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Robust Quantum Optical States for Quantum Sensing and Entanglement Tests Tesi di dottorato del dott. Nicolò Spagnolo

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List of abbreviations

ICT Information and Communication Technology **POVM** Positive Operator Valued Measurements EPR Einstein Podolski Rosen CHSH Clauser-Horny-Shimony-Holt LHV Local Hidden Variable QFI Quantum Fisher Information SQL Standard Quantum Limit SNL Shot-Noise Limit HL Heisenberg Limit SPDC Spontaneous Parametric Down Conversion **OPA** Optical Parametric Amplifier **QED** Quantum Electro-Dynamics **CV** Continuous-Variable **UV** UltraViolet **BS** Beam-Splitter **QIOPA** Quantum Injected Optical Parametric Amplifier **U-Not** Universal-Not **PBS** Polarizing Beam-Splitter **BBO** Beta Barium Borate HWP Half waveplate **QWP** Quarter waveplate **IF** Interferential Filter **DM** Dichroic Mirror **APD** Avalanche PhotoDiode **SHG** Second Harmonic Generation **OF** Orthogonality Filter **PM** PhotoMultipliers **UBS** Unbalanced Beam-Splitter **TD** Threshold Detector NOPA Non-Degenerate Optical Parametric Amplifier NL Non Linear

TTL Transistor-Transistor-Logic

CONTENTS

CP Completely PositiveHD Homodyne DetectionLO Local Oscillator

QCR Quantum Cramer Rao

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Introduction

In the last few decades the emergence of a new field of research dealing with information at the quantum level has led to a "second quantum revolution", that promises new technologies whose design is based on the principles of quantum mechanics. A relevant example is provided by the fascinating project of the quantum computer, originally proposed by R. Feynman [Fey82] and subsequently formulated by Deutsch [Deu85]. This proposal is based on developing a new theory of computation based on the rules of quantum mechanics, including peculiar features such as entanglement and superposition, which can lead to improved performances with respect to a classical approach. Several other contexts have been identified which can benefit from this quantum approach. These include quantum cryptography and communication, which deal with developing quantum strategies for secure sharing of a key between two or more distant parties for crypting and decrypting a secret meassage. Quantum measurements represent a crucial aspect of quantum theory since the amount of information which can be extracted from a system critically depends on the capability of performing efficient measurements. Among the different fields of quantum information, quantum principles have also found application in the field of measurement science. In particular, the ability to perform precise measurements of time, length, phase, position using the fewest possible resources represents one of the most relevant issues of information and communication technologies (ICT). Quantum mechanics has been identified as a powerful tool to perform measurements with precision beyond the classical limit established from the noise associated to any physical probe. Entanglement, which has no classical analogue, has been proposed as one possible way to overcome the classical limits and to reach the more fundamental Heisenberg limit [GLM06]. A paradigmatic example is given by the so-called "N00N states", i.e. maximally entangled N-particle states, which in ideal conditions allow the interferometrical estimation of an unknown phase with a precision scaling as N^{-1} : the Heisenberg limit.

In parallel, the tools developed within quantum information have found application in the investigation of the foundations of quantum mechanical theory. In this context, an open question is related with the fundamental mechanisms leading to the transition from the quantum dynamics of the microscopic world to the classical dynamics of the macroscopic world. The observation of quantum properties in systems of growing size has been the object of a great research effort in the last few years. Among quantum properties, entanglement has been recognized by Schrödinger [Sch35] as "the characteristic trait of quantum mechanics". Entanglement is given by the presence of correlations between two physical systems which have no corresponding classical analogue. The main experimental challenge to overcome in order to observe entanglement in multiparticle systems is the uncontrolled interaction with the environment, that is, *decoherence* [Zur03], which is responsible for the progressive reduction of quantum features and limits the capability of observing these properties. Indeed, the coupling of a physical system with an external environment becomes stronger and stronger as the size of the system increases, thus leading to a faster decrease in coherence properties of the state. Alongside, in order to observe quantum features it is necessary to employ quantum measurements with sufficient resolution to catch the quantumness of the investigated state. Despite the great theoretical and experimental progress obtained in the last few years, the observation of quantum properties is still limited to systems of only few particles.

Several experimental platforms have been developed to implement quantum systems for both fundamental tests and application to quantum information protocols. As relevant examples, quantum systems have been implemented by adopting as information carriers photons, cold atoms, trapped ions, or superconducting devices. Each platform presents its advantages and weak points. Quantum optics has represented a valuable tool for the practical implementation of quantum information tasks. Indeed, photonic field possessing quantum features can be easily generated, manipulated and detected with the present technology. For example, in order to generate entangled states, the process of spontaneous parametric down-conversion provides an efficient method to generate high purity entangled photon pairs in several degrees of freedom, including polarization, linear momentum or angular momentum. Furthermore, the measurement of photonic fields can be performed efficiently by adopting different techniques. Indeed, single-photons can be efficiently detected by single-photon counting modules commercially available, while the phase properties of an optical field can be probed with the homodyne detection technique. All these properties have led to the implementation of various tests of quantum mechanics, including the violation of Bell's inequalities, and to the implementation of several quantum information protocols.

The present thesis is aimed at investigating the possibility of performing both quantum mechanical tests and quantum information protocols with multiphoton states. We adopt as a platform the quantum states generated by an optical parametric amplification process. The main idea beyond this approach is given by the capability of the amplification process to broadcast the features of the input state into a system with a larger number of photons. This property will be applied to analyze the possibility of observing quantum effects when the number of particles in the system progressively increases. We begin by considering a single-photon input into the amplifier, and we show that the states generated in this configuration present a significant resilience with respect to the action of detection losses. From a fundamental point of view, the resilience to losses of such states represents a tool for the investigation of quantum phenomena in a system of increasing size, thus allowing to explore the transfer of quantum properties from a sin-

INTRODUCTION

gle particle state to a collective multi-particle one. As a second system, we analyze the bipartite system which is obtained by amplification of a single-photon belonging to an entangled pair, thus generating an hybrid microscopic-mesoscopic system. We consider the possibility of detecting the entanglement in this configuration when the number of photons in the mesoscopic part progressively increases. This investigation requires a detailed analysis on the various classes of entanglement and nonlocality tests that can be performed in a joint microscopic-macroscopic bipartite system. After the development of a first insight on this problem with a discrete variable approach, continuous-variables collective measurements will be investigated. Specific attention will be devoted to the entanglement criteria based on the quadrature phase-space operators. The tools developed with this continuous-variables approach will be applied in a different configuration, where both the two subsystems are composed by a multiphoton field. This system can be generated by adopting a parametric amplification process in a noncollinear configuration in the spontaneous emission regime. We investigate the possibility of observing nonlocal features in this class of states when both coarse-grained and high efficiency continuousvariables measurements are adopted. The possible applications in quantum information tasks of the quantum states generated through the optical parametric amplifier will be then investigated. Among the various fields, attention will be devoted to quantum metrology [GLM06] in presence of a lossy apparatus. We show that by performing an amplification process we can preserve the information on the optical phase to be measured from the action of losses, unavoidable in any experimental implementation. More specifically, this approach relies on amplifying the probe state after the phase information has been acquired, increasing its robustness with respect to losses.

The organization of the present thesis is reported in Fig. 1, which is mainly composed by three Parts. In Part I, we introduce the fundamental concepts of quantum information and quantum optics. In Chap. 1 we review the fundamental concepts of quantum information theory. Then, in Chap. 2 we introduce the process of parametric down conversion, and we discuss the two configurations adopted throughout the present work. Then, in Chap. 3 we introduce the basic concepts of continuous-variables quantum optics, by discussing the representation and the measurement of quantum states in the phase-space.

In Part II, we discuss the application of the optical parametric amplifier to perform fundamental tests of quantum mechanics. More specifically, the capability of the amplifier to produce an output field with increasing number of photons renders this device a suitable platform to investigate the possibility of observing quantum properties in multiparticle systems. In Chapter 4 we begin by considering the multiphoton optical field generated by optical parametric amplification when injected by a single photon. The resilience to decoherence of such states is investigated by adopting different criteria, relying on both discrete- and continuous-variables. Then, in Chapters 5 and 6 we analyze the amplification of a photon belonging to an entangled photon pair. We investigate how the initial entanglement between the two photons before the amplification is broadcasted by the action of the amplifier, and we analyze the detrimental effect of decoher-



Figure 1: Scheme of the thesis.

ence. This analysis considers different criteria based on discrete variables. Furthermore, we develop an hybrid approach relying on discrete-variables and continuous-variables measurements combined on the same system in order to exploit the advantages of both techniques. The results obtained for the hybrid approach motivated the analysis on a different platform, which consists in a parametric down-conversion source working in a bipartite multiphoton-multiphoton configuration. This is analyzed in Chapter 7, where the possibility of violating a Bell's inequality in a macroscopic-macroscopic configuration is considered by exploiting both low resolution and high resolution measurements. Finally, the analysis of the parametric amplifier within the context of fundamental theory is concluded in Chapter 8. We report the experimental implementation of a fundamental process, that is, single-photon addition, which provides a relevant resource for several continuous-variables quantum information protocols. This experiment, performed at the Quantum Optics Group of Institut d'Optique in Paris, has been focused on characterizing the nongaussianity of the photon addition process.

In Part III, we exploit the results obtained in Part II to apply the optical parametric amplifier to phase estimation tasks. Indeed, the resilience to losses of the multiphoton states generated by parametric amplification can provide a useful platform for real-world sensing applications. In Chapter 9 we report the experimental implementation of a phase estimation protocol performed with single photon probes, which permits to obtain a significant enhancement in presence of detection losses. This protocol is generalized in Chapter 10 for a coherent probe state.

Part I

Preliminary concepts

Chapter 1

Elements of quantum information theory

The aim of quantum information is to develop suitable strategies to exploit quantum mechanics in order to increase the performance of several tasks, such as quantum computation, communication, cryptography or sensing. In the present chapter we introduce the basic element of quantum mechanics and quantum information theory. We first discuss the representation of quantum states in terms of density operators, and we then brielfy review the general formalism of quantum maps which describes their time evolution. Then, after introducing the theory of quantum measurements, we discuss the quantum cloning task, which is permitted by quantum mechanics only in an approximate fashion. Among the different characteristic features of quantum theories, entanglement and nonlocality represent one of the most fascinating questions. In this chapter we introduce the concept of entangled states, and we discuss the Bell's theorem for a bipartite system. The detection of entanglement and nonlocality in multiphoton systems will be addressed in Part II by adopting a specific optical system. Finally, in Part III we discuss a specific application in quantum parameter estimation, where the aim is to estimate an unknown parameter by exploiting quantum resources, which allow for increased performances with respect to classical strategies. The discussed elements represent the building blocks for the analysis and the application of quantum mechanics in a quantum information context.

1.1 Representation of quantum states

In this section we briefly introduce the density operator formalism, which allows to describe any quantum mechanical state. According to the postulates of quantum mechanics, the pure state of a physical system is defined by a vector in the corresponding Hilbert space \mathscr{H} . Such postulate can be generalized in the case of statistical ensembles, where a general state is represented by an operator $\hat{\rho}$ in \mathscr{H} . The latter corresponds to the incoherent mixture of the state vectors of the ensemble elements.

1.1.1 State vector and the quantum bit

We begin by considering the pure state case. A general pure state $|\psi\rangle$ in an d-dimensional Hilbert space \mathscr{H} can be written as:

$$|\psi\rangle = \sum_{n=0}^{d} c_n |n\rangle,$$
 (1.1)

where $\{|n\rangle\}$ is an orthonormal set of state vectors in \mathcal{H} , c_n are the coefficient defined by the scalar product $c_n = \langle n | \psi \rangle$. The normalization condition for the state $|\psi\rangle$ is given by: $\langle \boldsymbol{\psi} | \boldsymbol{\psi} \rangle = \sum_{n=0}^{d} |c_n|^2 = 1.$

As a specific example, we consider a 2-dimensional system. A general state in this Hilbert space is defined as the quantum bit, or qubit, which represents the quantum extension of the classical bit: a two levels system performing as the building block of communication and computational protocols. A general pure state of this system reads:

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle, \qquad (1.2)$$

where $\{|0\rangle, |1\rangle\}$ is the orthonormal set in the 2-dimensional Hilbert space that defines the computational basis. The state of a bidimensional system admits a representation in terms of a spherical surface, called the Bloch sphere [NC00]. The state $|\psi\rangle$ can be represented as a point on the surface of a unitary radius, defined by the polar angle θ and by the azimuthal angle ϕ [see Fig. 1.1].



 $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\phi}|1\rangle)$ Figure 1.1: Quon representation Bloch sphere. Pure state are identified by the points in the surface of unitary radius, while mixed states are identified by the internal points of the sphere. The polar and azimuthal angles θ and ϕ determine the position in spherical coordinates with respect to the poles given by the vectors of the computational basis $\{|0\rangle, |1\rangle\}$.

1.1.2 Mixed states: the density matrix

The pure states formalism does not permit to describe statistical ensemble of quantum states. Suppose that the system under consideration is prepared in the state $|\psi_i\rangle$ with probability p_i . The general state of the system can be described by adopting the density operator $\hat{\rho}$ [Sak03]:

$$\hat{\rho} = \sum_{i} p_{i} |\psi\rangle_{i} \langle \psi|_{i}, \quad \text{where } \sum_{i} p_{i} = 1, \qquad (1.3)$$

which for pure state reduces to $\hat{\rho} = |\psi\rangle \langle \psi|$. The density operator is characterized by the following properties:

Quantum processes and time evolution

- 1. $\hat{\rho}$ is Hermitian, that is, $\hat{\rho}^{\dagger} = \hat{\rho}$.
- 2. The density operator is normalized: $Tr\hat{\rho} = 1$.
- 3. For pure states $\text{Tr}[\hat{\rho}^2] = 1$ holds, while for mixed state $\text{Tr}[\hat{\rho}^2] < 1$.

Average values of an observable \hat{O} on a state $\hat{\rho}$ can be evaluated as:

$$\langle \hat{O} \rangle = \operatorname{Tr}[\hat{\rho}\hat{O}]$$
 (1.4)

In the case of a d = 2 dimensional system, the density operator of a general state reads:

$$\hat{\boldsymbol{\rho}} = \frac{1}{2} \left(\hat{\mathbb{1}} + \mathbf{s} \cdot \hat{\boldsymbol{\sigma}} \right). \tag{1.5}$$

In this expression, $\mathbf{s} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ is the vector which defines the position on the sphere. The vector $\hat{\boldsymbol{\sigma}}$ has the Pauli operators $\hat{\sigma}_i$ (i = 1, ..., 3) and the identity operator $\hat{\mathbb{1}} = \hat{\sigma}_0$ (i = 0) as elements. Such operators represent a basis for any density operator in a d = 2-dimensional Hilbert space. According to this definition, the density operator $\hat{\rho}$ can be represented as a point in the Bloch sphere, with radius \mathbf{s} , polar angle θ and azimuthal angle ϕ [see Fig. 1.1].

The definition of density operator can be directly extended to a multipartite system in a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, by considering the $|\psi\rangle_i$ states in Eq. (1.3) as the state vectors in the full Hilbert space \mathcal{H} . When measuring a physical observable \hat{O}_j for the subsystem *j* alone, one can define the reduced density operator describing the state of subsystem *j* as:

$$\hat{\rho}_j = \mathrm{Tr}_{k \neq j}[\hat{\rho}] \tag{1.6}$$

that is, by tracing the density operator over all other Hilbert space $\mathscr{H}_{k\neq j}$.

1.2 Quantum processes and time evolution

In the present section we describe the formalism for the time evolution of a physical system. For closed systems, the time evolution is described by the Schrödinger equation, which permits to obtain the state vector of a physical system at time t according to the action of a unitary operator on the initial state. Such a description in terms of unitary operators cannot be adopted in the case of an open system, that is, interacting with an additional system not accessible by the observer. In this case, the time evolution of the system is described by a completely positive map acting on the density operator.

1.2.1 Unitary evolution of closed systems

The properties of a closed physical system are defined by the quantum mechanical extension $\hat{\mathscr{H}}$ of the classical Hamiltonian \mathscr{H} . The operator $\hat{\mathscr{H}}$ acts as the generator of the

time evolution of such a system according to the Schrödinger equation:

$$\imath\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{\mathscr{H}}|\psi\rangle; \quad \frac{\partial\hat{\rho}}{\partial t} = -\frac{\imath}{\hbar}[\hat{\mathscr{H}},\hat{\rho}], \tag{1.7}$$

where $[\hat{\mathcal{H}}, \hat{\rho}]$ is the commutator between the two operators. The time evolution at a fixed time *t* of a state vector in the initial state $|\psi(0)\rangle$ and on a density matrix $\hat{\rho}(0)$ at t = 0 can be obtained as:

$$|\boldsymbol{\psi}(t)\rangle = \hat{U}(t)|\boldsymbol{\psi}(0)\rangle; \quad \hat{\boldsymbol{\rho}}(t) = \hat{U}(t)\hat{\boldsymbol{\rho}}(0)\hat{U}^{\dagger}(t).$$
(1.8)

The expectation value at time t of a physical observable \hat{O} , can be then evaluated as the average value over the density matrix $\hat{\rho}(t)$, or equivalently, we can consider that the time evolution modifies the action of the observable \hat{O} without affecting the state $\hat{\rho}(0)$. We obtain the two equivalent formulations:

$$\langle \hat{O} \rangle(t) = \operatorname{Tr}[\hat{\rho}(t)\hat{O}] = \langle \hat{O} \rangle(t) = \operatorname{Tr}[\hat{\rho}\hat{O}(t)].$$
 (1.9)

Here, $\hat{O}(t) = \hat{U}^{\dagger}(t)\hat{O}\hat{U}(t)$ is the time evolution induced by the Heisenberg equation:

$$\frac{\partial \hat{O}}{\partial t} = \frac{\iota}{\hbar} [\hat{\mathscr{H}}, \hat{O}]. \tag{1.10}$$

The two representations are called the Schrödinger and the Heisenberg picture respectively.

1.2.2 Nonunitary evolution: quantum maps

Unitary operators with Hamiltonian generators describe the time evolution of closed physical systems. In general, for any open quantum system it is not possible to describe time evolution in terms of unitary operators acting on the system. However, such evolution can be described in terms of quantum maps \mathscr{E} , which must obey the following constraints [NC00]:

- (1) *Hermiticity* If $\hat{\rho}^{\dagger} = \hat{\rho}$, then $\hat{\rho}' = \mathscr{E}[\hat{\rho}]$ must satisfy $(\hat{\rho}')^{\dagger} = \hat{\rho}'$.
- (2) Trace preserving If $\operatorname{Tr}\hat{\rho} = 1$, then $\hat{\rho}' = \mathscr{E}[\hat{\rho}]$ must satisfy $\operatorname{Tr}\hat{\rho}' = 1$.
- (3) Complete positivity Consider a density matrix acting on a Hilbert space ℋ_A. A map ℰ is completely positive (CP) if for any extension of the Hilbert space ℋ_A ⊗ ℋ_B the map ℰ_A ⊗ 𝔅_B is positive. Recall that a map is positive if ρ̂' = ℰ[ρ̂] is nonnegative when ρ̂ is nonnegative.
- (4) *Linearity* If $\hat{\rho} = \lambda \hat{\rho}_1 + (1 \lambda) \hat{\rho}_2$, then $\mathscr{E}[\hat{\rho}] = \lambda \mathscr{E}[\hat{\rho}_1] + (1 \lambda) \mathscr{E}[\hat{\rho}_2]$.

It can be demonstrated that [Kra83] for a map \mathscr{E} which obeys the constraints (1) - (4), it is always possible to represent the map in the following form:

$$\mathscr{E}[\hat{\rho}] = \sum_{\mu} \hat{M}_{\mu} \hat{\rho} \hat{M}_{\mu}^{\dagger}, \qquad (1.11)$$

where $\{\hat{M}_{\mu}\}\$ is a set of operators satisfying $\sum_{\mu} \hat{M}_{\mu}^{\dagger} \hat{M}_{\mu} = \hat{1}$. Note that the number of operators in the set $\{\hat{M}_{\mu}\}\$ in general is not bounded by the dimension of the Hilbert space \mathscr{H}_A . Such theorem is known as the *Kraus representation theorem* [Kra83], and provides a powerful tool to represent the time evolution of a general open system. The action of the map \mathscr{E} in the Kraus representation can be also expressed in terms of the action of a rank-4 tensor on the density matrix $\hat{\rho}$. By choosing an orthonormal basis $\{|i\rangle\}$, the elements of the density matrix $\mathscr{E}[\hat{\rho}]$ can be evaluated as:

$$\left(\mathscr{E}[\hat{\rho}]\right)_{l,k} = \sum_{n,m} \mathscr{E}_{l,k}^{n,m} \rho_{n,m}, \qquad (1.12)$$

where $\hat{\rho} = \sum_{n,m} \rho_{n,m} |n\rangle \langle m|$, and:

$$\mathscr{E}_{l,k}^{n,m} = \sum_{\mu} \langle l | \hat{M}_{\mu} | n \rangle \langle m | \hat{M}_{\mu}^{\dagger} | k \rangle.$$
(1.13)

1.3 Quantum measurements

Within the theory of quantum mechanics, quantum measurements require a different treatment with respect to the unitary time evolution of states and operators. In this section we briefly describe the formulation of quantum measurements, and we discuss the general formalism to investigate the optimal extraction of information from a quantum system.

1.3.1 Measurement theory

In the previous section we analyzed the problem of the time evolution induced by quantum processes, starting from the unitary evolution of closed quantum systems and moving forward to a general treatment of quantum maps. In quantum mechanics, the measurement problem requires a different approach, since it includes the action of the external observer. To this end, the problem of quantum measurements can be described starting from one of the postulates of quantum theory.

The action of a measurement apparatus on a quantum state is described by a set of operators $\{\hat{M}_{\xi}\}\)$, where the index ξ stands for the possible different outcomes of the measurement. The general scheme for the measurement process is reported in Fig. 1.2. When the measurement described by the set of operators $\{\hat{M}_{\xi}\}\)$ is performed, the outcome ξ can occur with probability:

$$p(\xi) = \langle \psi | \hat{M}_{\xi}^{\dagger} \hat{M}_{\xi} | \psi \rangle; \quad p(\xi) = \operatorname{Tr}(\hat{M}_{\xi}^{\dagger} \hat{M}_{\xi} \hat{\rho}).$$
(1.14)

The quantum state of the system after the measurement when the outcome ξ has occurred is described by the action of the projection operator \hat{M}_{ξ} on the original state ($|\psi\rangle$ for pure state, $\hat{\rho}$ for mixed states) [NC00]:

$$|\psi_{\xi}\rangle = \frac{\hat{M}_{\xi}|\psi\rangle}{\sqrt{\langle\psi|\hat{M}_{\xi}^{\dagger}\hat{M}_{\xi}|\psi\rangle}}; \hat{\rho}_{\xi} = \frac{\hat{M}_{\xi}\hat{\rho}\hat{M}_{\xi}^{\dagger}}{\mathrm{Tr}(\hat{M}_{\xi}^{\dagger}\hat{M}_{\xi}\hat{\rho})}.$$
(1.15)

No restriction has to be imposed on the number of the operators, which can exceed the dimensionality of the Hilbert space. The only constraint is given by the completeness relation:

$$\sum_{\xi} \hat{M}_{\xi}^{\dagger} \hat{M}_{\xi} = \hat{\mathbb{1}}, \qquad (1.16)$$

which can be alternatively expressed as:

$$\sum_{\xi} p(\xi) = \sum_{\xi} \langle \psi | \hat{M}_{\xi}^{\dagger} \hat{M}_{\xi} | \psi \rangle = 1.$$
(1.17)

Such completeness relation corresponds to the law of conservation for probabilities.



Figure 1.2: General scheme corresponding to a quantum measurement. The measurement device is modeled by a set of operators $\{\hat{M}_{\xi}\}$ satisfying the properties reported in the text. The outcome ξ is obtained with probability $p(\xi)$, while the measured state is transformed to $\hat{\rho}_{\xi}$ conditioned to the outcome ξ .

1.3.2 Von Neumann projective measurements

A relevant class of quantum measurement is provided by the *Von Neumann* [vN55] measurement operators, which describe projective measurements. They are represented by an Hermitian operator \hat{M} with spectral decomposition:

$$\hat{M} = \sum_{\xi} \xi \hat{P}_{\xi}. \tag{1.18}$$

The \hat{P}_{ξ} operators are projection operators, corresponding to the subspace of \hat{M} defined by the eigenvalue ξ . Hence, these operators satisfy the identity $\hat{P}_{\xi}\hat{P}_{\zeta} = \delta_{\xi,\zeta}\hat{P}_{\xi}$. The Von Neumann projectors $\{\hat{P}_{\xi}\}$ can be recovered from the general measurement operators $\{\hat{M}_{\xi}\}$ by adding the latter constraint. Finally, the expectation value corresponding to the measurement of \hat{M} takes the form:

$$\mathbf{E}(\hat{M}) = \sum_{\xi} \xi p(\xi) = \langle \boldsymbol{\psi} | \hat{M} | \boldsymbol{\psi} \rangle.$$
(1.19)

1.3.3 Positive Operator-Valued Measurements

In some classes of measurements, the system is destroyed by the process itself. In these cases, it is possible to recover only the statistics of the measurement outcomes and the process is described by the positive operator-valued measurements (POVM) formalism. Let us now define the following set of operators:

$$\hat{E}_{\xi} = \hat{M}_{\xi}^{\dagger} \hat{M}_{\xi} \tag{1.20}$$

Such operators are defined as the elements of POVM and they represent a complete set of positive operators satisfying the completeness relation $\sum_{\xi} \hat{E}_{\xi} = \hat{1}$. The probability corresponding to the outcome ξ can be evaluated as:

$$p(\xi) = \langle \psi | \hat{E}_{\xi} | \psi \rangle. \tag{1.21}$$

Furthermore, such set $\{\hat{E}_{\xi}\}$ fully characterizes the measurement process. Starting from the \hat{E}_{ξ} operators the corresponding measurement operators \hat{M}_{ξ} can be obtained as $\hat{M}_{\xi} = \sqrt{\hat{E}_{\xi}}$ [NC00].

1.3.4 The Neumark's theorem

Here, we briefly discuss the connection between POVMs and Von Neumann projective measurements. More specifically, any POVM corresponds to a standard projective measurement in an extended Hilbert space.

Let us consider a bipartite system in the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Orthogonal measurement on the composed Hilbert space are defined by a set of projection operators satisfying:

$$\sum_{\xi} \hat{P}_{\xi} = \hat{\mathbb{1}}_{AB}. \tag{1.22}$$

Let us consider an initial state of the form:

$$\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B. \tag{1.23}$$

The probability of occurrence of outcome ξ can be evaluated according to Eq. (1.14):

$$p(\xi) = \operatorname{Tr}_{AB}[\hat{P}_{\xi}(\hat{\rho}_A \otimes \hat{\rho}_B)].$$
(1.24)

Suppose now, that the observer has access only to subsystem *A*. Then, the probability of outcome ξ reads:

$$p(\xi) = \operatorname{Tr}_{A}\{\operatorname{Tr}_{B}[\hat{P}_{\xi}(\hat{\rho}_{A}\otimes\hat{\rho}_{B})]\} = \operatorname{Tr}_{A}[\hat{E}_{\xi}\hat{\rho}_{A}].$$
(1.25)

The elements of the \hat{E}_{ξ} operators are defined according to:

$$(E_{\xi})_{ji} = \sum_{\mu,\nu} (P_{\xi})_{j\nu,i\mu} (\rho_B)_{\mu\nu}.$$
 (1.26)

where the operators are expanded, according to $(O)_{ab} = \langle a | \hat{O} | b \rangle$, in terms of the orthonormal bases $\{|i\rangle_A\}$ and $\{|\mu\rangle_B\}$ for the systems *A* and *B* respectively. It can be then demonstrated that the set of operators $\{\hat{E}_{\xi}\}$ satisfies the following properties:

- 1. Hermiticity: $(E_{\xi})_{ji}^* = (E_{\xi})_{ij}$.
- 2. Positivity: $_{A}\langle \psi | \hat{E}_{\xi} | \psi \rangle_{A} \geq 0.$
- 3. Completeness: $\sum_{\xi} \hat{E}_{\xi} = \hat{\mathbb{1}}_A$.

However, no constraints are imposed on the number of operators of the set $\{\hat{E}_{\xi}\}$ in terms of the dimension of the Hilbert space A accessed by the observer. Hence, a projective measurement on a bipartite Hilbert space is equivalent to a corresponding POVM in one of the two subsystem. The Neumark's theorem extends this results by demonstrating that any set of POVM operators $\{\hat{E}_{\xi}\}$ can be realized in terms of projective measurements on a larger Hilbert space. A detailed discussion on the Neumark's theorem can be found in Ref. [Per93].

1.3.5 Fidelity between quantum states

A relevant parameter in measurement theory is the definition of overlap between quantum states. For pure states $\{|\psi\rangle, |\chi\rangle\}$, the overlap is defined by the scalar product $|\langle\psi|\chi\rangle|^2$. For mixed state, the parameter which quantifies the overlap between two density matrices $\hat{\rho}$ and $\hat{\sigma}$ is given by the fidelity \mathscr{F} , satisfying the following properties:

- 1. $\mathscr{F}(\hat{\rho}, \hat{\sigma}) = |\langle \psi | \chi \rangle|^2$ for pure states, with $\hat{\rho} = |\psi\rangle \langle \psi|$ and $\hat{\sigma} = |\chi\rangle \langle \chi|$.
- 2. $0 \leq \mathscr{F}(\hat{\rho}, \hat{\sigma}) \leq 1$, where $\mathscr{F}(\hat{\rho}, \hat{\sigma}) = 1$ iff $\hat{\rho} = \hat{\sigma}$.
- 3. Symmetry: $\mathscr{F}(\hat{\rho}, \hat{\sigma}) = \mathscr{F}(\hat{\sigma}, \hat{\rho}).$
- 4. Convexity: if $\hat{\sigma}_1, \hat{\sigma}_2 \ge 0$ and $p_1 + p_2 = 1$ then the following holds: $\mathscr{F}(\hat{\rho}, p_1\hat{\sigma}_1 + p_2\hat{\sigma}_2) \ge p_1\mathscr{F}(\hat{\rho}, \hat{\sigma}_1) + p_2\mathscr{F}(\hat{\rho}, \hat{\sigma}_2)$.
- 5. $\mathscr{F}(\hat{\rho}, \hat{\sigma}) \geq \operatorname{Tr}(\hat{\rho}\hat{\sigma}).$

- 6. Multiplicativity: $\mathscr{F}(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = \mathscr{F}(\hat{\rho}_1, \hat{\sigma}_1) \mathscr{F}(\hat{\rho}_2, \hat{\sigma}_2)$ where 1 and 2 label two different Hilbert spaces.
- 7. Non-decreasing: $\mathscr{F}(\hat{\rho}, \hat{\sigma})$ is invariant under unitary operations. Furthermore, for any measurement process which transforms $\{\hat{\rho}, \hat{\sigma}\}$ in $\{\hat{\rho}', \hat{\sigma}'\}$, the following inequality holds $\mathscr{F}(\hat{\rho}', \hat{\sigma}') \ge \mathscr{F}(\hat{\rho}, \hat{\sigma})$.

The fidelity \mathscr{F} between two quantum states satisfying properties 1-7 is given by the following definition [Joz94]:

$$\mathscr{F}(\hat{\rho}, \hat{\sigma}) = \operatorname{Tr}^{2}\left(\sqrt{\hat{\rho}^{\frac{1}{2}} \hat{\sigma} \hat{\rho}^{\frac{1}{2}}}\right).$$
(1.27)

The previous definition reduces to $\mathscr{F}(|\psi\rangle, \hat{\rho}) = \langle \psi | \hat{\rho} | \psi \rangle$ when one of the system is in a pure state, and to $\mathscr{F}(|\psi\rangle, |\chi\rangle) = |\langle \psi | \chi \rangle|^2$ for the case of two pure states.

1.3.6 Distance between quantum states

In this section we review the definition of a metric in the mixed state's space which is based on the definition of fidelity of Eq. (1.27). In order to correctly define a true distance, the following properties have to be fullfilled:

- (i) Positivity: $d(x, y) \ge 0, \forall (x, y) \in \mathscr{H}$.
- (ii) Nondegenerate: d(x, y) = 0 iff x = y.
- (iii) Symmetry: $d(x, y) = d(y, x), \forall (x, y) \in \mathcal{H}$.
- (iv) Triangular inequalities: $d(x,y) \le d(x,z) + d(z,y), \forall (x,y,z) \in \mathcal{H}$.

Starting from these properties, several distances in the space of mixed states can be identified. The *Bures distance* is defined as [Bur69, Hub92, Hub93]:

$$\mathscr{D}_{B}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}}) = \sqrt{2 - 2[\mathscr{F}(\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\sigma}})]^{1/2}}.$$
(1.28)

The distance \mathscr{D}_B will be exploited in Chapter 4 in the analysis of the resilience to decoherence in macroscopic quantum superpositions. This quantity is bounded between 0 and $\sqrt{2}$, being $\mathscr{D}_B = 0$ when $\hat{\rho} = \hat{\sigma}$ and $\mathscr{D}_B = \sqrt{2}$ for orthogonal states. Finally, the *Bures distance* can be normalized to unity according to the definition:

$$\mathscr{D}(\hat{\rho}, \hat{\sigma}) = \frac{\mathscr{D}_B(\hat{\rho}, \hat{\sigma})}{\sqrt{2}} = \sqrt{1 - [\mathscr{F}(\hat{\rho}, \hat{\sigma})]^{1/2}}.$$
(1.29)

1.3.7 Optimal extraction of information

When dealing with a classical system, the measurement process can be performed assuming that the state of the system remains unperturbed. In the quantum case, the measurement process necessarily acts onto the state under analysis. Hence, a trade-off is established between the amount of information extracted and the amount of disturbance introduced by the process.

Let us consider a system of N identical d-level systems prepared in an input state $|\phi\rangle$ [MP95]. The measurement performed on the system can be modeled by a two-stage process, as shown in Fig. 1.3. As a first step [Fig. 1.3 (a)], the system in the state



Figure 1.3: General measurement scheme for the estimation of an unknown state. (a) The input state $|\phi\rangle$ evolves according to the interaction with the measurement device. (b) The observer reads the output state of the measurement device $|\psi_f^i\rangle_{MD}$ in order to extract information on the input state.

 $|\phi\rangle$ interacts with the measurement apparatus, while in a second step [Fig. 1.3 (b)] the state of the apparatus is read to retrieve the outcome of the measurement. The amount of information which can be extracted from the unknown state is quantified by the fidelity between the guessed state and the real state, averaged over all possible input states. The maximization over all POVMs leads to the optimal state estimation fidelity:

$$\mathscr{F}_{se}^{\text{opt}}(N) = \frac{N+1}{N+d}.$$
(1.30)

which represents the maximum amount of information which can be extracted from the system. The algorithm to determine the optimal POVM which permits to saturate the bound of Eq. (1.30) has been found in [DBE98] for d = 2, corresponding to a set of spin-1/2 particles. These results can be extended by considering a subset of all possible input states. A relevant example is provided by the equatorial qubits, which are defined

by a single parameter φ according to:

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + e^{i\varphi}|1\rangle\right). \tag{1.31}$$

For this subset the optimal estimation fidelity reads:

$$\mathscr{F}_{pe}^{\text{opt}}(N) = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{i=0}^{N-1} \sqrt{\binom{N}{i} \binom{N}{i+1}}.$$
(1.32)

We observe that the optimal fidelity for the complete set of spin-1/2 particles is lower than the fidelity for the subset of equatorial qubits [see Fig.1.4], since in the second case a partial *a-priori* knowledge on the input state is available. This knowledge reduces the number of states accesible to the system thus enhancing the fidelity obtained for a fixed amount of information gained on the state.



Figure 1.4: Comparison between the $1 \rightarrow N$ optimal estimation fidelities for the universal case $\mathscr{F}_{se}^{opt}(N)$ and the phase-covariant case $\mathscr{F}_{pe}^{opt}(N)$. Note that the fidelity for the phase-covariant case is always above the universal case due to the partial a-priori knowledge on the input state.

