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Asymptotics for 3D Venttsel' problems in fractal domains

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INTRODUCTION

The aim of this dissertation is to study a boundary value problem for a second order operator in divergence form with Venttsel's boundary conditions, which we can state formally :

$$(P)\begin{cases} u_t(t,P)-Lu(t,P)=f(t,P) & \text{in } [0,T]\times Q^{(\xi)}\\ u_t(t,P)-\Delta_S u(t,P)+b(P)u(t,P)=-\frac{\partial u}{\partial n_{\mathcal{A}}}+f(t,P) & \text{on } [0,T]\times S^{(\xi)}\\ u(0,P)=0 & \text{in } Q^{(\xi)}.\end{cases}$$

Here L is an operator in divergence form, $Lu=\mathrm{div}(\mathcal{A}Du)$, $[\mathcal{A}]_{ij}=a_{ij}(P)$, $i,j=1,2,3$; a_{ij} are symmetric, uniformly bounded functions in $Q^{(\xi)}$ satisfying suitable ellipticity conditions (see condition (H) in Section 3.2.2), $Q^{(\xi)}$ is the three-dimensional domain with lateral boundary $S^{(\xi)}=F^{(\xi)}\times[0,1]$, where $F^{(\xi)}$ is the Koch mixture snowflake; Δ_S is the fractal Laplacian on $S^{(\xi)}$ (see Theorem 3.2.6 in Section 3.2.2), b is a continuous strictly positive function on $\overline{Q}^{(\xi)}$, $\frac{\partial u}{\partial n_{\mathcal{A}}}$ is the co-normal derivative across $S^{(\xi)}$ to be defined in a suitable sense (see Theorem 4.3.1), $f(t,P)$ is a given function in $C^\theta([0,T];L^2(\overline{Q}^{(\xi)},m))$, $\theta\in(0,1)$ and m is the sum of the three-dimensional Lebesgue measure and of a suitable measure g supported on $S^{(\xi)}$ (see Section 3.2.2).

From the point of view of numerical analysis it is also crucial to study the corresponding approximating (prefractal) problems (P_h). To this aim the asymptotic behavior, as $h\to\infty$, of the approximating solutions is studied. More precisely, we consider for each $h\in\mathbb{N}$, the prefractal problems, here formally stated:

$$(P_h)\begin{cases} (u_h)_t(t,P)-L_h u_h(t,P)=f_h(t,P) & \text{in } [0,T]\times Q_h^{(\xi)}\\ \delta_h(u_h)_t(t,P)-\Delta_{S_h}u_h(t,P)+\delta_h b(P)u_h(t,P)=-\frac{\partial u}{\partial n_{\mathcal{A}_h}}+\delta_h f_h(t,P) & \text{on } [0,T]\times S_h^{(\xi)}\\ u_h(0,P)=0 & \text{in } Q_h^{(\xi)}.\end{cases}$$

We denote by $L_h u=\mathrm{div}(\mathcal{A}^h Du)$, $[\mathcal{A}^h]_{ij}=a^h_{ij}(P)$, $i,j=1,2,3$; a^h_{ij} are uniformly bounded functions in $Q^{(\xi)}$, satisfying suitable ellipticity conditions (see condition (H_h) in Section 3.2.1), $Q_h^{(\xi)}$ are a sequence of increasing (invading) domains approximating $Q^{(\xi)}$, $S_h^{(\xi)}=F_h^{(\xi)}\times[0,1]$ are the corresponding approximating polyhedral surfaces, where $F_h^{(\xi)}$ is a prefractal curve approximating F (see Section 1.4); Δ_{S_h} is the piecewise tangential Laplacian defined on S_h, $\frac{\partial u}{\partial n_{\mathcal{A}_h}}$ is the co-normal derivative across S_h to be defined in a suitable sense (see Theorem 4.3.2), $f_h(t, P)$ is a given function in $C^\theta([0, T]; L^2(Q, m_h))$, $\theta \in (0, 1)$; m_h is the sum of the three-dimensional Lebesgue measure and of the surface measure $\delta_h \sigma$ of S_h, where δ_h is a positive constant (see Section 3.2.1).

Venttsel' conditions are the most feasible boundary conditions for an elliptic or parabolic problem, they include Dirichlet, Neumann and general oblique boundary conditions as special cases.
They appeared for the first time in ([60]) in the framework of probability theory. From the point of view of applications they occur in different contexts such as three-dimensional water wave theory, models of heat transfer and hydraulic fracturing (see [28], [57], [8]).

In the framework of heat transfer, Venttsel' boundary conditions appear when considering the asymptotic behavior of heat flow problems for highly conductive coated structures, see [13] for details. The interest in studying the heat flow across irregular domains with fractal boundaries arises from the fact that a lot of industrial and natural processes lead to the formation of rough surfaces or take place across them.
For example the current flow across rough electrodes in chemistry (see [56]) and the diffusion processes in physiological membranes are transport phenomena taking place across irregular layers/boundaries.
The literature on Venttsel' problems in regular domains is huge, we refer to [14] and the references listed in, as to Venttsel problems in fractal domains the first results, to our knowledge, can be found in [38] where the two-dimensional case is considered.
In Venttsel' problems, the fractal set has both a static and a dynamical role, that is on one side it is the boundary of an Euclidean domain and on the other side it supports the notion of a Laplacian, (as e.g. in transmission problems [32]-[38]), from the point of view of PDEs this fact has a counterpart, since the associated energy functional is the sum of the bulk energy and of the boundary (fractal) energy.
We define the form $E[\cdot]$

$$E[u] = \int_{Q^{(\xi)}} \mathcal{A} Du \cdot Du d\mathcal{L}_3 + E_{S^{(\xi)}}[u|_{S^{(\xi)}}] + \int_{S^{(\xi)}} b |u|_{S^{(\xi)}}|^2 dg,$$

defined on the space

$$V(Q^{(\xi)}, S^{(\xi)}) = \{u \in H^1(Q^{(\xi)}), u|_{S^{(\xi)}} \in \mathcal{D}(S^{(\xi)})\},$$

where $d\mathcal{L}_3$ is the three-dimensional Lebesgue measure, $[\mathcal{A}]_{ij} = a_{ij}(P)$, $i, j = 1, 2, 3$, $E_{S^{(\xi)}}$ is the energy defined on the fractal boundary $S^{(\xi)}$ with domain $\mathcal{D}(S^{(\xi)})$ (see Section 3.2.2 for its definitions and properties), b is a continuous and strictly positive function defined on

$\overline{Q}^{(\xi)}$, g is the Hausdorff measure supported on $S^{(\xi)}$ (see Section 1.4) and $u|_{S^{(\xi)}}$ is the trace to $S^{(\xi)}$ to be properly defined (see Section 2.2).
We also define the form $E^{(h)}[\cdot]$

$$E^{(h)}[u] = \int\limits_{Q^{(\xi)}} \chi_{Q_h^{(\xi)}} \mathcal{A}^h Du\, Du d\mathcal{L}_3 + E_{S_h^{(\xi)}}[u|_{S_h^{(\xi)}}] + \delta_h \int\limits_{S_h^{(\xi)}} b|u|_{S_h^{(\xi)}}|^2 d\sigma$$

defined on the space

$$V(Q^{(\xi)}, S_h^{(\xi)}) = \{u \in H^1(Q^{(\xi)}), u|_{S_h^{(\xi)}} \in H^1(S_h^{(\xi)})\},$$

where δ_h is a positive constant, dσ the surface measure on $S_h^{(\xi)}$.

For classical fractal curves such as the Sierpiński gasket, the Koch curve, the snowflake and so on, which are nice self similar sets, energy forms can be obtained as limits of suitable approximating energies by exploiting the self-similarity of the underlying set (see e.g [17]). We remark that also on scale irregular (non self-similar) sets, known as fractal mixture sets, energy forms can be defined too (see [4] and [51]).

The extension to three-dimensional fractal case is not straightforward, in fact since fractal surfaces are typically non self-similar sets, to define energy forms on them is a difficult task. To our knowledge the first examples of energies on fractal surfaces can be found in [32],[34], [36], [37] and [53], where the fractal surface is obtained by the Cartesian product of a fractal set and a one dimensional interval, the corresponding energy forms are built taking into account the underlying geometry. Indeed this is the type of surfaces we consider.
We study these Venttsel' problems by a semigroup approach. In order to do this we consider suitable abstract Cauchy problems $(\overline{P}_h)$ and $(\overline{P})$. To this aim we consider the Venttsel' energy forms $E^{(h)}[\cdot]$ and $E[\cdot]$, proving that they are symmetric, closed, densely defined forms in suitable Hilbert spaces (see Section 3.2.1 and 3.2.2) and that they admit non positive, self-adjoint operators $A^{(h)}$and A respectively such that

$$E^{(h)}(u,v) = -(A^{(h)}u, v),\, u \in \mathcal{D}(A^{(h)}),\, v \in V(Q^{(\xi)}, S_h^{(\xi)}),$$
$$E(u,v) = -(Au, v),\, u \in \mathcal{D}(A),\, v \in V(Q^{(\xi)}, S^{(\xi)})$$

which are the infinitesimal generators of strongly continuous contraction semigroups $T^{(h)}(t)$ and $T(t)$ respectively (see Section 3.2.3). We prove existence and uniqueness result for the solutions of the abstract Cauchy problems $(\overline{P}_h)$ and $(\overline{P})$ respectively (see Section 4.2).
We also give the corresponding strong interpretations by proving that the solutions of $(\overline{P}_h)$ and $(\overline{P})$ satisfy the formally stated problems (P_h) and (P) (see Theorems 4.3.1 and 4.3.2).
As to the asymptotic behavior of the solutions, it is to be pointed out that the presence of

the time derivative in the boundary conditions has required, as a natural functional setting for these problems, the spaces $L^2(\overline{Q}, m)$ and $L^2(Q, m_h)$, respectively; thus leading us to the framework of varying Hilbert spaces, this is why we use the Mosco convergence (see [49] and [50]) adapted to this setting, studied by Kuwae and Shioya in [29] and in the following named as M-K-S convergence.

When studying the M-K-S convergence in our approach, a crucial role is played by the existence of a core of smooth functions dense in the domain $V(Q^{(\xi)}, S^{(\xi)})$.

In the two-dimensional case one can prove a complete characterization of the energy space on the fractal curve in terms of "fractal" Lipschitz spaces, which in turn are subsets of Hölder continuous functions on the fractal set (see Theorem 4.6 in [16], Theorem 3.1 in [39] for the case of Koch curve and Theorem 1 in [24] for the case of Sierpiński gasket). In the three-dimensional case, as far as we know, this characterization does not hold. Therefore it is of the utmost importance to approximate the functions in the energy form domains by "smooth" functions.

We prove density results for the energy spaces $\mathcal{D}(S^{(\xi)})$ and $V(Q^{(\xi)}, S^{(\xi)})$. In Theorem 3.3.3 we prove that the space $\mathcal{D}(S^{(\xi)})$ has a core, that is a subset dense in $\mathcal{D}(S^{(\xi)})$, with respect to the $\mathcal{D}(S^{(\xi)})$ norm; this in turn it is a crucial tool together with Proposition 3.3.5, where we prove a delicate extension result for functions in $\mathcal{D}(S^{(\xi)})$, by using the Whitney decomposition. In Theorem 3.3.4 we prove that there exists a subset of smooth functions dense in $V(Q^{(\xi)}, S^{(\xi)})$. These results are contained in [33].

When S is the equilateral surface, that is $S = F \times I$, with F the equilateral snowflake, we prove the Mosco-Kuwae-Shioya convergence of the energy forms $E^{(h)}$, which in turn implies the convergence of the associated semigroups (see Theorem 3.4.5). This property is crucial in proving the convergence of the solutions of problems $(\overline{P}_h)$ to the solution of problem $(\overline{P})$ (see Theorems 4.2.2 and 4.2.3).

This is the plan of the thesis. In Chapter 1 we recall some generalities on fractal sets; in particular we describe the construction of the Koch snowflake, of the fractal mixtures, and we describe the geometry of the three-dimensional domains $Q^{(\xi)}$, $Q_h^{(\xi)}$ and the geometry of their fractal boundaries. In Chapter 2 we introduce the functional spaces and trace theorems: we give the definition of d-sets and d-measures and we state trace theorems on d-sets and on piecewise regular sets. We introduce the Besov spaces $B^{2,2}_{\frac{d}{2}}(S^{(\xi)})$. In Chapter 3 we introduce the approximating energy forms $E^{(h)}[\cdot]$, the fractal energy form $E[\cdot]$, the related semigroups $T^{(h)}(t)$, $T(t)$, their generators $A^{(h)}$, A with their main properties. We state and prove the above mentioned density theorems on the domain of the fractal energy form. In order to prove the M-K-S convergence of the energy forms $E^{(h)}$ to E, one has also to take into account that there is a jump of dimension when passing from the prefractal surface to the limit fractal one. This is achieved by choosing suitably the factor δ_h and the constants σ_h^i, $i = 1, 2$ in the definition of the forms $E_{S_h}[\cdot]$ (see (3.2.1)). In Chapter 4 we prove existence and uniqueness results for the problems $(\overline{P}_h)$ and $(\overline{P})$ respectively. The convergence of the solutions of problems $(\overline{P}_h)$ to the solution of problem (P), follows from the M-K-S convergence of the forms, which in turn implies the convergence of semigroups (see Theorem 3.4.5). At last we give the strong interpretation of the solutions of the abstract problem $(\overline{P}_h)$ and $(\overline{P})$. Namely we prove that the solutions of the abstract Cauchy problems solve problems (P) and (P_h) in a suitable weak sense (see Theorems 4.3.1 and 4.3.2). In the Appendix we recall some definitions and properties of forms, semigroups. For the sake of completeness we introduce the Whitney decomposition and the diagonalization lemma and we recall the construction of the energy form on the equilateral snowflake.

1. GENERALITIES ON FRACTAL SETS

Definition 1.0.1. *Let Λ be an open subset of $\mathbb{R}^n$. Its boundary Γ is continuous (Lipschitz continuous, $C^{k,1}$) if for every $p \in \Gamma$ there exists an open neighborhood V of p in $\mathbb{R}^n$ and new orthogonal coordinates $\{y_1, ..., y_n\}$ such that*

1. *V is a hypercube in the new coordinates:*

$$V = \{(y_1, ..., y_n)| - a_j < y_i < a_j, 1 \leq j \leq n\};$$

2. *there exists a continuous function φ(respectively Lipschitz continuous, $C^{k,1}$, continuously differentiable) , defined in*

$$V' = \{(y_1, ..., y_n)| - a_j < y_i < a_j, 1 \leq j \leq n - 1\}$$

and such that

$$|\varphi(P')| \leq a_n/2, \text{for every } P' = (y_1, ..., y_{n-1}) \in V',$$
$$\Lambda \cap V = \{P = (P', y_n) \in V | y_n < \varphi(p')\},$$
$$\Gamma \cap V = \{p = (p', y_n) \in V | y_n = \varphi(P')\}.$$

Remark 1.0.2. *In other words it is requested that in a neighborhood of p, Γ is the graph of φ. The most important example of this definition is that of a subset of $\mathbb{R}^2$, whose boundary Γ is polygonal: this open set will have Lipschitz boundary, not continuously differentiable.*

Definition 1.0.3. *Let Λ be an open subset $\mathbb{R}^n$. Let's say that $\overline{\Lambda}$ is a continuous sub-manifold (respectively Lipschitz continuous, $C^{k,1}$, continuously differentiable) if for every $p \in \Gamma$ there exists a neighborhood V of p in $\mathbb{R}^n$ and an application ψ from V to $\mathbb{R}^n$ such that*

1. *ψ is injective*

2. *ψ together with ψ^{-1} (defined on $\psi(V)$) is continuous*

3. *$\Lambda \cap V = \{p \in \Lambda | \psi_n(p) < 0\}$, where $\psi_n(p)$ denotes the n-th component of $\psi(p)$*

Definition 1.0.4. *Let Λ be an open subset of $\mathbb{R}^n$. Let's say that Λ has the uniform property of segment (respectively cone property), if for every $P \in \Gamma$,there exists an open neighborhood V of P in $\mathbb{R}^n$ and new coordinates $\{y_1, ..., y_n\}$ such that*

1. *V is a hypercube in the new coordinates:*

$$V = \{(y_1, ..., y_n)| -a_j < y_i < a_j, 1 \leq j \leq n\}$$

2. *$p - z \in \Lambda$ when $p \in \overline{\Lambda} \cap V$ and $z \in C$, where C is the open segment $\{0, ..., 0, z_n | 0 < z_n < h\}$ (respectively C is the open cone $\{z = (z', z_n)|(cot\theta)|z'| < z_n < h\}$ for some $\theta \in (0, \pi/2]$) for some $h > 0$.*

Theorem 1.0.5. *A bounded and open subset of $\mathbb{R}^n$ has the uniform cone property if and only if its boundary is Lipschitz continuous.*

1.1 Self-similar sets

Definition 1.1.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be Lipschitz continuous on X if*

$$L = \sup_{x,y \in X, x \neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)} < \infty$$

The constant L is called the Lipschitz constant of f.

Definition 1.1.2. *(Contraction). Let (X, d) be a metric space. If $f : X \to X$ is Lipschitz continuous on X and its Lipschitz constant $L < 1$, then f is called contraction with respect to the metric d with contraction factor L; L is denoted also by $L = L(f)$. In particular, a contraction f with contraction factor r is called similitude if $d(f(x), f(y)) = rd(x, y)$ for every $x, y \in X$.* We denote by $B(P_0, r) = \{x \in X : d(x, P_0) < r\}$

Theorem 1.1.3. *(Contraction principle). Let X be a complete metric space and let $f : X \to X$ be a contraction with respect to the metric d. Then there exists a unique fixed point of f, that is, there exists a unique solution to the equation $f(x) = x$. Moreover if x_* is the fixed point of f, then $\{f^n(a)\}_{n \geq 0}$ converges to x_* for every $a \in X$ where f^n is the n-th iteration of f.*

Theorem 1.1.4. *Let (X, d) be a complete metric space. If $f_i : X \to X$ is a contraction with respect to the metric d for $i = 1, 2, ..., N$ then there exists a unique non empty compact subset K of X such that*

$$K = f_1(K) \cup ... \cup f_N(K).$$

K is called self-similar set with respect to $\{f_1, f_2, ..., f_N\}$.

Remark 1.1.5. *The contraction principle is a special case of Theorem 1.1.4 when $N = 1$.*

We define

$$F(A)=\cup_{1\leq j\leq N} f_j(A)$$

for $A \subset X$. The main idea is to show the existence of a fixed point of F. In order to do so, first a good domain for F has to be chosen:

$$\mathcal{C}(X) = \{A : A \text{ is a non empty compact subset of } X\}.$$

Obviously F is a mapping from $\mathcal{C}(X)$ to itself. We introduce now a metric δ on $\mathcal{C}(X)$, which is called the Hausdorff metric on $\mathcal{C}(X)$.

Proposition 1.1.6. *For $A, B \in \mathcal{C}(X)$ we define*

$$\delta(A, B) = \inf\{r > 0 : U_r(A) \supseteq B \text{ and } U_r(B) \supseteq A\},$$

where $U_r(A) = \{x \in X : d(x, P) \leq r \text{ for some } P \in A\}$= $\cup_{P\in A} B(P, y)$. Then δ is a metric on $\mathcal{C}(X)$. Moreover if (X, d) is complete, then $(\mathcal{C}(X), \delta)$is complete.

Proof. It is obvious that $\delta(A, B) = \delta(B, A) \geq 0$ and $\delta(A, A) = 0$.
$\delta(A, B) = 0 \Rightarrow A = B$: for any n, $U_{1/n}(B) \supseteq A$. Then for any $x \in A$, we can choose $x_n \in B$ such that $d(x, x_n) \leq 1/n$. Since B is closed, $x \in B$. Then $A \subseteq B$. $B \subseteq A$ is obtained in the same way.
Triangular inequality: if $r > \delta(A, B)$ and $s > \delta(B, C)$, then $U_{r+s}(A) \supseteq C$ and $U_{r+s}(C) \supseteq A$. Thus $r + s \geq \delta(A, C)$. This implies that $\delta(A, B) + \delta(B, C) \geq \delta(A, C)$.
It remains to prove that $(\mathcal{C}(X), \delta)$ is complete if (X, d) is complete. We consider a Cauchy sequence $\{A_n\}_{n\geq 1}$ in $(C(X), \delta)$, and we define $B_n = \overline{\cup_{k\geq n} A_k}$.
First we show that B_n is compact. Since B_n is a decreasing sequence of closed sets, it is enough to show that B_1 is compact. For every $r > 0$, it can be chosen m such that $U_{r/2}(A_m) \supseteq A_k$ such that $k \geq m$. Since A_m is compact, there exists a finite recover of A_m with sphere with ray $r/2$. We call Q this recover. It is immediate to verify that $\cup_{x\in P} B_r(x) \supseteq U_{r/2}(A_m) \supseteq \cup_{k\geq m} A_k$. Since $\cup_{P\in Q} B(P, r)$ is closed, Q is a finite recover of sphere with ray r for B_m. Adding to Q recovers with ray r $A_1, A_2, ..., A_{m-1}$, we obtain a recovering with sphere with ray r for B_1. Then B_1 is totally bounded. Moreover B_1 is complete because it is a closed subset of the complete metric space X. Then B_n is compact.
Since $\{B_n\}$ is a decreasing sequence of non-empty compact sets, $A = \cap_{n\geq 1} B_n$ is non empty and compact. For any $r > 0$, we can choose m so that $U_r(A_m) \supseteq A_k$ for all $k \geq m$. Then $U_r(A_m) \supseteq B_m \supseteq A$. On the other hand $U_r(A) \supseteq B_m \supseteq A_m$ for sufficiently large m. Then we have $\delta(A, A_m) \leq r$ for sufficiently large m. Hence $A_m \to A$ for $m \to \infty$ in the Hausdorff metric. Then $(\mathcal{C}(X), \delta)$ is complete. □

Theorem 1.1.4 can be proved in the following way using the Hausdorff metric:

Theorem 1.1.7. *Let (X, d) be a complete metric space and let $f_j : X \to X$ be a contraction for $j = 1, 2, ..., n$. We define $F : \mathcal{C}(X) \to \mathcal{C}(X)$ in this way*

$$F(A) = \cup_{1\leq j\leq N} f_j(A).$$

Then F has a unique fixed point K. Moreover, for every $A \in \mathcal{C}(X)$, $F^n(A)$ converges to K for $n \to \infty$ with respect to the Hausdorff metric. We first prove two preliminary lemma

Lemma 1.1.8. *Let $A_1, A_2, B_1, B_2 \in \mathcal{C}(X)$, then*

$$\delta(A_1 \cup A_2, B_1 \cup B_2) \leq \max\{\delta(A_1, B_1), \delta(A_2, B_2)\}$$

Proof. If $r > \max\{\delta(A_1, B_1), \delta(A_2, B_2)\}$, then $U_r(A_2) \supseteq B_2$ e $U_r(A_1) \supseteq B_1$. Hence $U_r(A_1 \cup A_2) \supseteq B_1 \cup B_2$. In a similar way it holds $U_r(B_1 \cup B_2) \supseteq A_1 \cup A_2$. Then $r \geq \delta(A_1 \cup A_2, B_1 \cup B_2)$. This completes the proof. □

Lemma 1.1.9. *If f is a contraction with contraction factor r, then for every $A, B \in \mathcal{C}(X)$, $\delta(f(A), f(B)) \leq r\delta(A, B)$.*

Proof. If $U_s(A) \supseteq B$ and $U_s(B) \supseteq A$, $U_{sr}(f(A)) \supseteq f(U_s(A)) \supseteq f(B)$. The same argument implies that $U_{sr}(f(B)) \supseteq f(A)$. Then, $\delta(f(A), f(B)) \leq rs$ and this complete the proof. □

Proof. Theorem 1.1.7.
Using Lemma 1.1.8, we get

$$\delta(F(A), F(B)) = \delta(\cup_{1\leq j\leq N} f_j(A), \cup_{1\leq j\leq N} f_j(B)) \leq \max_{1\leq j\leq N} \delta(f_j(A), f_j(B)).$$

From Lemma 1.1.9 $\delta(f_j(A), f_j(B)) \leq r_j\delta(A, B)$ where r_j is the contraction factor of f_j. If $r = \max_{1\leq j\leq N} r_j$, then $\delta(F(A), F(B)) \leq r\delta(A, B)$. Then F is a contraction with respect to the Hausdorff metric. From the Proposition 1.1.6 we deduce that $(\mathcal{C}(X), \delta)$ is complete. From the contraction principle it follows that F has a unique fixed point. □

1.2 The Koch curve and the snowflake

Let K_0 be a unit segment, having the endpoints $A = (0, 0)$ and $B = (1, 0)$. Let K_1 the curve obtained dividing K_0 in three segment of equal length, removing the central segment and replacing it by two sides of the equilateral triangle with base the segment removed. Then applying the same procedure to every side of the curve K_1, we get K_2. Iterating this construction, we obtain a sequence of polygonal prefractal curves K_h, one for every n in $\mathbb{N}_0$.
Let us consider a set of four contractive similitudes $\Psi = \{\psi_1, ..., \psi_4\}$, with the same contraction factor $l^{-1} = \dfrac{1}{3}$, defined in the following way

$$\psi_1(z) = \frac{z}{3},$$
$$\psi_2(z) = \frac{z}{3}e^{i\frac{\pi}{3}} + \frac{1}{3},$$
$$\psi_3(z) = \frac{z}{3}e^{-i\frac{\pi}{3}} + \frac{1}{2} + i\frac{\sqrt{3}}{6},$$
$$\psi_4(z) = \frac{z}{3} + \frac{2}{3}$$

where $\psi_i : \mathbb{C} \to \mathbb{C}$, $i = 1, ..., 4$. Given a set $E \subset \mathbb{R}^d$, we define

$$\Psi(E) = \cup_{i=1}^{4} \psi_i(E)$$

and, for every integer h, let us denote by $\Psi^h(E) = \Psi \circ ... \circ \Psi(E)$ the h-th composition of Ψ. Let K_0 be the segment above defined, then for every $h \in \mathbb{N}$ we set

$$K_1 = \Psi(K_0) = \cup_{i=1}^{4} \psi_i(K_0),$$
$$\vdots$$
$$K_{h+1} = \Psi(K_h) = \cup_{M \in F_h} \cup_{i=1}^{4} \psi_i(M)$$

where $F_h = \{M : M \text{ is a segment of } K_h\}$ is the set of the segments of the h-th prefractal curve K_h. The Koch curve is the unique compact set K invariant for Ψ, that is $K = \Psi(K) = \cup_{i=1}^{4} \psi_i(K)$. On the Koch curve K there exists an invariant measure μ that is

$$\int_K \phi d\mu = \sum_{i=1}^{4} \frac{1}{4} \int_K (\phi \circ \psi_i) d\mu, \ \phi \in C_0(K)$$

which is given by the normalized Hausdorff measure on K (see [21]). By the snowflake F we denote the union of three complanar Koch curves (see [12]). We assume that the junction points A_1, A_2, A_3 are the vertices of a regular triangle with unit side length, that is $|A_1 - A_3| = |A_1 - A_2| = |A_2 - A_3| = 1$. One can define, in a natural way, a finite Borel measure μ_F supported on F by

$$\mu_F := \mu_1 + \mu_2 + \mu_3 \tag{1.2.1}$$

where μ_i denotes the normalized d_f-dimensional Hausdorff measure, restricted to K_i, $i = 1, 2, 3$.

