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Isochronous Systems and Orthogonal Polynomials

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Foreword

The theory of orthogonal polynomials has seen many remarkable developments during the last two decades, due to its connections with integrable systems, spectral theory and random matrices. Indeed, in recent years the interest for this theory has often arisen from outside the orthogonal polynomial community after their connection with integrable systems was found. In this thesis the study has been restricted to classical *integrable* dynamical systems and a *new connection* with orthogonal polynomials is presented.

In fact, we have considered several many-body systems that have the fundamental property to be integrable. The main results in this thesis are related to *Diophantine* findings obtained from an important class of integrable systems: *Isochronous systems.* The most famous of these systems is the *harmonic oscillator*. The first remarkable Diophantine conjecture has been presented over a century and half ago by Sylvester [45] and revisited and proven recently by Askey [2] and Holtz [38]. An extension of it is presented in [12] . In the first chapter we explain in detail the properties of these systems and how we can construct isochronous systems from a large class of integrable systems. To clarify this issue we will present one example of this procedure in the case of the Toda system. In chapter 2, we take one Diophantine conjecture and we give the proof, so we will see the connection of the Diophantine properties with orthogonal polynomials and their *complete factorization.* The aim of chapter 3 is the identifications of classes of orthogonal polynomials defined by three term recursion relations depending on a parameter ν, which satisfy also a second recursion involving that parameter, and some of which feature zeros given by formulas involving integers. In chapter 4 we apply the machinery developed in chapter 3 to all the polynomials of the Askey scheme. For these polynomials we identify other, new, additional recursion relations involving a shift of some parameters that they feature. For several of these polynomials we obtain factorization formulas for special values of their parameters. In chapter 6 we connect our machinery with the discrete integrability. We compare the three term recursion relation with a spectral problem involving a discrete Schrödinger operator, and the second recursion with a discrete time evolution for the eigenfunctions. Following the Lax technique developed in the last three decades we will construct an entire hierarchy of equations, and we will see the relation of this hierarchy with the hierarchy of the discrete time Toda lattice.

In the last chapter we present another approach of the our machinery applied to integrable ODE. We consider the stationary KdV's hierarchy, but this general procedure could be extended to various soliton hierarchies.

The results reported and discussed in this thesis have been obtained working in collaboration with Prof. F. Calogero and Prof. M. Bruschi of Roma University "La Sapienza" and under the supervision of Prof.O. Ragnisco. They describe the research activity that I have carried out in the last three years, during my Ph.D. at the Rome 3 University.

Contents

Chapter 1

From Isochronous System to Diophantine properties and related Orthogonal Polynomials.

A classical dynamical system is called *isochronous* if exists in its phase space an open set of fully dimensionality, called "the isochrony region", where *all solutions are periodic in all their degrees of freedom with the same, fixed period,* independently on the initial data provided. A well-known isochronous N-body system is the Calogero-Moser model characterized by the Hamiltonian

$$H(\underline{z}, \underline{p}) = \frac{1}{2} \sum_{n=1}^{N} \left[p_n^2 + \left(\frac{\omega}{2} \right)^2 z_n^2 \right] + \frac{1}{4} \sum_{m,n=1; m \neq n}^{N} \frac{g^2}{(z_n - z_m)^2} , \qquad (1.1)$$

and correspondingly by the Newtonian equations of motion

$$\ddot{z}_n + \left(\frac{\omega}{2} \right)^2 z_n = \sum_{m=1; m \neq n}^{N} \frac{g^2}{(z_n - z_m)^3} , \qquad (1.2)$$

where ω is a *positive* constant, $\omega > 0$.

Indeed, in the *real* domain, *all* the solutions of these equations of motion are *isochronous*, namely completely periodic:

$$z_n (t + T) = z_n (t)$$

with

$$T = \frac{2\pi}{\omega}$$

provided $g \neq 0$.

We call *entirely isochronous* the dynamical systems which are *isochronous* in their *entire* (natural) phase space- possibly up to a *lower dimensional* set of *singular* solutions, and possibly featuring a *finite* (generally small) number of different periods- all of them *integer* multiples of a basic one- in different, *fully dimensional,* **r**egions of their phase space, separated from each other by the *lower dimensional* set of data yielding *singular* solutions.

Now we will introduce a *trick* that modifies a integrable dynamical system into a new system, again integrable, which exhibits the remarkable property of been *isochronous.* This trick is a *complex* transformation of the dependent and independent variables. It contains a *positive,* **real** parameter ω: the "ω-modified" model yielded by this transformation is isochronous because it possesses an open set of full dimensionality in its phase space where all solutions are completely periodic, i.e. periodic in all degree of freedom with the same period $\tilde{T}$ which is a finite multiple of the basic period $T = \frac{2\pi}{\omega}$. The $\omega-$modified system manufactured in this manner can be generally made autonomous: this being indeed the case we deem more interesting hence on which we focus hereafter. It obtains from (possibly nonautonomous) dynamical systems belonging to a quite large class, restricted mainly by the condition that it makes sense to extend by analytic continuation its time evolution to complex time, and moreover by a scaling requirement.

We take into account a quite general dynamical system possibly non autonomous, which we write

$$\underline{\zeta}' = \underline{F}(\underline{\zeta};\tau) \ . \qquad (1.3)$$

Here, $\underline{\zeta} = \underline{\zeta}(\tau) = (\zeta_1(\tau),\ \zeta_2(\tau), ...,\ \zeta_n(\tau))$ is the dependent variable which might be a scalar, a vector, a tensor, etc. The independent variable is τ and we assume that it is permissible to consider as complex. This requires that the derivative with respect to this complex variable τ appearing in the left-hand side of this evolution equation (1.3) makes sense, namely that the dynamical system (1.3) is analytic entailing in turn that the dependent variable $\underline{\zeta}$ is an *analytic* function of the complex variable τ. Notice, however, that this does not require $\underline{\zeta}(\tau)$ to be a *holomorphic* or a *meromorphic* function of τ; $\underline{\zeta}(\tau)$ might feature all sorts of singularities, including branch points, in the *complex* τ-plane, indeed this will generally happen since we assume the evolution equation (1.3) to be *nonlinear.*

Now we introduce the following transformations of the dependent and independent variables:

$$\underline{z}(t) = \exp(i\lambda\omega t)\underline{\zeta}(\tau) \ ; \qquad (1.4a)$$

$$\tau \equiv \tau(t) = \frac{\exp(i\omega t) - 1}{i\omega}. \qquad (1.4b)$$

This transformation is called "*The trick*" (It was first introduced in [8]). The essential part of it is the change of *independent* variable (1.4b): and let us emphasize that the new *independent* variable t is considered as the *real*, "physical" time variable.

Note that (1.4) implies

$$\tau(0) = 0 \quad , \; \underline{z}(0) = \underline{\zeta}(0) \quad , \; \underline{\dot{z}}(0) - i\lambda\omega\ \underline{z}(0) = \underline{\zeta}'(0) \ , \tag{1.5}$$

(where the symbol $' = \frac{d}{d\tau}$, and the dot symbol $\cdot = \frac{d}{dt}$) and most importantly, that $\tau(t)$ is a *periodic* function of t with period T. As the time t increase from zero onward, the *complex* variable $\tau(t)$ travels counterclockwise making a full tour in the time T on the circle C, whose diameter of length $\frac{2}{\omega}$ lies on the imaginary axis in the complex τ-plane with one extreme at the origin $\tau = 0$ and the other at the point $\tau = \frac{2i}{\omega}$. Hence, if $\underline{\zeta}(\tau)$ is *holomorphic* in τ in the disk D encircled by the circle C, then the trick relation implies that $\underline{z}(t)$ is "λ-periodic,"

$$\underline{z}(t + T) = \exp(2\pi i\lambda)\ \underline{z}(t) \ . \tag{1.6}$$

The "ω- modified" dynamical system is the one that obtains from (1.3) via the trick (1.4). The new equations of motion are

$$\underline{\dot{z}}(t) - i\lambda\omega\ \underline{z}(t) = \exp\left[i(\lambda+1)\omega t\right] \underline{F}\left(\exp(-i\lambda\omega t)\ \underline{z};\ \frac{\exp(i\omega t) - 1}{i\omega}\right) \ . \tag{1.7}$$

We are interested at the case of *autonomous* dynamical systems. Indeed, if (1.3) is autonomous $\underline{\zeta}' = \underline{F}(\underline{\zeta})$ and $\underline{F}$ satisfy the *scaling property*

$$\underline{F}(c\underline{\zeta}) = c^{\gamma}\underline{F}(\underline{\zeta}) \tag{1.8}$$

where c is a scalar and with γ a rational number ($\gamma \neq 1$), the "ω- modified" dynamical system (1.7) remain autonomous and we have the following equations of motion

$$\underline{\dot{z}}(t) - i\lambda\omega\ \underline{z}(t) = \underline{F}\left(\underline{z}\ \right) \ , \tag{1.9}$$

if the parameter λ satisfies the condition

$$\lambda = \frac{1}{\gamma - 1} \ ,$$

hence, we see that also the parameter λ is a rational number, because γ is a rational number.

Now we identify, if there exist, the equilibrium configurations of the system. The equilibrium positions are

$$\underline{z}(t) = \underline{\bar{z}} \ ; \tag{1.10}$$

$$\underline{\dot{z}}(t) = \underline{0}; \tag{1.11}$$

and we compute these positions by the following stationary equation:

$$-i\lambda\omega \; \underline{\bar{z}} = \underline{F} \left(\; \underline{\bar{z}} \right) \; . \tag{1.12}$$

After this step, we investigate the small oscillations of this system in the neighbourhood of its equilibrium configurations. We consider:

$$\underline{z}(t) = \underline{\bar{z}} + \epsilon \; \underline{x}(t) \; , \tag{1.13}$$

where $\underline{\bar{z}}$ is the equilibrium position, ϵ is a small parameter $\epsilon << 1$, and inserting it into the equation (1.9), we obtain the following equation

$$\epsilon \underline{\dot{x}}(t) - \epsilon i\lambda\omega \; \underline{x}(t) - i\lambda\omega \underline{\bar{z}} = \underline{F} \left(\; \underline{\bar{z}} + \epsilon \; \underline{x}(t) \right) \; . \tag{1.14}$$

If $\underline{F}$ is analytic then $\exists \; \underline{R}$ such that for $|\epsilon \underline{x}| < \underline{R}$; $\underline{F} \left(\; \underline{\bar{z}} + \epsilon \; \underline{x}(t) \right)$ can be written in Taylor series respect to the parameter ϵ, and it converges uniformly in this domain. The value of $\underline{R}$ depends by the field $\underline{F}$ and by the equilibrium positions $\underline{\bar{z}}$. By construction, the zero order in ϵ gives the equilibrium condition (1.12). At first order in ϵ we have the linearized equations of motion for the "ω-modified" system

$$\dot{x}_n(t) - i\lambda\omega x_n(t) - \sum_{m=1}^{n} \frac{\partial F_n}{\partial z_m}|_{\underline{z}=\underline{\bar{z}}} \; x_m(t) = 0 \; . \tag{1.15}$$

It is possible write the linearized equations only if the autonomous vectorial field $\underline{F}$ is analytic in the variables $\underline{z}$**.** The general solution of the ODE (1.15) is provided by formula

$$x_n(t) = a_n \exp(ip\omega t) \; \tilde{x}_n \; , \tag{1.16}$$

where a is a constant and p is the solution of the determinantal equation

$$\det[(ip\omega - i\lambda\omega) \, \mathbf{I} \; - \mathbf{F}] = 0, \tag{1.17}$$

$\mathbf{F}$ being the matrix defined componentwise as

$$(\mathbf{F})_{n,m} = \frac{\partial F_n}{\partial z_m}|_{\underline{z}=\underline{\bar{z}}} \; .$$

But we already known that the solution of the *isochronous* model (1.9) are *completely periodic* with period T. The same periodicity property must, therefore, characterize the behaviour of solution (1.16) describing the behaviour of the system in the neighbourhood of its equilibrium configuration.

We thus arrive at the following *Diophantine property: the roots p of the equation* (1.17) *are all integers.*

The original strategy to arrive at the *Diophantine* findings that provided the motivation for the developments reported in this thesis can be outlined as follows. (i) Identify an *integrable* dynamical system. (ii) Modify it so that it becomes *isochronous.* (iii) Identify an *equilibrium* configuration of the *isochronous* system. (iv) Investigate, via the standard linearization technique ("the theory of small oscillations around equilibria") the behavior of the *isochronous* system near its *equilibrium* configuration, which is then characterized by a set of basic oscillation frequencies whose values are provided by the *eigenvalues* of a matrix obtained from the equations of motion and evaluated at the *equilibrium* values of the dependent variables. (v) Observe that – because the *isochronous* nature of the dynamical system under consideration must also characterize its behavior around equilibrium – *all* these basic frequencies of oscillation must be *integer* multiples of a basic frequency. (vi) Infer that *all* the *eigenvalues* of the matrix characterizing the behavior around equilibrium must be *integers* (up to a common rescaling). This fact – that *all* the *eigenvalues* of a matrix, of arbitrary order and of reasonably neat appearance, are *integers* – constitute the *Diophantine* finding (which is nontrivial provided the similarity transformation diagonalizing the matrix in question is not obvious).

In the follows we provide – quite tersely – an example of this procedure.

An integrable dynamical system:

$$\eta_n' = \xi_n - \xi_{n-1} \ , \quad \xi_n' = \xi_n(\eta_{n+1} - \eta_n) \quad n = 1, ..., N. \tag{1.18}$$

These are the equations of motion (in the version more convenient for our purposes) of the classical Toda model [35] [36], whose integrability was noted by H. Henon [37] and demonstrated by H. Flaschka [39] [40] and by S. Manakov [42].

Free-end boundary conditions:

$$\xi_0 = \eta_{N+1} = 0 \ . \tag{1.19}$$

The trick:

$$y_n(t) = \exp(it)\eta_n(\tau) \ , \quad x_n(t) = \exp(2it)\xi_n(\tau) \ , \quad \tau = i\left[1 - \exp(it)\right] \ . \tag{1.20}$$

The isochronous version:

$$\dot{y}_n - i\omega\dot{y}_n = x_n - x_{n-1} \ , \ \ \dot{x}_n - 2i\omega x_n = x_n(y_{n+1} - y_n) \ , \ \ x_0 = y_{N+1} = 0 \ . \tag{1.21}$$

Equilibrium configuration (satisfying the free-end boundary conditions):

$$x_n(t) = \bar{x}_n = n(2N+1-n) \ , \quad y_n(t) = \bar{y}_n = 2i(N+1-n) \ . \quad (1.22)$$

Small oscillations around equilibrium:

$$x_n(t) = \bar{x}_n + \epsilon u_n(t) \ , \quad y_n(t) = \bar{y}_n + \epsilon w_n(t) \ , \quad \epsilon \approx 0 \ . \quad (1.23)$$

The linearized equations of motion:

$$\dot{w}_n - i w_n = u_n - u_{n-1} \ , \quad u_0 = 0 \ , \quad (1.24a)$$

$$\dot{u}_n = n(2N+1-n)(w_{n+1} - w_n) \ , \quad w_{N+1} = 0 \ , \quad (1.24b)$$

$$\ddot{w}_n - i\dot{w}_n - n(2N+1-n) w_{n+1} + 2[N(2n-1) - n + 1] w_n - (n-1)(2N-n+2) w_{n-1} = 0 \ , \quad w_{N+1} = 0 \ . \quad (1.24c)$$

The basic oscillations:

$$w_n(t) = \tilde{w}_n \exp(i\lambda t) \ . \quad (1.25)$$

The eigenvalue problem determining the eigenfrequencies λ:

$$\lambda(\lambda-1)\tilde{w}_n + n(2N+1-n)\tilde{w}_{n+1} - 2[N(2n-1) - n + 1]\tilde{w}_n + (n-1)(2N-n+2)\tilde{w}_{n-1} = 0 \ , \quad \tilde{w}_{N+1} = 0 \ . \quad (1.26)$$

Diophantine finding: setting $N = \mu$, defining the $n \times n$ matrix $L^{(\mu)}$ as follows,

$$L^{(\mu)}_{m,m} = m(2\mu - m + 1) + (m-1)(2\mu - m + 2) \ ] \ , \quad (1.27a)$$

$$L^{(\mu)}_{m,m-1} = -(m-1)(2\mu - m + 2) \ , \quad (1.27b)$$

$$L^{(\mu)}_{m,m+1} = -m(2\mu - m + 1) \ , \quad (1.27c)$$

one concludes that the solution of the following polynomial equation of degree $2n$ in λ must have *rational* solutions:.

$$\det[\lambda(\lambda-1) - L^{(n)}] = 0 \ . \quad (1.28)$$

Hence, setting $z = \lambda(\lambda-1)$ so that $\lambda = \left(1 \pm \sqrt{1+4z}\right)/2$, one infers that $1+4z$ must be the square of a *rational* number, say $1+4z = (4m-1)^2$ hence $z = 2m(2m-1)$ with m *rational*. Indeed one finds by direct calculation

$$\begin{vmatrix} z-4 & 4 \\ 4 & q-10 \end{vmatrix} = (z-2)(z-12) \ , \quad (1.29a)$$

$$\begin{vmatrix} z-6 & 6 & 0 \\ 6 & z-16 & 10 \\ 0 & 10 & z-22 \end{vmatrix} = (z-2)(z-12)(z-30) \ , \tag{1.29b}$$

$$\begin{vmatrix} z-8 & 8 & 0 & 0 \\ 8 & z-22 & 14 & 0 \\ 0 & 14 & z-32 & 18 \\ 0 & 0 & 18 & z-38 \end{vmatrix}$$
$$= \quad (z-2)(z-12)(z-30)(z-56) \ , \tag{1.29c}$$

$$\begin{vmatrix} z-10 & 10 & 0 & 0 & 0 \\ 10 & z-28 & 18 & 0 & 0 \\ 0 & 18 & z-42 & 24 & 0 \\ 0 & 0 & 24 & z-52 & 28 \\ 0 & 0 & 0 & 28 & z-58 \end{vmatrix}$$
$$= \quad (z-2)(z-12)(z-30)(z-56)(z-90) \ . \tag{1.29d}$$

One therefore sees that the numbers m are in fact *integers*, and infers that, if one defines the family of polynomials $P_n^{(\mu)}(z)$, of degree n, via the formula

$$\det\left[z - L^{(\mu)}\right] = P_n^{(\mu)}(z) \ , \tag{1.30}$$

there holds the *Diophantine* property

$$P_n^{(n)}(z) = \prod_{m=1}^{n} \left[z - 2m\,(2m-1)\right] \ . \tag{1.31}$$

Indeed the *tridiagonal* character of the $n \times n$ matrix $L^{(\mu)}$, see (1.27), entails that the family of (monic, orthogonal) polynomials (1.30) are characterized by the recursion relation

$$P_{n+1}^{(\mu)}(x) = (x - 2n^2 + 4n\mu + 2\mu)\ P_n^{(\mu)}(x) - n^2(2\mu - n + 1)^2\ P_{n-1}^{(\mu)}(z) \ , \tag{1.32a}$$

$$P_{-1}^{(\nu)}(x) = 0 \ , \qquad P_0^{(\nu)}(x) = 1 \ , \tag{1.32b}$$

and it is therefore immediately seen that they coincide with the polynomials $p_n^{(\nu)}(x)$ of [12] up to the identification $\nu = 2\mu + 1$, so that the *Diophantine* factorization (1.31) coincides with the Smet formula (see eq. (59) of [12])

$$p_n^{(2n+1)}(x) = \prod_{m=1}^{n} \left[x - 2m\,(2m-1)\right] \ . \tag{1.33}$$

We can see by the three terms recursion relation (1.32a) that the polynomials $P_n^{(\mu)}(x)$ *are included into the class of Dual Hahn polynomial for determined values of its parameters.*

Chapter 2

Proof of Diophantine Conjectures and Connection with classes of Orthogonal Polynomials.

In this chapter we want to give a proof of certain *Diophantine conjectures* studied in my Laurea thesis. They were suggested, as we have seen in the Chapter 1 , by the investigation of the behaviour of certain isochronous many-body problems in the neighbourhood of their equilibrium configurations. In this case the integrable lattice investigated was the so called Bruschi-Ragnisco-Ruijsenaars-Toda (see Ref. [19]). In order to do so we identify a class of polynomials satisfying three-term recursion relations (hence belonging to an orthogonal class) which seems of interest in their own right, at least inasmuch as they also yield *additional Diophantine* findings.

The first of these conjectures (see Ref. [19]) states that the *tridiagonal* $N \times N$ matrix $U(N)$ defined componentwise as follows,

$$\begin{aligned} U_{n,n}(N) &= N(N-1)-(n-1)^2-(N-n)^2 = -2\,n^2+(N+1)(2\,n-1)\ , \\ n &= 1,...,N\ , \end{aligned} \tag{2.1a}$$

$$U_{n,n-1}(N) = (n-1)^2\ , \quad n = 2,...,N\ , \tag{2.1b}$$

$$U_{n,n+1}(N) = (N-n)^2\ , \quad n = 1,...,N-1\ , \tag{2.1c}$$

(of course with all other elements vanishing) has the N eigenvalues $n\,(n-1)$, $n = 1,...,N$, i. e.

$$\det[x-U(N)] = \prod_{n=1}^{N}[x-n\,(n-1)]\ . \tag{2.2}$$

Note the symmetry property of this $N \times N$ matrix $U(N)$ under the exchange $n \Longleftrightarrow N+1-n$. And, more importantly, also note that the argument N of this matrix $U(N)$ plays a double role in its definition: it denotes the *order* of this matrix, and moreover it appears as a *parameter* in its definition.

To prove this conjecture we firstly introduce the (more general) *tridiagonal* $M \times M$ matrix $V(M,\nu)$ defined componentwise as follows:

$$\begin{aligned} V_{m,m}(M,\nu) &= \nu(\nu-1)-(m-1)^2-(\nu-m)^2 = -2\,m^2+(\nu+1)(2\,m-1)\ , \\ m &= 1,...,M\ , \end{aligned} \tag{2.3a}$$

$$V_{m,m-1}(M,\nu) = (m-1)^2\ , \quad m=2,...,M\ , \tag{2.3b}$$

$$V_{m,m+1}(M,\nu) = (\nu-m)^2\ , \quad m=1,...,M-1\ , \tag{2.3c}$$

(of course with all other elements vanishing), and the class of polynomials

$$p_n^{(\nu)}(x) = \det\left[x - V(n,\nu)\right]\ . \tag{2.4}$$

It is easily seen that $p_n^{(\nu)}(x)$ is a monic polynomial of degree n in the variable x and it is also a polynomial of degree n in the (*a priori* arbitrary) parameter ν, and that

$$p_1^{(\nu)}(x) = x-\nu+1\ , \quad p_2^{(\nu)}(x) = x^2-2\ (2\,\nu-3)\ x+2\ (\nu-1)\ (\nu-2)\ . \tag{2.5}$$

It is as well plain (see (2.1) and (2.3)) that

$$V(n,n) = U(n)\ , \tag{2.6}$$

hence that the conjecture reported above, see (2.2), amounts to the formula (see (2.4) and (2.6))

$$p_n^{(n)}(x) = \prod_{m=1}^{n}\left[x - m\ (m-1)\right]\ , \tag{2.7a}$$

which for future reference is complemented by the assignment

$$p_0^{(0)}(x) = 1\ . \tag{2.7b}$$

We will show that the polynomials $p_n^{(\nu)}(x)$ satisfy (and are in fact defined by) the three-term recursion relation

$$\begin{aligned} p_{n+1}^{(\nu)}(x) = &\left[x+2\,n^2-2\,(\nu-1)\,n-\nu+1\right]\, p_n^{(\nu)}(x) \\ &-n^2\ (n-\nu)^2\ p_{n-1}^{(\nu)}(x)\ , \quad n=1,2,... \end{aligned} \tag{2.8a}$$

with the "initial assignments"

$$p_{-1}^{(\nu)}(x) = 0\ , \quad p_0^{(\nu)}(x) = 1\ . \tag{2.8b}$$

This is in fact a rather trivial consequence of their definition (2.4) with (2.3) – and, as we show in the following section, it entails for these polynomials the rather explicit representation

$$p_n^{(\nu)}(x) = p_n^{(n)}(x) + \sum_{m=0}^{n-1} \left\{ \left(\frac{n!}{m!} \right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} . \qquad (2.9)$$

In the right hand side of this formula the expressions $p_n^{(n)}(x)$ (and of course likewise $p_m^{(m)}(x)$) are *a priori* given by (2.7) (see the derivation of this formula in the following section), but clearly setting $\nu = n$ in this formula (2.9) this notation gets validated because the sum in the right-hand side of this formula disappears due to the vanishing of the product appearing in it: namely the conjecture (2.7a) is thereby proven.

Clearly this expression, (2.9) with (2.7), of the polynomial $p_n^{(\nu)}(x)$, which is valid when n is a *positive* integer and ν is an *arbitrary* number, entails the relation

$$p_n^{(j)}(x) = p_n^{(n)}(x) + \sum_{m=j}^{n-1} \left\{ \left(\frac{n!}{m!} \right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) \right\} , \qquad (2.10)$$

provided j is a *positive* integer not exceeding n, $j \leq n$ (since the last product in the right-hand side of (2.9) vanishes for $m \leq j \leq n$). Moreover, since clearly (see (2.7a))

$$p_m^{(m)} [k(k-1)] = 0 \quad \text{for} \quad k = 1, 2, ..., m , \qquad (2.11)$$

(2.9) with (2.7) entail (provided k is a *positive integer*)

$$p_n^{(\nu)} [k(k-1)] = p_n^{(n)} [k(k-1)]$$
$$+ \sum_{m=0}^{\min[n-1,k-1]} \left\{ \left(\frac{n!}{m!} \right)^2 \frac{p_m^{(m)} [k(k-1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - \nu) \right\} \qquad (2.12a)$$

and (see (2.10))

$$p_n^{(j)} [k(k-1)] = p_n [k(k-1), n] + \sum_{m=j}^{\min[n-1,k-1]} \left(\frac{n!}{m!} \right)^2 \frac{p_m^{(m)} [k(k-1)]}{(n-m)!} \prod_{\ell=m+1}^{n} (\ell - j) . \qquad (2.12b)$$

Hence (see (2.11), and note that the sum in the right-hand side of (2.12b) vanishes if its lower limit exceeds its upper limit)

$$p_n^{(j)} [k(k-1)] = 0 \quad \text{if} \quad k = 1, 2, ..., j , \quad j = 1, 2, ..., n . \qquad (2.13)$$

In words: the polynomial (of degree n) $p_n^{(j)}(x)$ with j any *positive integer* not exceeding n $(j = 1, ..., n)$ has the j zeros $k\,(k-1)$ with $k = 1, ..., j$. This is a remarkable Diophantine property associated with the family of orthogonal polynomials characterized by the three-term recursion relation (2.8); it includes the result (2.7), which is reproduced for $j = n$.

Note that these j zeros of the polynomial $p_n^{(j)}(x)$ are indipendent of its order n.

It is moreover easily seen that (2.10) with (2.7) entails

$$p_n^{(n-1)}(x) = (x+n)\; p_{n-1}^{(n-1)}(x) = (x+n) \prod_{m=1}^{n-1} [x - m\;(m-1)] \;, \qquad (2.14)$$

$$\begin{aligned} p_n^{(n-2)}(x) &= \left[x^{\,2} + 2\;(2\,n-1)\;x + 2\,n\;(n-1)\right]\, p_{n-2}^{(n-2)}(x) \\ &= [x - x_+(n)]\;[x - x_-(n)] \prod_{m=1}^{n-2} [x - m\;(m-1)] \;, \end{aligned} \qquad (2.15\text{a})$$

of course with

$$x_\pm\,(n) = -2\,n + 1 \pm \left[n^{\,2} + (n-1)^{\,2}\right]^{\,1/2} \;. \qquad (2.15\text{b})$$

It is thus seen that, in addition to the $n-1$ integer zeros $k\;(k-1)$ implied by (2.13), the polynomial $p_n^{(n-1)}(x)$ also vanishes at $x = -n$,

$$p_n^{(n-1)}\,(-n) = 0 \;; \qquad (2.16)$$

hence also this polynomial $p_n^{(n-1)}(x)$, as well as the polynomial $p_n^{(n)}(x)$, has the remarkable *Diophantine* property that *all* its n zeros are *integer* numbers (see (2.14)). On the other hand, as shown by (2.15), for a generic (*positive integer*) value of $n \geq 3$ the polynomial $p_n^{(n-2)}(x)$ has $n-2$ integer zeros (see (2.13)), but its remaining 2 zeros are *not* integer numbers (see (2.15b)), except for n is the second entry in the sequence A001652 [33] of twin pythagorean triples. This confirms that the search for integer zeros of orthogonal polynomials is connected to the existence of perfect codes, and that in this context the results in [31] might point towards further applications of our results.

Note that the recursion relations (2.8a) entail that exist a weight function and an interval where the polynomials $p_n^{(\nu)}\,(x)$ are *orthogonal* (Spectral theorem for the orthogonal polynomials: see [26], or the "*Favard theorem*": see [24], and, for instance, p. 159 of ref. [23]).

It would of course be sufficient at this stage to simply *verify* that the polynomials given by the formulas (2.9) satisfy the recursion relations (2.8) ;

but it seems more appropriate to prove these formulas via a route that makes clear how they were obtained.