As said, the amount of information extracted from a system is related to the degree of disturbance introduced by the measurement [Fig. 1.5 (a)]. Let us consider a single copy *d*-level system in an input state $|\psi\rangle$. We can define a density matrix $\hat{\rho}_G$, corresponding to the estimated state, and the density matrix $\hat{\rho}_{out}$ at the output port of the measurement device. The information gain on the state of the system is defined by the fidelity:

$$\mathscr{G} = \langle \psi | \hat{\rho}_G | \psi \rangle. \tag{1.33}$$

In parallel, the degree of perturbation introduced on the system is quantified by:

$$\mathscr{F} = \langle \psi | \hat{\rho}_{out} | \psi \rangle. \tag{1.34}$$

If no operation is performed on the state the input state is left unperturbed, corresponding to $\mathscr{F} = 1$, while no knowledge is extracted about the system, correspoding to $\mathscr{G} = \frac{1}{d}$. On the contrary, an optimal measurement leads to the maximum information gain $\mathscr{G} = \frac{2}{1+d}$, with the impossibility of performing any additional operation on the output state:

 $\mathscr{F} = \frac{2}{1+d}$. In Ref. [Ban01] it has been demonstrated that in general the two quantities are related by the following inequality:

$$\sqrt{\mathscr{F} - \frac{1}{d+1}} \le \sqrt{\mathscr{G} - \frac{1}{d+1}} + \sqrt{(d-1)\left(\frac{2}{d+1} - \mathscr{G}\right)}.$$
(1.35)

Such bound, experimentally proven in Ref. [SRD⁺06], imposes a limit on the maximum amount of information attainable for a given degree of perturbation introduced by the measurement [Fig. 1.5 (b)].



Figure 1.5: (a) Conceptual scheme of the measurement process. The output quantum channel describes the disturbance introduced by the measurement, while the output classical channel describes the information gain. (b) Plot of the optimal curve expressing the trade-off between the information gain \mathscr{G} and the disturbance \mathscr{F} introduced by a quantum measurement. Shaded area corresponds to the forbidden region according to Eq. (1.35).

1.4 Quantum cloning

In quantum information theory, one of the main difference between classical and quantum physics concerns the possibility of copying unknown bits or qubits. Classically, it is always possible to produce an arbitrary number of exact copies of an unknown input bit. On the contrary, in the quantum domain it is not possible to produce two exact copies of an unknown arbitrary input qubit: the No-Cloning theorem [Ghi81, WZ82, Die82]. Such a feature is connected to the impossibility of deterministically estimating the quantum state when only a single copy is available. Indeed, the capability of perfectly copying an unknown state would imply the possibility of estimating the state with fidelity $\mathscr{F} \rightarrow 1$, violating the bound of Eq. (1.30). However, the cloning process can be still performed introducing some errors in the process, that is, the fidelities between the output copies and the input state are lower than 1.

1.4.1 Universal optimal cloning

As said, while the No-Cloning theorem demonstrates the impossibility of obtaining two identical copies of an unknown arbitrary state, such operation can be still performed by obtaining two output copies possessing a fidelity $\mathscr{F}' < 1$ with respect to the input state. The process saturating the maximum fidelity when no a-priori knowledge on the state is available is known as *universal optimal cloning machine*.

The map describing the universal $1 \rightarrow 2$ cloning machine has been reconstructed in Refs. [BH96, BVE⁺98] for spin-1/2 systems. The necessary resources for this process are the input qubit to be cloned $|\psi\rangle$, an empty target qubit $|0\rangle$ and an ancillary system *A* in the state $|X\rangle$. The action of the cloning is described by a unitary evolution of the form:

$$|\psi\rangle|0\rangle|X\rangle \to |\Psi\rangle = \hat{U}|\psi\rangle|0\rangle|X\rangle.$$
 (1.36)

We now perform the following assumptions:

- (i) The two output clones are symmetric. This means that the reduced density matrix for the two clones $\hat{\rho}_{(1,2)} = \operatorname{Tr}_{(2,1),A}(|\Psi\rangle \langle \Psi|)$ must satisfy $\hat{\rho}_1 = \hat{\rho}_2$.
- (ii)-(a) The cloning device does not change the direction in the Bloch sphere, that is, the Bloch vector of the clones satisfies $\mathbf{s}_{(1,2)} = \eta \mathbf{s}_{\Psi}$.
- (ii)-(b) The output fidelity of the clones is independent on the input state, that is, $\text{Tr}(\hat{\rho}_{\psi}\hat{\rho}_{1}) = \text{const.}$

Under these assumptions, the action of the cloning device reduces to the following map:

$$\hat{\rho}_{1,2} = \eta |\psi\rangle \langle \psi| + (1-\eta) \frac{\hat{\mathbb{1}}}{2}, \qquad (1.37)$$

where η acts as a shrinking factor of the original vector in the Bloch sphere, which evolves from the input state $|\psi\rangle \langle \psi| = \frac{1}{2} (\hat{\mathbb{1}} + \mathbf{s} \cdot \hat{\boldsymbol{\sigma}})$ to the output state $\hat{\rho} = \frac{1}{2} (\hat{\mathbb{1}} + \eta \mathbf{s} \cdot \hat{\boldsymbol{\sigma}})$. Note that such result is imposed by the assumptions (i)-(ii) on the rotational invariance of the cloning machine.

The unitary operation can be derived from maximizing the cloning fidelity $\mathscr{F} = \langle \psi | \hat{\rho}_1 | \psi \rangle$ under the constraints (i)-(ii) for a generic unitary operation. The cloning fidelity reads:

$$\mathscr{F} = \frac{1}{2}(1+\eta). \tag{1.38}$$

The optimal value for this quantity is $\mathscr{F} = \frac{5}{6}$, corresponding to $\eta = \frac{2}{3}$. Such result has been extended [BEM98] to the case where *M* copies are obtained by acting on *N* input states, leading to:

$$\mathscr{F}_{univ}^{\text{opt}}(N,M) = \frac{NM+N+M}{M(N+2)}.$$
(1.39)

A plot of the $1 \rightarrow M$ case is reported in Fig. 1.6. Such result has been generalized in Ref. [Wer98] for *d*-level systems.

We conclude by observing that the cloning fidelity can be further related to the state estimation fidelity according to [BEM98, BA06]:

$$\mathscr{F}_{univ}^{\text{opt}}(N,\infty) = \mathscr{F}_{se}^{\text{opt}}(N).$$
(1.40)

This shows that the cloning fidelity from N copies of a system is equivalent to the state estimation fidelities of the N copies.

1.4.2 Phase-covariant optimal cloning

We conclude the discussion on the quantum cloning problem by analyzing the case in which some a-priori knowledge on the input state is present. More specifically, we consider the case in which the input state in the cloning device is restricted to the subset of equatorial qubits of Eq. (1.31). Such a device, since the properties of the input state are defined by the phase factor φ , is called the *phase-covariant optimal cloning machine*. We expect, due to the reduced subset of possible input states, a higher cloning fidelity in this case with respect to the universal case.

By following an analogous approach to Ref. [BEM98], imposing the isotropy of the cloning machine, the output state reads [BCDM00]:

$$\hat{\rho} = \eta_{xy}(N,M) |\varphi\rangle \langle \varphi| + (1 - \eta_{xy}(N,M)), \frac{\hat{\mathbb{1}}}{2}$$
(1.41)

where $\eta_{xy}(N,M)$ is the phase-covariant shrinking factor. The optimal fidelity for this cloning device reads [BCDM00]:

$$\mathscr{F}_{pcc}^{\text{opt}}(N,M) = \frac{1}{2} + 2^{M-N-1} \frac{\sum_{l=0}^{N-1} \sqrt{\binom{N}{l}\binom{N}{l+1}}}{\sum_{j=0}^{M-1} \sqrt{\binom{M}{j}\binom{M}{j+1}}}.$$
(1.42)

In a following step, it was demonstrated [DP01, DM03] that a different map can lead to a cloning fidelity in the $1 \rightarrow 3$ case greater than the limit imposed by Eq. (1.42). The optimal fidelity in the $1 \rightarrow M$ case has been derived, and it reads:

$$\mathscr{F}_{pcc}^{\text{opt}}(1,M) = \frac{1}{2} \left(1 + \frac{M+1}{2M} \right), \quad \text{for } M \text{ odd}; \tag{1.43}$$

$$\mathscr{F}_{pcc}^{\text{opt}}(1,M) = \frac{1}{2} \left(1 + \frac{\sqrt{M(M+2)}}{2M} \right), \quad \text{for } M \text{ even.}$$
(1.44)

In Fig.1.6 we compare the $1 \to M$ fidelities of the universal and phase-covariant cases. As said, the fidelity \mathscr{F}_{pcc}^{opt} in the phase-covariant case is greater than the fidelity \mathscr{F}_{univ}^{opt} in universal case.



Figure 1.6: Optimal curves for the $1 \rightarrow M$ cloning machines in the universal case $\mathscr{F}_{univ}^{opt}(1,M)$ and in the phase covariant case $\mathscr{F}_{pcc}^{opt}(1,M)$. Note that the curve corresponding to the phase-covariant cloning machine is above the universal one due to the reduced subset of input states in the machine.

1.5 Quantum entanglement and nonlocality

Quantum mechanics presents some peculiar properties which do not have a corresponding counterpart in classical physics. Among these properties, entanglement was defined by Schrödinger as "the characteristic trait of quantum mechanics" [Sch35]. This characteristic of quantum mechanics was discovered by Einstein, Podolsky and Rosen in the famous EPR paper [EPR35], defining the emergence of quantum correlations in some classes of composite systems as "spooky action at distance". This property represents both a fundamental aspect of quantum mechanics and a valuable tool for the implementation of quantum-enhanced information protocols. In the last decades, a strong research effort has been devoted to the generation and the characterization of entangled states with increasing number of photons, as well as its exploitation in different areas such as quantum computation [VDB⁺10], communication [UTSM⁺07] or cryptography [BB84, Eke91].

In the EPR paper, the authors provide an argument to demonstrate the inconsistency of quantum mechanics under the assumptions of locality and realism. This argument led to the formulation of the Bell's theorem [Bel64] in terms of the so-called local hidden variables. The violation of the Bell's theorem provides the confutation of all hidden variables models. Several experimental implementations of nonlocality tests relying on different classes of inequalities have been performed with several platforms, such as photons [ADR82, AGR82] or atoms [MMM⁺08]. However, up to now no conclusive experiment has been reported since all these implementations rely on some supplementary assumptions, leading to *loopholes*.

1.5.1 Quantum entanglement of bipartite systems

A density matrix $\hat{\rho}_{AB}$ of a bipartite Hilbert space $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ is separable if and only if it can be written in the following form:

$$\hat{\rho}_{AB}^{\text{sep}} = \sum_{i} p_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i, \text{ with } \sum_{i} p_i = 1, \qquad (1.45)$$

otherwise is entangled. For pure states, the separability condition of Eq. (1.45) reduces to:

$$|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\chi\rangle_B. \tag{1.46}$$

This means that a pure state in a bipartite Hilbert space is separable when its wavefunction can be written as the product of two different uncorrelated functions for the single subspaces.

To better understand the underlying physics, let us consider a simple case of bipartite entangled state, that is, a singlet state of two spin-1/2 particles:

$$|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_{1}|\downarrow\rangle_{2} - |\downarrow\rangle_{1}|\uparrow\rangle_{2}), \qquad (1.47)$$

where $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$ are the projections along the $\hat{\sigma}_z$ quantization axis, and i = 1, 2 labels the particle. For any choice of the quantization axis for the two particles 1 and 2, such state cannot be written in the form of two separate contributions for the single particles alone, thus being an entangled state. From Eq. (1.47) we observe that such state is characterized by the presence of perfect anti-correlations in the spin degree of freedom. This means that simultaneous measurement of the projection of the spin along the same quantization axis for the particles always lead to opposite results. When measuring the spin projection for a single particle alone, the results of the measurement are described by the reduced density operator:

$$\hat{\rho}_{i}^{-} = \operatorname{Tr}_{j}[|\psi^{-}\rangle_{AB}\langle\psi^{-}|] = \frac{1}{2}[|\uparrow\rangle_{i}\langle\uparrow|+|\downarrow\rangle_{i}\langle\downarrow|], \qquad (1.48)$$

with i = 1, 2 and $j \neq i$. Such operator describes the state of a completely mixed spin-1/2 particle. As a general statement, this means that for entangled states it is not possible to describe the state of the joint system in terms of the states of the subsystems, but a common description in the enlarged Hilbert space is necessary.

In general, it is not straightforward to determine whether a state in entangled or not. A strong effort has been recently devoted to develop suitable criteria for the detection of entanglement in bipartite systems. All these criteria can be grouped in two classes. The first one relies on the complete reconstruction of the density matrix, and by subsequently identifying the presence of entanglement on the reconstructed state. For example, the Peres criterion [Per96], which is based on the partial transpose operation and which represents a necessary condition for separability, can be adopted as one of the possible tests to be run on the complete density matrix. The second class of criteria is based on a set of operators called *entanglement witnesses* [LKCH00]. In this case, one aims to define a criterion which does not require the complete reconstruction of the state, but that can be performed by applying a few measurements on the analyzed system. Several criteria relying on this concept have been defined, allowing to develop different classes of entanglement witness operator \hat{W} whose average value is measured on the input state. In order for an operator to be considered a witness, it must posses the following properties:

- 1. For all separable states $\hat{\sigma}$, the average value of the witness operator is nonnegative: Tr $[\hat{\sigma}\hat{\mathscr{W}}] \ge 0$.
- 2. It exists at least one entangled state $\hat{\rho}$ for which the average value of the witness operator is negative: $\text{Tr}[\hat{\rho}\hat{\mathscr{W}}] < 0$.

Hence, if one measures for a state $\hat{\rho}'$ a negative average $\text{Tr}[\hat{\rho}'\hat{\mathcal{W}}] < 0$, such state $\hat{\rho}'$ is entangled. As a direct consequence, a single witness operator $\hat{\mathcal{W}}$ in general does not permit to detect all classes of entangled states, but needs to be appropriately defined for the system under investigation. The main advantage of this approach is that entanglement of a quantum system can be detected by performing only a few measurements on the state and without the need of a complete reconstruction of the state.

1.5.2 Bell's inequalities for bipartite systems: the Bell's theorem

In the original EPR paper [EPR35], the authors formulated the following three hypotheses for any reasonable physical theory:

- (1) *Reality*. If without perturbing a system we can predict with certainty the value of a physical observable, then it exists an element of physical reality associated to the observable.
- (2) *Locality*. Consider two systems *A* and *B* separated by a space-like distance. Any action on *A* cannot influence the measurements performed on *B*, and conversely any action on *B* cannot influence the measurements performed on *A*.
- (3) Completeness. Any element of reality is represented in the physical theory.

EPR showed that for certain classes of states, that is, for certain entangled states, these three assumptions are not consistent. Bell's theorem [Bel64] is formulated in the attempt of restoring the completeness of the theory in terms of a mathematical theory aimed at the description of physical systems according to (1)-(2). Such theory is called *local hidden variable* (LHV) theory, since the outcomes of a measurement operator are defined in terms of a set of variables { λ } which are not accessible by the observer.

Let us now consider the case of two spin-1/2 particles described by a local hidden variables theory. The measurement of the spin projection $A = \boldsymbol{\sigma} \cdot \mathbf{a}$ along axis \mathbf{a} on a single particle is identified by the function $\mathscr{A}(\mathbf{a}, \lambda)$, where λ is the set of hidden variables for the system and $\mathscr{A}(\mathbf{a}, \lambda) = \pm 1$. This formulation describes a deterministic measurement process, whose outcome occurs according to the pre-assigned value of λ [assumption (1)]. The average value of $\langle A \rangle$ is then described by the following integral:

$$\langle A \rangle_{\psi}^{\text{LHV}} = \int d\lambda \, \mu_{\psi}(\lambda) \mathscr{A}(\mathbf{a}, \lambda),$$
 (1.49)

where $\mu(\lambda)$ represents the probability distribution for the hidden variable set λ , thus expressing its inaccessibility to the observer. Correlations between two particles *A* and *B* are expressed in a local hidden variable theory by:

$$E^{\text{LHV}}(\mathbf{a}, \mathbf{b}) = \int d\lambda \,\mu(\lambda) \mathscr{A}(\mathbf{a}, \lambda) \mathscr{B}(\mathbf{b}, \lambda).$$
(1.50)

Here, the measurement outcome for particle A is independent from the setting **b** of particle B, and conversely for the measurement outcome of particle B. This condition expresses the locality of the measurement process [assumption (2)]. Consider now the measurement of the following quantity in a system composed by two particles:

$$S = A \cdot B + A \cdot B' + A' \cdot B - A' \cdot B', \qquad (1.51)$$

where (A, A') measure the spin projection of particle *A* on the direction $(\mathbf{a}, \mathbf{a}')$, and analogously (B, B') measure the spin projection of particle *B* on the direction $(\mathbf{b}, \mathbf{b}')$ [Fig. 1.7]. In LHV theories, we obtain the following bound for $|\langle S \rangle^{\text{LHV}}|$:

$$|\langle S \rangle^{\text{LHV}}| \le 2. \tag{1.52}$$



Figure 1.7: Conceptual scheme for a Bell's inequality test in a CHSH form. The state source generates a two-particle spin state. Particles A and B are measured by two indipendent measurement apparatuses, with measurement settings $(\mathbf{a}, \mathbf{a}')$ and $(\mathbf{b}, \mathbf{b}')$ respectively.

Let us now consider the quantum version of the same experiment. The state of the system is identified by the state $|\psi\rangle$, while the average value of the correlations is obtained by the quantum mechanical average:

$$E^{\text{QM}}(\mathbf{a}, \mathbf{b}) = \langle \boldsymbol{\psi} | (\hat{\boldsymbol{\sigma}}_A \cdot \mathbf{a}) \otimes (\hat{\boldsymbol{\sigma}}_B \cdot \mathbf{b}) | \boldsymbol{\psi} \rangle.$$
(1.53)

If the state $|\psi\rangle$ is chosen to be the singlet-state of Eq. (1.47), by appropriately choosing the measurement settings $(\mathbf{a}, \mathbf{a}')$ and $(\mathbf{b}, \mathbf{b}')$ we obtain:

$$\langle \hat{S} \rangle = \langle \hat{A} \otimes \hat{B} \rangle + \langle \hat{A}' \otimes \hat{B} \rangle + \langle \hat{A} \otimes \hat{B}' \rangle - \langle \hat{A}' \otimes \hat{B}' \rangle = 2\sqrt{2}.$$
(1.54)

Hence, if the measured value of S is greater than 2, the inequality (1.52) is violated and hence LHV models are confuted. This means that one of the hypotheses (1) or (2), that is, realism or locality, has to be abandoned to describe physical systems.

When dealing with the practical realization, some issues may arise when all the hypotheses underlying the test are not satisfied. Three main classes of loopholes can be identified:

- (a) Detection loophole. In presence of a nonunitary detection efficiency η , the detectable correlations are decreased proportionally with η . This effect imposes a lower bound for the minimum detection efficiency η_{th} necessary to obtain a genuine violation of a Bell's inequality. When $\eta < \eta_{th}$, a violation can be still obtained by normalizing the correlations to the detected events. This can be performed by assuming that the fraction of detected events is representative of the whole set of data, that is, by a *fair-sampling assumption*. Experiments closing this loophole have been performed by exploiting atomic qubits [MMM⁺08], which can be measured with detection efficiency $\eta \sim 99\%$.
- (b) Locality loophole. The two observers A and B must not communicate during the measurement. Hence, the two measurement apparatuses have to be separated by a spacelike distance. Experiments closing this loophole have been performed by exploiting photonic qubits propagating up to space-like separation [WJS⁺98].
- (c) Freedom of choice loophole. The measurement settings (a, a') and (b, b') must be chosen independently in order to avoid communication between the observers A and B. This loophole has been closed by exploiting detection apparatuses with random choice of the measurement settings [WJS⁺98].

1.6 Quantum metrology and parameter estimation

In quantum information theory, a relevant task is the measurement of physical quantities. Quantum mechanics introduces some fundamental limits in the maximum precision achievable in measuring an unknown parameter. In this context, two quantities can be introduced to formulate the fundamental bounds: the classical and the quantum Fisher information. Such bounds can be obtained by considering only classical or quantum resources, showing that the employment of quantum probe states can lead to a significant increase in the achievable resolution.

1.6.1 The parameter estimation problem

In the parameter estimation problem [Hel76, Par09] (see Fig. 1.8), a crucial requirement is the development of suitable strategies which permits to obtain an accurate estimate $\overline{\lambda}$ converging to the true value λ of the parameter. To this end a probe system, prepared in a suitable state $\hat{\rho}$ interacts with the physical system under scrutiny. The probe state evolves into $\hat{\rho}_{\lambda} = \hat{U}_{\lambda} \hat{\rho} \hat{U}_{\lambda}^{\dagger}$ when the interaction is unitary, while in general the process is described by a completely positive map $\hat{\rho}_{\lambda} = \mathcal{M}_{\lambda}[\hat{\rho}]$. Finally, the probe state is measured through a detection apparatus described by POVM operators { $\hat{\Pi}_x$ }, being {x} the possible measurement outcomes [Fig. 1.8]. Such process is repeated M times, thus producing a vector of outcomes { x_k } $_{k=1}^M$. The value of the parameter is then retrieved by defining an estimator function $\bar{\lambda} = \bar{\lambda}(x_1, x_2, \dots, x_M)$. The error associated to the estimated value $\bar{\lambda}$ of the true value λ of the parameter is given by the mean square error:

$$V(\lambda) = E_{\lambda}[(\bar{\lambda}(\{x_k\}) - \lambda)^2], \qquad (1.55)$$

where $E_{\lambda}[\cdot]$ stands for the expectation value. Unbiased estimators are those where the estimated value $\overline{\lambda}$ converges to the real value λ of the parameter. In this case, the mean square error is equal to the variance:

$$\operatorname{Var}(\lambda) = E_{\lambda}[\bar{\lambda}^2] - E_{\lambda}[\bar{\lambda}]^2.$$
(1.56)

The aim of parameter estimation is to determine the ultimate bounds in the measurement of the unknown parameter λ and the corresponding optimal strategies, which permit to minimize the error Var(λ) associated the estimated value.



Figure 1.8: General theoretical framework for the parameter estimation problem. The input system in the state $\hat{\rho}$ acquires information on the parameter λ after the interaction (\hat{U}_{λ} or $\mathcal{M}_{\lambda}[\cdot]$), then is measured by a detection apparatus $\hat{\Pi}_x$, and finally the parameter λ is retrieved by a specific choice of the estimator $\bar{\lambda}$. The classical Fisher information $I(\lambda)$ is obtained by minimizing the variance $V(\lambda)$ over all possible choices of the estimator, while the quantum Fisher information $H(\lambda)$ is obtained by minimizing over all possible measurement apparata.

1.6.2 Classical Fisher information

In classical estimation theory, the Cramer-Rao inequality permits to define the lower bound for the variance $Var(\lambda)$ when fixing the measurement operators $\{\hat{\Pi}_x\}$. More specifically, one aims to optimize the choice of the estimator which allows to extract the maximum amount of information on λ from of measurement outcomes $\{x_k\}_{k=1}^M$. In this context, one needs to optimize the strategy for extracting the information from $p(x|\lambda)$, that is, the conditional probability distribution of obtaining the outcome x for a given value of the parameter λ . Such an optimized strategy is called an optimal estimator, and it represents the best choice of the function $\overline{\lambda}(\{x_k\})$. Optimal estimators are defined as those functions $\overline{\lambda}(\{x_k\})$ which saturate the Cramer-Rao inequality:

$$\operatorname{Var}(\lambda) \ge \frac{1}{MI(\lambda)}.$$
 (1.57)
Here *M* is the number of repeated experiments, and $I(\lambda)$ is the Fisher information:

$$I(\lambda) = \int dx p(x|\lambda) \left[\frac{\partial \ln p(x|\lambda)}{\partial \lambda}\right]^2 = \int dx \frac{1}{p(x|\lambda)} \left[\frac{\partial p(x|\lambda)}{\partial \lambda}\right]^2.$$
(1.58)

In quantum mechanics, the probability distribution is defined by: $p(x|\lambda) = \text{Tr}[\hat{\Pi}_x \hat{\rho}_{\lambda}]$. The Fisher information can be expressed in the following form:

$$I(\lambda) = \int dx \frac{(\mathrm{Tr}[\hat{\rho}_{\lambda}\hat{\Pi}_{x}\hat{L}_{\lambda}])^{2}}{\mathrm{Tr}[\hat{\rho}_{\lambda}\hat{\Pi}_{x}]}.$$
(1.59)

Here, \hat{L}_{λ} is the symmetric logarithmic derivative (SLD) such that:

$$\frac{\partial \hat{\rho}_{\lambda}}{\partial \lambda} = \frac{\hat{L}_{\lambda} \hat{\rho}_{\lambda} + \hat{\rho}_{\lambda} \hat{L}_{\lambda}}{2}.$$
(1.60)

A closed form for \hat{L}_{λ} can be obtained by expanding the density matrix $\hat{\rho}_{\lambda}$ in terms of its eigenvalues and eigenvectors $\hat{\rho}_{\lambda} = \sum_{n} \rho_{n} |\psi_{n}\rangle \langle \psi_{n}|$:

$$\hat{L}_{\lambda} = \sum_{n} \frac{\partial_{\lambda} \rho_{n}}{\rho_{n}} |\psi_{n}\rangle \langle\psi_{n}| + 2\sum_{n \neq m} \frac{\rho_{n} - \rho_{m}}{\rho_{n} + \rho_{m}} \langle\psi_{m}|\partial_{\lambda} \psi_{n}\rangle |\psi_{m}\rangle \langle\psi_{n}|, \qquad (1.61)$$

where $\sum_{n \neq m}$ is extended over $\rho_n + \rho_m \neq 0$. The Fisher information $I(\lambda)$ quantifies the amount of information encoded in the probability distribution $p(x|\lambda)$ of the measurement outcome for the specific choice of probe state $\hat{\rho}$ and of the measurement operators $\{\hat{\Pi}_x\}$. Recently, it has been shown that maximum likelihood estimators [LBC93, HMOB96] and Bayesian estimators [PSK⁺07, PS08] are examples of optimal unbiased estimators, which permit to correctly estimate the parameter since $\overline{\lambda} \rightarrow \lambda$ and to saturate asymptotically the Cramer-Rao inequality. Hence, they permit to efficiently analyze the probability distribution of the measurement outcomes $p(x|\lambda)$.

1.6.3 Quantum Fisher information

In order to evaluate the ultimate precision bound in the estimation of λ for a given probe state, it is necessary to maximize the Fisher information of Eq. (1.58) over all possible POVMs { $\hat{\Pi}_x$ }. Such maximization procedure can be performed by the following hierarchy of inequalities [Hel76, Par09]:

$$I(\lambda) \leq \int dx \left| \frac{\operatorname{Tr}[\hat{\rho}_{\lambda}\hat{\Pi}_{x}\hat{L}_{\lambda}]}{\sqrt{\operatorname{Tr}[\hat{\rho}_{\lambda}\hat{\Pi}_{x}]}} \right|^{2} = \int dx \left| \operatorname{Tr} \left[\frac{\sqrt{\hat{\rho}_{\lambda}}\sqrt{\hat{\Pi}_{x}}}{\sqrt{\operatorname{Tr}[\hat{\rho}_{\lambda}\hat{\Pi}_{x}]}} \sqrt{\hat{\Pi}_{x}}\hat{L}_{\lambda}\sqrt{\hat{\rho}_{\lambda}} \right] \right|^{2}$$

$$\leq \int dx \operatorname{Tr}[\hat{\Pi}_{x}\hat{L}_{\lambda}\hat{\rho}_{\lambda}\hat{L}_{\lambda}] = \operatorname{Tr}[\hat{L}_{\lambda}\hat{\rho}_{\lambda}\hat{L}_{\lambda}].$$

$$(1.62)$$

The Fisher information is upper bounded by:

$$I(\lambda) \le H(\lambda) = \operatorname{Tr}[\hat{\rho}_{\lambda}\hat{L}_{\lambda}^{2}] = \operatorname{Tr}[(\partial_{\lambda}\hat{\rho}_{\lambda})\hat{L}_{\lambda}].$$
(1.63)

Here $H(\lambda)$ is the quantum Fisher information (QFI), which sets the ultimate precision on the variance Var(λ) according to the quantum Cramer-Rao inequality:

$$\operatorname{Var}(\lambda) \ge \frac{1}{MH(\lambda)}.$$
 (1.64)

Note that, since the QFI is obtained by maximizing over all possible POVMs, it depends only on the geometry of the family of states $\{\hat{\rho}_{\lambda}\}$. An explicit form of the optimal estimator can be found in terms of the symmetric logarithmic derivative:

$$\hat{O}_{\lambda} = \lambda \,\hat{\mathbb{1}} + \frac{\hat{L}_{\lambda}}{H(\lambda)}.\tag{1.65}$$

By exploiting Eq. (1.61) the quantum Fisher information can be written in terms of the eigenvectors and the eigenvalues of $\hat{\rho}_{\lambda}$ as:

$$H(\lambda) = \sum_{n} \frac{(\partial_{\lambda} \rho_{n})^{2}}{\rho_{n}} + 2 \sum_{n \neq m} \varepsilon_{n,m} |\langle \psi_{m} | \partial_{\lambda} \psi_{n} \rangle|^{2}, \qquad (1.66)$$

where:

$$\varepsilon_{n,m} = \frac{(\rho_n - \rho_m)^2}{\rho_n + \rho_m}.$$
(1.67)

The expression for the quantum Fisher information can be further simplified when the time evolution is unitary $\hat{\rho}_{\lambda} = \hat{U}_{\lambda} \hat{\rho} \hat{U}_{\lambda}^{\dagger}$, and $\hat{U}_{\lambda} = e^{-i\hat{G}\lambda}$. In this case, the quantum Fisher information reads:

$$H(\lambda) = 2\sum_{n \neq m} \varepsilon_{n,m} G_{n,m}^2, \qquad (1.68)$$

where:

$$G_{n,m} = \langle \psi_n | \hat{G} | \psi_m \rangle. \tag{1.69}$$

Finally, in the case of a pure state $\hat{\rho}_{\lambda} = |\psi_{\lambda}\rangle \langle \psi_{\lambda}|$ we obtain the following simplified form in terms of the fluctuations of the generator \hat{G} on the unperturbed state $|\psi_{0}\rangle$:

$$H(\lambda) = 4 \langle \psi_0 | \Delta^2 \hat{G} | \psi_0 \rangle.$$
(1.70)

The quantum Fisher information is directly connection to the metrics in Hilbert spaces. Indeed, the capability of estimating an unknown parameter from a family of states $\{\hat{\rho}_{\lambda}\}$ is related to the distinguishability of these states. It can be shown that the quantum Fisher information is related to the Bures distance (see Sec. 1.3.6) between the states of the family. Let us consider an infinitesimal change $d\lambda$ of the parameter. The distance, and hence

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the distinguishability, between $\hat{\rho}_{\lambda}$ and $\hat{\rho}_{\lambda+d\lambda}$ is related to the quantum Fisher information $H(\lambda)$ according to:

$$\mathscr{D}_{B}^{2}(\hat{\rho}_{\lambda},\hat{\rho}_{\lambda+d\lambda}) = \frac{1}{4}H(\lambda)(d\lambda)^{2}.$$
(1.71)

In general the optimal measurement strategy, that is, the one saturating the quantum Cramer-Rao bound, may depend on the value of the parameter λ . When dealing with the detection of a small variation $d\lambda$, the quantum Cramer-Rao bound can be saturated since one can employ the system in the optimal operating regime, that is, by choosing the optimal measurement strategy for λ . In the general case, when no a-priori knowledge is available on λ , it is necessary to exploit an adaptive protocol. In this case, the first subset of measurement is typically exploited to obtain a rough estimate of λ , and then to apply the optimal estimation strategy depending on the results of the first step. In general, the limit imposed by the quantum Cramer-Rao bound may not be achievable, while in some cases Eq. (1.64) can be saturated asymptotically for large M [Nag88, OP09, GLM11]. Let us consider a specific example. The input probe state is a coherent state, generated by the application of the displacement operator $\hat{D}(\alpha) = \exp[\alpha \hat{a}^{\dagger} - \alpha^* \hat{a}]$ on an input vacuum state $|0\rangle$. For a more detailed discussion on the displaced vacuum state we refer to Sec. 3.2.4. The quantum Fisher information associated to the state $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ under a unitary evolution $\hat{U}_{\lambda} = \exp[-\iota(\hat{a}^{\dagger}\hat{a})\lambda]$ results to be:

$$H_{\alpha}(\lambda) = 4|\alpha|^2 \tag{1.72}$$

Let us consider the case in which the output state is detected by measuring the $\hat{X}_{\theta} = (\hat{a}e^{-i\theta} + \hat{a}^{\dagger}e^{i\theta})/2^{1/2}$ operator, that is, by exploiting a homodyne detection apparatus. In this case, the classical Fisher information associated to the measurement outcomes read:

$$I_{\alpha}(\lambda) = 4|\alpha|^2 \sin^2(\lambda + \theta)$$
(1.73)

Hence, the value of θ must be appropriately chosen according to the relation $\theta + \lambda = \pi/2$ in order to saturate the quantum Cramer-Rao bound. This can be obtained by employing a first fraction of the measurements to retrieve a rough estimate λ_0 of the parameter λ , and by sequently choosing the value of θ according to the condition $\theta + \lambda_0 = \pi/2$.

1.6.4 Quantum enhancement in parameter estimation

Having discussed the optimal bounds achievable when optimizing the measurement and the data-processing stages for a fixed probe state, then the last step is to develop the best strategies in terms of the choice of the probe states. Recently, it has been proposed [GLM04, GLM06] that the employment of quantum resources can lead to a significant enhancement in the achievable resolution. Let us consider the configurations described in Figs. 1.9 (a) and (b). A k-probe state is prepared before the interaction \hat{U}_{λ} . Then, the state after the interaction is measured with a certain choice of the measurement, including the possibility of performing an entangled measurement upon the k probes. Finally, the



Figure 1.9: Resume of the theoretical framework for parameter estimation theory. (a) Standard quantum limit $(\delta\lambda)_{SQL}$ achievable with a separable k-probe input state. (b) Heisenberg limit $(\delta\lambda)_{HL}$ achievable with an entangled k-probe input state.

experiment is repeated M times in order to improve the statistical significance. When classical resources are adopted at the probe stage [Fig. 1.9 (a)], the achievable resolution on the parameter λ reads:

$$(\delta\lambda)_{\text{SQL}} \ge \frac{1}{\sqrt{kM}}.$$
 (1.74)

In optical interferometry, that is, the measurement of an optical phase, such bound is the standard quantum limit (SQL), or shot-noise limit (SNL), which is obtained with Mrepeated measurements on a k-photon probe in a classical state. Such a limit is obtained for example by employing coherent states of the electromagnetic field. An enhancement in the resolution $\delta\lambda$ can be obtained when quantum properties, such as entanglement or squeezing, are present in the probe state [Fig. 1.9 (b)]. In this case, the resolution $\delta\lambda$ is limited by the following inequality:

$$(\delta\lambda)_{\rm HL} \ge \frac{1}{k\sqrt{M}}.\tag{1.75}$$

Such inequality is the Heisenberg limit (HL), which is basically due to the Heisenberg principle between pairs of conjugated variables, in this case photon number and phase. Note that the presence of entanglement is a crucial requirement only at the probe stage, and that in general entangled measurements are not necessary to obtain a quantum enhancement in the estimation of the parameter λ [GLM06].