The measure μ_F is a d-measure (see Definition 2.1.1), that is there exist two positive constants c_1, c_2

$$c_1 r^d \leq \mu_F(B(P, r) \cap F) \leq c_2 r^d, \forall P \in F$$

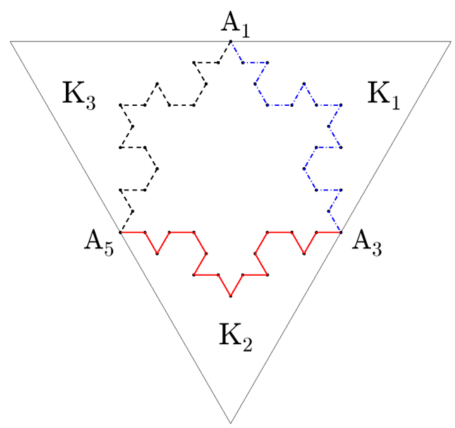

Fig. 1.1: Koch snowflake

where

$$d = d_f = \frac{log4}{log3} \tag{1.2.2}$$

and where $B(P, r)$ denotes the Euclidean ball in $\mathbb{R}^2$. K_1 is the uniquely determined self-similar set with respect to four suitable contractions $\psi^{(1)}, ..., \psi^{(4)}$, with respect to the same ratio $\frac{1}{3}$ (see [16]). Let $V_0^{(1)} := \{A_1, A_3\}$, $V_{j_1 \cdots j_h}^{(1)} := \psi_{j_1}^{(1)} \circ ... \circ \psi_{j_h}^{(1)}(V_0^{(1)})$ and

$$V_h^{(1)} := \bigcup_{j_1 \cdots j_h = 1}^{4} V_{j_1 \cdots j_h}^{(1)}.$$

On every V_h^i, $i = 1, 2, 3$, it can be defined a discrete measure μ_i^h, for any $h \geq 1$, by

$$\mu_i^h = \frac{1}{4^h} \sum_{P \in V_h^i} \delta_P \tag{1.2.3}$$

where δ_P denotes the Dirac measure at the point P. Note that $\mu_i^h(V_h^i) = 1 + \frac{1}{4^h}$. It can be proved (see [39]) that the sequence $(\mu_i^h)_{h \geq 1}$ weakly converge in $C(K_i)'$ to the measure μ_i.
We set $V_\star^{(1)} := \cup_{h \geq 0} V_h^{(1)}$. It holds that $K_1 = \overline{V_\star^{(1)}}$. Let $K_1^{(0)}$ denote the unit segment whose endpoints are A_1 and A_3 and $K_{j_1 \ldots j_h}^{(1)} := \psi_{j_1}^{(1)} \circ ... \circ \psi_{j_h}^{(1)}(K_1^{(0)})$. For $h > 0$ we denote

$$F_{(1)}^h = \{\psi_{j_1}^{(1)} \circ ... \circ \psi_{j_h}^{(1)}(K_1^{(0)}), j_1, ..., j_h = 1, ..., 4\}.$$

We set $K_1^{(1)} = \bigcup_{j=1}^{4} \psi_j^{(1)}(K_1^{(0)})$, $K_1^{(h+1)} = \bigcup_{M \in F_{(1)}^h} \bigcup_{j=1}^{4} \psi_j^{(1)}(M)$, where M denotes a segment of the $h+1$-th generation; $K_1^{(h+1)}$ the polygonal curve and $V_{h+1}^{(1)}$ the set of its vertices.
In a similar way, it is possible to approximate K_2, K_3 by the sequences $(V_h^{(2)})_{h \geq 0}$, $(V_h^{(3)})_{h \geq 0}$, and denote their limits by $V_\star^{(2)}, V_\star^{(3)}$, and the corresponding polygonal curves

$K_2^{(h+1)}, K_3^{(h+1)}$.
In order to approximate F, we define the increasing sequence of finite sets of points $\mathcal{V}_h := \cup_{i=1}^3 V_h^{(i)}, h \geq 1$ and $\mathcal{V}_\star := \cup_{h\geq 1}\mathcal{V}_h$. It holds that $\mathcal{V}_\star = \cup_{i=1}^3 V_\star^{(i)}$ and $F = \overline{V_\star}$.
In the following we denote by

$$F_{h+1} = \bigcup_{i=1}^{3} K_i^{(h+1)}$$

the closed polygonal curve approximating F at the $(h+1)-$th step.

1.3 Fractal mixtures

In this section we recall the definition of scale irregular Koch curves (Koch mixtures), following the construction described in [51] and in [4].
Let $A = \{1, 2\}$: for $a \in A$, we consider $2 < l_a < 4$, and for each $a \in A$ we set

$$\Psi^{(a)} = \{\psi_1^{(a)}, ..., \psi_4^{(a)}\}$$

the family of contractive similitudes $\psi_i^{(a)} : \mathbb{C} \to \mathbb{C}$, $i = 1, ..., 4$, with contraction factor l_a^{-1}

$$\psi_1^{(a)}(z) = \tfrac{z}{l_a}, \psi_2^{(a)}(z) = \tfrac{z}{l_a}e^{i\theta(l_a)} + \tfrac{1}{l_a},$$
$$\psi_3^{(a)}(z) = \tfrac{z}{l_a}e^{i\theta(l_a)} + \tfrac{1}{2} + i\sqrt{\tfrac{1}{l_a} - \tfrac{1}{4}}, \psi_4^{(a)}(z) = \tfrac{z-1}{l_a} + 1$$

where

$$\theta(l_a) = arcsin(\tfrac{\sqrt{l_a(4-l_a)}}{2}).$$

Let $\Xi = A^{\mathbb{N}}$; we call $\xi \in \Xi$ an environment. We define a left shift $\mathcal{S}$ on Ξ such that if $\xi = (\xi_1, \xi_2, ...)$, then $\mathcal{S}\xi = (\xi_2, \xi_3, ...)$. For $\mathcal{O} \subset \mathbb{R}^2$, we set

$$\Phi^{(a)}(\mathcal{O}) = \bigcup_{i=1}^{4} \psi_i^{(a)}(\mathcal{O})$$

and

$$\Phi_h^{(\xi)}(O) = \Phi^{(\xi_1)}(O) \circ ... \circ \Phi^{(\xi_h)}(O).$$

We consider the line segment of unit length K with endpoints $B = (0, 0)$ and $C = (1, 0)$. We set, for each $h \in \mathbb{N}$, $K^{(\xi),h} = \Phi_h^{(\xi)}(K)$: $K^{(\xi),h}$ is the h-th prefractal curve.
The fractal $K^{(\xi)}$ associated with the environment sequence ξ is defined by

$$K^{(\xi)} = \overline{\bigcup_{h=1}^{\infty} \Phi_h^{(\xi)}(\Gamma)}$$

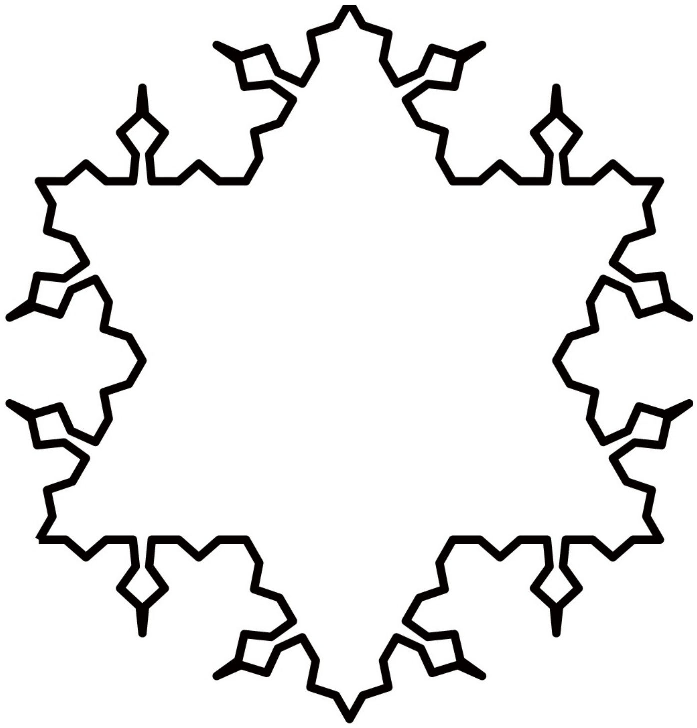

Fig. 1.2: Koch type snowflake

where $\Gamma = \{B, C\}$.

These fractals don't have any exact self-similarity, but $K^{(\xi)}, \xi \in \Xi$ satisfies

$$K^{(\xi)} = \Phi^{(\xi_1)}(K^{(\mathcal{S}\xi)}).$$

For $\xi \in \Xi$, we set $i|h = (i_1, i_2, ..., i_h)$ and $\psi_{i|h} = \psi_{i_1}^{(\xi_1)} \circ ... \circ \psi_{i_h}^{(\xi_h)}$ and for any $\mathcal{O} \subset \mathbb{R}^2$, $\psi_{i|h}(\mathcal{O}) = \mathcal{O}^{i|h}$. There exists a unique Radon measure $\mu^{(\xi)}$ on $K^{(\xi)}$ such that

$$\mu^{(\xi)}(\psi_{i|h}(K^{(S^h\xi)})) = \tfrac{1}{4^h}$$

(see Section 2 in [4]).

The fractal set $K^{(\xi)}$ and the measure $\mu^{(\xi)}$ depend on the structural constants of the families and the asymptotic frequency of the occurrence of each family. We denote by $c_a^{(\xi)}(h)$ the frequency of the occurrence of a in the finite sequence $\xi|_h$, $h \geq 1$:

$$c_a^{(\xi)}(h) = \tfrac{1}{h}\sum_{i=1}^{h} \mathbf{1}_{\xi_i = a},\, a = 1, 2$$

Let p_a be a probability distribution on A and suppose that ξ satisfies

$$c_a^{(\xi)}(h) \to p_a,\, h \to \infty,$$

where $0 \leq p_a \leq 1$, $p_1 + p_2 = 1$; it also holds

$$|c_a^{(\xi)}(h) - p_a| \leq \frac{f(h)}{h},$$

$a = 1, 2$ $(h \geq 1)$, where f is an increasing function on the real line, $f(0) = 1$, $f(h) \leq f_0 h^{\beta_0}$, $f_0 > 1$, $0 \leq \beta_0 < 1$.
If $\beta_0 = 0$, the measure $\mu^{(\xi)}$ is a $d^{(\xi)}$-measure in the sense of the Definition 3.1, that is there exist two positive constants C_1, C_2, such that

$$C_1 r^{d^{(\xi)}} \leq \mu^{(\xi)}(B(P,r) \bigcap K^{(\xi)}) \leq C_2 r^{d^{(\xi)}}, \forall P \in K^{(\xi)}$$

with

$$d^{(\xi)} = \frac{ln4}{p_1 ln l_1 + p_2 ln l_2} \tag{1.3.4}$$

where $B(P,r)$ denotes the Euclidean ball with center in P and radius $0 < r \leq 1$ and p_a is the probability distribution on A.
If $\beta_0 > 0$ instead

$$C_1 r^{d^{(\xi)}-i} \leq \mu^{(\xi)}(B(P,r) \bigcap K^{(\xi)}) \leq C_2 r^{d^{(\xi)}-i}, \forall P \in K^{(\xi)}$$

We will confine ourselves to the case $\beta_0 = 0$.
Following [16], we introduce the snowflake-type set $F^{(\xi)}$, obtained by the union of three Koch mixtures $K^{(\xi)}$ with the same structural constants, that is

$$F^{(\xi)} = \bigcup_{i=1}^{3} K_i^{(\xi)}$$

and we define a finite Radon measure supported on $F^{(\xi)}$

$$\mu_F^{(\xi)} := \mu_1^{(\xi)} + \mu_2^{(\xi)} + \mu_3^{(\xi)},$$

where $\mu_i^{(\xi)}$ denotes the $d^{(\xi)}$-dimensional normalized Hausdorff measure restricted to $K_i^{(\xi)}$, $i = 1, 2, 3$.
The dimension of $F^{(\xi)}$ is

$$d^{(\xi)} = d_f^{(\xi)}. \tag{1.3.5}$$

We denote by $\Omega^{(\xi)}$ the open bounded two-dimensional domain with boundary $F^{(\xi)}$.

1.4 Geometry of Q, $Q^{(\xi)}$, Q_h, S, $S^{(\xi)}$, $S_h^{(\xi)}$ and S_h

By S_h we denote

$$F_h \times I, \tag{1.4.6}$$

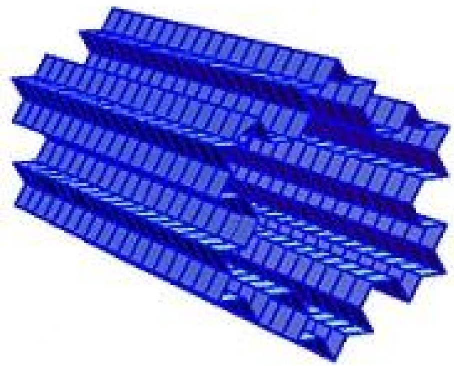

Fig. 1.3: Surface S

where F_h is the prefractal approximation of F at the step h, $I = [0,1]$. S_h is a surface of polyhedral type. We give a point $P \in S_h$ the Cartesian coordinates $P = (x, x_3)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plane containing F_h and x_3 is the coordinate of the orthogonal projection of P on the x_3-line containing the interval I.
By Ω_h we denote the open bounded two-dimensional domain with boundary F_h. By Q_h we denote the domain with S_h as lateral surface and $\Omega_h \times \{0\}$, $\Omega_h \times \{1\}$ as bases of Q_h. The measure on S_h is

$$d\sigma = dl \times dx_3,$$

where dl is arc-length measure on F_h and dx_3 is the one-dimensional Lebesgue measure on I. We introduce $S = F \times I$ the fractal surface given by the Cartesian product between F and I; S is a polyhedral surface. It can be defined on S the finite Borel measure

$$dg = d\mu_F \times dx_3$$

supported on S. The measure g is a d-measure (see Definition 2.1.1), that is there exist two positive constants c_1, c_2

$$c_1 r^d \leq g(B(P,r) \cap S) \leq c_2 r^d, \forall P \in S$$

where $d = d_f + 1 = \frac{log12}{log3}$ and where $B(P,r)$ denotes the Euclidean ball in $\mathbb{R}^3$. By Ω we denote the two-dimensional domain whose boundary is F. By Q we denote the open cylindrical domain where $S = F \times I$ is the "lateral surface" and where the sets $\Omega \times \{0\}$, $\Omega \times \{1\}$ are the bases. By $\mathcal{R}$ we denote the open equilateral triangle whose midpoints are the vertices A_1, A_2, A_3 and by $\mathcal{T}$ the open prism $\mathcal{R} \times [0,1]$ with bases $\mathcal{R} \times \{0\}$ and $\mathcal{R} \times \{1\}$
By $S^{(\xi)}$ we denote the cylindrical-type fractal surface

$$S^{(\xi)} = F^{(\xi)} \times I,$$

where $I = [0, 1]$.

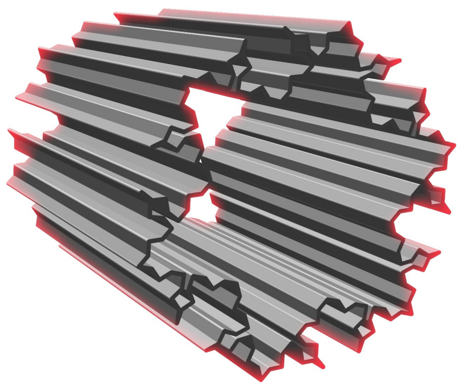

Fig. 1.4: Fractal mixture surface

We define on $S^{(\xi)}$ the following measure

$$dg^{(\xi)} = d\mu_F^{(\xi)} \times d\mathcal{L}_1 \tag{1.4.7}$$

supported on $S^{(\xi)}$, where $\mathcal{L}_1$ is the one dimensional Lebesgue measure on I, $g^{(\xi)}$ is a $d-$measure with $d = d_f^{(\xi)} + 1$.
By $Q^{(\xi)}$ we denote the open cylindrical domain where $S^{(\xi)} = F^{(\xi)} \times I$ is the "lateral surface" and where the sets $\Omega^{(\xi)} \times \{0\}$, $\Omega^{(\xi)} \times \{1\}$ are the bases.
We denote by $P \in S^{(\xi)}$, the couple (x, y), where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of P on the plain containing $F^{(\xi)}$ and y is the coordinate of the orthogonal projection of P on the interval $[0, 1]$: $(x_1, x_2) \in F^{(\xi)}$, $y \in I$.
Similarly we denote by $S_h^{(\xi)}$ the Cartesian product $F_h^{(\xi)} \times I$, where $F_h^{(\xi)}$ is the prefractal approximation of $F^{(\xi)}$ at the step h, $I = [0, 1]$. $S_h^{(\xi)}$ is a surface of polyhedral type.
Finally, by $Q_h^{(\xi)}$ we denote the open cylindrical domain where $S_h^{(\xi)} = F_h^{(\xi)} \times I$ is the "lateral surface" and where the sets $\Omega_h^{(\xi)} \times \{0\}$, $\Omega_h^{(\xi)} \times \{1\}$ are the bases.

2. FUNCTIONAL SPACES

By $L^2(\cdot)$ we denote the Lebesgue space with respect to the Lebesgue measure $\mathcal{L}_3$ on measurable subsets of $\mathbb{R}^3$, which will be left to the context whenever that does not create ambiguity. Let T be a closed set of $\mathbb{R}^3$, by $C(T)$ we denote the space of continuous functions on T and $C^{0,\beta}(T)$ is the space of Hölder continuous functions on T, $0 < \beta < 1$. Let G be an open set of $\mathbb{R}^3$, by $H^s(G)$, $s \in \mathbb{R}^+$ we denote the Sobolev spaces, possibly fractional (see [54]). $D(G)$ is the space of infinitely differentiable functions with compact support on G.
From now on we will refer to the sets Q, S, S_h, Q_h, $S_h^{(\xi)}$, $Q_h^{(\xi)}$, $Q^{(\xi)}$, $S^{(\xi)}$ as defined in Section 1.4.

2.1 Trace theorems on prefractal sets

Definition 2.1.1. *A closed set M is a d-set in $\mathbb{R}^n$, ($0 < d \leq n$), if there exist a Borel measure μ with supp$\mu = M$ and two positive constants c_1, c_2*

$$c_1 r^d \leq \mu_F(B(P,r) \cap M) \leq c_2 r^d, \forall P \in M$$

Remark 2.1.2. *F is a d_f-set. The measure μ_F is a d_f-measure. S is a $d_f + 1$-set. The measure g is a $d_f + 1$-measure.*

Definition 2.1.3. *Let $\mathcal{G}$ be an open subset in $\mathbb{R}^3$. If $f \in H^s(\mathcal{G})$, we call trace of f*

$$\gamma_0 f(P) = \lim_{r\to 0} \frac{1}{|B(P,r) \bigcap \mathcal{G}|} \int_{B(P,r)\bigcap \mathcal{G}} f(Q) d\mathcal{L}_3$$

Remark 2.1.4. *It is known that the limit exists at quasi every $P \in \overline{\mathcal{G}}$ with respect to the $(s,2)-$capacity (see [1]).*

The following result is the Theorem 3.1 in [22], specialized in the case of interest. We refer to [19] and [10] for a more general discussion.

Proposition 2.1.5. *Let $\mathcal{G}$ denote Q_h or $Q_h^{(\xi)}$ respectively and let Γ denote S_h and $S_h^{(\xi)}$ respectively.*
Let $\frac{1}{2} < s < \frac{3}{2}$. Then $H^{s-\frac{1}{2}}(\Gamma)$ is the trace space to Γ of $H^s(\mathcal{G})$ in the following sense:

1. *γ_0 is a continuous and linear operator from $H^s(\mathcal{G})$ to $H^{s-\frac{1}{2}}(\Gamma)$,*

2. *there is a continuous linear operator* Ext *from* $H^{s-\frac{1}{2}}(\Gamma)$ *to* $H^s(\mathcal{G})$*, such that* $\gamma_0 \circ Ext$ *is the identity operator in* $H^{s-\frac{1}{2}}(\Gamma)$.

From now on we denote by $u|_\Gamma$ the trace operator, that is $u|_\Gamma = \gamma_0 u$.
The following Theorem characterizes the trace on the polyhedral set S_h of a function belonging to the Sobolev space $H^\beta(\mathbb{R}^3)$.

Theorem 2.1.6. *Let* S_h *be as defined in (1.4.6). Let* $u \in H^\beta(\mathbb{R}^3)$ *and* $\delta_h = (3^{1-d_f})^h$*. Then for* $\frac{1}{2} \leq \beta \leq 1$,

$$\|u|_{S_h}\|^2_{L^2(S_h)} \leq \frac{C_\beta}{\delta_h}\|u\|^2_{H^\beta(\mathbb{R}^3)}, \tag{2.1.1}$$

where C_β *is independent from* h. In order to prove it, we recall the following lemma, (see [25] page 104):

Lemma 2.1.7. *Let* $0 < d \leq n$ *and let* μ *be a positive measure satisfying* $\mu(B(P, r)) \leq cr^d$, $r \leq r_0$, $x \in \mathbb{R}^n$*. Then*

$$\int\limits_{|P-t|\leq a} |P-t|^{-\gamma} d\mu(t) \leq ca^{d-\gamma},$$

if $d > \gamma$, $a \leq r_0$*, and*

$$\int\limits_{a\leq|P-t|\leq b} |P-t|^{-\gamma} \leq ca^{d-\gamma},$$

if $d < \gamma$, $b \leq r_0$.
Here c *is a constant depending on* c_1, γ, r_0, d. we also recall some estimates on Bessel kernels (see [59]):

Proposition 2.1.8. G_β *is a positive, decreasing function of* $|x|$*, analytic on* $\mathbb{R}^n \setminus 0$*, satisfying*

- $|D^j G_\beta(x)| \leq c|x|^{\alpha-|j|-n}$*, for* $\beta < n + |j|$
- $|D^j G_\beta(x)| \leq c log\frac{1}{|x|}$, $0 < |x| < 1$*, for* $\beta = n + |j|$
- $|D^j G_\beta(x)| \leq ce^{-c_1|x|}$*, for all* j, $|x| \geq 1$*, for some* $c_1 > 0$

Proof. Theorem 2.1.6.
We adapt the proof from the two dimensional case treated in [25]. Any $u \in H^\beta(\mathbb{R}^n)$ can be written in terms of Bessel kernels G_β, of order β, that is $u = G_\beta * g$, $g \in L^2(\mathbb{R}^3)$, (see [58]). Then

$$\|u|_{S_h}\|^2_{L^2(S_h)} = \int_{S_h} |\int_{\mathbb{R}^3} G_\beta(x-y)g(y)dy|^2 d\sigma \leq \int_{S_h}(\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2a}|g(y)|^2 dy) \\ (\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2(1-a)}dy)d\sigma,$$

where $0 < a < 1$ will be chosen later. By using the estimates for the Bessel kernels and Lemma 1 on page 104 in [25], we get

$$\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2(1-a)} dy \le C_1$$

if

$$3 > 2(3-\beta)(1-a), \tag{2.1.2}$$

where C_1 is independent of h.
Moreover, since S_h is a $2-$set with $C_2 = C_3 \delta_h^{-1}$, we get again from Lemma 1 on page 104 in [25]

$$\int_{S_h} |G_\beta(x-y)|^{2a} d\sigma \le C_4 \delta_h^{-1},$$

if

$$2 > 2a(3-\beta), \tag{2.1.3}$$

where C_4 is independent of h.
By choosing a in order to satisfy (2.1.2) and (2.1.3), we get

$$\|u|_{S_h}\|^2_{L^2(S_h)} \le C_1 \int_{S_h} (\int_{\mathbb{R}^3} |G_\beta(x-y)|^{2a} |g(y)|^2 dy) d\sigma = \int_{\mathbb{R}^3} (\int_{S_h} |G_\beta(x-y)|^{2a}) |g(y)|^2 dy \le \\ C_1 C_4 \delta_h^{-1} \int_{\mathbb{R}^3} |g(y)|^2 dy = C_1 C_4 \delta_h^{-1} \|g\|^2_{L^2(\mathbb{R}^3)} = C_\beta \delta_h^{-1} \|u\|_{H^\beta(\mathbb{R}^3)},$$

where $C_\beta = C_1 C_4$ is independent of h. □

Remark 2.1.9. *We note that the Theorem 2.1.6 holds also when the trace is taken on the polyhedral set $S_h^{(\xi)}$ (on the Sobolev spaces of the functions belonging to $H^\beta(\mathbb{R}^3)$) with $\delta_h = \delta_h^{(\xi)} = ((l_1 l_2)^{1-d_f^{(\xi)}})^h$.*

Let $\mathcal{T}$ denote the $(d+1)-$ sets S or $S^{(\xi)}$ equipped with their $(d+1)-$measures η. The following theorem that characterizes the trace on the set Γ of a function belonging to Sobolev spaces $H^\beta(\mathbb{R}^3)$ is a consequence of Theorem 1 in Chapter 5 of [25] as the fractal Γ is a $d-$set.