The diligent reader will verify that these polynomials, as given by (2.9) , can be identified with generalized hypergeometric functions [23] as follows:

$$p_n^{(\nu)}(x) = n!\ (1-\nu)_n\ {}_3F_2\left(-n, m_+(x), m_-(x); 1, 1-\nu; 1\right)\ , \tag{2.17a}$$

$$m_\pm(x) = \frac{1 \pm (1+4\,x)^{1/2}}{2}\ , \tag{2.17b}$$

Here and below the hypergeometric function is of course defined in the standard manner [23]:

$${}_nF_m\left[a_1, ..., a_n; c_1, ..., c_m; z\right] = \sum_{\ell=0}^{\infty} \frac{(a_1)_\ell \cdots (a_n)_\ell\ z^\ell}{(c_1)_\ell \cdots (c_m)_\ell\ \ell!}\ , \tag{2.18a}$$

with the Pochhammer symbol $(a)_\ell$ defined as follows,

$$(a)_\ell = \frac{\Gamma(a+\ell)}{\Gamma(a)}\ , \tag{2.19}$$

where Γ denotes the standard gamma function [23].

Hence, when the parameter ν is a *positive integer* larger than n, $\nu = N+1$, $N \geq n$, our polynomials $p_n^{(\nu)}(x)$ can be related to the "Dual Hahn polynomials" $R_n[x;\gamma,\delta,N]$ (see for instance [32]) via the formulas

$$p_n^{(N+1)}(x) = n!\ (-N)_n\ R_n[x\,;0,0,N] \tag{2.20}$$

Note that instead our Diophantine results (see (2.10) and the formulae following this equation) refer to the case when n and N are *indeed non-negative* integers but $N+1 \leq n$; and that in these cases these formulas must be interpreted with appropriate care.

Additional remark. Christophe Smet pointed out- on the basic of numerical evidences- that

$$p_n^{(2n+1)}(x) = \prod_{m=1}^{n} \left[x - 2m(2m-1)\right]\ , \tag{2.21}$$

so that also these polynomials $p_n^{(2n+1)}(x)$ *only* feature *integer* zeros.

Now we want prove the results reported previously.

Let us consider firstly the findings connected with the first conjecture [19]. As stated above, the fact that the polynomials defined by (2.4) with (2.3) satisfy the recursion relation (2.8) is an easy consequence of the determinantal definition (2.4) with (2.3): to verify it compute det $[x - V(n+1, \nu)]$ by multiplying the (only two nonvanishing) elements of the last line (or equivalently of the last column) of this determinant by their adjoint determinants, obtaining thereby the recursion (2.8a), and then check if need be that, for $n = 0$ and $n = 1$, (2.8) yields (2.5).

It is then convenient to renormalize the polynomials $p_n^{(\nu)}(x)$ via the definition

$$p_n^{(\nu)}(x) = (n!)^2 \, q_n^{(\nu)}(x) \; , \tag{2.22}$$

entailing that the polynomials $q_n^{(\nu)}(x)$ satisfy the three-term recursion relation

$$\begin{aligned}(n+1)^2 \, q_{n+1}^{(\nu)}(x) = \left[x + 2\,n^2 - 2\,(\nu - 1)\,n - (\nu - 1)\right] \, q_n^{(\nu)}(x) \\ - (\nu - n)^2 \, q_{n-1}^{(\nu)}(x) \; , \quad n = 1, 2, \dots \; ,\end{aligned} \tag{2.23a}$$

$$q_{-1}^{(\nu)}(x) = 0 \; , \quad q_0^{(\nu)}(x) = 1 \; , \quad q_1^{(\nu)}(x) = x - \nu + 1 \; . \tag{2.23b}$$

The purpose of this step is to obtain a recursion relation, see (2.23a), in which the index n only enters quadratically (rather than quartically, see (2.8a)).

Next, we introduce the generating function

$$Q(x, \nu; z) = \sum_{n=0}^{\infty} (z+1)^{-n} \, q_n^{(\nu)}(x) \; . \tag{2.24}$$

It is then rather easy to verify that the recursion relation (2.23) entails that this generating function satisfies the second-order ODE

$$\begin{aligned}(z+1)^2 \, z^2 \, Q'' + (z + 2 - 2\,\nu) \, (z+1) \, z \, Q' \\ + \left[(\nu - 1 - x) \, z + \nu \, (\nu - 1) - x\right] \, Q = 0 \; ,\end{aligned} \tag{2.25}$$

where (just above, and always below) the appended primes denote differentiations with respect to z. It is also plain that, at large values of $|z|$, via (2.23b) we get from (2.24)

$$Q(x, \nu; z) = 1 + \frac{x - \nu + 1}{z} + O\left(\frac{1}{|z|^2}\right) \; . \tag{2.26}$$

To solve the ODE (2.25) and thereby identify the generating function $Q(x, \nu; z)$, it is convenient to set

$$Q(x, \nu; z) = (z+1)^{1-\nu} \, z^{\nu - 1} \, F(x, \nu; z) \; , \tag{2.27}$$

entailing that $F(x,\nu;z)$ satisfies the ODE

$$(z+1)\ z^2\,F''+z^2\,F'-x\,F=0\ . \tag{2.28}$$

It is then plain that, by setting

$$F(x,\nu;z)=\sum_{n=0}^{\infty} C_n\,(x,\nu)\ z^{-n}\ , \tag{2.29}$$

one gets for the quantities $C_n\,(x,\nu)$ the recursion relation

$$C_n\,(x,\nu)=\frac{x-n\,(n-1)}{n^2}\,C_{n-1}\,(x,\nu) \tag{2.30a}$$

entailing

$$C_n\,(x,\nu)=\frac{p_n^{(n)}(x)}{(n!)^2}\,C_0\,(x,\nu)\ . \tag{2.30b}$$

Here and hereafter $p_n^{(n)}(x)$ is defined by (2.7).

Hence we conclude that a solution of (2.28) is provided by the formula

$$F(x,\nu;z)=C_0\,(x,\nu)\sum_{n=0}^{\infty}\frac{p_n^{(n)}(x)}{(n!)^2}\,z^{-n} \tag{2.31}$$

with $C_0\,(x,\nu)$ an arbitrary function of its two arguments. Via (2.27) this yields for the generating function $Q(x,\nu;z)$ the expression

$$Q(x,\nu;z)=C_0\,(x,\nu)\,\left(1+\frac{1}{z}\right)^{1-\nu}\sum_{n=0}^{\infty}\frac{p_n^{(n)}(x)}{(n!)^2}\,z^{-n}\ , \tag{2.32}$$

entailing at large value of $|z|$ (via (2.7))

$$Q(x,\nu;z)=\ C_0\,(x,\nu)\,\left(1+\frac{x+1-\nu}{z}\right)+O\left(\frac{1}{|z|^2}\right)\ . \tag{2.33}$$

Comparing with (2.26) we therefore conclude that $C_0\,(x,\nu)=1$, yielding for $Q(x,\nu;z)$ the final expression

$$Q(x,\nu;z)=\left(1+\frac{1}{z}\right)^{1-\nu}\sum_{n=0}^{\infty}\frac{p_n^{(n)}(x)}{(n!)^2}\,z^{-n}\ . \tag{2.34}$$

It is clear that (2.24) entails the following integral expression for the polynomials $q_n^{(\nu)}(x)$:

$$q_n^{(\nu)}(x)=(2\,\pi\,i)^{-1}\oint dz\ (z+1)^{n-1}\ Q(x,\nu;z)\ , \tag{2.35}$$

with the integral $\oint$ (see just above and always below) being performed, in the complex z-plane, counterclockwise on a closed contour encircling the point $z = -1$ (and not the point $z = 0$). Hence, via (2.34),

$$q_n^{(\nu)}(x) = \sum_{m=0}^{\infty} \frac{p_m^{(m)}(x)}{(m!)^2} \, (2\,\pi\,i)^{-1} \oint dz\, z^{\nu-1-m}\,(z+1)^{n-\nu} \ . \tag{2.36}$$

We now use the formula

$$(2\,\pi\,i)^{-1} \oint dz\, z^{\,\nu-1-m}\,(z+1)^{n-\nu} = \frac{\prod_{\ell=m+1}^{n}(\ell-\nu)}{(n-m)!} \ , \tag{2.37}$$

which is easily proven by expanding $z^{\,\nu-1-m}$ in inverse powers of $(z+1)$,

$$\begin{aligned} z^{\,\nu-1-m} &= (z+1)^{\,\nu-1-m}\left(1-\frac{1}{z+1}\right)^{\nu-1-m} \\ &= \sum_{j=0}^{\infty}(-)^{j}\binom{\nu-1-m}{j}(z+1)^{-j} \ . \end{aligned} \tag{2.38}$$

We thereby obtain the following expression of the polynomials $q_n^{(\nu)}(x)$:

$$q_n^{(\nu)}(x) = \frac{p_n^{(n)}(x)}{(n!)^2} + \sum_{m=0}^{n-1}\left(\frac{1}{m!}\right)^2 \frac{p_m^{(m)}(x)}{(n-m)!} \prod_{\ell=m+1}^{n}(\ell-\nu) \ . \tag{2.39}$$

Via (2.22) this yields (2.9). Q. E. D.

Chapter 3

Tridiagonal Matrices, Orthogonal Polynomials and Diophantine relations.

3.1 Main results

As we have seen in the previous chapter (and we can see in [12]) , these Diophantine conjectures entailed the identification of certain classes of orthogonal polynomials $p_n^{(\nu)}(x)$, of degree n in the variable x and depending (also polynomially) on a parameter ν, which feature zeros given by simple formulae involving integers (and join to sequence of integers: **see** [33]) when the parameter ν takes appropriate integer values.

Our results consist in the identification of classes, defined by *three-term* recursion relations (see (3.1)), of orthogonal polynomials some of which – of arbitrary degree n – feature *zeros* given by neat formulas involving *integers*, or equivalently in the identification of "remarkable" *tridiagonal* matrices – of arbitrary order n, see (3.3) – whose *eigenvalues* are likewise given by neat formulas involving *integers*. Another finding – which is instrumental to get our *Diophantine* results, but seems of interest in its own right (indeed, might possibly be deemed the most interesting finding of this paper) – identifies classes of orthogonal polynomials, defined by three-term recursion relations and depending on a parameter ν (see (3.1)), which moreover also satisfy a *second* recursion involving that parameter (see (3.6)) and possibly as well some remarkable *factorization* properties.

We now present our main results. Let the class of monic polynomials $p_n^{(\nu)}(x)$, of degree n in the variable x and depending on the parameter ν, be defined by the three-term recursion relation

$$p_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right) p_n^{(\nu)}(x) + b_n^{(\nu)} p_{n-1}^{(\nu)}(x) \qquad (3.1a)$$

with the "initial" assignment

$$p^{(\nu)}_{-1}(x) = 0 \ , \quad p^{(\nu)}_{0}(x) = 1 \ , \tag{3.1b}$$

clearly entailing

$$p^{(\nu)}_{1}(x) = x + a^{(\nu)}_{0} \ , \quad p^{(\nu)}_{2}(x) = \left(x + a^{(\nu)}_{1}\right)\left(x + a^{(\nu)}_{0}\right) + b^{(\nu)}_{1} \tag{3.1c}$$

and so on.

Notation: hereafter the index n is a nonnegative *integer* (but some of the formulas written below might make little sense for $n = 0$, requiring a – generally quite obvious – special interpretation), and $a^{(\nu)}_{n}$, $b^{(\nu)}_{n}$ are functions of this index n and of the parameter ν. These functions are hereafter assumed to be independent of the variable x; although a *linear* dependence of $a^{(\nu)}_{n}$ on x and a *quadratic* dependence of $b^{(\nu)}_{n}$ on x would not spoil the polynomial character (of degree n) of $p^{(\nu)}_{n}(x)$. They might also depend on other parameters besides ν (see below); but ν plays a special role, because in the following we shall mainly focus on special values of this parameter (generally simply related to the index n).

Remark 3.1. The polynomials $p^{(\nu)}_{n}(x)$ are generally orthogonal ("Favard theorem" [24] [23]) but this feature plays no role in the following.

Remark 3.2. The (monic, orthogonal) polynomials $p^{(\nu)}_{n}(x)$ defined by the *three-term* recursion relation (3.1) are related to *tridiagonal* matrices via the well-known formula

$$p^{(\nu)}_{n}(x) = \det\left[x - M^{(\nu)}\right] \tag{3.2}$$

with the *tridiagonal* $n \times n$ matrix $M^{(\nu)}$ defined componentwise as follows,

$$M^{(\nu)}_{m,m+1} = \frac{b^{(\nu)}_{m}}{c^{(\nu)}_{m}} \ , \quad m = 1, ..., n-1 \ , \tag{3.3a}$$

$$M^{(\nu)}_{m,m} = -a^{(\nu)}_{m-1} \ , \quad m = 1, ..., n \ , \tag{3.3b}$$

$$M^{(\nu)}_{m,m-1} = -c^{(\nu)}_{m-1} \ , \quad m = 2, ..., n \ , \tag{3.3c}$$

with *all* other elements vanishing. Here the $n-1$ quantities $c^{(\nu)}_{m}$, $m = 1, ..., n-1$ are *arbitrary* (of course *nonvanishing*, $c^{(\nu)}_{m} \neq 0$, see (3.3a)). These formulas entail that the n *zeros* of the polynomial $p^{(\nu)}_{n}(x)$ defined by the three-term recursion relation (3.1) coincide with the n *eigenvalues* of the tridiagonal $n \times n$ matrix $M^{(\nu)}$, see (3.3). $\boxdot$

Hence the *Diophantine* findings reported below, identifying polynomials belonging to *orthogonal* families that feature *zeros* given by neat formulas involving *integers*, might as well be reformulated as identifying *tridiagonal* matrices that are *remarkable* inasmuch as they feature *eigenvalues* given by neat formulas involving *integers.*

Second recursion relation

We now report a result concerning the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ defined by the three-term recursion relations (3.1). This finding is instrumental to obtain the *Diophantine* results detailed in the following, but – as already mentioned above – it seems of interest in itself.

Proposition 3.3. Assume that the quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ satisfy the nonlinear recursion relation

$$\begin{aligned} &\left[A_{n-1}^{(\nu)} - A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \alpha^{(\nu)}\right] \\ = \;&\left[A_{n-1}^{(\nu-1)} - A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)} - A_{n-2}^{(\nu-2)} + \alpha^{(\nu-1)}\right] \end{aligned} \qquad (3.4a)$$

with the boundary condition

$$A_0^{(\nu)} = A \qquad (3.4b)$$

where A is an arbitrary constant (independent of ν), and that the coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ are defined in terms of these quantities by the following formulas:

$$a_n^{(\nu)} = A_{n+1}^{(\nu)} - A_n^{(\nu)} \ , \qquad (3.5a)$$

$$b_n^{(\nu)} = \left[A_n^{(\nu)} - A_n^{(\nu-1)}\right]\left[A_n^{(\nu)} - A_{n-1}^{(\nu-1)} + \alpha^{(\nu)}\right] \ . \qquad (3.5b)$$

Then the polynomials $p_n^{(\nu)}(x)$ identified by the recursion relation (3.1) satisfy the following *additional* recursion relation (involving a shift both in the order n of the polynomials and in the parameter ν):

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)}\, p_{n-1}^{(\nu-1)}(x) \ , \qquad (3.6)$$

with

$$g_n^{(\nu)} = A_n^{(\nu)} - A_n^{(\nu-1)} \ . \ \boxdot \qquad (3.7)$$

A more general version of this *Proposition 3.3* can be formulated, but since we did not (yet) find any interesting application of it we relegate it to the end of the chapter.

Now we obtain some relations satisfied by the quantities $a_n^{(\nu)}$, $b_n^{(\nu)}$, $g_n^{(\nu)}$, as entailed by relations (3.5) and (3.7) with (3.4). These formulae are used in section 3.3 to prove these main results.

The main relations read as follows:

$$a_n^{(\nu)} - a_n^{(\nu-1)} = g_{n+1}^{(\nu)} - g_n^{(\nu)} \ , \tag{3.8a}$$

$$b_{n-1}^{(\nu-1)} \, g_n^{(\nu)} - b_n^{(\nu)} \, g_{n-1}^{(\nu)} = 0 \ , \tag{3.8b}$$

with

$$g_n^{(\nu)} = -\frac{b_n^{(\nu)} - b_n^{(\nu-1)}}{a_n^{(\nu)} - a_{n-1}^{(\nu-1)}} \ , \tag{3.8c}$$

and the "initial" condition

$$g_1^{(\nu)} = a_0^{(\nu)} - a_0^{(\nu-1)} \tag{3.8d}$$

entailing via (3.8c) (with $n = 1$)

$$b_1^{(\nu)} - b_1^{(\nu-1)} + \left(a_0^{(\nu)} - a_0^{(\nu-1)}\right) \left(a_1^{(\nu)} - a_0^{(\nu-1)}\right) = 0 \tag{3.8e}$$

and via (3.8a) (with $n = 0$)

$$g_0^{(\nu)} = 0 \ . \tag{3.8f}$$

The fact that these relations correspond to (3.5) and (3.7) with (3.4) is plain: indeed (3.8a) follows immediately from (3.5a) and (3.7), while (3.8b) follows from (3.5b) and (3.7) via (3.4a).

It would be interesting to find the *general* solution of the nonlinear relations (3.4a) with (or possibly without) (3.4b). We have not (yet) been able to do so, but nontrivial classes of quantities $A_n^{(\nu)}$ and $\alpha^{(\nu)}$ satisfying the nonlinear relations (3.4) are provided in the follows – as well as the corresponding coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ (see (3.5)) and $g_n^{(\nu)}$ (see (3.7)) defining, via the recursion relations (3.1), families of (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ satisfying – as entailed by this *Proposition 3.3* – also the *second* class of recursion relations (3.6).

Proposition 3.4. Assume that the class of (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ defined by the recursion (3.1) satisfies Proposition 3.3, hence that they also obey the ("second") recursion relation (3.6). Then there also holds the relations

$$p_n^{(\nu)}(x) = \left[x - x_n^{(1,\nu)}\right] p_{n-1}^{(\nu-1)}(x) + b_{n-1}^{(\nu-1)} \, p_{n-2}^{(\nu-1)}(x) \ , \tag{3.9a}$$

$$x_n^{(1,\nu)} = -\left[a_{n-1}^{(\nu-1)} + g_n^{(\nu)}\right] \ , \tag{3.9b}$$

as well as

$$p_n^{(\nu)}(x) = \left[x - x_n^{(2,\nu)}\right] p_{n-1}^{(\nu-2)}(x) + c_n^{(\nu)} \, p_{n-2}^{(\nu-2)}(x) \ , \tag{3.10a}$$

$$x_n^{(2,\nu)} = -\left[a_{n-1}^{(\nu-2)} + g_n^{(\nu)} + g_n^{(\nu-1)}\right] \ , \qquad (3.10b)$$

$$c_n^{(\nu)} = b_{n-1}^{(\nu-2)} + g_n^{(\nu)}\, g_{n-1}^{(\nu-1)} \ , \qquad (3.10c)$$

as well as

$$p_n^{(\nu)}(x) = \left[x - x_n^{(3,\nu)}\right] p_{n-1}^{(\nu-3)}(x) + d_n^{(\nu)}\, p_{n-2}^{(\nu-3)}(x) + e_n^{(\nu)}\, p_{n-3}^{(\nu-3)}(x) \ , \qquad (3.11a)$$

$$x_n^{(3,\nu)} = -\left[a_{n-1}^{(\nu-3)} + g_n^{(\nu)} + g_n^{(\nu-1)} + g_n^{(\nu-2)}\right] \ , \qquad (3.11b)$$

$$d_n^{(\nu)} = b_{n-1}^{(\nu-3)} + g_n^{(\nu)}\, g_{n-1}^{(\nu-2)} + g_n^{(\nu-1)}\, g_{n-1}^{(\nu-2)} + g_n^{(\nu)}\, g_{n-1}^{(\nu-1)} \ , \qquad (3.11c)$$

$$e_n^{(\nu)} = g_n^{(\nu)}\, g_{n-1}^{(\nu-1)}\, g_{n-2}^{(\nu-2)} \ . \ \boxdot \qquad (3.11d)$$

These findings correspond to Proposition 1 of Ref. [14].

Factorization

In the following we introduce a second parameter μ, but for notational simplicity we do *not* emphasize *explicitly* the dependence of the various quantities on this parameter.

Proposition 3.5. If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ are defined by the recursion relation (3.1) and the coefficients $b_n^{(\nu)}$ satisfy the relation

$$b_n^{(n+\mu)} = 0 \ , \qquad (3.12)$$

entailing that, for $\nu = n + \mu$, the recursion relation (3.1a) reads

$$p_{n+1}^{(n+\mu)}(x) = \left(x + a_n^{(n+\mu)}\right) p_n^{(n+\mu)}(x) \ , \qquad (3.13)$$

then there holds the *factorization*

$$p_n^{(m+\mu)}(x) = \tilde{p}_{n-m}^{(-m)}(x)\, p_m^{(m+\mu)}(x) \ , \quad m = 0, 1, ..., n \ , \qquad (3.14)$$

with the "complementary" polynomials $\tilde{p}_n^{(-m)}(x)$ (of course of degree n) defined by the following three-term recursion relation analogous (but not identical) to (3.1):

$$\tilde{p}_{n+1}^{(-m)}(x) = \left(x + a_{n+m}^{(m+\mu)}\right) \tilde{p}_n^{(-m)}(x) + b_{n+m}^{(m+\mu)}\, \tilde{p}_{n-1}^{(-m)}(x) \ , \qquad (3.15a)$$

$$\tilde{p}_{-1}^{(-m)}(x) = 0 \ , \quad \tilde{p}_0^{(-m)}(x) = 1 \ , \qquad (3.15b)$$

entailing

$$\tilde{p}_1^{(-m)}(x) = x + a_m^{(m+\mu)} \ , \qquad (3.15c)$$

$$\tilde{p}_2^{(-m)}(x) = \left(x + a_{m+1}^{(m+\mu)}\right)\left(x + a_m^{(m+\mu)}\right) + b_{m+1}^{(m+\mu)}$$
$$= \left(x - x_m^{(+)}\right)\left(x - x_m^{(-)}\right) \tag{3.15d}$$

with

$$x_m^{(\pm)} = \frac{1}{2}\left\{-a_m^{(m+\mu)} - a_{m+1}^{(m+\mu)} \pm \left[\left(a_m^{(m+\mu)} - a_{m+1}^{(m+\mu)}\right)^2 - 4\, b_{m+1}^{(m+\mu)}\right]^{1/2}\right\}, \tag{3.15e}$$

and so on. ⊡

Note incidentally that also the complementary polynomials $\tilde{p}_n^{(-m)}(x)$, being defined by three-terms recursion relations, see (3.15a), belong to orthogonal families, hence they shall have to be eventually investigated in such a context, perhaps applying also to them the kind of findings reported in this paper and in others of this series.

The following two results are immediate consequences of this Proposition 3.5**.**

Corollary 3.6. If (3.12) holds – entailing (3.13) and (3.14) with (3.15) – the polynomial $p_n^{(n-1)}(x)$ has the zero $-a_{n-1}^{(n-1)}$,

$$p_n^{(n-1+\mu)}\left(-a_{n-1}^{(n-1+\mu)}\right) = 0 \;, \tag{3.16a}$$

and the polynomial $p_n^{(n-2+\mu)}(x)$ has the two zeros $x_{n-2}^{(\pm)}$, see (3.15e),

$$p_n^{(n-2+\mu)}\left(x_{n-2}^{(\pm)}\right) = 0 \;. \tag{3.16b}$$

The first of these results is a trivial consequence of (3.13); the second is evident from (3.14) and (3.15d). Note moreover that from the factorization formula (3.14) one can likewise find *explicitly* 3 zeros of $p_n^{(n-3+\mu)}(x)$ and 4 zeros of $p_n^{(n-4+\mu)}(x)$, by evaluating from (3.15) $\tilde{p}_3^{(-m)}(x)$ and $\tilde{p}_4^{(-m)}(x)$ and by taking advantage of the *explicit* solvability of algebraic equations of degree 3 and 4. ⊡

These findings often have a *Diophantine* connotation, due to the neat expressions of the zeros $-a_{n-1}^{(n-1+\mu)}$ and $x_{n-2}^{(\pm)}$ in terms of *integers.*

Corollary 3.7. If (3.12) holds – entailing (3.13) and (3.14) with (3.15) – and moreover the quantities $a_n^{(m)}$ and $b_n^{(m)}$ satisfy the *properties*

$$a_{n-m}^{(-m+\mu)}\left(\underline{\rho}\right) = a_n^{(m+\tilde{\mu})}\left(\underline{\tilde{\rho}}\right) \;, \quad b_{n-m}^{(-m+\mu)}\left(\underline{\rho}\right) = b_n^{(m+\tilde{\mu})}\left(\underline{\tilde{\rho}}\right) \;, \tag{3.17}$$

then clearly

$$\tilde{p}_n^{(m)}\left(x;\underline{\rho}\right) = p_n^{(m+\tilde{\mu})}\left(x;\underline{\tilde{\rho}}\right) \;, \tag{3.18}$$

entailing that the *factorization* (3.14) takes the neat form

$$p_n^{(m+\mu)}\left(x;\underline{\rho}\right)=p_{n-m}^{(-m+\tilde{\mu})}\left(x;\underline{\tilde{\rho}}\right)\,p_m^{(m+\mu)}\left(x;\underline{\rho}\right)\ ,\quad m=0,1,...,n\ . \tag{3.19}$$

Note that – for future convenience, see below – we have emphasized explicitly the possibility that the polynomials depend on additional parameters (indicated with the vector variables $\underline{\rho}$ respectively $\underline{\tilde{\rho}}$; these additional parameters must of course be independent of n, but they might depend on m). ⊡

The following remark is relevant when both Propositions 3.3 and 3.4 hold.

Remark 3.8. As implied by (3.5b), the condition (3.12) can be enforced via the assignment

$$\alpha^{(\nu)}=A_{\nu-1}^{(\nu-1+\mu)}-A_{\nu}^{(\nu+\mu)}\ , \tag{3.20}$$

entailing that the nonlinear recursion relation (3.5a) reads

$$\left[A_{n-1}^{(\nu)}-A_{n-1}^{(\nu-1)}\right]\left[A_n^{(\nu)}-A_{n-1}^{(\nu-1)}+A_{\nu-1}^{(\nu-1+\mu)}-A_{\nu}^{(\nu+\mu)}\right]$$
$$=\left[A_{n-1}^{(\nu-1)}-A_{n-1}^{(\nu-2)}\right]\left[A_{n-1}^{(\nu-1)}-A_{n-2}^{(\nu-2)}+A_{\nu-2}^{(\nu-2+\mu)}-A_{\nu-1}^{(\nu-1+\mu)}\right]\ .\ ⊡ \tag{3.21}$$

Diophantine findings

The *Diophantine* character of the findings reported below is due to the generally neat expressions of the following zeros in term of *integers*.

Proposition 3.9. If the (monic, orthogonal) polynomials $p_n^{(\nu)}(x)$ are defined by the three-term recursion relations (3.1) with coefficients $a_n^{(\nu)}$ and $b_n^{(\nu)}$ satisfying the requirements sufficient for the validity of both Propositions 3.3 and Proposition 3.4 (namely (3.5), with (3.4) and (3.12), or just with (3.21)), then

$$p_n^{(n+\mu)}(x)=\prod_{m=1}^{n}\left[x-x_m^{(1,m+\mu)}\right]\ , \tag{3.22a}$$

with the expressions (3.9b) of the zeros $x_m^{(1,\nu)}$ and the standard convention according to which a product equals unity when its lower limit exceeds its upper limit. Note that these n zeros are *n-independent* (except for their number). In particular

$$p_0^{(\mu)}(x)=1\ ,\quad p_1^{(1+\mu)}(x)=x-x_1^{(1,1+\mu)}\ ,\quad p_2^{(2+\mu)}(x)=\left[x-x_1^{(1,2+\mu)}\right]\left[x-x_2^{(1,2+\mu)}\right]\ , \tag{3.22b}$$

and so on. ⊡

The following results are immediate consequences of this Proposition 3.9 and of Corollary 3.6.

Corollary 3.10. If proposition 3.9 holds, then also the polynomials $p_n^{(n-1+\mu)}(x)$ and $p_n^{(n-2+\mu)}(x)$ (in addition to $p_n^{(n+\mu)}(x)$, see (3.22)) can be written in the following completely factorized form (see (3.9b) and (3.15e)):

$$p_n^{(n-1+\mu)}(x) = \left[x + a_{n-1}^{(n-1)}\right] \prod_{m=1}^{n-1} \left[x - x_m^{(1,m+\mu)}\right] \ , \tag{3.23a}$$

$$p_n^{(n-2+\mu)}(x) = \left[x - x_m^{(+)}\right] \left[x - x_m^{(-)}\right] \prod_{m=1}^{n-2} \left[x - x_m^{(1,m+\mu)}\right] \ . \tag{3.23b}$$

Analogously complete factorizations can clearly be written for the polynomials $p_n^{(n-3+\mu)}(x)$ and $p_n^{(n-4+\mu)}(x)$, see the last part of Corollary 3.6.