Let us now consider a specific example to illustrate how quantum-enhanced protocols permit to achieve sub-SQL performances. As a first case, we consider a Mach-Zehnder interferometer for the estimation of an optical phase [Fig. 1.10 (a)]. The input probe state, given by a coherent state $|\psi_{in}\rangle = |\alpha\rangle_1 |0\rangle_2$ containing an average number of photons $\langle n \rangle_{\alpha} = |\alpha|^2$, is injected in one or both the two ports of the input beamsplitter. Then, a relative phase shift $\hat{U}_{\phi} = e^{i(\hat{a}_2^{\dagger}\hat{a}_2)\phi}$ is introduced between the two paths in the interferometer. Finally, the state is recombined at the second beam-splitter and the difference in the photon-number $I(\phi) = \langle (\hat{n}_1 - \hat{n}_2) \rangle$ is recorded. Suppose now we want to measure a small



Figure 1.10: (a) Mach-Zehnder interferometer for the estimation of an optical phase with coherent states. (b) Interferometric scheme for the estimation of an optical phase at the Heisenberg limit with a N00N state.

phase shift ϕ around the $\overline{\phi} = \pi/2$ point. The resolution on ϕ is limited by the fluctuations on the measured signal $I(\phi)$ and by the slope of the signal function according to:

$$\delta\phi = \frac{\delta I(\phi)}{\left|\frac{\partial I_{\phi}}{\partial \phi}\right|}.$$
(1.76)

The amount of detected signal is given by:

$$I_{\alpha}(\phi) = |\alpha|^2 \cos \phi. \tag{1.77}$$

The resolution on ϕ around the $\bar{\phi} = \pi/2$ point results to be:

$$\delta\phi = \frac{1}{\sqrt{\langle n \rangle}_{\alpha}},\tag{1.78}$$

which is the standard quantum limit of Eq. (1.74). The adoption of an entangled probe state, shown in Fig. 1.10 (b) permits to increase the achievable resolution on ϕ . Let us consider the injection of a *N*-photon input state $|\psi_{in}\rangle = \sum_{n=0}^{N} c_n |n\rangle_1 |N - n\rangle_2$. The c_n are chosen in such a way to obtain the following form for the state propagating inside the interferometer after the beam-splitter:

$$|\psi_{\text{N00N}}\rangle = \frac{1}{\sqrt{2}}(|N\rangle_1|0\rangle_2 - |0\rangle_1|N\rangle_2).$$
 (1.79)

Such class of states is known as N00N states [BKA⁺00, Dow08], corresponding to the presence of N photons distributed coherently either in arm 1 or 2. The action of the phase

shift in the state results to be:

$$|\psi_{\text{N00N}}\rangle = \frac{1}{\sqrt{2}} (|N\rangle_1|0\rangle_2 - e^{iN\phi}|0\rangle_1|N\rangle_2).$$
(1.80)

Note that the presence of N photons in a collective state is responsible for a $e^{iN\phi}$ phase term, while with classical fields the introduced phase shift term is $e^{i\phi}$. The output field is then analyzed by measuring the following operator:

$$\hat{\Sigma}_N = |N\rangle_1 \langle 0| \otimes |0\rangle_2 \langle N| + |0\rangle_1 \langle N| \otimes |N\rangle_2 \langle 0|.$$
(1.81)

The recorded signal is given by the following expression:

$$I_{\text{N00N}}(\phi) = \langle \psi_{\text{N00N}} | \hat{\Sigma}_N | \psi_{\text{N00N}} \rangle = \cos(N\phi).$$
(1.82)

The resolution achievable on ϕ is given by:

$$\delta\phi = \frac{1}{N},\tag{1.83}$$

thus reaching the Heisenberg limit of Eq. (1.75).

Chapter 2

Elements of quantum and nonlinear optics

Quantum optics represents a powerful platform for the implementation of quantum information protocols, due to the availability of sources of high quality entangled states, adopted as the information carriers. These properties, combined with the realization of optical detectors able to discriminate single photons, have led to the implementation of several quantum information protocols such as communication [UTSM⁺07], cryptography [THT⁺10] and computation [CVD⁺09, VDB⁺10]. All these experiments involving discrete-valued degrees of freedom of the photon are included in the general field of discrete quantum optics. In this Chapter we introduce the basic elements of quantum and nonlinear optics which will be exploited throughout this thesis. We discuss in details two different sources of quantum fields based on a nonlinear optical process called parametric down-conversion. We discuss their application for the generation of entangled states and squeezed light, and for the process of optimal quantum cloning. Finally, we briefly review the problem of direct detection of quantum fields by discussing the photon-counting technique and single-photon detection.

2.1 Noncollinear parametric down-conversion

Parametric down-conversion happens in nonlinear crystals having nonvanishing $\chi^{(2)}$ coefficient. This source, first introduced in Ref. [KMW⁺95] by Kwiat et al., is based on a particular geometric configuration which possesses full rotational invariance in the polarization degree of freedom, and is currently one of the most commonly exploited entangled state source in quantum optics experiments.

2.1.1 The optical configuration

The source under consideration is based on a second order nonlinear process known as parametric down-conversion, a three-wave interaction mediated by a nonlinear crystal with nonvanishing second order susceptibility tensor $\chi^{(2)}$ [Boy07]. A quantum description of the process is given by the annihilation of a photon in the pump beam and the creation of two photons at frequencies ω_1 and ω_2 with wave vectors \mathbf{k}_s and \mathbf{k}_i [Fig. 2.1 (a)]. The three photons involved in the interaction must obey $\omega_P = \omega_s + \omega_i$ and the phasematching condition $\mathbf{k}_P = \mathbf{k}_s + \mathbf{k}_i$, which represent respectively the conservation of energy and of the photon momentum in the crystal. Two configurations are possible for the linear polarization of the three photons. In type-I down-conversion, the two photons on modes \mathbf{k}_s and \mathbf{k}_i present the same polarization, while the photon from the pump presents the orthogonal polarization.

In order to obtain an entangled state in a type-II system it is necessary to choose a particular orientation for the crystal optical axis. Generally the down-converted photons are emitted along two distinct cones. By properly choosing the axis the two cones intersect along two particular directions \mathbf{k}_1 and \mathbf{k}_2 [Fig. 2.1 (a)]. Moreover, it is possible to obtain, along these directions, photons that are frequency degenerate so to make them indistinguishable both in energy and polarization [Fig. 2.1 (b)]. As a matter of fact, the



Figure 2.1: (a) Optical configuration for the noncollinear type-II parametric downconversion source. The relative orientation between the optical axis and the \mathbf{k}_P vector of the pump beam is set to obtain an intersection between the ordinary and extraordinary emission cones along two directions \mathbf{k}_1 and \mathbf{k}_2 . The output photons emitted along these directions are entangled due to the polarization and frequency indistinguishability of the spatial modes. (b) Diagrammatic scheme for the transition induced by the presence of the nonlinear medium.

presence of one photon with $\vec{\pi}_o$ polarization in one of the two modes implies the presence of one photon with $\vec{\pi}_e$ polarization on the twin mode. At first order, we then expect that the emitted state is given by the entangled singlet state:

$$|\psi^{-}\rangle_{\mathbf{k}_{1},\mathbf{k}_{2}} = \frac{1}{\sqrt{2}} \left(|o\rangle_{\mathbf{k}_{1}}|e\rangle_{\mathbf{k}_{2}} - |e\rangle_{\mathbf{k}_{1}}|o\rangle_{\mathbf{k}_{2}}\right).$$

$$(2.1)$$

2.1.2 Interaction Hamiltonian and time evolution

The interaction Hamiltonian of a down-conversion source can be evaluated as [Boy07]:

$$\hat{\mathscr{H}}_{\text{SPDC}} = \int d^3 \mathbf{r} \boldsymbol{\chi}^{(2)} \cdot \hat{\mathbf{E}}_p^{(+)}(\mathbf{r},t) \cdot \hat{\mathbf{E}}_A^{(s)}(\mathbf{r},t) \cdot \hat{\mathbf{E}}_B^{(+)}(\mathbf{r},t) + \text{h.c.}, \qquad (2.2)$$

where the integral is restricted to the crystal volume. Here, $\hat{\mathbf{E}}_j(\mathbf{r},t)$ are the electric field operators for the three fields involved in the interaction. For an intense pump, the field operator $\hat{\mathbf{E}}_p(\mathbf{r},t)$ can be replaced with the corresponding classical amplitude $\mathbf{E}_p(\mathbf{r},t)$. Starting from this expression, the interaction Hamiltonian (2.2) with the k-vector emission geometry shown in Fig. 2.1 in the limit of a monochromatic pump beam can be written as:

$$\hat{\mathscr{H}}_{\text{SPDC}} = \iota \hbar \chi (\hat{a}_{1\pi}^{\dagger} \hat{a}_{2\pi_{\perp}}^{\dagger} - \hat{a}_{1\pi_{\perp}}^{\dagger} \hat{a}_{2\pi}^{\dagger}) + \text{h.c.}, \qquad (2.3)$$

where $\{\pi, \pi_{\perp}\}$ stands for any set of orthogonal polarization modes and χ is the nonlinear constant that describes the strength of the interaction according to:

$$\chi \propto \chi_{p,s,i}^{(2)} E_P L_z \operatorname{sinc}\left(\Delta k_z \frac{L_z}{2}\right),$$
(2.4)

where $\chi_{p,s,i}^{(2)}$ is the element of the susceptibility tensor, E_P is the amplitude of the pump beam, L_z is the crystal length and sinc $\left(\Delta k_z \frac{L_z}{2}\right)$ is the phase-matching term which takes into account the phase-matching condition ($\Delta \mathbf{k} = \mathbf{k}_P - \mathbf{k}_1 - \mathbf{k}_2$).

The time evolution equation describing the action of the source in the interaction picture [Sak03] is obtained by considering the unitary evolution operator:

$$\hat{U}_{\text{SPDC}} = e^{g(\hat{a}_{1\pi}^{\dagger} \hat{a}_{2\pi_{\perp}}^{\dagger} - \hat{a}_{1\pi_{\perp}}^{\dagger} \hat{a}_{2\pi}^{\dagger}) - g(\hat{a}_{1\pi} \hat{a}_{2\pi_{\perp}} - \hat{a}_{1\pi_{\perp}} \hat{a}_{2\pi})},$$
(2.5)

where $g = \chi t$ is the nonlinear gain of the amplifier. Such operator can be expressed in a different form by exploiting the operatorial relation [Col88] $e^{g(\hat{\sigma}_+ + \hat{\sigma}_-)} = e^{\Gamma \hat{\sigma}_+} e^{-\ln C \hat{\sigma}_z} e^{\Gamma \hat{\sigma}_-}$, where $\Gamma = \tanh g$, $C = \cosh g$, and $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$:

$$\hat{U}_{\text{SPDC}} = e^{\Gamma(\hat{a}_{1\pi}^{\dagger} \hat{a}_{2\pi\perp}^{\dagger} - \hat{a}_{1\pi\perp}^{\dagger} \hat{a}_{2\pi}^{\dagger})} e^{-\ln C(2 + \hat{n}_{1\pi} + \hat{n}_{2\pi\perp} + \hat{n}_{1\pi\perp} + \hat{n}_{2\pi})} e^{\Gamma(\hat{a}_{1\pi\perp} \hat{a}_{2\pi} - \hat{a}_{1\pi} \hat{a}_{2\pi\perp})}, \qquad (2.6)$$

where we have made use of the following relations:

$$\hat{\sigma}_{+} = \hat{a}_{1\pi}^{\dagger} \hat{a}_{2\pi_{\perp}}^{\dagger}, \ \hat{\sigma}_{-} = -\hat{a}_{1\pi} \hat{a}_{2\pi_{\perp}}, \ \hat{\sigma}_{z} = 1 + \hat{n}_{1\pi} + \hat{n}_{2\pi_{\perp}}; \tag{2.7}$$

$$\hat{\sigma}'_{+} = -\hat{a}^{\dagger}_{1\pi_{\perp}}\hat{a}^{\dagger}_{2\pi}, \ \hat{\sigma}'_{-} = \hat{a}_{1\pi_{\perp}}\hat{a}_{2\pi}, \ \hat{\sigma}'_{z} = 1 + \hat{n}_{1\pi_{\perp}} + \hat{n}_{2\pi}.$$
(2.8)

Hence, for any input state $|\psi_{in}\rangle$, the output state of the source can be evaluated as: $|\Psi_{out}\rangle = \hat{U}_{SPDC} |\psi_{in}\rangle$.

The time evolution due to the interaction Hamiltonian $\hat{\mathscr{H}}_{SPDC}$ can also be described in the Heisenberg picture where the annihilation and creation operators evolve as:

$$\imath \hbar \frac{d\hat{a}_{\mathbf{k},\pi}}{dt} = \left[\hat{a}_{\mathbf{k},\pi}, \hat{\mathscr{H}}_{\text{SPDC}}\right],\tag{2.9}$$

and the corresponding Hermitian conjugate equation. This set of differential equations can be solved analitically, obtaining [Yar89, De 98a, DS05]:

$$\hat{a}_{1\pi}^{\dagger}(t) = \hat{a}_{1\pi}^{\dagger}C + \hat{a}_{2\pi\perp}S; \qquad \hat{a}_{1\pi\perp}^{\dagger}(t) = \hat{a}_{1\pi\perp}^{\dagger}C - \hat{a}_{2\pi}S, \qquad (2.10)$$

$$\hat{a}_{2\pi}^{\dagger}(t) = \hat{a}_{2\pi}^{\dagger}C - \hat{a}_{1\pi_{\perp}}S; \qquad \hat{a}_{2\pi_{\perp}}^{\dagger}(t) = \hat{a}_{2\pi_{\perp}}^{\dagger}C + \hat{a}_{1\pi}S.$$
(2.11)

where $C = \cosh g$ and $S = \sinh g$.

2.1.3 Generation of polarization entangled states

Such a source can be adopted to generate entangled states in the polarization degree of freedom in a spontaneous emission regime. In the low gain regime, the state emitted by the source is obtained by applying the first order expansion of the time evolution operator (2.5) to a two mode vacuum state:

$$\hat{U}_{\text{SPDC}} |0\pi, 0\pi_{\perp}\rangle_{1} |0\pi, 0\pi_{\perp}\rangle_{2} \sim [\hat{1} + (\hat{a}_{1\pi}^{\dagger} \hat{a}_{2\pi_{\perp}}^{\dagger} - \hat{a}_{1\pi_{\perp}}^{\dagger} \hat{a}_{2\pi}^{\dagger})] |0\pi, 0\pi_{\perp}\rangle_{1} |0\pi, 0\pi_{\perp}\rangle_{2} = = |0\pi, 0\pi_{\perp}\rangle_{1} |0\pi, 0\pi_{\perp}\rangle_{2} + g(|1\pi, 0\pi_{\perp}\rangle_{1} |0\pi, 1\pi_{\perp}\rangle_{2} - |0\pi, 1\pi_{\perp}\rangle_{1} |1\pi, 0\pi_{\perp}\rangle_{2}),$$

$$(2.12)$$

where $|n\pi, m\pi_{\perp}\rangle_i$ labels a Fock state with $n \pi$ -polarized photons and $m \pi_{\perp}$ -polarized photons on spatial mode \mathbf{k}_i . Hence, by removing the vacuum contribution the down conversion analyzed in this source permits to generate a singlet spin-1/2 entangled state in polarization. This result can be extended by considering the evolution induced by \hat{U}_{SPDC} , without any approximation, on the vacuum state, leading to an output state $|\Psi^-\rangle$ [KMW⁺95]:

$$|\Psi^{-}\rangle = \frac{1}{C^2} \sum_{n=0}^{\infty} \Gamma^n \sqrt{n+1} |\Psi_{-}^n\rangle, \qquad (2.13)$$

where $\Gamma = \tanh g$, $C = \cosh g$. Here, $|\psi_{-}^{n}\rangle$ is the singlet spin-n/2 polarization state, corresponding to the generation of *n* photon pairs:

$$|\psi_{-}^{n}\rangle = \frac{1}{\sqrt{n+1}} \sum_{m=0}^{n} (-1)^{m} |(n-m)\pi, m\pi_{\perp}\rangle_{1} |m\pi, (n-m)\pi_{\perp}\rangle_{2}.$$
(2.14)

Hence, at each order the generated state is entangled between the two spatial modes \mathbf{k}_1 and \mathbf{k}_2 . Such a property is due to the indistinguishability between the two emission cones of the source. The present source in the spontaneous regime will be further investigated in Chap. 7 to perform nonlocality tests in a multiphoton configuration.

2.1.4 Universal optimal cloning machine

The rotational invariance of the entangled source analyzed in this section, described by the Hamiltonian \mathscr{H}_{SPDC} of Eq. (2.3), can be applied to the problem of quantum cloning, analyzed in Sec. 1.4.1, in a stimulated emission regime.

Let us consider the injection on mode \mathbf{k}_1 , of a single photon state in the generic polarization state $|\pi\rangle_1$ in the low gain regime. By neglecting the identity contribution $\hat{1}$ in the expansion this state can be written as:

$$\hat{U}_{\text{SPDC}}|1\pi,0\pi_{\perp}\rangle_{1}|0\pi,0\pi_{\perp}\rangle_{2} \approx \sqrt{\frac{2}{3}}|2\pi,0\pi_{\perp}\rangle_{1}|0\pi,1\pi_{\perp}\rangle_{2} - \sqrt{\frac{1}{3}}|1\pi,1\pi_{\perp}\rangle_{1}|1\pi,0\pi_{\perp}\rangle_{2}.$$
(2.15)

Here, the Fock state $|1\pi, 1\pi_{\perp}\rangle_1$ is the symmetric combination of the two photons in mode **k**₁, labeled as 1*a* and 1*b*:

$$|1\pi, 1\pi_{\perp}\rangle_{1} = \frac{1}{\sqrt{2}} \left(|\pi\rangle_{1a} |\pi_{\perp}\rangle_{1b} + |\pi_{\perp}\rangle_{1a} |\pi\rangle_{1b} \right), \qquad (2.16)$$

where the latter equation is written in the first quantization formalism. The reduced density operator for any of the two photons 1x on spatial mode \mathbf{k}_1 reads:

$$\hat{\rho}_{1x} = \operatorname{Tr}_{1x',2}(\hat{\rho}_{12}) = \frac{5}{6} |\pi\rangle_{1x} \langle \pi| + \frac{1}{6} |\pi_{\perp}\rangle_{1x} \langle \pi_{\perp}|, \qquad (2.17)$$

with x = a, b and $x' \neq x$. The fidelity between the input state $|\pi\rangle_1$ and the reduced output states $\hat{\rho}_{1x}$, calculated from the definition (1.27), reads:

$$\mathscr{F}(|\pi\rangle_1, \hat{\rho}_{1x}) = {}_1\langle \pi | \hat{\rho}_{1x} | \pi \rangle_1 = \frac{5}{6}.$$
(2.18)

Such result shows that the noncollinear parametric source analyzed in this section performs the $1 \rightarrow 2$ universal optimal quantum cloning. Furthermore, this result can be generalized by considering an *N*-photon input state and by analyzing the *M*-photon contribution of the output state. Indeed, it can be shown that such device performs the optimal $N \rightarrow M$ universal quantum cloning operation [PSS⁺03, DPS04].

2.2 Collinear parametric down-conversion

A type-II crystal, with optical axis oriented so as to make the ordinary and extraordinary cones tangent, realizes a collinear optical parametric amplifier. This configuration due to the collinear operation is particularly suitable for the generation of multiphoton fields in the high gain regime.

2.2.1 The optical configuration

The optical parametric amplifier (OPA) working in a collinear regime is obtained by exploiting a type-II nonlinear crystal, where the relative orientation between the \mathbf{k}_P vector and the crystal optical axis is set so that the two emission cones are made to be tangent along one direction identified by the wave vector \mathbf{k} [Fig. 2.2]. The phase matching condition is again set to obtain a degenerate operating regime, that is, $\omega_i = \omega_P/2$ where i = o, e.

Such configuration presents the feature of having the pump beam and the generated field which propagate along the same direction inside the crystal. In this way, the generated photons act as further seeds for the process, allowing for an effective enhancement in the nonlinear gain of the amplifier with respect to the noncollinear configuration. Such feature of the collinear amplifier is suitable for the generation of multiphoton output fields up to $10^4 - 10^5$ particles [DSV08, VST⁺10a].

2.2.2 Interaction Hamiltonian and time evolution

The interaction Hamiltonian of the amplifier can be evaluated starting from Eq. (2.2) and by restricting *k*-vector emission geometry to the configuration shown in Fig. 2.2, leading



Figure 2.2: (a) Optical configuration for the noncollinear type-II parametric downconversion source. The two ordinary and extraordinary emission cones are tangent along one direction \mathbf{k} in this configuration. (b) Diagrammatic scheme for the transition induced by the presence of the nonlinear medium.

to [GW97]:

$$\hat{\mathscr{H}}_{\text{OPA}} = \imath \hbar \chi \int d\omega_o \int d\omega_e f(\omega_o, \omega_e) [\hat{a}_o^{\dagger}(\omega_o) \hat{a}_e^{\dagger}(\omega_e)] + \text{h.c.}, \qquad (2.19)$$

where $f(\omega_o, \omega_e)$ is a nonsymmetric function that takes into account the spectral correlations between the emitted photons, and is a function of the amplitude of the pump beam and of the geometry of the source. This expression can be further simplified in the limit of a monochromatic pump beam, leading to the following interaction Hamiltonian:

$$\hat{\mathscr{H}}_{\text{OPA}} = \imath \hbar \chi e^{\imath \lambda} (\hat{a}_{H}^{\dagger} \hat{a}_{V}^{\dagger}) + \text{h.c.} = \imath \hbar \chi e^{\imath (\lambda - \varphi)} \left(\frac{\hat{a}_{\varphi}^{\dagger 2}}{2} - \frac{\hat{a}_{\varphi_{\perp}}^{\dagger 2}}{2} \right) + \text{h.c.}$$
(2.20)

where λ is the phase of the pump beam. The latter expression is a good approximation for the interaction Hamiltonian of the source even in the case of a broadband pulsed pump beam.

In the interaction picture, the time evolution of an input state in the amplifier is described by the following unitary operator:

$$\hat{U}_{\text{OPA}} = e^{ge^{i\lambda}\hat{a}_{H}^{\dagger}\hat{a}_{V}^{\dagger} - ge^{-i\lambda}\hat{a}_{H}\hat{a}_{V}} = e^{ge^{i(\lambda-\varphi)} \left(\frac{\hat{a}_{\varphi}^{\dagger}2}{2} - \frac{\hat{a}_{\varphi_{\perp}}^{\dagger}2}{2}\right) - ge^{-i(\lambda-\varphi)} \left(\frac{\hat{a}_{\varphi}^{2}}{2} - \frac{\hat{a}_{\varphi_{\perp}}^{2}}{2}\right)}, \quad (2.21)$$

where $\{\hat{a}_{\varphi}, \hat{a}_{\varphi_{\perp}}\}\$ are the annihilation operators for the equatorial polarization modes $\vec{\pi}_{\varphi} = (\vec{\pi}_H + e^{i\varphi}\vec{\pi}_V)/\sqrt{2}$ and $\vec{\pi}_{\varphi_{\perp}} = (\vec{\pi}_{\varphi})_{\perp}$. In the $\{\vec{\pi}_H, \vec{\pi}_V\}$ polarization basis the time evolution can be written as:

$$\hat{U}_{\text{OPA}}^{(HV)} = e^{\Gamma e^{i\lambda} \hat{a}_{H}^{\dagger} \hat{a}_{V}^{\dagger}} e^{-\ln C(1+\hat{n}_{H}+\hat{n}_{V})} e^{-\Gamma e^{-i\lambda} \hat{a}_{H} \hat{a}_{V}}, \qquad (2.22)$$

with $\Gamma = \tanh g$ and $C = \cosh g$. For any equatorial polarization basis, the unitary evolution of the amplifier takes the form of two separate single-mode contributions:

$$\hat{U}_{\text{OPA}}^{(\phi)} = e^{g\left(e^{i(\lambda-\phi)}\frac{\hat{a}_{\phi}^{\dagger}}{2} - e^{-i(\lambda-\phi)}\frac{\hat{a}_{\phi}^{2}}{2}\right)}; \quad \hat{U}_{\text{OPA}}^{(\phi_{\perp})} = e^{-g\left(e^{i(\lambda-\phi)}\frac{\hat{a}_{\phi_{\perp}}^{\dagger}}{2} - e^{-i(\lambda-\phi)}\frac{\hat{a}_{\phi_{\perp}}^{2}}{2}\right)}.$$
(2.23)

Hence, the amplifier acts independently on the two orthogonal equatorial polarization modes. The two operators $\hat{U}_{\text{OPA}}^{(\phi)} \in \hat{U}_{\text{OPA}}^{(\phi_{\perp})}$ can be expressed separately as:

$$\hat{U}_{\text{OPA}}^{(\phi)} = e^{\Gamma e^{i(\lambda-\phi)}\frac{\hat{a}_{\phi}^{\dagger}}{2}} e^{-\ln C\left(\frac{1}{2}+\hat{n}_{\phi}\right)} e^{-\Gamma e^{-i(\lambda-\phi)}\frac{\hat{a}_{\phi}^{2}}{2}}, \qquad (2.24)$$

$$\hat{U}_{\text{OPA}}^{(\varphi_{\perp})} = e^{-\Gamma e^{i(\lambda-\varphi)}\frac{\hat{a}_{\varphi_{\perp}}^{\dagger\,2}}{2}} e^{-\ln C\left(\frac{1}{2}+\hat{n}_{\varphi_{\perp}}\right)} e^{\Gamma\left(e^{-i\varphi\frac{\hat{a}_{\varphi}}{2}}\right)}.$$
(2.25)

We can now proceed by solving the Heisenberg equations of the field operators. In the Heisenberg picture the evolution reads:

$$\iota \hbar \frac{d\hat{a}_{\mathbf{k},\pi}}{dt} = \left[\hat{a}_{\mathbf{k},\pi}, \hat{\mathscr{H}}_{\text{OPA}}\right].$$
(2.26)

For the $\{\vec{\pi}_H, \vec{\pi}_V\}$ polarization basis, one has:

$$\hat{a}_{H}^{\dagger}(t) = \hat{a}_{H}^{\dagger}C + e^{-\iota\lambda}\hat{a}_{V}S; \quad \hat{a}_{V}^{\dagger}(t) = \hat{a}_{V}^{\dagger}C + e^{-\iota\lambda}\hat{a}_{H}S,$$
(2.27)

while for any equatorial polarization basis:

$$\hat{a}^{\dagger}_{\varphi}(t) = \hat{a}^{\dagger}_{\varphi}C + e^{-\iota(\lambda-\varphi)}\hat{a}_{\varphi}S; \quad \hat{a}^{\dagger}_{\varphi_{\perp}}(t) = \hat{a}^{\dagger}_{\varphi_{\perp}}C - e^{-\iota(\lambda-\varphi)}\hat{a}_{\varphi_{\perp}}S, \qquad (2.28)$$

where $S = \sinh g$.

2.2.3 Field in the spontaneous emission regime

The field emitted in the spontaneous emission regime can be calculated as $|\Phi^0\rangle = \hat{U}_{\text{OPA}}^{(HV)}|0\rangle$ [see Eq. (2.22)], leading to:

$$|\Phi^{0}\rangle = \frac{1}{C} \sum_{n=0}^{\infty} (\Gamma e^{i\lambda})^{n} |nH, nV\rangle.$$
(2.29)

The photon-number distribution on a single polarization mode will then be:

$$P_0(m) = \frac{1}{C^2} \Gamma^{2m} = \frac{\langle \hat{n} \rangle^m}{\left(1 + \langle \hat{n} \rangle\right)^m},$$
(2.30)

where $\langle \hat{n} \rangle$ is the average number of photons:

$$\langle 0|\hat{a}_{H}^{\dagger}(t)\hat{a}_{H}(t)|0\rangle = \overline{n} = \sinh^{2}g.$$
(2.31)

The photon-number distribution of Eq. (2.30) is a Planckian distribution corresponding to a thermal state.

The output field in the generic equatorial basis $\{\vec{\pi}_{\varphi}, \vec{\pi}_{\varphi_{\perp}}\}$ reads:

$$|\Phi^{0}\rangle = \frac{1}{C} \sum_{j,k=0}^{\infty} \left[e^{i(\lambda-\varphi)}\right]^{j+k} \left(\frac{\Gamma}{2}\right)^{j} \left(-\frac{\Gamma}{2}\right)^{k} \frac{\sqrt{(2j)!}\sqrt{(2k)!}}{j!k!} |(2j)\varphi,(2k)\varphi_{\perp}\rangle.$$
(2.32)

Note that the photons are emitted in pairs along the same polarization mode, leading to an output photon-number distribution with only terms corresponding to an even number of photons. In the equatorial polarization bases, the time evolution induced by the collinear amplifier takes the form of a squeezing operation [WM95] on the single polarization mode quadrature variables, defined as: $\hat{X}^{\varphi}_{\theta} = (\hat{a}_{\varphi}e^{-\iota\theta} + \hat{a}^{\dagger}_{\varphi}e^{\iota\theta})/\sqrt{2}$. It can be shown that the variance of the $\hat{X}^{\varphi}_{\theta}$ operators takes the form [WM95]:

$$V(X_{\theta}^{\varphi}) = \frac{1}{2} [C^2 + S^2 + 2CS\cos(\lambda - \varphi - 2\theta)].$$
 (2.33)

Hence, for $\theta_+ = (\lambda - \varphi)/2$ and $\theta_- = \theta_+ + \pi/2$ one obtains respectively: $V(X_{\theta_+}^{\varphi}) = e^{2g}/2$ and $V(X_{\theta_-}^{\varphi}) = e^{-2g}/2$. Such property is known as squeezing and corresponds to the presence of reduced fluctuations on one quadrature variable $X_{\theta_-}^{\varphi}$, with corresponding magnified fluctuations on the conjugate variable $X_{\theta_+}^{\varphi}$, still satisfying the minimum uncertainty relation according to: $V(X_{\theta_+}^{\varphi})V(X_{\theta_-}^{\varphi}) = 1/4$. The squeezing property will be discussed later on in Sec. 3.2.4 within the context of continuous variables quantum optics.

2.2.4 Phase-covariant optimal cloning machine

In Sec. 2.1.4 we showed that the noncollinear amplifier can be applied within the context of quantum cloning. An analogous result holds for the collinear amplifier analyzed in this section. More specifically, such a device performs the optimal phase-covariant cloning in the $1 \rightarrow M$ case due to the symmetry properties of the Hamiltonian (2.20). In the specific case in which the amplifier is injected by a single photon in the $\vec{\pi}_{\varphi}$ polarization state the output state reads:

$$|\Phi^{\varphi}\rangle = \frac{1}{C^{2}} \sum_{j,k=0}^{\infty} [e^{i(\lambda-\varphi)}]^{j+k} \left(\frac{\Gamma}{2}\right)^{j} \left(-\frac{\Gamma}{2}\right)^{k} \frac{\sqrt{(2j+1)!}\sqrt{2k!}}{j!k!} |(2j+1)\varphi,(2k)\varphi_{\perp}\rangle.$$
(2.34)

Note that the photons are always emitted in pairs, and the number of photons in the two orthogonal polarization modes presents different parities. However, the presence of the injected seed leads to a strong unbalancement in the output photon-number distribution along the $\vec{\pi}_{\varphi}$ polarization mode due to the stimulated emission process. The average number of photons generated in the injected $\vec{\pi}_{\varphi}$ polarization mode is found to be $\langle \Phi^{\varphi} | \hat{n}_{\varphi} | \Phi^{\varphi} \rangle = 3\bar{n} + 1$, while in the orthogonal polarization mode we obtain $\langle \Phi^{\varphi} | \hat{n}_{\varphi_{\perp}} | \Phi^{\varphi} \rangle = \bar{n}$ due to the spontaneous emission contribution.

In the low gain limit, we can neglect all the contributions with order greater that 2 with respect to g. The first order term, written in the first quantization formalism by labeling the 3 photons with x = a, b, c, can be written as:

$$|\Phi^{\varphi}\rangle \approx \frac{\sqrt{3}}{2} e^{i(\lambda-\varphi)} |\varphi\rangle_{a} |\varphi\rangle_{b} |\varphi\rangle_{c} - \frac{1}{2} e^{i(\lambda-\varphi)} |\{\varphi,\varphi_{\perp},\varphi_{\perp}\}\rangle, \qquad (2.35)$$

where $|\{\varphi, \varphi_{\perp}, \varphi_{\perp}\}\rangle$ is the symmetric combination:

$$|\{\varphi,\varphi_{\perp},\varphi_{\perp}\}\rangle = \frac{1}{\sqrt{3}} (|\varphi\rangle_{a}|\varphi_{\perp}\rangle_{b}|\varphi_{\perp}\rangle_{c} + |\varphi_{\perp}\rangle_{a}|\varphi\rangle_{b}|\varphi_{\perp}\rangle_{c} + |\varphi_{\perp}\rangle_{a}|\varphi_{\perp}\rangle_{b}|\varphi\rangle_{c}). \quad (2.36)$$

The quantum state for any of the three photons present in the output state can be calculated by evaluating the partial trace on the overall state $\hat{\rho}_{abc}$, leading to:

$$\hat{\rho}_{x} = \operatorname{Tr}_{x',x''}(\hat{\rho}_{abc}) = \frac{5}{6} |\varphi\rangle_{x} \langle \varphi| + \frac{1}{6} |\varphi_{\perp}\rangle_{x} \langle \varphi_{\perp}|, \qquad (2.37)$$

for x = a, b, c and $x \neq x' \neq x''$. The fidelity between any of the output photons and the input photon $|\varphi\rangle$ then reads:

$$\mathscr{F}(\hat{\rho}_x) = \langle \varphi | \hat{\rho}_x | \varphi \rangle = \frac{5}{6}.$$
 (2.38)

Such value of the fidelity corresponds to the optimal cloning fidelity of Eq. (1.43) for the phase-covariant cloning machine in the $1 \rightarrow 3$ case. Such result can be generalized to the $1 \rightarrow M$ case by analyzing the *M*-photon contribution of Eq. (2.34), showing that the cloning operation is optimal in the $1 \rightarrow M$ case [SD05, NDSD07].

2.3 Detection of photonics fields with discrete variable techniques

In this section we briefly discuss two methods for the direct detection of photonic fields. The first case is given by photon-counting apparata, such as photomultipliers and photodiodes, which are capable of producing an output photocurrent proportional to the number of impinging photons. When dealing with single photons, avalanche photodiodes (APD) are exploited due to their capability of generating a macroscopic current when only a single photon is absorbed.

2.3.1 Photon-counting

In order to detect light, it is necessary to exploit the interaction between light fields and matter. Indeed, the absorption of each photon extracts a single electron from a solid state device, so that all the emitted electrons produce an output current proportional to the number of incident photons. However, such absorption process is probabilistic and cannot be performed with unitary efficiency. Hence, each photodetection process is characterized by a parameter η , that is, the quantum efficiency, which quatifies the fraction of impinging photons which lead to a photoelectron. Typical examples of devices which permit photon-counting detection are provided by photomultipliers and photodiodes.

Let us consider the case of a photodetector with quantum efficiency η , with detection window of duration *T*. The probability of obtaining *m* counts from an input density matrix $\hat{\rho}$ of a stationary field of frequency ω reads:

$$p_m(\eta) = \sum_{k=m}^{\infty} \rho_{kk} \binom{k}{m} \eta^m (1-\eta)^{k-m}.$$
(2.39)

The POVM operators, decribing the occurence of *m* counts according to standard measurement theory $p_m(\eta) = \text{Tr}[\hat{\rho}\hat{\Pi}_m(\eta)]$, can be written as [FOP05]:

$$\hat{\Pi}_m(\eta) = \eta^m \sum_{k=m}^{\infty} (1-\eta)^{k-m} \binom{k}{m} |k\rangle \langle k|.$$
(2.40)

2.3.2 Single-photon detection

Most of the detectors adopted for this technique (photomultipiers, photodiodes) are not able to detect the presence of a single-photon, being a single electron too weak to produce a detectable photocurrent. In order to produce a macroscopic photocurrent when only single-photons are absorbed an avalanche processes is required. All this means that such detectors cannot discriminate the number of impinging photons, because they lead to equal output currents when the number of photons absorbed is different from zero. Single photon detection can occur only with limited efficiency η . Such an effect can be modeled by the insertion of a beam-splitter with transmittivity η in the transmission path of the field before an ideal detector. The POVM operators describing the action of lossy singlephoton detectors can be written as [FOP05]:

$$\hat{\Pi}_{0}(\eta) = \sum_{k=0}^{\infty} (1-\eta)^{k} |k\rangle \langle k|; \quad \hat{\Pi}_{1}(\eta) = \hat{\mathbb{1}} - \hat{\Pi}_{0}(\eta).$$
(2.41)

Chapter 3 Continuous variables quantum optics

Since the development of the first protocols and optical sources, single photons and few photon states have represented a valuable tool for the implementation of several quantum information tasks. In parallel, an alternative quantum optical approach based on the continuous quantum variables has been developed. The two approaches present different strong and weak points, which can potentially lead to a hybrid platform in order to exploit the advantages of both approaches. In the present chapter we review the elements of continuous-variables quantum optics, by discussing the tools adopted for the representation and the measurement of quantum states. As shown in this Chapter, continuousvariables are suitable for the description and the measurement of multiphoton fields. For this reason, these techniques will be applied throughout the thesis to analyze quantum properties of multiphoton fields generated by the process of optical parametric amplification.