Theorem 2.1.10. *Let $u \in H^\beta(\mathbb{R}^3)$. Then, for $1 - \frac{d}{2} < \beta$,*

$$\|u\|^2_{L^2(\mathcal{T})} \le C^*_\beta \|u\|^2_{H^\beta(\mathbb{R}^3)}. \tag{2.1.4}$$

It is possible to prove that the domains Ω_h are (ϵ, δ) domains with parameter indipendent of the increasing number of sides F_h and, taking into account the underlying Cartesian structure of $Q_h = \Omega_h \times I$, this result holds for Q_h.
The following theorem, consequence of extension Theorem for (ϵ, δ) domains (see [23]) holds:

Theorem 2.1.11. *There exists a bounded linear extension operator $Ext_J : H^1(Q_h) \to H^1(\mathbb{R}^3)$, such that*

$$\|Ext_J v\|^2_{H^1(\mathbb{R}^3)} \leq C_J \|v\|^2_{H^1(Q_h)}, \tag{2.1.5}$$

with C_j independent of h.

Theorem 2.1.12. *There exists a linear extension operator $Ext : H^\beta(Q) \to H^\beta(\mathbb{R}^3)$, such that, for any $\beta > 0$,*

$$\|Extv\|_{H^\beta(\mathbb{R}^3)} \leq \overline{C}_\beta \|v\|_{H^\beta(Q)} \tag{2.1.6}$$

with $\overline{C}_\beta$ depending on β.

2.2 Besov spaces

We recall that F is a d_f-set, the measure μ_F is a d_f-measure, S is a $d_f + 1$-set and the measure g is a $d_f + 1$-measure.

We define the Besov space on S: we recall here the definition which best fits our aims and we restrict ourselves to the case $p = q = 2$ and $\beta = \frac{d}{2}$; for a general treatment see [25].

Definition 2.2.1. *We say that $f \in B^{2,2}_{\frac{d}{2}}(\mathcal{T})$ if $f \in L^2(\mathcal{T}, \eta)$ and it holds*

$$\|f\|_{B^{2,2}_{\frac{d}{2}}(\mathcal{T})} < +\infty,$$

where

$$\|f\|_{B^{2,2}_{\frac{d}{2}}(\mathcal{T})} = \|f\|_{L^2(\mathcal{T},g)} + \left(\int \int_{|P-P'|<1} \frac{|f(P) - f(P')|^2}{|P - P'|^{2d+1}} d\eta(P) d\eta(P') \right)^{\frac{1}{2}} \tag{2.2.7}$$

Theorem 2.2.2. *Let $\mathcal{G}$ denote Q, $Q^{(\xi)}$ and let $\mathcal{T}$ denote S or $S^{(\xi)}$ respectively, then $B^{2,2}_{\frac{d}{2}}(\mathcal{T})$ is the trace space of $H^1(\mathcal{G})$ that is:*

1. *There exists a linear and continuous operator $\gamma_0 : H^1(\mathcal{G}) \to B^{2,2}_{\frac{d}{2}}(\mathcal{T})$.*
2. *There exists a linear and continuous operator $Ext : B^{2,2}_{\frac{d}{2}}(\mathcal{T}) \to H^1(\mathcal{G})$, such that $\gamma_0 \circ Ext$ is the identity operator on $B^{2,2}_{\frac{d}{2}}(\mathcal{T})$, that is*

$$\gamma_0 \circ Ext = Id_{B^{2,2}_{\frac{d}{2}}(\mathcal{T})}$$

For the proof see Chapter V page 103 in [25].
In the following we denote by the symbol $u|_{\mathcal{T}}$ the trace $\gamma_0 u$ to $\mathcal{T}$.

2.3 Varying Hilbert spaces

We introduce the notion of convergence in varying Hilbert spaces; for more details, see [29].

Definition 2.3.1. *A sequence of Hilbert spaces $\{H_h\}_{h\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{\Phi_h\}_{h\in\mathbb{N}}$ of linear operators $\Phi_h : C \to H_h$ such that*

$$\lim_{h\to\infty} \|\Phi_h u\|_{H_h} = \|u\|_H \text{ for any } u \in C$$

We set $\mathcal{H} = \bigcup H_h \bigcup H$.
We now provide the definitions of strong and weak convergence in $\mathcal{H}$.

Definition 2.3.2. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if $u_h \in H_h$, $u \in H$ and there exists a sequence $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \in C$ tending to u in H such that*

$$\lim_{m\to\infty} \overline{\lim}_{h\to\infty} \|\Phi_h \widetilde{u}_m - u_h\|_{H_h} = 0$$

Definition 2.3.3. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ weakly converges to u in $\mathcal{H}$, if $u_h \in H_h$, $u \in H$ and*

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ strongly tending to v in $\mathcal{H}$.

Remark 2.3.4. *Strong convergence implies weak convergence.*

Lemma 2.3.5. *Let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence weakly convergent to u in H, then*

- $\sup_h \|u_h\|_{H_h} < \infty$.
- $\|u\|_H \leq \underline{\lim}_{h\to\infty} \|u_h\|_{H_h}$.
- $u_h \to u$ *if and only if* $\|u\|_H = \lim_{h\to\infty} \|u_h\|_{H_h}$.

Now we state other characterizations of strong convergence in $\mathcal{H}$.

Lemma 2.3.6. *Let $u \in H$ and let $\{u_h\}_{h\in\mathbb{N}}$ be a sequence of vectors $u_h \in H_h$. Then $\{u_h\}_{h\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$, if and only if*

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}_{h\in\mathbb{N}}$ with $v_h \in H_h$ weakly converging to a vector v in $\mathcal{H}$.

Lemma 2.3.7. *A sequence of vectors $\{u_h\}_{h\in\mathbb{N}}$ with $u_h \in H_h$ strongly converges to u in $\mathcal{H}$ if and only if*

- $\|u_h\|_{H_h} \to \|u\|_H$
- $(u_h, \Phi_h\varphi)_{H_h} \to (u, \varphi)_H$ *for every* $\varphi \in C$.

Lemma 2.3.8. *Let* $\{u_h\}_{h\in\mathbb{N}}$ *be a sequence with* $u_h \in H_h$*. If* $\|u_h\|_{H_h}$ *is uniformly bounded, there exists a subsequence of* $\{u_h\}_{h\in\mathbb{N}}$ *which weakly converges in* $\mathcal{H}$*.*

Lemma 2.3.9. *For every* $u \in H$ *there exists a sequence* $\{u_h\}_{h\in\mathbb{N}}$*,* $u_h \in H_h$ *strongly converging to* u *in* $\mathcal{H}$*.*

Definition 2.3.10. *A sequence of bounded operators* $\{B_h\}_{h\in\mathbb{N}}$*,* $B_h \in \mathcal{L}(H_h)$ *strongly converges to an operator* $B \in \mathcal{L}(H)$*, if for every sequence of vectors* $\{u_h\}_{h\in\mathbb{N}}$ *with* $u_h \in H_h$ *strongly converging to* u *in* $\mathcal{H}$*, the sequence* $\{B_h u_h\}_{h\in\mathbb{N}}$ *strongly converges to* Bu *in* $\mathcal{H}$*.*

2.3.1 Convergence of spaces

From now on we put $H = L^2(\overline{Q}, m)$, where m is the measure defined in (3.2.13), and the sequence $\{H_h\}_{h\in\mathbb{N}} = \{L^2(Q, m_h)\}_{h\in\mathbb{N}}$, dove m_h is the measure defined in (3.2.6), with norms

$$\|u\|_H^2 = \|u\|^2_{L^2(Q)} + \|u|_S\|^2_{L^2(S,g)}, \|u\|^2_{H_h} = \|u\|^2_{L^2(Q_h)} + \|u|_{S_h}\|^2_{L^2(S_h,\delta_h)}$$

Proposition 2.3.11. *Let* $\delta_h = (3^{1-d_f})^h$*. The sequence of Hilbert spaces* $\{H_h\}_{h\in\mathbb{N}}$ *converges to the Hilbert space* H*.*

Proof. We put $C = C(\overline{Q})$ and Φ_h the identical operator on $C(\overline{Q})$. We have to prove that

$$\lim_{h\to\infty} \|u\|_{H_h} = \|u\|_H, \text{ for any } u \in C.$$

So we have to prove that

$$\lim_{h\to\infty} \int\limits_Q \chi_{Q_h} |u|^2 d\mathcal{L}_3 = \int\limits_Q |u|^2 d\mathcal{L}_3 \tag{2.3.8}$$

and

$$\lim_{h\to\infty} \delta_h \int\limits_I \int\limits_{F_h} |u|^2 dl dx_3 = \int\limits_I \int\limits_F |u|^2 dg \tag{2.3.9}$$

and hence

$$\lim_{h\to\infty} \delta_h \int\limits_{F_h} |u|^2 dl = \int\limits_F |u|^2 d\mu. \tag{2.3.10}$$

$$\delta_h \int_{F_h} |u|^2\, dl = \sum_{j=1}^{3\cdot 4^h} \delta_h \int_{M_j} |u|^2\, dl,$$

where M_j denotes a segment of h-generation.
Since $u(\cdot, x_3)$ is continuous on F_h for each $x_3 \in [0, 1]$, by the mean value Theorem, there exists $\xi_j \in M_j$ such that

$$\delta_h \int_{F_h} |u|^2 \, dl = \sum_{j=1}^{3\cdot 4^h} \delta_h |u(\xi_j, x_3)|^2 3^{-h}.$$

We can write $| \int_F |u(x,x_3)|^2 d\mu - \delta_h \int_{F_h} |u(x,x_3)|^2 dl|$

$$\leq | \int\limits_F |u(x,x_3)|^2 d\mu - \sum_{j=1}^{3\cdot 4^h} \frac{|u(P_j,x_3)|^2}{4^h} | + | \sum_{j=1}^{3\cdot 4^h} \delta_h 3^{-h} (|u(P_j,x_3)|^2 - |u(\xi_j,x_3)|^2)|, \quad (2.3.11)$$

where P_j is one of the endpoints of M_j. The first term of right-hand side of the inequality tends to zero as $h \to \infty$ from the Corollary 3.4 in [40], while the second vanishes since $|u|^2$ is uniformly continuous in every M_j. Since

$$\sup_{x_3 \in [0,1]} \delta_h \int_{F_h} |u|^2 \, dl \leq 3 \|u^2\|_{C(Q)}$$

the thesis follows from dominated convergence theorem. $\square$

Remark 2.3.12. *We note that the Theorem 2.3.11 holds also with* $\delta_h = \delta_h^{(\xi)} = ((l_1 l_2)^{1-d_f^{(\xi)}})^h$.

3. VENTTSEL' ENERGY FORMS

3.1 Introduction

The aim of this chapter is to introduce the approximating energy forms $E^{(h)}[\cdot]$ and the fractal energy form $E[\cdot]$ related to the Venttsel' problems we will study in the following chapter; in particular we are interested to asymptotic behavior for h tending to $+\infty$ of $E^{(h)}[\cdot]$: we will prove the Mosco-convergence of the approximating energy forms to the fractal one in the framework of varying Hilbert spaces (see Theorem 3.4.4). This will allow us to deduce the convergence of the related resolvents and semigroups and then the convergence in a suitable sense of the solutions of the approximating problem to the limit one (see Chapter 4).
To this purpose we prove the existence of a core of smooth functions in the domains of $E_S[\cdot]$, $E[\cdot]$ respectively (see Theorems 3.3.3 and 3.3.4). In order to prove these results , the main tool is Whitney type argument. These results are contained in [33].
We point out that we can prove the Mosco-convergence only when Q is the open cylindrical domain with lateral surface S, it is still an open problem in the general case of $Q^{(\xi)}$. The density results hold also for the case of $Q^{(\xi)}$ too.

3.2 Energy forms

In this chapter we consider $Q^{(\xi)}$, $Q_h^{(\xi)}$, $S_h^{(\xi)}$, $S^{(\xi)}$ defined as in Section 1.4 and we suppress all the superscripts ξ.

3.2.1 Approximating energy forms

We introduce now the energy forms $E_{S_h}[\cdot]$ on $S_h = F_h \times I$, $h \in \mathbb{N}$. By l we denote the arc-length coordinate on each edge F_h and we introduce the coordinate $x_1 = x_1(l)$, $x_2 = x_2(l)$, $x_3 = x_3$ on every affine face $S_h^{(j)}$ of S_h. By dl we denote the 1-dimensional measure given by the arc-length l, and by $d\sigma$ the surface measure on $S_h^{(j)}$, $d\sigma = dl dx_3$. $E_{S_h}[\cdot]$ is defined by

$$E_{S_h}[u] = \sum_j \left(\int_{S_h^{(j)}} \sigma_h^1 |D_l u|^2 + \sigma_h^2 |\partial_3 u|^2 \right) d\sigma, \tag{3.2.1}$$

where σ_h^1, σ_h^2 are positive constants, D_l denotes the tangential derivative along the prefractal F_h, and $u \in H^1(S_h)$. By the Fubini Theorem, E_{S_h} can be written in the form

$$E_{S_h}[u] = \sigma_h^1 \int\limits_I \left(\int\limits_{F_h} |D_l u|^2 dl \right) dx_3 + \sigma_h^2 \int\limits_{F_h} \left(\int\limits_I |\partial_3 u|^2 dx_3 \right) dl. \tag{3.2.2}$$

We denote by $E_{S_h}(u,v)$ the corresponding bilinear form defined by polarization.
Let us consider now the function space

$$V(Q,S_h) = \{u \in H^1(Q) : u|_{S_h} \in H^1(S_h)\} \tag{3.2.3}$$

and the energy form

$$E^{(h)}[u] = \int\limits_Q \chi_{Q_h} \mathcal{A}^h Du\, Du d\mathcal{L}_3 + E_{S_h}[u|_{S_h}] + \delta_h \int\limits_{S_h} b|u|_{S_h}|^2 d\sigma \tag{3.2.4}$$

defined on $V(Q,S_h)$, where $b \in C(\overline{Q})$, $b > 0$, χ_{Q_h} denotes the characteristic function of Q_h, δ_h is a positive constant, where $\mathcal{A}^h = [a_{ij}^h], i,j = 1,2,3$; a_{ij}^h are uniformly bounded functions in $\overline{Q}$,

$$(H_h) \qquad \begin{cases} a_{ij}^h = a_{ji}^h, & \forall i,j = 1,2,3 \\ \exists \lambda > 0 : & \\ \sum_{i,j=1}^3 a_{ij}^h \xi_i \xi_j \geq \lambda \sum_{i=1}^3 |\xi_i|^2 & \forall (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 \end{cases}$$

The corresponding bilinear form, obtained by polarization is

$$E^h(u,v) = \int\limits_Q \chi_{Q_h} \mathcal{A}^h Du \cdot Dv d\mathcal{L}_3 + E_{S_h}(u|_{S_h}, v|_{S_h}) + \delta_h \int\limits_{S_h} b u|_{S_h} v|_{S_h} d\sigma \tag{3.2.5}$$

defined on $V(Q,S_h) \times V(Q,S_h)$.
We introduce now the space $L^2(Q,m_h)$, where m_h is the measure defined as

$$dm_h = \chi_{Q_h} d\mathcal{L}_3 + \chi_{S_h} \delta_h d\sigma, \tag{3.2.6}$$

where χ_{S_h} denotes the characteristic function of S_h and δ_h is a positive constant.

Theorem 3.2.1. *The form $E^{(h)}$, defined in (3.2.4) with dense domain $V(Q,S_h)$, is a Dirichlet form in $L^2(Q,m_h)$, and the space $V(Q,S_h)$ is a Hilbert space equipped with the scalar product*

$$(u,v)_{V(Q,S_h)} = \int_Q \chi_{Q_h} Du Dv d\mathcal{L}_3 + E_{S_h}(u|_{S_h}, v|_{S_h}) + (u,v)_{L^2(Q,m_h)}.$$

3.2.2 Fractal energy form

By proceeding as in [16] we construct an energy form on F, by defining a Lagrangian measure $\mathcal{L}_F$ on F, which has the role of the Euclidean Lagrangian $d\mathcal{L}(u,v) = Du\,Dv\,dx$. The corresponding energy form on F is given by

$$\mathcal{E}_F(u,v) = \int_F d\mathcal{L}_F(u,v)$$

with domain $\mathcal{D}(F) = \{u \in L^2(F,\mu_F) : \mathcal{E}_F[u] < +\infty\}$ dense in $L^2(F,\mu_F)$, (see Section 5.3 in the Appendix and the references therein).

Proposition 3.2.2. *$\mathcal{D}(F)$ is a Hilbert space equipped with the following norm*

$$\|u\|_{\mathcal{D}(F)} = (\|u\|^2_{L^2(F)} + \mathcal{E}_F[u])^{\frac{1}{2}}. \tag{3.2.7}$$

As in [42], Lemma 6.2.2 page 43, it can be proved that

Proposition 3.2.3. *$\mathcal{D}(F)$ is embedded in $C^{0,\beta}(F)$, with $\beta = \frac{ln4}{2ln(\min(l_1,l_2))}$.*
We now define the energy form on S and the fractal Laplacian Δ_S.

$$E_S[u] = \int\limits_I \mathcal{E}_F[u]\mathrm{d}x_3 + \int\limits_F \int\limits_I |\partial_3 u|^2 \mathrm{d}x_3 d\mu_F, \tag{3.2.8}$$

where ∂_3 denotes the derivative with respect the direction x_3.
The form E_S is defined for $u \in \mathcal{D}(S)$,

$$\mathcal{D}(S) = \overline{C(S) \cap L^2(0,1;\mathcal{D}(F)) \cap H^1(0,1;L^2(F))}^{\|\cdot\|_{\mathcal{D}(S)}}, \tag{3.2.9}$$

where $\|\cdot\|_{\mathcal{D}(S)}$ is the intrinsic norm

$$\|u\||_{\mathcal{D}(S)} = (E_S[u] + \|u\|^2_{L^2(S,g)})^{\frac{1}{2}}. \tag{3.2.10}$$

Proposition 3.2.4. *$E_S(u,v)$ with domain $\mathcal{D}(S) \times \mathcal{D}(S)$ is a Dirichlet form in $L^2(S,g)$ and $\mathcal{D}(S)$ is a Hilbert space equipped with the intrinsic norm.*

Proof. For the proof see [53]. □

We now give an embedding result for the domain $\mathcal{D}(S)$. Unlike the two dimensional case where there is a characterization of the functions in $\mathcal{D}(F)$ in terms of the so-called Lipschitz spaces (see Theorem 3.1 in [39]), for $\mathcal{D}(S)$ we do not have a characterization , but the following result holds:

Proposition 3.2.5. *$\mathcal{D}(S) \subset B^{2,2}_\beta(S)$, for any $0 < \beta < 1$.*

Proof. We follow the proof in [32], adapted to the present case.
We recall that

$$\mathcal{D}(S) := \overline{C(S) \bigcap L^2([0,1]; \mathcal{D}(F)) \bigcap H^1([0,1]; L^2(F))}^{\|\cdot\|_{\mathcal{D}(S)}}$$

Following [43] we define $B^{2,2}_{D_f-\varepsilon,1}(S) := L^2([0,1]; B^{2,2}_{D_f-\varepsilon}(F)) \bigcap H^1([0,1]; L^2(F))$ for $\varepsilon > 0$.

For any Banach space X and for any $0 < \beta < 1$, $H^1([0,1]; X) \subset H^\beta([0,1]; X)$, moreover if $p = q = 2$ and β is not integer, it holds

$$H^\beta([0,1]; X) \equiv B^{2,2}_\beta([0,1]; X).$$

Hence if $0 < \beta < 1$

$$B^{2,2}_{D_f-\varepsilon,1}(S) \subset L^2([0,1]; B^{2,2}_{D_f-\varepsilon}(F)) \bigcap B^{2,2}_\beta([0,1]; L^2(F)) \subset$$

$$L^2([0,1]; B^{2,2}_\beta(F)) \bigcap B^{2,2}_\beta([0,1]; L^2(F)) = B^{2,2}_\beta(S),$$

the last equivalence can be proved following [43]. □

From Proposition 3.2.4 and Theorem 5.2.10 in the Appendix, we have

Theorem 3.2.6. *There exists a unique non positive self-adjoint operator Δ_S on $L^2(S, g)$ with domain $\mathcal{D}(\Delta_S) := \{u \in L^2(S, g) : \Delta_S u \in L^2(S, g)\} \subseteq \mathcal{D}(S)$ dense in $L^2(S, g)$ such that*

$$E_S(u, v) = -\int_S \Delta_S u \, v dg, \text{ for each } u \in \mathcal{D}(\Delta_S), v \in \mathcal{D}(S).$$

Now we introduce the energy form on Q. Let us consider the space

$$V(Q, S) = \big\{u \in H^1(Q) : u|_S \in \mathcal{D}(S)\big\} \tag{3.2.11}$$

and the energy form

$$E[u] = \int\limits_Q \mathcal{A} Du \cdot Du d\mathcal{L}_3 + E_S[u|_S] + \int\limits_S b|u|_S|^2 dg, \tag{3.2.12}$$

defined on $V(Q, S)$, where $b \in C(\overline{Q})$, $b > 0$, $[\mathcal{A}]_{ij} = a_{ij}$, where a_{ij} are uniformly bounded functions in $\overline{Q}$,

$$(H) \qquad \begin{cases} a_{ij} = a_{ji}, & \forall i, j = 1, 2, 3 \\ \exists \lambda > 0 : & \\ \sum_{i,j=1}^3 a_{ij}\xi_i\xi_j \geq \lambda \sum_{i=1}^3 |\xi_i|^2 & \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \end{cases}$$

We denote by $L^2(\overline{Q}, m)$ the Lebesgue space with respect to the measure

$$dm = d\mathcal{L}_3 + dg, \tag{3.2.13}$$

where dg is defined in (1.4.7).
By $E(u,v)$ we denote the bilinear form, obtained by polarization

$$E(u,v) = \int_Q \mathcal{A}Du \cdot Dvd\mathcal{L}_3 + E_S(u|_S, v|_S) + \int_S bu|_S v|_S dg, \tag{3.2.14}$$

defined on $V(Q,S) \times V(Q,S)$.

Proposition 3.2.7. *The form E is a Dirichlet on $L^2(\overline{Q}, m)$ and $V(Q,S)$ is a Hilbert space equipped with the scalar product*

$$(u,v)_{V(Q,S)} = (u,v)_{H^1(Q)} + E_S(u|_S, v|_S) + (u|_S, v|_S)_{L^2(S,g)} \tag{3.2.15}$$

with norm

$$\|u\|_{V(Q,S)} = (\|u\|^2_{H^1(Q)} + \|u|_S\|^2_{\mathcal{D}(S)})^{\frac{1}{2}}. \tag{3.2.16}$$

Proof. We start proving that $V(Q,S)$ is a Hilbert space: let $\{u_n\}$ be a Cauchy sequence in $V(Q,S)$. Then $\{u_n\}$ is a Cauchy sequence in $H^1(Q)$ and $\{u_n|_S\}$ is a Cauchy sequence in $\mathcal{D}(S)$; hence there exists $u \in H^1(Q)$ and $v \in \mathcal{D}(S)$ such that

$$\lim_{n\to\infty} \|u_n - u\|_{H^1(Q)} = 0$$

$$\lim_{n\to\infty} \|u_n|_S - v\|_{\mathcal{D}(S)} = 0$$

From the Theorem 2.2.2 it follows that $u|_S \in B^{2,2}_{\frac{d_f}{2}}(S)$. Moreover we have

$$\|u|_S - v\|_{B^{2,2}_{\frac{d_f}{2}}(S)} \leq \|u|_S - u_n|_S\|_{B^{2,2}_{\frac{d_f}{2}}(S)} + \|u_n|_S - v\|_{B^{2,2}_{\frac{d_f}{2}}(S)} \leq c_1\|u_n - u\|_{H^1(Q)} + c_2\|u_n|_S - v\|_{\mathcal{D}(S)},$$

where the last inequality follows from Theorem 2.2.2 and 3.2.5. Then

$$\|u|_S - v\|_{B^{2,2}_{\frac{d_f}{2}}(S)} = 0$$

and thus $u|_S = v$ in $B^{2,2}_{\frac{d_f}{2}}(S)$. By Theorem 3.2.5 and since $v \in \mathcal{D}(S)$, it follows that $u|_S \in \mathcal{D}(S)$ and then $u \in V(Q,S)$.