And of course the factorization (3.14) together with (3.22a) entails the (generally *Diophantine*) finding that the polynomial $p_n^{(m+\mu)}(x)$ with $m = 1, ..., n$ features the m zeros $x_\ell^{(1,\ell+\mu)}$, $\ell = 1, ..., m$, see (3.9b):

$$p_n^{(m+\mu)}\left(x_\ell^{(1,\ell+\mu)}\right) = 0 \ , \quad \ell = 1, ..., m \ , \quad m = 1, ..., n \ . \ \boxdot \tag{3.24}$$

Proposition 3.11. Assume that for the class of polynomials $p_n^{(\nu)}(x)$ there holds the preceding Proposition 3.3, and moreover that, for some value of the parameter μ (and of course for all *nonnegative integer* values of n), the coefficients $c_n^{(2n+\mu)}$ vanish (see (3.10a) and (3.10c)),

$$c_n^{(2n+\mu)} = b_{n-1}^{(2n+\mu-2)} + g_n^{(2n+\mu)} \, g_{n-1}^{(2n+\mu-1)} = 0 \ , \tag{3.25a}$$

then the polynomials $p_n^{(2n+\mu)}(x)$ factorize as follows:

$$p_n^{(2n+\mu)}(x) = \prod_{m=1}^{n} \left[x - x_m^{(2,2m+\mu)}\right] \ , \tag{3.25b}$$

entailing

$$p_0^{(\mu)}(x) = 1 \ , \quad p_1^{(2+\mu)}(x) = x - x_1^{(2,2+\mu)} \ , \quad p_2^{(4+\mu)}(x) = \left[x - x_1^{(2,2+\mu)}\right] \left[x - x_2^{(2,4+\mu)}\right] \ , \tag{3.25c}$$

and so on.

Likewise, if for all *nonnegative integer* values of n, the following *two* properties hold (see (3.11a), (3.11c) and (3.11d)),

$$d_n^{(3n+\mu)} = b_{n-1}^{(3n+\mu-3)} + g_n^{(3n+\mu)} g_{n-1}^{(3n+\mu-2)} + g_n^{(3n+\mu-1)} g_{n-1}^{(3n+\mu-2)} + g_n^{(3n+\mu)} g_{n-1}^{(3n+\mu-1)} = 0 \ , \tag{3.26a}$$

$$e_n^{(3n+\mu)} = 0 \quad \text{i.e.} \quad g_n^{(3n+\mu)} = 0 \quad \text{or} \quad g_{n-1}^{(3n+\mu-1)} = 0 \quad \text{or} \quad g_{n-2}^{(3n+\mu-2)} = 0 \,, \tag{3.26b}$$

then the polynomials $p_n^{(3n+\mu)}(x)$ factorize as follows:

$$p_n^{(3n+\mu)}(x) = \prod_{m=1}^{n} \left[x - x_m^{(3,3m+\mu)} \right] \,, \tag{3.26c}$$

entailing

$$p_0^{(\mu)}(x) = 1 \,, \quad p_1^{(3+\mu)}(x) = x - x_1^{(3,3+\mu)} \,, \quad p_2^{(6+\mu)}(x) = \left[x - x_1^{(3,3+\mu)} \right] \left[x - x_2^{(3,6+\mu)} \right] \,, \tag{3.26d}$$

and so on.

Here of course the n (n-independent!) zeros $x_m^{(2,2m+\mu)}$ respectively $x_m^{(3,3m+\mu)}$ are defined by (3.10b) respectively (3.11b). ⊡

3.2 Examples

In this Section we report some assignments of the quantities $A_n^{(\nu)}$, $\alpha^{(\nu)}$ – hence correspondingly of the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$, see (3.5) and (3.7) – guaranteeing the validity of *Proposition* 3.3, and often as well of the other results reported in the preceding section when $\mu = 0$; and whenever appropriate we tersely discuss the corresponding polynomials, which are often related to known ("named") ones. But before delving into the exhibition of various examples, let us report the following, rather obvious

Remark 3.12. If a set of coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ satisfy the requirements sufficient to guarantee the validity of the results reported on the previous section, the following extension of it,

$$\check{a}_n^{(\nu)} = \gamma a_n^{(\nu)} + \delta \,, \quad \check{b}_n^{(\nu)} = \gamma^2 b_n^{(\nu)} \,, \quad \check{g}_n^{(\nu)} = \gamma g_n^{(\nu)} \tag{3.27a}$$

with δ and γ two *arbitrary* parameters, also satisfy the same conditions, this extension being clearly related to the following transformation of the corresponding polynomials:

$$\check{p}_n^{(\nu)}(x) = \gamma^n \, p_n^{(\nu)} \left(\frac{x+\delta}{\gamma} \right) \,. \tag{3.27b}$$

Note that the polynomials $\check{p}_n^{(\nu)}(x)$ are as well *monic*. ⊡

In the examples presented below we generally refrain from reducing the number of free parameters by exploiting systematically this *Remark 3.12*, since this might obfuscate rather than highlight the transparency of our findings. The diligent reader is welcome to verify the consistency of all the findings reported below with the validity of this *Remark 3.12*.

Polynomial solution of (3.4)

The following assignment satisfies the nonlinear conditions (3.4):

$$A_n^{(\nu)} = k_0 + k_1 n + k_2 n^2 + k_3 n^3 + \left(k_4 n - \frac{3}{2} k_3 n^2 \right) \nu \ , \tag{3.28a}$$

with

$$\alpha^{(\nu)} = -k_1 + k_2 + \frac{1}{2} k_3 + k_4 + k_5 - \left(2k_2 + \frac{3}{2} k_3 + 2k_4 \right) \nu + \frac{3}{2} k_3 \nu^2 \ , \tag{3.28b}$$

$$A = k_0 \ . \tag{3.28c}$$

Here the 5 parameters $k_j \ , \ j = 1, .., 5$ are *arbitrary.*

The corresponding expressions of the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ read

$$a_n^{(\nu)} = k_1 + k_2 + k_3 + \left(-\frac{3}{2} k_3 + k_4 \right) \nu + \left[2k_2 + 3k_3 \left(1 - \nu \right) \right] n + 3k_3 n^2 \ , \tag{3.28d}$$

$$b_n^{(\nu)} = -\frac{1}{4} n \left(3k_3 n - 2k_4 \right) \left[2k_5 + 2 \left(2k_2 + k_4 \right) \left(n - \nu \right) + 3k_3 \left(n - \nu \right)^2 \right] \ , \tag{3.28e}$$

$$g_n^{(\nu)} = -\frac{1}{2} n \left(3k_3 n - 2k_4 \right) \ . \tag{3.28f}$$

Hereafter we identify our polynomials $p_n^{(\nu)}(x)$ belonging to this class – hence satisfying *Proposition 3.3* – as $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$.

Remark 3.13. It is plain (see (3.28e) and (3.12)) that the subclass $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, 0)$ of these polynomials also satisfy *Propositions 3.5* and *3.9* when $\mu = 0$, entailing the *factorization*

$$p_n^{(n)}(x; k_1, k_2, k_3, k_4, 0) = \prod_{m=1}^{n} \left(x - x_m^{(1,m)} \right) \tag{3.29a}$$

with

$$x_m^{(1,m)} = \alpha^{(m)} = -k_1 + k_2 + \frac{1}{2} k_3 + k_4 - \left(2k_2 + \frac{3}{2} k_3 + 2k_4 \right) m + \frac{3}{2} k_3 m^2 \ , \tag{3.29b}$$

as well as

$$p_n^{(n-1)}(x;k_1,k_{2,}k_{3,}k_{4,}0)=(x-\hat{x}_n)\prod_{m=1}^{n-1}\left(x-x_m^{(1,m)}\right) \tag{3.30a}$$

with

$$\hat{x}_n=-k_1+k_2+\frac{1}{2}k_3+k_4-\left(2k_2+\frac{3}{2}k_3+k_4\right)n\ , \tag{3.30b}$$

and

$$p_n^{(n-2)}(x;k_1,k_{2,}k_{3,}k_{4,}0)=\left(x-\hat{x}_n^{(+)}\right)\left(x-\hat{x}_n^{(-)}\right)\prod_{m=1}^{n-2}\left(x-x_m^{(1,m)}\right) \tag{3.31a}$$

with

$$\hat{x}_n^{(\pm)}=-k_1+2\left(k_2+k_3+k_4\right)-(2k_2+3k_3+k_4)n\ \pm\frac{1}{2}\sqrt{z_n}) \tag{3.32}$$

where

$$\begin{aligned} z_n &= \left(2k_2+3k_3+2k_4\right)^2-2\left(3k_3+2k_4\right)\left(2k_2+3k_3+k_4\right)n \\ &\quad +6k_3\left(2k_2+3k_3+k_4\right)n^2\ . \end{aligned} \tag{3.33}$$

Obviously there are many special cases in which z_n becomes a perfect square, for instance

$$k_3=0\ ,\ k_4=-2k_2,\quad z_n=\left(2k_2\right)^2 \tag{3.34a}$$

yielding

$$\hat{x}_n^{(+)}=-k_1-k_2\ ,\ \hat{x}_n^{(-)}=-k_1-3k_2\ ; \tag{3.34b}$$

$$k_2=0,\ k_4=-\frac{3}{2}k_3\ ,\quad z_n=\left(3k_3n\right)^2 \tag{3.35a}$$

yielding

$$\hat{x}_n^{(+)}=-k_1-k_3\ ,\ \hat{x}_n^{(-)}=-k_1-k_3-3k_3n; \tag{3.35b}$$

$$k_2=-\frac{1}{2}k\ ,\ k_3=\frac{1}{3}\ ,\ k_4=-\frac{1}{2}+k\ ,\quad z_n=\left(n-k\right)^2 \tag{3.36a}$$

yielding

$$\hat{x}_n^{(+)}=-k_1-\frac{1}{3}+\frac{1}{2}k\ ,\ \hat{x}_n^{(-)}=-k_1-\frac{1}{3}+\frac{3}{2}k-n\ ; \tag{3.36b}$$

$$k_2=\frac{k\left(2k-3\right)}{2\left(2k-1\right)}\ ,\ k_3=\frac{1}{3\left(2k-1\right)}\ ,\ k_4=\frac{1}{2}\ ,\quad z_n=\left(n-k\right)^2 \tag{3.37a}$$

yielding

$$\hat{x}_n^{(\pm)} = -k_1 + \frac{(2 \pm 1)}{2} k - \frac{1}{3\,(2k-1)} + \left(\frac{1 \pm 1}{2} - k \right) n \ . \qquad (3.37b)$$

There moreover holds the *factorization* (3.14) and, for the subclass of polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, -k_2, 0)$, the *factorization* (3.19),

$$\begin{aligned} p_n^{(m)}(x; k_1, k_2, k_3, -k_2, 0) &= p_{n-m}^{(-m)}(x; k_1, k_2, k_3, -k_2, 0)\, p_m^{(m)}(x; k_1, k_2, k_3, -k_2, 0) \ , \\ m &= 0, 1, ..., n \ . \ \boxdot \end{aligned} \qquad (3.38)$$

In the following subsections we report a few specific examples involving "named" polynomials; examples involving other named polynomials are in hand and they are presented in the next chapter.

Laguerre polynomials

The "normalized Laguerre polynomials" $\pounds_n^{(\alpha)}(x)$ – related to the usual generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ by the formula

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} \pounds_n^{(\alpha)}(x) \qquad (3.39)$$

– are the following special case of the polynomials $p_n^{(\nu)}(x; k_1, k_2, k_3, k_4, k_5)$,

$$\pounds_n^{(\alpha)}(x) = p_n^{(-\alpha)}(x; 0, -1, 0, 1, 0) \ , \qquad (3.40)$$

as seen by comparing the recursion relation (1.11.4) of
http://aw.twi.tudelft.nl/˜koekoek/askey/ch1/par11/par11.html [32]
with our recursion relation (3.1) with (3.28d) and (3.28e). Note that it was actually unnecessary to set $k_5 = 0$ in the right-hand side of this formula, (3.40), since – as can be easily seen – any value of k_5 yields in this case the same outcome; by setting $k_5 = 0$ we made it evident that these polynomials satisfy not only *Proposition 3.3*, but as well *Propositions 3.5* and *3.10*. Hence the normalized Laguerre polynomials $\pounds_n^{(\alpha)}(x)$ satisfy the second recursion relation (see (3.7))

$$\pounds_n^{(\alpha)}(x) = \pounds_n^{(\alpha+1)}(x) + n \pounds_{n-1}^{(\alpha+1)} \ , \qquad (3.41a)$$

and correspondingly the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ satisfy the (well-known) second recursion relation

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)} \ . \qquad (3.41b)$$

Likewise the normalized Laguerre polynomials satisfy the factorization

$$\begin{aligned}\pounds_n^{(-m)}(x) &= \pounds_{n-m}^{(m)}(x)\pounds_m^{(-m)}(x)\\ &= x^m \pounds_{n-m}^{(m)}(x)\ , \quad m=0,1,..,n\ ,\end{aligned} \tag{3.42a}$$

entailing for the generalized Laguerre polynomials the formula

$$L_n^{(-m)}(x) = \frac{m!\,(n-m)!}{n!} x^m L_{n-m}^{(m)}(x)\ , \quad m=0,1,..,n\ . \tag{3.42b}$$

And the previous findings entail that the generalized Laguerre polynomials $L_n^{(-m)}(x)$ satisfy the following properties (displaying the *Diophantine* character of their zeros):

$$L_n^{(-n)}(x) = \frac{(-1)^n}{n!} x^n\ , \tag{3.43a}$$

$$L_n^{(-n+1))}(x) = \frac{(-1)^n}{n!} x^{n-1}\,(x-n)\ \ , \tag{3.43b}$$

$$L_n^{(-n+2))}(x) = \frac{(-1)^n}{n!}\left(x-n-\sqrt{n}\right)\left(x-n+\sqrt{n}\right)x^{n-2}\ , \tag{3.43c}$$

implying, for instance, the additional *Diophantine* finding

$$L_{n^2}^{\left(-n^2+2)\right)}(x) = \frac{(-1)^{n^2}}{n^2!}\,[x-n(n+1)]\,[x-n(n-1)]\,x^{n^2-2}\ . \tag{3.43d}$$

Some (but not all) of these formulas are reported in the standard compilations [23] [25] [1] [32].

Meixner polynomials

The "normalized Meixner polynomials" $\tilde{M}_n(x;\beta,c)$ – related to the usual Meixner polynomials $M_n^{(\alpha)}(x;\beta,c)$ by the formula

$$M_n(x;\beta,c) = \frac{1}{(\beta)_n}\left(\frac{c-1}{c}\right)^n \tilde{M}_n(x;\beta,c) \tag{3.44}$$

– are the following special case of the polynomials $p_n^{(\nu)}(x;k_1,k_{2,}k_{3,}k_{4,}k_5)$,

$$\tilde{M}_n(x;\beta,c) = p_n^{(-\beta)}(x;-\frac{1}{2}\frac{c+1}{c-1},\frac{1}{2}\frac{c+1}{c-1},0,-\frac{c}{c-1},-\frac{1}{c-1})\ , \tag{3.45}$$

as seen by comparing the recursion relation (1.9.4) of http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par9/par9.html [32] with our recursion relation (3.1) (with (3.28d) and (3.28e)). Note that in this case the condition $k_5 = 0$ cannot be enforced (except for $c = \infty$), so these polynomials satisfy *Proposition 3.3* but not *Propositions 3.5* and *3.9*. Hence the normalized Meixner polynomials $\tilde{M}_n(x;\beta,c)$ satisfy the second recursion relation (see (3.7))

$$\tilde{M}_n(x;\beta,c) = \tilde{M}_n(x;\beta+1,c) - \frac{c}{c-1} n\ \tilde{M}_{n-1}(x;\beta+1,c)\ , \qquad (3.46a)$$

and correspondingly the usual Meixner polynomials $M_n(x;\beta,c)$ satisfy the second recursion relation

$$\beta M_n(x;\beta,c) = (\beta+n)\, M_n(x;\beta+1,c) - n\,(\beta+n)\, M_{n-1}(x;\beta+1,c)\ , \qquad (3.46b)$$

which is not reported in the standard compilations [23] [25] [1] [32].

Askey's B polynomials

Let us introduce the following modified version of the polynomials $B_n\,(x;a,\eta)$ introduced by R. Askey [2], via the position

$$\hat{B}_n(x;a,\eta) = B_n(x+a\eta;a,\eta)\ . \qquad (3.47)$$

The motivation for modifying in this manner Askey's B-polynomials will be clear below. Let us moreover emphasize that we allow the parameter η to be an *arbitrary* number (while it was restricted to be an integer in Ref. [2]).

It is easily seen that these polynomials $\hat{B}_n(x;a,\eta)$ are a subclass of our polynomials $p_n^{(\nu)}(x;k_1,k_2,k_3,k_4,k_5)$:

$$\hat{B}_n(x;a,\eta) = p_n^{(\eta+1+k)}\left[x; -a(1+k)+\frac{1}{2}, -\frac{1}{2}, 0, a, k\,(a-1)\right]\ . \qquad (3.48a)$$

It is moreover plain that the parameter k appearing in the right-hand side of this formula plays no role, hence hereafter we set it to zero:

$$\hat{B}_n(x;a,\eta) = p_n^{(\eta+1)}(x; -a+\frac{1}{2}, -\frac{1}{2}, 0, a, 0)\ . \qquad (3.48b)$$

It is thereby clear that these polynomials satisfy the condition (3.12) hence satisfy *Proposition 3.5* (see *Remark 3.13*), in addition of course to *Propositions 3.3* and *3.9* (while clearly the *factorization* (3.19) only holds for $a = 1/2$).

Hence these polynomials satisfy the second recurrence relation,

$$\hat{B}_n(x;a,\eta) = \hat{B}_n\,(x;a,\eta-1) + an\hat{B}_{n-1}\,(x;a,\eta-1)\ \ , \qquad (3.49)$$

and there holds for their subclass with $a = 1/2$ the *factorization*

$$\hat{B}_n(x; \frac{1}{2}, m-1) = \hat{B}_{n-m}(x; \frac{1}{2}, -m-1)) \cdot \hat{B}_m(x; \frac{1}{2}, m-1) \ ,$$
$$m = 1, 2, ..n \ . \tag{3.50}$$

Let us emphasize that, due to the definition (3.47), a shift in the parameter η of the polynomials $\hat{B}_n(x; a, \eta)$ also entails a shift in the variable x for the polynomials $B_n(x; a, \eta)$.

Moreover for these polynomials there hold the *Diophantine factorizations*

$$\hat{B}_n(x; a, n-1) = \prod_{m=1}^{n} \left(x - x_m^{(1,m)} \right) \ , \tag{3.51a}$$

$$x_m^{(1,m)} = (2a-1)(1-m) \ ; \tag{3.51b}$$

$$\hat{B}_n(x; a, n-2) = (x - \hat{x}_n) \prod_{m=1}^{n-1} \left(x - x_m^{(1,m)} \right) \ , \tag{3.52a}$$

$$\hat{x}_n = a - (1-a)(1-n) \ ; \tag{3.52b}$$

$$\hat{B}_n(x; a, n-3) = \left(x - \hat{x}_n^{(+)} \right) \left(x - \hat{x}_n^{(-)} \right) \prod_{m=1}^{n-2} \left(x - x_m^{(1,m)} \right) \ , \tag{3.53a}$$

$$\hat{x}_n^{(\pm)} = 3 \left(a - \frac{1}{2} \right) + (1-a) n \pm \frac{1}{2} \sqrt{(1-2a)^2 + 4a(1-a) n} \ . \tag{3.53b}$$

Hence, in addition to the (already known [2]) simple cases ($a = 0, a = 1$) when the original three-term relation becomes a two-term relation, additional *Diophantine* (i.e., *integer* respectively *rational*) zeros occur, for instance, for $n = m^2, a = 1/2$ entailing $\hat{x}_n^{(\pm)} = m(m \pm 1)/2$ respectively for $n = m^2, a = (4m^2 - 1) / [2(2m^2 - 1)]$ entailing $\hat{x}_n^{(\pm)} = m(5m \pm 1) / [2(2m^2 - 1)]$.

Rational solution of (3.4)

The following assignment satisfies the nonlinear conditions (3.4):

$$A_n^{(\nu)} = \frac{n \left(c_0 c_1 + (c_1 - c_2 + c_0 c_3 + 3 c_0^2 c_4)\nu + (c_2 + c_3 \nu) n + c_4 (2c_0 - 2\nu + n) n^2 \right))}{(c_0 + 2n - \nu)} \ , \tag{3.54a}$$

with

$$\alpha^{(\nu)} = -c_1 + c_3 + c_4 (1 + 3c_0) - (2c_3 + 3c_4 (1 + 2c_0)) \nu + 3 c_4 \nu^2 \ , \tag{3.54b}$$

$$A = 0 \ . \tag{3.54c}$$

Here the 5 parameters c_j , $j = 0, ..., 4$ are *arbitrary.*

The corresponding expressions of the coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ and $g_n^{(\nu)}$ read

$$a_n^{(\nu)} = \frac{a\,(n,\nu)}{(c_0 + 2n - \nu)\,(c_0 + 2 + 2n - \nu)} \ , \tag{3.55a}$$

$$\begin{aligned} a\,(n,\nu) \ = \ & c_0\,[c_2 + c_4 + c_0\,(c_1 + 2c_4)] \\ & -(1 + c_0)\,\{c_2 + c_4 - c_0\,[c_3 + 3c_4\,(c_0 - 1)]\}\,\nu \\ & -[c_0\,(c_3 + 3c_0c_4) + c_1 - c_2 + c_3 - 2c_4]\,\nu^2 \\ & +(1 + c_0)\,[c_2 + c_4\,(1 + 3c_0)]\,n \\ & +2\,[c_0\,(c_3 - 6c_4) - c_2 + c_3 - 4c_4]\,n\nu \\ & -2\,(c_3 - 3c_4)\,n\nu^2 \\ & +2\left[c_2 + c_4\left(4 + 9c_0 + 3c_0^2\right)\right]n^2 \\ & +2\,[c_3 - 3c_4\,(3 + 2c_0)]\,n^2\nu + 6c_4n^2\nu^2 \\ & +12\,(1 + c_0)\,c_4n^3 - 12c_4n^3\nu + 6c_4n^4 \ ; \end{aligned} \tag{3.55b}$$

$$b_n^{(\nu)} = \frac{n(n - \nu)\,(c_0 + n)\,(c_0 + n - \nu)\,\tilde{b}\,(n,\nu)\,\hat{b}\,(n,\nu)}{(c_0 + 2n - \nu)^2\,(c_0 + 1 + 2n - \nu)\,(c_0 - 1 + 2n - \nu)} \ , \tag{3.56a}$$

$$\tilde{b}\,(n,\nu) = c_0\,(c_3 + 3c_0c_4) + 2c_1 - c_2 + (2c_3 + 3c_0c_4)\,n - 3c_4n^2 \ , \tag{3.56b}$$

$$\begin{aligned} \hat{b}\,(n,\nu) \ = \ & c_0\,(c_3 + 3c_0c_4) - 2c_1 + c_2 + 3c_4\nu^2 \\ & +(2c_3 + 9c_0c_4)\,(n - \nu) - 6c_4\nu n + 3c_4n^2 \ , \end{aligned} \tag{3.56c}$$

$$g_n^{(\nu)} = \frac{n\,(c_0 + n)\,(c_0\,(c_3 + 3c_0c_4) + 2c_1 - c_2 + (2c_3 + 3c_0c_4)\,n - 3c_4n^2)}{(c_0 + 2n - \nu)\,(c_0 + 1 + 2n - \nu)} \ . \tag{3.57}$$

Hereafter we identify our polynomials $p_n^{(\nu)}(x)$ belonging to this class – hence satisfying *Proposition 3.3* – as $p_n^{(\nu)}(x; c_0, c_1, c_2, c_3, c_4)$. Of course they should not be confused with the polynomials solution introduced in the preceding subsection.

It is plain (see (3.56a) and (3.12)) that these polynomials also satisfy *Propositions 3.5* and *3.9* when $\mu = 0$, entailing the *factorizations*

$$p_n^{(n)}(x; c_0, c_1, c_2, c_3, c_4) = \prod_{m=1}^{n} \left(x - x_m^{(1,m)}\right) \ , \tag{3.58a}$$

$$x_m^{(1,m)} = \alpha^{(m)} = -c_1 + c_3 + c_4\left(1 + 3c_0\right) - \left[2c_3 + 3c_4\left(1 + 2c_0\right)\right] m + 3c_4 m^2 \ ; \tag{3.58b}$$

$$p_n^{(n-1)}(x; c_0, c_1, c_2, c_3, c_4) = (x - \hat{x}_n) \prod_{m=1}^{n-1} \left(x - x_m^{(1,m)}\right) \ , \tag{3.59a}$$

$$\hat{x}_n = -\frac{(1 + c_0)\left[c_1 - c_3 - c_4\left(1 + 3c_0\right)\right] + \left[c_0(c_3 + 3c_4\left(2 + c_0\right)) - c_1 + c_2 + c_3 + 2c_4\right] n}{1 + c_0 + n} \ . \tag{3.59b}$$

Three interesting cases that deserve to be highlighted read as follows:

$$p_n^{(n-1)}(x; -1, c_1, c_2, c_3, c_4) = (x - \hat{x}_n) \prod_{m=1}^{n-1} \left(x - x_m^{(1,m)}\right) \ , \tag{3.60a}$$

$$\hat{x}_n = c_1 - c_2 + c_4 \ , \tag{3.60b}$$

$$x_m^{(1,m)} = -c_1 + c_3 - 2c_4 - \left(2c_3 - 3c_4\right) m + 3c_4 m^2 \ ; \tag{3.60c}$$

$$p_n^{(n-1)}(x; 0, c_2 + c_3 + 2c_4, c_2, c_3, c_4) = (x - \hat{x}_n) \prod_{m=1}^{n-1} \left(x - x_m^{(1,m)}\right) \ , \tag{3.61a}$$

$$\hat{x}_n = -\frac{c_2 + c_4}{n + 1} \ , \tag{3.61b}$$

$$x_m^{(1,m)} = -\left(c_2 + c_4 + \left(2c_3 + 3c_4\right) m - 3c_4 m^2\right) \ ; \tag{3.61c}$$

$$p_n^{(n-1)}(x; 0, -\frac{1 + 3\sigma}{2} c_4, -c_4 - \rho, -\frac{3}{2}\left(1 + \sigma\right) c_4, c_4) = (x - \hat{x}_n) \prod_{m=1}^{n-1} \left(x - x_m^{(1,m)}\right) \ , \tag{3.62a}$$

$$\hat{x}_n = \rho \frac{n}{n + 1} \ , \tag{3.62b}$$

$$x_m^{(1,m)} = 3c_4 m(m + \sigma) \ . \tag{3.62c}$$

Moreover

$$p_n^{(n-2)}(x; c_0, c_1, c_2, c_3, c_4) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right) \prod_{m=1}^{n-2} \left(x - x_m^{(1,m)}\right) \ . \tag{3.63}$$

We do not report the (rather complicated) expressions of the two zeros $\hat{x}_n^{(\pm)}$, except in the following special cases:

$$p_n^{(n-2)}(x; -1, c_1, c_2, c_3, c_4) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right) \prod_{m=1}^{n-2} \left(x - x_m^{(1,m)}\right) \ , \tag{3.64a}$$

$$\hat{x}_n^{(+)} = \frac{-3c_1 + c_2 + 4c_3 - 5c_4 + (c_1 - c_2 - 2c_3 + c_4)\, n}{n+1} \ , \qquad (3.64b)$$

$$\hat{x}_n^{(-)} = c_1 - c_2 + c_4 \ , \qquad (3.64c)$$

with the *zeros* $x_m^{(1,m)}$ given by (3.60c);

$$p_n^{(n-2)}(x; -2, c_1, c_2, c_3, 0) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right) \prod_{m=1}^{n-2} \left(x - x_m^{(1,m)}\right) \ , \qquad (3.65a)$$

$$\hat{x}_n^{(\pm)} = c_1 - c_2 \pm c_3 \ , \qquad (3.65b)$$

$$x_m^{(1,m)} = -c_1 + c_3 - 2c_3 m \ . \qquad (3.65c)$$

The enterprising reader will surely identify several other remarkable cases.