3.1 Continuous-variables quantum optics and quantum information

Discrete-variables quantum optics, which exploits degrees of freedom presenting only a finite set of possible values, lead to several implementations of quantum information protocols in the single-photon and few-photon regime. Alongside, an alternative quantum optical approach, based on continuous-variables (CV), has been developed. In this case, quantum information is encoded in a pair of conjugated field variables (X, P), that is, its quadratures.

The discrete- and continuous-variables approaches present different features in both the generation and the measurement stages. Optical sources for the generation of discrete variables quantum states rely on conditional configurations, since the vacuum state has to be ruled out from the output state. This allows to generate quantum states with high values of purity, at the cost of a nonunitary generation probability. Within the context of continuous variables, two main classes of states can be identified depending on whether their representation in the quadrature space (X, P) presents gaussian distribution or not. Continuous-variables gaussian states, such as squeezed light, can be generated with unconditional sources at the cost of introducing a certain amount of noise. However, different continuous variables quantum information protocols [OKW00, CRM02, OPB03, RGM⁺03, CKN⁺05, CLP07, LRH08] require the presence of nongaussianity. An example is provided by quantum error correction [NFC09], which cannot be performed with only gaussian resources. Nongaussianity has to be achieved with Kerr-type interactions, or by exploiting heralded schemes which reduce the success rates. Discrete-variables quantum states are typically measured by exploiting single-photon counting methods, which provide information on the photon number of the field. With the current technology, such techniques can be still performed with a limited detection efficiency, thus reducing the successful events rate. Continuous-variables quantum states are measured by homodyne detection apparatuses, which permit to obtain information on the phase properties of the field and can be performed with high efficiency. In summary, discretevariables quantum information protocols can be implemented with high purity, but with a limited events rate. On the other side, gaussian continuous-variable protocols can be performed in an unconditional fashion but with a limited purity [see Fig. 3.1].

In this section, we highlight the key features of continuous-variables quantum infor-



Figure 3.1: Schematic comparison between discrete- and continuous- variables quantum optics and quantum information.

mation, that is, generation and detection of quantum states. We first describe the general formalism for describing quantum states in the (X,P) space in terms of the quasiprobability distributions, and we provide some examples of commonly adopted states. Then, we discuss the homodyne detection technique for the measurement and the reconstruction of optical states in the (X,P) space.

3.2 Representation of quantum states in the quantum phase space

In Chap. 1 we reviewed the representation of quantum states through the density operator $\hat{\rho}$. This representation can be exploited to describe the state of the electromagnetic field in the photon-number basis, that is, the space spanned by the Fock vectors $|n\rangle$ defined by the presence of *n* excitations in the optical mode. In this section we describe a complementary approach to represent a generic state in the quantum analogue of the phase-space. Such a representation is obtained through the quasi-probability distributions, a mathematical tool which allows us to calculate the average values of physical observables.

3.2.1 Quadrature operators and the quantum phase-space

We begin by defining the quadrature operators for the quantized electromagnetic field. In a parametric amplifier, the interaction Hamiltonian for the free field, where the quantization is performed by assuming the confinement in a cavity of volume V, can be written as the sum of quantum harmonic oscillators, one for each mode of the field, according to:

$$\hat{\mathscr{H}}_{R} = \sum_{\mathbf{k}} \sum_{\pi} \hbar \omega_{\mathbf{k}} \left(\hat{a}_{\mathbf{k},\pi}^{\dagger} \hat{a}_{\mathbf{k},\pi} + \frac{1}{2} \right).$$
(3.1)

The quadrature operators for the mode (\mathbf{k}, π) are defined as:

$$\hat{X}_{\mathbf{k},\pi} = \frac{1}{\sqrt{2}} (\hat{a}_{\mathbf{k},\pi} + \hat{a}_{\mathbf{k},\pi}^{\dagger}); \quad \hat{P}_{\mathbf{k},\pi} = \frac{\iota}{\sqrt{2}} (\hat{a}_{\mathbf{k},\pi}^{\dagger} - \hat{a}_{\mathbf{k},\pi}).$$
(3.2)

Such operators obey to the canonical commutation relation, and consequently to the Heisenberg principle:

$$\left[\hat{X}_{\mathbf{k},\pi},\hat{P}_{\mathbf{k},\pi}\right] = \iota; \quad \Delta^2 X_{\mathbf{k},\pi} \Delta^2 P_{\mathbf{k},\pi} \ge \frac{1}{4}.$$
(3.3)

The $(\hat{X}_{\mathbf{k},\pi}, \hat{P}_{\mathbf{k},\pi})$ operators represent an analogue of the position and the momentum operator for a quantum mechanical oscillator. In this formalism, the Hamiltonian can be written as:

$$\hat{\mathscr{H}}_{R} = \sum_{\mathbf{k}} \sum_{\pi} \frac{\hbar \omega_{\mathbf{k}}}{2} \left(\hat{X}_{\mathbf{k},\pi}^{2} + \hat{P}_{\mathbf{k},\pi}^{2} \right).$$
(3.4)

Furthermore, they are associated to the cosine and sine oscillating term of the electric field operator. This can be shown by writing the expression of the transverse component of electric field in terms of the quadrature operators:

$$\hat{\mathbf{E}}_{T}(\mathbf{r},t) = \sum_{\mathbf{k}} \sum_{\pi} \mathbf{e}_{\mathbf{k},\pi} \left(\frac{\hbar \omega_{\mathbf{k}}}{2\varepsilon_{0} V} \right)^{1/2} \left\{ \hat{X}_{\mathbf{k},\pi} \cos \left[\chi_{\mathbf{k}}(\mathbf{r},t) \right] + \hat{P}_{\mathbf{k},\pi} \sin \left[\chi_{\mathbf{k}}(\mathbf{r},t) \right] \right\}, \quad (3.5)$$

where $\chi_{\mathbf{k}}(\mathbf{r},t) = \omega_{\mathbf{k}}t - \mathbf{k} \cdot r - \pi/2$ is the phase angle and $\mathbf{e}_{\mathbf{k},\pi}$ is the polarization vector.

3.2.2 Wigner function

In classical optics, the state of a coherent electromagnetic field is perfectly defined by its classical amplitude $|\alpha|$ and by its optical phase ϕ , or equivalently by its position X and momentum P. This means that it is possible to define a classical phase-space (X, P), in which any state (coherent or incoherent) is characterized by a probability distribution $\mathscr{P}(X,P)$, which can be interpreted as the probability of finding a certain pair (X,P) for the position and the momentum when a simultaneous measurement of these two quantities is performed. In quantum mechanics, such a definition of probability distribution $\mathscr{P}(X,P)$ in the space defined by the operators (\hat{X},\hat{P}) cannot be provided. More specifically, a probability distribution in the strict sense cannot be defined since no meaning can be attributed to the probability of finding a certain value of (X, P) when a simultaneous measurement of (\hat{X}, \hat{P}) is performed. The reason for this can be found in the Heisenberg uncertainty principle, which sets a lower bound for the uncertainty achievable in a simultaneous measurement of (\hat{X}, \hat{P}) according to $\Delta^2 X \Delta^2 P > 1/4$. However, a description of quantum states in the quantum analogue of the classical phase-space is still possible in terms of quasi-probability distributions. Such distributions can be interpreted as mathematical tools for describing a general state in the quantum phase-space, allowing us to calculate average values of operators as classical probability distributions. Furthermore, to describe a quantum state there is no unique choice for a quasi-probability distribution, but a whole class of functions. Among the possible choices of the quasi-probability distributions in the quantum phase-space, the Wigner function [Wig32] represents the most widely exploited in the field of quantum optics. Other relevant examples are the Q function and the Glauber-Sudarshan P function [CG69a], being respectively a regular and a highly singular quasi-probability distribution.

The Wigner function is defined as the Fourier transform of the characteristic function:

$$\tilde{W}(\mu,\nu) = \operatorname{Tr}\left[\hat{\rho}e^{-\iota\mu\hat{X}-\iota\nu\hat{P}}\right].$$
(3.6)

The exponential operator can be expressed as $e^{-\iota \nu \hat{P}} e^{-\iota \mu \hat{X}} e^{-\iota \mu \nu/2}$ according to the Baker-Campbell-Hausdorff formula $e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2}$, leading to:

$$\tilde{W}(\mu,\nu) = e^{-\iota\mu\nu/2} \operatorname{Tr}\left[\hat{\rho}e^{-\iota\nu\hat{P}}e^{-\iota\mu\hat{X}}\right].$$
(3.7)

By expanding the trace in the position operator eigenbasis $\{|q\rangle\}$, by considering that the action of the $e^{-\iota v \hat{P}}$ on the position eigenstate $|q\rangle$ leads to a shift $|q\rangle \rightarrow |\xi + v\rangle$, and by changing the integration variables in $q = \xi - v/2$, we find:

$$\tilde{W}(\mu, \mathbf{v}) = \int_{-\infty}^{\infty} d\xi e^{-\iota\mu\xi} \langle \xi - \mathbf{v}/2|\hat{\rho}|\xi + \mathbf{v}/2\rangle.$$
(3.8)

Finally, the Wigner function is obtained as the Fourier transform of the characteristic function, leading to [Leo98]:

$$W(X,P) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi e^{i\nu P} \langle X - \nu/2 | \hat{\rho} | X + \nu/2 \rangle.$$
(3.9)

As already discussed, while the Wigner function does not represent a trusted probability distribution, it can be still exploited to calculate the statistical momenta of the operators \hat{X} and \hat{P} . Indeed, the Wigner function is normalized to unity according to:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX dP W(X, P) = 1.$$
(3.10)

Furthermore, the marginal distributions of the \hat{X} and \hat{P} operators are well defined and present all the properties of a trusted probability distribution. The marginals of the two variables can be evaluated as:

$$\mathscr{P}(X) = \int_{-\infty}^{\infty} dP W(X, P), \text{ and: } \mathscr{P}(P) = \int_{-\infty}^{\infty} dX W(X, P),$$
 (3.11)

and the momenta of X and P can be evaluated by standard statistical techniques.

In general, we can define the rotated quadrature set $\{\hat{X}_{\theta}\}$:

$$\hat{X}_{\theta} = \frac{1}{\sqrt{2}} (\hat{a}e^{-\iota\theta} + \hat{a}^{\dagger}e^{\iota\theta}) = \hat{X}\cos\theta + \hat{P}\sin\theta, \qquad (3.12)$$

and the marginal distributions of the rotated quadratures can be evaluated as:

$$\mathscr{P}(X_{\theta},\theta) = \int_{-\infty}^{\infty} dP_{\theta} W(X_{\theta}\cos\theta - P_{\theta}\sin\theta, X_{\theta}\sin\theta + P_{\theta}\cos\theta).$$
(3.13)

The overlap between two pure states $|\psi_1\rangle$ and $|\psi_2\rangle$ can be calculated as:

$$|\langle \psi_1 | \psi_2 \rangle|^2 = 2\pi \int_{-\infty}^{\infty} dX dP W_{\psi_1}(X, P) W_{\psi_2}(X, P).$$
 (3.14)

In general, it is possible to associate to any operator \hat{O} a corresponding Wigner function by replacing $\hat{\rho}$ with \hat{O} in Eq. (3.9). One then obtains the overlap formula:

$$\operatorname{Tr}[\hat{O}_{1}\hat{O}_{2}] = 2\pi \int_{-\infty}^{\infty} dX dP W_{1}(X, P) W_{2}(X, P), \qquad (3.15)$$

where (\hat{O}_1, \hat{O}_2) are two arbitrary operators, and (W_1, W_2) are the corresponding Wigner functions. The Wigner function can be then exploited to evaluate the average values of any operator \hat{O} on a density matrix $\hat{\rho}$ as:

$$\operatorname{Tr}\left[\hat{\rho}\hat{O}\right] = 2\pi \int_{-\infty}^{\infty} dX dP W_{\hat{\rho}}(X, P) W_{\hat{O}}(X, P).$$
(3.16)

The latter expression includes also the operator $|n\rangle\langle m|$. This permits us to calculate the elements of a density matrix $\hat{\rho}$ in a chosen basis $\{|n\rangle\}$ according to:

$$\langle m|\hat{\rho}|n\rangle = 2\pi \int_{-\infty}^{\infty} dX dP W_{\hat{\rho}}(X,P) W_{n,m}(X,P).$$
(3.17)

We conclude this section by observing that the definition of the position operator \hat{X} and of the momentum operator \hat{P} is made up to a factor κ :

$$\hat{X} = \frac{1}{\sqrt{2\kappa}}(\hat{a} + \hat{a}^{\dagger}); \quad \hat{P} = \frac{\iota}{\sqrt{2\kappa}}(\hat{a}^{\dagger} - \hat{a}).$$
 (3.18)

This factor κ can take the values $\{1/2, 1, 2\}$, and the Heisenberg uncertainty principle defining the minimum uncertainty for the set of noncompatible operators $\{\hat{X}, \hat{P}\}$ reads: $\Delta^2 X \Delta^2 P = 1/(4\kappa^2)$. The definition of the Wigner function has to be modified appropriately in order to ensure the normalization condition of Eq. (3.10). In the case $\kappa = 2$, by defining $\eta = (\nu - \iota \mu)/2$ and $\alpha = X + \iota P$ the characteristic function $\chi(\eta)$ and the Wigner function for a state $\hat{\rho}$ can be evaluated respectively as:

$$\tilde{W}(\eta) = \operatorname{Tr}\left[\hat{\rho}e^{\eta\hat{a}^{\dagger}-\eta^{*}\hat{a}}\right], \qquad (3.19)$$

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 \eta \, \tilde{W}(\eta) e^{\eta^* \alpha - \eta \alpha^*}. \tag{3.20}$$

3.2.3 Generalized s-parametrized quasi-probability distributions

We now briefly review the properties of a more general class of quasi-probability distributions, parametrized by a real parameter s and which includes the Wigner function as a subcase. This class of generalized quasi-probability distribution includes the Glauber-Sudarshan P-distribution [Sud63, Gla63] and the Husimi Q-distribution [Hus40]. The choice of the adopted distribution may depend from the specific context, such as for example the measurement of cavity fields [RMB⁺05] or the characterization of the nonclassicality of an optical field [KVBZ11]. It is then necessary to choose appropriately the measurement apparatus, such as homodyne detection for the W-function [VR89], heterodyne detection for the Q-function [YS80], or by opportune nonclassicality filters on the outcome of homodyne detection for the P-function [KVP⁺08, KVH⁺09, KVBZ11]. The class of s-parametrized quasi-probability distributions is defined from its characteristic function:

$$\tilde{W}(\mu, \nu; s) = \tilde{W}(\mu, \nu) \exp\left[\frac{s}{4}(u^2 + \nu^2)\right], \qquad (3.21)$$

where $\tilde{W}(\mu, \nu)$ is the Fourier transform of the Wigner function defined in the previous section. Finally, the s-parametrized quasi-probability distribution W(X, P; s) is obtained as [Leo98]:

$$W(X,P;s) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\mu d\nu \,\tilde{W}(\mu,\nu;s) e^{i\mu X + i\nu P}.$$
(3.22)

The Wigner function corresponds to the case s = 0.

The s-parametrized quasi-probability distributions are related to the action of a lossy channel with efficiency η on a generic state $\hat{\rho}$. The action of losses can be modeled by combining the input state $\hat{\rho}$ with a vacuum state in a beam-splitter with transmittivity η . It can be shown that in the lossy regime the Wigner function of the output state can be written as [Leo93, LR09]:

$$W_{\eta}(X,P) = \frac{1}{\pi(1-\eta)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX' dP' W(X',P') e^{-\frac{(X-\sqrt{\eta}X')^2 + (P-\sqrt{\eta}P')^2}{1-\eta}}.$$
 (3.23)

The Wigner function after losses η can be then evaluated as the s-parametrized quasiprobability distribution for $s = -(1 - \eta)/\eta$ and can be interpreted as a gaussification, or smoothing, of the original Wigner function under the action of gaussian noise. Eq. (3.23) can be generalized to obtain a hierarchy between W(X, P; s) and W(X, P; s') according to [Leo93]:

$$W(X,P;s) = \frac{1}{\pi(s'-s)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX' dP' W(X',P';s') e^{-\frac{(X-X')^2 + (P-P')^2}{s'-s}},$$
(3.24)

where the integral converges for s < s'.

3.2.4 Gaussian states

Gaussian states provide a relevant class of continuous variables quantum states. Such class of states can be exploited as a useful resource for teleportation [BK98, FSB⁺98], cloning [BCI⁺01, CRD04, AJL05], or dense coding [Ban99, BK00, LPJ⁺02]. Furthermore, such states can be described and handled mathematically since they can be easily described in terms of gaussian functions, possessing by definition a gaussian Wigner function. By defining the vector of the quadrature variables $\mathbf{R} = (X, P)^T$, the vector of the quadrature operators $\hat{\mathbf{R}} = (\hat{X}, \hat{P})^T$, and the vector of the average values $\overline{\mathbf{R}} = (\langle \hat{X} \rangle, \langle \hat{P} \rangle)^T$, we obtain that the Wigner function for a general single-mode gaussian state takes the form:

$$W_{\hat{\rho}}(X,P) = \frac{1}{\pi\kappa \text{Det}[\mathbf{V}]} \exp\left\{-\frac{1}{2}(\mathbf{R}-\overline{\mathbf{R}})^T(\mathbf{V})^{-1}(\mathbf{R}-\overline{\mathbf{R}})\right\}.$$
 (3.25)

Here, V is the covariance matrix, defined as:

$$V_{ij} = [\mathbf{V}]_{ij} = \frac{1}{2} \langle \{\hat{R}_i, \hat{R}_j\} \rangle - \langle \hat{R}_i \rangle \langle \hat{R}_j \rangle, \qquad (3.26)$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. The corresponding characteristic function presents an analogue gaussian form in terms of the $\boldsymbol{\xi} = (\mu, \nu)^T$ vector:

$$\tilde{W}(\mu, \nu) = \exp\left\{-\frac{1}{2}\boldsymbol{\xi}^{T}(\boldsymbol{\Omega}\boldsymbol{V}\boldsymbol{\Omega}^{T})\boldsymbol{\xi} - \iota(\boldsymbol{\Omega}\overline{\mathbf{R}})^{T}\boldsymbol{\xi}\right\},$$
(3.27)

where ω is the symplectic matrix:

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix}. \tag{3.28}$$

Such results can be further extended to a general *n*-modes gaussian state by defining the vectors $\mathbf{R}_n = (X_1, P_1, X_2, P_2, \dots, X_n, P_n)^T$, $\overline{\mathbf{R}}_n = (\langle \hat{X}_1 \rangle, \langle \hat{P}_1 \rangle, \langle \hat{X}_2 \rangle, \langle \hat{P}_2 \rangle, \dots, \langle \hat{X}_n \rangle, \langle \hat{P}_n \rangle)^T$, $\boldsymbol{\xi}_n = (\mu_1, \nu_1, \mu_2, \nu_2, \dots, \mu_n, \nu_n)^T$, and the corresponding *n*-modes covariance matrix. We obtain:

$$W_{\hat{\rho}}(\mathbf{R}_n) = \frac{1}{(\pi\kappa)^n \text{Det}[\mathbf{V}_n]} \exp\left\{-\frac{1}{2}(\mathbf{R}_n - \overline{\mathbf{R}}_n)^T (\mathbf{V}_n)^{-1} (\mathbf{R}_n - \overline{\mathbf{R}}_n)\right\}, \quad (3.29)$$

$$\tilde{W}(\boldsymbol{\xi}_n) = \exp\left\{-\frac{1}{2}\boldsymbol{\xi}_n^T(\boldsymbol{\Omega}_n \mathbf{V}_n \boldsymbol{\Omega}_n^T)\boldsymbol{\xi}_n - \iota(\boldsymbol{\Omega}_n \overline{\mathbf{R}}_n)^T \boldsymbol{\xi}_n\right\},\tag{3.30}$$

where $\mathbf{\Omega}_n$ is the *n*-modes symplectic matrix:

$$\boldsymbol{\Omega}_n = \bigoplus_{i=1}^n \boldsymbol{\omega}_i; \quad \boldsymbol{\omega}_i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
(3.31)

In the following, we discuss some relevant examples of gaussian states, including coherent states, squeezed vacuum states, and thermal states.

Coherent states

The Glauber coherent α -states are defined as eigenstates of the annihilation operator \hat{a} , and are parametrized by a complex eigenvalue α according to: $\hat{a}|\alpha\rangle = \alpha |\alpha\rangle$. The coherent states are obtained by application of the displacement operator $\hat{D}(\alpha)$ to the vacuum state as:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle; \text{ with } \hat{D}(\alpha) = \exp(\alpha \hat{a}^{\dagger} - \alpha^* \hat{a}).$$
 (3.32)

By exploiting the Baker-Campbell-Hausdorff formula, we obtain the following expression for the coherent state in the Fock space:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$
(3.33)

The evaluation of the Wigner function of $|\alpha\rangle$ state is:

$$W_{\alpha}(X,P) = \frac{1}{\pi} e^{-(X-X_0)^2 - (P-P_0)^2},$$
(3.34)

corresponding to a gaussian distribution satisfying the minimum uncertainty relation given by $\Delta^2 X \Delta^2 P = 1/4$, and centered in the phase-space at $X_0 = \sqrt{2} \alpha \cos \varphi$ and $P_0 = \sqrt{2} \alpha \sin \varphi$, where φ is the phase of complex amplitude $\alpha = |\alpha|e^{i\varphi}$ [see Fig. 3.2 (c)]. Hence, the action of the displacement operator on the vacuum state is to shift the average values of the quadrature operators without affecting their fluctuations.

Squeezed vacuum states

Here we consider the squeezed vacuum gaussian state, which is defined by the action of the squeezing operator $\hat{S}(\tau)$ on the vacuum state:

$$|\tau\rangle = \hat{S}(\tau)|0\rangle; \text{ with } \hat{S}(\tau) = \exp\left\{-\frac{\tau}{2}\hat{a}^{\dagger 2} + \frac{\tau^*}{2}\hat{a}^2\right\},$$
(3.35)

where τ is the complex squeezing parameter, with amplitude $r = |\tau|$ and phase $\lambda = \arg \tau$. The Wigner function of the state reads:

$$W_{\tau}(X,P) = \frac{1}{\pi} e^{-\left\{X^{2}[\cosh(2r) + \cos\lambda\sinh(2r)] + P^{2}[\cosh(2r) - \cos\lambda\sinh(2r)] + 2XP\sin\lambda\sinh(2r)\right\}} = \frac{1}{\pi} e^{-\frac{X_{\theta}^{2}}{e^{-2r}} - \frac{P_{\theta}^{2}}{e^{2r}}}.$$
(3.36)

We note that, depending on the value of the squeezing phase λ , the quadrature X_{θ} with $\theta = \frac{\lambda}{2}$ presents a reduced fluctuation $\Delta^2 X_{\theta} = (2e^{-2r})^{-1}$, while the corresponding $X_{\theta+\pi/2} = P_{\theta}$ quadrature presents an increased fluctuation $\Delta^2 P_{\theta} = (2e^{2r})^{-1}$ [see Fig. 3.2 (b)]. However, the minimum uncertainty relation is preserved after the squeezing operator according to $\Delta^2 X_{\theta} \Delta^2 P_{\theta} = \frac{1}{4}$. Hence, the action of the squeezing operator is to stretch the Wigner function along a certain direction identified by its phase λ .

Thermal states

Thermal states are defined according to:

$$\hat{\rho}^{\text{th}}(N) = \sum_{n=0}^{\infty} \frac{N^n}{(1+N)^{1+n}} |n\rangle \langle n|, \qquad (3.37)$$

where N is the thermal noise parameter. Such class of density matrices describes chaotic light, and is written in the form of an incoherent mixture of photon-number states with a Planckian distribution. The Wigner function of the state reads:

$$W^{\text{th}}(X,P) = \frac{1}{\pi(1+2N)} e^{-\frac{1}{1+2N}(X^2+P^2)}.$$
(3.38)

Hence, the Wigner function of a thermal state is a gaussian function with center in the origin of the phase-space, with increased fluctuations on the quadrature variables $\Delta^2 X = \Delta^2 P = [2(1+2N)]^{-1}$ depending on the value of the thermal noise N [see Fig. 3.2 (c)]. Note that this state does not satisfy anymore the minimum uncertainty relation $\Delta^2 X \Delta^2 P = 1/4$.



Figure 3.2: Bidimensional representation of gaussian states. (a) Action of the displacement operator on the vacuum state. (b) Action of the squeezing operator on the vacuum state. (c) Thermal states (larger circle) present bigger fluctuations in the (\hat{X}, \hat{P}) operators with respect to the vacuum state (smaller circle).

Single-mode gaussian states

It can be shown that [Bv05, FOP05] a general single-mode gaussian state can be obtained starting from a thermal state, and performing linear and bilinear operations, such as the displacement and squeezing operations, on it. More specifically, the following form holds:

$$\hat{\rho}_G = \hat{D}(\alpha)\hat{S}(\tau)\hat{\rho}^{\text{th}}(N)\hat{S}^{\dagger}(\tau)\hat{D}^{\dagger}(\alpha).$$
(3.39)

This means that a general gaussian state can be obtained by squeezing along a certain direction θ a thermal state, and by subsequently applying a displacement in the phase-space. The Wigner function of this gaussian state can be defined from the vector of first order moments $\mathbf{R} = (\sqrt{2} \Re(\alpha), \sqrt{2} \Im(\alpha))$ and from the variance matrix [Eq. (3.26)]:

$$V_{11} = \frac{2N+1}{2} \left[\cosh(2r) - \cos\lambda \sinh(2r) \right], \qquad (3.40)$$

$$V_{22} = \frac{2N+1}{2} \left[\cosh(2r) + \cos\lambda \sinh(2r) \right], \qquad (3.41)$$

$$V_{12} = V_{21} = -\frac{2N+1}{2}\sin\lambda\sinh(2r).$$
 (3.42)

3.2.5 Nongaussian and nonclassical states

While gaussian states provide a powerful resource in some quantum information tasks, other protocols require the employment of nongaussian resources. There is an ongoing effort to study which protocols are allowed by nongaussian resources. The most no-table example is certainly their use for an optical quantum computer [RGM⁺03, LRH08], where gaussian states does not allow to obtain a significant increase in the computational



Figure 3.3: Diagram for the relation between nongaussianity and nonclassicality for pure and mixed states. Note that the implication nongaussian \rightarrow nonclassical holds only for pure states.

power. Furthermore, some computational steps such as error correction cannot be performed with gaussian resources only [NFC09], thus rendering nongaussianity a necessary requirement for this task. Other examples of quantum information tasks where the employment of nongaussian resources can lead to substantial benefits are teleportation [OKW00, CRM02, OPB03], cloning [CKN⁺05], and storage [CLP07]. Furthermore, it has been recently proposed that nongaussianity, either at the generation stage or at the detection stage, is a crucial requirement for the violation of a Bell's inequality with continuous-variables [RMMJ05]. This can be explained by considering that the outcomes of gaussian measurements on gaussian states can be interpreted in terms of a classical probability distribution.

The presence of nongaussian features is also related to the emergence of nonclassical features. In general, the Wigner function of a state is not completely positive over all the phase space. The emergence of negative values for the Wigner function has been identified as a peculiar feature of quantum physics, and has been connected to the quantum superposition properties of the state [Bar44]. Hence, the presence of negative regions in the Wigner function of a quantum state has been recognized as a sufficient (but not necessary) condition for its nonclassicality. For pure states, the connection between the nonpositivity and the nongaussianity of a Wigner function has been extablished by the Hudson-Piquet theorem [Hud74]. Indeed, for any nongaussian pure state a Wigner function is necessarily non completely positive, and conversely a non completely positive Wigner function is necessarily nongaussian. Hence, any pure nongaussian states present nonglassical features. However, care should be taken when dealing with mixed states. In this case, the Hudson-Piquet theorem does not hold and no direct connection between nongaussianity and nonclassicality can be extablished [see Fig. 3.3]. As an example, in Fig. 3.4 are reported the Wigner function for different states: classical and gaussian state [the vacuum state (a)], nonclassical and gaussian state [the squeezed vacuum state (b)], classical and nongaussian state [a mixture of two coherent states (c)], nonclassical and



Figure 3.4: Different examples of continuous-variables states. (a) The vacuum state. (b) Squeezed vacuum state. (c) Mixture of two coherent states $(|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|)/2$. (d) Single-photon state.

nongaussian state [a pure single-photon state (d)].

According to the previous considerations, it becomes necessary to define opportune nongaussianity and nonclassicality measures in order to characterize both properties in the investigated state. Different measures of nongaussianity [GPB07, GPB08, GP10] and nonclassicality [DMMW00, DR03, MMS04] have been recently proposed. In Chap. 8 we analyze in more details two specific measures, and we apply them to a relevant conditional process, that is, single-photon addition, suitable for the generation of nongaussian resources.

3.3 Detection of quantum states in the quantum phase space: homodyne detection

In this section we describe in details the homodyne detection, an experimental technique to measure the quadrature operators (\hat{X}, \hat{P}) . Homodyning relies on the interference on a

symmetric beam-splitter between the optical signal and a classical coherent state. This technique can be performed with high efficiency, rendering it suitable for the analysis of multiphoton states.

3.3.1 Theory of the pulsed homodyne detection technique

The optical scheme for a homodyne detection apparatus is shown in Fig. 3.5. The signal on mode \mathbf{k}_S is combined in a 50/50 beam-splitter with a coherent state $|\alpha_L\rangle$ on mode \mathbf{k}_L , dubbed as the local oscillator. The beams after the mixing process on modes \mathbf{k}_1 and \mathbf{k}_2 are detected by means of two photodiodes, which give a photocurrent proportional to the number of impinging photons. Finally, the two output photocurrents are subtracted electronically to measure the difference in the photon-number \hat{n}_- . Thanks to the interference the observed mode is naturally selected by the mode of the local oscillator without requiring additional filtering. This feature, combined with the high quantum efficiency of the photodiodes, allows to perform this measurement with extremely high efficiency.



Figure 3.5: Optical scheme for homodyne detection. The signal and the local oscillator are mixed in a 50/50 beam-splitter, and the difference in the photon-number of the two output modes is detected.

Measurement of the rotated quadratures

It is possible to prove that the difference in the photocurrents \hat{N}_{-} is directly related to the measurement of the quadrature operator \hat{X}_{θ} . The measured photon-number in a detection time *T* in a photodiode is obtained as [Lou00]:

$$\hat{N}_{j} = \int_{0}^{T} dt \hat{I}_{j}(t).$$
(3.43)

The photon-flux at the detector's plane, assumed to be at z = 0, can be evaluated as:

$$\hat{I}_{j} = \int_{\text{Det}} d^{2} \boldsymbol{\rho} \, \hat{\boldsymbol{\Phi}}_{j}^{(-)}(\boldsymbol{\rho}, 0, t) \cdot \hat{\boldsymbol{\Phi}}_{j}^{(+)}(\boldsymbol{\rho}, 0, t).$$
(3.44)

where $\boldsymbol{\rho} = (x, y)$ are the transverse coordinates. The spatial integral is performed over the detector area, and $(\hat{\boldsymbol{\Phi}}_{j}^{(+)}, \hat{\boldsymbol{\Phi}}_{j}^{(-)})$ are respectively the positive and negative frequency part

of the photon-flux operator in the paraxial approximation:

$$\hat{\boldsymbol{\Phi}}_{j}^{(+)}(\boldsymbol{\rho},z,t) = \iota \sqrt{c} \sum_{k} \hat{a}_{k} \mathbf{v}_{k}(\boldsymbol{\rho},0,t), \qquad (3.45)$$

$$\hat{\boldsymbol{\Phi}}_{j}^{(-)}(\boldsymbol{\rho},z,t) = -\iota \sqrt{c} \sum_{k} \hat{a}_{k}^{\dagger} \mathbf{v}_{k}^{*}(\boldsymbol{\rho},0,t).$$
(3.46)

Here j = S,L labels the signal and the local oscillator modes. When T is large, the orthogonality relation for the mode functions at the detector plane reads:

$$\int_0^T dt \int_{\text{Det}} d^2 \boldsymbol{\rho} \, \mathbf{v}_k^*(\boldsymbol{\rho}, 0, t) v_m(\boldsymbol{\rho}, 0, t) = \delta_{k,m}. \tag{3.47}$$

The measured photon-number difference $\hat{N}_{-} = \hat{N}_{1} - \hat{N}_{2}$ is obtained by evaluating the difference in the photon-number for the two output modes of the beam-splitter \hat{N}_{i} , where the beam-splitter transformations for the photon-flux operators read:

$$\hat{\boldsymbol{\Phi}}_1 = \frac{1}{\sqrt{2}} (\hat{\boldsymbol{\Phi}}_S + \hat{\boldsymbol{\Phi}}_L), \quad \hat{\boldsymbol{\Phi}}_2 = \frac{1}{\sqrt{2}} (\hat{\boldsymbol{\Phi}}_S - \hat{\boldsymbol{\Phi}}_L).$$
(3.48)

By exploiting the latter expressions we obtain:

$$\hat{N}_{-} = \int_{0}^{T} dt \int_{\text{Det}} d^{2} \boldsymbol{\rho} \hat{\boldsymbol{\Phi}}_{L}^{(-)}(\boldsymbol{\rho}, 0, t) \cdot \hat{\boldsymbol{\Phi}}_{S}^{(+)}(\boldsymbol{\rho}, 0, t) + h.c.$$
(3.49)

Now, the local oscillator photon-flux operator, generated in a high photon-number coherent state, can be replaced by its classical counterpart:

$$\boldsymbol{\Phi}_{L}^{(+)}(\boldsymbol{\rho},0,t) = \iota \sqrt{c} \alpha_{L} \mathbf{v}_{L}(\boldsymbol{\rho},0,t), \qquad (3.50)$$

where α_L is the complex coherent amplitude of the local oscillator, and $\mathbf{v}_L(\boldsymbol{\rho}, 0, t)$ defines its optical mode at the detector plane. By replacing Eqs. (3.45-3.46) and (3.50) in Eq. (3.49), we obtain [Leo98, LR09]:

$$\hat{N}_{-} = \alpha_L^* \hat{a} + \alpha_L \hat{a}^{\dagger} = \sqrt{2} |\alpha_L| \hat{X}_{\theta}.$$
(3.51)

Here, θ is the local oscillator phase difference with the signal beam, and \hat{a} is the annihilation operator of the detected mode, which is defined by the local oscillator mode according to:

$$\hat{a} = \sum_{k} \hat{a}_{k} c \int_{0}^{T} dt \int_{\text{Det}} d^{2} \boldsymbol{\rho} \mathbf{v}_{L}^{*}(\boldsymbol{\rho}, 0, t) \mathbf{v}_{S}(\boldsymbol{\rho}, 0, t).$$
(3.52)

This result shows that the output photocurrent of a homodyne apparatus corresponds to the measurement of \hat{X}_{θ} , where the phase of the observed quadrature is defined by the local oscillator phase. Furthermore, the mode of the electromagnetic field measured by the apparatus is selected by the mode of the local oscillator according to Eq. (3.52).

Mode matching and homodyne detection efficiency

The homodyne apparatus performs the measurement of the \hat{X}_{θ} quadrature operator in the optical mode defined by local oscillator. Hence, an accurate mode-matching (including spectral, spatial and temporal profiles) must be performed between the signal and the local oscillator, in order to measure the \hat{X}_{θ} quadrature of the desired state. Any mismatch between the optical modes of the signal and of the local oscillator is responsible for additional noise in the measurement process, which appears as a reduced value of the overall detection efficiency.