Now we prove that the form $E[u]$ is closed, that is, following the Definition 5.2.4, we want to prove that if $u_n \in V(Q,S)$, $u_n \to u$ in $L^2(\overline{Q}, m)$ and $E[u_n - u_m] \to 0$ then $u \in V(Q,S)$ and $E[u_n - u] \to 0$: if $u_n \to u$ in $L^2(\overline{Q}, m)$, then $\|u_n - u_m\|_{L^2(\overline{Q},m)} \to 0$, hence

$$\|u_n - u_m\|^2_{L^2(\overline{Q},m)} + E[u_n - u_m] \to 0$$

The square root of $\|\cdot\|^2_{L^2(\overline{Q},m)} + E[\cdot]$ is a norm in $V(Q,S)$ equivalent to (3.2.16), in fact

$$\int_Q |u|^2 d\mathcal{L}_3 + \int_Q \mathcal{A}Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg \leq$$

$$\int_Q |u|^2 d\mathcal{L}_3 + |\mathcal{A}|\int_Q Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg \leq$$

$$C\left(\int_Q |u|^2 d\mathcal{L}_3 + \int_Q Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S |u|_S|^2 dg\right)$$

where $|\mathcal{A}|$ is the norm of the matrix $\mathcal{A}$ and $C=\max\{|\mathcal{A}|, \max_S b+1\}$. On the other hand, by the ellipticity of $\mathcal{A}$ we have

$$\lambda\int_Q |Du|^2 d\mathcal{L}_3 + \int_Q |u|^2 d\mathcal{L}_3 + E_S[u|_S] + (\min_S b+1)\int_S |u|_S|^2 dg \leq$$

$$\int_Q |u|^2 d\mathcal{L}_3 + \int_Q \mathcal{A}Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg,$$

and choosing $c=\min\{\lambda, \min_S(b+1)\}$ we get

$$c\left(\int_Q |u|^2 d\mathcal{L}_3 + \int_Q Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg\right) \leq$$

$$\int_Q |u|^2 d\mathcal{L}_3 + \int_Q \mathcal{A}Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg$$

Then we proved that there exist two constants c and C such that $c\|u\|^2_{V(Q,S)} \leq \int_Q |u|^2 d\mathcal{L}_3 + \int_Q \mathcal{A}Du\cdot Dud\mathcal{L}_3 + E_S[u|_S] + \int_S b|u|_S|^2 dg + \int_S |u|_S|^2 dg \leq C\|u\|^2_{V(Q,S)}$. Then we have a Cauchy sequence in $V(Q,S)$, then $u\in V(Q,S)$ and $E[u_n-u]\to 0$.
We now prove that the form E is Markovian following the Proposition 5.2.6: let u be a function in $V(Q,S)$ and let $v=\min(\max(u,0),1)$: we have to prove that $v\in V(Q,S)$ and that $E(v,v)\leq E(u,u)$. The proof that $E_S[\cdot]$ is a Dirichlet form follows from the Proposition 3.2.4.
We note that, from definition $0\leq v\leq 1$ a.e. in $\overline{Q}$ then $v\in L^2(\overline{Q},m)$; moreover

$$\int_Q |Dv|^2 d\mathcal{L}_3 = \int_Q \chi_{\{0\leq u\leq 1\}}|Du|^2 d\mathcal{L}_3, \tag{3.2.17}$$

in fact where $u\leq 0$ then $v=0$ (a.e) and where $u\geq 1$ then $v=1$ (a.e) and in these two cases $Dv=0$ a.e.; where $0\leq u\leq 1$ a.e., then $v=u$ a.e and thus $Dv=Du$ a.e. Then

$v \in H^1(Q)$ and from $\int_Q \mathcal{A}Du \cdot Dud\mathcal{L}_3 \leq |\mathcal{A}| \int_Q |Du|^2 d\mathcal{L}_3$, from 3.2.17 it follows that $\int_Q \mathcal{A}Dv \cdot Dvd\mathcal{L}_3 < +\infty$. This proves that $u \in V(Q,S)$.
We finally prove that

$$\int_Q \mathcal{A}Dv \cdot Dvd\mathcal{L}_3 + \int_S b|v|_S|^2 dg \leq \int_Q \mathcal{A}Du \cdot Dud\mathcal{L}_3 + \int_S b|u|_S|^2 dg.$$

In fact

$$\int_Q \mathcal{A}Dv \cdot Dvd\mathcal{L}_3 = \int_Q \mathcal{A}\chi_{\{0 \leq u \leq 1\}} Du \cdot Dud\mathcal{L}_3 \leq \int_Q \mathcal{A}Du \cdot Dud\mathcal{L}_3$$

and

$$\int_S b|v|_S|^2 dg = \int_S \chi_{\{u \leq 0\}} 0 dg + \int_S b\chi_{\{0 \leq u \leq 1\}} |u|_S|^2 dg + \int_S \chi_{\{u \geq 1\}} b \cdot 1 dg \leq \int_S b|u|_S|^2 dg$$

and summing we get the thesis.
Thus this proves that E is a Dirichlet form. $\square$

3.2.3 Semigroups associated with E and E$^{(h)}$

In this subsection we will mainly refer to Kato's Theorem and Lumer-Phillips Theorem, which we recall in the Appendix for sake of completeness.

Proposition 3.2.8. *$E(u,v)$ is a Dirichlet form in $L^2(\overline{Q},m)$ with domain $V(Q,S)$ dense in $L^2(\overline{Q},m)$, hence there exists a unique non positive, self-adjoint operator A on $L^2(\overline{Q},m)$ with $\mathcal{D}(A) \subseteq V(Q,S)$ dense in $L^2(Q,m)$, such that*

$$E(u,v) = -\int_Q Au \cdot vdm, u \in \mathcal{D}(A), v \in V(Q,S). \tag{3.2.18}$$

Proof. From Proposition 3.2.7 it follows that $E[\cdot]$ is closed in $L^2(\overline{Q},m)$, hence from Theorem 5.2.10 in the Appendix we get the thesis. $\square$

Since $E[\cdot]$ is a Dirichlet form, it follows that A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \geq 0}$.
Moreover it holds

Proposition 3.2.9. *$\{T(t)\}_{t \geq 0}$ is a contraction analytic semigroup on $L^2(\overline{Q},m)$.*

Proof. The contraction property follows from the Lumer-Phillips Theorem (see Theorem 5.4.16 in the Appendix). In order to prove the analyticity, it will be enough to prove that there exists a positive α and λ_0 such that

$$E[u] + \lambda_0 \|u\|^2_{L^2(\overline{Q},m)} \geq \alpha \|u\|_{V(Q,S)}$$

(see Proposition 3 Section 6 in Chapter 17 of [11]).

□

Proposition 3.2.10. *$E^{(h)}(u, v)$ is a Dirichlet form in $L^2(Q, m_h)$ with domain $V(Q, S_h)$ dense in $L^2(Q, m_h)$, hence there exists a unique non positive, self-adjoint operator A^h on $L^2(Q, m_h)$ with $\mathcal{D}(A^h) \subseteq V(Q, S_h)$ dense in $L^2(Q, m_h)$, such that*

$$E^h(u, v) = -\int_Q A^h u \cdot v dm_h, u \in \mathcal{D}(A^h), v \in V(Q, S_h).$$

Proof. From Proposition 3.2.1 it follows that $E^{(h)}[\cdot]$ is closed in $L^2(\overline{Q}, m)$, hence from Theorem 5.2.10 in the Appendix we get the thesis. □

Proposition 3.2.11. *Let $\{T_h(t)\}_{t\geq 0}$ be the semigroup associated with A^h. Then $\left\{T^h(t)\right\}_{t\geq 0}$ is a contraction analytic semigroup on $L^2(Q, m_h)$.*

Proof. The contraction property follows from the Lumer-Phillips Theorem (see Theorem 5.4.16 in the Appendix). In order to prove the existence of strongly continuous semigroups and its analyticity, it will be enough to prove that there exists a positive α and λ_0 such that

$$E^{(h)}[u] + \lambda_0 \|u\|^2_{L^2(Q, m_h)} \geq \alpha \|u\|_{V(Q, S_h)}$$

The proof follows from Chapter 17, Section 6 in [11]. □

3.3 Density theorems

In this section we prove two important density theorems for the energy spaces $\mathcal{D}(S)$ and $V(Q, S)$ respectively.

3.3.1 Density theorem for $\mathcal{D}(S)$

Following the notations of [43] page 8, we denote by $W(0, 1)$ the following space:

$$W(0, 1) := L^2([0, 1]; \mathcal{D}(F)) \bigcap H^1([0, 1]; L^2(F)). \tag{3.3.1}$$

This is a Hilbert space equipped with the norm

$$\|u\|_{W(0,1)} = (\|u\|^2_{L^2([0,1];\mathcal{D}(F))} + \|\partial_3 u\|^2_{L^2([0,1];L^2(F))})^{\frac{1}{2}}. \tag{3.3.2}$$

From [43] Theorem 2.1 page 11, the following result holds

Proposition 3.3.1. *The space $D([0, 1]; \mathcal{D}(F))$ is densely embedded in $W(0, 1)$, that is*

$$\overline{D([0, 1]; \mathcal{D}(F))}^{\|\cdot\|_{W(0,1)}} = W(0, 1) \tag{3.3.3}$$

We now prove that

Proposition 3.3.2. $D(0,1;\mathcal{D}(F)) \subset C(S)$.

Proof. From Proposition 3.2.3 it holds that $\mathcal{D}(F) \subset C^{0,\beta}(F)$, in particular $\mathcal{D}(F) \subset C(F)$, then

$$D([0,1];\mathcal{D}(F)) \subset C([0,1];\mathcal{D}(F)) \subset C([0,1];C(F)).$$

It remains to prove

$$C([0,1];C(F)) \equiv C(S).$$

We follow the lines of the proof given in [5] pages 68-70. If $u \in C(S)$, then for every $y \in [0,1]$ $u(\cdot,y) \in C(F)$, for every $x \in F$ $u(x,\cdot) \in C([0,1])$ and $\sup_{y\in[0,1]} \sup_{x\in F} |u(x,y)| < \infty$, hence

$$C(S) \subseteq C([0,1];C(F)).$$

If $u \in C([0,1];C(F))$, then $u(\cdot,y) \in C(F)$ for every fixed y in $[0,1]$ and from the continuity of u in $[0,1]$ for every x in F it follows that

$$\sup_{x\in F} |u(x,y) - u(x,y_n)\ | \to 0$$

for every $\{y_n\} \subset I$, $y_n \to y$ when $n \to \infty$. Therefore $C([0,1];C(F)) \equiv C(S)$.

□

Theorem 3.3.3. *The space $D(0,1;\mathcal{D}(F))$ is dense in $\mathcal{D}(S)$ with respect to the intrinsic norm $\|\cdot\|_{\mathcal{D}(S)}$.*

Proof. From Proposition 3.3.2 and (3.3.3), it holds that

$$D([0,1];\mathcal{D}(F)) \subset C(S) \bigcap L^2([0,1];\mathcal{D}(F)) \bigcap H^1([0,1];L^2(F))$$

which amounts to say that $D([0,1];\mathcal{D}(F)) \subset C(S) \bigcap W(0,1)$; from the definition of $\mathcal{D}(S)$ we have

$$C(S) \bigcap W(0,1) \subset \mathcal{D}(S).$$

It follows that

$$D([0,1];\mathcal{D}(F)) \subset \mathcal{D}(S). \tag{3.3.4}$$

Now let f be a function in $\mathcal{D}(S)$, then from the definition of $\mathcal{D}(S)$ it follows that there exists $\{\varphi_n\} \subset W(0,1) \bigcap C(S)$ such that

$$\|\varphi_n - f\|_{\mathcal{D}(S)} \to 0$$

for $n \to \infty$.
On the other hand $\{\varphi_n\} \subset W(0,1)$, and from Proposition 3.3.1, there exists $\{\psi_{m,n}\}_{m\in\mathbb{N}} \subset D([0,1];\mathcal{D}(F))$ such that, for every fixed n

$$\|\psi_{m,n} - \varphi_n\|_{W(0,1)} \to 0 \tag{3.3.5}$$

when $m \to \infty$. From Fubini Theorem for measure valued functions it follows that $\|\cdot\|_{\mathcal{D}(S)} = \|\cdot\|_{W(0,1)}$ and hence for every fixed n

$$\|\psi_{m,n} - \varphi_n\|_{\mathcal{D}(S)} \to 0 \tag{3.3.6}$$

for $m \to \infty$.
We now use a diagonalization argument. From [2] Corollary 1.16 there exists an increasing mapping

$$m \to n(m),$$

that tends to ∞ for $m \to \infty$, such that

$$\overline{\lim}_{m\to\infty}\|\psi_{m,n(m)} - \varphi_{n(m)}\|_{\mathcal{D}(S)} \leq \overline{\lim}_{n\to\infty}\lim_{m\to\infty}\|\psi_{m,n} - \varphi_n\|_{\mathcal{D}(S)}. \tag{3.3.7}$$

The right hand of (3.3.7) tends to zero when m $\to \infty$ and from this it follows that $\overline{\lim}_{m\to\infty}\|\psi_{m,n(m)} - \varphi_{n(m)}\|_{\mathcal{D}(S)} = 0$. Hence also

$$\underline{\lim}_{m\to\infty}\|\psi_{m,n(m)} - \varphi_{n(m)}\|_{\mathcal{D}(S)} = 0,$$

This proves that $\lim_{m\to\infty}\|\psi_{m,n(m)} - \varphi_{n(m)}\|_{\mathcal{D}(S)} = 0$.
Finally $\|\psi_{n(m),m} - f\|_{\mathcal{D}(S)} \leq \|\psi_{n(m),m} - \varphi_{n(m)}\|_{\mathcal{D}(S)} + \|\varphi_{n(m)} - f\|_{\mathcal{D}(S)} \to 0$ for $m \to \infty$. $\square$

3.3.2 Density Theorem for $V(Q,S)$

We now state the main Theorem of the section, which allow us to approximate functions in $V(Q,S)$ by continuous functions, and this will be crucial in the proof of the Mosco-convergence of the energy forms $E^{(h)}[\cdot]$.

Theorem 3.3.4. *For every $u \in V(Q,S)$, there exists $\{\psi_n\} \subset V(Q,S) \bigcap C(\overline{Q})$ such that:*

1. $\|\psi_n - u\|_{H^1(Q)} \to 0$, *for* $n \to \infty$
2. $\|\psi_n - u\|_{L^2(\overline{Q},m)} \to 0$, *for* $n \to \infty$
3. $E_S[\psi_n - u] \to 0$, *for* $n \to \infty$.

In order to prove this Theorem, we need a preliminary proposition on trace and extension operators.

Proposition 3.3.5. *Let $\beta = \frac{D_f}{2}$. Let γ_0 and Ext be the trace and the extension operators defined in Theorem 2.2.2 respectively. Then*
(1) If $u \in C(\mathbb{R}^3) \bigcap H^1(\mathbb{R}^3)$ then $\gamma_0 u \in C(S) \bigcap B^{2,2}_\beta(S)$.
(2) If $u \in C(S) \bigcap B^{2,2}_\beta(S)$ then $Ext(u) \in C(\mathbb{R}^3) \bigcap H^1(\mathbb{R}^3)$.

Proof. We start proving (1). Since $u \in H^1(\mathbb{R}^3)$, then for $P \in S$, $\gamma_0 u(P)$ exists and from Theorem 2.2.2 $\gamma_0 u$ belongs to $B^{2,2}_\beta(S)$ with $\beta = \frac{D_f}{2}$; since u is also in $C(\mathbb{R}^3)$, in particular u is in $C(S)$.
By the mean value Theorem there exists $\zeta \in B(P,r) \bigcap S$ such that

$$\frac{1}{m(B(P,r) \bigcap S)} \int_{B(P,r) \bigcap S} u(\mathcal{P}) d\mathcal{L}_3 = u(\zeta).$$

Hence when $r \to 0$

$$u(\zeta) \to u(P).$$

In order to prove (2) we make use of Whitney decomposition. We refer to the Appendix, Section 5.1 and [25] page 23 for details. Let Q_i be the cubes in $\mathbb{R}^3 \backslash S$ such that $\bigcup\limits_i Q_i = \mathbb{R}^3 \backslash S$, with centers P_i, $l_i =$diamQ_i and $\{\phi_i\}$ the associated unity partition. From [25], page 109, we define for $P \in \mathbb{R}^3 \setminus S$

$$Ext(u)(P) = \sum_{i \in I} \phi_i(P) c_i \int_{|t-P_i| \leq 6l_i} u(t) dg(t),$$

where $c_i = (g(|t - P_i| \leq 6l_i))^{-1}$.
In our assumptions $u \in B^{2,2}_\beta(S)$, then from Theorem 2.2.2 $Ext(u) \in H^1(\mathbb{R}^3)$ and $\gamma_0(Ext(u)) = u$ on S. It results, by construction, that $Ext(u)$ is in particular continuous in $\mathbb{R}^3 \setminus S$ (see Appendix). Since $u \in C(S) \bigcap B^{2,2}_\beta(S)$, it remains to prove that for every $P_0 \in S$

$$|Ext(u)(P) - u(P_0)| \to 0$$

when $P \to P_0$, that is for every $\varepsilon > 0$ $\exists \delta_\varepsilon$: $|P - P_0| < \delta_\varepsilon$:$|Ext(u)(P) - u(P_0)| < \varepsilon$.
We now estimate $|Ext(u)(P) - u(P_0)|$.

$$|Ext(u)(P) - u(P_0)| = |\sum_{i \in I} \phi_i(P) c_i \int_{|t-P_i| \leq 6l_i} u(t) dg - u(P_0)| =$$
$$|\sum_{i \in I} \phi_i(P) c_i \int_{|t-P_i| \leq 6l_i} (u(t) - u(P_0)) dg| \leq c(l_i)^{\frac{-(d_f+1)}{2}} (\int_{|t-P_i| \leq 6l_i} |u(t) - u(P_0)|^2 dg)^{\frac{1}{2}},$$

where the last inequality is obtained from Hölder inequality. Since g is a $(d_f + 1)-$measure supported on S and since $|P - P_0| \leq \delta$, we obtain

$$c(l_i)^{\frac{-(d_f+1)}{2}} (\int_{|t-P_i| \leq 6l_i} |u(t) - u(P_0)|^2 dg)^{\frac{1}{2}} =$$
$$c(l_i)^{\frac{-(d_f+1)}{2}} \int_{\{|t-P_i| \leq 6l_i\} \bigcap \{|t-P_0| \leq \delta\}} |u(t) - u(P_0)|^2 dg)^{\frac{1}{2}}.$$

As $u \in C(S)$ we get

$$c(l_i)^{\frac{-(d_f+1)}{2}} \int_{\{|t-P_i|\leq 6l_i\}\bigcap\{|t-P_0|\leq\delta\}} |u(t)-u(P_0)|^2 dg)^{\frac{1}{2}} \leq$$

$$c(l_i)^{\frac{-(D_f+1)}{2}} \sup_{\{|(x,y)-P_i|\leq 6l_i\}\bigcap\{|(x,y)-P_0|\leq\delta\}} |u(x,y) - u(P_0)|(\int_{\{|t-P_i|\leq 6l_i\}\bigcap\{|t-P_0|\leq\delta\}} dg)^{\frac{1}{2}} \leq$$

$$cl_i^{\frac{-(d_f+1)}{2}} l_i^{\frac{d_f+1}{2}} \varepsilon = c\varepsilon,$$

where the last inequality follows from the continuity of u on S. □

We are now ready to prove Theorem 3.3.4.

Proof. We start proving (1).
Let us consider $u \in V(Q,S)$, then $u|_S \in \mathcal{D}(S)$. From Theorem 3.3.3 there exists $\{\varphi_n\} \subset D(0,1;\mathcal{D}(F))$ such that

$$\|\varphi_n - u|_S\|_{\mathcal{D}(S)} \to 0, \text{ when } n \to \infty.$$

We note that since $\varphi_n \in D(0,1;\mathcal{D}(F)) \subset \mathcal{D}(S) \subset B^{2,2}_\alpha(S)$ and $D(0,1;\mathcal{D}(F)) \subset C(S)$, it follows that $\varphi_n \in B^{2,2}_\alpha(S)\bigcap C(S)$. Let $\widehat{\varphi_n}$ be the function defined as $Ext(\varphi_n)$ and let $\widehat{u}$ be the function defined as $Ext(u|_S)$. Then from (2) of Proposition 3.3.5 $\widehat{\varphi_n} \in H^1(Q)\bigcap C(\overline{Q})$ and $\widehat{u} \in H^1(Q)$ (see [25]).
We prove that $\|\widehat{\varphi_n} - \widehat{u}\|_{H^1(Q)} \to 0$; in fact from Theorem 2.2.2 and the inclusion of $\mathcal{D}(S)$ in $B^{2,2}_{\frac{D_f}{2}}(S)$ (see Theorem 3.2.5),

$$\|\widehat{\varphi_n} - \widehat{u}\|_{H^1(Q)} \leq C_1\|\varphi_n - u|_S\|_{B^{2,2}_{\frac{d_f}{2}}(S)} \leq \|\varphi_n - u|_S\|_{\mathcal{D}(S)}$$

From the density Theorem 3.3.3 $\|\widehat{\varphi_n} - \widehat{u}\|_{H^1(Q)} \to 0$.
Now let us consider $u - \widehat{u}$: this is a function in $H^1(Q)$ and $(u-\widehat{u})|_S = 0$, then $u - \widehat{u} \in H^1_0(Q)$, (see Theorem 3 in [61]); there exists $\{\eta_m\}_{m\in\mathbb{N}} \subset C^1_0(\overline{Q})$ such that

$$\|\eta_m - (u-\widehat{u})\|_{H^1(Q)} \to 0. \tag{3.3.8}$$

Let $\{\psi_{n,m}\}$ denote the doubly indexed sequence of functions $\{\widehat{\varphi_n} - \eta_m\}$. The sequence $\{\psi_{n,m}\} \subset H^1(Q)\bigcap C(\overline{Q})$. From Corollary 1.16 in [2] we deduce that $\{\psi_{m,n}\}$ converges to u in $H^1(Q)$ as $n \to \infty$. In fact there exists an increasing mapping $n \to m(n)$, tending to ∞ as $n \to \infty$, such that

$$\overline{\lim}_{n\to\infty}\|u - \psi_{n,m(n)}\|_{H^1(Q)} = \overline{\lim}_{n\to\infty}\|u - \widehat{\varphi_n} - \eta_{m(n)}\|_{H^1(Q)} \leq \\ \overline{\lim}_{n\to\infty}(\|u - \widehat{u} - \eta_{m(n)}\|_{H^1(Q)} + \|\widehat{\varphi_n} - \widehat{u}\|_{H^1(Q)}),$$

then by applying Corollary 1.16 in [2] to the right hand side of the above inequality it follows that

$$\overline{\lim}_{n\to\infty}\|u - \psi_{n,m(n)}\|_{H^1(Q)} \leq \lim_{m\to\infty}\lim_{n\to\infty}\{\ \|u - \widehat{u} - \eta_m\|_{H^1(Q)} + \|\widehat{\varphi_n} - \widehat{u}\|_{H^1(Q)}\ \}.$$

The two terms in the sum tend to 0 when $m, n \to \infty$, then

$$\overline{\lim}_{n\to\infty} \|\psi_{n,m(n)} - u\|_{H^1(Q)} = 0, \tag{3.3.9}$$

and also $\underline{\lim}_{n\to\infty} \|\psi_{n,m(n)} - u\|_{H^1(Q)} = 0$, hence we conclude that

$$\|\psi_{n,m(n)} - u\|_{H^1(Q)} \to 0,\, n \to \infty.$$

From now on we denote by

$$\psi_n = \psi_{n,m(n)}.$$

Now we prove (2), that is

$$\|\psi_n - u\|_{L^2(Q,m)} = \|\psi_n - u\|_{L^2(Q)} + \|\psi_n - u\|_{L^2(S)} \to 0. \tag{3.3.10}$$

The first term in the right hand side of (3.3.10) tends to 0 when $n \to \infty$ since

$$\|\psi_n - u\|_{L^2(Q)} \leq \|\psi_n - u\|_{H^1(Q)}.$$

We now prove that the second term in (3.3.10) tends to 0.

$$\begin{aligned} \|\psi_n - u\|_{L^2(S)} &= \|\widehat{\varphi_n}|_S - \eta_n|_S - u|_S\|_{L^2(S)} \\ &\equiv \|\varphi_n - u|_S\|_{L^2(S)} \leq \|\varphi_n - u|_S\|_{\mathcal{D}(S)}, \end{aligned}$$

and the last term vanishes since $D(0, 1; \mathcal{D}(F))$ is dense in $\mathcal{D}(S)$ (see Proposition 3.3.3). This proves that $\psi_n \to u$ in $L^2(\overline{Q}, m)$.
Now we prove (3):

$$E_S[(u - \psi_n)|_S] = E_S[u|_S - \psi_n|_S] \equiv E_S[u|_S - \varphi_n] \leq \|u|_S - \varphi_n\|_{\mathcal{D}(S)} \to 0.$$

□

3.4 M-convergence of the energy forms

In this section we study the convergence of the approximating energy forms $E^{(h)}$ to the fractal energy E. More precisely we prove the Mosco-convergence of the energy forms in the case of varying Hilbert spaces. The proof relies on the density results for the functions of Section 3.3 . We will follow the notations of Section 2.3.1 and we will use the results therein. We note that this result holds when Q is the cylindrical domain whose lateral boundary is the surface S, with $S = F \times I$, F is the equilateral snowflake.
In this asymptotic behavior the factors σ_h^1 and σ_h^2 have a key role and can be regarded as sort of renormalization factors of the approximating energies. These factors take into account the non rectifiability of the curve F and hence the irregularity of the surface S, and in particular the effect of the d-dimensional length intrinsic to the curve; for details, see [40]. We now

give the definition of M-convergence of forms (see [50]) in the case of varying Hilbert space, by using the definition of Kuwae and Shioya in [29].
We extend the forms E and $E^{(h)}$ on the whole spaces H and H_h respectively as follows:

$$E[u] = +\infty, \text{ for every } u \in H \setminus V(Q, S)$$

and

$$E^{(h)}[u] = +\infty, \text{ for every } u \in H_h \setminus V(Q, S_h)$$

Definition 3.4.1. *A sequence of forms $\{E^{(h)}\}$ M-converges to a form E if*

1. *for every $v_h \in H_h$ weakly converging to $u \in H$ in $\mathcal{H}$*

$$\underline{\lim}_{h\to\infty} E^{(h)}[v_h] \geq E[u] \tag{3.4.11}$$

2. *for every $u \in H$ there exists $\{w_h\}$, with $w_h \in H_h$ strongly converging to $u \in \mathcal{H}$ such that*

$$\overline{\lim}_{h\to\infty} E^{(h)}[w_h] \leq E[u]. \tag{3.4.12}$$

Proposition 3.4.2. *Let $\{v_h\}_{h\in\mathbb{N}}$ be a sequence weakly converging to a vector $u \in H$ in $\mathcal{H}$, then $\{v_h\}_{h\in\mathbb{N}}$ weakly converges to u in $L^2(Q)$ and $\lim_h \delta_h \int_{S_h} \varphi v_h d\sigma = \int_S \varphi u dg$, for every $\varphi \in C$.*