Moreover there holds the *factorization* (3.14) and, for the subclass of polynomials with

$$c_3 = -3c_0 c_4 \qquad (3.66a)$$

the *factorization* (3.19):

$$\begin{aligned} & p_n^{(m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4) \\ = \ & p_{n-m}^{(-m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4)\, p_m^{(m)}(x; c_0, c_1, c_2, -3c_0c_4, c_4) \ , \\ m = \ & 0, 1, ..., n \ . \end{aligned} \qquad (3.66b)$$

Jacobi polynomials

The "normalized Jacobi polynomials" $\tilde{P}_n^{(\alpha,\beta)}(x)$ – related to the usual Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(n + \alpha + \beta + 1)_n}{2^n n!} \tilde{P}_n^{(\alpha,\beta)}(x) \qquad (3.67)$$

– are the following special case of the polynomials $p_n^{(\nu)}(x; c_0, c_1, c_2, c_3, c_4)$:

$$\tilde{P}_n^{(\alpha,\beta)}(x) = p_n^{(-\beta)}(x; \alpha, 1, 0, 0, 0) \ , \qquad (3.68)$$

as seen by comparing the recursion relation (1.8.4) of

http://aw.twi.tudelft.nl/~koekoek/askey/ch1/par8/par8.html#par1 [32] with our recursion relation (3.1). Here, and always in the following, additional relations are implied by the well-known symmetry of Jacobi polynomials under the exchange of the two parameters they feature,

$$P_n^{(\alpha,\beta)}(x) = P_n^{(\beta,\alpha)}(-x) \ , \quad \tilde{P}_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\beta,\alpha)}(-x) \ . \qquad (3.69)$$

It is evident that these polynomials (see (3.68)) satisfy *Propositions 3.3, 3.5* and *3.9*. Hence the normalized Jacobi polynomials $\tilde{P}_n^{(\alpha,\beta)}(x)$ satisfy the second recursion relation (see (3.7))

$$\tilde{P}_n^{(\alpha,\beta)}(x) = \tilde{P}_n^{(\alpha,\beta+1)}(x) + \frac{2n\,(n+\alpha)}{(2n+\alpha+\beta)\,(2n+\alpha+\beta+1)}\tilde{P}_n^{(\alpha,\beta+1)}(x) \ , \tag{3.70a}$$

and correspondingly the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the (well-known) second recursion relation

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) = (n+\alpha+\beta+1)P_n^{(\alpha,\beta+1)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}(x) \ . \tag{3.70b}$$

There holds moreover the following (well known) *Diophantine* factorization formula

$$\tilde{P}_n^{(\alpha,-n)}(x) = (x+1)^n \ , \tag{3.71}$$

as well as (see (3.59b))

$$\tilde{P}_n^{(\alpha,-n+1)}(x) = \left(x + \frac{\alpha+1-n}{\alpha+1+n}\right)(x+1)^{n-1} \ , \tag{3.72}$$

$$\tilde{P}_n^{(\alpha,-n+2)}(x) = \left(x - \hat{x}_n^{(+)}\right)\left(x - \hat{x}_n^{(-)}\right)(x+1)^{n-2} \ , \tag{3.73a}$$

$$\hat{x}_n^{(\pm)} = \frac{n\,(n-1) - (\alpha+1)\,(\alpha+2) \pm 2\sqrt{(\alpha+2)\,n\,(n+\alpha+1)}}{n\,(n+2\alpha+3) + (\alpha+1)\,(\alpha+2)} \ . \tag{3.73b}$$

In particular for $\alpha = -2$

$$\tilde{P}_n^{(-2,-n+2)}(x) = (x-1)^2\,(x+1)^{n-2} \ , \tag{3.73c}$$

and for $\alpha = -1$

$$\tilde{P}_n^{(-1,-n+2)}(x) = \left(x - \frac{n-3}{n+1}\right)(x-1)\,(x+1)^{n-2} \ . \tag{3.73d}$$

And clearly there are additional *Diophantine zeros* whenever $(\alpha+2)\,n\,(n+\alpha+1)$ is a perfect square, for instance

$$\tilde{P}_n^{(\alpha,-n+2)}(x) = \left[x - \frac{n-\delta\,(2\delta+1)}{n+\delta}\right] \cdot$$

$$\cdot \left[x - \frac{n^2 - (2\delta^2+3\delta+3)\,n - \delta}{n^2 + (3\delta+1)\,n + \delta\,(2\delta+1)}\right](x+1)^{n-2} \ , \tag{3.73e}$$

$$\alpha = \frac{(\delta^2+2\delta-1)\,n + 2\delta^2}{n-\delta^2} \ , \tag{3.73f}$$

$$\tilde{P}^{(\alpha,\beta)}_{1+2k(k-1)}(x) = \left(x - \frac{2k-1}{4k(k-1)+1}\right)^2 (x+1)^{-1+2k(k-1)} \quad (3.73g)$$

$$\alpha = -1+2k(k-1)\ , \quad \beta = -1-2k(k-1)\ , \quad k = 2,3,\ldots\ . \quad (3.73h)$$

Perhaps (some of) these formulas deserve to be included in the standard compilations.

Remark 3.14. We report a solution, involving an *arbitrary* function $f(z)$, of conditions (3.8a) and (3.8b) with (3.8c), which also satisfies the symmetry property (3.17):

$$a_n^{(\nu)} = f(2n-\nu) + f(2n-\nu+1)\ , \quad (3.74a)$$

$$b_n^{(\nu)} = -f(2n-\nu)\, f(2n-\nu-1)\ , \quad (3.74b)$$

$$g_n^{(\nu)} = -f(2n-\nu)\ . \quad (3.74c)$$

But it is plain that this solution disappears altogether ("$f(z)=0$") if one requires it to satisfy either the additional "initial" condition (3.8d) (or equivalently (3.8f)) also required for the validity of *Proposition 3.3* or the hypothesis (3.12) required for the validity of *Corollary 3.7.*

3.3 Proofs

In this section we prove the three *Propositions* reported in the preceding Section 3.1.

The proof of *Proposition 3.3* (i. e. of (3.6)) is by induction. Clearly this relation holds for $n=1$ (via (3.1c) and (3.8d)). Let us assume that it holds up to n, and prove that it then holds for $n+1$. Indeed using (3.6) in the right-hand side of the recursion relation (3.1a) we get

$$\begin{aligned} p_{n+1}^{(\nu)}(x) &= \left(x+a_n^{(\nu)}\right)\left[p_n^{(\nu-1)}(x) + g_n^{(\nu)}\, p_{n-1}^{(\nu-1)}(x)\right] \\ &\quad + b_n^{(\nu)}\left[p_{n-1}^{(\nu-1)}(x) + g_{n-1}^{(\nu)}\, p_{n-2}^{(\nu-1)}(x)\right]\ . \end{aligned} \quad (3.75)$$

We then note that the recursion relation (3.1a) entails the formulas

$$p_{n-1}^{(\nu-1)}(x) = \frac{p_{n+1}^{(\nu-1)}(x) - \left(x+a_n^{(\nu-1)}\right) p_n^{(\nu-1)}(x)}{b_n^{(\nu-1)}}\ , \quad (3.76a)$$

$$p_{n-2}^{(\nu-1)}(x) = \frac{\left[b_n^{(\nu-1)} + \left(x+a_{n-1}^{(\nu-1)}\right)\left(x+a_n^{(\nu-1)}\right)\right] p_n^{(\nu-1)}(x)}{b_n^{(\nu-1)}\, b_{n-1}^{(\nu-1)}} - \frac{\left(x+a_{n-1}^{(\nu-1)}\right) p_{n+1}^{(\nu-1)}(x)}{b_n^{(\nu-1)}\, b_{n-1}^{(\nu-1)}} . \tag{3.76b}$$

The second, (3.76b), of these two formulas is of course obtained by replacing n with $n-1$ in the first, (3.76a), and then by using again (3.76a) to eliminate $p_{n-1}^{(\nu-1)}(x)$.

The proof of *Proposition 3.5* (i. e., of the *factorization* formula (3.14)) is again by induction. Clearly (3.14) holds for $n=0$ (hence $m=0$), see (3.1b) and (3.15b). Let us now assume that it holds up to n, and show that it then holds for $n+1$. Indeed, by using it in the right-hand side of the relation (3.1a) with $\nu = m$ we get

$$p_{n+1}^{(m)}(x) = \left[\left(x+a_n^{(m)}\right) \tilde{p}_{n-m}^{(-m)}(x) + b_n^{(m)}\, \tilde{p}_{n-1-m}^{(-m)}(x)\right] p_m^{(m)}(x) ,$$
$$m = 0,1,...,n-1 , \tag{3.77a}$$

and clearly using the recursion relation (3.15a) the square bracket in the right-hand side of this equation can be replaced by $\tilde{p}_{n+1-m}^{(-m)}(x)$, yielding

$$p_{n+1}^{(m)}(x) = \tilde{p}_{n+1-m}^{(-m)}(x)\, p_m^{(m)}(x) , \quad m = 0,1,...,n+1 . \tag{3.77b}$$

Note that for $m = n+1$ this formula is an identity, since $\tilde{p}_0^{(-m)}(x) = 1$, see (3.15b); likewise, this formula clearly also holds for $m=n$, provided (3.12) holds, see (3.1a) with $m=n$ and (3.15c).

But this is just the formula (3.14) with n replaced by $n+1$. Q. E. D.

Remark 3.15. The hypothesis (3.12) has been used above, in the proof of *Proposition 3.5*, only to prove the validity of the final formula, (3.77b), for $m=n$. Hence one might wonder whether this hypothesis, (3.12), was redundant, since the validity of the final formula (3.77b) for $m=n$ seems to be implied by (3.77a) with (3.15c) and (3.15b), without the need to invoke (3.12). But in fact, by setting $m=n$ in the basic recurrence relation (3.1a) it is clear that (3.15c) and (3.15b) hold only provided (3.12) also holds. ⊡

Finally, let us prove *Proposition 3.9*, namely the validity of the *factorization* formula (3.22). For $\nu = n$ the relation (3.6) yields

$$p_n^{(n)}(x) = p_n^{(n-1)}(x) + g_n^{(n)}\, p_{n-1}^{(n-1)}(x) , \tag{3.78}$$

and via (3.13) (with n replaced by $n-1$) this can be rewritten as follows:

$$p_n^{(n)}(x) = \left(x + a_{n-1}^{(n-1)} + g_n^{(n)}\right) p_{n-1}^{(n-1)}(x) , \tag{3.79}$$

clearly entailing (together with the initial condition $p_0^{(0)}(x) = 1$, see (3.1b)), the factorization formula (3.22). Q. E. D.

Chapter 4

Factorizations and Diophantine properties associated with the polynomials of the Askey-Scheme.

In this chapter we apply to (almost) all the "named" polynomials of the Askey scheme, as defined by their standard three-term recursion relations, the machinery developed in chapter 3. For each of these polynomials we identify at least one additional recursion relation involving a shift in some of the parameters they feature, and for several of these polynomials characterized by special values of their parameters factorizations are identified yielding some or all of their zeros – generally given by simple expressions in terms of *integers* (*Diophantine* relations). We then apply this theoretical machinery to the "named" polynomials of the Askey scheme [32], as defined by the basic three-term recursion relation they satisfy: this entails the identification of the parameter ν- which can often be done in more than one way, especially for the named polynomials involving several parameters- and yields the identification of additional recursion relations satisfied by (most of) these polynomials. These our results could also be obtained by other routes- for instance, by exploiting the relations of these polynomials with hypergeometric functions: as we will see in the next chapter.

Again, most of these results seem *new* and deserving to be eventually recorder in the standard compilations although they generally require that the parameters of the named polynomials did *not* satisfy the standard restrictions required for the orthogonality property. To clarify this restriction let us remark that an elementary example of such factorizations- which night be considered the *prototype* of formulas reported below for many of the polynomials of the Askey scheme- reads as follows:

$$L_n^{(-n)}(x) = \frac{(-x)^n}{n!} \ , \quad n = 0, 1, 2, ..., \tag{4.1a}$$

where $L_n^{(-n)}(x)$ is the standard (generalized) Laguerre polynomial of order n, for whose orthogonality,

$$\int_0^\infty dx\ x^\alpha \exp(-x) L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) = \delta_{nm} \frac{\Gamma(n+\alpha+1)}{n!} \ , \tag{4.2a}$$

it is, however, generally required that $\mathbb{R}(\alpha) > -1$. This formula (4.1a) is well known and it is indeed displayed in some of the standard compilations reporting results for classical orthogonal polynomials (see, e.g., page 109 of the classical book by Magnus and Oberhettinger [43] or [25], Equation 8.973.4). And this remark applies as well to the following neat generalization of this formula, reading

$$L_n^{(-m)}(x) = (-1)^m \frac{(n-m)!}{n!} x^m L_{n-m}^{(m)}(x) \ , \quad m = 0, 1, ..., n, \quad n = 0, 1, 2, ..., \tag{4.3a}$$

which qualifies as well as the prototype of formulas reported below for many of the polynomials of the Askey scheme. Note that this formula can be inserted without difficulty in the standard orthogonality relation for generalized Laguerre polynomials, (4.2a), reproducing the standard relation: the singularity of the weight function gets indeed compensated by the term x^m appearing in the right-hand side of (4.3a). Presumably, this property- and the analogous version for Jacobi polynomials- is well known to most experts on orthogonal polynomials. Most of the formulas (analogous to (4.3a) and (4.1a)) for the named polynomials of the Askey scheme that are reported below are instead, to the best of our knowledge, *new*. And let us also note that, as it is generally done in the standard treatments of "named" polynomials, we have treated separately each of the differently "named" classes of these polynomials, even though " in principle" it would be sufficient to only treat the most general class of them- Wilson polynomials- that encompasses all the other classes via appropriate assignments (including limiting ones) of the 4 parameters it features.

4.1 Results for the polynomials of the Askey scheme

In this section we apply to the polynomials of the Askey scheme [32] the results reviewed in the previous chapter. This class of polynomials (including the classical polynomials) may be introduced in various manners: via generating functions, Rodriguez-type formulas, their connections with hypergeometric formulas,... . In order to apply our machinery, as described in the preceding section, we introduce them via the three-term recursion relation they satisfy:

$$p_{n+1}\left(x;\underline{\eta}\right)=\left[x+a_n\left(\underline{\eta}\right)\right]\,p_n\left(x;\underline{\eta}\right)+b_n\left(\underline{\eta}\right)\,p_{n-1}\left(x;\underline{\eta}\right) \tag{4.4a}$$

with the "initial" assignments

$$p_{-1}\left(x;\underline{\eta}\right)=0\ ,\quad p_0\left(x;\underline{\eta}\right)=1\ , \tag{4.4b}$$

clearly entailing

$$p_1(x;\underline{\eta})=x+a_0\left(\underline{\eta}\right)\ ,\quad p_2\left(x;\underline{\eta}\right)=\left[x+a_1\left(\underline{\eta}\right)\right]\,\left[x+a_0\left(\underline{\eta}\right)\right]+b_1\left(\underline{\eta}\right) \tag{4.4c}$$

and so on. Here the components of the vector $\underline{\eta}$ denote the additional *parameters* generally featured by these polynomials.

Let us emphasize that in this manner we introduced the *monic* (or "normalized" [32]) version of these polynomials: below we always also display the relation of this version with the more standard version [32].

To apply our machinery we must identify, among the parameters characterizing these polynomials, the single parameter ν playing a special role in our approach. This can be generally done in several ways (even for the same class of polynomials, see below). Once such identification has been made, i. e. the assignment $\underline{\eta}\equiv\underline{\eta}\,(\nu)$, the recursion relations (4.4) coincide with the relations (3.1) via the self-evident notational identification

$$p_n^{(\nu)}\left(x\right)\equiv p_n\left(x;\underline{\eta}\left(\nu\right)\right)\ ,\quad a_n^{(\nu)}\equiv a_n\left(\underline{\eta}\left(\nu\right)\right)\ ,\quad b_n^{(\nu)}\equiv b_n\left(\underline{\eta}\left(\nu\right)\right)\ . \tag{4.5}$$

Before proceeding with the report of our results, let us also emphasize that, when the polynomials considered below feature symmetries regarding the dependence on their parameters – for instance they are invariant under exchanges of some of them – obviously *all* the properties of these polynomials reported below can be duplicated by using such symmetry properties; but it would be a waste of space for us to report explicitly the corresponding formulae, hence such duplications are hereafter omitted (except that sometimes results arrived at by different routes can be recognized as trivially related via such symmetries: when this happens this fact is explicitly noted). We will use systematically the notation of [32] – up to obvious changes made whenever necessary in order to avoid interferences with our previous notation. When we obtain a result that we deem interesting but is not reported in the standard compilations [32] [23] [25] [1] we identify it as **new**. And let us reiterate that, even though we performed an extensive search for such results, this investigation cannot be considered "exhaustive": additional results might for instance be found via assignments of the ν-dependence $\underline{\eta}\,(\nu)$ different from those considered below.

4.1.1 Wilson

The monic Wilson polynomials (see [32], and note the notational replacement of the 4 parameters a, b, c, d used there with $\alpha, \beta, \gamma, \delta$)

$$p_n\left(x; \alpha, \beta, \gamma, \delta\right) \equiv p_n\left(x; \underline{\eta}\right) \tag{4.6a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = \alpha^2 - \tilde{A}_n - \tilde{C}_n \ , \quad b_n\left(\underline{\eta}\right) = -\tilde{A}_{n-1}\tilde{C}_n \ , \tag{4.6b}$$

where

$$\tilde{A}_n = \frac{\left(n+\alpha+\beta\right)\left(n+\alpha+\gamma\right)\left(n+\alpha+\delta\right)\left(n-1+\alpha+\sigma\right)}{\left(2\,n-1+\alpha+\sigma\right)\left(2\,n+\alpha+\sigma\right)} \ , \tag{4.6c}$$

$$\tilde{C}_n = \frac{n\left(n-1+\beta+\gamma\right)\left(n-1+\beta+\delta\right)\left(n-1+\gamma+\delta\right)}{\left(2\,n-2+\alpha+\sigma\right)\left(2\,n-1+\alpha+\sigma\right)} \ , \tag{4.6d}$$

$$\sigma \equiv \beta+\gamma+\delta \ , \quad \rho \equiv \beta\,\gamma+\beta\,\delta+\gamma\,\delta \ , \quad \tau \equiv \beta\,\gamma\,\delta \ . \tag{4.6e}$$

The standard version of these polynomials reads (see [32]):

$$W_n\left(x; \alpha, \beta, \gamma, \delta\right) = \left(-1\right)^n \left(n-1+\alpha+\beta+\gamma+\delta\right)_n\, p_n\left(x; \alpha, \beta, \gamma, \delta\right) \ . \tag{4.7}$$

Let us also recall that these polynomials $p_n\left(x; \alpha, \beta, \gamma, \delta\right)$ are invariant under any permutation of the 4 parameters $\alpha, \beta, \gamma, \delta$.

As for the identification of the parameter ν, see (4.5), several possibilities are listed in the following subsections.

First assignment

$$\alpha = -\nu \ . \tag{4.8}$$

With this assignment one can set, consistently with our previous treatment,

$$\begin{aligned} A_n^{(\nu)} \;=\; & \left[6\left(2\,n-2-\nu+\sigma\right)\right]^{-1} n\,\{4-5\,\sigma+6\,\rho-6\,\tau \\ & +\left(5-6\,\sigma+6\,\rho\right)\,\nu+\left[-10+9\,\sigma-6\,\rho+\left(-9+6\,\sigma\right)\,\nu\right]\,n \\ & +\left(8-4\,\sigma+4\,\nu\right)\,n^2-2\,n^3\} \ , \end{aligned} \tag{4.9a}$$

$$\omega^{(\nu)} = -\nu^2 \ , \tag{4.9b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Wilson polynomials (4.6):

$$p_n^{(\nu)}(x) = p_n(x; -\nu, \beta, \gamma, \delta) \ . \tag{4.10}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Wilson polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{n\,(n-1+\beta+\gamma)\,(n-1+\beta+\delta)\,(n-1+\gamma+\delta)}{(2\,n-2-\nu+\sigma)\,(2\,n-1-\nu+\sigma)} \ . \tag{4.11}$$

Note that this finding is obtained without requiring any limitation on the 4 parameters of the Wilson polynomials $p_n(x; \alpha, \beta, \gamma, \delta)$.

It is moreover plain that, with the assignment

$$\nu = n-1+\beta \quad \text{namely} \quad \alpha = -n+1-\beta \ , \tag{4.12}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = \beta - 1$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7,* they are reported in subsection *Factorizations* And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n(x; -n+1-\beta, \beta, \gamma, \delta) = \prod_{m=1}^{n} \left[x + (m-1+\beta)^2\right] \ , \tag{4.13}$$

while *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[-(\ell-1+\beta)^2; -m+1-\beta, \beta, \gamma, \delta\right] = 0 \ , \quad \ell = 1, ..., m \ , \quad m = 1, ..., n \ . \tag{4.14}$$

Remark 4.1. A look at the formulae (4.6) suggests other possible assignments of the parameter ν satisfying (3.12), such as, say, $\nu = n-2+\sigma$ namely $\alpha = 2-n-\sigma$. But these assignments actually fail to satisfy (3.12) for *all* values of n, because for this to happen it is not sufficient that the numerator in the expression of $b_n^{(\nu+\mu)}$ vanish, it is moreover required that the denominator in that expression *never* vanish. In the following we shall consider only assignments of the parameter ν in terms of n that satisfy these requirements. $\boxdot$

Second assignment

$$\alpha = -\frac{\nu}{2} \ , \quad \beta = \frac{1-\nu}{2} \ . \tag{4.15}$$

With this assignment one can set, consistently with our previous treatment,

$$\begin{aligned} A_n^{(\nu)} \ = \ & [6\ (4\,n - 3 - 2\,\nu + 2\,\gamma + 2\,\delta)]^{-1}\, n\ \{3 - 4\,\gamma - 4\,\delta + 6\,\gamma\,\delta \\ & + (7 - 9\,\gamma - 9\,\delta + 12\,\gamma\,\delta)\,\nu + 3\ (1 - \gamma - \delta)\,\nu^2 \\ & - \left[11 - 12\,\gamma - 12\,\delta + 12\,\gamma\,\delta + 3\ (5 - 4\,\gamma - 4\,\delta)\ \nu + 3\,\nu^2\right]\ n \\ & + 4\ (3 + 2\,\nu - 2\,\gamma - 2\,\delta)\ n^2 - 4\,n^3\big\} \ , \end{aligned} \tag{4.16a}$$

$$\omega^{(\nu)} = -\frac{\nu^2}{4} \ , \tag{4.16b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Wilson polynomials (4.6):

$$p_n^{(\nu)}(x) = p_n\left(x; -\frac{\nu}{2}, \frac{1-\nu}{2}, \gamma, \delta\right) . \tag{4.17}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Wilson polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{n\,(n - 1 + \gamma + \delta)\,(2\,n - 1 - \nu + 2\,\gamma)\,(2\,n - 1 - \nu + 2\,\delta)}{(4\,n - 3 - 2\,\nu + 2\,\gamma + 2\,\delta)\,(4\,n - 1 - 2\,\nu + 2\,\gamma + 2\,\delta)} \ . \tag{4.18}$$

Note that this assignment entails now the (single) restriction $\beta = \alpha + 1/2$ on the 4 parameters of the Wilson polynomials $p_n(x; \alpha, \beta, \gamma, \delta)$.

It is moreover plain that, with the assignments

$$\nu = n - \frac{1}{2} \ , \quad \text{hence} \quad \alpha = -\frac{n}{2} + \frac{1}{4} \ , \quad \beta = -\frac{n}{2} + \frac{3}{4}, \tag{4.19a}$$

$$\nu = n - 2 + 2\delta \ , \quad \gamma = \delta - \frac{1}{2} \ , \quad \alpha = -\frac{n}{2} + 1 - \delta \ , \quad \beta = -\frac{n}{2} + \frac{3}{2} - \delta \ , \tag{4.19b}$$

respectively

$$\nu = n - 1 + 2\delta \ , \quad \gamma = \delta + \frac{1}{2} \ , \quad \alpha = -\frac{n}{2} + \frac{1}{2} - \delta \ , \quad \beta = -\frac{n}{2} + 1 - \delta \ , \tag{4.19c}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1/2$, $\mu = -2 + 2\delta$ respectively $\mu = -1 + 2\delta$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations.* And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\left(x;-\frac{n}{2}+\frac{1}{4},-\frac{n}{2}+\frac{3}{4},\gamma,\delta\right)=\prod_{m=1}^{n}\left[x+\left(\frac{2m-1}{4}\right)^2\right] , \tag{4.20a}$$

$$p_n\left(x;-\frac{n}{2}+1-\delta,-\frac{n}{2}+\frac{3}{2}-\delta,\delta-\frac{1}{2},\delta\right)=\prod_{m=1}^{n}\left[x+\left(\frac{m-2+2\delta}{2}\right)^2\right] , \tag{4.20b}$$

respectively

$$p_n\left(x;-\frac{n}{2}+\frac{1}{2}-\delta,-\frac{n}{2}+1-\delta,\delta+\frac{1}{2},\delta\right)=\prod_{m=1}^{n}\left[x+\left(\frac{m-1+2\delta}{2}\right)^2\right] . \tag{4.20c}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[-\left(\frac{2\ell-1}{4}\right)^2;-\frac{m}{2}+\frac{1}{4},-\frac{m}{2}+\frac{3}{4},\gamma,\delta\right]=0 ,$$
$$\ell = 1,...,m , \quad m=1,...,n , \tag{4.21a}$$

$$p_n\left[-\left(\frac{\ell-2+2\delta}{2}\right)^2;-\frac{m}{2}+1-\delta,-\frac{m}{2}+\frac{3}{2}-\delta,\delta-\frac{1}{2},\delta\right]=0 ,$$
$$\ell = 1,...,m , \quad m=1,...,n , \tag{4.21b}$$

respectively

$$p_n\left[-\left(\frac{\ell-1+2\delta}{2}\right)^2;-\frac{m}{2}+\frac{1}{2}-\delta,-\frac{m}{2}+1-\delta,\delta+\frac{1}{2},\delta\right]=0 ,$$
$$\ell = 1,...,m , \quad m=1,...,n . \tag{4.21c}$$

Moreover, with the assignments

$$\nu=2n-2+2\delta , \quad \alpha=-n+1-\delta , \quad \beta=-n+\frac{3}{2}-\delta , \tag{4.22a}$$

respectively

$$\nu = 2n - 1 + 2\delta \ , \quad \alpha = -n + \frac{1}{2} - \delta \ , \quad \beta = -n + 1 - \delta \ , \qquad (4.22b)$$

Proposition 3.11 becomes applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\left(x; -n + 1 - \delta, -n + \frac{3}{2} - \delta, \gamma, \delta\right) = \prod_{m=1}^{n} \left[x + (m - 1 + \delta)^2\right] \ , \qquad (4.23a)$$

respectively

$$p_n\left(x; -n + \frac{1}{2} - \delta, -n + 1 - \delta, \gamma, \delta\right) = \prod_{m=1}^{n} \left[x + (m - 1 + \delta)^2\right] \ , \qquad (4.23b)$$

obviously implying the relation

$$p_n\left(x; -n + 1 - \delta, -n + \frac{3}{2} - \delta, \gamma, \delta\right) = p_n\left(x; -n + \frac{1}{2} - \delta, -n + 1 - \delta, \gamma, \delta\right) \ . \qquad (4.23c)$$

Factorizations

The following **new** relations among monic Wilson polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$\begin{aligned} & p_n\left(x; -m + 1 - \beta, \beta, \gamma, \delta\right) \\ = \ & p_{n-m}\left(x; m + \beta, \gamma, 1 - \beta, \delta\right) p_m\left(x; -m + 1 - \beta, \beta, \gamma, \delta\right) \ , \\ m = \ & 0, 1, ..., n \ , \end{aligned} \qquad (4.24a)$$

$$\begin{aligned} & p_n\left(x; -\frac{m}{2} + 1 - \delta, -\frac{m}{2} + \frac{3}{2} - \delta, \delta - \frac{1}{2}, \delta\right) \\ = \ & p_{n-m}\left(x; \frac{m}{2} - \frac{1}{2} + \delta, \frac{m}{2} + \delta, 1 - \delta, -\delta + \frac{3}{2}\right) \cdot \\ & \cdot p_m\left(x; -\frac{m}{2} + 1 - \delta, -\frac{m}{2} + \frac{3}{2} - \delta, \delta - \frac{1}{2}, \delta\right) \ , \\ m = \ & 0, 1, ..., n \ . \end{aligned} \qquad (4.24b)$$

4.1.2 Racah

The Racah polynomials are a family of orthogonal polynomials introduced by James Wilson in 1978. He used this name in onour of Giulio Racah (a italian mathematical physicist, with israelian origin), by the fact that the orthogonality relations for these polynomials are equivalent to the orthogonality relations for the Racah coefficients.

The monic Racah polynomials (see [32])

$$p_n(x;\alpha,\beta,\gamma,\delta) \equiv p_n\left(x;\underline{\eta}\right) \tag{4.25a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = \tilde{A}_n + \tilde{C}_n \ , \quad b_n\left(\underline{\eta}\right) = -\tilde{A}_{n-1}\tilde{C}_n \ , \tag{4.25b}$$

where

$$\tilde{A}_n = \frac{(n+1+\alpha)(n+1+\alpha+\beta)(n+1+\beta+\delta)(n+1+\gamma)}{(2n+1+\alpha+\beta)(2n+2+\alpha+\beta)} \ , \tag{4.25c}$$

$$\tilde{C}_n = \frac{n(n+\alpha+\beta-\gamma)(n+\alpha-\delta)(n+\beta)}{(2n+\alpha+\beta)(2n+1+\alpha+\beta)} \ . \tag{4.25d}$$

The standard version of these polynomials reads (see [32]):

$$R_n(x;\alpha,\beta,\gamma,\delta) = \frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\, p_n(x;\alpha,\beta,\gamma,\delta) \ . \tag{4.26a}$$

Note however that in the following we do *not* restrict the parameters of these polynomials to satisfy one of the restrictions $\alpha = -N$ or $\beta + \delta = -N$ or $\gamma = -N$, with N a positive integer and $n = 0, 1, ..., N$, whose validity is instead required for the standard Racah polynomials [32].

Let us recall that these polynomials are invariant under various shufflings of their parameters:

$$\begin{aligned} p_n(x;\alpha,\beta,\gamma,\delta) &= p_n(x;\alpha,\beta,\beta+\delta,\gamma-\beta) \\ &= p_n(x;\beta+\delta,\alpha-\delta,\gamma,\delta) \\ &= p_n(x;\gamma,\alpha+\beta-\gamma,\alpha,-\alpha+\gamma+\delta) \ . \end{aligned} \tag{4.26b}$$

Let us now proceed and provide various identifications of the parameter ν, see (4.5).