The role of a mode-mismatch between the signal to be detected and the local oscillator can be modeled by the action of beam-splitter of transmittivity η_H combining the desired signal with a vacuum state injected in the other input port. This can be shown by the following consideration. We can write the photon-flux operator of the signal as:

$$\hat{\boldsymbol{\Phi}}_{S}(\boldsymbol{\rho},0,t) = \iota \sqrt{c} \, \hat{a}_{S} \mathbf{v}_{S}(\boldsymbol{\rho},0,t) + \hat{\boldsymbol{\Phi}}_{0}(\boldsymbol{\rho},0,t). \tag{3.53}$$

Here, $\mathbf{v}_{S}(\boldsymbol{\rho}, 0, t)$ defines the profile of the signal mode described by the operator \hat{a}_{S} , while $\hat{\boldsymbol{\Phi}}_{0}(\boldsymbol{\rho}, 0, t)$ is the photon-flux operator for all the other optical modes populated by the vacuum state. By exploiting this definition, the annihilation operator of the measured mode \hat{a} can be written as the sum of two terms:

$$\hat{a} = \sqrt{\eta_H} \, \hat{a}_S + \sqrt{1 - \eta_H} \, \hat{a}_0. \tag{3.54}$$

Here, \hat{a}_0 is the annihilation operator for the effective vacuum-injected mode, while η_H represents the overlap between the modes of the signal and the local oscillator according to [Leo98]:

$$\sqrt{\eta}_{H} = c \int_{0}^{T} dt \int_{\text{Det}} d^{2} \boldsymbol{\rho} \mathbf{v}_{L}^{*}(\boldsymbol{\rho}, 0, t) \mathbf{v}_{S}(\boldsymbol{\rho}, 0, t).$$
(3.55)

Here, η_H takes the role of an effective detection efficiency for the homodyne apparatus. Typical values of the homodyne efficiency in experimental implementations range from 60 - 70% in the pulsed regime up to $\sim 90 - 95\%$ in the continuous-wave regime. Hence, the measured Wigner function corresponds to the convolution of Eq. (3.23), and the output probability distribution reads [LR09]:

$$\mathscr{P}(X_{\theta}, \theta) = \langle : \frac{\exp\left[-(X_{\theta}/\eta - \hat{X}_{\theta})^2\right]/(2\sigma^2)}{\sqrt{2\pi\sigma^2}} : \rangle.$$
(3.56)

where X_{θ} is the realization of the variable and \hat{X}_{θ} is the quadrature operator.

3.3.2 Reconstruction of a quantum state through homodyne tomography

Within the context of quantum optics and quantum information, a crucial requirement for the characterization of any platform is the capability to completely reconstruct the density matrix of the generated states. Homodyne detection represents a relevant technique to characterize the Wigner function of a physical state. The main idea at the basis of this method is embodied in the Radon transform of Eq. (3.13). Such mathematical expression relates the Wigner function of a state $\hat{\rho}$, which includes all the relevant information on its properties, with the probability distributions of the quadrature operator \hat{X}_{θ} . The latter can be measured with a homodyne apparatus, as shown in the previous section.

A typical homodyne apparatus for the tomographics reconstruction of a single-mode state is shown in Fig. 3.6. The input state is analyzed by a conventional balanced homodyne detection apparatus. By scanning over different values θ_m of the local oscillator phase, for instance through a piezoelectric translational stage, several distributions of $\{\mathscr{P}(X_{\theta_m}, \theta_m)\}$ are recorded. Finally, the output data are processed through a suitable algorithm to reconstruct the Wigner function of the state.



Figure 3.6: Optical scheme for homodyne tomography. The quadrature distribution for \hat{X}_{θ} is measured for different values of θ by scanning over the local oscillator phase. The Wigner function is then reconstructed by processing the data through a suitable algorithm as described in the text.

Maximum likelihood estimation

In order to reconstruct the Wigner function of the measured state starting from the probability distributions $\mathscr{P}(X_{\theta}, \theta)$ obtained through homodyne detection, a maximum likelihood approach can be exploited. This approach reduces some artifacts that may arise in some reconstructed states by adopting the inverse Radon transform method [VR89], which is based on performing the direct inversion of Eq. (3.13).

As for all measurement apparatuses, a POVM set of operators $\hat{\Pi}_j$ can be associated to the homodyne detection system. The *j*-th outcome for the measurement occurs with probability $\mathcal{P}(j)$:

$$\mathscr{P}_{\hat{\rho}}(j) = \operatorname{Tr}[\hat{\Pi}_{j}\hat{\rho}]. \tag{3.57}$$

The tomographic problem is then shifted to an inversion problem for the latter relation. We can introduce a nonnegative operator \hat{R} , defined as:

$$\hat{R}[\hat{\rho}] = \frac{1}{N} \sum_{j} \frac{f_j}{\mathscr{P}_{\hat{\rho}}(j)} \hat{\Pi}_j, \qquad (3.58)$$

where f_j is the frequency of the *j*-th outcome of the measurement. For homodyne detection, the operator \hat{R} is given by

$$\hat{R}[\hat{\rho}] = \sum_{j} \frac{f_j}{\operatorname{pr}_{\hat{\rho}}(X_j, \theta_j)} \hat{\Pi}(X_j, \theta_j), \qquad (3.59)$$

where the measurement operator $\hat{\Pi}(X_j, \theta_j)$ is the projector over the eigenstate $|X_j, \theta_j\rangle$ of the quadrature \hat{X}_{θ_j} . The principle of maximum likelihood algorithms is based on the definition of a likelihood function, which finds the physical state which maximizes the probability of obtaining the measured data set.

The likelihood function for this problem is defined as:

$$\mathscr{L}(\hat{\boldsymbol{\rho}}) = \prod_{j} [\mathscr{P}_{\hat{\boldsymbol{\rho}}}(j)]^{f_j}.$$
(3.60)

The state $\hat{\rho}_0$ that maximizes the function \mathscr{L} must satisfy the following conditions:

$$\hat{R}[\hat{\rho}_0]\hat{\rho}_0 = \hat{\rho}_0, \quad \hat{\rho}_0\hat{R}[\hat{\rho}_0] = \hat{\rho}_0, \quad \hat{R}[\hat{\rho}_0]\hat{\rho}_0\hat{R}[\hat{\rho}_0] = \hat{\rho}_0.$$
(3.61)

The next step is then to apply an iterative algorithm to determine the matrix $\hat{\rho}_0$ which maximizes \mathscr{L} . As a starting point, a common approach is to initialize at step 0 the density matrix to $\hat{\rho}^{(0)} = \mathscr{N}[\hat{1}]$, dove $\hat{1}$ is the identity operator and $\mathscr{N}[\cdot]$ imposes the normalization to a unitary trace. The iteration of the algorithm from step *k* to step *k* + 1 is chosen as:

$$\hat{\boldsymbol{\rho}}^{(k+1)} = \mathscr{N}\left[\hat{\boldsymbol{R}}[\hat{\boldsymbol{\rho}}^{(k)}]\hat{\boldsymbol{\rho}}^{(k)}\hat{\boldsymbol{R}}[\hat{\boldsymbol{\rho}}^{(k)}]\right].$$
(3.62)

In some cases, such algorithm does not lead to a monotonical increase of \mathscr{L} for each step of the protocol. In this cases, a different iteration rule is exploited:

$$\hat{\rho}^{(k+1)} = \mathcal{N}\left[\frac{\hat{\mathbb{1}} + \varepsilon \hat{R}}{1 + \varepsilon} \hat{\rho}^{(k)} \frac{\hat{\mathbb{1}} + \varepsilon \hat{R}}{1 + \varepsilon}\right].$$
(3.63)

The precision of this reconstruction algorithm can be increased by increasing the number of sampled quadrature phases θ_m .

Continuous variables quantum optics

Part II

Fundamental tests of Quantum Mechanics with multiphoton states
Chapter 4

Resilience to decoherence and nonclassicality of the multiphoton quantum superpositions generated by amplification of single-photon states

The observation of quantum properties in systems of growing size is limited by the uncontrolled interaction with the environment of any physical system, which hence cannot be completely described by a unitary evolution. For this reason, the production and detection of quantum states possessing a large number of particles is still an open challenge. In this Chapter we analyze in details the properties of the optical parametric amplifier when such a system is exploited to broadcast the properties of a single particle into a multiphoton state. Such transfer of the properties from a microscopic system to a multiphoton one is due to the cloning features of the amplifier, which produce output photons as similar as possible to the input one. Here we show that the produced states present a significant resilience to losses, and that the nonclassicality of the state is preserved until half of the particle are lost. The obtained results can be found in Refs. [DSS09b, DSS09a, SVD⁺09, DSSV10, SVSD10], and open the way to the application of the multiphoton states generated by parametric amplification to both fundamental test of quantum mechanics and application to quantum information protocols. This will be the subject of the rest of this thesis.

4.1 Observation of quantum properties in multiphoton systems

In the last decades the physical implementation of multiphoton quantum superpositions involving a large number of particles has attracted a great deal of attention. Indeed it was generally understood that the experimental realization of a multiphoton quantum superposition is very difficult and, in several instances, practically impossible owing to the extremely short persistence of quantum coherence, that is, of the extremely rapid decoherence due to the entanglement established between the macroscopic system and the environment [NC00, Zur91, Zur03, Zur07]. Formally, the irreversible decay towards a probabilistic classical mixture is implied theoretically by the tracing operation of the overall state over the environmental variables [DSC02, DB04]. In the framework of quantum information different schemes based on optical systems have been undertaken to generate and to detect multiphoton quantum superposition states involving an increasing number of particles. A Cavity OED scheme based on the interaction between Rydberg atoms and a high-Q cavity has lead to the indirect observation of quantum superposition of coherent states and of their temporal evolutions [BHR⁺92, RBH01]. A different approach able to generate freely propagating beams adopts photon-subtracted squeezed states; experimental implementations of quantum states with an average number of photons of around four have been reported both in the pulsed and in the continuous wave regimes [NNNH⁺06, OTBL⁺06, OJTBG07, OFTBG09]. These states exhibit nongaussian characteristics and open new perspectives for quantum computing based on continuous-variable systems, entanglement distillation protocols [ESP02, DLH⁺08] and loophole-free tests of Bell's inequality.

Recently, a different class of multiphoton states based on the process of optical parametric amplification has been realized in order to establish the entanglement between a single photon and a multiphoton state given by an average of many thousands of photons. More specifically, such class of field is generated through a high-gain cloning machine seeded by a single-photon belonging to an entangled pair [De 98a, De 98b, DS05, DSS05, SD05]. A first theoretical insight on the dynamical features of this amplification-based multiphoton states and a thorough experimental characterization of the quantum correlations were recently reported [NDSD07, DSV08]. In the present chapter we perform a thorough theoretical analysis of the class of states based on parametric amplification of single-photons. First, we analyze the evolution of the density matrix after the action of a lossy channel, applying two criteria to assess the quantum superposition properties of these states in the Fock space. Then, we perform a complete quantum phase-space analysis able to recognize the persistence of nonclassical properties after the action of losses [Sch01, Wig32]. Among the different representation of quantum states in the continuousvariables space [CG69a], the Wigner quasi-probability representation has been widely exploited as an evidence of non-classical properties, such as squeezing [WM95] and EPR non-locality [BW98]. In particular, as shown in Chap. 3 the presence of negative quasiprobability regions has been considered as a consequence of the quantum superposition of distinct physical states [Bar44]. By exploiting the properties of this distribution, we focus our interest on the effects of decoherence on the multiphoton states and on the emergence of the classical regime. For both analysis, the results are compared with the paradigmatic example of the superposition of coherent states, $|\alpha\rangle \pm |-\alpha\rangle$.

The present chapter is organized as follows. First, in Sec. 4.2 we introduce and

analyze the class of states under investigation. More specifically, we consider two different configurations for the optical parametric amplification process of single photons; a collinear and a noncollinear one. We calculate both the density matrix and the Wigner function of these states after the action of a lossy channel. Then, in Sec. 4.3 we investigate the resilience to losses of these superposition states by adopting two different criteria. Finally, in Sec. 4.4 we analyze the persistance of nonclassical properties in the Wigner function associated to the parametrically amplified single-photon states, when these states undergo a lossy process.



Figure 4.1: (a) Scheme for the phase-covariant cloning of a single-photon state with a collinear optical parametric amplifier. The beam-splitter [BS(T)] is inserted to simulate the propagation over lossy channels of the output field. (b) Scheme for the universal cloning of a single-photon state with a non-collinear optical parametric amplifier (beam-splitters on spatial modes \mathbf{k}_1 [BS(T_1)] and \mathbf{k}_2 [BS(T_2)]).

4.2 Optical parametric amplification of a single-photon state

As a first step we consider the generation of a multiphoton quantum field, obtained by parametric amplification. Let us briefly describe the conceptual scheme. An entangled pair of two photons in the singlet state $|\Psi^-\rangle_{A,B}=2^{-\frac{1}{2}}(|H\rangle_A|V\rangle_B-|V\rangle_A|H\rangle_B)$ is produced through a spontaneous parametric down-conversion (SPDC) by crystal 1 pumped by a pulsed ultraviolet (UV) pump beam: Fig. 4.1 (a-b). There $|H\rangle$ and $|V\rangle$ stands, respectively, for a single photon with horizontal and vertical polarization ($\vec{\pi}$) while the labels A, B refer to particles associated respectively with the spatial modes \mathbf{k}_A and \mathbf{k}_B . The photon belonging to \mathbf{k}_B , together with a strong UV pump beam, is fed into an optical parametric

amplifier consisting of a second non-linear crystal pumped by the beam \mathbf{k}'_{P} . We consider two different configurations for the amplifier. The first one is the collinear configuration [Fig. 4.1 (a)], already described in Sec. 2.2. In this case, the pairs of amplified photons are emitted over the same spatial mode in two orthogonal $\vec{\pi}$ modes, respectively horizontal and vertical. The second configuration is the non-collinear one [Fig. 4.1 (b)], that is, the configuration adopted for the generation of entangled single-photon pairs. In this case, the pairs of amplified photons are emitted over two different spatial modes.

The main idea beyond this approach is that the action of the process of optical parametric amplification is to broadcast the properties of the input state into a state with a larger number of particles. Indeed, by increasing the gain of the amplifier it is possible to modulate the number of generated photons. This property of the optical parametric amplification process provides a natural approach to progressively increase the number of photons in the state, thus allowing to investigate how the decoherence rate is modified by the presence of a larger number of particles.

4.2.1 Phase-covariant amplifier

Let us now consider the collinear optical configuration leading to the phase-covariant optimal quantum cloning machine: Fig. 4.1 (a) [DSV08].



Figure 4.2: Equivalent model for the collinear phase-covariant optical parametric amplifier. The two polarization modes $\vec{\pi}_{\phi}$ and $\vec{\pi}_{\phi_{\perp}}$ undergo a separate single mode amplification process $\hat{U}_{\text{OPA}}^{(\phi)}$ and $\hat{U}_{\text{OPA}}^{(\phi_{\perp})}$.

By exploiting the results of previous Section and of Sec. 2.2.2, the amplified state for an injected equatorial qubit with $\vec{\pi}_{\phi}$ polarization reads:

$$|\Phi_{\text{OPA}}^{\phi}\rangle = \sum_{i,j=0}^{\infty} \gamma_{ij} |(2i+1)\phi, (2j)\phi_{\perp}\rangle, \qquad (4.1)$$

where $\gamma_{ij} = \frac{1}{C^2} \left(e^{-\iota\phi} \frac{\Gamma}{2} \right)^i \left(-e^{-\iota\phi} \frac{\Gamma}{2} \right)^j \frac{\sqrt{(2i+1)!} \sqrt{(2j)!}}{i!j!}$, $C = \cosh g$, $\Gamma = \tanh g$. Hereafter, the state $|p\psi, q\psi_{\perp}\rangle_i$ stands for a Fock state with p photons polarized $\vec{\pi}_{\psi}$ and q photons polarized $\vec{\pi}_{\psi_{\perp}}$ on spatial mode \mathbf{k}_i . The average number of generated photons with polarization $\vec{\pi}_{\phi}$ is given by $\langle \hat{n}_{\phi} \rangle = 3\bar{n} + 1$, corresponding to a stimulated emission process. where $\bar{n} = \sinh^2 g$. Furthermore, we observe that the generated field presents a well defined parity, corresponding to an odd number of photons in the injected polarization and to an even number of photons in the orthogonal polarization.

Photon-number distribution

Before the lossy process, the density matrix of the state $\hat{\rho}_{\text{OPA}}^{\phi} = |\Phi_{\text{OPA}}^{\phi}\rangle \langle \Phi_{\text{OPA}}^{\phi}|$ is:

$$\hat{\rho}_{\text{OPA}}^{\phi} = \sum_{i,j,k,q=0}^{\infty} \gamma_{ij} \gamma_{kq}^{*} |(2i+1)\phi, (2j)\phi_{\perp}\rangle \langle (2k+1)\phi, (2q)\phi_{\perp}|.$$
(4.2)

We note from this expression that only elements with an odd number of photons in the $\vec{\pi}_{\phi}$ and an even number in the $\vec{\pi}_{\phi_{\perp}}$ polarization are present. Furthermore, in Fig. 4.3 (a) we note that the photon number distribution presents a strong unbalancement due to the quantum injection of the $\vec{\pi}_{\phi}$ single photon. Indeed, the amplifier seeded by a photon with equatorial polarization acts as a phase-covariant optimal cloning machine, and is stimulated to generate an output field containing more photons in the polarization of the injected seed.

Let us now analyze the effects of the transmission in a lossy channel for the equatorial amplified qubits by plotting the photon number distributions. The output density matrix after the transmission over the lossy channel is the sum of four terms with different parities [DSS09b]:

$$\hat{\rho}_{\eta}^{\phi} = \sum_{i,j,k,q=0}^{\infty} \left\{ \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i+1)\phi, (2j)\phi_{\perp} \rangle \langle (2k+1)\phi, (2q)\phi_{\perp} | + \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i)\phi, (2j)\phi_{\perp} \rangle \langle (2k)\phi, (2q)\phi_{\perp} | + \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i+1)\phi, (2j+1)\phi_{\perp} \rangle \langle (2k+1)\phi, (2q+1)\phi_{\perp} | + \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i)\phi, (2j+1)\phi_{\perp} \rangle \langle (2k)\phi, (2q+1)\phi_{\perp} | \right\}.$$

$$(4.3)$$

The details on the calculation and on the expressions of the coefficients are reported in App. A.1.2. When the original state propagates through a lossy channel, the first effect at low values of *R* is the cancellation of the peculiar comb structure [Fig. 4.3 (a)] given by the presence in the density matrix (4.2) only of terms with a specific parity $|(2i+1)\phi, (2j)\phi_{\perp}\rangle \langle (2k+1)\phi, (2q)\phi_{\perp}|$. However, at progressively higher values of *R*, the distributions in the Fock space remain unbalanced in the polarization of the injected photon [Fig. 4.3 (a)]. The resilience of this unbalancement allows to distinguish the orthogonal macro-qubits $\{|\Phi_{OPA}^{\phi}\rangle, |\Phi_{OPA}^{\phi_{\perp}}\rangle\}$ even after the propagation over the lossy channel, by exploiting this property with a suitable detection scheme, such as the orthogonality filter (OF) device reported in [DSV08]. All these considerations will be discussed and quantified later in Sec. 4.3.3 by analyzing the distinguishability of such states as a function of the lossy channel efficiency η .

For the sake of completeness, we analyze the evolution of $|\Phi_{OPA}^H\rangle$ and $|\Phi_{OPA}^V\rangle$ amplified states. As a first remark, we note that the collinear optical parametric amplifier is not an optimal cloner for states with $\vec{\pi}_H$ and $\vec{\pi}_V$ polarization, and the output states do



Figure 4.3: (a) Probability distribution in the Fock space $(n_{\phi}, n_{\phi_{\perp}})$ for the amplified $|\Phi^{\phi}\rangle$ state of a generic equatorial qubit for different values of the transmittivity. (b) Probability distribution in the Fock space (n_H, n_V) for the amplified $|\Phi^H\rangle$ state for different values of the transmittivity. All distributions refer to a gain value of g = 1.5, corresponding to an average number of photons $\langle n \rangle \approx 19$.

not possess the same peculiar properties obtained with an equatorial injected qubit. The density matrix of the $|\Phi^H\rangle$ amplified state is:

$$\hat{\rho}_{\text{OPA}}^{H} = |\Phi_{\text{OPA}}^{H}\rangle \langle \Phi_{\text{OPA}}^{H}| = \frac{1}{C^4} \sum_{n,m=0}^{\infty} \Gamma^{n+m} \sqrt{n+1} \sqrt{m+1} |(n+1)H, nV\rangle \langle (m+1)H, mV|.$$

$$(4.4)$$

In Fig. 4.3 (b) we plotted the photon number distribution of this state (R=0). We note that the $\vec{\pi}_H$ amplified state does not possess the same unbalancement of the equatorial macro-qubits $|\Phi^{\varphi}\rangle$ analyzed in the previous section. After the propagation over the lossy

channel, the density matrix reads:

$$\hat{\rho}_{\eta}^{H} = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=0}^{\infty} \left(\hat{\rho}_{\eta}^{H} \right)_{ijk} |iH, jV\rangle \langle kH, (k+j-i)V| + \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} \left(\hat{\rho}_{\eta}^{H} \right)_{ijk} |iH, jV\rangle \langle kH, (k+j-i)V|,$$
(4.5)

where details on the calculation and on the coefficients are reported in App. A.1.3. The effect of the propagation over the lossy channel is shown in Fig. 4.3 (b). The original distribution for R = 0 is pseudo-diagonal, corresponding to the presence only of terms $|(n+1)H, nV\rangle$. Here the difference of one photon between the two polarization is due to the injection of the seed. For values of *R* different from 0, the distribution is no longer pseudo-diagonal and this characteristic becomes progressively smoothed. Furthermore, the absence of the unbalancement in the photon-number distribution typical of the equatorial macro-qubits does not allow to exploit this feature to discriminate among the orthogonal states $\{|\Phi_{OPA}^H\rangle, |\Phi_{OPA}^V\rangle\}$. We then expect that this couple of states possess a lower resilience to losses than the equatorial $|\Phi_{OPA}^{\phi}\rangle$ macro-states.

Wigner function

In this section we calculate the Wigner function associated with the output state of the collinear optical parametric amplifier in absence and in presence of a certain amount of losses $R = 1 - \eta$ [SVD⁺09]. Here, η is the channel efficiency, with $\eta = 1$ corresponding to no losses in the state. For the sake of simplicity, let us first consider the Hamiltonian of a degenerate amplifier acting on a single *k*-mode with polarization $\vec{\pi}_+$:

$$\hat{\mathscr{H}}_{\rm s.m.} = i\hbar \frac{\chi}{2} \left(\hat{a}_{+}^{\dagger 2} - \hat{a}_{+}^{2} \right).$$
(4.6)

The corresponding time evolution operator is:

$$\hat{U}_{+}(g) = \exp[\frac{g}{2}(\hat{a}_{+}^{\dagger})^{2} - (\hat{a}_{+})^{2}].$$
(4.7)

When no seed is injected, the amplifier operates in the regime of spontaneous emission and the characteristic function evaluated from Eq. (3.19) reads:

$$\chi_{0}(\xi,g) = \langle 0+|\exp[\xi\hat{a}^{\dagger}_{+}(g) - \xi^{*}\hat{a}_{+}(g)]|0+\rangle = \langle 0+|\exp[\xi(g)\hat{a}^{\dagger}_{+} - \xi^{*}(g)\hat{a}_{+}]|0+\rangle,$$
(4.8)

with:

$$\hat{a}_{+}(g) = \hat{a}_{+}\cosh g + \hat{a}_{+}^{\dagger}\sinh g, \quad \xi(g) = \xi\cosh g - \xi^{*}\sinh g, \quad (4.9)$$

and $g = \chi t$. Hereafter, we explicitly report the dependence of the Wigner function from the interaction time t. We obtain the following expression by using the operatorial relation $\exp(\hat{A} + \hat{B}) = \exp \hat{A} \exp \hat{B} \exp(-1/2[\hat{A}, \hat{B}])$:

$$\chi_0(\xi,g) = \exp\left(-\frac{1}{2}|\xi(g)|^2\right).$$
 (4.10)

The calculation then proceeds as follows. Starting from the definition of the Wigner function (3.19-3.20), we perform the two subsequent transformations of the integration variables $d^2\xi \rightarrow d^2\xi(g) \rightarrow xdxd\varphi$, where $\xi \rightarrow \xi(g)$ has been defined in Eq. (4.9) and can be expressed as $\xi(g) = xe^{i\varphi}$. The Wigner function is then calculated as:

$$W_{|0+\rangle}(\alpha,g) = \frac{1}{\pi^2} \int e^{x[(\bar{\alpha}-\bar{\alpha}^*)\cos\varphi + i(-\bar{\alpha}-\bar{\alpha}^*)\sin\varphi] - \frac{1}{2}x^2} x dx d\varphi$$

= $\frac{2}{\pi} \int_0^\infty J_0(-2|\bar{\alpha}|x) \exp\left(-\frac{1}{2}x^2\right) x dx = \frac{2}{\pi} \exp\left[-2|\bar{\alpha}|^2\right],$ (4.11)

where $\bar{\alpha} = \alpha \cosh(g) - \alpha^* \sinh(g)$, and $J_0(x)$ is the Bessel function of order 0. We can now write $\bar{\alpha} = \Re e(\alpha)e^g + \iota \mathscr{I}m(\alpha)e^{-g}$ as a function of the *X*, *P* quadrature operators, defined by the expression $\alpha = X + \iota P$. By substituting such variables Eq. (4.11) becomes

$$W_{|0+\rangle}(X,P,g) = \frac{2}{\pi} \exp\left[-2\left(X^2 e^{-2g} + P^2 e^{2g}\right)\right].$$
(4.12)

that corresponds to the Wigner function of a squeezed vacuum state.

When we consider the case in which a single photon with polarization $\vec{\pi}_+$ is injected: $|\psi_{in}\rangle = |1+\rangle$, analogous calculations leads to the characteristic function:

$$\chi_1(\xi,g) = \langle 1 + |\exp[\xi(t)\hat{a}_+^{\dagger} - \xi^*(t)\hat{a}_+] | 1 + \rangle = \left(1 - |\xi(t)|^2 \right) \exp\left(-\frac{1}{2}|\xi(t)|^2\right).$$
(4.13)

The Wigner function reads:

$$W_{|1+\rangle}(X,P,g) = -\frac{2}{\pi} \left[1 - 4 \left(X^2 e^{-2g} + P^2 e^{2g} \right) \right] \exp\left[-2(X^2 e^{-2g} + P^2 e^{2g}) \right].$$
(4.14)

Note that the Wigner function when a single photon is injected presents a negative value in the origin $W_{|1+\rangle}(0,0,g) = -2/\pi$.

Let us now consider the action of a certain amount of losses *R* on the calculated Wigner functions. The effect of such process can be evaluated by considering that the action of a lossy channel in the phase-space on a generic Wigner function W(X,P) can be written in the form of a gaussian convolution of Eq. (3.23) [Leo93]. As a first case, we analyze the evolution of the Wigner function of the squeezed vacuum state. Analogously to the unperturbed case, the quadrature variables for the single-mode OPA are defined by $\alpha = X + \iota P$. The Wigner function for the squeezed vacuum after losses then reads:

$$W_{|0+\rangle}(X,P,R,g) = \frac{2}{\pi} \frac{1}{\sqrt{1+4(1-R)RS^2}} \times \exp\left[-2\frac{(X^2e^{-2g}+P^2e^{2g})+2RS(X^2e^{-g}-P^2e^g)}{1+4(1-R)RS^2}\right],$$
(4.15)

with $S = \sinh g$. The same calculation can be performed on the Wigner function for the single-photon amplified state, which reads:

$$W_{|1+\rangle}(X,P,R,g) = \frac{2}{\pi} \frac{1}{\sqrt{1+4(1-R)RS^2}} P_{|1+\rangle}(X,P,R,g) \times \\ \times \exp\left[-2\frac{\left(X^2 e^{-2g} + P^2 e^{2g}\right) + 2RS\left(X^2 e^{-g} - P^2 e^g\right)}{1+4(1-R)RS^2}\right],$$
(4.16)

where the polynomial $P_{|1+\rangle}(X, P, R, g)$ has the form:

$$P_{|1+\rangle}(X,P,R,g) = 1 - \frac{4(1-R)}{1+4(1-R)RS^2} \left[\frac{1}{2} \left(1+2RS^2 \right) + \left(X^2 e^{-2g} + P^2 e^{2g} \right) + -2 \left(1+2RS^2 \right) \frac{\left(X^2 e^{-2g} + P^2 e^{2g} \right) + 2RS \left(X^2 e^{-g} - P^2 e^g \right)}{1+4(1-R)RS^2} \right].$$

$$(4.17)$$

The plots of the Wigner function will be reported in Sec. 4.4.2, when the amount of negativity is studied as a function of the losses parameter R.

We can now proceed to evalute the two-mode Wigner function corresponding to the Hamiltonian $\hat{\mathscr{H}}_{OPA}$ for the collinear amplifier. Indeed, the unitary evolution of the Hamiltonian expressed in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis can be written in the separable form $\hat{\mathscr{H}}_{OPA} = \hat{U}_+(g) \otimes \hat{U}_-(-g)$ [Fig. 4.2]. Hence, for a separable single-photon input state in the $\vec{\pi}_+$ polarization state $|1+,0-\rangle$, the Wigner function can be recovered from the single-mode terms (4.15-4.17) as:

$$W_{|1+,0-\rangle}(X_{+},P_{+},X_{-},P_{-},R,g) = W_{|1+\rangle}(X_{+},P_{+},R,g) \times W_{|0-\rangle}(X_{-},P_{-},R,-g).$$
(4.18)

Here, $\{X_+, P_+\}$ and $\{X_-, P_-\}$ are the quadrature variables for the two polarization modes, and the term for the $\vec{\pi}_-$ polarization is evaluated for a gain (-g) due to the opposite sign in the Hamiltonian $\hat{\mathscr{H}}_{OPA}$. In Sec. 4.4.2 we analyze in details the properties of the $W_{|1+,0-\rangle}(X_+, P_+, X_-, P_-, R, g)$ function by considering the amount of negativity as a function of the gain g and of the losses parameter R.

Finally, we conclude by considering that the Wigner function for a generic polarization basis $\{\vec{\pi}_{\psi}, \vec{\pi}_{\psi_{\perp}}\}\)$ can be recovered from Eq. (4.18) by changing the quadrature variables with the appropriate unitary matrix $R_{\psi,\psi_{\perp}}^{+,-}$ describing the corresponding rotation in the Bloch sphere.

4.2.2 Universal amplifier

When the amplifier is exploited in a noncollinear configuration [Fig. 4.1 (b)], it acts as an universal $N \rightarrow M$ quantum cloning machine [De 98a, PSS⁺03, DPS04] as well as a Universal - Not (U-Not) quantum machine [DBSS02]. The interaction Hamiltonian for the amplifier is given by Eq. (2.3). The output state of the amplifier after injection of a single photon reads:

$$\begin{split} |\Phi_{\text{SPDC}}^{1\psi}\rangle &= \hat{U}_{\text{SPDC}}|1\psi\rangle_1 = \frac{1}{C^3}\sum_{n,m=0}^{\infty}\Gamma^{n+m}(-1)^m\sqrt{n+1}\\ |(n+1)\psi,m\psi_{\perp}\rangle_1 \otimes |m\psi,n\psi_{\perp}\rangle_2. \end{split}$$
(4.19)

where $\vec{\pi}_{\psi} = \cos(\theta/2)\vec{\pi}_H + e^{i\phi}\sin(\theta/2)\vec{\pi}_V$ is a generic polarization state and $\vec{\pi}_{\psi_{\perp}} = (\vec{\pi}_{\psi})^{\perp}$. We note that the multiphoton quantum superposition state $|\Phi_{SPDC}^{1\psi}\rangle = \cos(\theta/2)|\Phi_{SPDC}^{1H}\rangle + e^{i\phi}\sin(\theta/2)|\Phi_{SPDC}^{1V}\rangle$ and $|\Phi_{SPDC}^{1\psi_{\perp}}\rangle$ lives in the joint system composed by the \mathbf{k}_1 and \mathbf{k}_2 spatial modes. More specifically, the photons generated on mode \mathbf{k}_1 , to which we refer as the cloning mode, present the property to be as close as possible, according to the laws of quantum mechanics, to the injected single-photon state. On the contrary, the photons generated on mode \mathbf{k}_2 , to which we refer as the anti-cloning mode, present a polarization state as close as possible to $|\psi_{\perp}\rangle$ [DBSS02]. Hence, both spatial modes carry out information on the single-photon injected in the amplifier.

Photon-number distribution

We begin the analysis of the universal amplifier by considering the photon number distribution of the amplified single-photon states $|\Phi_{SPDC}^{1\psi}\rangle$. In order to investigate the features of the state of Eq. (4.19), Fig. 4.4 reports the photon-number distribution for the reduced states $\hat{\rho}_{\mathbf{k}_i}^{1\psi(1\psi_\perp)} = \text{Tr}_{\mathbf{k}_i} \left[|\Phi_{SPDC}^{1\psi(1\psi_\perp)}\rangle \langle \Phi_{SPDC}^{1\psi(1\psi_\perp)}| \right]$ associated to the output spatial modes \mathbf{k}_1 and \mathbf{k}_2 . The photon-number distributions in the \mathbf{k}_1 spatial mode [Figs. 4.4 (a) and (c)], i.e. the cloning mode, show a strong unbalancement along the direction of the injected polarization state. The anticloning \mathbf{k}_2 mode [Figs. 4.4 (b) and (d)] presents the opposite unbalancement along the direction of the orthogonal polarization, since on that spatial mode the amplifier works as a U-Not machine [DBSS02]. This feature is also enlightened by the contour plots of Figs.4.4 (e-h), where the white regions represent the Fock-space zones where the photon-number distributions are more densely populated. Furthermore, we note that at variance with the phase-covariant amplifier [DSS09b, DSS09a], the output states do not exhibit any comb structure in their photon number distributions.

We can now investigate the action of detection losses. Due to the properties of $\hat{\mathscr{H}}_{SPDC}$, the time evolution operator in the interaction picture $\hat{U}_{SPDC} = \exp(-i\hat{\mathscr{H}}_{SPDC}t/\hbar)$ can be decomposed as the product of two independent operators $\hat{U}_{SPDC} = \hat{U}_{\mathscr{A}} \otimes \hat{U}_{\mathscr{A}'}$, acting on two different Hilbert spaces corresponding to the two sets of modes $\mathscr{A} \equiv \{(\mathbf{k}_1, \vec{\pi}_{\psi}), (\mathbf{k}_2, \vec{\pi}_{\psi_{\perp}})\}$ and $\mathscr{A}' \equiv \{(\mathbf{k}_1, \vec{\pi}_{\psi_{\perp}}), (\mathbf{k}_2, \vec{\pi}_{\psi})\}$ (Fig. 4.5) [PSS⁺03, DPS04]:

$$\hat{U}_{\mathscr{A}} = \exp\left[\chi t (\hat{a}_{1\psi}^{\dagger} \hat{a}_{2\psi_{\perp}}^{\dagger} - \hat{a}_{1\psi} \hat{a}_{2\psi_{\perp}})\right]; \quad \hat{U}_{\mathscr{A}'} = \exp\left[-\chi t (\hat{a}_{1\psi_{\perp}}^{\dagger} \hat{a}_{2\psi}^{\dagger} - \hat{a}_{1\psi_{\perp}} \hat{a}_{2\psi})\right].$$
(4.20)



Figure 4.4: Probability distribution (a-d) and contour plots (e-h) of the reduced density matrices $\hat{\rho}_{\mathbf{k}_1}^{1\psi}$ (a)-(e), $\hat{\rho}_{\mathbf{k}_2}^{1\psi}$ (b)-(f), $\hat{\rho}_{\mathbf{k}_1}^{1\psi_{\perp}}$ (c)-(g) and $\hat{\rho}_{\mathbf{k}_2}^{1\psi_{\perp}}$ (d)-(h). All plots correspond to the gain value g = 1.5.

