3.5.9. *The F-function ring H coincides with the projective Kronecker function ring* $\mathrm{PKr}(S, sat \circ b)$.

3.6 Discussion and questions

These results on projective star operations are a work in progress with O. Heubo. Although we could build a natural correspondence between the set of projective star operations over S and classical star operations defined on the underlying domains R_i's, we are trying to generalize our work in the following directions:

Question 3.6.1. When $S := K[X_0, \ldots, X_n]/P$ is the quotient of a polynomial ring over a homogeneous prime ideal, is it possible to make a similar construction? (In this case F can be chosen to be an algebraic function field of K, rather than a function field).

The main obstacle for such a generalization is to define properly the operations of dehomogenization and homogenization. The same problem occurs when going in the following direction:

Question 3.6.2. A projective model is a topological space. In our case we cover it with the $n+1$ affine models $V(R_i)$. Does a projective star operation depend on the choice of the R_i's? From the theory of projective schemes we know it should not, since coherent sheaves of ideals are then defined on each open set. How could we define for each open subset U of $\mathrm{Proj}(S)$ a star operation on $\mathcal{O}(U)$ (where $\mathcal{O}$ denotes the structure sheaf of $\mathrm{Proj}(S)$)?

Even in our more restrictive case, we were not able to find an example of a mapping behaving like a star operation, but just on homogeneous ideals. Hence we could not answer to the following question.

Question 3.6.3. We presented in Example 3.3.6 a star operation on S that is not a projective star operation. What about the reverse situation? Does there exist a projective star operation on S that is not a star operation in the classical sense on S? Or is it always possible to lift a projective star operation to a star operation?

Bibliography

[1] S. S. Abhyankar. *Resolution of singularities of embedded algebraic surfaces*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 1998.

[2] D. D. Anderson and D. F. Anderson. Generalized GCD domains. *Comment. Math. Univ. St. Paul.*, 28(2):215–221, 1980.

[3] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[4] M. B. Boisen, Jr. and P. B. Sheldon. CPI-extensions: overrings of integral domains with special prime spectrums. *Canad. J. Math.*, 29(4):722–737, 1977.

[5] D. E. Dobbs, R. Fedder, and M. Fontana. Abstract Riemann surfaces of integral domains and spectral spaces. *Ann. Mat. Pura Appl. (4)*, 148:101–115, 1987.

[6] D. E. Dobbs and M. Fontana. Locally pseudovaluation domains. *Ann. Mat. Pura Appl. (4)*, 134:147–168, 1983.

[7] D. E. Dobbs and M. Fontana. Kronecker function rings and abstract Riemann surfaces. *J. Algebra*, 99(1):263–274, 1986.

[8] D. Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[9] D. Eisenbud and J. Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[10] S. El Baghdadi, M. Fontana, and G. Picozza. Semistar Dedekind domains. *J. Pure Appl. Algebra*, 193(1-3):27–60, 2004.

[11] A. J. Engler and A. Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[12] M. Fontana. Topologically defined classes of commutative rings. *Ann. Mat. Pura Appl. (4)*, 123:331–355, 1980.

[13] M. Fontana, J. A. Huckaba, and I. J. Papick. *Prüfer domains*, volume 203 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1997.

[14] M. Fontana, P. Jara, and E. Santos. Prüfer $\star$-multiplication domains and semistar operations. *J. Algebra Appl.*, 2(1):21–50, 2003.

[15] M. Fontana and K. A. Loper. Kronecker function rings: a general approach. In *Ideal theoretic methods in commutative algebra (Columbia, MO, 1999)*, volume 220 of *Lecture Notes in Pure and Appl. Math.*, pages 189–205. Dekker, New York, 2001.

[16] M. Fontana and K. A. Loper. Nagata rings, Kronecker function rings, and related semistar operations. *Comm. Algebra*, 31(10):4775–4805, 2003.

[17] M. Fontana and K. A. Loper. An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations. In *Multiplicative ideal theory in commutative algebra*, pages 169–187. Springer, New York, 2006.

[18] M. Fontana and K. A. Loper. A generalization of Kronecker function rings and Nagata rings. *Forum Math.*, 19(6):971–1004, 2007.

[19] M. Fontana and K. A. Loper. Cancellation properties in ideal systems: a classification of *e.a.b.* semistar operations. *J. Pure Appl. Algebra*, 213(11):2095–2103, 2009.

[20] M. Fontana and G. Picozza. Prüfer $\star$-multiplication domains and $\star$-coherence. *Ric. Mat.*, 55(1):145–170, 2006.

[21] S. Gabelli and E. Houston. Ideal theory in pullbacks. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 199–227. Kluwer Acad. Publ., Dordrecht, 2000.

[22] R. Gilmer. *Multiplicative ideal theory*, volume 90 of *Queen's Papers in Pure and Applied Mathematics*. Queen's University, Kingston, ON, 1992. Corrected reprint of the 1972 edition.