First assignment

$$\alpha = -\nu \ . \qquad (4.27)$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = [6\,(2\,n - \nu + \beta)]^{-1}\, n\, \{\beta\ (2 + 3\,\gamma + 3\,\delta) - [2 + 3\ (\gamma + \delta) + 6\,\gamma\ (\beta + \delta)]\ \nu$$
$$+ [4 + 6\ (\gamma + \delta) + 3\ (\beta\gamma - \beta\delta + 2\gamma\delta) - 3\,(2\,\beta + \gamma + \delta)\,\nu]\ n$$
$$+4\,(-\nu + \beta)\ n^2 + 2\,n^3\} \ , \qquad (4.28a)$$

$$\omega^{(\nu)} = (\nu - 1)\,(\nu + \gamma + \delta) \ , \qquad (4.28b)$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Racah polynomials (4.25):

$$p_n^{(\nu)}(x) = p_n\,(x; -\nu, \beta, \gamma, \delta) \ . \qquad (4.29)$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Racah polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -\frac{n\,(n + \beta)\,(n + \beta + \delta)\,(n + \gamma)}{(2\,n - \nu + \beta)\,(2\,n + 1 - \nu + \beta)} \ . \qquad (4.30)$$

Note that this finding is obtained without requiring any limitation on the 4 parameters of the Racah polynomials $p_n\,(x; \alpha, \beta, \gamma, \delta)$.

It is moreover plain that, with the assignments

$$\nu = n \ , \quad \text{hence} \quad \alpha = -n \ , \qquad (4.31a)$$

$$\nu = n - \delta \ , \quad \text{hence} \quad \alpha = -n + \delta \ , \qquad (4.31b)$$

respectively

$$\nu = n + \beta - \gamma \ , \quad \text{hence} \quad \alpha = -n - \beta + \gamma \ , \qquad (4.31c)$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 0$, $\mu = -\delta$ respectively $\mu = \beta - \gamma$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations.* And *Proposition*

3.9 becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\left(x;-n,\beta,\gamma,\delta\right)=\prod_{m=1}^{n}\left[x-\left(m-1\right)\left(m+\gamma+\delta\right)\right]\ , \tag{4.32a}$$

$$p_n\left(x;-n+\delta,\beta,\gamma,\delta\right)=\prod_{m=1}^{n}\left[x-\left(m+\gamma\right)\left(m-\delta-1\right)\right]\ , \tag{4.32b}$$

respectively

$$p_n\left(x;-n-\beta+\gamma,\beta,\gamma,\delta\right)=\prod_{m=1}^{n}\left[x-\left(m-1+\beta-\gamma\right)\left(m+\beta+\delta\right)\right]\ . \tag{4.32c}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n\left[\left(\ell-1\right)\left(\ell+\gamma+\delta\right);-m,\beta,\gamma,\delta\right]=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ , \tag{4.33a}$$

$$p_n\left[\left(\ell+\gamma\right)\left(\ell-\delta-1\right);-m+\delta,\beta,\gamma,\delta\right]=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ , \tag{4.33b}$$

respectively

$$\begin{aligned} & p_n\left[\left(\ell-1+\beta-\gamma\right)\left(\ell+\beta+\delta\right);-m-\beta+\gamma,\beta,\gamma,\delta\right]=0\ , \\ \ell &= 1,...,m\ ,\quad m=1,...,n\ . \end{aligned} \tag{4.33c}$$

Factorizations

The following **new** relations among Racah polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$\begin{aligned} & p_n\left(x;-m,\beta,-1,1\right)=p_{n-m}\left(x;m,\beta,-1,1\right)\ p_m\left(x;-m,\beta,-1,1\right)\ , \\ & m=0,1,...,n\ , \end{aligned} \tag{4.34a}$$

$$\begin{aligned} & p_n\left(x;-m+\delta,\beta,-\delta,\delta\right) \\ = \ & p_{n-m}\left(x;m-\delta,2\delta+\beta,\delta,-\delta\right)\ p_m\left(x;-m+\delta,\beta,-\delta,\delta\right)\ , \\ & m=0,1,...,n\ , \end{aligned} \tag{4.34b}$$

$$p_n\left(x;-m-\beta+\gamma,\beta,\gamma,c-\gamma\right)=$$
$$p_{n-m}\left(x;m+\beta-\gamma+c,-\beta+2\gamma-c,\gamma,c-\gamma\right)\,p_m\left(x;-m-\beta+\gamma,\beta,\gamma,c-\gamma\right)\ ,$$
$$m=0,1,...,n\ , \tag{4.34c}$$

$$p_n\left(x;\alpha,-m,\gamma,\delta\right)=p_{n-m}\left(x;\alpha,m,\delta,\gamma\right)\,p_m\left(x;\alpha,-m,\gamma,\delta\right)\ ,$$
$$m=0,1,...,n\ , \tag{4.34d}$$

$$\begin{aligned}&p_n\left(x,\alpha,-m-\alpha+\eta,\eta,\delta\right)\\=\ &p_{n-m}\left(x,\eta,m,\eta+\delta-\alpha,\alpha\right)\,p_m\left(x,\alpha,-m-\alpha+\eta,\eta,\delta\right)\ ,\\&m=0,1,...,n\ .\end{aligned} \tag{4.34e}$$

4.1.3 Continuous Dual Hahn (CDH)

The monic Continuous Dual Hahn (CDH) polynomials $p_n\left(x;\alpha,\beta,\gamma\right)$ (see [32], and note the notational replacement of the 3 parameters a,b,c used there with α,β,γ),

$$p_n\left(x;\alpha,\beta,\gamma\right)\equiv p_n\left(x;\underline{\eta}\right)\ , \tag{4.35a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right)=\alpha^2-\left(n+\alpha+\beta\right)\left(n+\alpha+\gamma\right)-n\left(n-1+\beta+\gamma\right)\ , \tag{4.35b}$$

$$b_n\left(\underline{\eta}\right)=-n\ \left(n-1+\alpha+\beta\right)\left(n-1+\alpha+\gamma\right)\left(n-1+\beta+\gamma\right)\ . \tag{4.35c}$$

The standard version of these polynomials reads (see [32]):

$$S_n\left(x;\alpha,\beta,\gamma\right)=\left(-1\right)^n\ p_n\left(x;\alpha,\beta,\gamma\right)\ . \tag{4.36}$$

Let us recall that these polynomials $p_n\left(x;\alpha,\beta,\gamma\right)$ are invariant under any permutation of the three parameters α,β,γ.

Let us now proceed and provide various identifications of the parameter ν, see (4.5).

First assignment

$$\alpha = -\nu \ . \tag{4.37}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = n \left[-\frac{5}{6} + \beta + \gamma - \beta\,\gamma + (\beta + \gamma - 1)\,\nu + \left(\frac{3}{2} - \beta - \gamma + \nu \right) n - \frac{2}{3}\,n^2 \right] \ , \tag{4.38a}$$

$$\omega^{(\nu)} = -\nu^2 \ , \tag{4.38b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized CDH polynomials (4.35):

$$p_n^{(\nu)}(x) = p_n(x; -\nu, \beta, \gamma) \ . \tag{4.39}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized CDH polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = n\ (n - 1 + \beta + \gamma) \ . \tag{4.40}$$

Note that this finding is obtained without requiring any limitation on the 3 parameters of the CDH polynomials $p_n(x; \alpha, \beta, \gamma)$.

It is moreover plain that, with the assignment

$$\nu = n - 1 + \beta \ , \quad \text{hence} \quad \alpha = -n + 1 - \beta \ , \tag{4.41}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1 + \beta$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations*. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorization

$$p_n(x; -n + 1 - \beta, \beta, \gamma) = \prod_{m=1}^{n} \left[x + (m - 1 + \beta)^2 \right] \ . \tag{4.42}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[-(\ell - 1 + \beta)^2 ; -m + 1 - \beta, \beta, \gamma \right] = 0 \ , \quad \ell = 1, ..., m \ , \quad m = 1, ..., n \ . \tag{4.43}$$

Likewise, with the assignment

$$\nu = 2n + \beta \ , \quad \alpha = -2n - \beta \ , \quad \gamma = \frac{1}{2} \ , \tag{4.44}$$

Proposition 3.11 becomes applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n \left(x; -2n - \beta, \beta, \frac{1}{2} \right) = \prod_{m=1}^{n} \left[x + (2m - 1 + \beta)^2 \right] \ . \tag{4.45}$$

Second assignment

$$\alpha = -\frac{1}{2}\nu + c \ , \quad \beta = -\frac{1}{2} (\nu + 1) + c \tag{4.46}$$

where c is an *a priori* arbitrary parameter.

With this assignment one can set, consistently with our previous treatment,

$$\begin{aligned} A_n^{(\nu)} &= n \left[-\frac{4}{3} + \frac{3}{2} \gamma + \frac{5}{2} c - c^2 - 2 \gamma c + \left(-\frac{5}{4} + \gamma + c \right) \nu - \frac{1}{4} \nu^2 \right. \\ &\left. \qquad + (2 - \gamma - 2 c + \nu) \, n - \frac{2}{3} n^2 \right] \ , \end{aligned} \tag{4.47a}$$

$$\omega^{(\nu)} = -\frac{1}{4} (1 - 2c + \nu)^2 \ , \tag{4.47b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized CDH polynomials (4.35):

$$p_n^{(\nu)}(x) = p_n \left(x; c - \frac{\nu}{2}, c - \frac{\nu}{2} - \frac{1}{2}, \gamma \right) \ . \tag{4.48}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized CDH polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = n \left(n - 1 - \frac{\nu}{2} + \gamma + c \right) \ . \tag{4.49}$$

Note that this assignment entails the (single) limitation $\beta = \alpha - 1/2$ on the parameters of the CDH polynomials.

It is moreover plain that, with the assignment

$$\nu = n + 2c - \frac{3}{2}\ , \quad \text{hence} \quad \alpha = -\frac{n}{2} + \frac{3}{4}\ , \quad \beta = -\frac{n}{2} + \frac{1}{4}\ , \tag{4.50}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 2c - 3/2$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations*. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorization

$$p_n\left(x; -\frac{n}{2} + \frac{3}{4}, -\frac{n}{2} + \frac{1}{4}, \gamma\right) = \prod_{m=1}^{n}\left[x + \left(\frac{2m-1}{4}\right)^2\right]\ . \tag{4.51}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[-\left(\frac{2\ell-1}{4}\right)^2; -\frac{m}{2} + \frac{3}{4}, -\frac{m}{2} + \frac{1}{4}, \gamma\right] = 0\ , \quad \ell = 1, ..., m\ , \quad m = 1, ..., n\ . \tag{4.52}$$

Likewise with the assignments

$$\nu = 2\,(n - 1 + c + \gamma)\ , \quad \text{hence} \quad \alpha = -n + 1 - \gamma\ , \quad \beta = -n + \frac{1}{2} - \gamma\ , \tag{4.53a}$$

respectively

$$\nu = 2\left(n - \frac{3}{2} + c + \gamma\right)\ , \quad \text{hence} \quad \alpha = -n + \frac{3}{2} - \gamma\ , \quad \beta = -n + 1 - \gamma\ , \tag{4.53b}$$

Proposition 3.11 becomes applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\left(x; -n + 1 - \gamma, -n + \frac{1}{2} - \gamma, \gamma\right) = \prod_{m=1}^{n}\left[x + (m - 1 + \gamma)^2\right]\ , \tag{4.54a}$$

respectively

$$p_n\left(x; -n + \frac{3}{2} - \gamma, -n + 1 - \gamma, \gamma\right) = \prod_{m=1}^{n}\left[x + (m - 1 + \gamma)^2\right]\ . \tag{4.54b}$$

Note that the right-hand sides of the last two formulas coincide; this implies (**new** finding!) that the left-hand side coincide as well.

Factorizations

The following **new** relations among Continuous Dual Hahn polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$\begin{aligned} & p_n\left(x;-m+1-\beta,\beta,\gamma\right) \\ = \ & p_{n-m}\left(x;m+\beta,1-\beta,\gamma\right)\, p_m\left(x;-m+1-\beta,\beta,\gamma\right)\ , \\ & m=0,1,...,n\ . \end{aligned} \tag{4.55}$$

4.1.4 Continuous Hahn (CH)

The monic Continuous Hahn (CH) polynomials $p_n\left(x;\alpha,\beta,\gamma,\delta\right)$ (see [32], and note the notational replacement of the 4 parameters a,b,c,d used there with $\alpha,\beta,\gamma,\delta$),

$$p_n\left(x;\alpha,\beta,\gamma,\delta\right)\equiv p_n\left(x;\underline{\eta}\right)\ , \tag{4.56a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right)=-i\left(\alpha+\tilde{A}_n+\tilde{C}_n\right)\ ,\quad b_n\left(\underline{\eta}\right)=\tilde{A}_{n-1}\tilde{C}_n\ , \tag{4.56b}$$

where

$$\tilde{A}_n=-\frac{\left(n-1+\alpha+\beta+\gamma+\delta\right)\left(n+\alpha+\gamma\right)\left(n+\alpha+\delta\right)}{\left(2\,n-1+\alpha+\beta+\gamma+\delta\right)\left(2\,n+\alpha+\beta+\gamma+\delta\right)}\ , \tag{4.56c}$$

$$\tilde{C}_n=\frac{n\left(n-1+\beta+\gamma\right)\left(n-1+\beta+\delta\right)}{\left(2\,n+\alpha+\beta+\gamma+\delta-1\right)\left(2\,n+\alpha+\beta+\gamma+\delta-2\right)}\ . \tag{4.56d}$$

The standard version of these polynomials reads (see [32]):

$$S_n\left(x;\alpha,\beta,\gamma,\delta\right)=\left(-1\right)^n\,p_n\left(x;\alpha,\beta,\gamma,\delta\right)\ . \tag{4.57a}$$

Let us recall that these polynomials are symmetrical under the exchange of the first two and last two parameters,

$$p_n\left(x;\alpha,\beta,\gamma,\delta\right)=p_n\left(x;\beta,\alpha,\gamma,\delta\right)=p_n\left(x;\alpha,\beta,\delta,\gamma\right)=p_n\left(x;\beta,\alpha,\delta,\gamma\right)\ . \tag{4.57b}$$

Let us now proceed and provide various identifications of the parameter ν, see (4.5).

First assignment

$$\alpha = -\nu \ . \tag{4.58}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = i\,n\,\frac{-\beta+\gamma+\delta-2\gamma\delta+(1-2\beta)\,\nu+(\beta-\gamma-\delta-\nu)\,n}{2\,(2-\beta-\gamma-\delta+\nu-2n)} \ , \tag{4.59a}$$

$$\omega^{(\nu)} = -i\,\nu \ , \tag{4.59b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized CH polynomials (4.56):

$$p_n^{(\nu)}(x) = p_n\,(x;-\nu,\beta,\gamma,\delta) \ . \tag{4.60}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized CH polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{i\,n\,(n-1+\beta+\gamma)\,(n-1+\beta+\delta)}{(2\,n-2-\nu+\beta+\gamma+\delta)\,(2\,n-1-\nu+\beta+\gamma+\delta)} \ . \tag{4.61}$$

Note that this assignment entails no restriction on the 4 parameters of the CH polynomials $p_n\,(x;\alpha,\beta,\gamma,\delta)$.

It is moreover plain that, with the assignment

$$\nu = n-1+\gamma \ , \quad \text{hence} \quad \alpha = -n+1-\gamma \ , \tag{4.62}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1+\gamma$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorization

$$p_n\,(x;-n+1-\gamma,\beta,\gamma,\delta) = \prod_{m=1}^{n}\left[x+i(m-1+\gamma)\right] \ . \tag{4.63}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n\left[-i(\ell-1+\gamma);-m+1-\gamma,\beta,\gamma,\delta\right] = 0 \ , \quad \ell = 1,...,m \ , \quad m = 1,...,n \ . \tag{4.64}$$

Second assignment

Analogous results also obtain from the assignment

$$\gamma = -\nu \ . \tag{4.65}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = -\frac{in\left[n\left(\alpha+\beta-\delta+\nu\right)+\left(2\delta-1\right)\nu+\alpha\left(2\beta-1\right)-\beta+\delta\right]}{2\left(2n-2+\alpha+\beta+\delta-\nu\right)} \ , \tag{4.66a}$$

$$\omega^{(\nu)} = i\nu \ , \tag{4.66b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized CH polynomials (4.56):

$$p_n^{(\nu)}(x) = p_n\left(x;\alpha,\beta,-\nu,\delta\right) \ . \tag{4.67}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized CH polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{i\,n\,\left(n-1+\alpha+\beta\right)\left(n-1+\beta+\delta\right)}{\left(2\,n-2-\nu+\alpha+\beta+\delta\right)\left(2\,n-1-\nu+\alpha+\beta+\delta\right)} \ . \tag{4.68}$$

Note that this assignment entails no restriction on the 4 parameters of the CH polynomials $p_n\left(x;\alpha,\beta,\gamma,\delta\right)$.

It is moreover plain that, with the assignment

$$\nu = n-1+\alpha \ , \quad \text{hence} \quad \gamma = -n+1-\alpha \ , \tag{4.69}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1+\alpha$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n\left(x;\alpha,\beta,-n+1-\alpha,\delta\right) = \prod_{m=1}^{n}\left[x-i\left(m-1+\alpha\right)\right] \ . \tag{4.70}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[i\left(\ell-1+\alpha\right);\alpha,\beta,-m+1-\alpha,\delta\right] = 0 \ , \quad \ell = 1,...,m \ , \quad m = 1,...,n \ . \tag{4.71}$$

4.1.5 Hahn

In this subsection we introduce a somewhat generalized version of the standard (monic) Hahn polynomials. These (generalized) monic Hahn polynomials $p_n(x;\alpha,\beta,\gamma)$ (see [32], and note the replacement of the integer parameter N with the arbitrary parameter γ: hence the standard Hahn polynomials are only obtained for $\gamma = N$ with N a *positive integer* and $n = 1, 2, .., N$),

$$p_n(x;\alpha,\beta,\gamma) \equiv p_n\left(x;\underline{\eta}\right) \;, \tag{4.72a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = -\left(\tilde{A}_n + \tilde{C}_n\right) \;, \quad b_n\left(\underline{\eta}\right) = -\tilde{A}_{n-1}\tilde{C}_n \;, \tag{4.72b}$$

where

$$\tilde{A}_n = \frac{(n+1+\alpha)(n+1+\alpha+\beta)(-n+\gamma)}{(2\,n+1+\alpha+\beta)(2\,n+2+\alpha+\beta)} \;, \tag{4.72c}$$

$$\tilde{C}_n = \frac{n\,(n+1+\alpha+\beta+\gamma)(n+\beta)}{(2\,n+\alpha+\beta)(2\,n+1+\alpha+\beta)} \;. \tag{4.72d}$$

The standard version of these polynomials reads (see [32]):

$$Q_n(x;\alpha,\beta,\gamma) = \frac{(n+1+\alpha+\beta)_n}{(1+\alpha)_n\,(-\gamma)_n}\,p_n(x;\alpha,\beta,\gamma) \;. \tag{4.73}$$

Let us now proceed and provide various identifications of the parameter ν, see (4.5).

First assignment

$$\alpha = -\nu \;. \tag{4.74}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = \frac{n\,[\beta + (1+2\,\gamma)\,\nu - (\beta+2\,\gamma+\nu)\,n]}{2\,(2\,n-\nu+\beta)} \;, \tag{4.75a}$$

$$\omega^{(\nu)} = \nu - 1 \;, \tag{4.75b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Hahn polynomials (4.72):

$$p_n^{(\nu)}(x) = p_n(x;-\nu,\beta,\gamma) \;. \tag{4.76}$$

Hence, with this identification, Proposition 2.1 becomes applicable, entailing (**new** finding!) that these normalized Hahn polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -\frac{n\,(n+\beta)\,(n-1-\gamma)}{(2\,n-\nu+\beta)\,(2\,n+1-\nu+\beta)}\ . \tag{4.77}$$

Note that this assignment entails no restriction on the 3 parameters of the Hahn polynomials $p_n\,(x;\alpha,\beta,\gamma)$.

It is moreover plain that, with the assignments

$$\nu = n \tag{4.78a}$$

respectively

$$\nu = n+1+\beta+\gamma\ , \tag{4.78b}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 1+\beta+\gamma$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\,(x;n,\beta,\gamma) = \prod_{m=1}^{n}(x-m+1)\ \ , \tag{4.79a}$$

respectively

$$p_n\,(x;n+1+\beta+\gamma,\beta,\gamma) = \prod_{m=1}^{n}(x-m-\beta-\gamma)\ \ . \tag{4.79b}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n\,(\ell-1;m,\beta,\gamma) = 0\ ,\quad \ell = 1,...,m\ ,\quad m = 1,...,n\ , \tag{4.80a}$$

respectively

$$p_n\,(\ell+\beta+\gamma;m+1+\beta+\gamma,\beta,\gamma) = 0\ ,\quad \ell = 1,...,m\ ,\quad m = 1,...,n\ . \tag{4.80b}$$

Second assignment

$$\beta = -\nu + \gamma + c \ , \tag{4.81}$$

where c is an arbitrary parameter.

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = -n \frac{\alpha - \gamma - c + \nu + 2\alpha\gamma + (-\alpha + c - \nu + 3\gamma)\, n}{2\,(\alpha + \gamma + c - \nu + 2n)} \ , \tag{4.82a}$$

$$\omega^{(\nu)} = 1 - \nu + 2\gamma + c \ , \tag{4.82b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Hahn polynomials (4.72):

$$p_n^{(\nu)}(x) = p_n\,(x; \alpha, -\nu + \gamma + c, \gamma) \ . \tag{4.83}$$

Hence, with this identification, Proposition 2.1 becomes applicable, entailing (**new** finding!) that these normalized Hahn polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{n\ (n + \alpha)\ (n - 1 - \gamma)}{(2\,n - \nu + \alpha + \gamma + c)\ (2\,n + 1 - \nu + \alpha + \gamma + c)} \ . \tag{4.84}$$

Note that this assignment entails no restriction on the 3 parameters of the Hahn polynomials $p_n\,(x; \alpha, \beta, \gamma)$.

It is moreover plain that, with the assignments

$$\nu = n + \gamma + c \ , \quad \text{hence} \quad \beta = -n \ , \tag{4.85a}$$

respectively

$$\nu = n + 1 + \alpha + 2\gamma + c \ , \quad \text{hence} \quad \beta = -\,(n + 1 + \alpha + \gamma) \ , \tag{4.85b}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = \gamma + c$ respectively $\mu = 1 + \alpha + 2\gamma + c$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\,(x; \alpha, -n, \gamma) = \prod_{m=1}^{n} (x + m - 1 - \gamma) \ , \tag{4.86a}$$

respectively

$$p_n\left(x;\alpha,-n-1-\alpha-\gamma,\gamma\right)=\prod_{m=1}^{n}\left(x+m+\alpha\right)\ . \tag{4.86b}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n\left(-\ell+1+\gamma;\alpha,-m,\gamma\right)=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ , \tag{4.87a}$$

respectively

$$p_n\left(-\ell-\alpha;\alpha,-m-1-\alpha-\gamma,\gamma\right)=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ . \tag{4.87b}$$

Third assignment

$$\beta=-\nu+c\ ,\quad \gamma=\nu\ , \tag{4.88}$$

where c is an arbitrary parameter.

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)}=-\frac{n\ \left[\nu+\alpha-c+2\,\alpha\,\nu+\left(\nu-\alpha+c\right)\,n\right]}{2\ \left(2\,n-\nu+\alpha+c\right)}\ , \tag{4.89a}$$

$$\omega^{(\nu)}=\nu\ , \tag{4.89b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}\left(x\right)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Hahn polynomials (4.72):

$$p_n^{(\nu)}\left(x\right)=p_n\left(x;\alpha,-\nu+c,\nu\right)\ . \tag{4.90}$$

Hence, with this identification, Proposition 2.1 becomes applicable, entailing (**new** finding!) that these normalized Hahn polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)}=-\frac{n\left(n+\alpha\right)\left(n+1+\alpha+c\right)}{\left(2\,n-\nu+\alpha+c\right)\left(2\,n+1-\nu+\alpha+c\right)}\ . \tag{4.91}$$

Note that this assignment entails no restriction on the 4 parameters of the Hahn polynomials $p_n\left(x;\alpha,\beta,\gamma\right)$.

It is moreover plain that, with the assignment

$$\nu=n+c\ ,\quad \beta=-n\ ,\quad \gamma=n+c\ , \tag{4.92}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = c$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n\left(x;\alpha,-n,n+c\right)=\prod_{m=1}^{n}\left(x-m-c\right)\ . \tag{4.93}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left(\ell+c;\alpha,-m,m+c\right)=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ . \tag{4.94}$$

4.1.6 Dual Hahn

In this subsection, we introduce a somewhat generalized version of the standard (monic) Dual Hahn polynomials. These (generalized) monic Dual Hahn polynomials $p_n\left(x;\gamma,\delta,\eta\right)$ (see [32], and note the replacement of the integer parameter N with the arbitrary parameter γ: hence the standard Hahn polynomials are only obtained for $\eta = N$ with N a *positive integer* and $n = 1, 2, .., N$),

$$p_n\left(x;\gamma,\delta,\eta\right)\equiv p_n\left(x;\underline{\eta}\right)\ , \tag{4.95a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right)=\tilde{A}_n+\tilde{C}_n\ ,\quad b_n\left(\underline{\eta}\right)=-\tilde{A}_{n-1}\tilde{C}_n \tag{4.95b}$$

where

$$\tilde{A}_n=\left(n+1+\gamma\right)\left(n-\eta\right)\ ,\quad \tilde{C}_n=n\left(n-1-\delta-\eta\right)\ . \tag{4.95c}$$

The standard version of these polynomials reads (see [32]):

$$R_n\left(x;\gamma,\delta,\eta\right)=\frac{1}{\left(1+\gamma\right)_n\left(-\eta\right)_n}\,p_n\left(x;\gamma,\delta,\eta\right)\ . \tag{4.96}$$

Let us now proceed and provide various identifications of the parameter ν.

First assignment

$$\eta = \nu \ . \tag{4.97}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = n \left[\frac{1}{3} + \frac{-\gamma + \delta}{2} - \gamma \, \nu - \left(1 + \nu + \frac{-\gamma + \delta}{2} \right) n + \frac{2}{3} n^2 \right] \ , \tag{4.98a}$$

$$\omega^{(\nu)} = \nu \ (1 + \nu + \gamma + \delta) \ , \tag{4.98b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Dual Hahn polynomials (4.95):

$$p_n^{(\nu)}(x) = p_n(x; \gamma, \delta, \nu) \ . \tag{4.99}$$

Hence, with this identification, Proposition 2.1 becomes applicable, entailing (**new** finding!) that these normalized Dual Hahn polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -n \, (n + \gamma) \ . \tag{4.100}$$

Note that this assignment entails no restriction on the 3 parameters of the Dual Hahn polynomials $p_n(x; \gamma, \delta, \eta)$.

It is moreover plain that, with the assignments

$$\nu = n - 1 \ , \quad \text{hence} \quad \eta = n - 1 \tag{4.101a}$$

(which is however incompatible with the requirement characterizing the standard Dual Hahn polynomials: $\eta = N$ with N a *positive integer* and $n = 1, 2, .., N$), respectively

$$\nu = n - 1 - \delta \ , \quad \text{hence} \quad \eta = n - 1 - \delta \ , \tag{4.101b}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1$ respectively $\mu = -1 - \delta$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations.* And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n(x; \gamma, \delta, n-1) = \prod_{m=1}^{n} \left[x - (m-1)(m + \gamma + \delta) \right] \ , \tag{4.102a}$$

respectively

$$p_n\left(x;\gamma,\delta,n-1-\delta\right)=\prod_{m=1}^{n}\left[x-\left(m+\gamma\right)\left(m-1-\delta\right)\right]\ . \tag{4.102b}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n\left[\left(\ell-1\right)\left(\ell+\gamma+\delta\right);\gamma,\delta,m-1\right]=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ , \tag{4.103a}$$

respectively

$$p_n\left[\left(\ell+\gamma\right)\left(\ell-1-\delta\right);\gamma,\delta,m-1-\delta\right]=0\ ,\quad \ell=1,...,m\ ,\quad m=1,...,n\ . \tag{4.103b}$$

While for

$$\nu=2n\ ,\quad \text{hence}\quad \eta=2n\ ,\quad \text{and moreover}\quad \delta=\gamma\ , \tag{4.104a}$$

respectively

$$\nu=2n-\delta\ ,\quad \text{hence}\quad \eta=2n-\delta\ ,\quad \text{and moreover}\quad \delta=-\gamma\ , \tag{4.104b}$$

Proposition 3.11 becomes applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n\left(x;\gamma,\gamma,2n\right)=\prod_{m=1}^{n}\left[x-2\left(2m-1\right)\left(m+\gamma\right)\right]\ , \tag{4.105a}$$

respectively

$$p_n\left(x;\gamma,-\gamma,2n+\gamma\right)=\prod_{m=1}^{n}\left[x-\left(2m-1+\gamma\right)\left(2m+\gamma\right)\right]\ . \tag{4.105b}$$

Second assignment

$$\gamma=-\nu\ ,\quad \delta=\nu+c\ . \tag{4.106}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)}=n\left[\frac{1}{3}+\left(1+\eta\right)\nu+\frac{1}{2}\,c-\left(1+\nu+\eta+\frac{1}{2}\,c\right)n+\frac{2}{3}\,n^2\right]\ , \tag{4.107a}$$

$$\omega^{(\nu)} = (\nu - 1)(\nu + c) \ , \tag{4.107b}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Dual Hahn polynomials (4.95):

$$p_n^{(\nu)}(x) = p_n(x; -\nu, \nu + c, \eta) \ . \tag{4.108}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Dual Hahn polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -n(n - 1 - \eta) \ . \tag{4.109}$$

Note that this assignment entails no restriction on the 3 parameters of the Dual Hahn polynomials $p_n(x; \gamma, \delta, \eta)$.