In the case of a separable input state in the amplifier $\hat{\rho} = \hat{\rho}_{\mathscr{A}} \otimes \hat{\rho}_{\mathscr{A}'}$, the amplified states can be written in a separable form:

$$\hat{\rho}(t) = \hat{U}\hat{\rho}\hat{U}^{\dagger} = \left(\hat{U}_{\mathscr{A}}\hat{\rho}_{\mathscr{A}}\hat{U}_{\mathscr{A}}^{\dagger}\right) \otimes \left(\hat{U}_{\mathscr{A}'}\hat{\rho}_{\mathscr{A}'}\hat{U}_{\mathscr{A}'}^{\dagger}\right).$$
(4.21)

Since the single-photon input $|1\psi\rangle_1$ is a separable state, the two amplifiers \mathscr{A} and \mathscr{A}' can be analyzed separately.



Figure 4.5: Equivalent model describing the separability property for the noncollinear universal optical parametric amplifier. The amplifiers acts as two independent two-modes amplifiers \mathscr{A} and \mathscr{A}' for the set of modes $\mathscr{A} =$ $\{(\mathbf{k}_1, \vec{\pi}_{\psi\perp}), (\mathbf{k}_2, \vec{\pi}_{\psi\perp})\}$ and $\mathscr{A}' =$ $\{(\mathbf{k}_1, \vec{\pi}_{\psi\perp}), (\mathbf{k}_2, \vec{\pi}_{\psi\perp})\}$. A separable input states produces at the output a separable state between \mathscr{A} and \mathscr{A}' .

The quantum state for the subsystem \mathscr{A} in the spontaneous emission regime is given

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by:

$$\hat{U}_{\mathscr{A}}|0\rangle = \frac{1}{C}\sum_{n=0}^{\infty}\Gamma^{n}|n\psi\rangle_{1}\otimes|m\psi_{\perp}\rangle_{2}.$$
(4.22)

The output state after the transmission over the lossy channel is obtained by applying the lossy channel map (A.1) to the density matrix of the state $\hat{\rho}_{\mathscr{A}}^{0} = \hat{U}_{\mathscr{A}}|0\rangle\langle 0|\hat{U}_{\mathscr{A}}^{\dagger}$. After direct application of the lossy channel map on the density matrix, the following expression is obtained [SSD10]:

$$\hat{\rho}^{0}_{\mathscr{A}}(\eta_{1},\eta_{2}) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=i-j}^{\infty} \left[\hat{\rho}^{0}_{\mathscr{A}}(\eta_{1},\eta_{2}) \right]^{(i\geq j)}_{ijk} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}| + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \sum_{k=0}^{\infty} \left[\hat{\rho}^{0}_{\mathscr{A}}(\eta_{1},\eta_{2}) \right]^{(i

$$(4.23)$$$$

where the expressions for the coefficients are reported in App. A.2.

The same procedure has been applied to the stimulated case, where the seed of the amplifier \mathscr{A} is the single-photon state $|1\psi\rangle_1$. In this case, the input state in the lossy channel has the following expression:

$$\hat{U}_{\mathscr{A}}|1\psi\rangle_{1} = \frac{1}{C^{2}}\sum_{n=0}^{\infty}\Gamma^{n}\sqrt{n+1}|(n+1)\psi\rangle_{1}\otimes|m\psi_{\perp}\rangle_{2}.$$
(4.24)

By applying the lossy channel map over the density matrix $\hat{\rho}_{\mathscr{A}}^{1\psi}$ of the state, we find:

$$\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) = \sum_{i=0}^{\infty} \sum_{j=0}^{i-1} \sum_{k=i-j}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j+1)} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}| + \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j)} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}|.$$

$$(4.25)$$

According to previous considerations and to the form of the interaction Hamiltonian, the density matrices of the states $\hat{\rho}^0_{\mathscr{A}'}(\eta_1,\eta_2)$ and $\hat{\rho}^{1\psi_{\perp}}_{\mathscr{A}'}(\eta_1,\eta_2)$ for amplifier \mathscr{A}' can be directly derived from Eqs. (4.23) and (4.25) by substituting (Γ) with ($-\Gamma$) and by relabelling the indexes describing the spatial and polarization modes. Finally, the complete output state can be reconstructed as:

$$\hat{\boldsymbol{\rho}}_{\text{SPDC}}^{1\psi}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) = \hat{\boldsymbol{\rho}}_{\mathscr{A}}^{1\psi}(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2) \otimes \hat{\boldsymbol{\rho}}_{\mathscr{A}'}^0(\boldsymbol{\eta}_1, \boldsymbol{\eta}_2).$$
(4.26)

Wigner function

In order to evaluate the Wigner function for the optical parametric amplifier in a noncollinear configuration when injected by a single-photon state [SVD⁺09], we exploit the separability property of Eq. (4.20) of the Hamiltonian of the amplifier. We then perform

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the calculation for the amplifier \mathscr{A} , while the corresponding term for the amplifier \mathscr{A}' can be recovered by changing $g \to -g$ and by re-labeling the optical modes. Furthermore, we fix the polarization basis in $\{\vec{\pi}_+, \vec{\pi}_-\}$. The expression of the Wigner function in the rotated quadratures in a different polarization basis can be recovered by applying the appropriate rotation matrix $R_{\psi,\psi_\perp}^{+,-}$.

We begin with the spontaneous emission term, corresponding to the injection of the vacuum state $|\psi_0\rangle = |0+\rangle_1 |0-\rangle_2$. The characteristic function is then evaluated starting from the definition:

$$\chi_{0,0}^{\mathscr{A}}(\xi_{1+},\xi_{2-},g) = \langle \psi_{0} | \exp[\xi_{1+}\hat{a}_{1+}^{\dagger}(g) - \xi_{1+}^{*}\hat{a}_{1+}(g) + \xi_{2-}\hat{a}_{2-}^{\dagger}(g) - \xi_{2-}^{*}\hat{a}_{2-}(g)] | \psi_{0} \rangle = = \exp\left[-\frac{1}{2}(|\xi_{1+}(g)|^{2} + |\xi_{2-}(g)|^{2})\right],$$
(4.27)

where $\xi_{1+}(g) = \xi_{1+} \cosh g - \xi_{2-}^* \sinh g$ and $\xi_{2-}(g) = \xi_{2-} \cosh g - \xi_{1+}^* \sinh g$.

The Wigner function of the amplified field can be then expressed as the 4-dimensional Fourier transform of the characteristic function:

$$W^{\mathscr{A}}_{|0+\rangle_{1}|0-\rangle_{2}}(\alpha_{1+},\alpha_{2-},g) = \frac{1}{\pi^{4}} \int d^{2}\xi_{1+} \int d^{2}\xi_{2-}\chi^{\mathscr{A}}_{0,0}(\xi_{1+},\xi_{2-},g) \times e^{\xi_{1+}^{*}\alpha_{1+}-\xi_{1+}\alpha_{1+}^{*}}e^{\xi_{2-}^{*}\alpha_{2-}-\xi_{2-}\alpha_{2-}^{*}}.$$
(4.28)

By following the same procedure of the collinear amplifier we find:

$$W_{|0+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},g) = \left(\frac{2}{\pi}\right)^{2} \exp\left[-8CS\left(P_{1+}P_{2-}-X_{1+}X_{2-}\right)\right] \times \exp\left[-2(1+2S^{2})\left(X_{1+}^{2}+X_{2-}^{2}+P_{1+}^{2}+P_{2-}^{2}\right)\right].$$
(4.29)

We now consider the injection of a single photon with polarization state $|\psi_{1+}\rangle = |1+\rangle_1|0-\rangle_2$. The characteristic function with this input state reads:

$$\chi_{1,0}^{\mathscr{A}}(\xi_{1+},\xi_{2-},g) = (1-|\xi_{1+}(g)|^2) \exp\left[-\frac{1}{2}(|\xi_{1+}(g)|^2+|\xi_{2-}(g)|^2)\right].$$
(4.30)

The corresponding Wigner function evaluated according to the definition reads:

$$W_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},g) = \left(\frac{2}{\pi}\right)^{2} \exp\left[-8CS\left(P_{1+}P_{2-}-X_{1+}X_{2-}\right)\right] \times \exp\left[-2(1+2S^{2})\left(X_{1+}^{2}+X_{2-}^{2}+P_{1+}^{2}+P_{2-}^{2}\right)\right] P_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},g),$$
(4.31)

where:

$$P_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},g) = 1 - \left[C^{2}(X_{1+}^{2}+P_{1+}^{2}) + S^{2}(X_{2-}^{2}+P_{2-}^{2})\right] + 2CS(P_{1+}P_{2-}-X_{1+}X_{2-})\right].$$
(4.32)

As for the collinear case, we observe that the Wigner function is negative in the origin of the space $W_{|1+\rangle_1|0-\rangle_2}^{\mathscr{A}}(0,0,0,0,g) = -(2/\pi)^2$.

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We can now proceed to insert the action of a lossy channel on both spatial modes with transmittivity η , which we assume to be equal for both channels. As in Sec. 4.2.1, this process in the phase-space can be treated as the convolution of the Wigner function with a gaussian kernel [see Eq. (3.23)]. The Wigner function for the amplifier when the vacuum state is injected then reads:

$$W_{|0+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},R,g) = \left(\frac{2}{\pi\sigma(R,g)}\right)^{2} \times \exp\left[-\frac{8CS(1-R)}{\sigma^{2}(R,g)}(P_{1+}P_{2-}-X_{1+}X_{2-}) - \frac{2h(R,g)}{\sigma^{2}(R,g)}(X_{1+}^{2}+X_{2-}^{2}+P_{1+}^{2}+P_{2-}^{2})\right],$$
(4.33)

where:

$$\sigma(R,g) = 1 + 4R(1-R)S^2, \quad h(R,g) = R + (1-R)S^2.$$
(4.34)

By following an analogous procedure we obtain the following expression for the Wigner function associated to the amplification of a single-photon state after losses R:

$$W_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},R,g) = \left(\frac{2}{\pi\sigma(R,g)}\right)^{2} P_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},R,g) \\ \times \exp\left[-\frac{8CS(1-R)}{\sigma^{2}(R,g)}\left(P_{1+}P_{2-}-X_{1+}X_{2-}\right) - \frac{2h(R,g)}{\sigma^{2}(R,g)}\left(X_{1+}^{2}+X_{2-}^{2}+P_{1+}^{2}+P_{2-}^{2}\right)\right],$$
(4.35)

where the polynomial $P_{|1+\rangle_1|0-\rangle_2}^{\mathscr{A}}$ reads:

$$P_{|1+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}}(X_{1+},P_{1+},X_{2-},P_{2-},R,g) = \frac{1}{\sigma^{2}(R,g)} \left\{ 2R - 1 + \frac{4(1-R)}{\sigma^{2}(R,g)} \left[C^{2}(X_{1+}^{2} + P_{1+}^{2}) + S^{2}(2R-1)^{2}(X_{2-}^{2} + P_{2-}^{2}) + 2CS(2R-1)(P_{1+}P_{2-} - X_{1+}X_{2-}) \right] \right\}.$$

$$(4.36)$$

Finally, the complete 4-modes Wigner function can be calculated as:

$$W_{|1+,0-\rangle_{1}|0+,0-\rangle_{2}}(X_{1+},P_{1+},X_{1-},P_{1-},X_{2+},P_{2+},X_{2-},P_{2-},R,g) = W_{|1+\rangle_{1}|0-\rangle_{2}}(X_{1+},P_{1+},X_{2-},P_{2-},R,g)W_{|0+\rangle_{1}|0-\rangle_{2}}^{\mathscr{A}'}(X_{1-},P_{1-},X_{2+},P_{2+},R,-g).$$
(4.37)

In Sec. 4.4.3 we report and analyze the plots corresponding to the negativity of the Wigner function in the origin of the phase-space.

4.3 Resilience to decoherence after the action of a lossy channel

In this section we discuss the resilience to decoherence of the multiphoton quantum superpositions generated by amplification of single-photon states. More specifically, this analysis is based on a Fock space analysis of the density matrices of this class of states after the action of a lossy channel. The adopted criteria are based on the concept of distinguishability between two orthogonal quantum states and the related degree of coherence of a quantum superposition involving the same states. The adopted merit figure is the Bures distance, already introduced in Sec. 1.3.6. When exploited in this context, this quantity expresses the persistance of quantum effects when a decoherence process progressively randomizes the relative phase between the two components of the superposition state.

4.3.1 Criteria for multiphoton quantum superpositions

In order to distinguish between two different quantum states, we exploit the definition of the normalized Bures distance: $\mathscr{D}(\hat{\rho}, \hat{\sigma}) = \sqrt{1 - [\mathscr{F}(\hat{\rho}, \hat{\sigma})]^{1/2}}$ [Bur69, Hub92], where \mathscr{F} is the fidelity between the two states [Joz94]. This quantity can be calculated for two generic multiphoton states $|\phi_1\rangle$ and $|\phi_2\rangle$ and the corresponding quantum superpositions: $|\phi^{\pm}\rangle = \frac{\mathscr{N}_{\pm}}{\sqrt{2}} (|\phi_1\rangle \pm |\phi_2\rangle)$. More specifically, we can define the two following criteria [DSS09b, DSS09a]:

- (I) The *distinguishability* between the two component states $|\phi_1\rangle$ and $|\phi_2\rangle$ is expressed by: $\mathscr{D}(|\phi_1\rangle, |\phi_2\rangle)$. This parameter quantifies the capability of an observer to discriminate among the two states by exploiting the appropriate measurement.
- (II) The *visibility*, that is, the degree of orthogonality, of the two superpositions $|\phi^{\pm}\rangle$ is expressed by $\mathscr{D}(|\phi^{+}\rangle, |\phi^{-}\rangle)$. Indeed, the value of the visibility depends on the relative phase between the component states: $|\phi_1\rangle$ and $|\phi_2\rangle$. The parameter \mathscr{D} then expresses the ability of an observer to discriminate between two initially orthogonal states, $\mathscr{D}(|\phi^{+}\rangle, |\phi^{-}\rangle) = 1$, after propagation in a lossy channel where the relative phase of $|\phi_1\rangle$ and $|\phi_2\rangle$ progressively randomizes leading to a fully mixed state: $\mathscr{D}(|\phi^{+}\rangle, |\phi^{-}\rangle) \to 0$.

The physical interpretation of $\mathscr{D}(|\phi^+\rangle, |\phi^-\rangle)$ as the visibility of a superposition $|\phi^{\pm}\rangle$ is legitimate insofar as the component states of the corresponding superposition, $|\phi_1\rangle$ and $|\phi_2\rangle$ may be defined, at least approximately, as *pointer states* [Zur03]. The latter are defined as the set of eigenstates of a quantum system least affected by the external noise and that are highly resilient to decoherence. In other words, the pointer states are quasi classical states which realize the minimum flow of information from (or to) the system to (or from) the environment.

In the following sections, we apply these criteria on the class of multiphoton states generated by parametric amplification of single photons, and we compare the obtained results with a reference class of states represented by the quantum superposition of coherent states.

4.3.2 Quantum superposition of coherent states as a reference

The quantum superpositions of coherent states is defined as [SPL91]:

$$|\Psi_{\varphi}^{\pm}\rangle = \mathscr{N}_{\varphi}^{\pm} \frac{1}{\sqrt{2}} \left(|\alpha e^{i\varphi}\rangle \pm |\alpha e^{-i\varphi}\rangle \right), \qquad (4.38)$$

with α real and $\mathcal{N}_{\varphi}^{\pm} = \left(1 \pm e^{-2|\alpha|^2 \sin^2 \varphi} \cos\left[|\alpha|^2 \sin 2\varphi\right]\right)^{-\frac{1}{2}}$ is an appropriate normalization factor. The two states with opposite relative phases $|\Psi_{\varphi}^+\rangle$ and $|\Psi_{\varphi}^-\rangle$ are orthogonal when $|\alpha|^2 \sin^2 \varphi > 1$. In such case the two components $|\alpha e^{i\varphi}\rangle$ and $|\alpha e^{-i\varphi}\rangle$ are distinguishable. This class of quantum superposition states presents several peculiar properties, such as squeezing and sub-poissonian statistics. Such properties are due to the superposition form of the $|\Psi_{\varphi}^{\pm}\rangle$ states, and cannot be explained by the characteristics of the component coherent $|\alpha\rangle$ states.

We can now proceed to apply the two criteria (I) and (II) to this class of states. We begin by analyzing the distinguishability between the states $|\alpha e^{\pm i\varphi}\rangle$. In this case, the application of the loss model to the output coherent state density matrix leads to: $\hat{\rho}_{\sqrt{\eta}\alpha e^{\pm i\varphi}} = |\sqrt{\eta}\alpha e^{\pm i\varphi}\rangle \langle \sqrt{\eta}\alpha e^{\pm i\varphi} |$, thus not changing the structure of the state. The distance between the two states with opposite phase is easily found [Lou00]:

$$\mathscr{D}\left(|\sqrt{\eta}\alpha e^{\imath\varphi}\rangle, |\sqrt{\eta}\alpha e^{-\imath\varphi}\rangle\right) = \sqrt{1 - e^{-2\eta|\alpha|^2 \sin^2\varphi}},\tag{4.39}$$

a value almost close to 1 for $\eta |\alpha|^2 \sin^2 \varphi > 1$ [Fig. 4.6 (a)]. In this regime the coherent states $|\alpha e^{\pm i\varphi}\rangle$ keep their mutual distinguishability through the lossy channel and comply with the definition of pointer states.



Figure 4.6: (a) Bures distance $\mathscr{D}(x)$ between the component coherent states $|\alpha e^{i\varphi}\rangle$ and $|\alpha e^{-i\varphi}\rangle$ as a function of the losses parameter *R* for different values of $\langle n \rangle = |\alpha|^2$. (b) Bures distance $\mathscr{D}(x)$ between the superpositions $\hat{\rho}^{\eta}_{\Psi^+_{\varphi}}$ and $\hat{\rho}^{\eta}_{\Psi^-_{\varphi}}$ as a function of the number of lost photons $x = R\langle n \rangle$.

We now proceed with the analysis of the density matrix after the propagation over the lossy channel. In the following we assume $|\alpha|^2 \sin^2 \varphi > 1$, hence $\mathcal{N}_{\varphi}^{\pm} \sim 1$. The density matrix of the quantum state after the lossy channel reads:

$$\hat{\rho}^{\eta}_{\Psi^{\pm}_{\varphi}} = \frac{1}{2} \left[|\beta e^{i\varphi}\rangle \langle \beta e^{i\varphi}| + |\beta e^{-i\varphi}\rangle \langle \beta e^{-i\varphi}| \pm \\ \pm e^{-2R|\alpha|^{2}\sin^{2}\varphi} \left(e^{iR|\alpha|^{2}\sin^{2}\varphi} |\beta e^{i\varphi}\rangle \langle \beta e^{-i\varphi}| + e^{-iR|\alpha|^{2}\sin^{2}\varphi} |\beta e^{-i\varphi}\rangle \langle \beta e^{i\varphi}| \right) \right],$$
(4.40)

with $\beta = \sqrt{1-R} \alpha$. Let us analyze the case $\varphi = \frac{\pi}{2}$. The ortoghonality between $|\Psi_+\rangle$ and $|\Psi_-\rangle$ quickly decrease as soon as *R* differs from 0, since the phase relation between the components $|\alpha\rangle$ and $|-\alpha\rangle$ becomes undefined. The visibility of these superposition states defined by the criterion (II) gives:

$$\mathscr{D}(\hat{\rho}^{\eta}_{\Psi^{+}_{\varphi}}, \hat{\rho}^{\eta}_{\Psi^{-}_{\varphi}}) = \sqrt{1 - \sqrt{1 - e^{-4R|\alpha|^{2}\sin^{2}\varphi}}}.$$
(4.41)

Hence $\mathscr{D}(x) \simeq e^{-2x}$ is exponentially decaying with $x \propto R |\alpha|^2$, that is, the average number of lost photons [Fig. 4.6 (b)]. Note that the decay in the function $\mathscr{D}(x)$ does not depend singularly from the number of photons nor from the channel efficiency, but is a function only of the amount of lost photons inpendently from the size of the system. The loss of 1 photon, on the average, leads to a visibility value: $\mathscr{D} \sim 0.096$, and then to the practical loss of any detectable interference effects in the superpositions $\hat{\rho}_{\Psi_{\phi}^{\pm}}^{\eta}$. This is fully consistent with the experimental observations [BHR+92, RBH01]. Note that the function $\mathscr{D}(x)$ approaches its minimum value with zero slope: $Sl = \lim_{R\to 1} |d\mathscr{D}(x)/dx| = 0$. These results are confirmed by the analysis of the photon-number distribution. The distribution in the Fock space exhibits only elements with an even number of photons for $|\Psi^+\rangle$ or an odd number of photons for $|\Psi^-\rangle$. This peculiar comb structure is very fragile under the effect of losses, as shown in Fig. 4.7. We observe that for a loss parameter *R* corresponding to about ~ 1.5 photon lost in average, the distribution resembles closely the Poisson distribution associated to the coherent states.

4.3.3 Resilience to decoherence of the phase-covariant amplified singlephoton states

As a following step, we have applied the criteria (I)-(II) to the quantum states generated by collinear parametric amplification of single photons [DSS09b, DSS09a]. As a first consideration, we observe that in virtue of the phase-covariance of the process, the distinguishability of $\{|\Phi_{OPA}^{+,-}\rangle\}$ through the distance $\mathscr{D}(|\Phi_{OPA}^{+}\rangle, |\Phi_{OPA}^{-}\rangle)$ coincides with the visibility of their equatorial quantum superpositions of the form:

$$|\Psi_{\text{OPA}}^{+}\rangle = e^{-\iota\phi/2} \left[\cos(\phi/2)|\Phi_{\text{OPA}}^{+}\rangle + \iota\sin(\phi/2)|\Phi_{\text{OPA}}^{-}\rangle\right], \quad (4.42)$$

$$|\Psi_{\text{OPA}}^{-}\rangle = (|\Psi_{\text{OPA}}^{+}\rangle)_{\perp}. \tag{4.43}$$



Figure 4.7: (a)-(d): Plot of the distribution of the number of photons in the $|\Psi_{+}^{\frac{n}{2}}\rangle$ state for $\alpha = 4$, corresponding to an average number of photons $\langle n \rangle = 16$, for reflectivities R = 0 (a), R = 0.1 (b), R = 0.5 (c) and R = 0.8 (d).

The following property then holds:

$$\mathscr{D}(|\Psi_{\text{OPA}}^{+}\rangle,|\Psi_{\text{OPA}}^{-}\rangle) = \mathscr{D}(|\Phi_{\text{OPA}}^{R}\rangle,|\Phi_{\text{OPA}}^{L}\rangle) = \mathscr{D}(|\Phi_{\text{OPA}}^{+}\rangle,|\Phi_{\text{OPA}}^{-}\rangle).$$
(4.44)

We then evaluated numerically the distinguishability of $\{|\Phi_{OPA}^{+,-}\rangle\}$, and the corresponding visibility of $\{|\Psi_{OPA}^{+,-}\rangle\}$, through the distance $\mathcal{D}(|\Phi_{OPA}^{+}\rangle, |\Phi_{OPA}^{-}\rangle)$ as a function of the average lost photons: $x \equiv R\langle n \rangle$. This calculation have been performed by taking the complete expression of the density matrix, reported in Sec. 4.2.1 and App. A.1.2, and by performing an approximate calculation of the fidelity through numerical algebraic matrix routines. This algorithm has been tested by evaluating numerically the Bures distance between the quantum superposition of coherent states $|\alpha\rangle \pm |-\alpha\rangle$. The comparison with the analytical result of Eq. (4.41) gave a high confidence level for the approximate results. The results for different values of the gain for equatorial macroqubits are reported in Fig. 4.8 (a).

Note that for small values of x the decay of $\mathscr{D}(x)$ is far slower than for the coherent state case. Furthermore, after a common inflection point at $\mathscr{D} \sim 0.6$ the slope of the set of functions $\mathscr{D}(x)$ for $R \to 1$ increases with the value of $\langle n \rangle$. The latter property can be shown with a perturbative approach on the density matrix. We find that, in the low η and high gain limit where $\mathscr{D}(x) \sim 0$, the slope $\frac{\partial \mathscr{D}(\hat{\rho}_{\eta}^{\phi}, \hat{\rho}_{\eta}^{\phi})}{\partial \eta}$ tends to:

$$\lim_{g \to \infty} \lim_{\eta \to 0} \frac{\partial \mathscr{D}(\hat{\rho}^{\varphi}_{\eta}, \hat{\rho}^{\varphi}_{\eta})}{\partial \eta} = \lim_{g \to \infty} (1 + 4C^2 + 2C^2\Gamma\sqrt{1 + 2\Gamma^2}) = \infty.$$
(4.45)



Figure 4.8: (a) Numerical evaluation of the distance $\mathscr{D}(x)$ between two orthogonal equatorial macro-qubits $|\Phi^{\phi,\phi_{\perp}}\rangle$ as function of the average lost particle $x = R\langle n \rangle$. Black line corresponds to the distance $\mathscr{D}(x)$ for the reference quantum superposition of coherent states. (b) Numerical evaluation of the distance $\mathscr{D}(x)$ between two orthogonal linear macro-qubits $|\Phi^{H,V}\rangle$ as function of the average lost particle $x = R\langle n \rangle$.

This means that the visibility can be significant even if the average number x of lost particles is close to the initial total number $\langle n \rangle$. This behavior is opposite to the case of the quantum superposition of coherent states where the function $\mathscr{D}(x)$ approaches the zero value with an exponential decay: Figs. 4.6 and 4.8. This analysis performed on the amplified multiphoton states shows that the amplification process provides a tool to obtain multiphoton states robust under the action of a lossy channel. This makes these states suitable for the investigation of quantum effects in multiphoton systems, as shown later in Chap. 5. Furthermore, they can find application in different quantum information context such as quantum sensing, as shown later in Chaps. 9-10.

For sake of completeness, we then performed the same calculation for the multiphoton states corresponding to the injection of a photon with horizontal (vertical) polarization $|\Phi_{\text{OPA}}^{H,V}\rangle$. The results are reported in Fig. 4.8 (b). For this injected qubit, not lying in the equatorial plane of the Bloch sphere, the amplification process does not correspond to an optimal cloning machine. For this reason the output states possess a faster decoherence rate. Indeed, the output distributions, as shown in Fig. 4.3 (b), do not possess the strong unbalancement in polarization of the equatorial states $|\Phi_{\text{OPA}}^{\phi}\rangle$ that is responsible of their resilience structure.

4.3.4 Resilience to decoherence of the universally amplified singlephoton states

Finally, we apply the criteria (I)-(II) to the class of states generated by noncollinear amplification of single photons [SSD10]. In agreement with the universality property of the source, we expect that the Bures distance between the superposition states $|\Phi_{SPDC}^{1\psi}\rangle =$

 $\cos(\theta/2)|\Phi_{\text{SPDC}}^{1H}\rangle + e^{i\phi}\sin(\theta/2)|\Phi_{\text{SPDC}}^{1V}\rangle$ and $|\Phi_{\text{SPDC}}^{1\psi_{\perp}}\rangle$ is independent on the choice of (θ,ϕ) :

$$\mathscr{D}(\hat{\rho}_{\mathrm{SPDC}}^{1\psi}, \hat{\rho}_{\mathrm{SPDC}}^{1\psi_{\perp}}) = \mathscr{D}(\hat{\rho}_{\mathrm{SPDC}}^{1\psi'}, \hat{\rho}_{\mathrm{SPDC}}^{1\psi'_{\perp}}), \qquad (4.46)$$

for any basis $\{\vec{\pi}_{\psi}, \vec{\pi}_{\psi'}\}$. This feature is the extension of the phase-covariance property of



Figure 4.9: Bures distance between the superposition states $|\Phi^{1\psi}\rangle$ and $|\Phi^{1\psi_{\perp}}\rangle$ after losses. (a) 3-dimensional surface $\mathscr{D}(R_1, R_2)$ as a function of the two spatial mode parameters $R_1 = 1 - T_1$ and $R_2 = 1 - T_2$. (b) $\mathscr{D}(R_1, R_2)$ with R_2 fixed, as a function of R_1 . (c) $\mathscr{D}(R_1, R_2)$ with R_1 fixed, as a function of R_2 . (b)-(c) Straight curves correspond to $R_{2(1)} = 0.9$, the dashed curves to $R_{2(1)} = 0.75$, dotted one to $R_{2(1)} = 0.5$, the dash-dotted curves to $R_{2(1)} = 0.2$ and dash-dot-dotted curves to $R_{2(1)} = 0.05$.

the collinear quantum cloning machine [DSS09a] to the full set of polarization states on the output Bloch sphere.

The Bures distance $\mathscr{D}(\hat{\rho}_{\text{SPDC}}^{1\psi}, \hat{\rho}_{\text{SPDC}}^{1\psi_{\perp}})$ has been evaluated by considering the joint cloning-anticloning multiphoton state, that is, by considering the full two-mode density matrix. In analogy with the previous case, we evaluated by standard algebraic numerical routines the distance between the orthogonal macrostates $|\Phi_{\text{SPDC}}^{1\psi}\rangle$ and $|\Phi_{\text{SPDC}}^{1\psi_{\perp}}\rangle$ as a function of the corresponding transmission parameters: η_1 and η_2 . In Fig. 4.4 (a) we report the 3-dimensional plot of the function $\mathscr{D}(R_1, R_2) = \mathscr{D}(\hat{\rho}_{\text{SPDC}}^{1\psi_{\perp}}, \hat{\rho}_{\text{SPDC}}^{1\psi_{\perp}})$ for a gain value of g = 1.2, corresponding to an overall average number of photons $\langle \hat{n} \rangle = \sum_{i=1}^{2} [\langle \hat{n}_{i\psi} \rangle + \langle \hat{n}_{i\psi_{\perp}} \rangle] \approx 15$. The figure shows that the visibility possesses a resilient structure in presence of losses, since the Bures distance does not decrease exponentially with the lossy parameters $\{R_1, R_2\}$. In Figs. 4.9 (b-c) we then report several sections of the 3-dimensional surface of Fig. 4.9 (a) by fixing either R_1 or R_2 . We note that the $|\Phi_{\text{SPDC}}^{1\psi}\rangle$ and

 $|\Phi_{SPDC}^{1\psi_{\perp}}\rangle$ states are more sensitive to losses in the cloning mode \mathbf{k}_1 than in the anticloning one \mathbf{k}_2 . This can be explained by considering that the distance between these orthogonal multiphoton states depends on the unbalancement in the corresponding photon-number distributions. Since this feature is pronounced in the spatial cloning mode \mathbf{k}_1 , losses acting on this mode cancel more rapidly the orthogonality between $|\Phi_{SPDC}^{1\psi}\rangle$ and $|\Phi_{SPDC}^{1\psi_{\perp}}\rangle$. As the number of photons present in the state is increased, the visibility keeps large up to a value $V \approx 0.5$ in a larger range of the number of reflected photons. All this shows that, in analogy with the phase-covariant case, the superposition states generated by quantum cloning become more resilient to losses since the capability to discriminate orthogonal superpositions can survive the loss of a larger number of photons. Furthermore, in this case the amplifier permits to broadcast the properties of an input state in a state with a larger number of particle for the complete set of polarization state.

4.4 Wigner function theory and nonclassicality after the action of a lossy channel

The nonclassicality of this class of states can be evaluated starting from the properties of the relative Wigner function, more specifically, to the presence of negative regions. Indeed, as already discussed in Sec. 3.2.5, the negativity of the Wigner function represents a signature of nonclassical properties in the state. For this reason, while we shall adopt the negativity of the Wigner function as a sufficient condition for the nonclassicality of the investigated state, we stress that the absence of a negative region in the Wigner function does not directly imply its classicality, and hence more investigations are required in this regime.

4.4.1 Wigner function for the quantum superposition of coherent states

For the sake of clarity, we briefly review previous results on the Wigner functions associated to coherent superposition states after the propagation over a lossy channel. At variance with the rest of the chapter, in this section the quadrature variables X and P are defined with fluctuations $\Delta^2 X = 1/2$ on the vacuum state. By applying this definition to the density matrix of the superposition of coherent states after losses (4.40) we obtain:

$$W_{\hat{\rho}_{\Psi_{\phi}^{\pm}}^{\eta}}(X,P,R) = \frac{(\mathscr{N}_{\phi}^{\pm})^2}{2} \left(W_{|\beta e^{i\varphi}\rangle}(X,P,R) + W_{|\beta e^{-i\varphi}\rangle}(X,P,R) \pm W_{\hat{\rho}_{\Psi_{\phi}^{\pm}}^{\eta}}^{\text{int}}(X,P,R) \right).$$

$$(4.47)$$

In the last expression, the first two components can be written as :

$$W_{|\beta e^{\pm i\varphi}\rangle}(X, P, R) = \frac{1}{\pi} e^{-\left(X - \sqrt{\eta}\overline{X}_{\varphi}\right)^2} e^{-\left(P \mp \sqrt{\eta}\overline{P}_{\varphi}\right)^2}, \qquad (4.48)$$

where $\overline{X}_{\varphi}^2 = 2|\alpha|^2 \cos^2 \varphi$ and $\overline{P}_{\varphi}^2 = 2|\alpha|^2 \sin^2 \varphi$. Hence losses reduce the average value of the quadratures \hat{X} and \hat{P} .

The interference contribution reads:

$$W_{\hat{\rho}_{\Psi_{\varphi}^{\pm}}^{\eta}}^{\text{int}}(X,P,R) = \frac{2}{\pi} e^{-P^2} e^{-\left(X - \sqrt{\eta}\overline{X}_{\varphi}\right)^2} e^{-R\overline{P}_{\varphi}^2} \cos\left[2\sqrt{2}\alpha\sqrt{\eta}\sin\varphi\left(X - \frac{\alpha(2\eta - 1)}{\sqrt{2\eta}}\cos\varphi\right)\right]$$
(4.49)

which is strongly reduced in amplitude by a factor proportional to $e^{-R\overline{P}_{\varphi}^{2}}$.

In Fig. 4.10 are plotted the Wigner functions and the corresponding projections on the P = 0 axis for the $\hat{\rho}^{\eta}_{\Psi^{\pm}_{\varphi}}$ associated to different values of *R*, for the same initial conditions $\varphi = \frac{\pi}{2}$ and $\alpha = 6$. As expected, by increasing the degree of losses the central peak is



Figure 4.10: Wigner functions (a-c) and P = 0 section (d-f) of the quantum superposition of coherent state for $|\alpha| = 36$ and $\phi = \pi/2$. (a-d) (R=0) Unperturbed case. (b-e) (R=0.005) For small reflectivity, the Wigner function remains negative in the central region. (c-f) (R=0.5) The Wigner function progressively evolves into a positive function in all the phase-space. Note that the interference term in the P = 0 section for R = 0.5 is almost negligible ($\sim 10^{-16}$).

progressively attenuated up to a complete deletion of the quantum features associated to the negativity of the Wigner functions. We observe that the damping factor $e^{-2R|\alpha|^2 \sin^2 \varphi}$ of the coherence terms derives from the exponential decrease of the non-diagonal terms of the density matrix (4.40). However, when *R* approaches the 0.5 value, the interference pattern is progressively shifted towards positive values in all the X-axis range, and at R = 0.5 it ceases to be non-positive, in agreement with the experimental implementation [RBH01]. This quantity has been evaluated by calculating the value of the Wigner function in the first minimum of the cosine term, corresponding to:

$$W_{\hat{\rho}_{\Psi_{\varphi}^{\pm}}^{\eta}}(X_{0},0,R) = \frac{\mathscr{N}_{+}^{\frac{n}{2}}}{\pi} e^{-2(1-R)|\alpha|^{2}} \left(e^{-2|\alpha|^{2}(1-R)} - e^{-2|\alpha|^{2}R} \right) = \begin{cases} < 0 & \text{if } R < \frac{1}{2}; \\ > 0 & \text{if } R > \frac{1}{2}, \end{cases}$$
(4.50)

where $X_0 = \pi/[2\sqrt{2}\sqrt{1-R}\alpha]$. This result implies that when half of the particles are lost in the state, the nonclassicality of the system cannot be infered anymore by the presence of negative regions in the Wigner function.