Proof. From Definition 2.3.3 it follows that for every $\varphi_h \in H_h$ strongly converging to $\varphi \in H$

$$\lim_{h\to\infty} \left(\int_{Q_h} v_h \varphi_h d\mathcal{L}_3 + \delta_h \int_{S_h} v_h \varphi_h d\sigma \right) = \int_Q u\varphi d\mathcal{L}_3 + \int_S u\varphi dg. \tag{3.4.13}$$

For every $w \in C$ we set $\varphi_h = w\chi_{Q_h}$ and $\varphi = w\chi_Q$: $\varphi_h \in H_h$ and $\varphi \in H$. We prove that φ_h strongly converges to φ in $\mathcal{H}$. This result follows from Lemma 2.3.7, in fact the first claim holds since

$$\|\varphi_h\|^2_{H_h} = \int_{Q_h} |w|^2 d\mathcal{L}_3, \ \|\varphi\|^2_H = \int_Q |w|^2 d\mathcal{L}_3$$

and Q_h is a family of sets invading Q. By the same argument it follows that

$$(g, \varphi_h)_{H_h} \to (g, \varphi)_H \ \forall g \in C.$$

From (3.4.13) and the choice of φ_h and φ

$$\lim_{h\to\infty} \int_{Q_h} v_h w d\mathcal{L}_3 = \int_Q uw d\mathcal{L}_3, \forall w \in C \tag{3.4.14}$$

The constant sequence $\{w\}$ strongly converges to w in $\mathcal{H}$; choosing $\varphi_h = w$ in (3.4.13) and taking into account (3.4.14), by difference we obtain

$$\lim_{h\to\infty} \delta_h \int_{S_h} wv_h d\sigma = \int_S wudg.$$

We now prove the weak convergence of v_h to u in $L^2(Q)$. We first prove the convergence for every $\phi \in C(Q)$, then the claim will follow by density.

$$\lim_{h\to\infty} \int_Q v_h\phi d\mathcal{L}_3 = \lim_{h\to\infty}(\int_Q v_h\phi\chi_{Q_h} d\mathcal{L}_3 + \int_Q v_h\phi\chi_{Q\setminus Q_h} d\mathcal{L}_3) = \int_Q u\phi d\mathcal{L}_3,$$

since $\phi\chi_{Q\setminus Q_h}$ strongly tends to zero in $\mathcal{H}$ and $\phi\chi_{Q_h}$ strongly converges to $\phi\chi_Q$ in $\mathcal{H}$. $\square$

Proposition 3.4.3. *If v_h weakly converges to u in $H^1(Q)$ and $b \in C(\overline{Q})$, then $\delta_h \int_{S_h} b|v_h|^2 d\sigma \to \int_S b|u|^2 dg$.*

Proof. $|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \int_S b|u|^2 dg| \leq$

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| + |\delta_h \int_{S_h} b|u|^2 d\sigma - \int_S b|u|^2 dg|.$$

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| \leq \delta_h \|b\|_{C(\overline{Q})}(\|v_h - u\|_{L^2(S_h)})(\|v_h + u\|_{L^2(S_h)}) \leq$$
$$\delta_h \|b\|_{C(\overline{Q})}(\|v_h - u\|_{L^2(S_h)})(\|v_h\|_{L^2(S_h)} + \|u\|_{L^2(S_h)}).$$

Since v_h weakly converges in $H^1(Q)$ to u, then v_h strongly converges to u in $H^\alpha(Q)$ for every $\alpha \in (0,1)$. Considering the extension of $(v_h - u)$ to $H^\alpha(\mathbb{R}^3)$, it follows from Theorems 2.1.6 and 2.1.12

$$\delta_h \|v_h - u\|_{L^2(S_h)} \leq C_\alpha \|Ext(v_h - u)\|_{H^\alpha(\mathbb{R}^3)} \leq c\|v_h - u\|_{H^\alpha(Q)}.$$

From these inequalities it follows that

$$|\delta_h \int_{S_h} b|v_h|^2 d\sigma - \delta_h \int_{S_h} b|u|^2 d\sigma| \to 0.$$

Since $u \in H^1(Q)$ there exists a sequence $\{g_n\} \in H^1(Q) \bigcap C(\overline{Q})$ such that $\|g_n - u\|_{H^1(Q)} \to 0$ (see Proposition 4.4 in [23]).

$$|\delta_h \int_{S_h} b|u|^2 d\sigma - \int_S b|u|^2 dg| \leq |\delta_h \int_{S_h} b|u|^2 d\sigma - \delta_h \int_{S_h} b|g_n|^2 d\sigma|$$
$$+ |\delta_h \int_{S_h} b|g_n|^2 d\sigma - \int_S b|g_n|^2 dg| + |\int_S b|g_n|^2 dg - \int_S b|u|^2 dg|.$$

It is possible to estimate from above the first and the third term of the right hand side of this inequality with $\|g_n - u\|_{H^1(Q)}$, and hence we conclude that for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that these two terms are less than $c\varepsilon$.
If we choose $n > n_\varepsilon$, the second term in the right-hand side goes to 0 for h tending to $+\infty$, since H_h converges to H. $\square$

Now we state and proof the main theorem of this Section.

Theorem 3.4.4. *Let $\delta_h = (3^{1-d_f})^h$, $\sigma_h^1 = \sigma_1 c_0 (\delta_h)^{-1}$, $\sigma_h^2 = \sigma_2 c_0 \delta_h$. Let us assume that there exists M > 0 such that $\|a_{ij}^h\|_{L^\infty(Q)} \leq M$, for every $h \in \mathbb{N}$, $i, j = 1, 2, 3$ and that a_{ij}^h converge a.e. in Q to a_{ij}, then the sequence $E^{(h)}$ converges in the sense of Mosco, Kuwae, Shioya to the form E.*

Proof. Condition 1.
We can assume that $v_h \in V(Q, S_h)$, otherwise the inequality (3.4.11) becomes trivial.
Let $v_h \in V(Q, S_h)$, there exists a c independent from h such that

$$\|v_h\|_{H^1(Q_h)} + E_{S_h}[v_h|_{S_h}] + \delta_h \|v_h\|_{L^2(S_h)} \leq C$$

and then $\|v_h\|_{H^1(Q_h)} < C$. For every $h \in \mathbb{N}$ from Theorem 2.1.11 there exists a continuous linear operator $Ext : H^1(Q_h) \to H^1(\mathbb{R}^3)$ such that

$$\|Extv_h\|_{H^1(\mathbb{R}^3)} \leq c\|v_h\|_{H^1(Q_h)} \leq cC.$$

Let $\widehat{v}_h = Extv_h|_Q$, $\widehat{v}_h \in H^1(Q)$ and $\|\widehat{v}_h\|_{H^1(Q)} \leq cC$, thus there exists a subsequence, still denoted by $\widehat{v}_h$ weakly converging to $\widehat{v}$ in $H^1(Q)$ and hence strongly in $L^2(Q)$. By Proposition 3.4.2 it follows that v_h weakly converges to u in $L^2(Q)$.
We want to prove that $\widehat{v} = u$ a.e. that is $\int_Q (\widehat{v} - u)\varphi d\mathcal{L}_3 = 0$ for each $\varphi \in L^2(Q)$.

$$\int_Q (\widehat{v} - u)\varphi d\mathcal{L}_3 = \int_Q (\widehat{v} - \widehat{v}_h + \widehat{v}_h - u)\varphi d\mathcal{L}_3 =$$
$$\int_Q (\widehat{v} - \widehat{v}_h)\varphi d\mathcal{L}_3 + \int_{Q_h} (v_h - u)\varphi d\mathcal{L}_3 + \int_{Q-Q_h} (\widehat{v}_h - u)\varphi d\mathcal{L}_3.$$

Since $\widehat{v}_h \to \widehat{v}$ in $L^2(Q)$ and v_h weakly converges to u in $L^2(Q)$, it follows that the first two terms of right hand side vanish. Moreover, from Holder inequality and since $|Q - Q_h| \to 0$ for $h \to \infty$, $\int_{Q-Q_h} (\widehat{v}_h - u)\varphi d\mathcal{L}_3 \leq \|\varphi\|_{L^2(Q-Q_h)}(\|\widehat{v}_h\|_{L^2(Q)} + \|u\|_{L^2(Q)}) \to 0$.
Now we prove that

$$\underline{\lim}_{h\to\infty} \int_Q \chi_{Q_h} \mathcal{A}^h Dv_h \cdot Dv_h d\mathcal{L}_3 \geq \int_Q \mathcal{A} Du \cdot Du d\mathcal{L}_3.$$

We set $\sqrt{\mathcal{A}} = [c_{ij}]$ and $\sqrt{\mathcal{A}^h} = [c^h_{ij}]$. From the assumptions it follows that

$$|c^h_{ij}| \leq M_1 \text{ for every } i, j,\ c^h_{ij} \to c_{ij} \text{ a.e.}$$

From Severini-Egorov Theorem it follows that $\sum_{i,j=1}^3 c^h_{ij}\chi_{Q_h}$ converges quasi-uniformly to $\sum_{i,j=1}^3 c_{ij}\chi_Q$ and from the weakly convergence of v_h to u in $H^1(Q)$ we deduce that $\sum_{i,j=1}^3 c^h_{ij}\chi_{Q_h}\partial_j v_h$ weakly converges in $L^2(Q)$ to $\sum_{i,j=1}^3 c_{ij}\chi_Q\partial_j u$. Then

$$\underline{\lim}_{h\to\infty} \int_Q \chi_{Q_h}\mathcal{A}^h Dv_h \cdot Dv_h d\mathcal{L}_3 = \underline{\lim}_{h\to\infty} \int_Q |\chi_{Q_h}\sqrt{\mathcal{A}^h} Dv_h|^2 d\mathcal{L}_3 =$$
$$\underline{\lim}_{h\to\infty} \sum_{i=1}^3 \|\sum_{j=1}^3 c^h_{ij}\chi_{Q_h}\partial_j v_h\|^2_{L^2(Q)} \geq \sum_{i=1}^3 \|\sum_{j=1}^3 c_{ij}\chi_Q\partial_j u\|^2_{L^2(Q)}$$

The proof that $\underline{\lim}_{h\to\infty} E_{S_h}[v_h] \geq E_S[u]$ follows from Remark 5.1 in [40].
Thesis follows from the liminf properties of the sum.
Condition 2.
We suppose that $u \in V(Q, S)$, otherwise the inequality (3.4.12) becomes trivial.
Step 1.
We suppose that $u \in C(\overline{Q})$, hence $u \in H$. We extend by continuity u to $\overline{\mathcal{T}}$ and we denote by $\widehat{u}$ this extension.
Following the same approach of [35], we introduce a quasi uniform triangulation τ_h of $\mathcal{T}$

made by equilateral tetrahedrons T_h^j such that the vertices of the prefractal surface S_h are nodes of the triangulation at the $h-th$ level. Let $\mathcal{S}_h$ be the space of all the functions being continuous on $\overline{\mathcal{T}}$ and affine on the tetrahedrons of τ_h. We indicate by $\mathcal{M}_h$ the nodes of τ_h, that is the set of the vertices of all T_h^j. For a given continuous function u, we denote by $I_h u$ the function which is affine on every $T_h^j \in \tau_h$ and which interpolates u in the nodes $P_{j,i} \in \mathcal{M}_h \bigcap \overline{Q}_h$. We put $w_h = I_h \widehat{u}$, and we prove that $\{w_h\}$ strongly converges in $\mathcal{H}$, using the Lemma 2.3.6: we have to prove that $(w_h, v_h)_{H_h} \to (u,v)_H$ for every $\{v_h\}$ weakly converging to v in $\mathcal{H}$. It holds that

$$\|w_h - u\|_{H^1(\mathcal{T})} \to 0$$

for h tending to ∞ (see [20]) and hence $\|w_h - u\|_{H^1(Q)} \to 0$. From Theorem 2.1.6, there exists c indipendent from h such that $\|w_h - u\|_{L^2(S_h)} \leq c(\delta_h)^{-1/2}\|w_h - u\|_{H^1(Q)}$.

$$\begin{gathered} 0 \leq |(w_h, v_h)_{H_h} - (u,v)_H| = |\int_{Q_h} w_h v_h d\mathcal{L}_3 + \delta_h \int_{S_h} w_h v_h d\sigma - \int_Q uv d\mathcal{L}_3 - \int_S uv dg| = \\ |(w_h - u, v_h)_{L^2(Q_h)} + \delta_h \int_{S_h} (w_h - u) v_h d\sigma + (u, v_h)_{H_h} - (u,v)_H| \leq \\ |(w_h - u, v_h)_{L^2(Q_h)}| + |((w_h - u)\sqrt{\delta_h}, \sqrt{\delta_h} v_h)_{L^2(S_h)}| + |(u, v_h)_{H_h} - (u,v)_H| \leq \\ \|w_h - u\|_{L^2(Q)} \|v_h\|_{L^2(Q)} + \sqrt{\delta_h}\|w_h - u\|_{L^2(S_h)} \sqrt{\delta_h}\|v_h\|_{L^2(S_h)} + |(u, v_h)_{H_h} - (u,v)_H|. \end{gathered}$$

Taking into account that v_h weakly converges to v in $\mathcal{H}$, w_h strongly converges to u in $H^1(Q)$ and from the fact that $\sqrt{\delta_h}\|w_h - u\|_{L^2(S_h)} \leq c\|w_h - u\|_{H^1(Q)}$, it follows that right hand side of the above inequality vanishes.

Now we show that the sequence $\{w_h\}$ satisfies the condition 2) of M-convergence. It holds

$$\lim_{h\to\infty} \delta_h \int_{S_h} b|w_h|^2 d\sigma = \int_S b|u|^2 dg.$$

From [34] we have $\overline{\lim}_{h\to\infty} E_{S_h}[w_h] \leq E_S[u]$.

We prove that

$$\overline{\lim}_{h\to\infty} \int_Q \chi_{Q_h} \mathcal{A}^h D w_h \cdot D w_h d\mathcal{L}_3 \leq \int_Q \chi_Q \mathcal{A} D u \cdot D u d\mathcal{L}_3.$$

The thesis follows since

$$\overline{\lim}_{h\to\infty} \int_Q \chi_{Q_h} \mathcal{A}^h D w_h \cdot D w_h d\mathcal{L}_3 = \overline{\lim}_{h\to\infty} \sum_{i=1}^3 \| \sum_{j=1}^3 c_{ij}^h \chi_{Q_h} \partial_j w_h \|_{L^2(Q)}^2$$

and, from the assumptions on c_{ij}^h and on w_h, we deduce that $\sum_{j=1}^3 c_{ij}^h \chi_{Q_h} \partial_j w_h$ converges to $\sum_{j=1}^3 c_{ij} \chi_Q \partial_j u$ in $L^2(Q)$. Then we get

$$\overline{\lim}_{h\to\infty} \int_Q \chi_{Q_h} \mathcal{A}^h D w_h \cdot D w_h d\mathcal{L}_3 = \int_Q \mathcal{A} D u \cdot D u d\mathcal{L}_3.$$

Thesis follows from the limsup properties of the sum.

Step 2.

If $u \in V(Q,S)$, but u is not continuous, from Theorem 3.3.4 there exists $\{\psi_n\} \subset V(Q,S) \bigcap C(\overline{Q})$ such that $\psi_n \to u$ in H, $\|\psi_n - u\|_{V(Q,S)} \to 0$. Let $n \in \mathbb{N}$ fixed such that $\|\psi_n - u\|_{V(Q,S)} \leq \frac{1}{n}$ and $\|\psi_n - u\|_H \leq \frac{1}{n}$. By $\tilde{\psi_n}$ we denote a continuous extension in $\overline{\mathcal{T}}$.

From Step 1 we have that for every fixed $n \in \mathbb{N}$ $I_h\tilde{\psi_n}$ strongly converges to $\tilde{\psi_n}$ in $\mathcal{H}$,$I_h\tilde{\psi_n}$ converges to $\tilde{\psi_n}$ in $H^1(\mathcal{T})$ when $h \to \infty$ and

$$\overline{\lim}_{h\to\infty} E^{(h)}[I_h\tilde{\psi_n}] \leq E[\tilde{\psi_n}].$$

Applying the upper limit for $n \to \infty$ to both sides of the above inequality we obtain

$$\overline{\lim}_{n\to\infty}(\overline{\lim}_{h\to\infty} E^{(h)}[I_h\tilde{\psi_n}]) \leq \overline{\lim}_{n\to\infty} E[\tilde{\psi_n}] = E[u]. \tag{3.4.15}$$

Now we want to apply Corollary 1.16 in [2] for proving that there exists an increasing mapping $h \to n(h)$, such that, denoting $w_h = I_h\tilde{\psi}_{n(h)}$, we have that w_h converges to u in $\mathcal{H}$ and $\overline{\lim}_{h\to\infty} E^{(h)}[w_h] \leq E[u]$. To this aim we have to prove that

$$\overline{\lim}_{n\to\infty}\overline{\lim}_{h\to\infty}|(w_{h,n}, v_h)_{H_h} - (u,v)_H| \leq 0, \tag{3.4.16}$$

for every $\{v_h\}$ weakly converging to v in $\mathcal{H}$.
$|(w_{h,n}, v_h)_{H_h} - (u,v)_H| \leq |(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H + (\tilde{\psi_n} - u, v)_H| \leq$
$|(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H| + \|\tilde{\psi_n} - u\|_H\|v\|_H \leq |(w_{h,n}, v_h)_{H_h} - (\tilde{\psi_n}, v)_H| + \frac{c}{n}$.
Passing to the upper limit for $h \to \infty$, we obtain

$$\overline{\lim}_{h\to\infty}|(w_{h,n}, v_h)_{H_h} - (u,v)_H| \to 0.$$

Then Corollary 1.16 in [2] provides the thesis. □

Now we state a Theorem that follows from Theorem 3.4.4, which is a generalization of Theorem 2.4.1 in [50].

Theorem 3.4.5. *Let $E^{(h)}$ and E be the energy forms defined in 3.2.4 and in 3.2.12, respectively; then the semigroups $\{T_h(t)\}$ associated with the form E^h converge, for every $t > 0$, to the semigroup $T(t)$ associated with the form E, in the sense of Definition 2.3.10.*

4. EVOLUTION VENTTSEL' PROBLEMS

In this chapter we will prove the existence and uniqueness, via a semigroup approach, of the abstract Cauchy problems $(\overline{P})$ and $(\overline{P}_h)$. Then we prove in Theorems 4.2.2 and 4.2.3, that the solutions of $(\overline{P}_h)$ converge in a suitable sense to the solution of $(\overline{P})$; finally we show that the solutions of the abstract problems $(\overline{P})$ and $(\overline{P}_h)$ solve the Venttsel' problems (P), (P_h) formally stated in the Introduction, proved in the Theorems 4.3.1 and 4.3.2 respectively.

4.1 Existence results for the Cauchy problems

Let us consider

$$(\overline{P}) \qquad \begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Au(t) + f(t), & 0 \leq t \leq T \\ u(0) = 0 \end{cases} \tag{4.1.1}$$

and for every $h \in \mathbb{N}$

$$(\overline{P_h}) \qquad \begin{cases} \frac{\mathrm{d}u_h(t)}{\mathrm{d}t} = A_h u_h(t) + f_h(t), & 0 \leq t \leq T \\ u_h(0) = 0 \end{cases} \tag{4.1.2}$$

where $A : \mathcal{D}(A) \subset H \to H$ and $A_h : \mathcal{D}(A_h) \subset H_h \to H_h$ are the infinitesimal generators associated with the energy form E and $E^{(h)}$ respectively. From Theorem 4.3.1 page 149 in [46] we deduce the following existence results.

Theorem 4.1.1. *Let* $0 < \theta < 1$, $f \in C^\theta([0,T]; L^2(\overline{Q}, m))$ *and let*

$$u(t) = \int_0^t T(t-s) f(s) ds, \tag{4.1.3}$$

where $T(t)$ *is the analytic semigroup generated from* A*. Then* u *is the unique strict solution of (4.1.1) , that is*

$$u \in C^1([0,T]; L^2(\overline{Q}, m)) \bigcap C([0,T]; D(A)),$$
$$\frac{du(t)}{dt} = Au(t) + f(t),\ \forall t \in [0,T] \text{ and } u(0) = 0.$$

and there exists c *such that the following inequality holds:*

$$\|u\|_{C^1([0,T];L^2(\overline{Q},m))} + \|u\|_{C([0,T];D(A))} \leq c \|f\|_{C^\theta([0,T];L^2(\overline{Q},m))}. \tag{4.1.4}$$

Theorem 4.1.2. *Let $0 < \theta < 1$, $f_h \in C^\theta([0,T]; H_h)$ and let*

$$u_h(t) = \int_0^t T_h(t-s) f_h(s) ds, \forall h \in \mathbb{N} \tag{4.1.5}$$

where $T_h(t)$ is the analytic semigroup generated by A_h. Then u_h is the unique strict solution of (4.1.2), that is

$$u_h \in C^1([0,T]; L^2(Q, m_h)) \bigcap C([0,T]; D(A_h)),$$
$$\frac{du_h(t)}{dt} = A_h u_h(t) + f_h(t), \forall t \in [0,T] \text{ and } u_h(0) = 0,$$

and there exists C, independent from h, such that the following inequality holds:

$$\|u_h\|_{C^1([0,T];L^2(Q,m_h))} + \|u_h\|_{C([0,T];D(A_h))} \leq C \|f_h\|_{C^\theta([0,T];L^2(Q,m_h))}. \tag{4.1.6}$$

4.2 Convergence of the solutions

This section is devoted to the study of the behavior of u_h when $h \to \infty$. We denote $K_h = L^2([0,T]; H_h)$ and $K = L^2([0,T]; H)$. It holds that K_h converges to K in the sense of definition 2.3.1, where the set $C = C([0,T] \times \overline{Q})$ and Φ_h is the identical operator on C. We denote $\mathcal{K} = \bigcup K_h \bigcup K$. Now we give a characterization of the strong convergence in $\mathcal{K}$.

Proposition 4.2.1. *A sequence $\{u_h\}$ strongly converges to u in $\mathcal{K}$ if one of the following conditions holds:*

1. $\begin{cases} \int_0^T \|u_h(t)\|^2_{H_h} dt \to \int_0^T \|u(t)\|^2_H dt, \\ \int_0^T (u_h(t), \varphi(t))_{H_h} dt \to \int_0^T (u(t), \varphi(t))_H dt \end{cases} \quad \forall \varphi \in C([0,T] \times \overline{Q}).$

2. $\int_0^T (u_h(t), v_h(t))_{H_h} dt \to \int_0^T (u(t), v(t))_H dt,$

for all $\{v_h\}$ weakly converging to v in $\mathcal{K}$.

Theorem 4.2.2. *Let u and u_h be the solutions of the problems $(\overline{P})$ and $(\overline{P_h})$ respectively. Let δ_h be as in Theorem 3.4.4. If for every $t \in [0,T]$, $\{f_h(t)\}$ strongly converges to $f(t)$ in $\mathcal{H}$ and there exists a costant c such that*

$$\|f_h\|_{C^\theta([0,T];H_h)} < c, \forall h \in \mathbb{N} \tag{4.2.7}$$

then

1. *$\{u_h(t)\}$ converges to $u(t)$ in $\mathcal{H}$, for every fixed $t \in [0,T]$*

2. $\{u_h\}$ *converges to* u *in* $\mathcal{K}$.

Proof. In order to prove 1) we use Lemma 2.3.6, hence we have to see that for every $t \in [0,T]$

$$(u_h, v_h)_{H_h} \to (u, v)_H$$

for every sequence $\{v_h\}$, with $v_h \in H_h$ weakly convergent in $\mathcal{H}$ to $v \in H$.
We have

$$(u_h, v_h)_{H_h} = \int_{Q_h} \int_0^t T_h(t-s) f_h(s,P) ds v_h(P) d\mathcal{L}_3 +$$
$$\delta_h \int_{S_h} \int_0^t T_h(t-s) f_h(s,P) ds v_h(P) d\sigma = \int_0^t (T_h(t-s) f_h(s), v_h)_{H_h} ds.$$

From Theorem 3.4.5, since for every $t \in [0,T]$, $f_h(t) \to f(t)$ in $\mathcal{H}$, then

$$T_h(t) f_h(t) \to T(t) f(t) \text{ in } \mathcal{H};$$

Moreover, since v_h weakly converges to v in $\mathcal{H}$ for every $t \in [0,T]$, it follows that

$$(T_h(t-s) f_h(s), v_h)_{H_h} \to (T(t-s) f(s), v)_H.$$

From Lemma 2.3.5, the contraction property of T_h and the assumption (4.2.7) $\|f_h\|_{C^\theta([0,T];H_h)} < c$, we have that there exists a constant c independent from h such that

$$|(T_h(t-s) f_h(s), v_h)_{H_h}| \leq c.$$

The claim follows from dominated convergence Theorem.
Now we prove 2). We note that

$$\|u_h(t)\|_{H_h} \leq c_1 \|f_h\|_{C^\theta([0,T];H_h)} \leq c, \forall t \in [0,T]$$

where the last inequality follows from (4.1.6) and (4.2.7).
Thus the sequence $\{u_h\}$ is equibounded in $[0,T]$ and from 1)

$$\|u_h\|_{H_h} \to \|u(t)\|_H.$$

By applying dominated convergence Theorem we obtain that

$$\|u_h\|_{K_h} \to \|u\|_K.$$

From 1) it follows in particular that for every $t \in [0,T]$

$$(u_h(t), \psi(t))_{H_h} \to (u(t), \psi(t))_H, \forall \psi \in C([0,T] \times \overline{Q}).$$

Since

$$|(u_h(t), \psi(t))_{H_h}| \leq c \|\psi\|_{C([0,T] \times \overline{Q})}.$$

From the dominated convergence Theorem we have

$$(u_h, \psi)_{K_h} \to (u, \psi)_K \ \forall \psi \in C([0,T] \times \overline{Q}).$$

From Proposition 4.2.1 we proved 2). □

Theorem 4.2.3. *With the same assumptions as in Theorem 4.2.2 we have*

1. $\{\frac{du_h}{dt}\}$ *weakly converges to* $\frac{du}{dt}$ *in* $\mathcal{K}$,
2. $\{A_h u_h\}$ *weakly converges to* Au *in* $\mathcal{K}$.