It is moreover plain that, with the assignments

$$\nu = n \ , \quad \text{hence} \quad \gamma = -n \ , \quad \delta = n + c \ , \tag{4.110a}$$

respectively

$$\nu = n - 1 - \eta - c \ , \quad \text{hence} \quad \gamma = -n + 1 + \eta + c \ , \quad \delta = n - 1 - \eta \ , \tag{4.110b}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 0$ respectively $\mu = -1 - \eta - c$. These are **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations*. And *Proposition 3.9* becomes as well applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n(x; -n, n + c, \eta) = \prod_{m=1}^{n} [x - (m - 1)(m + c)] \ , \tag{4.111a}$$

respectively

$$p_n(x; -n + 1 + \eta + c, n - 1 - \eta, \eta) = \prod_{m=1}^{n} [x - (m - 1 - \eta)(m - 2 - \eta - c)] \ . \tag{4.111b}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n[(\ell - 1)(\ell + c); -m, m + c, \eta] = 0 \ , \quad \ell = 1, ..., m \ , \quad m = 1, ..., n \ , \tag{4.112a}$$

respectively

$$p_n\left[(\ell-1-\eta)(\ell-2-\eta-c);-m+1+\eta+c,-m-1-\eta,\eta\right]=0\ ,$$
$$\ell = 1,...,m\ ,\quad m=1,...,n\ . \tag{4.112b}$$

While for

$$\nu=2n-\eta\ ,\quad \text{hence}\quad \gamma=-2n+\eta\ ,$$
$$\text{and moreover}\quad c=0\ ,\quad \text{hence}\quad \delta=2n-\eta\ , \tag{4.113a}$$

respectively

$$\nu=2n+1\ ,\quad \text{hence}\quad \gamma=-2n-1\ ,$$
$$\text{and moreover}\quad c=-2(\eta+1)\ ,\quad \text{hence}\quad \delta=2n-1-2\eta\ , \tag{4.113b}$$

Proposition 3.11 becomes applicable, entailing (**new** findings!) the remarkable *Diophantine* factorizations

$$p_n(x;-2n+\eta,2n-\eta,\eta)=\prod_{m=1}^{n}\left[x-(2m-2-\eta)(2m-1-\eta)\right]\ , \tag{4.114a}$$

respectively

$$p_n(x;-2n-1,2n-1-2\eta,\eta)=\prod_{m=1}^{n}\left[x-2(2m-1)(m-1-\eta)\right]\ . \tag{4.114b}$$

Factorizations

The following **new** relations among Dual Hahn polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$p_n(x;\gamma,-\gamma,m-1)$$
$$=p_{n-m}(x;\gamma,-\gamma,-m-1)\,p_m(x;\gamma,-\gamma,m-1)\ ,$$
$$m=0,1,...,n\ , \tag{4.115a}$$

$$p_n(x;\gamma,\delta,m-1-\delta)$$
$$=p_{n-m}(x;\delta,\gamma,-m-1-\gamma)\,p_m(x;\gamma,\delta,m-1-\delta)\ ,$$
$$m=0,1,...,n\ , \tag{4.115b}$$

$$\begin{aligned} & p_n\left(x;-m,m,\eta\right) \\ = \; & p_{n-m}\left(x;m,-m,\eta\right)\,p_m\left(x;-m,m,\eta\right)\ , \\ & m=0,1,...,n\ , \end{aligned} \tag{4.115c}$$

$$\begin{aligned} & p_n\left(x;-m+1+\eta+c,m-1-\eta,\eta\right)= \\ & p_{n-m}\left(x;m-1-\eta,-m+1+\eta+c,-\eta-c-2\right)\cdot \\ & \cdot p_m\left(x;-m+1+\eta+c,m-1-\eta,\eta\right)\ , \\ & m=0,1,...,n\ . \end{aligned} \tag{4.115d}$$

4.1.7 Shifted Meixner-Pollaczek (sMP)

In this subsection we introduce and treat a modified version of the standard (monic) Meixner-Pollaczek polynomials. The standard (monic) Meixner-Pollaczek (MP) polynomials $p_n\left(x;\alpha,\lambda\right)$ (see [32]),

$$p_n\left(x;\alpha,\lambda\right)\equiv p_n\left(x;\underline{\eta}\right)\ , \tag{4.116a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right)=\alpha\left(n+\lambda\right)=\frac{n+\lambda}{\tan\phi}\ , \tag{4.116b}$$

$$b_n\left(\underline{\eta}\right)=-\frac{1}{4}\left(1+\alpha^2\right)\,n\,\left(n-1+2\lambda\right)=-\frac{n\,\left(n-1+2\lambda\right)}{4\,\sin^2\phi}\ . \tag{4.116c}$$

The standard version of these polynomials reads (see [32]):

$$P_n^{(\lambda)}\left(x;\tan\phi\right)=\frac{\left(2\sin\phi\right)^n}{n!}\,p_n\left(x;\alpha,\lambda\right)\ ,\quad \alpha\equiv\frac{1}{\tan\phi}\ . \tag{4.117a}$$

But we have not found any assignment of the parameters α and λ in terms of ν allowing the application of our machinery. We therefore consider the (monic) "shifted Meixner-Pollaczek" (sMP) polynomials

$$p_n\left(x;\alpha,\lambda,\beta\right)=p_n\left(x+\beta;\alpha,\lambda\right)\ , \tag{4.118}$$

defined of course via the three-term recursion relation (4.4) with

$$a_n\left(\underline{\eta}\right)=\alpha\left(n+\lambda\right)+\beta=\frac{n+\lambda}{\tan\phi}\ +\beta, \tag{4.119a}$$

$$b_n\left(\underline{\eta}\right)=-\frac{1}{4}\left(1+\alpha^2\right)\,n\,\left(n-1+2\lambda\right)=-\frac{n\,\left(n-1+2\lambda\right)}{4\,\sin^2\phi}\ . \tag{4.119b}$$

Then with the assignment

$$\lambda=-\frac{1}{2}\left(\nu+c\right)\ ,\quad \beta=-\frac{1}{2}i\left(\nu+C\right) \tag{4.120}$$

(entailing no restriction on the parameters α, λ, β, inasmuch as the two parameters c and C are arbitrary), one can set, consistently with our previous treatment,

$$A_n^{(\nu)}=\frac{1}{2}n\left(\alpha n-\alpha\nu-i\nu-iC-\alpha c-\alpha\right)\ ,\quad \omega^{(\nu)}=\frac{1}{2}i\left(2\nu+c+C\right)\ , \tag{4.121}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized shifted Meixner-Pollaczek polynomials:

$$p_n^{(\nu)}\left(x\right)=p_n\left(x;\alpha,-\frac{1}{2}\left(\nu+c\right),-\frac{1}{2}i\left(\nu+C\right)\right)\ . \tag{4.122}$$

Hence, with this identification, P*roposition 3.3* becomes applicable, entailing (**new** finding!) that these (normalized) shifted Meixner-Pollaczek polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)}=-\frac{1}{2}\ n\left(\alpha+i\right)\ . \tag{4.123}$$

It is moreover plain that, with the assignment

$$\nu=n-1-c\ \ \text{hence}\ \ \lambda=-\frac{1}{2}\left(n-1\right)\ ,\quad \beta=-\frac{1}{2}i\left(n-1-c+C\right)\ , \tag{4.124}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu=-1-c$. And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n\left(x;\alpha,-\frac{1}{2}\left(n-1\right),-\frac{1}{2}i\left(n-1-c+C\right)\right)=\prod_{m=1}^{n}\left[x-i\left(m-1+\frac{C-c}{2}\right)\right]\ . \tag{4.125}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n\left[i\left(l-1+\frac{C-c}{2}\right);\alpha,-\frac{1}{2}(m-1),-\frac{1}{2}i(m-1-c+C)\right]=0\,,\quad \ell=1,...,m\,,\quad m=1,...,n\,. \tag{4.126}$$

4.1.8 Meixner

Josef Meixner was a German theoretical physicist and the Meixner polynomials, introduced in 1934, are a family of orthogonal polynomials that are orthogonal with respect to a certain *discrete* measure .

The monic Meixner polynomials $p_n(x;\beta,c)$ (see [32]),

$$p_n(x;\beta,c)\equiv p_n\left(x;\underline{\eta}\right)\ , \tag{4.127a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right)=\frac{\beta\,c+(1+c)\,n}{c-1}\ ,\quad b_n\left(\underline{\eta}\right)=-\frac{c\,n\,(n-1+\beta)}{(c-1)^2}\ . \tag{4.127b}$$

The standard version of these polynomials reads (see [32]):

$$M_n(x;\beta,c)=\frac{1}{(\beta)_n}\left(\frac{c-1}{c}\right)^n p_n(x;\beta,c)\ . \tag{4.128a}$$

We now identify the parameter ν via the assignment

$$\beta=-\nu\ . \tag{4.129}$$

One can then set, consistently with our previous treatment,

$$A_n^{(\nu)}=\frac{n\,[1+c+2\,c\,\nu-(1+c)\,n]}{2\,(1-c)}\ ,\quad \omega^{(\nu)}=\nu\ , \tag{4.130}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Meixner polynomials (4.128):

$$p_n^{(\nu)}(x)=p_n(x;-\nu,c)\ . \tag{4.131}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Meixner polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = \frac{cn}{1-c} \ . \tag{4.132}$$

Note that this assignment entails no restriction on the 2 parameters of the Meixner polynomials $p_n(x;\beta,c)$.

It is moreover plain that, with the assignment

$$\nu = n-1 \ , \quad \text{hence} \quad \beta = 1-n \ , \tag{4.133}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n(x;1-n,c) = \prod_{m=1}^{n} (x-m+1) \ . \tag{4.134}$$

And *Corollary 3.10* entails even more general properties, such as (**new** finding!)

$$p_n(\ell-1;1-m,c) = 0 \ , \quad \ell = 1,...,m \ , \quad m = 1,...,n \ . \tag{4.135}$$

Likewise for

$$\nu = 2n \quad \text{hence} \quad \beta = -2n \quad \text{and moreover} \quad c = -1 \ , \tag{4.136}$$

Proposition 3.11 becomes applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n(x;-2n,-1) = \prod_{m=1}^{n} (x-2m+1) \ . \tag{4.137}$$

4.1.9 Krawtchouk

Mikhail Krawtchouk was a Ukraine mathematician. In 1929 published his most famous work, *Sur une generalisation des polynomes d'Hermite.* In this paper he introduced a new system of orthogonal polynomials now known as the Krawtchouk polynomials,which are polynomials associated with the binomials distribution.

The monic Krawtchouk polynomials $p_n(x;\alpha,\beta)$ (see [32]: and note the notational replacement of the parameters p and N used there with the parameters α and β used here, implying that only when $\beta = N$ and $n = 1, 2, .., N$ with N a *positive integer* these polynomials $p_n(x;\alpha,\beta)$ coincide with the standard Krawtchouk polynomials),

$$p_n(x;\alpha,\beta) \equiv p_n\left(x;\underline{\eta}\right) \ , \tag{4.138a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = -\alpha\beta + n(2\alpha - 1) \ , \quad b_n\left(\underline{\eta}\right) = \alpha\,(1-\alpha)\,n\,(n-1-\beta) \ . \tag{4.138b}$$

The standard version of these polynomials reads (see [32]):

$$K_n(x;\alpha,\beta) = \frac{1}{\alpha^n(-\beta)_n} p_n(x;\alpha,\beta) \ . \tag{4.139a}$$

We now identify the parameter ν via the assignment

$$\beta = \nu \ . \tag{4.140}$$

One can then set, consistently with our previous treatment,

$$A_n^{(\nu)} = n\left[\frac{1}{2} - \alpha - \alpha\,\nu + \left(-\frac{1}{2} + \alpha\right)n\right] \ , \quad \omega^{(\nu)} = \nu \ , \tag{4.141}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Krawtchouk polynomials (4.138):

$$p_n^{(\nu)}(x) = p_n(x;\alpha,\nu) \ . \tag{4.142}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (**new** finding!) that these normalized Krawtchouk polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -\alpha n \ . \tag{4.143}$$

Note that this assignment entails no restriction on the 2 parameters of the Krawtchouk polynomials $p_n(x;\alpha,\beta)$.

It is moreover plain that, with the assignment

$$\nu = n-1 \ , \quad \text{hence} \quad \beta = n-1 \tag{4.144}$$

(which is however incompatible with the definition of the standard Krawtchouk polynomials: $\beta = N$ and $n = 1, 2, .., N$ with N a *positive integer*), the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = -1$. These are **new** findings. And *Proposition 3.9* becomes as well applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n(x;\alpha, n-1) = \prod_{m=1}^{n} (x - m + 1) \ . \tag{4.145}$$

And *Corollary 3.10* entails even more general properties, such as (**new** findings!)

$$p_n(\ell - 1; \alpha, m-1) = 0 \ , \quad \ell = 1, ..., m \ , \quad m = 1, ..., n \ . \tag{4.146}$$

Likewise for

$$\nu = 2n \quad \text{hence} \quad \beta = 2n \quad \text{and moreover} \quad \alpha = \frac{1}{2} \tag{4.147}$$

(which is also incompatible with the definition of the standard Krawtchouk polynomials: $\beta = N$ and $n = 1, 2, .., N$ with N a *positive integer*), *Proposition 3.11* becomes applicable, entailing (**new** finding!) the remarkable *Diophantine* factorization

$$p_n(x; -2n, -1) = \prod_{m=1}^{n} (x - 2m + 1) \ . \tag{4.148}$$

4.1.10 Jacobi

The monic Jacobi polynomials $p_n(x; \alpha, \beta)$ (see [32]),

$$p_n(x;\alpha,\beta) \equiv p_n\left(x; \underline{\eta}\right) \ , \tag{4.149a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = \frac{(\alpha+\beta)(\alpha-\beta)}{(2\,n+\alpha+\beta)(2\,n+\alpha+\beta+2)} \ , \tag{4.149b}$$

$$b_n\left(\underline{\eta}\right) = -\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2\,n+\alpha+\beta-1)(2\,n+\alpha+\beta+1)(2\,n+\alpha+\beta)^2} \ . \tag{4.149c}$$

The standard version of these polynomials reads (see [32]):

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)_n}{2^n\, n!}\, p_n(x;\alpha,\beta) \ . \tag{4.150a}$$

Let us recall that for the Jacobi polynomials there holds the symmetry relation

$$p_n(-x;\beta,\alpha) = p_n(x;\alpha,\beta) \ . \tag{4.150b}$$

We now identify the parameter ν as follows:

$$\alpha = -\nu \ . \tag{4.151}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = -\frac{n(\nu+\beta)}{2n-\nu+\beta} \ , \quad \omega^{(\nu)} = 1 \ , \tag{4.152}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Jacobi polynomials (4.149):

$$p_n^{(\nu)}(x) = p_n(x;-\nu,\beta) \ . \tag{4.153}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (*well known* result!) that these normalized Jacobi polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -\frac{2n\,(n+\beta)}{(2n-\nu+\beta)(2n-\nu+\beta+1)} \ . \tag{4.154}$$

It is moreover plain that, with the assignment

$$\nu = n \ , \quad \text{hence} \quad \alpha = -n \ , \tag{4.155}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 0$. These seem **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations*. And *Proposition 3.9* becomes as well applicable, entailing (*well known* result!) the remarkable *Diophantine* factorization

$$p_n(x;-n,\beta) = (x-1)^n \ . \tag{4.156}$$

And *Corollary 3.10* entails even more general properties, such as the fact that the m Jacobi polynomials $p_n(x;-m,\beta)$, $m = 1,...,n$, feature $x = 1$ as a zero of order m.

Factorizations

The following (*not new*) relations among Jacobi polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$p_n(x;-m,\beta) = p_{n-m}(x;m,\beta)\, p_m(x;-m,\beta) \ , \quad m = 0,1,...,n \ . \tag{4.157}$$

4.1.11 Laguerre

The monic Laguerre polynomials $p_n(x;\alpha)$ (see [32]),

$$p_n(x;\alpha) \equiv p_n\left(x;\underline{\eta}\right) \ , \tag{4.158a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = -(2n+1+\alpha) \ , \quad b_n\left(\underline{\eta}\right) = -n(n+\alpha) \ . \tag{4.158b}$$

The standard version of these polynomials reads (see [32]):

$$L_n^{(\alpha)}(x) = \frac{1}{n!}\, p_n(x;\alpha) \ . \tag{4.159a}$$

We now identify the parameter ν as follows:

$$\alpha = -\nu \ . \tag{4.160}$$

With this assignment one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = -n(n-\nu) \ , \quad \omega^{(\nu)} = 0 \ , \tag{4.161}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with the normalized Laguerre polynomials (4.158):

$$p_n^{(\nu)}(x) = p_n(x;-\nu) \ . \tag{4.162}$$

Hence, with this identification, *Proposition 3.3* becomes applicable, entailing (*well known* result!) that the normalized Laguerre polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = n \ . \tag{4.163}$$

It is moreover plain that, with the assignment

$$\nu = n \ , \quad \text{hence} \quad \alpha = -n \ , \tag{4.164}$$

the factorizations implied by *Proposition 3.5*, and the properties implied by *Corollary 3.6*, become applicable with $\mu = 0$. These seem **new** findings. As for the additional findings entailed by *Corollary 3.7*, they are reported in Subsection *Factorizations*. And *Proposition 3.9* becomes as well applicable, entailing (*well known* result!) the remarkable *formula*

$$p_n(x; -n) = x^n \ . \tag{4.165}$$

And *Corollary 3.10* entails even more general properties, such as the fact that the m Laguerre polynomials $p_n(x; -m)$, $m = 1, ..., n$, feature $x = 0$ as a zero of order m.

Factorizations

The following (*not new*) relations among Laguerre polynomials are implied by *Proposition 3.5* with *Corollary 3.7*:

$$p_n(x; -m) = p_{n-m}(x; m)\, p_m(x; -m) \ , \quad m = 0, 1, ..., n \ . \tag{4.166}$$

4.1.12 Modified Charlier

In this subsection we introduce and treat a modified version of the standard (monic) Charlier polynomials. The standard (monic) Charlier polynomials $p_n(x; \alpha)$ (see [32]),

$$p_n(x; \alpha, \lambda) \equiv p_n\left(x; \underline{\eta}\right) \ , \tag{4.167a}$$

are defined by the three-term recursion relations (4.4) with

$$a_n\left(\underline{\eta}\right) = -n - \alpha \ , \quad b_n\left(\underline{\eta}\right) = -n\alpha \ . \tag{4.167b}$$

The standard version of these polynomials reads (see [32]):

$$C_n(x; \alpha) = (-\alpha)^{-n}\, p_n(x; \alpha) \ . \tag{4.168a}$$

But we have not found any assignment of the parameters α in terms of ν allowing the application of our machinery. To nevertheless proceed we introduce the class of (monic) "modified Charlier" polynomials $p_n(x; \alpha, \beta, \gamma)$ characterized by the three-term recursion relation (4.4) with

$$a_n\left(\underline{\eta}\right) = -\gamma(n + \alpha) + \beta \ , \quad b_n\left(\underline{\eta}\right) = -\gamma^2 n\alpha \ , \tag{4.169}$$

that obviously reduce to the monic Charlier polynomials for $\beta = 0,\ \gamma = 1$. Assigning instead

$$\beta = -\nu\ , \quad \gamma = -1\ , \tag{4.170}$$

one can set, consistently with our previous treatment,

$$A_n^{(\nu)} = \frac{1}{2} n \left(n - 1 - 2\nu + 2\alpha\right)\ ,\ \ \omega^{(\nu)} = \nu\ , \tag{4.171}$$

implying, via (3.4) and (3.5), that the polynomials $p_n^{(\nu)}(x)$ defined by the three-term recurrence relations (3.1) coincide with these (monic) modified Charlier polynomials:

$$p_n^{(\nu)}(x) = p_n\left(x; \alpha, -\nu, -1\right)\ . \tag{4.172}$$

Hence, with this identification, *Proposition 2.1* becomes applicable, entailing (**new** finding!) that these (monic) modified Charlier polynomials satisfy the second recursion relation (3.6) with

$$g_n^{(\nu)} = -n\ . \tag{4.173}$$

There does not seem to be any interesting results for the zeros of these polynomials.

Chapter 5

Hypergeometric Origin of Diophantine properties.

When the polynomials treated are given in their usual hypergeometric representation, some of the results given in the previous chapter are consequences of transformation formula which date back the 19th century in some case and the early twentieth century in others. Indeed, in a recent paper [21], M.E.H. Ismail and Y.Chen have proved that the *Diophantine* property of certain polynomials in the Askey scheme, reported in the previous chapters, is explained, with suitably chosen parameter value, in terms of the summation theorem of hypergeometric series. They have also generalized this procedure to polynomials arising from the basic hypergeometric series. They found, with suitably chosen of parameters and certain q-analogue of the summation theorems, zeros of these polynomials explicitly, which are no longer integer valued. Probably almost all of the results on zeros can be extended to basic hypergeometric polynomials when there is a need for them in physics.

We have considered polynomials defined by a secular equation

$$P_N(x) = \det(x_N - A_N) \tag{5.1}$$

where A_N is a tridiagonal matrix of size N. Hence, $P_N(x)$ maybe interpreted as orthogonal polynomials if the super-diagonal elements of A_N are real and none of them vanishes. The motivation, for Chen and Ismail, of considering these Diophantine property is the fact that a hypergeometric polynomial of degree n in the variable x, is factored as $f_m(x)g_{n-m}(x)$, here $f_m(x)$ has degree m, $g_{n-m}(x)$ has degree $n-m$ in the variable x. Moreover the zeros of f_m are equispaced. This hold for all m, $1 \leq m \leq n$. They are proved that all Diophantine results in the previous chapters follow from summation theorems for hypergeometric functions, and in their article they give q-analogues version.

In fact, in the Appendix A of [14] we have already considered the connection of the polynomials obtained with the hypergeometric functions, and we have mentioned that the factorizations are conseguence of Saalschütz formula.

Chapter 6

Connection with the discrete integrability.

The aim of this chapter is the connection of the study of the three term recursion relation

$$p_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right) p_n^{(\nu)}(x) + b_n^{(\nu)}\, p_{n-1}^{(\nu)}(x) \tag{6.1}$$

and the second recursion

$$p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)}\, p_{n-1}^{(\nu-1)}(x) \ , \tag{6.2}$$

with the theory of the integrable discrete systems developed in last three decades. Indeed, we use the Lax technique developed in [18] in the discrete case.

We consider the three term recursion as a spectral problem:

$$\hat{L}\ p_n^{(\nu)}(x) = x\ p_n^{(\nu)}(x) \ , \tag{6.3}$$

where the operator $\hat{L}$ has the form

$$\hat{L} = \hat{L}_n^{(\nu)} = \hat{E}_+ - a_n^{(\nu)} \hat{I} - b_n^{(\nu)}\ \hat{E}_- \ , \tag{6.4}$$

and the variable x is regarded as a n-independent spectral parameter.

The shift operators $\hat{E}_\pm$ act on a generic function $f = f_n^{(\nu)}$ as

$$\hat{E}_\pm\ f_n^{(\nu)} = f_{n\pm1}^{(\nu)} \ .$$

Moreover, we will consider the second recursion as a discrete-time evolution of the "wave function". For a general treatment we initially introduce the discrete time evolution equation

$$\hat{T} p_n^{(\nu)}(x) = \left(\hat{I} + \hat{H}^{[k]}\right) p_n^{(\nu)}(x) \ ; \tag{6.5}$$

where the "time-shift" operator $\hat{T}$ acts on a generic function as

$$\hat{T} f_n^{(\nu)} = f_n^{(\nu+1)} \ ;$$

and $\hat{H}^{[k]} = \hat{H}_n^{(\nu)\ [k]}$ will be determined. We will see later that the index k is related to the maximum order of the operator $\hat{E}_-$.

The compatibility condition for the two linear problems (6.3),(6.5) reads:

$$\begin{aligned}
\hat{T}\ \hat{L} p_n^{(\nu)}(x) &= \hat{L}^+ \hat{T} p_n^{(\nu)}(x) = \hat{L}^+ \left(\hat{I} + \hat{H}^{[k]}\right) p_n^{(\nu)}(x) = \hat{T} x p_n^{(\nu)}(x) = x \hat{T} p_n^{(\nu)}(x) \\
&= x \left(\hat{I} + \hat{H}^{[k]}\right) p_n^{(\nu)}(x) = \left(\hat{I} + \hat{H}^{[k]}\right) x p_n^{(\nu)}(x) = \left(\hat{I} + \hat{H}^{[k]}\right) \hat{L} p_n^{(\nu)}(x)
\end{aligned} \tag{6.6}$$

yelding the operator equation:

$$\hat{L}^+ \left(\hat{I} + \hat{H}^{[k]}\right) - \left(\hat{I} + \hat{H}^{[k]}\right) \hat{L} = 0 \ , \tag{6.7}$$

where $\hat{L}^+$is defined as

$$\hat{L}^+ = \hat{E}_+ - a_n^{(\nu+1)} \hat{I} - b_n^{(\nu+1)}\ \hat{E}_- \quad . \tag{6.8}$$

6.1 The complete hierarchy

In this section we will present a general procedure to construct a hierarchy of nonlinear discrete-discrete equations by requiring that an operator

$$\hat{H}^{[k]} := \hat{H}_n^{(\nu)[k]} = \sum_{l=1}^{k} h_n^{(\nu)[l,k]} \left(\hat{E}_-\right)^l \ , \tag{6.9}$$

with

$$h_n^{(\nu)[l,k]} = 0 \ \text{ if } \ l \leq 0 \text{ and } l > k \ , \tag{6.10}$$

and $k+1$ scalar functions $w_n^{(\nu)[l,k]}$ $\ l = 0, 1, ..., k$ satisfies the equation

$$\hat{L}^{(+)} \hat{H}^{[k]} - \hat{H}^{[k]} \hat{L} = \sum_{l=0}^{k} w_n^{(\nu)[l,k]} \left(\hat{E}_-\right)^l \ . \tag{6.11}$$

Explicitly, we require:

$$\left(\hat{E}_+ - a_n^{(\nu+1)}\hat{I} - b_n^{(\nu+1)}\hat{E}_-\right)\left[\sum_{l=1}^{k} h_n^{(\nu)[l,k]}\left(\hat{E}_-\right)^l\right] -$$
$$\left[\sum_{l=1}^{k} h_n^{(\nu)[l,k]}\left(\hat{E}_-\right)^l\right]\left(\hat{E}_+ - a_n^{(\nu)}\hat{I} - b_n^{(\nu)}\hat{E}_-\right) =$$
$$= \sum_{l=0}^{k} w_n^{(\nu)[l,k]}\left(\hat{E}_-\right)^l \ , \tag{6.12}$$

so that the functions $w_n^{(\nu)[l,k]}$ take the form

$$\begin{aligned} w_n^{(\nu)[l,k]} &= h_{n+1}^{(\nu)[l+1,k]} - h_n^{(\nu)[l+1,k]} - \left(a_n^{(\nu+1)} - a_{n-l}^{(\nu)}\right) h_n^{(\nu)[l,k]} \\ &\quad - b_n^{(\nu+1)} h_{n-1}^{(\nu)[l-1,k]} + b_{n-l+1}^{(\nu)} h_n^{(\nu)[l-1,k]} \\ l &= 1, ..., k \ , \end{aligned} \tag{6.13}$$

with

$$w_n^{(\nu)[l,k]} = 0 \ , \quad \text{for } l < 0 \text{ and } l > k+1; \tag{6.14}$$

$$w_n^{(\nu)[0,k]} = h_{n+1}^{(\nu)[1,k]} - h_n^{(\nu)[1,k]} \ ; \tag{6.15}$$

$$-b_n^{(\nu+1)} h_{n-1}^{(\nu)[k,k]} + b_{n-k}^{(\nu)} h_n^{(\nu)[k,k]} = 0. \tag{6.16}$$

We then construct a new operator $\hat{H}^{[k+1]}$ using the following *ansatz*:

$$\begin{aligned} \hat{H}^{[k+1]} &= \hat{H}^{[k]}\hat{L} + \sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]}\left(\hat{E}_-\right)^l \\ &= \left[\sum_{l=1}^{k} h_n^{(\nu)[l,k]}\left(\hat{E}_-\right)^l\right]\left(\hat{E}_+ - a_n^{(\nu)}\hat{I} - b_n^{(\nu)}\hat{E}_-\right) + \sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]}\left(\hat{E}_-\right)^l \\ &= \sum_{l=1}^{k+1} h_n^{(\nu)[l,k+1]}\left(\hat{E}_-\right)^l \ , \end{aligned} \tag{6.17}$$

where the functions $q_n^{(\nu)[l,k+1]}$ must be determined.