4.4.2 Wigner function for the phase-covariant amplified single-photon state

The negativity of the Wigner function adopted as a sufficient criterion for nonclassicality permits to discuss the quantum properties of single-photon states after phase-covariant collinear amplification [SVD⁺09]. In Fig. 4.11 we report the plots of $W_{|1+\rangle}(X, P, R, g)$ for



Figure 4.11: Wigner functions (a-c) and P = 0 section (d-f) of a single-photon amplified state in a single-mode degenerate OPA for g = 3. (a-d) (R=0) Unperturbed case. (b-e) (R=0.005) For small reflectivity, the Wigner function remains negative in the central region. (c-f) (R=0.5) The Wigner function progressively evolves into a positive function in all the phase-space.

the single-mode case, evaluated in Sec. 4.2.1 in Eqs. (4.15-4.18), for different values of

the reflectivity *R*. As a first effect, the negative region is deleted for a reflectivity R = 1/2: Fig. 4.11 (c). Then, the form of the distribution remains unchanged until the reflectivity becomes close to 1 and all the photons present in the states are lost: $R\langle n \rangle \simeq \langle n \rangle$.

We then consider the value of the Wigner function at the origin $X_+ = X_- = 0$ and $P_+ = P_- = 0$ of the phase-space:

$$W_{|1+,0-\rangle}(\{0\},R,g) = \frac{4}{\pi^2} \frac{2R-1}{\left[1+4R(1-R)S^2\right]^2} = \begin{cases} <0 & \text{if } R < \frac{1}{2};\\ >0 & \text{if } R > \frac{1}{2}. \end{cases}$$
(4.51)

 $W_{|1+,0-\rangle}(\{0\}, R, g) < 0$ for $R \le 1/2$, showing that the negativity is maintained in that range of the lossy channel efficiency, as for the quantum superposition of coherent states. Hence, the multiphoton quantum superpositions generated by phase-covariant cloning of a single-photon with equatorial polarization surely presents nonclassical features up to a value of the losses parameter R < 0.5, that is, when half of the particles are lost. Note that, since the negativity of the Wigner function does not represent a necessary criterion of nonclassicality, this result does not directly imply that the analyzed multiphoton states are classical for $R \ge 0.5$. More investigation based on different nonclassicality measures are necessary in this losses regime.

4.4.3 Wigner function for the universally amplified single-photon state

As for the collinear case, the Wigner function for the amplified single-photon states in a non collinear configuration [SVD $^+09$] has a minimum in the origin of the phase-space:

$$W_{|1+,0-\rangle_1|0+,0-\rangle_2}(\{0\},R,g) = \frac{16}{\pi^4} \frac{2R-1}{\left[1+4R(1-R)S^2\right]^3} = \begin{cases} <0 & \text{if } R < \frac{1}{2};\\ >0 & \text{if } R > \frac{1}{2}. \end{cases}$$
(4.52)

We note that the negativity of the Wigner function is maintained for R > 1/2, consistently with the other classes of states analyzed in this chapter. Analogously with the phase covariant case, the multiphoton quantum superpositions generated by universal cloning of a single photon with equatorial polarization surely presents nonclassical features up to a value of the losses parameter R < 0.5. More investigation based on different nonclassicality measures are still necessary in the $R \ge 0.5$ losses regime.

To conclude this analysis in Fig. 4.12 we compare the negativity of the Wigner function for the different classes of states analyzed in this chapter. We note that the trends for the collinear and noncollinear amplifier are analogous, and that all the analyzed states present negative region in the same losses range up to R = 0.5. Moreover, the absolute value of the negativity is smaller in the noncollinear case due to the higher number of vacuum injected optical modes. This behaviour can be related to the smaller cloning fidelity in the universal case. Finally, we observe that the decrease in the negativity is faster in the quantum superposition of coherent states with respect to single-photon amplified states. This means that the multiphoton system generated by quantum cloning of a microscopic state present more robust nonclassical features in presence of optical losses.



Figure 4.12: Trend of the minimum of the Wigner functions as a function of the losses parameter *R* for the different classes of multiphoton states. Green lines: superposition of coherent states. Red lines: collinear amplification of single photons. Blue lines: non-collinear amplification of single photons. (a) $\langle n \rangle = 4$. (b) $\langle n \rangle = 8$. (c) $\langle n \rangle = 12$.

4.5 Conclusions and perspectives

In this chapter we investigated the quantum properties of a class of multiphoton states generated by parametric amplification of single photons. Such analysis has been performed both in the Fock space and in the phase-space, focusing on the resilience of nonclassical properties in presence of a decohering-lossy system-environment interaction. The obtained results have been compared with the quantum superpositions of coherent states, chosen as a reference. The amplified single-photon states in a lossy configuration were investigated, allowing to observe the persistence of the non-positivity of the Wigner function in a certain range of the losses interaction parameter *R*. The same behaviour was found for the superposition of coherent $|\alpha\rangle$ states, which possesses a non-positive W-representation in the same interval of the interaction parameter *R*. This analysis, combined with the slower decreasing rate of the Bures distance between orthogonal states, shows that the amplified single-photon states present more robust nonclassical features in the same losses range up to R = 0.5.

As a further perspectives, we note that the negativity of the W-representation is a sufficient but not a necessary condition for the nonclassicality of any physical system. Hence, future investigations could be aimed to the analysis of the decoherence regime in which the Wigner function is completely positive, analyzing the presence of quantum properties from a different point of view. Furthermore, the nonclassical features of the Wigner distribution suggest as a possible direction the development, and subsequently the application, of entanglement and nonlocality tests based on phase-space measurements to such class of multiphoton states.

Decoherence on multiphoton quantum superpositions

Chapter 5

Entanglement detection and manipulation in a bipartite microscopic-macroscopic system

The observation of entangled states in systems with a large number of particles is still an open challenge. Moreover, the complexity of the entanglement criteria increases with the size of the system, thus rendering the development of suitable experimentally feasible criteria an essential task. In the previous chapter we discussed the properties of the multiphoton states generated by parametric amplification of single photons. We showed that such process leads to the generation of an output multiphoton field which can be efficiently discriminated in presence of losses, and which presents nonclassical features even for significant losses. These properties, combined with the possibility of progressively increasing the number of generated photons by tuning the nonlinear gain, suggests the possibility of adopting this device to investigate entanglement in a multiphoton system.

In this chapter, we investigate the possibility of observing the entanglement in realistic conditions in a bipartite system obtained by amplification of a photon belonging to an entangled pair. We discuss several dichotomic detection techniques, some of them relying on supplementary assumptions on the optical source. Then, we consider several schemes to manipulate the generated multiphoton field in order to increase the distinguishability of such states, for applications in entanglement and nonlocality tests, as well as in quantum information protocols. The obtained results are reported in Refs. [SVSD10, VSSD10]. The analysis of this system motivated further investigation in the field of detection and manipulation of multiphoton entangled states [SSB+11, RSS11, STS+11]. Furthermore, the results reported in this chapter suggest that in order to observe the entanglement in this multiphoton system a high efficiency measurement is required, since it is necessary to detect most of the involved particles.

5.1 Entanglement and manipulation of multiphoton systems

The observation of quantum phenomena, such as quantum entanglement [HHHH09], has been mainly limited to systems of only few particles. In order to observe the presence of entanglement in large systems, a large effort has been devoted in the last few years to construct suitable criteria for the assertion of entanglement in multiphoton systems [JPR09, Wod00, GKLC01, SJR07, LJJ09, LJJ10]. For bipartite systems of a large number of particles, this approach has been further investigated considering the possibility to exploit collective measurements on the multiparticle state. Within this context, Duan et al. proposed a general criterion [DGCZ00] based on continuous-variables observables, further extended later on to different classes of operators [KLL+02, SBT+03, KL05, SB03, CPHZ02]. An experimental application of this criteria based on collective spin measurements has been performed in a bipartite system of two atomic gas samples [JKP01]. However, an experimental realization of most of these criteria in the quantum optical domain requires photon-number resolving detectors with nearly unitary efficiency, which is beyond the current technology. A feasible approach for the analysis of multiphoton fields has been developed in the last few years, and is based on the deliberate attenuation of the analyzed system up to the single-photon level. In this way, standard single-photon techniques and criteria can be used to investigate the properties of the field. The verification of the entanglement in the high losses regime is an evidence of the presence of entanglement before the attenuation, since no entanglement can be generated by local operations. Such approach has been exploited in [EKD⁺04, CDP⁺06] to demonstrate the presence of entanglement in a high gain spontaneous parametric down-conversion source up to 12 photons. Analogous conclusion has been theoretically obtained in Ref. [DSEB04] on the same system by exploiting symmetry considerations of the source. The attenuation method has been also applied to a different system, allowing to obtain an experimental proof of the presence of entanglement between a single-photon state and a multiphoton state generated through the process of optical parametric amplification in an universal cloning configuration with up to 12 cloned photons [DSS05].

The present chapter is organized as follows. In Sec. 5.2 we investigate a specific optical micro-macro system, that is, the quantum state generated by parametric amplification of a single photon belonging to an entangled pair. We consider different approaches to detect the presence of entanglement in lossy conditions. First, we consider the application of the method based on the deliberate attenuation of the multiphoton field up to the single-photon level. Then, we discuss the test performed in Ref. [DSV08], by focusing on the assumptions necessary for the validity of the test. We then consider a modified version of this entanglement inequality which can be adopted with any dichotomic measurement operator and does not require any supplementary assumption. Finally, in Sec. 5.3 we consider the possibility of manipulating the multiphoton states in order to increase the distinguishability of the detected states, in the perspective of a possible application in entanglement and nonlocality tests.

5.2 Benchmark state: amplification of an entangled photon pair

In this section we describe a benchmark state for the analysis of entanglement between a single photon and a multiphoton system. The chosen system is obtained by optical parametric amplification of a single photon belonging to a polarization entangled photon pair.

5.2.1 The optical configuration

An entangled pair of two photons in the singlet state $|\psi^-\rangle_{A,B}=2^{-\frac{1}{2}}(|H\rangle_A|V\rangle_B-|V\rangle_A|H\rangle_B)$ is produced through spontaneous parametric down-conversion by crystal 1 (C1) pumped by a pulsed UV pump beam: Fig.5.1. There $|H\rangle$ and $|V\rangle$ stands, respectively, for a single photon with horizontal and vertical polarization $(\vec{\pi}_{H,V})$ while the labels A, B refer to particles associated respectively with the spatial modes \mathbf{k}_A and \mathbf{k}_B . The photon belonging to \mathbf{k}_B , together with a strong UV pump beam, is injected into an optical parametric amplifier consisting of a non-linear crystal 2 (C2) pumped by the beam \mathbf{k}'_P . The crystal 2 is oriented for collinear operation, i.e., emitting pairs of amplified photons over the same spatial mode which supports two orthogonal $\vec{\pi}$ modes, respectively horizontal and vertical. The overall state after the amplification process reads:

$$|\Psi^{-}\rangle_{AB} = \frac{1}{\sqrt{2}} (|\phi\rangle_{A} |\Phi^{\phi_{\perp}}\rangle_{B} - |\phi_{\perp}\rangle_{A} |\Phi^{\phi}\rangle_{B}).$$
(5.1)

From now on, such state will be dubbed as micro-macro. The output multiphoton states are defined as $|\Phi^{\phi}\rangle = \hat{U}_{\text{OPA}}|\phi\rangle$, where $|\phi\rangle$ labels the injection of single-photon state with equatorial polarization. For a detailed discussion on the properties of such states we refer to Secs. 4.2.1, 4.3.3 and 4.4.2, and to Refs. [DSV08, DSS09b].

5.2.2 Entanglement witness in a highly attenuated scenario

As a first step, we consider a discrete-variables approach that can be used to demonstrate the presence of entanglement in the optical bipartite microscopic-macroscopic system described in the previous section. This approach is based on the introduction of a deliberate attenuation up to the single-photon regime in the multiphoton subsystem. Standard criteria for microscopic bipartite systems can be then applied in this condition, such as the Peres-Horodecki criterion [Per96] or the calcution of the concurrence [Woo98]. Since the action of losses is a local operation, no entanglement can be generated by introducting any amount of controlled attenuation. Hence, if the state is entangled after the lossy process, it must have been entangled before losses. Such method has been exploited to demonstrate



Figure 5.1: Scheme of the optical setup. The main UV laser beam provides the OPA excitation field beam at $\lambda_P = 397.5$ nm. A type II BBO (Beta Barium Borate) crystal (crystal 1: C1) generates pair of photons with $\lambda = 795$ nm. In virtue of the non-local correlations established between the modes \mathbf{k}_A and \mathbf{k}_B , the preparation of a single-photon on mode \mathbf{k}_B with polarization state $\vec{\pi}_{\varphi}$ is conditionally determined by detecting a single-photon after proper polarization analysis on the mode \mathbf{k}_A [polarizing beamsplitter (PBS), $\lambda/2$ and $\lambda/4$ waveplates, Soleil-Babinet compensator, interferential filter (IF), avalanche photodiodes (D_A, D_A^*)]. The photon belonging to \mathbf{k}_B , together with the pump laser beam \mathbf{k}'_p , is fed into an high gain optical parametric amplifier consisting of a NL crystal 2 (C2), cut for collinear type-II phase matching. Finally, the output field on the multiphoton mode \mathbf{k}_B is sent to the detection stage.

the entanglement up to 12 photons in a spontaneous parametric down-conversion source [EKD⁺04], or in a micro-macro configuration [DSS05]. The average number of photons impinging onto the detector in this regime is $\eta \langle n \rangle \leq 1$, where η is the overall quantum efficiency of the channel. In this condition, the probability of detecting more than one photon becomes negligible.

Let us now focus on the optical system described in the previous section. The density matrix of the macroscopic state can be reduced to a 1-photon subspace, and the joint micro-macro system is defined in a 2 × 2 polarization Hilbert space spanned by the basis vectors $\{|H\rangle_A|H\rangle_B, |H\rangle_A|V\rangle_B, |V\rangle_A, |H\rangle_B, |V\rangle_A|V\rangle_B\}$. The complete state $\hat{\rho}_{\eta}^{AB}$ can be then evaluated by applying the map describing a lossy channel [DSEB04] to the micro-macro amplified state:

$$\hat{\rho}_{\eta}^{AB} = (\hat{l}^A \otimes \mathscr{L}_{\eta}^B) \left[(\hat{l}^A \otimes \hat{U}_{\text{OPA}}^B) | \psi^- \rangle_{AB} \langle \psi^- | (\hat{l}^A \otimes \hat{U}_{\text{OPA}}^B) \right].$$
(5.2)

We obtain the following expression:

$$\hat{\rho}_{\eta}^{AB} = \frac{1}{1+3t^2} \begin{pmatrix} t^2 & 0 & 0 & 0\\ 0 & \frac{1}{2}(1+t^2) & -\frac{1}{2}(1+t^2) & 0\\ 0 & -\frac{1}{2}(1+t^2) & \frac{1}{2}(1+t^2) & 0\\ 0 & 0 & 0 & t^2 \end{pmatrix},$$
(5.3)

where:

$$t = (1 - \eta)\Gamma. \tag{5.4}$$

In Fig. 5.2 (a) we show the density matrix of the joint micro-macro system for a value of g = 3 and $\eta = 10^{-4}$, showing the presence of the off-diagonal terms even in the high losses regime. This system is entangled for any value of the nonlinear gain g. This property can be tested by application of the Peres criterion or by direct calculation of the concurrence, which reads:

$$C[\hat{\rho}_{\eta}^{AB}] = \left(\frac{1-t^2}{1+3t^2}\right) > 0.$$
(5.5)

This quantity is always positive, as plotted in Fig. 5.2 (b), showing the presence of entanglement for any value of the gain.



Figure 5.2: (a) Density matrix of joint micro-macro system in the high losses regime, for a gain value of g = 3 and a value of the losses parameter $\eta = 10^{-4}$. (b) Plot of the concurrence $C[\hat{\rho}_{\eta}^{AB}]$ as a function of the parameter $t = \Gamma(1 - \eta)$. We note the persistance of the off-diagonal terms and entanglement for all values of g and η .

This criterion allows us to discuss an important feature of the micro-macro system based on optical parametric amplification. The entanglement of this system is generated in the micro-micro source, where the singlet polarization state $|\Psi^-\rangle$ is produced. The action of the amplifier is to broadcast the properties of the injected seed to the multiparticle state. In particular, the entanglement present in the original photon pair after the amplification process is transfered and shared among the generated particles (see Fig. 5.3). If a certain amount of losses is introduced in the macro-state and ε is the percentage of photons that



Figure 5.3: Diagramatic scheme of the entanglement broadcasting from the single photon pair to the multiparticle state. In presence of losses, the entanglement is reduced of a factor ε .

survive such decoherence process, the amount of entanglement detected after losses is reduced of a factor ε but drops to 0 only if all particles are lost. Analytically, this feature is obtained by analyzing the expression (5.5) for $C[\hat{\rho}_{\eta}^{AB}]$. In the high gain limit ($\Gamma \sim 1$), the concurrence of our system in the highly attenuated regime becomes:

$$C[\hat{\rho}_{\eta}^{AB}] \sim \frac{1 - \Gamma^2}{1 + 3\Gamma^2} + \eta \frac{8\Gamma^2}{(1 + 3\Gamma^2)^2} \stackrel{\Gamma \to 1}{\to} \frac{\eta}{2} \propto \eta, \qquad (5.6)$$

being directly proportional to η , that is, the fraction of detected photons.

To conclude these considerations, we extend the analysis of the micro-macro amplified system in this highly attenuated scenario to the case where the injection of the single photon in the optical parametric amplifier occur with a non unitary efficiency p < 1. Such parameter represents the amount of matching (spectral, spatial, and temporal) between the optical mode of the amplifier and the optical mode of the injected single-photon. To model this source of experimental imperfection, the joint state between the two modes \mathbf{k}_A and \mathbf{k}_B before amplification is described by $\hat{\rho}_p^- = p |\Psi^-\rangle_{AB} \langle \Psi^-| + (1-p)\frac{\hat{I}_A}{2} \otimes |0\rangle_B \langle 0|$, where $\hat{I}_A =$ $|H\rangle_A \langle H| + |V\rangle_A \langle V|$ stands for a completely mixed polarization state and $|0\rangle_B \langle 0|$ represents the vacuum input state. By following the same procedure described for the p = 1 case, the density matrix of the joint micro-macro system after amplification and losses in the highly attenuated regime reads:

$$\hat{\rho}_{\eta,p}^{AB} = \mathcal{N}_{\eta,p}^{-1} \left\{ \frac{2p}{C^2} \frac{1}{1-t^2} \begin{pmatrix} t^2 & 0 & 0 & 0\\ 0 & \frac{1}{2} \left(1+t^2\right) & -\frac{1}{2} \left(1+t^2\right) & 0\\ 0 & -\frac{1}{2} \left(1+t^2\right) & \frac{1}{2} \left(1+t^2\right) & 0\\ 0 & 0 & 0 & t^2 \end{pmatrix} + (1-p)\Gamma \begin{pmatrix} t & 0 & 0 & 0\\ 0 & t & 0 & 0\\ 0 & 0 & t & 0\\ 0 & 0 & 0 & t \end{pmatrix} \right\},$$
(5.7)

where $\mathcal{N}_{\eta,p}$ is the normalization constant. In Figs. 5.4 (a) and (b) we show the density



Figure 5.4: (a)-(b) Density matrix of the micro-macro system in the high losses regime, for a gain value of g = 3 and a value of the losses parameter $\eta = 10^{-4}$. (a) Injection probability p = 0.5 and (b) injection probability of p = 0.15. (c) Concurrence $C[\hat{\rho}_{\eta,p}^{AB}]$ as a function of the gain g for $\eta = 10^{-4}$. Red solid line corresponds to an injection probability p = 1, green long dashed line to p = 0.5, blue short dashed line to p = 0.25 and black dotted line to p = 0.05. (d) 3-dimensional plot for the critical injection probability p_{crit} as a function of the gain g and the transmission coefficient η .

matrix for a gain value g = 3, for $\eta = 10^{-4}$ and injection probabilities of p = 0.5 and p = 0.15. The effect of a decreasing injection probability p is the reduction of the offdiagonal terms and hence of the coherence terms. The application of the Peres criterion on this density matrix gives a critical value of the injection probability $p_{\text{crit}} = \frac{S^2(1-\eta)}{1+S^2(1-\eta)}$. For $p > p_{crit}$, the micro-macro system in this highly attenuated regime is entangled, while for $p \le p_{crit}$ the system is separable. The same result is confirmed by the calculation of the concurrence, which reads:

$$C[\hat{\rho}_{\eta,p}^{AB}] = \begin{cases} \frac{p(1-t^2) - (1-p)tS^2(1-t^2)}{p(1+3t^2) + 2(1-p)tS^2(1-t^2)} & \text{for } p > p_{\text{crit}}, \\ 0 & \text{for } p \le p_{\text{crit}}. \end{cases}$$
(5.8)

In Fig. 5.4 (c) we report the plot of the concurrence as a function of the gain g for several values of the injection probability p and $\eta = 10^{-4}$. For decreasing p, the concurrence drops to 0 for a lower value of the gain. Furthermore, in Fig. 5.4 (d) we report the plot of the critical injection probability p_{crit} as a function of the gain g and the transmission efficiency η . As the gain g is increased, the value of the critical injection probability increases up to a value close to 1. This means that, for high values of the gain, an high injection efficiency is requested to detect the entanglement with such measurement strategy.

5.2.3 Dichotomic measurements: Orthogonality filter and threshold detector

The first dichotomic measurement technique we analyze in this section is based on the O-Filter (OF) device introduced in [NDSD07, DSV08]. In this scheme the incident radiation is analyzed in polarization by a couple of photon-number resolving detectors on each spatial mode { $\mathbf{k}_1, \mathbf{k}_2$ }. In the ideal case, this measurement corresponds to the projection of the impinging field onto the Von Neumann operators: $\hat{\Pi}_{n,m} = |n\pi, m\pi_{\perp}\rangle \langle n\pi, m\pi_{\perp}|$, where $|n\pi, m\pi_{\perp}\rangle$ represents a quantum state with *n* photons with polarization π and *m* photons with polarization π_{\perp} . Subsequently, the dichotomization of the measurement corresponds to assign the value (+1) if $n_{\pi} - m_{\pi_{\perp}} > k$, (-1) if $m_{\pi_{\perp}} - n_{\pi} > k$, and (0) otherwise [Fig. 5.5 (a)]. This choice of the detection scheme corresponds to the POVM operators:

$$\hat{F}_{\pi,\pi_{\perp}}^{(+1)}(k) = \sum_{n=k}^{\infty} \sum_{m=0}^{n-k} \hat{\Pi}_{n,m}, \qquad (5.9)$$

$$\hat{F}_{\pi,\pi_{\perp}}^{(-1)}(k) = \sum_{m=k}^{\infty} \sum_{n=0}^{m-k} \hat{\Pi}_{n,m}, \qquad (5.10)$$

$$\hat{F}_{\pi,\pi_{\perp}}^{(0)}(k) = \hat{I} - \hat{F}_{\pi,\pi_{\perp}}^{(+1)} - \hat{F}_{\pi,\pi_{\perp}}^{(-1)}.$$
(5.11)

The discarded outcome turns out to be state dependent. This property renders this kind of dichotomic measurement unfeasible for applications in Bell's inequalities test.

An analogue measurement scheme, based on *N*-fold coincidences, is shown in Fig. 5.5 (b). The field is analyzed in polarization, and each branch is equally divided among a set of single-photon detectors (APD). Coincidences between the output TTL signals are recorded for each analyzed polarization, and the (+1) or the (-1) outcomes are assigned depending on which of the two analyzed sets of APDs record the *N*-fold coincidence.

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If no N-fold coincidences are recorded, the (0) inconclusive outcome is assigned to the event. This scheme performs the measurement of the N-th order correlation function of the field, where N is the number of detectors. We note that the O-Filter based and the multi-detector based schemes select analogous regions of the Fock space.



Figure 5.5: (a) O-Filter based detection apparatus. The field is analyzed in polarization [$\lambda/4$ and $\lambda/2$ wave-plates, polarizing beam-splitter] and the intensities are measured by two photomultipliers (PM). Right figure: diagram of the two-mode Fock space's region selected by the O-Filter measurement scheme. Green region (+1) corresponds to the condition $n_{\pi} - m_{\pi_{\perp}} > k$, red region (-1) corresponds to the condition $m_{\pi_{\perp}} - n_{\pi} > k$, grey region (0) corresponds to the condition $|n_{\pi} - m_{\pi_{\perp}}| < k$. (b) Multi-detector measurement strategies. The field should be analyzed in polarization [$\lambda/4$ and $\lambda/2$ wave-plates, polarizing beam-splitter]. Each polarization state should be divided in equal parts by a sequence of 50/50 beam-splitters (BS) and then detected by a set of APD's (Avalanche photo-diodes): the coincidences between all the detectors trigger the successful events. Right figure: diagram of the two-mode Fock space's region selected by the multi-detector measurement scheme. Green region (+1) corresponds to the presence of a coincidence only between all π polarization detectors, red region (-1) corresponds to presence of a coincidence only between all π_{\perp} polarization detectors, grey region (0) corresponds to the inconclusive outcome. In this case, k is the number of detectors.

Let us now introduce a different dichotomic measurement method which is based on a threshold detection (TD) scheme. Let us consider the following apparatus. As in the OF case, the incident field is analyzed in polarization on each spatial mode by photon-counting detectors, and the Von Neumann operators that describe this intensity measurement are again the $\hat{\Pi}_{n,m}$ projectors. The dichotomization of the measurement then proceeds as follows. The (+1) outcome is assigned when the threshold condition $n_{\pi} + m_{\pi_{\perp}} > h$ is satisfied and when $n_{\pi} > m_{\pi_{\perp}}$. Analogously, the (-1) outcome is assigned in the opposite case $n_{\pi} < m_{\pi_{\perp}}$ conditionally to the satisfaction of the threshold condition $n_{\pi} + m_{\pi_{\perp}} > h$. If $n_{\pi} = m_{\pi_{\perp}}$, one of the two outputs (±1) is assigned with equal probability p = 1/2. The POVM operators that describe the measurement can then be written in the form:

$$\hat{T}_{\pi,\pi_{\perp}}^{(+1)}(h) = \sum_{n=h}^{\infty} \sum_{m < \frac{n}{2}} \hat{\Pi}_{n-m,m}, \qquad (5.12)$$

$$\hat{T}_{\pi,\pi_{\perp}}^{(-1)}(h) = \sum_{n=h}^{\infty} \sum_{m>\frac{n}{2}} \hat{\Pi}_{n-m,m}, \qquad (5.13)$$

$$\hat{T}_{\pi,\pi_{\perp}}^{(0)}(h) = \hat{I} - \hat{T}_{\pi,\pi_{\perp}}^{(+1)} - \hat{T}_{\pi,\pi_{\perp}}^{(-1)}.$$
(5.14)

We note that the choice of the threshold *h* is made independently from the input state, and it is an intrinsic property of the detection apparatus. Furthermore, this scheme has the peculiar property of selecting an invariant region of the Fock space with respect to rotations of the polarization basis. More specifically, let us consider the case in which the measurement is performed choosing a polarization basis π, π_{\perp} . With that choice, all the pulses for which $n_{\pi} + m_{\pi_{\perp}} \leq h$ are not detected. Rotating the basis to π', π'_{\perp} , the undetected part of the wave function still corresponds to the application of the same threshold condition in the new basis $n_{\pi'} + m_{\pi'_{\perp}} > h$. Hence, the filtered Fock-space region is independent on the choice of the polarization basis but is a function only of the threshold *h*, which is an intrinsic property of the detection apparatus. This feature is the main difference with the OF device discussed in previous section, and renders the TD-based detection strategy feasible for its implementation in Bell's inequalities tests.

5.2.4 Dichotomic variables entanglement test with supplementary assumptions

In this section we analyze the hypothesis underlying a recent entanglement test performed in Ref. [DSV08]. The system under investigation is the micro-macro source discussed in the previous section. We focus our analysis on the exploited entanglement criterion, obtained as the extension of a spin-based criterion for a bipartite microscopic-microscopic system [EKD⁺04].
Micro-micro entanglement witness

For a two-photon separable state $|\psi\rangle$, defined on two different modes *a* and *b* the following inequality holds [Dur04, EKD⁺04]:

$$\psi \langle \hat{\sigma}_1^{(a)} \otimes \hat{\sigma}_1^{(b)} \rangle_{\psi} + \psi \langle \hat{\sigma}_2^{(a)} \otimes \hat{\sigma}_2^{(b)} \rangle_{\psi} + \psi \langle \hat{\sigma}_3^{(a)} \otimes \hat{\sigma}_3^{(b)} \rangle_{\psi} \le 1,$$
(5.15)

where $\hat{\sigma}_{1,2,3}$ are the Pauli operators and $\psi \langle \cdot \rangle \psi$ stands for the average on the state $|\psi\rangle$. A violation of this bound for an input state $|\psi\rangle$ witnesses the entanglement properties of the measured state.

Micro-macro entanglement witness in the ideal case

The same criterion can be extended to a micro-macro scenario, by measuring the pseudo spin operators $\hat{\Sigma}_i$ on the macro state, obtained through an unitary transformation upon the micro-micro state. Here, the $\hat{\Sigma}_i$ operators are the time evolution of the Pauli operators according to $\hat{\Sigma}_i = \hat{U}_{\text{OPA}} \hat{\sigma}_i \hat{U}_{\text{OPA}}^{\dagger}$, where \hat{U}_{OPA} is the time evolution operator of the amplifier and i = 1, 2, 3 refer to the polarization basis $1 \rightarrow \{H, V\}, 2 \rightarrow \{R, L\}, 3 \rightarrow \{+, -\}$. Since the operators $\hat{\Sigma}_i$ are built from the unitary evolution of eigenstates of $\hat{\sigma}_i$, they satisfy the same commutation rules of the single-particle 1/2-spin: $[\hat{\Sigma}_i, \hat{\Sigma}_j] = 2\imath\varepsilon_{ijk}\hat{\Sigma}_k$, where ε_{ijk} is the Levi-Civita tensor. The measurement of these operators require parity detection on the output field, and their complete expressions can be found in Ref. [DSV08].

Let us now consider the state $|\Psi\rangle$ obtained by the amplification of the state $|\psi\rangle$ over the single spatial mode **k**_B. Such state can be identified as a two-qubit state of a micro and a macro system. In the ideal case, the following map holds:

$$\begin{aligned} |\pm\rangle &\to |\Phi^{\pm}\rangle = \hat{U}_{\text{OPA}}|\pm\rangle, \\ |R/L\rangle &\to |\Phi^{R,L}\rangle = \hat{U}_{\text{OPA}}|R/L\rangle. \end{aligned}$$
(5.16)

According to the properties of pseudo-Pauli operators $\{\hat{\Sigma}_i\}$ the following inequality holds for any separable state $|\Psi\rangle$ built through the process of parametric amplification of the micro-micro state $|\psi\rangle$:

$$\Psi \langle \hat{\Sigma} \rangle \Psi = \Psi \langle \hat{\sigma}_1^{(a)} \otimes \hat{\Sigma}_1^{(b)} \rangle \Psi + \Psi \langle \hat{\sigma}_2^{(a)} \otimes \hat{\Sigma}_2^{(b)} \rangle \Psi + \Psi \langle \hat{\sigma}_3^{(a)} \otimes \hat{\Sigma}_3^{(b)} \rangle \Psi \le 1.$$
(5.17)

A violation of this bound witnesses the presence of entanglement in the microscopicmacroscopic two-qubit state $|\Psi\rangle$. A direct calculation of this inequality on the two-qubit micro-macro state $|\Psi^-\rangle$ of Eq. (5.1) in absence of losses gives $_{\Psi^-}\langle \hat{\Sigma} \rangle_{\Psi^-} = 3$, witnessing the entanglement of the state in ideal conditions.

Micro-macro entanglement in the lossy case

The micro-macro entanglement test of Eq. (5.17) has been applied to the optical source descibed in Sec. 5.2.1 by adopting the OF device described above. This approach has

been adopted since the measurement of the pseudo-Pauli operators, requiring the perfect discrimination of the number of photons present in the detected state, is out of reach by current technology.

The entanglement test performed on the investigated system in Ref. [DSV08] is given by Eq. (5.17). In that test, the $\{\hat{\Sigma}_i\}$ operators of the original inequality for two-dimensional micro-macro systems are replaced with the $\{\hat{F}_{\pi,\pi_{,|}}^{(\pm 1)}\}$ operators of the O-Filter:

$$\Psi \langle \hat{\sigma}_{1}^{(a)} \otimes \hat{\Pi}_{1}^{(b)} \rangle \Psi + \Psi \langle \hat{\sigma}_{2}^{(a)} \otimes \hat{\Pi}_{2}^{(b)} \rangle \Psi + \Psi \langle \hat{\sigma}_{3}^{(a)} \otimes \hat{\Pi}_{3}^{(b)} \rangle \Psi \leq 1.$$
(5.18)

where i = 1, 2, 3 corresponds to three different polarization bases and:

$$\hat{\Pi}_{i}^{(b)} = \hat{F}_{\pi_{i},\pi_{i}^{\perp}}^{(+1)} - \hat{F}_{\pi_{i},\pi_{i}^{\perp}}^{(-1)}$$
(5.19)

It is worth noting that, in general, the resulting Eq.(5.18) is no longer an entanglement witness. Without any assumption on the investigated system the inequality (5.18), that is, the original pseudo-Pauli criterion (5.17) where the $\{\hat{\Sigma}_i\}$ operators have been replaced by the $\{\hat{F}_{\pi,\pi_{\perp}}^{(\pm 1)}\}$ ones, does not represent anymore a bound for entangled states. It is satisfied by separable states of the form [SBB⁺09]:

$$\hat{\rho}_{\rm sep} = \frac{1}{2\pi} \int_0^{2\pi} d\phi \hat{U}(\phi) |1\pi_i, 0\pi_i^{\perp}\rangle_a |0\pi_i, N\pi_i^{\perp}\rangle_{b\,a} \langle 1\pi_i, 0\pi_i^{\perp}|_b \langle 0\pi_i, N\pi_i^{\perp}|\hat{U}(\phi)^{\dagger}, \quad (5.20)$$

where $\hat{U}(\phi)$ is a rotation of the whole system polarization around the *z* axis by an angle ϕ .

The bound of Eq. (5.18) can be recovered as an entanglement witness by making a supplementary assumption on the micro-macro source: the macro state has to be generated by an amplification process upon a micro-micro entangled pair (Fig. 5.6). In this case, one is entitled to rule out separable states of the form (5.20) since they cannot be generated by amplification of a micro-micro-state, thus recovering the validity of Eq. (5.18) as an entanglement witness. Furthermore, a careful analysis of the OF properties show that, for asymptotically high threshold, the mean values of the $\{\hat{\Sigma}_i\}$ tend to the mean values of the corresponding $\hat{F}_{\pi,\pi_{\perp}}^{(\pm 1)}(k)$ OF operators [SVD⁺09].