Proof. It holds

$$\sup_{t\in[0,T]} \|\tfrac{du_h}{dt}\|_{H_h} \leq c$$

in particular $\frac{du_h}{dt} \in L^2([0,T];H_h)$ and there exists c independent from h such that $\|\frac{du_h}{dt}\|_{L^2([0,T];H_h)} \leq c, \forall h \in \mathbb{N}$.
From Lemma 2.3.8 there exists a subsequence, still denoted by $\frac{du_h}{dt}$, which weakly converges in $\mathcal{K}$ to a function $v \in \mathcal{K}$.
We have to prove that $v = \frac{du}{dt}$.
From definition of weak convergence we can write

$$(\tfrac{du_h}{dt}, w_h)_{K_h} \to (v, w)_K$$

for every sequence$\{w_h\} \in K_h$, $w_h \to w$ in $\mathcal{K}$.
Choosing $\{w_h\} = \{\varphi(t,P)\}$, where $\varphi \in C^1([0,T];C(\overline{Q}))$, we have

$$\lim_{h\to\infty} \int_Q \int_0^T \tfrac{du_h(t,P)}{dt}\varphi(t,P)dtdm_h = \int_Q \int_0^T v(t,P)\varphi(t,P)dtdm.$$

We integrate by parts and we obtain

$$\int_Q \int_0^T \tfrac{du_h(t,P)}{dt}\varphi(t,P)dtdm_h =$$
$$\int_Q (u_h(T,P)\varphi(T,P) - u_h(0,P)\varphi(0,P))dm_h - \int_Q \int_0^T u_h(t,P)\tfrac{d\varphi(t,P)}{dt}dtdm_h.$$

Passing to the limit in the first term in the right hand side of this equality for $h \to \infty$, we obtain, by 1) in Theorem 4.2.2

$\int_Q (u_h(T,P)\varphi(T,P) - u_h(0,P)\varphi(0,P))dm_h \to \int_Q (u(T,P)\varphi(T,P) - u(0,P)\varphi(0,P))dm.$

It remains to study

$$\lim_{h\to\infty} \int_0^T \int_Q u_h(t,P)\frac{d\varphi(t,P)}{dt}dtdm_h. \tag{4.2.8}$$

It holds that

$$\int_0^T \int_Q u_h(t,P)\tfrac{d\varphi(t,P)}{dt}dtdm_h = (u_h(t), \tfrac{d\varphi(t)}{dt})_{K_h}$$

From 2) in Theorem 4.2.2

$$(u_h(t), \tfrac{d\varphi(t)}{dt})_{K_h} \to (u(t), \tfrac{d\varphi(t)}{dt})_K,$$

hence

$$\int_Q \int_0^T v(t,P)\varphi(t,P)dtdm =$$
$$\int_Q (u(T,P)\varphi(T,P) - u(0,P)\varphi(0,P))dm - \int_Q \int_0^T u(t,P)\frac{d\varphi(t,P)}{dt}dtdm,$$

which implies $v = \frac{du}{dt}$. It remains to prove 2): we recall that

$$A_h u_h = \frac{du_h}{dt} - f.$$

Choosing as in 1) a test sequence $\{w_h\} = \{\varphi\}$, with $\varphi(t,P) \in C^1([0,T]; C(\overline{Q}))$ we get

$$(A_h u_h, \varphi)_{K_h} = (\frac{du_h}{dt} - f, \varphi)_{K_h}.$$

Recalling that $\frac{du_h}{dt}$ weakly converges to $\frac{du}{dt}$ in $\mathcal{K}$, we get the thesis. □

4.3 Strong interpretation

4.3.1 The fractal case

Theorem 4.3.1. *Let u be the solution of the problem (4.1.1) Then for every fixed $t \in [0,T]$*

$$\begin{cases} u_t(t,P) - Lu(t,P) = f(t,P) & for a.e. P \in Q \\ \frac{\partial u}{\partial n_{\mathcal{A}}} \in (B^{2,2}_{\beta}(S))', & \beta = \frac{d_f}{2} \\ u(0,P) = 0 & for P \in S \end{cases}$$

and for every $z \in \mathcal{D}(S)$

$$\langle u_t, z\rangle_{(\mathcal{D}(S))',\mathcal{D}(S)} = -E_S(u|_S, z) - \left\langle \frac{\partial u}{\partial n_{\mathcal{A}}}, z \right\rangle_{(\mathcal{D}(S))',\mathcal{D}(S)} + \langle f, z\rangle_{(\mathcal{D}(S))',\mathcal{D}(S)} - \int_S bu|_S z dg \tag{4.3.9}$$

where $\frac{\partial u}{\partial n_{\mathcal{A}}}$, is the co-normal derivative defined as an element of $(B^{2,2}_{\beta}(S))'$. Moreover $\frac{\partial u}{\partial n_{\mathcal{A}}} \in C([0,T]; (B^{2,2}_{\beta}(S))')$.

Proof. Let us consider $L^2(\overline{Q}, m)$, $dm = d\mathcal{L}_3 + dg$, equipped with the norm $\|u\|_{L^2(\overline{Q},m)} = \|u\|_{L^2(Q,d\mathcal{L}_3)} + \|u\|_{L^2(S,g)}$.
Given $\varphi \in C_0^\infty(Q)$, multiplying both members of (4.1.1) and integrating over Q we obtain

$$\int_Q u_t\, \varphi dm = \int_Q Au\, \varphi dm + \int_Q f\, \varphi dm. \tag{4.3.10}$$

From (3.2.8) we have

$$\int_Q u_t\, \varphi dm = -E(u,\varphi) + \int_Q f\, \varphi dm. \tag{4.3.11}$$

Since φ is compactly supported on Q, then

$$\int_Q \mathcal{A}Du \cdot D\varphi d\mathcal{L}_3 = \int_Q f\, \varphi d\mathcal{L}_3 - \int_Q u_t\, \varphi d\mathcal{L}_3. \tag{4.3.12}$$

Hence it follows that for every fixed $t \in [0,T]$

$$\sum_{i,j=1}^{3} \partial_i(a_{ij}(P)\,\partial_j u(t,P)) = u_t(t,P) - f(t,P), \tag{4.3.13}$$

holds in $D'(Q)$. From the density of $D(Q)$ in $L^2(Q)$ and since the right hand side of (4.3.13) belongs to $L^2(Q)$ for every fixed t in $[0,T]$, we obtain that (4.3.13) holds almost everywhere in Q. Taking into the right hand side belongs to $C([0,T];L^2(Q))$, we deduce that $\sum_{i=1,j}^{3} \partial_i(a_{ij}(P)\partial_j u(t,P)) \in C([0,T];L^2(Q))$, hence $u \in C([0,T];V(Q))$, where

$$V(Q) = \Big\{u \in H^1(Q) : \textstyle\sum_{i,j=1}^{3} \partial_i(a_{ij}\,\partial_j u) \in L^2(Q)\Big\}.$$

Here the derivatives are intended in the distributional sense.
We can prove, proceeding as in [38], that $\frac{\partial u}{\partial n_{\mathcal{A}}} \in C([0,T];(B^{2,2}_\beta(S))')$. The Green formula yields for every $t \in [0,T]$ and for every $\varphi \in H^1(Q)$

$$\left\langle \frac{\partial u}{\partial n_{\mathcal{A}}}, \varphi|_S \right\rangle_{(B^{2,2}_\beta(S))', B^{2,2}_\beta(S)} = \int_Q \mathcal{A}Du(t,P)\cdot D\varphi(P)d\mathcal{L}_3 + \int_Q \sum_{i,j=1}^{3} \partial_i(a_{ij}(P)\,\partial_j u(t,P))\,\varphi d\mathcal{L}_3. \tag{4.3.14}$$

Fix t_0 in $[0,T]$ and consider

$$\|\tfrac{\partial u(t)}{\partial_{n_{\mathcal{A}}}} - \tfrac{\partial u(t_0)}{\partial_{\mathcal{A}}}\|_{(B^{2,2}_\beta(S))'} = \sup_{\theta \in B^{2,2}_\beta(S):\|\theta\|_{(B^{2,2}_\beta(S))'} \le 1} |\langle \tfrac{\partial u(t)}{\partial_{n_{\mathcal{A}}}} - \tfrac{\partial u(t_0)}{\partial_{n_{\mathcal{A}}}}, \theta\rangle_{(B^{2,2}_\beta(S))', B^{2,2}_\beta(S)}|.$$

From (4.3.14) and Schwartz inequality we obtain that

$$\|\tfrac{\partial u(t)}{\partial_{n_{\mathcal{A}}}} - \tfrac{\partial u(t_0)}{\partial_{\mathcal{A}}}\|_{(B^{2,2}_\beta(S))'} \le \|w\|_{H^1(Q)}(|\mathcal{A}|\|D(u(t)-u(t_0))\|_{L^2(Q)} + \|L(u(t)-u(t_0))\|_{L^2(Q)})$$

where $w \in H^1(Q)$ and $w|_S = \theta$, m-a.e. The thesis follows since $u \in C([0,T];V(Q))$.

Now let ψ be in $V(Q,S)$ for every fixed t in $[0,T]$. Multiplying (4.1.1) and integrating over Q, we obtain

$$\int_Q u_t\,\psi d\mathcal{L}_3 + \int_S u_t\,\psi dg =$$
$$-\int_Q \mathcal{A}Du\,D\psi d\mathcal{L}_3 - E_S(u|_S,\psi|_S) - \int_S bu|_S\,\psi|_S dg + \int_Q f\,\psi d\mathcal{L}_3 + \int_S f|_S\,\psi|_S dg.$$

Taking into account (4.3.14), we get

$$\begin{aligned}&\int_Q u_t\,\psi d\mathcal{L}_3 + \int_S u_t\,\psi dg = \\ &-\left\langle \tfrac{\partial u}{\partial n_{\mathcal{A}}}, \psi|_S \right\rangle_{(B^{2,2}_\beta(S))', B^{2,2}_\beta(S)} + \int_Q \textstyle\sum_{i,j=1}^{3} \partial_i(a_{ij}\partial_j u)\,\psi d\mathcal{L}_3 \\ &-E_S(u|_S,\psi|_S) - \int_S bu|_S\,\psi|_S dg + \int_Q f\,\psi d\mathcal{L}_3 + \int_S f|_S\,\psi|_S dg.\end{aligned} \tag{4.3.15}$$

Since $u_t - \sum_{i,j=1}^{3} \partial_i(a_{ij}\partial_j u) - f = 0$ a.e. in Q, we have

$$\int_S u_t\,\psi dg = -\left\langle \frac{\partial u}{\partial n_{\mathcal{A}}}, \psi|_S \right\rangle_{(B^{2,2}_\beta(S))', B^{2,2}_\beta(S)} - E_S(u|_S, \psi|_S) - \int_S bu|_S\,\psi|_S dg + \int_S f|_S\,\psi|_S dg \tag{4.3.16}$$

From Proposition 3.2.5, by proceeding as in Section 6 of [31], we have

$$u_t - \Delta_S u + bu = -\frac{\partial u}{\partial n_{\mathcal{A}}} + f \tag{4.3.17}$$

in $(\mathcal{D}(S))'$. □

4.3.2 The prefractal case

Theorem 4.3.2. *Let u_h be the solution of problem (4.1.2) Then we have for every fixed $t \in [0,T]$*

$$\begin{cases} (u_h)_t(t,P) - L_h u_h(t,P) = f_h(t,P) & for a.e. P \in Q \\ \frac{\partial u}{\partial n_{\mathcal{A}_h}} \in (H^{\frac{1}{2}}(S_h))', & \\ u(0,P) = 0 & in H^{\frac{1}{2}}(S_h) \end{cases}$$

and

$$\delta_h(u_h)_t - \Delta_{S_h} u_h + \delta_h b u_h = -\frac{\partial u_h}{\partial n_{\mathcal{A}_h}} + \delta_h f_h, \tag{4.3.18}$$

in $(H^{\frac{1}{2}}(S_h))'$.
$\frac{\partial u_h}{\partial n_{\mathcal{A}_h}}$ is the inward co-normal derivative and Δ_{S_h} is the piece-wise tangential Laplacian associated to the Dirichlet form E_{S_h}. Moreover $\frac{\partial u_h}{\partial n_{\mathcal{A}_h}} \in C([0,T];(H^{\frac{1}{2}}(S_h))')$.

Proof. The first equality follows by proceeding as in Theorem 4.3.1. From this it follows that for every $t \in [0,T]$

$$u_h(t,\cdot) \in V(Q_h) = \{u_h \in H^1(Q) : \textstyle\sum_{i,j=1}^{3} \partial_i(a^h_{ij}\partial_j u_h) \in L^2(Q_h).\}$$

Proceeding as in section 6.2 of [38] we prove that for every $t \in [0,T]$, $\frac{\partial u_h}{\partial n_{\mathcal{A}_h}} \in (H^{\frac{1}{2}}(S_h))'$. By proceeding as in Theorem 4.3.1 we can prove that for every $t \in [0,T]$ and for every $z \in V(Q,S_h)$

$$\begin{gathered} \delta_h((u_h(t))_t, z)_{L^2(S_h)} - \langle \Delta_{S_h} u_h(t), z\rangle_{(H^{\frac{1}{2}}(S_h))', H^{\frac{1}{2}}(S_h)} + \delta_h(bu_h(t), z)_{L^2(S_h)} = \\ -\left\langle \frac{\partial u_h(t)}{\partial n_{\mathcal{A}_h}}, z \right\rangle_{(H^{\frac{1}{2}}(S_h))', H^{\frac{1}{2}}(S_h)} + \delta_h(f_h(t), z)_{L^2(S_h)} \end{gathered}$$

that is the boundary condition

$$\delta_h(u_h)_t - \Delta_{S_h} u_h + \delta_h b u_h = -\frac{\partial u_h}{\partial n_{\mathcal{A}_h}} + \delta_h f_h$$

holds in the dual of $H^{\frac{1}{2}}(S_h)$ (see [32]). □

4.4 Future works

A possible generalization of the present work could be to study the case of operators in non divergence form, this is a natural extension of the present case, since in Venttsel' problems, appeared for the first time in [60], such operators are involved.
The presence of these operators change completely the framework, the corresponding associated energy forms associated are not symmetric, neither positive. We hope to use the theory developed in [47] for non-symmetric forms and the Mosco-convergence for non symmetric forms (see e.g. [48]) suitably extended to varying Hilbert spaces.

5. APPENDIX

5.1 Whitney decomposition

In this Section we recall the main properties of the Whitney decomposition and we refer to [58] for more details.
In what follows, G will denote an arbitrary non-empty closed set in $\mathbb{R}^n$, $\Omega = \mathcal{C}(G)$ its complement. By a cube we mean a closed cube in R^n, with sides parallel to the axes, and two such cubes will be said to be disjoint if their interiors are disjoint. For such a cube Q, diam(Q) denotes its diameter, and dist(Q, G) its distance from G.

Theorem 5.1.1. *Let G be a closed set in $\mathbb{R}^n$. Then there exists a collection of cubes $\mathcal{G}$ $= \{Q_1, Q_2, ...Q_k, ...\}$ such that*

1. $\bigcup_k Q_k = \Omega$
2. *The Q_k are mutually disjoint,*
3. a_1 *diam*$(Q_k) < dist(Q_k, G) < a_2$ *diam*(Q_k).

The constants a_1 and a_2 are independent of G, in fact we may take $a_1 = 1$ and $a_2 = 4$.

Proof. Consider the lattice of points in $\mathbb{R}^n$ whose coordinates are integer. This lattice determines a mesh $\mathcal{M}_0$, which is a collection of cubes: namely all cubes of unit length, whose vertices are points of the above lattice.
The mesh $\mathcal{M}_0$ leads to a two-way infinite chain of such meshes $\{\mathcal{M}_k\}_{-\infty}^{\infty}$ with $\mathcal{M}_k = 2^{-k}\mathcal{M}_0$. Thus each cube in the mesh $\mathcal{M}_k$ gives rise to 2^n cubes in the mesh $\mathcal{M}_{k+1}$ by bisecting the sides. Each cube in the mesh $\mathcal{M}_k$ has sides of length 2^{-k} and thus of diameter $\sqrt{n}2^{-k}$.
In addition to the meshes $\mathcal{M}_k$ we consider the layers Ω_k, defined by

$$\Omega_k = \{x : c2^{-k} \le dist(x, G) \le c2^{-k+1}\};$$

c is a positive constant to be fixed later. Obviously $\Omega = \bigcup_{k=-\infty}^{\infty} \Omega_k$.
We now make an initial choice of cubes, and denote the resulting collection by $\mathcal{G}_0$. Our choice is made as follows: we consider the cubes of the mesh $\mathcal{M}_k$ (each such cube is of size 2^{-k}), and include a cube of this mesh in $\mathcal{G}_0$ if it intersects Ω_k (the points of the latter are all approximately at a distance 2^{-k} from $\mathcal{G}$). That is we take

$$\mathcal{G}_0 = \bigcup_k \{Q \in \mathcal{M}_k : Q \cap \Omega_k \neq \emptyset\}.$$

We then have

$$\bigcup_{Q \in \mathcal{G}_0} Q = \Omega.$$

For appropriate choice of c

$$diam(Q) \leq dist(Q, G) \leq 4diam(Q), Q \in \mathcal{G}_0 \tag{5.1.1}$$

Let us prove (5.1.1) first. Suppose $Q \in \mathcal{M}_k$; then the diameter of $Q = \sqrt{n}2^{-k}$.
Since $Q \in \mathcal{G}_0$ there exists $x \in Q \cap \Omega_k$. Thus

$$\text{dist}(Q, G) \leq \text{dist}(x, G) \leq c2^{-k+1},$$

and

$$\text{dist}(Q, G) \geq \text{dist}(x, G) \text{ -diam}(Q) > c2^{-k} - \sqrt{n}2^{-k}.$$

If we choose $c = 2\sqrt{n}$ we get (5.1.1).
Then by (5.1.1), the cubes $Q \in \mathcal{G}_0$ are disjoint from G and clearly cover Ω. Therefore (1) is also proved.
Notice that the collection $\mathcal{G}_0$ has all our required properties, except that the cubes in it are not necessarily disjoint. To finish the proof of the theorem we need to refine our choice leading to $\mathcal{G}_0$, eliminating those cubes which were really unnecessary. We require the following simple observation.
Suppose Q_1 and Q_2 are two cubes (taken respectively from the mesh $\mathcal{M}_{k_1}$ and $\mathcal{M}_{k_2}$. Then if Q_1 and Q_2 are not disjoint, one of the two must be contained in the other. (In particular $Q_1 \subset Q_2$ if $k_1 > k_2$.)
Start now with any cube $Q \in \mathcal{G}_0$, and consider the maximal cube in $\mathcal{G}_0$ which contains it. In view of the inequality (5.1.1) for any cube $Q' \in \mathcal{G}_0$, which contains Q in $\mathcal{G}_0$ we have

$$\text{diam}(Q') \leq 4 \text{ diam}(Q)$$

Moreover any two cubes Q' and Q" which contain Q have obviously a non-trivial intersection. Thus by the observation made above each cube $Q \in \mathcal{G}_0$ has a unique maximal cube in $\mathcal{G}_0$ which contains it. By the same token these maximal cubes are also disjoint. We let $\mathcal{G}$ denote the collection of maximal cubes of $\mathcal{G}_0$. Then obviously

1. $\bigcup_{Q \in \mathcal{G}} Q = \Omega$
2. The cubes of $\mathcal{G}$ are disjoint,
3. $\text{diam}(Q) \leq \text{dist}(Q, G) \leq 4 \text{ diam}(Q_k)$.

The Theorem is therefore proved. $\square$

We shall now make a few observations about the family $\mathcal{G}$ of cubes whose existence is guaranteed by Theorem 5.1.1.
Let us say that two distinct cubes of $\mathcal{G}$, Q_1 and Q_2, touch if their boundaries have a common point. (We remind the reader that two distinct cubes of $\mathcal{G}$ always have disjoint interiors.)

Proposition 5.1.2. *Suppose Q_1 and Q_2 touch. Then*

$$(1/4)\, \mathit{diam}(Q_2) \leq \mathit{diam}(Q_1) \leq 4\, \mathit{diam}(Q_2).$$

Proof. We know that $\operatorname{dist}(Q_1, G) \leq 4 \operatorname{diam}(Q_1)$. Then $\operatorname{dist}(Q_2, G) \leq 4 \operatorname{diam}(Q_1) + \operatorname{diam}(Q_1) = 5 \operatorname{diam}(Q_1) >$, since Q_1 and Q_2 touch. But $\operatorname{diam}(Q_2) \leq \operatorname{dist}(Q_2, G)$, therefore $\operatorname{diam}(Q_1) \leq 5 \operatorname{diam}(Q_2)$.
However $\operatorname{diam}(Q_2) = 2^k \operatorname{diam}(Q_1)$ for some integer k, thus

$$\operatorname{diam}(Q_1) \leq 4 \operatorname{diam}(Q_2),$$

and the symmetrical implication proves the proposition. □

We now set $N = (12)^n$. The exact size of N needed in what follows is of no importance; what matters is that it can be chosen to depend only on the dimension N, and in particular to be independent of the closed set G.

Proposition 5.1.3. *Suppose $Q \in \mathcal{G}$. Then there are at most N cubes in $\mathcal{G}$ which touch Q.*

Proof. If the cube Q belongs to the mesh $\mathcal{M}_k$ then as is easily seen, there are 3^n cubes (including Q) which belong to the mesh $\mathcal{M}_k$ and touch Q. Next, each cube in the mesh $\mathcal{M}_k$ can contain at most 4^n cubes of $\mathcal{G}$ of diameter $\geq (1/4) \operatorname{diam}(Q)$. If we combine this with Proposition 5.1.2 we get the proof of Proposition 5.1.3. □

Let now Q_k denote any cube in G. Write x^k as the center of this cube and l_k the common length of its sides. Then of course $\operatorname{diam}(Q_k) = \sqrt{n} l_k$. For any ε, $0 < \varepsilon < 1/4$, which is arbitrary but will be kept fixed in what follows, denote by Q_k^* the cube which has the same center as Q_k but is expanded by the factor $1 + \varepsilon$, that is,

$$Q_k^* = (1 + \varepsilon)[Q_k - x^k] + x^k.$$

Clearly $Q_k \subset Q_k^*$ and the cubes Q_k^* no longer have disjoint interiors.
However the following holds:

Proposition 5.1.4. *Each point of Ω is contained in at most N of the cubes Q_k^*.*

Proof. Let Q and Q_k be two cubes of $\mathcal{G}$. We claim that Q_k^* intersects Q only if Q_k touches Q. In fact consider the union of Q_k with all the cubes in $\mathcal{F}$ which touch Q_k; since the diameters of these cubes are all $\geq (1/4)$ diam(Q_k), it is clear that this union contains Q_k^*. Therefore Q intersects Q_k^* only if Q touches Q_k. However any point $x \in \Omega$, belongs to some cube Q and therefore by Proposition 5.1.3 there are at most N cubes Q_k^* which contain x.
The proof also shows that every point of Ω is contained in a small neighborhood intersecting at most N cubes Q_k^*. □

Now let Q_0 denote the cube of unit length centered at the origin. Fix a C^∞ function φ with the following properties:

1. $0 \leq \varphi \leq 1$;
2. $\varphi(x) = 1, x \in Q_0$;
3. $\varphi(x) = 0, x \notin (1+\varepsilon)Q_0$.

Let φ_k denote the function φ adjusted to the cube Q_k, that is

$$\varphi_k(x) = \varphi(\tfrac{x-x^k}{l_k}).$$

Recall that x^k is the center of Q_k and l_k is the common length of its sides. Notice that therefore

1. $\varphi_k(x) = 1$ if $x \in Q_k$,
2. $\varphi_k(x) = 0$ if $x \notin Q_k^*$.

It is to be observed that for every multi-index $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\alpha_i \in \mathbb{N}$, with $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, we have

$$|(\frac{\partial}{\partial x})^\alpha \varphi_k(x)| \leq A_\alpha(diam(Q_k))^{-|\alpha|}.$$

We now define $\varphi_k^*(x)$ for $x \in \Omega$ by

$$\varphi_k^*(x) = \frac{\varphi(x)}{\Phi(x)}$$

where $\Phi(x) = \sum_k \varphi_k(x)$.
The obvious identity

$$\sum_k \varphi_k^*(x) \equiv 1, x \in \Omega$$

then defines our required partition of unity.

5.2 Bilinear forms and representation of closed bilinear forms

In this section we follow [17], [26] and [55] to which we refer for more details.