In order that the coefficient of the identity operator $\hat{I}$ vanishes, must hold the equation:

$$h_n^{(\nu)[1,k]} = -q_n^{(\nu)[0,k+1]} \ , \tag{6.18}$$

and for the others order of the operator $\hat{E}_-$ we have:

$$h_n^{(\nu)[l,k+1]} = h_n^{(\nu)[l+1,k]} - a_{n-l}^{(\nu)} h_n^{(\nu)[l,k]} - b_{n-l+1}^{(\nu)} h_n^{(\nu)[l-1,k]} + q_n^{(\nu)[l,k+1]} \ . \tag{6.19}$$

For the iterated operator must be true the following equation containing new functions $w_n^{(\nu)[l,k+1]}$ with $l = 0, 1, ..., k+1$

$$\hat{L}^{(+)}\hat{H}^{[k+1]} - \hat{H}^{[k+1]}\hat{L} = \sum_{l=0}^{k+1} w_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l \ , \tag{6.20}$$

hence, explicitly

$$\begin{aligned}
&\left[\sum_{l=0}^{k} w_n^{(\nu)[l,k]} \left(\hat{E}_-\right)^l \right] \left(\hat{E}_+ - a_n^{(\nu)}\hat{I} - b_n^{(\nu)}\hat{E}_-\right) \\
&+\left(\hat{E}_+ - a_n^{(\nu+1)}\hat{I} - b_n^{(\nu+1)}\hat{E}_-\right) \left[\sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l\right] - \\
&\left[\sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l\right] \left(\hat{E}_+ - a_n^{(\nu)}\hat{I} - b_n^{(\nu)}\hat{E}_-\right) = \\
&\left(\sum_{l=0}^{k} w_n^{(\nu)[l,k]} - \sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]}\right) \left[\left(\hat{E}_-\right)^{l-1} - a_{n-l}^{(\nu)} \left(\hat{E}_-\right)^l - b_{n-l}^{(\nu)} \left(\hat{E}_-\right)^{l+1} \right] + \\
&\left[\sum_{l=0}^{k+1} q_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^{l-1} - \sum_{l=0}^{k+1} a_n^{(\nu+1)} q_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l - \sum_{l=0}^{k+1} b_n^{(\nu+1)} q_{n-1}^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^{l+1}\right] \\
= \ &\sum_{l=0}^{k+1} w_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l
\end{aligned} \tag{6.21}$$

$$\begin{aligned}
&\sum_{l=0}^{k}\{\left[w_n^{(\nu)[l,k]} + q_{n+1}^{(\nu)[l,k+1]} - q_n^{(\nu)[l,k+1]}\right] \left(\hat{E}_-\right)^{l-1} \\
&+\left[\left(a_{n-l}^{(\nu)} - a_n^{(\nu+1)}\right) q_n^{(\nu)[l,k+1]} - a_{n-l}^{(\nu)} w_n^{(\nu)[l,k]}\right] \left(\hat{E}_-\right)^l \\
&+\left[b_{n-l}^{(\nu)} q_n^{(\nu)[l,k+1]} - b_n^{(\nu+1)} q_{n-1}^{(\nu)[l,k+1]} - b_{n-l}^{(\nu)} w_n^{(\nu)[l,k+1]}\right] \left(\hat{E}_-\right)^{l+1}\} \\
&+\{q_{n+1}^{(\nu)[k+1,k+1]} \left(\hat{E}_-\right)^k - a_n^{(\nu+1)} q_n^{(\nu)[k+1,k+1]} \left(\hat{E}_-\right)^{k+1} \\
&-b_n^{(\nu+1)} q_{n-1}^{(\nu)[k+1,k+1]} \left(\hat{E}_-\right)^{k+2} \\
&-q_n^{(\nu)[k+1,k+1]} \left[\left(\hat{E}_-\right)^k - a_{n-k-1}^{(\nu)} \left(\hat{E}_-\right)^{k+1} - b_{n-k-1}^{(\nu)} \left(\hat{E}_-\right)^{k+2}\right]\} \\
= \ &\sum_{l=0}^{k+1} w_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l \ ,
\end{aligned} \tag{6.22}$$

with the identity

$$w_n^{(\nu)[0,k]} + q_{n+1}^{(\nu)[0,k+1]} - q_n^{(\nu)[0,k+1]} = 0 \ ; \tag{6.23}$$

and the equations

$$b_{n-k-1}^{(\nu)} q_n^{(\nu)[k+1,k+1]} - b_n^{(\nu+1)} q_{n-1}^{(\nu)[k+1,k+1]} = 0 \tag{6.24}$$

$$\begin{aligned}
w_n^{(\nu)[k+1,k+1]} &= -b_{n-k}^{(\nu)} w_n^{(\nu)[k,k]} - b_n^{(\nu+1)} q_{n-1}^{(\nu)[k,k+1]} - b_{n-k}^{(\nu)} q_n^{(\nu)[k,k+1]} \\
&\quad - \left(a_n^{(\nu+1)} - a_{n-k-1}^{(\nu)}\right) q_n^{(\nu)[k+1,k+1]} && (6.25a) \\
w_n^{(\nu)[k,k+1]} &= -a_{n-k}^{(\nu)} w_n^{(\nu)[k,k]} + q_{n+1}^{(\nu)[k+1,k+1]} - q_n^{(\nu)[k+1,k+1]} \\
&\quad - \left(a_n^{(\nu+1)} - a_{n-k}^{(\nu)}\right) q_n^{(\nu)[k,k+1]} && (6.25b) \\
w_n^{(\nu)[l,k+1]} &= w_n^{(\nu)[l+1,k]} - a_{n-l}^{(\nu)} w_n^{(\nu)[l,k]} + q_{n+1}^{(\nu)[l+1,k+1]} - q_n^{(\nu)[l+1,k+1]} \\
&\quad - \left(a_n^{(\nu+1)} - a_{n-l}^{(\nu)}\right) q_n^{(\nu)[l,k+1]} - b_{n-l+1}^{(\nu)} w_n^{(\nu)[l-1,k]} \\
&\quad -b_n^{(\nu+1)} q_{n-1}^{(\nu)[l-1,k+1]} + b_{n-l+1}^{(\nu)} q_n^{(\nu)[l-1,k+1]} && (6.25c) \\
\text{for } l &= 0, 1, .., k-1.
\end{aligned}$$

It is easily seen that

$$w_n^{(\nu)[0,k+1]} = h_{n+1}^{(\nu)[1,k+1]} - h_n^{(\nu)[1,k+1]} \ ; \tag{6.26}$$

(*i.e. after the iteration the fields* $w_n^{(\nu)[0,k+1]}$ *preserve the structure of difference*).

Now we consider the operator $(\hat{I} + \hat{H}^{[k+1]})$ and the following equation

$$\hat{L}^{(+)}(\hat{I} + \hat{H}^{[k+1]}) - (\hat{I} + \hat{H}^{[k+1]})\hat{L} = \sum_{l=0}^{k+1} r_n^{(\nu)[l,k+1]} \left(\hat{E}_-\right)^l \ , \tag{6.27}$$

where $\hat{I}$ is the identity operator.

Imposing the compatibility condition of equations (6.3), (6.5) and (6.27), the fields $r_n^{(\nu)[l,k+1]}$ must be zeros for all values of $l = 0, 1, ..., k+1$. So that we obtain the following hierarchy of equations ($k+4$ equations for the $k+4$ unknown quantity $a_n^{(\nu)}$, $b_n^{(\nu)}$and $q_n^{(\nu)[l,k+1]}$ with $l = 0, 1, ..., k+1$):

$$a_n^{(\nu+1)} - a_n^{(\nu)} = w_n^{(\nu)[0,k+1]} \ ; \tag{6.28}$$

$$b_n^{(\nu+1)} - b_n^{(\nu)} = w_n^{(\nu)[1,k+1]} \; ; \tag{6.29}$$

$$w_n^{(\nu)[l,k+1]} = 0 \quad \text{for} \;\; l = 2, ..., k+1 \; ; \tag{6.30}$$

and

$$q_n^{(\nu)[0,k+1]} + h_n^{(\nu)[1,k]} = 0 \; ; \tag{6.31}$$

$$b_{n-k-1}^{(\nu)} q_n^{(\nu)[k+1,k+1]} - b_n^{(\nu+1)} q_{n-1}^{(\nu)[k+1,k+1]} = 0 \; . \tag{6.32}$$

6.2 Case $k = 0$

In this section we will show the first equations of the hierarchy. The case when $k = 0$. This case is related to the previous treatment of the chapter 3, and the compatibility equations are in some way the discrete-time Toda equations (see [41]). A comparable procedure is given by Spiridonov and Zhedanov considering discrete Darboux transformation for the discrete time Toda lattice (see [44]).

We have by (6.9)

$$\hat{H}^{[0]} = 0. \tag{6.33}$$

By (6.11) we obtain

$$w_n^{(\nu)[0,0]} = 0 \; , \tag{6.34}$$

By (6.17)

$$\hat{H}^{[1]} = q_n^{(\nu)[1,1]} \hat{E}_- \; , \tag{6.35}$$

imposing by (6.18)

$$q_n^{(\nu)[0,1]} = 0.$$

The equations of the hierarchy (6.28),(6.29) become

$$a_n^{(\nu+1)} - a_n^{(\nu)} = w_n^{(\nu)[0,1]} \; ; \tag{6.36}$$

$$b_n^{(\nu+1)} - b_n^{(\nu)} = w_n^{(\nu)[1,1]} \; , \tag{6.37}$$

where by (6.25b)

$$w_n^{(\nu)[0,1]} = h_{n+1}^{(\nu)[1,1]} - h_n^{(\nu)[1,1]} = q_{n+1}^{(\nu)[1,1]} - q_n^{(\nu)[1,1]} \; , \tag{6.38}$$

and by (6.25a)

$$w_n^{(\nu)[1,1]} = -\left(a_n^{(\nu+1)} - a_{n-1}^{(\nu)}\right) q_n^{(\nu)[1,1]} \; .$$

Moreover, we have the other equation

$$b_{n-1}^{(\nu)} q_n^{(\nu)[1,1]} - b_n^{(\nu+1)} q_{n-1}^{(\nu)[1,1]} = 0 \ . \tag{6.39}$$

Recapitulating,

$$a_n^{(\nu+1)} - a_n^{(\nu)} = q_{n+1}^{(\nu)[1,1]} - q_n^{(\nu)[1,1]} \ , \tag{6.40a}$$

$$b_n^{(\nu+1)} - b_n^{(\nu)} = -q_n^{(\nu)[1,1]} \left(a_n^{(\nu+1)} - a_{n-1}^{(\nu)} \right) \ , \tag{6.40b}$$

$$b_{n-1}^{(\nu)} q_n^{(\nu)[1,1]} = b_n^{(\nu+1)} q_{n-1}^{(\nu)[1,1]} \ . \tag{6.40c}$$

are related to the equations of the discrete time Toda lattice (see [41]).

In this case,the second recursion relation (3.6) is related to $I + \hat{H}^{[1]}$ when

$$q_n^{(\nu)[1,1]} = g_n^{(\nu+1)}. \tag{6.41}$$

And the equations (6.40) are the same of (3.8a), (3.8b) and (3.8c).

6.3 Appendix 6

In this appendix we show the connection between the spectral problem previously seen

$$p_{n+1}^{\nu}(x) - a_n^{\nu} \ p_n^{\nu}(x) - b_n^{\nu} \ p_{n-1}^{\nu}(x) = L_n^{\nu} \ p_n^{\nu}(x) = x \ p_n^{\nu}(x) \ , \tag{6.42}$$

and the discrete Schrödinger spectral problem

$$\psi_{n-1,m} + a_{n,m}\psi_{n+1,m} + b_{n,m}\psi_{n,m} = L_{n,m} \ \psi_{n,m} = x \ \psi_{n,m} \ . \tag{6.43}$$

We make the following ansatz for the wave function:

$$\psi_{n,m} = q_{n,m} \ p_{n,m} \ ; \tag{6.44}$$

and we substitute it in (6.43), moreover divide for $q_{n,m}$

$$\frac{q_{n-1,m}}{q_{n,m}} \ p_{n-1,m} + a_{n,m} \ \frac{q_{n+1,m}}{q_{n,m}} \ p_{n+1,m} + (b_{n,m} - x) \ \ p_{n,m} = 0 \ . \tag{6.45}$$

We impose

$$a_{n,m} \ \frac{q_{n+1,m}}{q_{n,m}} = 1 \ , \tag{6.46a}$$

$$\frac{q_{n-1,m}}{q_{n,m}} = -\tilde{b}_{n,m} \ , \tag{6.46b}$$

$$b_{n,m} = -\tilde{a}_{n,m} \ . \tag{6.46c}$$

where $\tilde{a}_{n,m}$, $\tilde{b}_{n,m}$ are two fields connected with the fields a_n^ν and b_n^ν in the case where the index m is not integer.

By (6.46a) we obtain

$$q_{n,m} = q_{0,m} \prod_{l=0}^{n-1} \frac{1}{a_{l,m}} \ .$$

(6.46b) and (6.46c) identify the fields $\tilde{a}_{n,m}$ and $\tilde{b}_{n,m}$:

$$\begin{aligned} \tilde{a}_{n,m} &= -b_{n,m} \ ; \\ \tilde{b}_{n,m} &= -a_{n-1,m} \ . \end{aligned}$$

Chapter 7

Diophantine relations from the Stationary KdV Hierarchies.

Now we will consider isochronous systems obtained modifying the N-th ODE of one stationary hierarchy of integrable equations. In our case we'll investigate the stationary KdV hierarchy. The complete integrability of this hierarchy is proved in the famous article of O.I. Bogoyavlenskij and S.P. Novikov [11]. Further relevant results on integrable finite-dimensional dynamical systems related to soliton hierarchies have been derived in from [3] to [7] and from [27] to [30].

The general approach to arrive at the findings reported in this chapter can be described as follows (see for instance [9]). One starts from an *integrable* ODE of (*arbitrary*) order $N+1$, *all* solutions of which are *meromorphic* functions of its independent ("time") variable **(the Painlevé property)**. One then modifies it (via an appropriate change of dependent and independent variables: the "Trick") so that—thanks to the analyticity properties in *complex* time of the solutions of the original *integrable* ODE—the modified ODE becomes *entirely isochronous*: its solutions are *all* periodic with the same *fixed* period. One then use the technique developed in the first chapter, i.e. identifies the equilibrium solutions of the *isochronous* ODE and investigates their infinitesimally small oscillations. In this manner one arrives at *Diophantine* relations: *polynomials* are identified which factorize in terms of *integer* zeros.

Our route to arrive at these findings is not new, and it might appear contrived: indeed, in the context treated below, its formulation via an *isochronous* ODE could be replaced by other, equivalent approaches of a more algebraico-geometrical character. We prefer this route because its "physical" significance is quite transparent and its application has already yielded interesting findings (for a review see [9], including its Appendix C entitled "Diophantine findings and conjectures"). The application of this approach to the *inte-*

grable ODE treated herein is new, hence the corresponding findings are as well new. And it is plain that analogous results are obtainable by applying the same approach to other *integrable* ODEs, this being perhaps the most interesting aspect of the findings reported below.

7.1 Results from the Stationary KdV equations: Main results

It is well-known that the following nonlinear ODE,

$$\{Ł\,[\zeta]\}^M \cdot \zeta' = 0 \ , \tag{7.1}$$

of order

$$N = 2M + 1 \ , \tag{7.2}$$

is *integrable*, and in particular that *all* its solutions $\zeta(\tau)$ possess the ("Painlevé") property to be *meromorphic* functions of the independent variable τ, considered as a *complex* variable. Here and throughout the integro-differential operator $Ł\,[\zeta]$ acts as follows on functions $\varphi(\tau)$:

$$\begin{aligned} Ł\,[\zeta] \cdot \varphi(\tau) &= \left[D^2 - 4\zeta - 2\zeta' D^{-1}\right] \cdot \varphi(\tau) \\ &= \varphi''(\tau) - 4\zeta(\tau)\,\varphi(\tau) - 2\zeta'(\tau) \int^{\tau} d\tau'\ \varphi(\tau') \ . \end{aligned} \tag{7.3}$$

Here $D \equiv d/d\tau$, primes appended to functions denote differentiations with respect to the independent variable τ, and the integration is meant to be performed *omitting* the contribution from the lower end of the integration range. The notation $\{Ł\,[\zeta]\}^M \cdot$ indicates of course the iterated application M times of the operator $Ł\,[\zeta]$; accordingly, here M is a fixed *positive integer*, and N is the corresponding *odd positive integer*, see (7.2). In the following we will freely use N and M (sometimes even in the same formula, to write it in neater form), on the understanding that they are always related by (7.2).

The fact that the ODE (7.1) is *integrable*—as well as the very fact that it is indeed an ODE rather than an integro-differential equation, as it might at first sight appear to be, see (7.3)—is of course well-known: this ODE is just the *stationary* version of the M-th PDE of the KdV hierarchy of *integrable* PDEs (with the "spatial" independent variable denoted here as τ), see for instance [10] [34].

For instance for $M = 1$, $M = 2$ respectively $M = 3$ the ODE (7.1) reads

$$\zeta^{(3)} = 6\zeta^{(1)}\zeta \ , \tag{7.4a}$$

$$\zeta^{(5)} = 10\left(\zeta^{(3)}\zeta + 2\zeta^{(2)}\zeta^{(1)} - 3\zeta^{(1)}\zeta^{2}\right) \ , \tag{7.4b}$$

respectively

$$\begin{aligned} \zeta^{(7)} \ &= \ 14\left[\zeta^{(5)}\zeta + 3\zeta^{(4)}\zeta^{(1)} + 5\zeta^{(3)}\zeta^{(2)}\right. \\ &\left. -5\left(\zeta^{(3)}\zeta^{2} + 4\zeta^{(2)}\zeta^{(1)}\zeta + \zeta^{(1)3}\right) + 10\zeta^{(1)}\zeta^{3}\right] \ . \end{aligned} \tag{7.4c}$$

Here (and throughout) we use the notation

$$\zeta^{(n)} \equiv \left(\frac{d}{d\tau}\right)^{n} \zeta\left(\tau\right) \ , \quad n = 1, 2, 3, ... \ . \tag{7.5a}$$

Via the convenient definition (entailing $\zeta_1(\tau) = \zeta(\tau)$)

$$\zeta^{(n-1)} \equiv \left(\frac{d}{d\tau}\right)^{n-1} \zeta\left(\tau\right) = \zeta_n\left(\tau\right) \ , \tag{7.5b}$$

with, here and hereafter (unless otherwise indicated), $n = 1, 2, 3, ..., N$, the single third-order ODE (7.4a) is seen to be equivalent to the system of 3 first-order ODEs

$$\zeta_1' = \zeta_2 \ , \quad \zeta_2' = \zeta_3 \ , \quad \zeta_3' = 6\zeta_2\zeta_1 \ , \tag{7.6a}$$

the single fifth-order ODE (7.4b) is seen to be equivalent to the system of 5 first-order ODEs

$$\zeta_n' = \zeta_{n+1} \ , \quad n = 1, 2, 3, 4 \ ; \quad \zeta_5' = 10\left(\zeta_4\zeta_1 + 2\zeta_3\zeta_2 - 3\zeta_2\zeta_1^{2}\right) \ , \tag{7.6b}$$

and the single seventh-order ODE (7.4c) is seen to be equivalent to the system of 7 first-order ODEs

$$\begin{aligned} \zeta_n' \ &= \ \zeta_{n+1} \ , \quad n = 1, 2, 3, 4, 5, 6 \ ; \\ \zeta_7' \ &= \ 14\left[\zeta_6\zeta_1 + 3\zeta_5\zeta_2 + 5\zeta_4\zeta_3\right. \\ &\left. -5\left(\zeta_4\zeta_1^{2} + 4\zeta_3\zeta_2\zeta_1 + \zeta_2^{3}\right) + 10\zeta_2\zeta_1^{3}\right] \ . \end{aligned} \tag{7.6c}$$

Likewise, the single ODE, of order N, satisfied by the dependent variable $\zeta(\tau)$, reading

$$\zeta_N' = f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right) \tag{7.7}$$

with the polynomial function $f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right)$ defined by identifying via (7.5b) this ODE with the N-th order ODE (7.1), is equivalent to the system of N first-order ODEs

$$\zeta_n' = \zeta_{n+1} \ , \quad n = 1, 2, ..., N-1 \ ; \quad \zeta_N' = f_N\left(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1\right) \ . \tag{7.8}$$

For instance this definition of $f_N(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1)$ entails (see (7.4) or (7.6))

$$f_3(\zeta_2, \zeta_1) = 6\zeta_2\zeta_1 \ , \tag{7.9a}$$

$$f_5(\zeta_4, \zeta_3, \zeta_2, \zeta_1) = 10\left(\zeta_4\zeta_1 + 2\zeta_3\zeta_2 - 3\zeta_2\zeta_1^2\right) \ , \tag{7.9b}$$

respectively

$$f_7(\zeta_6, \zeta_5, \zeta_4, \zeta_3, \zeta_2, \zeta_1) =$$
$$14\left[\zeta_6\zeta_1 + 3\zeta_5\zeta_2 + 5\zeta_4\zeta_3 - 5\left(\zeta_4\zeta_1^2 + 4\zeta_3\zeta_2\zeta_1 + \zeta_2^3\right) + 10\zeta_2\zeta_1^3\right] \ . \tag{7.9c}$$

The *integrable* dynamical system (7.8)—with the N functions $\zeta_n \equiv \zeta_n(\tau)$ considered as N dependent variables—is our starting point. This choice represents the main novelty of our treatment; the possibility of analogous developments, using the same methodology, see below, but with different points of departure (say, other hierarchies of *integrable* nonlinear PDEs), is obvious.

The fact that this system of ODEs, (7.8), is *integrable* entails that it possesses the Painlevé property: *all* its solutions $\zeta_n(\tau)$ are *meromorphic* functions of the complex variable τ (see for instance [34]).

It is moreover well-known—and in any case clear from its definition, see (7.5b), (7.7) and (7.8)—that the function $f_N(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1)$ features the following *scaling property*:

$$f_N\left(\alpha^N\zeta_{N-1}, \alpha^{N-1}\zeta_{N-2}, ..., \alpha^2\zeta_1\right) = \alpha^{N+2}\ f_N(\zeta_{N-1}, \zeta_{N-2}, ..., \zeta_1) \ . \tag{7.10}$$

It is therefore possible (see for instance [9]), via the following change of dependent and independent variables,

$$z_n(t) = \exp\left[i\ \ (n+1)\ \ t\right]\ \zeta_n(\tau) \ , \tag{7.11a}$$

$$\tau = i\ \left[1 - \exp(i\ t)\right] \ , \tag{7.11b}$$

to transform the (*autonomous* and *integrable*) dynamical system (7.8) into the following system,

$$\dot{z}_n - i\ \ (n+1)\ \ z_n = z_{n+1} \ , \quad n = 1, ..., N-1 \ , \tag{7.12a}$$

$$\dot{z}_N - i\ \ (N+1)\ \ z_N = f_N(z_{N-1}, z_{N-2}, ..., z_1) \ , \tag{7.12b}$$

which is as well *autonomous* and of course *integrable*, and is moreover *isochronous*, so that *all* its solutions feature the periodicity property

$$z_n(t + 2\pi) = z_n(t) \ . \tag{7.13}$$

Here and below i is the imaginary unit, $i^2 = -1$, and a superimposed dot denotes differentiation with respect to the independent, *real* "time" variable t, so that in particular $\dot{\tau}(t) = \exp(i\ t)$ (see (7.11b), and note that this relation also entails $\tau(0) = 0$ hence, via (7.11a), $z_n(0) = \zeta_n(0)$; this simplifies the relation among the initial data of the two dynamical systems (7.8) and (7.12), but plays no role in the following developments).

The fact that the system (7.12) is *isochronous*, see (7.13), is an obvious consequence [9] of the change of variables (7.11) together with the *meromorphic* character of all the solutions $\zeta_n(\tau)$ of the integrable dynamical system (7.8).

Let now

$$z_n(t) = \bar{z}_n \tag{7.14}$$

denote an *equilibrium* configuration of the dynamical system (7.12), so that the N (time-independent!) numbers $\bar{z}_n$ satisfy the set of N algebraic equations

$$-i\ \ (n+1)\ \ \bar{z}_n = \bar{z}_{n+1}\ , \quad n = 1, ..., N-1\ , \tag{7.15a}$$

$$-i\ \ (N+1)\ \ \bar{z}_N = f_N(\bar{z}_{N-1}, \bar{z}_{N-2}, ..., \bar{z}_1)\ . \tag{7.15b}$$

It is then clearly convenient to set

$$\bar{z}_n = n!\ \ (-i)^{n+1}\ \ y\ , \tag{7.16}$$

which guarantees that the $N-1$ equations (7.15a) are all automatically satisfied, while, to also satisfy the remaining equation (7.15b), the number y is required to be one of the $M+1$ roots of the following polynomial equation of order $M+1 = (N+1)/2$:

$$(N+1)!\ y = f_N((N-1)!\ y, (N-2)!\ y, ..., 3!\ y, 2\ y, y)\ . \tag{7.17}$$

Note that, to write this equation in a neater way, we took advantage of the scaling property (7.10). The fact that this is a polynomial equation of degree $M+1 = (N+1)/2$ in the unknown y is a clear consequence of the definition of the function $f_N(z_{N-1}, z_{N-2}, ..., z_1)$, as given above: see for instance (7.9), of course with (7.11), (7.14) and (7.16). Hence this polynomial equation has $M+1 = (N+1)/2$ solutions, including the trivial solution $y_0 = 0$ (which clearly is always featured by this equation: see, for instance, again (7.9), or below).

A simple calculation shows that for $M = 1$, $N = 3$, this equation, (7.17), reads (see (7.9a))

$$4!\ y - f_3(2\ y, y) = -12\ y\ \ (y-2) = 0\ , \tag{7.18}$$

entailing for y the 2 values $y_0 = 0$ and $y_1 = 2$; likewise, it is easily seen that for $M = 2$, $N = 5$ the 3 values of y are $y_0 = 0, y_1 = 2$ and $y_2 = 6$, and for $M = 3$, $N = 7$ the 4 values of y are $y_0 = 0,\ y_1 = 2$, $y_2 = 6$ and $y_3 = 12$. This suggest the conjecture that, for arbitrary (*positive integer*) M, the $M + 1$ values of y satisfying (7.17) are

$$y_k = k\,(k+1)\ ,\quad k = 0, 1, ..., M\ , \tag{7.19a}$$

implying the factorization

$$\begin{aligned}
&(N+1)!\ y - f_N\left((N-1)!\ y, (N-2)!\ y, ..., 3!\ y, 2\ y, y\right)\\
= \ &K_M \prod_{k=0}^{M}\ [y - k\,(k+1)]\ ,
\end{aligned} \tag{7.19b}$$

with K_M an appropriate normalization constant (independent of y). *This conjecture is validated in the following Section.* Hence the $M+1$ equilibrium configurations (7.16) are characterized by the relations

$$\bar{z}_n^{(k)} = n!\ (-i)^{n+1}\ k\ \ (k+1)\ ,\quad n = 1, ..., N\ ;\quad k = 0, 1, ..., M\ . \tag{7.20}$$

Let us note that, in the context of the algebraico-geometrical approach applied directly to the original dynamical system, this is a well-known result (see for instance [34]).

The next step is to linearize the *isochronous* system of ODEs (7.12) near these equilibria. Hence we set in (7.12)

$$z_n\,(t) = \bar{z}_n + \varepsilon\ w_n\,(t)\ ,\quad n = 1, ..., N\ , \tag{7.21}$$

and in the limit of infinitesimal ε we obtain, for the N dependent variables $w_n\,(t)$, the *linear* system of N ODEs

$$\dot{w}_n - i\ \ (n+1)\ \ w_n = w_{n+1}\ ,\quad n = 1, ..., N-1\ , \tag{7.22a}$$

$$\dot{w}_N - i\ \ (N+1)\ \ w_N = \sum_{n=1}^{N-1} f_{N,n}\ w_n\ , \tag{7.22b}$$

where clearly

$$f_{N,n} = \left.\frac{\partial\ f_N\,(z_{N-1}, z_{N-2}, ..., z_1)}{\partial\ z_n}\right|_{z_{N-1}=\bar{z}_{N-1},\ ...,\ z_1=\bar{z}_1}\ . \tag{7.23}$$

These formulas refer of course to a specific equilibrium configuration; in particular in (7.21) and (7.23) the quantities $\bar{z}_n$ are given by (7.16) or (7.20). To emphasize this dependence we also use below, whenever appropriate, in place of the notation $f_{N,n}$ the alternative notations $f_{N,n}(y)$ corresponding to (7.23) with (7.16), or $f_{N,n}^{(k)}$ corresponding to (7.23) with (7.20) (but, for notational simplicity, we do not explicitly highlight the corresponding dependence on y or k of the functions $w_n(t)$). We also set, for notational convenience (see below),

$$f_{N,n}(y) = (i)^n \ g_{N,n}(y) \ , \quad f_{N,n}^{(k)} = (i)^n \ g_{N,n}^{(k)} \ . \tag{7.24}$$

So, for instance, for $M = 1, \ N = 3$, from (7.9a) one easily gets

$$g_{3,1}(y) = 12 \ y \ , \quad g_{3,2}(y) = 6 \ y \ , \tag{7.25a}$$

and

$$g_{3,1}^{(1)} = 24 \ , \quad g_{3,2}^{(1)} = 12 \ . \tag{7.25b}$$

The N basic solutions of the *linear* system of ODEs (7.22) read of course

$$\underline{w}^{(m)}(t) = \underline{\bar{w}}^{(m)} \ \exp(-i \ x_m \ t) \ , \quad m = 1, ..., N \ , \tag{7.26}$$

with $\underline{w}^{(m)}(t)$ indicating the N-vector of components $w_n^{(m)}(t)$ while the N numbers x_m, respectively the N constant N-vectors $\underline{\bar{w}}^{(m)}$, are the N eigenvalues, respectively the N corresponding eigenvectors, of the $N \times N$ matrix $\underline{A}$ defined componentwise as follows:

$$A_{n,n} = -(n+1) \ , \quad A_{n,n+1} = i \ , \quad n = 1, ..., N-1 \ ; \tag{7.27a}$$

$$A_{N,n} = (i)^{n+1} \ g_{N,n} \ , \quad n = 1, ..., N-1 \ ; \ A_{N,N} = -(N+1) \ , \tag{7.27b}$$

with all other matrix elements vanishing. Note that, for notational simplicity, we omit here (and, likewise, sometimes below) to indicate explicitly the dependence on the specific equilibrium configuration under consideration, namely on y (see (7.16)) or k (see (7.20)). Equivalently, the N numbers x_n are the N roots of the following N-th degree *monic* polynomial in x:

$$P_N(x) = \det\left[x \ \underline{I} - \underline{\tilde{A}}\right] \ , \tag{7.28}$$

where of course $\underline{I}$ is the $N \times N$ identity matrix and we replaced the matrix $\underline{A}$ with the matrix $\underline{\tilde{A}}$ defined componentwise as follows:

$$\tilde{A}_{n,n} = -(n+1) \ , \quad \tilde{A}_{n,n+1} = -1 \ , \quad n = 1, ..., N-1 \ ; \tag{7.29a}$$

$$\tilde{A}_{N,n} = (-1)^{n-M} \ g_{N,n} \ , \quad n = 1, ..., N-1 \ ; \ \tilde{A}_{N,N} = -(N+1) \ . \tag{7.29b}$$

This matrix $\underline{\tilde{A}}$ is obtained from the matrix $\underline{A}$ by multiplying its nm-th matrix element by i^{m-n}, an operation that does not affect the determinant in the right-hand side of (7.28) as it amounts to multiplying the matrix $\underline{A}$ from the right by $\underline{J}$ =diag(i^n) and from the left by $\underline{J}^{-1}$.