[23] R. Gilmer and W. Heinzer. Irredundant intersections of valuation rings. *Math. Z.*, 103:306–317, 1968.

[24] F. Halter-Koch. Kronecker function rings and generalized integral closures. *Comm. Algebra*, 31(1):45–59, 2003.

[25] J. R. Hedstrom and E. G. Houston. Pseudo-valuation domains. *Pacific J. Math.*, 75(1):137–147, 1978.

[26] W. Heinzer. Integral domains in which each non-zero ideal is divisorial. *Mathematika*, 15:164–170, 1968.

[27] W. Heinzer. Noetherian intersections of integral domains. II. In *Conference on Commutative Algebra (Univ. Kansas, Lawrence, Kan., 1972)*, pages 107–119. Lecture Notes in Math., Vol. 311. Springer, Berlin, 1973.

[28] W. J. Heinzer, J. A. Huckaba, and I. J. Papick. m-canonical ideals in integral domains. *Comm. Algebra*, 26(9):3021–3043, 1998.

[29] O. Heubo Kwegna. Kronecker function rings of algebraic extensions, preprint.

[30] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[31] P. Jaffard. *Les systèmes d'idéaux*. Travaux et Recherches Mathématiques, IV. Dunod, Paris, 1960.

[32] I. Kaplansky. *Commutative rings*. The University of Chicago Press, Chicago, Ill.-London, revised edition, 1974.

[33] K. A. Loper. Constructing examples of integral domains by intersecting valuation domains. In *Non-Noetherian commutative ring theory*, volume 520 of *Math. Appl.*, pages 325–340. Kluwer Acad. Publ., Dordrecht, 2000.

[34] K. A. Loper and F. Tartarone. A classification of the integrally closed rings of polynomials containing $\mathbb{Z}[X]$. *J. Commut. Algebra*, 1(1):91–157, 2009.

[35] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.

[36] A. Mimouni. Integral domains in which each ideal is a W-ideal. *Comm. Algebra*, 33(5):1345–1355, 2005.

[37] A. Okabe and R. Matsuda. Semistar-operations on integral domains. *Math. J. Toyama Univ.*, 17:1–21, 1994.

[38] A. Okabe and R. Matsuda. Kronecker function rings of semistar-operations. *Tsukuba J. Math.*, 21(2):529–540, 1997.

[39] A. Okabe and R. Matsuda. On characterizations of a Prüfer domain. *Math. J. Ibaraki Univ.*, 39:1–10, 2007.

[40] B. Olberding. Noetherian spaces of integrally closed overrings with an application to intersections of valuation overrings, to appear.

[41] B. Olberding. Irredundant intersections of valuation overrings of two-dimensional Noetherian domains. *J. Algebra*, 318(2):834–855, 2007.

[42] B. Olberding. Overrings of two-dimensional Noetherian domains representable by Noetherian spaces of valuation rings. *J. Pure Appl. Algebra*, 212(7):1797–1821, 2008.

[43] G. Picozza and F. Tartarone. Flat ideals and stability, preprint.

[44] G. Picozza and F. Tartarone. When the semistar operation $\tilde{\star}$ is the identity. *Comm. Algebra*, 36(5):1954–1975, 2008.

[45] F. Richman. Generalized quotient rings. *Proc. Amer. Math. Soc.*, 16:794–799, 1965.

[46] I. Swanson and C. Huneke. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006.

[47] M. Zafrullah. On finite conductor domains. *Manuscripta Math.*, 24(2):191–204, 1978.

[48] M. Zafrullah. Flatness and invertibility of an ideal. *Comm. Algebra*, 18(7):2151–2158, 1990.

[49] O. Zariski and P. Samuel. *Commutative algebra. Vol. II.* Springer-Verlag, New York, 1975. Reprint of the 1960 edition, Graduate Texts in Mathematics, Vol. 29.

Acknowledgments

I wish to express my deepest gratitude to Prof. Bruce Olberding, for suggesting me the very interesting topics of my thesis and for helping me with his precious advices through these years. I also thank him for the encouragement, moral support and for the great hospitality when I was a visitor at New Mexico State University.

I wish to thank Prof. Marco Fontana for putting his trust in me, for supporting me and for being present with many useful discussions.

I am grateful to Prof. Stefania Gabelli, Dr. Florida Girolami and Prof. Francesca Tartarone for their constant encouragement and moral support.

Thanks to Olivier Heubo for his collaboration. Also I feel thankful to him and to all the Department of Mathematical Sciences for their hospitality when I was a visitor at New Mexico State University.

I wish to thank Prof. Stefania Gabelli, Prof. Valentina Barucci and Prof. Luigi Salce for accepting to be members in my thesis committee.

Grazie a tutti i miei colleghi dottorandi, in particolare alle mie compagne d'ufficio Elisa, Silvia e Cristiana, per aver allietato questi anni di vita insieme.