Definition 5.2.1. *Let H be a Hilbert space. E is called symmetric bilinear form in H if the following properties hold:*

$$\begin{gathered} E : D(E) \times D(E) \to \mathbb{R},\ D(E) \text{ subspace of } H \\ E(u+v,w) = E(u,w) + E(v,w),\ E(u,v+w) = E(u,v) + E(u,w) \\ aE(u,v) = E(au,v) \\ E(u,v) = E(v,u) \end{gathered} \tag{5.2.2}$$

$a \in \mathbb{R}$, $u, v \in D(E) := \{u \in H : E[u] < \infty\}$. *$D(E)$ is called domain of the form E.*

Definition 5.2.2. *A function $F : H \to [0, +\infty]$ is called quadratic forms if there exists a susbspace D of H and a bilinear form $\varepsilon : D \times D \to [0, +\infty]$ such that*

$$F(u) = E(u,u) \tag{5.2.3}$$

if $u \in D$ e

$$F(u) = +\infty \tag{5.2.4}$$

if $u \in H \setminus D(E)$ The form F it is said generated by ε.
From a quadratic form F it is possible to define a bilinear form E by polarization:

$$\begin{gathered} D(E) = \{u \in H : F(u) < \infty\} \\ E(u,v) = \frac{1}{2}(F(u+v) - F(u) - F(v))\ \forall u, v \in D(E). \end{gathered}$$

Following [26] Chapter 6, Section 1.3 we give the following definition:

Definition 5.2.3. *Let E be a bilinear form in H. A sequence $\{u_n\}$, is said E-convergent to $u \in H(u_n \to_E u)$ if*

$$u_n \in D(E)\ u_n \to u \text{ in } H \text{ and } E[u_n - u_m] \to 0.$$

for $n, m \to \infty$
We note that u is not necessarily an element of $D(E)$.

Definition 5.2.4. *A form E in H is said closed if*

$$u_n \to_E u \Rightarrow u \in D(E) \text{ and } E[u_n - u] \to 0.$$

Definition 5.2.5. *A symmetric form E is said Markovian if the following conditions hold: For each $\varepsilon > 0$, there exists a real function $\phi_\varepsilon(t)$, $t \in \mathbb{R}$, such that*

$$\phi_\varepsilon(t) = t, \forall t \in [0,1], -\varepsilon \leq \phi_\varepsilon(t) \leq 1+\varepsilon, \forall t \in \mathbb{R}$$

$$0 \leq \phi_\varepsilon(t^{'}) - \phi_\varepsilon(t) \leq t^{'} - t, \forall t < t^{'}$$

$$u \in D(E) \Longrightarrow \phi_\varepsilon(u) \in D(E), E(\phi_\varepsilon(u), \phi_\varepsilon(u)) \leq E(u,u).$$

We say that a symmetric form is a Dirichlet form if it is a bilinear, closed and Markovian form.
We now state a stronger condition which implies the condition in Definition 5.2.5:

Proposition 5.2.6. *If the following condition holds:*

$$u \in D(E), v = \inf(\sup(u,0),1) \Longrightarrow v \in D(E), E(v,v) \leq E(u,u)$$

then E is a Markovian form.
We note that if E is a Dirichlet form, then $D(E)$ is a pre-Hilbert space with the intrinsic norm $||u||^2_{D(E)} = ||u||^2_H + E[u]$.

Remark 5.2.7. *$u_n \in D(E)$ is E-converging if and only if u_n is a Cauchy sequence in $(D(E), (.)_E)$.*

From this we have that the Definition 5.2.4 is equivalent to the following one:

Definition 5.2.8. *A form E in H is said closed if*

$u_n \in D(E)$, $(u_n - u_m, u_n - u_m)_E \to 0$, *when* $n, m \to \infty$ *implies* $\exists u \in D(E)$ *such that* $||u_n - u||_E \to 0$ *when* $n \to \infty$.

Now we recall the representations of closed, symmetric, bilinear forms (see Theorem 2.1 in Chapter 6 of [26]). We start recalling the representation Theorem for bounded closed forms (see Chapter 5, Section 2.1 in [26]): Let H be a Hilbert space with scalar product $(\cdot,\cdot)_H$ and norm $||_H$

Theorem 5.2.9. *Let $E(u,v)$ be a bilinear symmetric bounded form in H. Then there exists a unique, bounded linear, operator such that*

$$E(u,v) = (Au, v)_H$$

for $u, v \in H$

Proof. It is a straightforward consequence of Riesz-Frechet Theorem. □

From this Theorem we deduce the representation Theorem for closed, bilinear forms.

Theorem 5.2.10. *Let $E(u, v)$ be a densely defined, symmetric, closed, bilinear form in H. There exists a positive, self-adjoint operator A such that*

1. *$\mathcal{D}(A) \subset \mathcal{D}(E)$ and*
$$E(u, v) = (Au, v)_H \tag{5.2.5}$$
for every $u \in \mathcal{D}(A)$ and $v \in \mathcal{D}(E)$;

2. *$\mathcal{D}(A)$ is a core of $\mathcal{D}(E)$;*

3. *if $u \in \mathcal{D}(E)$, $w \in H$ and*
$$E(u, v) = (w, v)_H \tag{5.2.6}$$
holds for every v belonging to a core of $\mathcal{D}(E)$, then $u \in D(A)$ and $Au = w$. The operator A is uniquely determined by the condition 1.

Proof. Let H_E be the associated Hilbert space into which $\mathcal{D}(E)$ is converted by introducing the inner product
$$(u, v)_E = E(u, v) + (u, v)_H$$
Consider the form $E_1 = E + I$, where I is the identity operator on H. E_1 as well as E is a bounded form on H_E. There is a closed, bounded operator B: $\mathcal{D}(B) \subset H_E$ such that
$$E_1(u, v) = (Bu, v)_E, \tag{5.2.7}$$
$u \in \mathcal{D}(B), v \in H_E = \mathcal{D}(E)$. Since $\|u\|_E^2 = E_1[u] = (Bu, u)_E \leq \|Bu\|_E \|u\|_E$, we have
$$\|u\|_E \leq \|Bu\|_E.$$
Hence B has a bounded inverse $B^{(-1)}$ with closed domain in H_E. This domain is the whole of H_E so that $B^{(-1)} \in \mathcal{B}(H_E)$ with $\|B^{(-1)}\|_E \leq 1$. To prove this, it suffices to show that $u \in H_E$ orthogonal in H_E to $D(B^{(-1)} = R(B)$ is zero. This is obvious from $\|u\|_E^2 = (Bu, u)_E$= 0.

For any fixed $u \in H$, consider the semilinear form $v \to l_u(v) = (u, v)$ defined for $v \in H_E$. l_u is a bounded form on H_E with
$$|l_u(v)| \leq \|u\| \|v\| \leq \|u\| \|v\|_E.$$
By the Riesz Theorem, there is a unique $u' \in H_E$ such that $(u, v) = l_u(v) = (u', v)_E$, $\|u'\|_E \leq \|u\|$.

We now define an operator T by
$$Tu = B^{-1}u'.$$

T is a linear operator with domain H and range in H_E. Regarded as an operator in H, T belongs to $\mathcal{B}(H)$ with $\|T\| \leq 1$, for $\|Tu\| = \|B^{-1}u'\| \leq \|B^{-1}u'\|_E \leq \|u'\|_E \leq \|u\|$.
It follows from the definition of T that

$$(u,v) = (u',v)_E = (BTu,v)_E = E_1(Tu,v) = (E+I)(Tu,v). \tag{5.2.8}$$

Hence

$$E(Tu,v) = (u - Tu, v), \tag{5.2.9}$$

$u \in H, v \in H_E = \mathcal{D}(E)$.
T is invertible, for $Tu = 0$ implies by (5.2.8) that $(u,v) = 0$ for all $v \in \mathcal{D}(E)$ and $\mathcal{D}(E)$ is dense in H. On writing $w = Tu$, $u = T^{-1}w$ in (5.2.9), we get

$$E(w,v) = ((T^{-1} - I)w, v) = (Aw, v),$$

where $A = T^{-1} - 1$, for every $w \in \mathcal{D}(A) = R(T) \subset \mathcal{D}(E)$ and $v \in \mathcal{D}(E)$. This proves 1) of the Theorem.
A is a closed operator in H since $T \in \mathcal{B}(H)$.
To prove 2) of Theorem, it suffices to show that $\mathcal{D}(A) = R(T)$ is dense in H_E. Since B maps H_E onto itself bicontinuously, it suffices to show that $BR(T) = R(BT)$ is dense in H_E. Let $v \in H_E$ be orthogonal in H_E to $R(BT)$. Then (5.2.8) shows that $(u,v) = 0$ for all $u \in H$ and so $v = 0$. Hence $R(BT)$ is dense in H_E. It is convenient at this point to consider E^*, the adjoint form of E. Since E^* is also densely defined and closed, we can construct a linear operator A', associated to E^* in the same way as we constructed T from E.
For any $u \in \mathcal{D}(E^*) = \mathcal{D}(E)$ and $v \in \mathcal{D}(A')$, we have then

$$E^*(v,u) = (A^{'}v, u) or E(u,v) = (u, A^{'}v). \tag{5.2.10}$$

In particular let $u \in \mathcal{D}(A) \subset \mathcal{D}(E)$ and $v \in \mathcal{D}(A^{'}) \subset \mathcal{D}(E)$.
(5.2.5) and (5.2.10) give $(Au, v) = (u, A^{'}v)$. This implies that $A^{'} \subset A^*$. But since A^* and $A^{'}$ are both m-sectorial (which implies that they are maximal accretive), we must have $A^{'} = A^*$ and hence $A^{'*} = A$ too. This leads to a simple proof of 3) of the Theorem. If (5.2.6) holds for all v of a core of E, it can be extended to all $v \in \mathcal{D}(E)$ by continuity. Specializing v to elements of $D(A^{'})$, we have then $(u, A'v) = E(u,v) = (w,v)$. Hence $u \in \mathcal{D}(A^{'*}) = \mathcal{D}(A)$ and $w = A^{'*}u = Au$ by the definition of $A^{'*}$. □

5.3 Energy form and Lagrangian on the equilateral snowflake

5.3.1 Energy form on the snowflake

In this section we recall the construction of the energy form on the snowflake; the main reference for this construction is [16]. For the case of scale irregular sets, we mainly refer to

[51] and the references therein.
In this section we use the notations of the section 1.2.
For any function $u : V_* \to \mathbb{R}$ we define

$$\mathcal{E}_h[u] = \frac{1}{2} 4^h \sum_{P \in V_h} \sum_{Q \sim_h P} (u(P) - u(Q))^2 \tag{5.3.11}$$

where $P \sim_h Q$ means that Q is a h-neighbor of P, that is there exists a $h-$tuple of indices $j_1, ..., j_h \in \{1, ..., 4\}$ such that P, $Q \in V_{j_1,...,j_h}$. It can be shown (see [30]) that the sequence $\{\mathcal{E}_n[u]\}_{n \geq 0}$ is non-decreasing, the limit of the right-hand side of (5.3.11) exists and the limit form

$$\mathcal{E}[u] = \lim_{h \to \infty} \mathcal{E}_h[u] \tag{5.3.12}$$

is non trivial with domain

$$\mathcal{D}_*(\mathcal{E}) = \{u : V_* \to \mathbb{R} | \mathcal{E}[u] < \infty\}$$

. Every function $u \in \mathcal{D}_*(\mathcal{E})$ can be uniquely extended to an element of $\mathcal{C}(K)$. We denote this extension still by u and set

$$\mathcal{D} = \{u \in \mathcal{C}(K) : \mathcal{E}[u] < \infty\}$$

where $\mathcal{E}[u] = \mathcal{E}[u|_{V_*}]$. Hence $\mathcal{D} \subseteq \mathcal{C}(K) \subseteq L^2(K, \mu)$, where $L^2(K, \mu)$ is the Hilbert space of square summable functions on K with respect to the self-similar measure μ.
We define the space $\mathcal{D}(\mathcal{E})$ as completion of $\mathcal{D}$ in the norm

$$\|u\|_{\mathcal{E}} = (\|u\|^2_{L^2(K,\mu)} + \mathcal{E}[u])^{1/2}. \tag{5.3.13}$$

$\mathcal{D}(\mathcal{E})$ is injected in $L^2(K, \mu)$ and is a Hilbert space with scalar product associated to norm (5.3.13). Then we extend $\mathcal{E}$ as usual on the completed space $\mathcal{D}(\mathcal{E})$.
By $\mathcal{E}(\cdot, \cdot)$ we denote the bilinear form defined on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ by polarization

$$\mathcal{E}(u, v) = \frac{1}{2}(\mathcal{E}[u + v] - \mathcal{E}[u] - \mathcal{E}[v])(u, v \in \mathcal{D}(\mathcal{E}))$$

. It is easy to see that, for any pair $u, v \in \mathcal{D}(\mathcal{E})$, $\mathcal{E}(u, v)$ is the limit of the sequence $(\mathcal{E}_n(u, v))$ given by

$$\mathcal{E}_h(u, v) = \frac{1}{2} 4^h \sum_{P \in V_h} \sum_{Q \sim_h P} [u(P) - u(Q)][v(P) - v(Q)]. \tag{5.3.14}$$

5.3.2 Lagrangian on Koch curve

We observe that the approximating energy forms $\mathcal{E}_h$ on V_h, defined in 5.3.14, can be written as

$$\mathcal{E}_h(u, v) = \int\limits_{V_h} \nabla_h u \cdot \nabla_h v d\mu^h \tag{5.3.15}$$

where μ^h is the discrete measure given in (1.2.3). For every $h \geq 0$, μ^h is a measure on K supported on V_h,and for any $P \in V_h$ the discrete gradient is given by

$$\nabla_h u \cdot \nabla_h v(P) = \tfrac{1}{2} \textstyle\sum_{Q \sim_h P} \frac{u(P)-u(Q)}{|P-Q|^\delta} \frac{v(P)-v(Q)}{|P-Q|^\delta},$$

$u, v \in \mathcal{D}(\mathcal{E})$, $\delta = \frac{\ln 4}{\ln 3}$ (see [52]).

Proposition 5.3.1. *Let A be any subset of K. For every $u, v \in \mathcal{D}(\mathcal{E})$ the sequence of measures given by*

$$\mathcal{L}_K^{(h)}(u,v)(A) = \int\limits_{A \cap V_h} \nabla_h u \cdot \nabla_h v d\mu^h, \tag{5.3.16}$$

$h \geq 0$, weakly converges in $(\mathcal{C}(K))'$ to a signed finite Radon measure $\mathcal{L}_K(u,v)$ on K as $h \to \infty$, called the Lagrangian measure on K. Moreover

$$\mathcal{E}(u,v) = \textstyle\int_K d\mathcal{L}(u,v),\ u, v \in \mathcal{D}(\mathcal{E}).$$

Proof. Let us restrict ourselves to the quadratic case. Fix $u \in \mathcal{D}(\mathcal{E})$, and set $\mathcal{L}_K^{(n)}[u] = \mathcal{L}_K^{(h)}(u,u)$, $n \geq 0$. From (5.3.15) and (5.3.12) it follows that $(\mathcal{L}_K^{(h)}[u](K))_{h \geq 0}$ is a uniformly bounded sequence, in fact

$$\mathcal{L}_K^{(h)}[u](K) = \int\limits_K d\mathcal{L}_K^{(h)}[u] = \mathcal{E}_h[u] \leq \mathcal{E}[u] < \infty,$$

$h \geq 0$. Let $h \in \mathbb{N}$ be fixed. It can be easily proved that, for every $u \in \mathcal{D}(\mathcal{E})$ and for every $\varphi \in \mathcal{D}(\mathcal{E}) \bigcap C_0(K)$, the following identity holds:

$$\int\limits_{V_h} \varphi d\mathcal{L}_K^{(h)}[K] = \mathcal{E}_h(\varphi u, u) - \frac{1}{2}\mathcal{E}_h(\varphi, u^2). \tag{5.3.17}$$

As the energy form $\mathcal{E}[u]$ is a Dirichlet form of diffusion type, it admits an integral representation (see [41]): there exists a unique positive Radon measure, which we call $\mathcal{L}[u]$, such that $\mathcal{E}[u] = \int_K d\mathcal{L}_K[u]$ and which is uniquely defined by

$$\int\limits_K \varphi d\mathcal{L}_K[K] = \mathcal{E}(\varphi u, u) - \frac{1}{2}\mathcal{E}(\varphi, u^2), \tag{5.3.18}$$

$\varphi \in \mathcal{D}(\mathcal{E}) \bigcap C_0(K)$ (see [50]). Passing to the limit as $n \to \infty$ in (5.3.17), from (5.3.12), taking into account the regularity of the form, it follows that the right-hand of (5.3.17) tends to the right-hand side of (5.3.18). Hence we have proved that

$$\mathcal{L}_K^{(h)}[u] \rightharpoonup \mathcal{L}_K[u], \tag{5.3.19}$$

$h \to \infty$. The signed Radon measure $\mathcal{L}_K^{(h)}(u,v)$ is given by polarization:

$$\mathcal{L}_K^{(h)}(u,v) = \frac{1}{2}\left\{\mathcal{L}_K^{(h)}[u+v] - \mathcal{L}_K^{(h)}[u] - \mathcal{L}_K^{(h)}[v]\right\}.$$

These are Radon measures on K uniquely associated with every $u, v \in \mathcal{D}(\mathcal{E})$. The weak convergence of the sequence $\mathcal{L}_K^{(h)}(u,v)_{h\geq 0}$ to the signed Radon measure $\mathcal{L}_K(u,v)$ for any $u, v \in \mathcal{D}(\mathcal{E})$ follows from the polarization formula and (5.3.19) (see [50]).

□

Remark 5.3.2. *The measure-valued map $\mathcal{L}_K$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ is bilinear, symmetric and positive. This measure-valued Lagrangian takes on the fractal K the role of the Euclidean Lagrangian $d\mathcal{L}(u,v) = Du \cdot Dvdx$. We note that in the case of the Koch curve the Lagrangian $\mathcal{L}_K$ is absolutely continuous with respect to the volume measure μ (see [9]).*

5.3.3 Lagrangian and energy form on the snowflake

We assume that we are given a Koch snowflake F as described in section 1.2 of the Chapter 1. We want to regard F as a "fractal manifold". We cover the snowflake by sets $U_i (i \geq 1)$, which are open subsets of the snowflake and which can be mapped by a corresponding set of homeomorphism $\{\phi_i\}_{i\geq 1}$ to certain "fractal reference sets ". Here "open in the snowflake" means open with respect to the trace toplogy on F of the Euclidean one on $\mathbb{R}^2$. We choose $U_i = \overset{\circ}{K}_i$, $i = 1, ..., 6$ and we define the mappings $\phi_i : \mathbb{R}^2 \to \mathbb{R}^2$ as uniquely determined orientation preserving Euclidean motions such that every ϕ_i maps the set K_i to the reference Koch curve K. A such map ϕ_i is given as a composition of a rotation and a translation of the plane: $\phi_i(P) = e^{i\theta_i} + b_i$ $i = 1, ..., 6$, where θ_i is the rotational angle and $b_i \in \mathbb{R}^2$ is a vector; we note $\phi_i(V_0) = V_0$. By means of these functions we choose the maps $\psi_j^{(i)}$, $(j = 1, ..., 4; i = 1, ..., 6)$ as

$$\psi_j^{(i)}(\cdot) = \phi_i^{-1}(\psi_j(\phi_i(\cdot))).$$

Lemma 5.3.3. *For any $h \geq 1$ and $i = 1, ..., 6$ the following holds: P and Q are h-neighbors in $V_h^{(i)}$ if and only if $\phi_i(P)$ and $\phi_i(Q)$ are h-neighbors in V_h. Moreover, for any $h \geq 1$ and $i = 1, ..., 6$, the map $\phi_i^{-1} : K \to K_i$ preserves the property on h-neighborhood in V_h.* Let $\mathcal{L}_K$ be the Lagrangian on the Koch curve. We introduce the space

$$\mathcal{D}_F = \left\{ w : F \to \mathbb{R} | w \circ \phi_i^{-1} \in \mathcal{D}(\mathcal{E}) \forall i = 1, ..., 6 \right\} \tag{5.3.20}$$

Let w, z be two given functions in $\mathcal{D}(F)$ defined on F. We want to define a measure $\mathcal{L}_F(w,z)$ on F.

Definition 5.3.4. *Let A be a Borel set of K_i. We introduce the measure valued Lagrangian $\mathcal{L}_F(u,v)$ of the set A as image measure (see [15]) of the measure $\mathcal{L}_F(w \circ \varphi_i^{-1}, z \circ \varphi_i^{-1})$ under the map φ_i^{-1}, that is*

$$\mathcal{L}_F(w,z)(A) = \mathcal{L}_K(w \circ \varphi_i^{-1}, z \circ \varphi_i^{-1})(\varphi_i(A))$$

Proposition 5.3.5. *The above definition of the Lagrangian $\mathcal{L}_F$ is independent of the choice of the sets K_i, i.e. if $A \subset K_i \cap K_j (i, j = 1, ..., 6; i \neq j)$, then*

$$\mathcal{L}_K(w \circ {\phi_i}^{-1}, z \circ {\phi_i}^{-1})(\phi_i(A)) = \mathcal{L}_K(w \circ {\phi_j}^{-1}, z \circ {\phi_j}^{-1})(\phi_j(A)) \tag{5.3.21}$$

for all $w, z \in \mathcal{D}(F)$

Proof. Choose two functions $w, z \in \mathcal{D}(F)$ and two indices $i \neq j$. From Proposition 5.3.1 it follows that $\mathcal{L}_K$ is the weak limit of ${\mathcal{L}_K}^{(h)}$. In order to prove (5.3.21) it is sufficient to show that, for any $h \geq 1$ and for any $P \in K_i \cap K_j \cap \mathcal{L}_h$, the discrete gradient satisfies

$$\nabla_h(w \circ {\phi_i}^{-1}) \cdot \nabla_h(z \circ {\phi_i}^{-1})(\phi_i(P)) = \nabla_h(w \circ {\phi_j}^{-1}) \cdot \nabla_h(z \circ {\phi_j}^{-1})(\phi_j(P)).$$

From (5.3.20) we have that the functions $u = w \circ {\phi_i}^{-1}$ and $v = z \circ {\phi_i}^{-1}$, acting from K to $\mathbb{R}$, are in $\mathcal{D}(\mathcal{E})$.Set $R = \phi_i(P)$. Then $R \in K \cap V_h$, and we have to show that, for any $h \geq 1$,

$$\nabla_h(u) \cdot \nabla_h(v)(R) = \nabla_h(u \circ (\phi_i \circ {\phi_j}^{-1})) \cdot \nabla_h(v \circ (\phi_i \circ {\phi_j}^{-1}))((\phi_j \circ {\phi_i}^{-1})(R)) \tag{5.3.22}$$

holds. Setting $k = \phi_j \circ {\phi_i}^{-1}$, the right-hand side of (5.3.22) is given by

$$\sum_{Q \sim_h k(R)} \frac{(u \circ k^{-1})(k(R)) - (u \circ k^{-1})(Q)}{|k(R) - Q|^\delta} \frac{(v \circ k^{-1})(k(R)) - (v \circ k^{-1})(Q)}{|k(R) - Q|^\delta}$$

$$= \sum_{Q' : k(Q') \sim_h k(R)} \frac{u(R) - u(Q')}{|k(R) - k(Q')|^\delta} \frac{v(R) - v(Q')}{|k(R) - k(Q')|^\delta}$$

$$= \sum_{Q' \sim_h R} \frac{u(R) - u(Q')}{|R - Q'|^\delta} \frac{v(R) - v(Q')}{|R - Q'|^\delta}$$

where the last two equalities follow from Lemma 5.3.3. The last sum equals to the left-hand side of 5.3.22. □

Definition 5.3.6. *If B is an arbitrary Borel subset of F, it can be regarded as disjoint union of sets $B_1, ..., B_6$ defined by $B_i = B \cap C_{i,i+1} (i = 1, ..., 5)$ and $B_6 = B \cap C_{6,1}$, where $C_{i,i+1}$ denotes the set of all points of F located between x_i and x_{i+1}, including x_i and excluding x_{i+1} and $C_{6,1}$ denotes the set of all points between x_6 and x_1, including x_6 and excluding x_1. Then any of the sets B_i is contained in one of the sets $K_1, ..., K_6$, and we define*

$$\mathcal{L}_F(w, z)(B) = \sum_{i=1}^{6} \mathcal{L}_F(w, z)(B_i).$$

$\mathcal{L}_F$ is defined on $\mathcal{D}(F) \times \mathcal{D}(F)$.

We define the energy form on the fractal snowflake F in terms of its local energy measure $\mathcal{L}_F$.

Definition 5.3.7. *We introduce on $\mathcal{D}(F) \times \mathcal{D}(F)$ the symmetric bilinear form*

$$\mathcal{E}_F(u,v) = \int_F d\mathcal{L}_F(u,v)(u,v \in \mathcal{D}_F). \tag{5.3.23}$$

We note that

$$\mathcal{E}_F(u,v) = \sum_{i=1}^{3} \int_{K_i} d\mathcal{L}_F(u,v) = \sum_{i=4}^{6} \int_{K_i} d\mathcal{L}_F(u,v)$$

as follows from Remark 5.3.2 in this simpler situation.