Note that, by expanding the determinant in the right-hand side of (7.28) along its last column one obtains the following expressions of the polynomial $P_N(x) = P_N(x;y) = P_N^{(k)}(x)$:

$$P_N(x;y) \equiv P_{2M+1}(x;y) = \prod_{\ell=2}^{N+1}(x+\ell) + (-1)^M \sum_{j=1}^{N-1} g_{N,j}(y) \prod_{\ell=2}^{j}(x+\ell) \ , \tag{7.30}$$

$$P_N^{(k)}(x) \equiv P_{2M+1}^{(k)}(x) = \prod_{\ell=2}^{N+1}(x+\ell) + (-1)^M \sum_{j=1}^{N-1} g_{N,j}^{(k)} \prod_{\ell=2}^{j}(x+\ell) \ . \tag{7.31}$$

We trust the notation used here to be self-explanatory; clearly these two formulas are equivalent via (7.19a). And note that, here and hereafter, we use the usual convention according to which a sum vanishes if its upper limit is smaller than its lower limit,

$$\sum_{\ell=m}^{n} \varphi_\ell = 0 \ , \quad \text{if} \quad n < m \ , \tag{7.32}$$

while a product takes *unit* value if its lower limit exceeds by *one unit* its upper limit,

$$\prod_{\ell=n+1}^{n} \varphi_\ell = 1 \ , \tag{7.33a}$$

and gets redefined as follows if its lower limit exceeds by *more than one unit* its upper limit,

$$\prod_{\ell=m}^{n} \varphi_\ell = \prod_{\ell=n+1}^{m-1} \frac{1}{\varphi_\ell} \quad \text{if} \quad m > n+1 \ . \tag{7.33b}$$

In these formulas and throughout n and m are of course *integers*. Note that the convention (7.33a) entails that the coefficient of $g_{N,1}(y)$ in the right-hand side of (7.30), and likewise of $g_{N,1}^{(k)}$ in the right-hand side of (7.31), is just *unity*. The convention (7.33b) will play a role below.

Let us turn to the derivation of the expression of the coefficients $g_{N,n}^{(k)}$. To this end we set, in (7.31), $x = -p-1$ with $p = 1, ..., N-1$, getting thereby (recalling the convention (7.33)) the following recursive system for these coefficients:

$$g_{N,p}^{(k)} = \frac{(-1)^{p-1}}{(p-1)!} \left\{ \sum_{j=1}^{p-1} \left[(-1)^j \frac{(p-1)!}{(p-j)!} \, g_{N,j}^{(k)} \right] + (-1)^M P_N^{(k)}(-p-1) \right\} \ ,$$

$$p \ = \ 1, 2, ..., N-1 \ . \tag{7.34}$$

It is then easy to verify that the solution of this recursion reads as follows:

$$g_{N,n}^{(k)}=(-1)^{M}\sum_{m=0}^{n-1}\left[\frac{(-1)^{m}\ P_{N}^{(k)}(-2-m)}{m!\ (n-1-m)!}\right]\ ,\quad n=1,...,N-1\ . \tag{7.35a}$$

In the following section we obtain a more explicit representation of the polynomials $P_N(x;y)$ respectively $P_N^{(k)}(x)$, reading

$$P_N(x;y)=(x+2)\prod_{\ell=0}^{M-1}\left[\frac{(x+3+2\ell)\ \ [(x+4+2\ell)(x+2+2\ell)-4y]}{x+2+2\ell}\right]$$
$$-y\sum_{m=0}^{M-1}\left\{\frac{2^{m+1}\ (x+6+4m)}{(m+1)}\prod_{\ell=0}^{m-1}\left[\frac{(3+2\ell)\,[(\ell+1)(\ell+2)-y]}{\ell+1}\right]\cdot\right.$$
$$\left.\cdot\prod_{\ell=0}^{M-m-2}\left[\frac{(x+5+2m+2\ell)\,[(x+6+2m+2\ell)(x+4+2m+2\ell)-4y]}{(x+4+2m+2\ell)}\right]\right\}\ , \tag{7.36}$$

respectively (via (7.19a)),

$$P_N^{(k)}(x)=(x+2)\prod_{\ell=0}^{M-1}\left[\frac{(x+3+2\ell)\ \ (x+4+2\ell+2k)\ \ (x+2+2\ell-2k)}{x+2+2\ell}\right]$$
$$+\sum_{m=0}^{k-1}\left\{(-2)^{m+1}\ \ (x+6+4m)\,\frac{(2m+1)!!\ \ (k+2)_m\ \ (1-k)_m}{(m+1)!}\cdot\right.$$
$$\left.\cdot\prod_{\ell=0}^{M-m-2}\left[\frac{(x+5+2m+2\ell)\ \ (x+6+2m+2\ell+2k)\ \ (x+4+2m+2\ell-2k)}{(x+4+2m+2\ell)}\right]\right\}\ ,$$
$$k\ =\ 0,1,...,M\ . \tag{7.37}$$

Here and hereafter the symbols $(m+1)!\equiv\prod_{\ell=0}^{m}(\ell+1)$ and $(2m+1)!!\equiv\prod_{\ell=0}^{m}(2\ell+1)$ denote the standard factorial and double factorial, and the Pochhammer symbol has the standard definition (see for instance page 56 of [23])

$$(z)_0=1\ ,\quad (z)_m=z\ (z+1)\cdots(z+m-1)=\frac{\Gamma(z+m)}{\Gamma(z)}\ . \tag{7.38}$$

And, via (7.33b), these formulas can be rewritten as follows:

$$P_N(x;y)\ =\ \sum_{m=0}^{M}\frac{2^m\,(x+4m+2)}{m!}\prod_{\ell=0}^{m-1}\{(2\ell+1)\,[\ell(\ell+1)-y]\}$$
$$\prod_{\ell=m+1}^{M}\frac{(x+2\ell+1)\,[(x+2\ell+2)(x+2\ell)-4y]}{(x+2\ell)}\ , \tag{7.39}$$

respectively

$$P_N^{(k)}(x) = \sum_{m=0}^{k} \left\{ 2^m (x+4m+2) \frac{(2m-1)!! \ (-k)_m \ (k+1)_m}{m!} \prod_{\ell=m+1}^{M} \frac{(x+2\ell+1)(x+2\ell-2k)(x+2\ell+2k+2)}{(x+2\ell)} \right\} \quad (7.40)$$

The fact that $P_N(x;y)$ is a polynomial in y of degree M is plain from its definition (7.36); the fact that $P_N^{(k)}(x)$ is a *monic* polynomial of degree N in x is not immediately apparent from the formula (7.37) but is implied by the following developments.

The above treatment entails the *Diophantine* observation that the N zeros $x_n^{(k)}$ of the polynomial $P_N^{(k)}(x)$, see (7.37), must *all* be *integers*, and *all* different among themselves, for *all* the $M+1=(N+1)/2$ values of the integer $k=0,1,...,M$. Indeed the fact that the N zeros $x_n^{(k)}$ of the monic polynomial $P_N^{(k)}(x)$ are *all integers* is clearly implied by the observation that *all* the solutions (7.26) must satisfy the *isochrony* property (7.13). And let us note that this is trivially true for $k=0$ entailing, see (7.37) and (7.32),

$$P_N^{(0)}(x) = \prod_{m=2}^{N+1} (x+m) \equiv (x+2)_N \ , \quad (7.41)$$

hence for $k=0$ the N zeros are given by the simple rule $x_n^{(0)} = -(n+1)$ with $n=1,2,...,N$.

The main result of this chapter consists in the identification—for all values of k in the range $k=0,1,...,M$—of the (*integer*!) values of these $N = 2M+1$ zeros $x_n^{(k)}$, as displayed by the following factorization of the polynomial $P_N^{(k)}(x)$:

$$P_N^{(k)}(x) \equiv P_{2M+1}^{(k)}(x) = \left[\prod_{\ell=0}^{k-1}(x-1-2\ell)\right]\left[\prod_{\ell=k+1}^{M}(x+1+2\ell)\right] \cdot \left[\prod_{\ell=1}^{M-k}(x+2\ell)\right]\left[\prod_{\ell=M+1}^{M+1+k}(x+2\ell)\right] \ , \quad k=0,1,...,M \ , \quad (7.42a)$$

or, equivalently,

$$P_N^{(k)}(x) \equiv P_{2M+1}^{(k)}(x) = (x+2+2M+2k) \cdot$$
$$\cdot \left[\prod_{j=1}^{k} \left[(x+2M+2j)(x-2j+1) \right] \right] \cdot$$
$$\cdot \left[\prod_{j=1}^{M-k} \left[(x+2j)(x+1+2k+2j) \right] \right] \;, \quad k=0,1,...,M \;. \quad (7.42b)$$

Note that these formulas also hold for $k=0$ and for $k=M$ via the convention (7.33a). The first version, (7.42a), shows clearly that M of the $N=2M+1$ zeros x_n are *odd integer* numbers (k of which are *positive integers* and $M-k$ are *negative integers*), and the remaining $M+1$ are *even integers* (all of them *negative*).

This formula is proven in the following section.

7.2 Proofs, and some interesting formulas

In this section **(and Appendix B)** we derive the main result reported in the previous section. We also report and prove some additional results.

Our first step is to obtain the modified version of the ODE (7.1) that only features *isochronous* solutions. To this end we set, consistently with our previous treatment,

$$z(t) = \exp(2it)\ \zeta(\tau) \;, \qquad \tau = i\left[1-\exp(it)\right] \;, \quad \dot{\tau}(t) = \exp(it) \;, \quad (7.43a)$$

$$\dot{z}(t) = 2iz(t) + \exp(3it)\ \zeta'(\tau) \;, \quad \zeta'(\tau) = \left[\dot{z}(t) - 2iz(t)\right]\ \exp(-3it) \;, \quad (7.43b)$$

so that

$$\frac{d}{d\tau} = \exp(-it)\ \frac{d}{dt} \;, \quad \frac{d^2}{d\tau^2} = \exp(-2it)\ \left(\frac{d^2}{dt^2} - i\frac{d}{dt}\right) \;. \quad (7.44)$$

The integro-differential operator (7.3) gets thereby reformulated to act as follows on a generic function $\varphi(t)$ of the independent variable t:

$$\pounds\left[z\right] \cdot \varphi(t) = \exp(-2it) \cdot$$
$$\cdot \left\{ \ddot{\varphi}(t) - i\dot{\varphi}(t) - 4z(t)\varphi(t) - 2\left[\dot{z}(t) - 2iz(t)\right]\exp(-it)\ \int^t dt'\ \exp(it')\ \varphi(t') \right\} (7.45)$$

where of course superimposed dots denote differentiations with respect to the independent variable t, and in the integration no contribution must be inserted at the lower end of the integration range.

Next, we focus attention on the immediate neighborhood of an equilibrium configuration, via the assignment (again, consistent with our previous treatment: see (7.16) and recall that $\bar{z} = \bar{z}_1$, see (7.43a) and (7.5)):

$$z(t) = -y + \varepsilon \ \exp(-ixt) \ , \quad \dot{z}(t) = -i \ x \ \varepsilon \ \exp(-ixt) \ , \tag{7.46}$$

entailing (see (7.43))

$$\zeta(\tau) = \exp(2it) \ \ [-y + \varepsilon \ \exp(-ixt)] \ , \tag{7.47a}$$

$$\zeta'(\tau) = i\,[2y - \varepsilon\,(x+2)] \exp(-3it) \ , \tag{7.47b}$$

as well as

$$\pounds\,[-y + \varepsilon \ \exp(-ixt)] = \pounds_0 + \varepsilon \ \pounds_1 \ , \tag{7.48a}$$

with the following definitions:

$$\pounds_0 = \exp(-2it) \ \left\{ \frac{d^2}{dt^2} - i\frac{d}{dt} + 4y\left[1 - i \ \exp(-it) \ \int^t dt' \ \exp(it')\right]\right\} \ , \tag{7.48b}$$

$$\pounds_1 = -2\exp[-(x+2)\,it]\left[2 - i\,(x+2)\exp(-it) \ \int^t dt' \ \exp(it')\right] \ . \tag{7.48c}$$

Hence

$$\{\pounds\,[-y + \varepsilon \ \exp(-ixt)]\}^M = \pounds_0^M + \varepsilon \ \sum_{\ell=0}^{M-1} \pounds_0^{M-1-\ell} \ \pounds_1 \ \pounds_0^{\ell} + O\left(\varepsilon^2\right) \ , \tag{7.49}$$

so that our basic ODE (7.1) now reads

$$\begin{aligned} \{Ł\,[\zeta]\}^M \cdot \zeta' &= i\,\{\pounds\,[-y + \varepsilon \ \exp(-ixt)]\}^M \cdot \{2y\exp(-3it) - \varepsilon \ (x+2)\exp[-i\,(x+3)\,t]\} \\ &= 2iy\pounds_0^M \exp(-3it) \\ &\quad +\varepsilon \ \left\{(x+2)\,\pounds_0^M \exp[-i\,(x+3)\,t] + 2y\sum_{\ell=0}^{M-1} \pounds_0^{M-1-\ell} \ \pounds_1 \ \pounds_0^{\ell}\exp(-3it)\right\} \\ &\quad +O\left(\varepsilon^2\right) = 0 \ . \end{aligned} \tag{7.50}$$

We now note that the definition (7.48b) of $\pounds_0$ implies

$$\pounds_0 \cdot \exp(-i\mu t) = \exp[-i\,(\mu+2)\,t] \ \frac{\mu \ [4y - (\mu+1)\,(\mu-1)]}{\mu - 1} \ , \tag{7.51a}$$

where (here and below) μ is an *arbitrary* constant, hence (with m an *arbitrary* nonnegative integer)

$$\pounds_0^m \cdot \exp(-i\mu t) = \exp[-i\,(\mu+2m)\,t] \cdot \\ \cdot \prod_{\ell=0}^{m-1}\left\{\frac{(\mu+2\ell) \ \ [4y_k - (\mu+2\ell+1)\,(\mu+2\ell-1)]}{\mu+2\ell-1}\right\} \ . \tag{7.51b}$$

Likewise the definition (7.48c) of $\pounds_1$ implies

$$\pounds_1 \cdot \exp(-i\mu t) = -2\exp[-i(\mu+x+2)t]\,\frac{2\mu+x}{\mu-1}\ . \tag{7.52}$$

Hence, to order $\varepsilon^0 = 1$, our basic ODE, (7.50), reading

$$\pounds_0^M \cdot \exp(-3it) = 0\ , \tag{7.53a}$$

entails, via (7.51b),

$$\prod_{m=0}^{M} [y_k - m(m+1)] = 0\ . \tag{7.53b}$$

The expression (7.19a) of y_k is thereby proven.

Next, to order ε, (7.50) yields the equation:

$$(x+2)\,\pounds_0^M \cdot \exp[-i(x+3)t] + 2y \sum_{\ell=0}^{M-1} \pounds_0^{M-1-\ell}\ \pounds_1\ \pounds_0^{\ell} \exp(-3it) = 0\ , \tag{7.54}$$

entailing (via (7.51), (7.52) and (7.19a))

$$P_N(x;y) = 0\ , \tag{7.55a}$$

$$P_N^{(k)}(x) = 0\ , \qquad k = 0, 1, ..., M\ , \tag{7.55b}$$

with $P_N(x;y)$ respectively $P_N^{(k)}(x)$ defined by (7.36) respectively (7.37). The derivation of these expressions is thereby achieved.

Next, let us prove that the polynomial $P_N^{(k)}(x)$ admits the factorization (7.42). We take as starting point its formulation (7.40). It is then easy to rewrite it as follows, by noting that in fact the upper limit k of the sum over the index m may be replaced by ∞ since all terms with $m > k$ vanish:

$$P_N^{(k)}(x) = 4^M\ (x+2)\ \frac{\left(\frac{x}{2}+\frac{3}{2}\right)_M\ \left(\frac{x}{2}+1-k\right)_M\ \left(\frac{x}{2}+2+k\right)_M}{\left(\frac{x}{2}+1\right)_M} \cdot$$
$$\cdot \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}\right)_m\ (-k)_m\ (k+1)_m\ \left(\frac{x}{2}+1\right)_m\ \left(\frac{x}{4}+\frac{3}{2}\right)_m}{m!\ \left(\frac{x}{4}+\frac{1}{2}\right)_m\ \left(\frac{x}{2}+\frac{3}{2}\right)_m\ \left(\frac{x}{2}+1-k\right)_m\ \left(\frac{x}{2}+2+k\right)_m}\ . \tag{7.56}$$

Here and below we use of course the Pochhammer notation, see (7.38).

Hence the polynomial $P_N^{(k)}(x)$ can be rewritten as follows in terms of the ${}_5F_4$ generalized hypergeometric series of *unit* argument (for the definition of the generalized hypergeometric series see page 182 of [23]):

$$P_N^{(k)}(x) = 4^M\ (x+2)\ \frac{\left(\frac{x}{2}+\frac{3}{2}\right)_M\ \left(\frac{x}{2}+1-k\right)_M\ \left(\frac{x}{2}+2+k\right)_M}{\left(\frac{x}{2}+1\right)_M}\ \cdot$$

$$\cdot\ {}_5F_4\left[\begin{array}{ccccc} \frac{x}{2}+1\ , & \frac{x}{4}+\frac{3}{2}\ , & \frac{1}{2}\ , & k+1\ , & -k\ ;\ 1 \\ \frac{x}{4}+\frac{1}{2}\ , & \frac{x}{2}+\frac{3}{2}\ , & \frac{x}{2}+1-k\ , & \frac{x}{2}+2+k\ , & \end{array}\right]\ . \tag{7.57}$$

And it is then immediately seen, via the formula 4.5(6) on page 191 of [23],

$${}_5F_4\left[\begin{array}{ccccc} a\ , & 1+\frac{a}{2}\ , & c\ , & d\ , & e\ ;\ 1 \\ \frac{a}{2}\ , & 1+a-c\ , & 1+a-d\ , & 1+a-e\ , & \end{array}\right]$$
$$= \frac{\Gamma(1+a-c)\,\Gamma(1+a-d)\,\Gamma(1+a-e)\,\Gamma(1+a-c-d-e)}{\Gamma(1+a)\,\Gamma(1+a-d-e)\,\Gamma(1+a-c-e)\,\Gamma(1+a-c-d)}, \tag{7.58}$$

and a few standard manipulations, that the factorization (7.42) is proven.

Let us also note that the formulas (7.36) respectively (7.37) can be conveniently re-written as follows:

$$P_N(x;y) = P_{2M+1}(x;y)$$
$$= (x+2)\ \prod_{\ell=0}^{M-1} u(x;y;\ell) + \sum_{m=0}^{M-1} U(x;y;m)\ \prod_{\ell=0}^{M-m-2} u(x;y;m+\ell+1)\ , \tag{7.59a}$$

with

$$u(x;y;n) = \frac{(x+3+2n)\ \ [(x+4+2n)(x+2+2n)-4y]}{x+2+2n}, \tag{7.59b}$$

$$U(x;y;m) = (x+6+4m)\ \ V(y;m)\ \ , \tag{7.59c}$$

$$V(y;m) = -\frac{2^{m+1}\ y}{(m+1)}\ \prod_{\ell=0}^{m-1}\left[\frac{(3+2\ell)\ \ [(\ell+1)(\ell+2)-y]}{\ell+1}\right]\ , \tag{7.59d}$$

respectively

$$\begin{aligned} P_N^{(k)}(x) &= P_{2M+1}^{(k)}(x) \\ &= (x+2)\ \prod_{\ell=0}^{M-1} u^{(k)}(x;\ell) + \sum_{m=0}^{k-1} U^{(k)}(x;m)\ \prod_{\ell=0}^{M-m-2} u^{(k)}(x;m+\ell+1)\ , \\ k &= 0,1,...,M\ , \end{aligned} \tag{7.60a}$$

with

$$u^{(k)}(x;n) = \frac{(x+3+2n)\ \ (x+4+2n+2k)\ \ (x+2+2n-2k)}{x+2+2n}\ , \tag{7.60b}$$

$$U^{(k)}(x;m) = (x+6+4m)\ V_m^{(k)}\ , \tag{7.60c}$$

$$V_m^{(k)} = k(k+1)(-2)^{m+1} \frac{(2m+1)!!\ (k+2)_m\ (1-k)_m}{(m+1)!} \quad \text{for} \quad m<k\ ,$$
$$V_m^{(k)} = 0 \quad \text{for} \quad m \geq k\ . \tag{7.60d}$$

It is then easily seen that the formula (7.59) entails the recursion

$$P_{N+2}(x;y) = u(x;y;M)\ P_N(x;y) + U(x;y;M)\ , \tag{7.61a}$$

and likewise the formula (7.60) entails the recursion

$$P_{2M+3}^{(k)}(x) = u^{(k)}(x;M)\ P_{2M+1}^{(k)}(x) + U^{(k)}(x;M)\ , \tag{7.61b}$$

while there clearly holds the relation

$$U^{(k)}(x;m) = 0 \quad \text{for} \quad m \geq k\ . \tag{7.62}$$

Note that this property, (7.62), entails that, for the standard set of values $k = 0, 1, ..., M$, the relation (7.61b) becomes the two-term recursion relation

$$P_{2M+3}^{(k)}(x) = u^{(k)}(x;M)\ P_{2M+1}^{(k)}(x)\ ; \tag{7.63}$$

and in fact, although the quantity $U^{(k)}(x;m)$ does not vanish for $k > M$, remarkably this two-term recursion relation also holds (via the convention (7.33)) for $k = M+1, M+2, ...$.

7.3 Appendix 7

In this appendix—throughout which we assume M and N to be *positive integers*, see (7.2)—we detail the transition from the formulations (7.36) and (7.37) of the polynomials $P_N(x;y)$ and $P_N^{(k)}(x)$ to their versions (7.39) and (7.40). Hence our starting point is the formula (7.36),

$$\begin{aligned} P_N(x;y) &= (x+2)\prod_{\ell=0}^{M-1} \frac{(x+3+2\ell)\left[(x+4+2\ell)(x+2+2\ell)-4y\right]}{(x+2+2\ell)} \\ &\quad -y\sum_{m=0}^{M-1} \frac{2^{m+1}(x+6+4m)}{m+1} \prod_{\ell=0}^{m-1} \frac{(2\ell+3)\left[(\ell+1)(\ell+2)-y\right]}{(\ell+1)} \\ &\quad \prod_{\ell=0}^{M-m-2} \frac{(x+2\ell+2m+5)\left[(x+2\ell+2m+6)(x+2\ell+2m+4)-4y\right]}{(x+2\ell+2m+4)}\ . \end{aligned} \tag{7.64a}$$

We then replace, in the first two products in the right-hand side of this formula, the index ℓ with the index j by setting $\ell = j - 1$, and in the third product by setting $\ell = j - m - 2$, getting thereby

$$P_N(x;y) = (x+2)\prod_{j=1}^{M}\frac{(x+2j+1)\left[(x+2j+2)(x+2j)-4y\right]}{(x+2j)}$$
$$-y\sum_{m=0}^{M-1}\frac{2^{m+1}(x+6+4m)}{m+1}\prod_{j=1}^{m}\frac{(2j+1)\left[j(j+1)-y\right]}{j}$$
$$\prod_{j=m+2}^{M}\frac{(x+2j+1)\left[(x+2j+2)(x+2j)-4y\right]}{(x+2j)} . \quad (7.64b)$$

Next, we replace the index m by setting $m = r - 1$, getting thereby

$$P_N(x;y) = (x+2)\prod_{j=1}^{M}\frac{(x+2j+1)\left[(x+2j+2)(x+2j)-4y\right]}{(x+2j)}$$
$$-y\sum_{r=1}^{M}\frac{2^{r}(x+4r+2)}{r!}\prod_{j=1}^{r-1}\left\{(2j+1)\left[j(j+1)-y\right]\right\}$$
$$\prod_{j=r+1}^{M}\frac{(x+2j+1)\left[(x+2j+2)(x+2j)-4y\right]}{(x+2j)} , \quad (7.64c)$$

hence, via (7.33b) entailing

$$\prod_{j=1}^{-1}\left\{(2j+1)\left[j(j+1)-y\right]\right\} = \prod_{j=0}^{0}\left\{(2j+1)\left[j(j+1)-y\right]\right\}^{-1} = -1/y , \quad (7.65)$$

we obtain precisely the expression (7.39)—where we also replaced, for notational convenience, the index r with m and the index j with ℓ.

Via (7.19a) (with k a *nonnegative integer*) and via the two identities

$$\ell(\ell+1) - k(k+1) = -(k-\ell)(\ell+k+1) , \quad (7.66a)$$

$$(x+2\ell+2)(x+2\ell) - 4k(k+1) = (2\ell+2k+x+2)(2\ell-2k+x) , \quad (7.66b)$$

this formula becomes (7.40), where we replaced the upper limit of the sum with k rather than M because clearly all terms with $m > k$ vanish due to the vanishing of the first product appearing in the right-hand side inside the sum over the index m.

Concluding remarks

During these last three years I have studied several isochronous dynamical systems obtained by modifying integrable dynamical systems via the "Trick". We have seen that the isochrony of the system provide Diophantine relations for the linearized system in the neighborhood of the equilibrium positions. Indeed, we obtain $N \times N$ matrices (where N can be understood as the number of particles of the system) with integer eigenvalues. In the proofs of these conjectures we have noticed a connection with known classes of orthogonal polynomials. In fact, the determinantal equation satisfied from these matrices is, by construction, a three term recurrence relation $p_{n+1}^{(\nu)}(x) = \left(x + a_n^{(\nu)}\right) p_n^{(\nu)}(x) + b_n^{(\nu)} p_{n-1}^{(\nu)}(x)$. Moreover, these polynomials satisfy a second recursion relation $p_n^{(\nu)}(x) = p_n^{(\nu-1)}(x) + g_n^{(\nu)} p_{n-1}^{(\nu-1)}(x)$, that involve a shift of a new parameter ν.

In addition, have seen a connection between the equilibrium positions of the isochronous system and the recurrence coefficients $a_n^{(\nu)}$, $b_n^{(\nu)}$ that define the family of orthogonal polynomials. One of the remarkable properties implied by this connection is that all orthogonal polynomials obtained from all integrable systems of the Toda-type studied in the literature are included in Dual Hahn family. A natural extension of this property could be the study of systems whose equilibrium positions provide orthogonal polynomials included in other classes of "named" orthogonal polynomials . In collaboration with O. Ragnisco and F. Calogero I have obtained interesting results in the study of the isochronous version of the Relativistic Toda lattice (see [22]). Indeed, the polynomials obtained are completely factorized, for all values of their parameters.

The main results of this thesis are about the factorization formulae for the family of orthogonal polynomials obtained. As we have seen, all the zeros of these polynomials are integers. These factorizations, straight deriving by the isochrony property of the system, have been generalized and applied to the complete class of orthogonal polynomials included in the Askey-scheme, obtaining in some case, new, factorizations which zeros are all integers. A number of factorizations are new and not present in the literature. We hope that such formulas will be inserted in future handbooks.

We can see in the work of Y. Chen and M.H. Ismail [21], that such procedure can be extended to the q-polynomials, for example to the Askey-Wilson polynomials. However in this case the zeros obtained are not integers. In chapter 6 we have connected our machinery to the theory of the discrete integrability, and construct a whole hierarchy of integrable discrete equations. Hence, by this iterative method we are able to generate *new* further recurrence relations for these orthogonal polynomials. In Appendix 6 we have presented the connection between the compatibility of the two recurrence relations and the discrete time equations of motion for the Toda lattice. It is very easy to see that the second equation of the hierarchy constructed, in the case $a_n^{(\nu)} = 0$, is related to the discrete time Volterra lattice.

Moreover, the compatibility between the two recurrence relation can be rewritten removing one of the fields in the recurrence relation. We have rewritten it using the quantity $A_n^{(\nu)}$ and $\omega^{(\nu)}$. This property for the compatibility is already known from the theory of the discrete-time Toda lattice. We can easily note that also for the second equation of the hierarchy we can remove the dependence on one of the fields.

One future interesting development could be to prove the validity of this property for all equations of the hierarchy.

Another important field of research arises from chapter 7. There we have studied the isochronous version of the stationary KdV's hierarchy obtaining *new* Diophantine relations and further already known results. An interesting perspective is the investigation of other hierarchies with the same machinery, for example the Boussinesq's hierarchy,the AKNS's hierarchy etc. The study of the stationary Burger's hierarchy is now in progress with the collaboration of F. Calogero and M.Bruschi.

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