5.3.4 A different definition of the energy form on F

Now we think the set F as the union of three Koch curves.
We recall that the energy form on one of these curves, for example K_1, is the following: for any function u: $V_* \to \mathbb{R}$ we set

$$\mathcal{E}_h^1[u] = \frac{1}{2} 4^h \sum_{P \in V_h^{(1)}} \sum_{Q \sim_h P} (u(P) - u(Q))^2.$$

On

$$\mathcal{D}_*(\mathcal{E}^{(1)}) = \left\{ u : V_*^{(1)} \to \mathbb{R} | \lim_{h \to \infty} \mathcal{E}_h^{(1)}[u] < \infty \right\},$$

we set

$$\mathcal{E}^{(1)}[u] = \lim_{h \to \infty} \mathcal{E}_h^1[u].$$

It can be proved that $(\mathcal{E}^{(1)}, \mathcal{D}(\mathcal{E}^{(1)}))$ is a strongly local Dirichlet form on $L^2(K_1, \mu_1)$ and $\mathcal{D}(\mathcal{E}^{(1)})$ is a Hilbert space equipped with the norm $(\| \cdot \|^2_{L^2(K_1,\mu_1)} + \mathcal{E}^{(1)}[\cdot])^{\frac{1}{2}}$.
In a similar way, the energy forms $\mathcal{E}^{(2)}, ..., \mathcal{E}^{(6)}$ on $K_2, ..., K_6$ can be obtained as the limits of $(\mathcal{E}_h^{(2)})_{h \geq 1}, ..., (\mathcal{E}_h^{(6)})_{h \geq 1}$. The domains of these strongly local Dirichlet energy forms are denoted by $\mathcal{D}(\mathcal{E}^{(2)}), ..., \mathcal{D}(\mathcal{E}^{(6)})$ and the corresponding Lagrangian on K_i by $\mathcal{L}_{K_i}[\cdot]$.
We define now the energy form on F: for any $u : \mathcal{V}_* \to \mathbb{R}$

$$\tilde{\mathcal{E}}_h[u] = \frac{1}{2} 4^h \sum_{P \in \mathcal{V}_h} \sum_{Q \sim_h P} (u(P) - u(Q))^2$$

$h \geq 1$. $(\tilde{\mathcal{E}}_h[u])_{h \geq 1})$ is a sequence non-decreasing in h. We introduce the domain

$$\tilde{\mathcal{D}} = \left\{ u \in C(F) | \tilde{\mathcal{E}}_F[u] := \lim_{h \to \infty} \tilde{\mathcal{E}}_h[u] < \infty \right\}.$$

Hence $\tilde{\mathcal{D}} \subseteq C(F) \subseteq L^2(F, \mu_F)$. We define the space $\mathcal{D}(\tilde{\mathcal{E}})$ as the completion of $\tilde{\mathcal{D}}$ in the norm

$$\|u\|_{\mathcal{D}(\tilde{\mathcal{E}}_F)} = \|u\|^2_{L^2(F,\mu_F)} + \tilde{\mathcal{E}}_F[u])^{\frac{1}{2}}. \tag{5.3.24}$$

$\mathcal{D}(\tilde{\mathcal{E}}_F)$ is injected into $L^2(F, \mu_F)$ and is a Hilbert space with scalar product associated to the norm (5.3.24).

Theorem 5.3.8. *A function u is in $\mathcal{D}(\tilde{\mathcal{E}}_F)$ if and only if $u \in C(F)$ and $u|_{K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$ $(i = 1, ..., 6)$. Moreover, it holds*

$$\tilde{\mathcal{E}}_F[u] = \sum_{i=1}^{3} \mathcal{E}^{(i)}[u|_{K_i}] = \sum_{i=4}^{6} \mathcal{E}^{(i)}[u|_{K_i}] \tag{5.3.25}$$

Proposition 5.3.9. *$(\tilde{\mathcal{E}}_F, \mathcal{D}(\tilde{\mathcal{E}}_F))$ is a strongly local, closed, regular Dirichlet form on $L^2(F, \mu_F)$.*

Proof. The result follows from Theorem 5.3.8 and the corresponding properties of $\mathcal{E}^{(i)}$ on K_i. □

Lemma 5.3.10. *For any $u \in \mathcal{D}_F$ we have $u|_{K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$,*

$$\int_{K_i} d\mathcal{L}_F[u] = \mathcal{E}^{(i)}[u|_{K_i}] \tag{5.3.26}$$

and $\mathcal{L}_{K_i}[u] = \mathcal{L}_F[u]|_{K_i}$, $i = 1, ..., 6$.

Proof. We prove the Lemma only for the case $i = 1$.
We consider $\mathcal{L}_F[u]|_{K_i}$ which is given by $\mathcal{L}_K[u \circ \phi_1^{-1}]$. We recall that, for $u \circ \phi_1^{-1}\mathcal{D}(\mathcal{E})$, $\mathcal{L}_K[u \circ \phi_1^{-1}]$ is the weak limit of the sequence $(\mathcal{L}_K^{(h)}[u \circ \phi_1^{-1}])$ defined in (5.3.16). Hence it can be written

$$\int_{K_i} d\mathcal{L}_F[u] = \int_K d\mathcal{L}_K[u \circ \phi^{-1}] = \lim_{h\to\infty} \int_{V_h} d\mathcal{L}_K^{(h)}[u \circ \phi^{-1}]$$

$$= \frac{1}{2} \lim_{h\to\infty} \sum_{P\in V_h} \sum_{Q\in V_h : Q\sim_h P} \frac{(u(\phi_1^{-1}(P)) - u(\phi_1^{-1}(Q)))^2}{|P-Q|^{2\delta}} = \frac{1}{2} \lim_{h\to\infty} \sum_{P'\in V_h} \sum_{Q'\in V_h : Q'\sim_h P'} \frac{(u(P') - u(Q'))^2}{|P'-Q'|^{2\delta}}$$

where the last equality follows from the fact that $\phi^{-1} : K \to K_1$ preserves h-neighborhood. The last limit is finite and from this it can be deduced that $u|_{K_1} \in \mathcal{E}^{()}$ and that

$$\frac{1}{2} \lim_{h\to\infty} \sum_{P'\in V_h} \sum_{Q'\in V_h : Q'\sim_h P'} \frac{(u(P') - u(Q'))^2}{|P'-Q'|^{2\delta}} = \mathcal{E}^{(1)}[u|_{K_1}].$$

□

Theorem 5.3.11. *A function $u : F \to \mathbb{R}$ belongs to $\mathcal{D}(F)$ if and only if it belongs to $\mathcal{D}(\tilde{\mathcal{E}}_F)$. In this case,*

$$\mathcal{E}_F[u] = \tilde{\mathcal{E}}_F[u]. \tag{5.3.27}$$

Proof. Let u be in $\mathcal{D}(F)$. Every $u \in \mathcal{D}(F)$ is continuous on F: from Lemma 5.3.10 and Theorem 5.3.8 it follows that $u \in \mathcal{D}(\tilde{\mathcal{E}}_F)$. $\mathcal{E}_F$ can be written, for $u \in \mathcal{D}(F)$ as

$$\mathcal{E}_F[u] = \sum_{i=1}^{3} \int_{K_i} d\mathcal{L}_F[u].$$

From Theorem 5.3.8 it follows, for $u \in \mathcal{D}(\tilde{\mathcal{E}}_F)$

$$\tilde{\mathcal{E}}_F[u] = \sum_{i=1}^{3} \mathcal{E}^{(i)}[u|_{K_i}]$$

This with Lemma 5.3.10 implies 5.3.27.
Now, if $u \in \mathcal{D}(\tilde{\mathcal{E}}_F)$, from Theorem 5.3.8 it follows that $u|_{K_i} \in \mathcal{D}(\mathcal{E}^{(i)})$. □

Theorem 5.3.12. *A function $u : F \to \mathbb{R}$ belongs to $\mathcal{D}(F)$ if and only if it belongs to $\mathcal{D}(\tilde{\mathcal{E}}_F)$. In this case*

$$\mathcal{E}_F[u] = \tilde{\mathcal{E}}_F[u].$$

5.4 Essentials on semigroups and generators

In this Section we recall the main properties of the semigroups and related generators. For more details we refer to [55].

Definition 5.4.1. *Let X be a Banach space. A one parameter family $T(t)$, $0 \le t \le +\infty$, of bounded linear operators from X to X is a semigroup on X if*

- *$T(0) = I$, where I is the identity operator on X;*
- *$T(t+s) = T(t)T(s)$, for every $t, s \ge 0$.*

Definition 5.4.2. *A semigroup $T(t)$, is uniformly continuous if*

$$\lim_{t\to 0} \|T(t) - I\| = 0 \tag{5.4.28}$$

Definition 5.4.3. *The linear operator A defined by*

$$D(A) = \left\{ x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} exists \right\} \tag{5.4.29}$$

and

$$Ax = \lim_{t\to 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0}, x \in D(A) \tag{5.4.30}$$

is the infinitesimal generator of the semigroup $T(t)$, $D(A)$ is the domain of A

Theorem 5.4.4. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 5.4.5. *Let $T(t)$ and $S(t)$ be uniformly continuous semigroups of bounded linear operators. If*

$$\lim_{t\to 0} \frac{T(t) - I}{t} = A = \lim_{t\to 0} \frac{S(t) - I}{t} \tag{5.4.31}$$

then $T(t) = S(t)$ for $t \geq 0$.

Definition 5.4.6. *A semigroup $T(t)$, $0 \leq t < +\infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t\to 0} T(t)x = x, \forall x \in X. \tag{5.4.32}$$

Theorem 5.4.7. *Let $T(t)$ be a strongly continuous semigroup. There exist constants $\omega \geq 0$ and $M \geq 1$, such that*

$$\|T(t)\| \leq Me^{\omega t}, 0 \leq t < +\infty. \tag{5.4.33}$$

As a consequence we have the following

Proposition 5.4.8. *If $T(t)$ is a strongly continuous semigroup then for every $x \in X$, the mapping $t \to T(t)x$ is a continuous function from $\mathbb{R}^+$ into X.*

Theorem 5.4.9. *Let $T(t)$ be a strongly continuous semigroup and let A be its infinitesimal generator. Then*

1. *For $x \in X$*

$$\lim_{t\to 0} \frac{1}{h} \int_t^{t+h} T(s)xds = T(t)x \tag{5.4.34}$$

2. *For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and*

$$A(\int_0^t T(s)xds) = T(t)x - x. \tag{5.4.35}$$

3. *For $x \in D(A)$, $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax \tag{5.4.36}$$

4. *For $x \in D(A)$,*

$$T(t)x - T(s)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)d\tau. \tag{5.4.37}$$

Proposition 5.4.10. *If A is the infinitesimal generator of a strongly continuous semigroups $T(t)$, then $D(A)$ is dense in X and A is a closed linear operator.*

Theorem 5.4.11. *Let $T(t)$ and $S(t)$ be two strongly continuous semigroups whose infinitesimal generators are A and B respectively. If $A = B$, then $T(t) = S(t)$, for $t \geq 0$.*

Definition 5.4.12. *A strongly continuous semigroup $T(t)$ is called semigroup of contraction if*

$$\|T(t)\|_X \leq 1. \tag{5.4.38}$$

Definition 5.4.13. *If A is linear (unbounded) operator on X the resolvent set $\rho(A)$ is the set of the complex numbers λ for which $\lambda I - A$ is invertible, that is $(\lambda I - A)^{-1}$ is a bounded linear operator. The family $R(\lambda; A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$, of bounded linear operators is called resolvent of A.*

Theorem 5.4.14 (Hille-Yosida)**.** *A linear (unbounded) operator A is the infinitesimal generator of a strongly continuous semigroup of contractions $T(t)$, $t \geq 0$ if and only if*

1. *A is closed and $\overline{D(A)} = X$.*

2. *The resolvent set $\rho(A)$ contains $\mathbb{R}^+$ and for every $\lambda > 0$*

$$\|R(\lambda; A)\| \leq \frac{1}{\lambda}. \tag{5.4.39}$$

Definition 5.4.15. *A linear operator A is dissipative if*

$$\|(\lambda I - A)x\| \geq \lambda \|x\| \tag{5.4.40}$$

$\forall x \in \mathcal{D}(A)$ and $\lambda > 0$.

Theorem 5.4.16 (Lumer-Phillips)**.** *Let A be a linear operator with dense domain $\mathcal{D}(A)$ in H.*
If A is a dissipative operator and there exists λ_0 such that the range of $\lambda_0 I - A$, $R(\lambda_0 I - A)$, is H, then A is the infinitesimal generator of a continuous semigroup of contractions on H.

Proof. Since A is dissipative and $R(\lambda_0 I - A) = H$, it follows that $(\lambda_0 I - A)^{-1}$ is a linear and bounded operator, then $(\lambda_0 I - A)$ is closed and also A is closed. If $R(\lambda I - A) = H$ for every $\lambda > 0$, then $\rho(A) \supseteq (0, \infty)$ and $\|R(\lambda; A)\| \leq \lambda^{-1}$, from (5.4.40).Then from the Hille-Yosida Theorem it follows that A is the infinitesimal generator of a continuous semigroup of contraction on H.
To complete the proof it remains to show that $R(\lambda I - A) = H$ for every $\lambda > 0$.
Let

$$\Lambda = \{\lambda > 0 : R(\lambda I - A) = H\}.$$

Let $\lambda \in \Lambda$. From the dissipativeness of A, it follows that $\lambda \in \rho(A)$. Since $\rho(A)$ is open, it contains a neighborhood of λ. The intersection of this neighborhood with the real line is in Λ and then Λ is open. Let $\lambda_n \in \Lambda$ such that $\lambda_n \to \lambda > 0$. For every $y \in H$ there exists $x_n \in \mathcal{D}(A)$ such that

$$\lambda_n x_n - A x_n = y. \tag{5.4.41}$$

From the dissipativeness it follows $\|x_n\| \leq \lambda^{-1}\|y\| \leq C$, for some $C > 0$.

$$\lambda_m \|x_n - x_m\| \leq \|\lambda_m (x_n - x_m) - A(x_n - x_m)\| = |\lambda_n - \lambda_m| \|x_n\| \leq C|\lambda_n - \lambda_m|.$$

Hence x_n is a Cauchy sequence and thus it converges to an element x. Then from (5.4.41) it follows $Ax_n \to \lambda x - y$. Since A is closed and $x \in \mathcal{D}(A)$ then $Ax = \lambda x - y$. From this it follows that $R(\lambda I - A) = H$ and $\lambda \in \Lambda$. Hence Λ is closed and open and is non empty by assumption ($\lambda_0 \in \Lambda$), then $\Lambda = (0, \infty)$. □

5.5 Diagonalization lemmas

In this Section we recall two diagonalization lemmas for doubly indexed sequence. We refer to [2], page 32-33 (Lemma 1.15 and Corollary 1.16 respectively).

Lemma 5.5.1. *Let $\{a_{n,m}, n = 1, 2, ..., m = 1, 2, ...\}$ be a doubly indexed family in $\mathbb{R}$. Then, there exists a mapping $n \to m(n)$ increasing to $+\infty$, such that:*

$$\liminf_{n \to +\infty} a_{n,m(n)} > \liminf_{m \to +\infty} (\liminf_{n \to +\infty} a_{n,m}). \tag{5.5.42}$$

Proof. Let $a = \liminf_{n \to +\infty} a_{n,m}$ and $a = \liminf_{m \to +\infty} a_m$. If $a = -\infty$, there is nothing to prove. Hence, let us assume $a > -\infty$ and take $(a_p)_p \in \mathbb{N}$ a sequence of real numbers strictly increasing to a.
If $a < +\infty$, take $a_p = a - 2^{-p}$.
If $a = +\infty$, take $a_p = p$.
By definition of a, there exists an increasing sequence $(m_p)_{p \in \mathbb{N}}$, $m_p \to +\infty$, such that

$$a_m > a_p, \text{ for all } m > m_p$$

This can be condensed in:

$$a_m > \inf_p (a - 2^{-p}) \tag{5.5.43}$$

for all $m > m_p$.
In the same way, there exists an increasing sequence $(n_p)_{p\in\mathbb{N}}$, $n_p \to +\infty$ such that

$$a_{n,m_p} > \inf_p (a_{m_p} - 2^{-p}) \tag{5.5.44}$$

for all $n > n_p$.
We set $m(n) = m_p$ if $n_p < n < n_{p+1}$ and prove that (5.5.42) is satisfied: when $n_p < n < n_{p+1}$, from (5.5.43) and (5.5.44)

$$a_{n,m(n)} > \inf_p (a_{m_p} - 2^{-p}) > \inf_p [\inf_p (a - 2^{-p}) - 2^{-p}]$$

If follows that

$$\liminf_{n\to+\infty} a_{n,m(n)} > \inf_p [\inf_p (a - 2^{-p}) - 2^{-p}].$$

This being true for any $p \in \mathbb{R}$, using the fact that for any $a \in \mathbb{R}$, $\inf_p [\inf_p (a - 2^{-p}) - 2^{-p}]$ increases to a as p goes to $+\infty$, we get:

$$\liminf_{n\to+\infty} a_{n,m(n)} > \liminf_{m\to+\infty} (\liminf_{n\to+\infty} a_{n,m})$$

□

Lemma 5.5.2. *Let $\{a_{n,m}, n = 1, 2...m, = 1, 2, ...\}$ be a doubly indexed family in $\mathbb{R}$. Then, there exists a mapping $n \to m(n)$, increasing to $+\infty$, such that:*

$$\limsup_{n\to+\infty} a_{n,m(n)} > \limsup_{m\to+\infty} (\limsup_{n\to+\infty} a_{n,m}) \tag{5.5.45}$$

BIBLIOGRAPHY

[1] Adams, D. R. and Hedberg, L. I. *Function spaces and potential theory*, Springer-Verlag, Berlin, 1966.

[2] Attouch, H *Variational Convergence for function and operators*, Eds. Pitman Advanced Publishing Program, London 1984.

[3] Baiocchi, C. and Capelo, A. *Variational and quasivariational inequalities:applications to free boundary value problems*, Wiley, New York, 1984.

[4] Barlow M.T. and Hambly B.M. *Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets*, Ann. Inst. Henri Poincare', Vol 33, 5, (1997), 531-557.

[5] Brezzi, F. and Gilardi G. *Fundamentals of P.D.E. for numerical analysis*, In Finite Element Handbook (ed: H. Kardenstuncer and D.H. Norrie), McGraw-Hill Book Co., New York, 1987.

[6] Brezis, H. *Analisi funzionale*, Liguori Ed., Napoli, 1986.

[7] Buffa, A. and Ciarlet, P. *On traces for functional spaces related to Maxwell's equations, part 1: an integration by parts formula in Lipschitz polyhedra*, Math. Meth. Appli. Sci., 21, (1), 9-30, 2001.

[8] Cannon, J. R. and Meyer, G. H. *On a diffusion in a fractured medium*, SIAM J. Appl. Math., 3, pp. 434-448, 1971.

[9] Capitanelli, R. *Lagrangians on homogeneous spaces*, PhD Thesis, Roma (Italy), Univ. di Roma "La Sapienza", 2001.

[10] Dauge, M. *Elliptic boundary value problems on corner domains*, Lecture notes in Mathematics, 1988.

[11] Dautray R. and Lions J.L. *Mathematical analysis and numerical methods for science and technology, volume 5: Evolution problem 1*, Springer-Verlag.

[12] Falconer, K. *The geometry of fractal sets*, 2nd ed. Cambridge Univ. Press, 1990.

[13] Favini, Labbas, Lemrabet, Maingot *Study of limit of transmission problems in a thin layer by the sum theory of linear operators*, Rev. mat. complut., 18, 143-176,(2005).

[14] Favini A., Goldstein, Ruiz G., Goldstein J. A., Romanelli S., *The heat equation with generalized Wentzell boundary condition*,J. Evol. Equ., 2,1,pp.1–19, 2002.

[15] Federer, H. *Geometric measure theory*, (Repr. of the 1969 ed. Classics in mathematics), Berlin, Springer-Verlag, 1996.

[16] Freiberg, U. and Lancia, M. R. *Energy form on a closed fractal curve*, Z. Anal. Anwendungen.23, 1, 115-135, 2004.

[17] Fukushima, M *Dirichlet forms, diffusion processes and spectral dimension for nested fractals*, Ideas and met. in Math. Anal., Stoch. and Appl., Cambridge Univ. Press, 1992, 151-161 .

[18] Grisvard, P. *Elliptic problems in non smooth domains*, Pitman, Boston, 1985.

[19] Grisvard, P. *Theoremes de traces relatifs à un polyèdre*, C.R.A. Acad. Sc. Paris, 278, Série, 1581-1583, 1974.

[20] Hackbush, W. *Elliptic partial differential equations, theory and numerical treatment*, Springer Series in Computational Mathematics 18, Springer-Verlag, Berlin, 1992.

[21] Hutchinson, J.E.*Fractals and self-similarity*, Indiana Univ. Math. J., 30, 713-747, 1981.

[22] Jerison, D. and Kenig, C. E. *The Neumann problem in Lipschitz domains*, Bull. Amer. Math. Soc., 4, 71-88, 1981.

[23] Jones, P.W. *Quasiconformal mapping and extendability of functions in Sobolev spaces*, Acta Math., 147, 71-88, 1981.

[24] Jonsson, A. *Brownian motion on fractals and function spaces*, Math. Zeith., 6, (1996), 495-504.

[25] Jonsson, A and Wallin, H *Function spaces on subset of* $\mathbb{R}^n$, Part 1. Math. Reports, vol.2, Harwood Acad. Publ., London, 1984.

[26] Kato, T. *Perturbation theory for linear operators*, Springer-Verlag, New York, 1966.

[27] Kolesnikov, A.V. *Convergence of Dirichlet forms with changing speed measures on* $\mathbb{R}^d$, Forum Math. 17, 225-259, 2005.

[28] Korman, P., *Existence of periodic solutions for a class of non linear problems*, Non linear Anal.,7, 873-879, (1983).

[29] Kuwae, K. and Shioya, T. *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Communications in analysis and geometry, Vol. 11, number 4, 599-673, 2003.

[30] Kusuoka, S *Diffusion processes on nested fractals*, Lect. Notes Math. 1567, 1993.

[31] Lancia, M.R. *A transmission problem with a fractal interface*, Z. Anal. Anwendungen, 21, 1, 113-133, 2002.

[32] Lancia, M.R. *Second order transmission problems across a fractal surface*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl.(5), 27,191-213, 2003.

[33] Lancia, M.R, Regis Durante V., Vernole P. *Density results for energy spaces on fractafold*, to appear on Z. Anal. Anwendungen.

[34] Lancia, M.R. and Vernole P. *Convergence results for parabolic transmission problems across higly conductive layers with small capacity*, Adv. Math. Sci. Appl. 16, 2, 411-445, (2006)

[35] Lancia M.R. and Vernole P., *Irregular heat flow problems,* SIAM Journal on Mathematical Analysis, 42,pp.1539–1567, 2010.

[36] Lancia, M.R. and Vernole P.G. *Semilinear evolution transmission problems across fractal layers*, Nonlinear Anal. 75, (2012), 4222-4240.

[37] Lancia, M.R. and Vernole P.G. *Semilinear fractal problems: approximation and regularity results*, Nonlinear Anal., 80, (2013), 216-232.

[38] Lancia, M.R. and Vernole P. *Venttsel' problems in fractal domains*, Jour. of Evol. Eq., 14, pp 681-712, 2014.

[39] Lancia, M.R. and Vivaldi M.A. *Lipschitz spaces and Besov traces on self similar fractals*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl., 23, pp 101-106, 1999.

[40] Lancia, M.R. and Vivaldi M.A. *Asymptotic convergence of transmission energy forms*, Adv. Math. Sc. Appl., 13, 315-341, 2003.

[41] Le Jean, Y. *Measures associeés á une forme de Dirichlet*, Appl., Bull. Soc. Math. France, 106, 61-112, 1978.

[42] Liang, H. *On the construction of certain fractal mixtures*, Thesis in applied mathematics, Worcester Polytechnic institute, 2009.

[43] Lions, J. and Magenes, E. *Non-Homogeneous boundary valued problems and applications*, vol. 1. Berlin, Springer-Verlag 1972.

[44] Lindstrom, T. *Brownian Motion on Nested Fractals*, Memoirs Amer. Math. Soc., 83, 420, 1990.

[45] Lions, J. L. *Equations differentielles operationelles et problemes aux limites*, Grundleheren Vol. 111, Springer, 1961.

[46] Lunardi, A. *Analytic semigroups and optimal regularity in parabolic problems*, Progress in non linear differential equations and their applications, 16, Birkhauser-Verlag, Basel, 1995.

[47] Ma, Z. M. and Rockner M. *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag Berlin, 1992.

[48] Mataloni, S. *On a type of convergence for non-symmetric Dirichlet forms*, Adv. in Math. 9, 749-773, 1999.

[49] Mosco, U. *Convergence of convex sets and of solutions of variational inequalities*, Adv. in Math. 3, 510-585, 1969.

[50] Mosco, U. *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal., 123, 2, 368-421, 1994.

[51] Mosco, U. *Harnack inequalities on scale irregular Sierpinski gaskets*, Non linear problems in Mathematical Physics and Related Topics II, Birman et al., Kluwer Acad. Plen. Publ., New York, 2002.

[52] Mosco, U. *Energy functionals on certain fractal structures*, J. Convex Anal. 9, 581-600, 2002.

[53] Mosco, U. and Vivaldi M.A. *Variational problems with fractal layers*, Rend. Acc. Naz. Sci. XL Mem. Mat. Appl. (5), 27, (2003) , 247-251.

[54] Necas, J. *Les methodes directes en theorie des èquationes elliptiques*, Masson, Paris, 1967.

[55] Pazy, A. *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. 44, Springer-Verlag, New York, 1983.

[56] Sapoval, B. *General formulation of Laplacian transfer across irregular surfaces*, Phys. Rev. Lett., 73, 3314-3316, 1994.

[57] Shinbrot, M.,*Watern waves over periodic bottoms in three dimensions*, J. Inst. Math. Appl.,25,4, 367-385, 1980.

[58] Stein, E. M. *Singular integrals and differentiability properties of functions*, Princeton Univ. Press., Princeton, 1970.

[59] Tableison, M. H. *On the theory of LIpschitz spaces of distributions on euclidean n-space*, I, J. Math. Mech., 13, 407-480, 1964.

[60] Venttsel, A. D. *On boundary conditions for multidimensional diffusion processes*, Teor. Veroyatnost. i Primenen. 4, 172-185, 1959; English transl. Theor. Probability Appl. 4,164-177, 1959.

[61] Wallin H., *The trace to the boundary of Sobolev spaces on a snowflake*, Mskr. Math. 73, 117-125, 1991.