5.2.5 Dichotomic variables entanglement test in absence of supplementary assumptions

The so far presented test is based on the inequality (5.17) for the pseudo-Pauli spin operators. That inequality is written for the case of a micro-macro state obtained via amplification process. When the $\{\hat{\Sigma}_i\}$ operators are replaced by a set $\{\hat{D}_i\}$ of more general dichotomic operators, the bound to be violated in order to demonstrate the entanglement of the overall micro-macro system without making any supplementary assumption must

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Figure 5.6: (a) Micro-macro system source in a black box configuration: no assumption is made about the source. (b) Micro-macro amplified system: the macroscopic state is generated by a coherent amplification process of a single photon, belonging to an entangled pair in the singlet polarization state $|\psi^-\rangle$.

be modified with respect to Eq. (5.17). It can be shown that a necessary condition for separable states is given by the following inequality:

$$S = \langle \hat{\sigma}_1^{(a)} \otimes \hat{D}_1^{(b)} \rangle_{\Psi} + \langle \hat{\sigma}_2^{(a)} \otimes \hat{D}_2^{(b)} \rangle_{\Psi} + \langle \hat{\sigma}_3^{(a)} \otimes \hat{D}_3^{(b)} \rangle_{\Psi} \le \sqrt{3}.$$
(5.21)

Details over the derivation of this criterion are reported in App. B. Such criterion presents the feature of not requiring any knowledge of the Hilbert space where the analyzed states live. Indeed, in the derivation of the bound (5.21) the only necessary assumption concerns the measurement operators, which can have only two possible outcomes (± 1) . We then



Figure 5.7: Numerical evaluation of the witness S for the specific choice of the Pauli pseudo-spin operators $\{\hat{\Sigma}_i\}$ adopted as measurement operators $\{\hat{D}_i\}$. The witness S is plotted as a function of the detection losses η , calculated for several values of the gain of the amplifier. The upper horizontal solid line corresponds to the bound for separables states of the general criterion (5.21), while the lower solid line corresponds to the bound for separa-1.0 ble states (5.17) where the commutation properties of the operators have been exploited and a standard assumption on the Hilbert space is necessary.

applied the obtained criterion to evaluate the quantity *S* for the micro-macro state generated through the process of optical parametric amplification, for the specific choice of the Pauli pseudo-spin operators $\{\hat{\Sigma}_i\}$ as the measurement operators. More specifically, we evaluated the value of *S* as a function of the transmission efficiency η of the multiphoton

mode \mathbf{k}_B for several values of the gain g [Fig. 5.7]. The value of S is then compared to the bound for separable states $S_{\text{gen}}^{\text{sep}} = \sqrt{3}$. We observe that this entanglement measurement is fragile under losses, since the value of S falls below the bound for separable states when the number of lost photons is $R\langle n \rangle \sim 1$. Such result is expected since the Pauli operators allows to distinguish the $|\Phi^{\phi}\rangle$ states exploiting the well-defined parity in the number of photon generated by the amplifier depending on the polarization of the input states. In presence of losses, such well-defined parity is quickly cancelled, thus not allowing to discriminate among the macro-states with this kind of measurement.

5.3 Manipulation of the multiphoton states by measurement induced quantum operations

In this section we consider several strategies for the realization of measurement-induced quantum operations on the multiphoton states generated thought the process of optical parametric amplification. We investigate theoretically how the measurement strategies, applied on a part of the multiphoton state before the final identification measurement, affect the distinguishability of orthogonal multiphoton states. Starting from the original



Figure 5.8: (a) Scheme of the measurement-induced quantum operation process. The field is split by an unbalanced beam-splitter, and the reflected portion is measured to conditionally active the optical shutter placed in the path of the transmitted portion of the field.

proposal of a preselection apparatus in a different configuration of Ref. [De 11], we consider the particular case in which a macro-state generated by the QIOPA is split by an unbalanced beam-splitter and manipulated by measuring the state on the reflected mode. The conceptual scheme underlying the present investigation is shown in Fig. 5.3: a part of the wave-function is measured and the results of the measurement are exploited to conditionally activate an optical shutter placed in the transmitted part. Such shutter, whose realization has been recently reported in Ref. [SVG⁺08], is used to allow the transmission of the optical beam only in presence of a trigger event, i.e. in this case the results of the measurement performed in the reflected part of the state. The interest in improving the capability of identifying the state generated by the quantum injected optical parametric amplifier system mainly relies in two motivations: the first one concerns the development of a discrimination method able to increase the transmission fidelity of the state after the propagation over a lossy channel, and hence to overcome the imperfections related to the practical implementation. Such increased discrimination capability in lossy conditions could find applications within the quantum communication context. The second

reason concerns the scenario in which an appropriate pre-selection of the multiphoton states could be adopted to demonstrate the nonlocality of the system, free from the auxiliary assumptions requested if the filtering procedure was applied at the final measurement stage.

5.3.1 Filtering of the macro-qubit

In this section we discuss a first pre-selection scheme, sketched in Fig. 5.9 (a). A similar scheme has been proposed in Ref. $[SHB^+09]$ and adopted to investigate the possibility of observing nonlocality in a different scenario. As previously discussed, one of the main experimental challenge for the realization of the bipartite single-photon and multiphoton system of Fig. 5.3 is the achievement of spectral, spatial and temporal matching between the optical mode of the injected single photon state and the optical mode of the amplifier. This results to be a source of additional noise in the amplified state. The filtering method here presented is adopted to reduce the noise introduced by the spontaneous emission of the amplifier.

Let us now discuss the propagation of the multiphoton field produced by the amplifier and the pre-selection procedure obtained through an intensity threshold detector and the shutter device. As shown in Fig. 5.9 (a), the amplified state is split by an unbalanced beam splitter (UBS) 0.90 - 0.10 in two parts: the smaller portion on mode $\mathbf{k}_{\mathbf{D}}$ is analyzed by the TD, and the larger one on mode $\mathbf{k}_{\mathbf{C}}$ is conditionally pushed through a polarization preserving shutter [SVG⁺08], and measured in polarization by a dichotomic measurement. The TD based filtering strategy allows then to obtain a better discrimina-



Figure 5.9: (a) Filtering of the macro-qubit: the shutter activation is conditioned to an intensity measurement on the reflected portion of the macro state. (b) Trend of the injection probability as a function of the TD threshold, for different initial values of p. The nonlinear gain of the amplifier is set at g = 1.5.

tion between the orthogonal macro states, by minimizing the noise related to the vacuum injection into the amplifier. This is performed by increasing the threshold h of activation of the TD device, which activates the shutter on the transmitted UBS mode, ensuring the

presence of the higher, i.e. correctly injected, pulses. It is worth nothing that, at variance with the techniques which will be introduced in the following sections, the TD action is invariant for rotation on the Fock space since it selects the same region of the multiphoton state either in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis either in the $\{\vec{\pi}_R, \vec{\pi}_L\}$ one. These considerations can be quantified introducing the injection probability p_{cond} conditioned to the activation of the shutter given by the threshold condition of the TD. We then evaluated numerically this quantity for several values of the un-conditioned injection probability p. It turns out that the value of p_{cond} is increased as shown in Fig. 5.9 (b), in which we report the trend of the conditional injection probability p_{cond} as a function of the TD threshold h.

5.3.2 Deterministic transmitted state identification

Here we investigate a pre-selection strategy based on a comparison between orthogonally polarized signals. This configuration is illustrated in Fig. 5.10 (a) and is based on a



Figure 5.10: (a) Preselection strategy: the small reflected part of the macro state is analyzed in polarization and detected through an OF based measurement strategy. (b) Measurement scheme adopted for the conditional activation of the shutter: if the OF, on the reflected mode, measures the state on the green regions, the shutter, on the transmitted mode, is conditionally activated. The green regions correspond to the state for which the signals belonging to orthogonal polarizations are unbalanced over a certain threshold k, i.e. $|p - q| \ge k$. (c) Scheme for the final detection of the output state: Conditioned on a measurement result in the ON region on the reflected mode, the transmitted mode is identified by a dichotomic measurement in the $\{\pi, \pi_{\perp}\}$ basis. The diagonal contribution to the quantum state is assigned randomly to the state $|\Phi^{\pi}\rangle$ or $|\Phi^{\pi\perp}\rangle$.

peculiar feature of the equatorial macro states. Indeed, any multiphoton states belonging to the injection of an equatorial qubit, can be discriminated efficiently through the OF measurement. Indeed, if analyzed in the same polarization basis of the injected qubit, the two signals will be unbalanced with an high probability. This can be explained by analyzing the probability distribution of the amplified states, reported in Fig. 5.11: (a) in the mutually unbiased equatorial polarization basis with respect to the injected state and (b) in the same basis as the injected qubit one.

We will address two cases in which the state generated by the amplifier is either $|\Phi^+\rangle$ or $|\Phi^R\rangle$, obtained by the amplification of a single photon polarized $\vec{\pi}_+ = \frac{\vec{\pi}_H + \vec{\pi}_V}{\sqrt{2}}$ and $\vec{\pi}_R =$



Figure 5.11: (a) Probability distribution of the state $|\Phi^R\rangle$ as a function of the number of photons $\{\vec{\pi}_+, \vec{\pi}_-\}$. (b) Probability distribution of the state $|\Phi^R\rangle$ as a function of the number of photons $\{\vec{\pi}_R, \vec{\pi}_L\}$. In both distributions g = 1.5.

 $\frac{\vec{\pi}_H + i\vec{\pi}_V}{\sqrt{2}}$, respectively. In both cases the analysis basis corresponding to the UBS reflected mode is fixed to $\{\vec{\pi}_+, \vec{\pi}_-\}$, while the transmitted mode is analyzed in the same basis in which the injected qubit has been encoded. Let us discuss the experimental setup shown in Fig. 5.10 (a). The multiphoton state $|\Phi^+\rangle$ (or $|\Phi^R\rangle$) generated by the QIOPA impinges on the UBS. A small portion of the field is reflected on mode \mathbf{k}_d and measured on the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis. The two signals belonging to orthogonal polarizations are then compared by an orthogonality filter. When the two signals are unbalanced, i.e. |p - q| > k, being p, q the number of photons $\vec{\pi}_+, \vec{\pi}_-$ polarized and k an appropriate threshold, the shutter on mode \mathbf{k}_c is activated and the field on that mode is conditionally transmitted [see Figs. 5.10 (b) and (c)]. The multiphoton state $|\Phi^+\rangle$ ($|\Phi^R\rangle$) is then analyzed in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ (or $\{\vec{\pi}_R, \vec{\pi}_L\}$) basis. In the following sections we will address the problem of discriminating the final multiphoton state, given the acquired information on the small portion of the reflected field.

Probability of shutter activation

Let us first evaluate the probability *P* of activating the shutter when the impinging state is detected on the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis, depending on the value of *k*, with an OF technique. As shown in Fig. 5.12, the probability of activating the shutter is the same for the two output fields $|\Phi^+\rangle$ and $|\Phi^R\rangle$. This result can be explained by considering the probability distributions of the state $|\Phi^R\rangle$ in the two mutually unbiased equatorial bases shown in Fig. 5.11. Due to the linearity of the quantum mechanics, the state $|\Phi^R\rangle$ can be written as $|\Phi^R\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + \iota |\Phi^-\rangle)$. Hence, due to the peculiar features of the two multiphoton states $|\Phi^{\pm}\rangle$, that have non-zero contributions for terms with different parity, the probability distribution of the macro-state $|\Phi^R\rangle$ in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis is given as the sum of the two probability distributions of the states $|\Phi^+\rangle$ and $|\Phi^-\rangle$ in the same basis. Since shot



Figure 5.12: (a) Probability of activating the shutter when the state $|\Phi^R\rangle$ is analyzed in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis versus the threshold k of the OF. (b) Probability of activating the shutter when the state $|\Phi^+\rangle$ is analyzed on the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis. The nonlinear gain of the amplifier is set at g = 1.5.

by shot the OF identifies the state $|\Phi^+\rangle$ or $|\Phi^-\rangle$ with the same probability, the activation of the shutter has the same probability of occurrence for any linear combination of $|\Phi^-\rangle$ and $|\Phi^+\rangle$ impinging on the UBS. Note that the shutter activation probability increases in steps since the amplifier emits photons in pairs.

Analysis of the $|\Phi^+ angle$ state

Let us analyze the evolution of the state $|\Phi^+\rangle$ passing through the pre-selection apparatus. We are interested in investigating the distinguishability between orthogonal macro-states by varying the pre-selection performed over the multiphoton state itself. This can be quantified by the visibility of the transmitted mode as a function of the unbalancement between $\vec{\pi}_+$ and $\vec{\pi}_-$ photons, detected on the reflected mode. Such quantity is evaluated as the normalized difference between the probabilities of correct and incorrect identification of the input state after the filtering process as:

$$V(k) = \frac{\sum_{m,n} \sum_{p,q} \left(P_{m,n}^{p,q+}(k) - P_{m,n}^{p,q-}(k) \right)}{\sum_{m,n} \sum_{p,q} \left(P_{m,n}^{p,q+}(k) + P_{m,n}^{p,q-}(k) \right)},$$
(5.22)

Here, $P_{m,n}^{p,q+}$ is the probability that, if the state $|p+,q-\rangle_d$ is detected on spatial mode \mathbf{k}_d , m > n is obtained on spatial mode \mathbf{k}_c , and hence the macro-state $|\Phi^+\rangle$ is identified. Conversely, $P_{m,n}^{p,q-}$ is the probability that, given the detection of the state $|p+,q-\rangle_d$ on spatial mode \mathbf{k}_d , n > m is obtained on spatial mode \mathbf{k}_c , and hence the macro-state $|\Phi^-\rangle$ is identified, even if the initial state impinging on the UBS was $|\Phi^+\rangle$. Here, m,n is the number of photons $\vec{\pi}_+$ and $\vec{\pi}_-$ polarized, and $|p+,q-\rangle_d$ is the state vector corresponding to p photons with $\vec{\pi}_+$ polarization and q photons with $\vec{\pi}_-$ polarization.

The trend of visibility as a function of k is reported on Fig. 5.13-(a). We observe that, as expected, we obtain higher visibilities by increasing the value of k.



Figure 5.13: (a)-(b) Trend of the visibility of the state $|\Phi^+\rangle$ measured in the basis $\{\vec{\pi}_+, \vec{\pi}_-\}$ and $\{\vec{\pi}_R, \vec{\pi}_L\}$ respectively as a function of the threshold *k*. (c)-(d) Trend of the visibility of the state $|\Phi^R\rangle$ measured in the basis $\{\vec{\pi}_+, \vec{\pi}_-\}$ and $\{\vec{\pi}_R, \vec{\pi}_L\}$ respectively as a function of the threshold *k*. The numerical results have been obtained for the value of the gain parameter g = 1.1.

Analysis of the Macro-state $|\Phi^R\rangle$

In analogy with the previous case, the visibility of the macro-state $|\Phi^R\rangle$ after the preselection stage reads:

$$V(k) = \frac{\sum_{m,n} \sum_{p,q} \left(P_{m,n}^{p,q\,R}(k) - P_{m,n}^{p,q\,L}(k) \right)}{\sum_{m,n} \sum_{p,q} \left(P_{m,n}^{p,q\,R}(k) + P_{m,n}^{p,q\,L}(k) \right)},$$
(5.23)

where $P_{m,n}^{p,q,R,L}(k)$ are the probabilities defined in full analogy with the previous case. The behaviour of the visibility (5.23) is reported in Fig. 5.13 (d).

It is interesting to analyze the case of the visibility for $|\Phi^R\rangle$ choosing $\{\vec{\pi}_+, \vec{\pi}_-\}$ as the measurement basis, or, analogously, the case of $|\Phi^+\rangle$ with $\{\vec{\pi}_R, \vec{\pi}_L\}$. In these cases the visibilities become decreasing functions of the threshold [see Fig. 5.13 (b) and (c)]. The decreasing trend can be explained by considering that the measurements in the two polarization basis correspond to two non-commuting operators acting on the same initial state. Indeed, for asymptotically high values of the threshold $k \to \infty$, the measurement of the $\hat{F}_{\pi,\pi_{\perp}}^{(\pm 1)}$ operators that describe the OF tends to the measurement of the pseudo-spin operators $\hat{\Sigma}_i$: that is, $\hat{\Sigma}_1 = |\Phi^+\rangle\langle\Phi^+| - |\Phi^-\rangle\langle\Phi^-|$ or $\hat{\Sigma}_2 = |\Phi^R\rangle\langle\Phi^R| - |\Phi^L\rangle\langle\Phi^L|$. Hence, the measurement on the \mathbf{k}_C mode corresponds to the measurement of the $\hat{\Sigma}_i$ operators. The information gained on this mode about one of the two pseudo-spin operator acting on the macro qubit does not allow to gain information about orthogonal pseudo-spin operator. As a further remark, let us stress that this feature of the OF measurement is related to the filtering of different regions of the Fock space depending on the analyzed basis. The portion of the state that survives the action of the OF is indeed different if measured on the $\{\vec{\pi}_{+}, \vec{\pi}_{-}\}$ basis or in the $\{\vec{\pi}_{R}, \vec{\pi}_{L}\}$ one and is shown in Fig. 5.14.



Figure 5.14: Selected region for the $|\Phi^+\rangle$ state after the measurement with an OF in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis. (a) Photon number distribution in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis. (b) Photon number distribution in the $\{\vec{\pi}_R, \vec{\pi}_L\}$ basis. In both cases k = 10 and g = 1.2.

5.3.3 Probabilistic transmitted state identification

We now consider the case of the field split in two equal parts by a 0.5/0.5 beam-splitter and both the reflected and the transmitted states detected through the OF device. In such a way, the measurement apparatus is tailored to extract information on the state in two different polarization bases. This analysis permits to discuss the possibility of adopting the multiphoton states here analyzed for quantum cryptography protocols. The measurement schemes are shown in Fig. 5.15: the OF technique is applied in order to extract the maximum information available from the two states.

We consider the case in which the portion on the reflected mode is analyzed in the polarization basis orthogonal to the codification one. In Fig. 5.16 (a) is reported the trend of visibility as a function of the thresholds h on the transmitted mode and k on the reflected one. The two polarization analysis basis are chosen to be mutually unbiased. It can be seen that for equal values of the two thresholds h = k the visibility reaches a value around 0.64, the same obtained through a pure dichotomic measurement, without any pre-selection procedure on the multiphoton state. In Fig. 5.16 (b) is reported the trend of the threshold on the reflected mode, keeping fixed the value of the threshold on the transmitted one. We can see that the visibility of the transmitted state decreases when the threshold on the reflected mode increases. If the threshold on the transmitted mode is greater than the one on the reflected mode, the visibility results to be



Figure 5.15: (a) Probabilistic transmitted state identification: the macro state is split in two equal parts, and both the reflected and the transmitted components are detected through an OF device. (b) Conditional activation of the shutter: if the OF acting on the reflected mode measures the state on the green regions, the shutter, on the transmitted mode, is conditionally activated. The green regions correspond to the state for which the signals belonging to orthogonal polarizations are unbalanced over a certain threshold *k*, i.e. $|p - q| \ge k$. (c) Corresponding to the ON region on the reflected mode, the transmitted mode is identified by a probabilistic measurement in the $\{\pi, \pi_{\perp}\}$ basis. The identification condition is $|m - n| \ge h$.

higher than 0.64, as expected by the action of the OF, which allows a better discrimination of the multiphoton state, measured in the codification polarization basis. Otherwise it can be seen how, decreasing the threshold h below the threshold k, the visibility decreases below the "no filtering value".



Figure 5.16: (a) Trend of the visibility of the state $|\Phi^R\rangle$ for different values of the threshold *h* on the transmitted mode and of the threshold *k* on the reflected one. The numerical result has been obtained for a value of the parameter g = 1.2. (b) Trend of the macro-state visibility as a function of the threshold *k* on the reflected mode, fixed the threshold *h* on the transmitted one.

From this analysis we can conclude that the macro states are not suitable for quantum cryptography. The action on a portion of the state can indeed be seen as an eavesdropping attack. If the state is measured in the codification basis, the visibility of the final state results to increase as shown in Figs. 5.13 (a)-(d). This means that the conclusive results for the eavesdropper would coincide with the conclusive results for the receiver, and the eavesdropper can gain information on the macrostates without introducing noise. Otherwise if the state is measured by the eavesdropper in the wrong basis, the visibility at the receiver is not affected if the state is measured above a certain filtering threshold. According to these considerations, an eavesdropper could then develop a strategy in which he measures its part of the transmitted state in two bases. With this approach he could gain information on the transmitted signal by considering only the measurement outcome in the right basis, and only a small amount of noise is introduced by keeping the filtering thresholds above a certain value. Related to the security of the multiphoton states is the possibility of performing a nonlocality tests upon them. As a final remark for this section, we remind that the adoption of the OF device at the measurement stage is not suitable for a nonlocality test, since the filtered portion of the state is *dependent* on the measurement basis [VST⁺10b]. We will then address the nonlocality task in the following section.

5.3.4 Pre-selection for entanglement and non-locality tests

Here we investigate a pre-selection scheme based on a conditional operation driven by the measurement of a portion of the multiphoton state in two different polarization bases [see Fig. 5.17]. A small portion of the generated multiphoton state is reflected by an unbalanced beam-splitter of transmittivity T = 0.9 and subsequently split by a 50/50 beamsplitter in two equal parts. One of the two parts is measured in an equatorial $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}$ basis by two photomultipliers, and the photocurrents $\{I_{\beta}, I_{\beta_{\perp}}\}$ are analyzed by an OF device [Fig. 5.10]. The other part undergoes the same measurement process in a different equatorial basis $\{\vec{\pi}_{\beta'}, \vec{\pi}_{\beta'_{\perp}}\}$.



Figure 5.17: Pre-selection for entanglement and non-locality tests: a double basis measurement is performed on the small reflected portion of the macro qubit.

When the threshold condition $|I_{\pi} - I_{\pi_{\perp}}| > k$ [Fig. 5.10] is realized in both branches, measured respectively in the polarization basis $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}\$ and $\{\vec{\pi}_{\beta'}, \vec{\pi}_{\beta'_{\perp}}\}\$, a TTL electronic signal is sent to conditionally activate the optical shutter placed in the optical path of the remaining part of the multiphoton state. Then, the field is analyzed at the measurement stage with the dichotomic strategy discussed in the previous paragraphs. For this pre-selection method, the relevant parameter is the angle ϕ between the two bases $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}$ and $\{\vec{\pi}_{\beta'}, \vec{\pi}_{\beta'_{\perp}}\}$ in which the small portion of the beam is analyzed. The angle ϕ is defined according to the relations between the two polarization bases:

$$\vec{\pi}_{\beta'} = e^{i\frac{\phi}{2}} \left[\cos\left(\frac{\phi}{2}\right) \vec{\pi}_{\beta} - i \sin\left(\frac{\phi}{2}\right) \vec{\pi}_{\beta\perp} \right], \qquad (5.24)$$

$$\vec{\pi}_{\beta'_{\perp}} = e^{\iota \frac{\phi}{2}} \left[-\iota \sin\left(\frac{\phi}{2}\right) \vec{\pi}_{\beta} + \cos\left(\frac{\phi}{2}\right) \vec{\pi}_{\beta_{\perp}} \right].$$
(5.25)

We analyze how the visibility changes as a function of the angle ϕ between the two bases of the pre-selection branch once the equatorial polarization of the injected state $\vec{\pi}_{\alpha}$ is optimized (see below). In Fig. 5.18 we show the numerical results obtained by calculating the visibility according to the standard definition:

$$V(k) = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}},$$
(5.26)

where:

$$I_{\max} = \sum_{m > n} P_{\overline{\alpha}} \left[m, n \left| (|I_{\beta} - I_{\beta_{\perp}}| > k) \cap (|I_{\beta'} - I_{\beta'_{\perp}}| > k) \right],$$
(5.27)

$$I_{\min} = \sum_{m < n} P_{\overline{\alpha}} \left[m, n \left| (|I_{\beta} - I_{\beta_{\perp}}| > k) \cap (|I_{\beta'} - I_{\beta'_{\perp}}| > k) \right].$$
(5.28)

Here $P_{\overline{\alpha}}\left[m,n\left|\left(|I_{\beta}-I_{\beta_{\perp}}|>k\right)\cap\left(|I_{\beta'}-I_{\beta'_{\perp}}|>k\right)\right]$ is the photon-number distribution of the state $|\Phi^{\overline{\alpha}}\rangle$ after the pre-selection stage. The value of $\overline{\alpha}$ is chosen in order to maximize the contribution of the $\sum_{m>n}$ term and minimize the contribution of the $\sum_{m<n}$ term. Eq. (5.26) then coincides with the usual definition of visibility. We note that the visibility is higher for smaller angles ϕ , since in that case a strong projection of the state is performed in two close bases. This condition is equivalent to the scheme of Fig. 5.10, where the OF measurement performed in one basis allows to obtain a better discrimination of the detected state only in the polarization basis of the pre-selection measurement [Fig. 5.13 (a)-(b)]. When ϕ is high, a lower visibility can be achieved since the projection of the pre-selection in one basis on the visibility is in contrast with the decreasing effect of the pre-selection in the other basis.

We conclude this section by discussing the feasibility of a nonlocality test by exploiting the proposed pre-selection method. We consider the case of a CHSH inequality [CHSH69] [see Sec. 1.5.2]. As said, the output field is measured by a pure dichotomic detection apparatus, possessing only two measurement outcomes (± 1). All local hidden variable models must satisfy the following inequality:

$$S_{CHSH} = E^{\rho}(a,b) + E^{\rho}(a,b') + E^{\rho}(a',b) - E^{\rho}(a',b') \le 2,$$
(5.29)



Figure 5.18: (a) Trend of the visibility for the double-filtering technique as a function of the angle ϕ between the two polarization bases $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}\)$ and $\{\vec{\pi}_{\beta'}, \vec{\pi}_{\beta'_{\perp}}\}\)$ of the preselection measurement. Square black points correspond to k = 3, circular red points to k = 5 and triangular green points to k = 7. (b) Filtering probability of the scheme as a function of the threshold k at the pre-selection measurement stage. All graphs correspond to g = 1.2.

where $E^{\rho}(x, y) = p_{++}(x, y) + p_{--}(x, y) - p_{+-}(x, y) - p_{-+}(x, y)$ are the correlations between the different outcomes measured in the x and y polarization respectively for systems A and B. Hence, in order to violate the CHSH inequality correlations must be present in different polarization bases. We consider the case in which the angle ϕ between the two bases $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}\$ and $\{\vec{\pi}_{\beta'}, \vec{\pi}_{\beta'_{\perp}}\}\$ is set at $\phi = \pi/4$. This choice is motivated by the following considerations. On one side, low values of ϕ would lead to a micro-macro state possessing strong correlations only in one polarization basis, thus not allowing to violate a Bell's inequality. On the other side, high values of ϕ does not allow to obtain the necessary enhancement in the correlations of the micro-macro system to violate a Bell's inequality. The obtained fringe patterns for the chosen case are reported in Fig. 5.19 and corresponds to the following conditions. The (+1) outcome of the dichotomic measurement is recorded as a function of the polarization $\vec{\pi}_{\alpha}$ of the injected single photon state. We then analyzed three different choices for the threshold k at the pre-selection stage. When the threshold k is set to 0, the fringe pattern corresponding to the two basis $\beta = 0$ and $\beta = \frac{\pi}{4}$ are mutually shifted of an angle $\frac{\pi}{4}$, since no filtering and no pre-selection is performed on the state. When the threshold k is increased, the mutual shift between the fringe pattern is progressively reduced and cancelled, since a strong filtering of the state is performed. In particular, the maximum of both the fringe pattern in the $\beta = 0$ and $\beta = \frac{\pi}{4}$ bases is obtained for the $|\Phi^{\overline{\alpha}}\rangle$ state with $\overline{\alpha} = \frac{\pi}{8}$. This means that this pre-selection strategy for sufficiently high value of k enhances the correlations in the micro-macro system in a specific polarization basis and suppresses the correlations in the other bases. For this reason, the proposed strategy does not allow to observe the violation of a Bell's inequal-



Figure 5.19: Fringe pattern as a function of the angle α of the polarization basis at the single-photon site. The angle ϕ between the two bases of the pre-selection stage is set at $\phi = \pi/4$, while g = 1.2. (a) Threshold k = 0. (b) Threshold k = 3. (c) Threshold k = 5. Square black points: fringe patterns obtained by recording the (+1) outcome at the measurement stage, where the measurement basis $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}$ is set at $\beta = 0$. Circle red points: fringe patterns obtained by recording the (+1) outcome at the measurement stage, where the measurement basis $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}$ is set at $\beta = 0$. Circle red points: fringe patterns obtained by recording the (+1) outcome at the measurement stage, where the measurement basis $\{\vec{\pi}_{\beta}, \vec{\pi}_{\beta_{\perp}}\}$ is set at $\beta = \frac{\pi}{4}$.

ity in the micro-macro system here analyzed. The enhanced value of the visibility could nevertheless be employed in quantum lithography and quantum metrology schemes, in which high visibility correlations pattern and high photon number regimes are required. Recently it has indeed been shown how the amplification process of a single photon probe can beat the detrimental effect of losses which happen in the transmission and detection stages (see Ref. [VSSD10] and Chaps. 9). Such a scheme for non invasive quantum metrology could benefits from the presented filtering procedures in order to improve the visibility value of the interference fringe pattern.

5.4 Conclusion and perspectives

In this chapter we analyzed several classes of entanglement criteria for bipartite systems of a large number of particles. As experimental benchmark, we considered the bipartite state obtained by amplification of a single photon belonging to an entangled singlet state. We discussed different entanglement criteria which do not require any supplementary assumption on the source, and applied these approaches to the micro-macro system based on optical parametric amplification. We first considered an approach based on deliberate attenuation of the multiphoton field to the single-photon regime, already introduced in Ref. [EKD⁺04], and we applied this approach to the investigated system. This analysis allowed us to show that a fraction ε of the original entanglement of the entangled photon pair exists even in presence of losses, where ε is proportional to the amount of lost particles. Then, we analyzed in details the conclusions that can be drawn on a recent experimental entanglement test performed on this system and reported in Ref. [DSV08]. The adopted entanglement criterion allowed to infer the presence of entanglement after the amplification process before losses in the detection apparatus. The validity of the test however requires a specific assumption on the system that generates the micro-macro pair. An a-priori knowledge of the source is necessary in order to exclude a class of separable states that can reproduce the obtained experimental results. One of the reasons for the necessity of this assumption is given by the exploited detection strategy, which presents the feature of a POVM with an inconclusive outcome which depends on the measurement basis. We then considered a generalization of the test performed in Ref. [DSV08], by proposing an entanglement inequality which can be adopted with any dichotomic measurement operators without supplementary assumptions on the optical source.

As a second step, we analyzed theoretically in details several schemes for the realization of conditioned measurement-induced operations. All these strategies are aimed at the manipulation and distillation of the macro-states for their applications in different contexts, such as the realization of entanglement and nonlocality tests or quantum sensing. We identified different strategies able to minimize the effects of the noise due to the vacuum injection into the amplifier, and to increase the distinguishability among the detected multiphoton states.

Several open points remain to be investigated. The entanglement tests discussed in this chapter relies on dichotomic measurements. Such measurements present a low resolution, and hence an open question still remains whether detection methods with higher resolution should be adopted in this context. Indeed, high resolution measurements have to be employed in order to detect all the particles present in the multiphoton state. This will be addressed in the next chapter by considering an hybrid detection method involving both discrete and continuous variables measurements. At the same time, the measurement-induced operations analyzed in this paper are all based on dichotomic detection schemes. Other approaches, such as the ones based on continuous variables measurements or on the processes of coherent photon-addition and photon-subtraction, can lead to a different manipulation of the QIOPA multiphoton states. Systems with different properties from the one analyzed in this paper could be obtained with these methods.

Chapter 6

Hybrid entanglement criteria in bipartite microscopic-macroscopic systems with discrete- and continuousvariables methods

The expertimental observation of entanglement between a microscopic and a mesoscopic system is still an open challenge. The main challenges rely on the difficulty of isolating any physical system from the environment, and on the absence of suitable criteria for this hybrid scenario. In the previous chapter we analyzed an optical source of a bipartite state composed by a single photon on one mode and a multimode field on the other mode. We analyzed in which regime of the systems' parameters entanglement could be detected by exploiting dichotomic measurements, showing the necessity of performing a high resolution measurement. In this chapter we propose to exploit a hybrid detection system as a possibile approach. Such approach is based on a hybrid measurement apparatus, employing discrete-variables on the microscopic part and continuous-variables on the macroscopic part. In this way, the advantages of both techniques is combined on the same platform. The obtained results, reported in Ref. [SVP+11], can open the way to further investigation in the field of micro-macro entanglement.

6.1 Entanglemement in hybrid microscopic-macroscopic systems

An open challenge for fundamental quantum physics is to affirm the quantum nature of a system that puts together a microscopic part and a mesoscopic one. This hybrid scenario can emerge in completely different experimental platforms ranging from individual spin systems interacting with multi-mode cavity fields (such as transmon qubits in coplanar transmission-line resonators) [WSB⁺04, ADW⁺06], to ionic impurities embedded in

ultra-cold atomic samples, such as the systems considered in some recent experiments reported in [ZPSK10, SHD10]. Another possible physical approach exploits a massive tiny mirror optomechanically interacting with a single photon within a Michelson interferometer [GBP⁺06, ACB⁺06, KB06, DMSVC10, MSPB03]. This endeavour could contribute to challenge the observability of quantum features at the macroscopic level which is one of the most fascinating open problems in quantum physics. The difficulties inherent in such a quest are manifold, and are related on one hand to the unavoidable interaction of the system with the surrounding environment [Zur03, KBLSG01, PGU⁺03, PAB⁺04]. On the other hand, one faces the debated problem of achieving a measurement-precision sufficient to observe quantum effects at such macro-scales [KB08, JPR09]. In this context, it has been experimentally proven that a dichotomic measurement performed upon a multiphoton entangled state is not sufficient to catch quantumness [VST⁺10b]: the accuracy of the measurement is crucial for the observation of quantum features.

Alongside with the problem of achieving the sufficient measurement resolution, one of the main open challenge for an experimental test in systems of large size is the construction of suitable criteria for the detection of entanglement in bipartite macroscopic systems [JPR09, Wod00, GKLC01, SJR07, LJJ09, LJJ10]. In Sec. 6.2 we introduce a hybrid method to experimentally demonstrate the truly quantum mechanical features of a general micro-macro system beyond any assumption on its state and without the necessity of any a priori state-knowledge [SVP⁺11]. We infer the entanglement properties by means of an hybrid approach that combines dichotomic measurements on a bidimensional system and phase-space inferences through the Wigner distribution associated with the macroscopic component of the state [Wod00]. At variance with previous proposals [BW99, Wod00], the approach presented in this chapter is tailored to fully exploit the polarization-spin degree of freedom on both the microscopic and the macroscopic subsystems. We analyze the effects of losses on a CHSH-like inequality test [CHSH69] and show that maximum violation is achieved when losses are absent, regardless of the size of the macroscopic part of the state. This is not the case under non-ideal conditions. However, we show how losses can be efficiently taken into account so as to infer entanglement of our multiphoton state. As a paradigmatic microscopic-macroscopic system, we investigate in Sec. 6.3 the state obtained from a fully microscopic entangled system through an amplification process [DSV08, SVSD10] discussed in the previous chapter.

6.2 Hybrid entanglement test for microscopic-macroscopic systems

In the proposed hybrid method, the microscopic part of the state is measured using spin-1/2 projection operators. On the other side, the macroscopic counterpart undergoes phase-space measurements based on the properties of its Wigner function [Wod00].



Figure 6.1: Hybrid entanglement test on an optical microscopic-macroscopic state generated by a "black-box". The single-photon mode \mathbf{k}_A is measured by a polarization detection apparatus, while the multiphoton mode \mathbf{k}_B undergoes polarization projection and the measurement of the displaced parity operators. (a) Direct measurement of the $\hat{\Pi}(\alpha)$ displaced parity operators. (b) Indirect measurement of the average value $\langle \hat{\Pi}(\alpha) \rangle$ of the displaced parity operators by exploiting a homodyne detection apparatus.

6.2.1 Hybrid entanglement test based on a CHSH Bell's inequality

Let us consider a general micro-macro state with its microscopic part embodied by a single-photon polarization state (a qubit). We take the macroscopic part, on the other hand, as encoded in the multiphoton state of a continuous-variable system. The two subsystems are supposed to be entangled by a mechanism whose details are inessential for our tasks. Polarization measurements performed over state of the single-photon mode \mathbf{k}_A are described by the Pauli spin operator:

$$\hat{\sigma}^{A}(\phi) = |\phi\rangle_{A} \langle \phi| - |\phi_{\perp}\rangle_{A} \langle \phi_{\perp}|, \qquad (6.1)$$

where ϕ is the direction identifying the polarization state in the Poincaré sphere and ϕ_{\perp} is its orthogonal direction. The CV measurements, on the other hand, are given by the

following measurement operators:

$$\hat{\Pi}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi) = \hat{\Pi}^{B}_{\chi}(\alpha_{\chi}) \otimes \hat{\mathbb{1}}^{B}_{\chi_{\perp}}, \qquad (6.2)$$

where $\hat{\Pi}_{i}^{B}(\alpha_{i}) = \hat{D}_{i}^{B}(\alpha_{i})(-1)^{\hat{n}_{i}^{B}} \hat{D}_{i}^{B\dagger}(\alpha_{i})$ is the displaced parity operator built from the displacement $\hat{D}_{i}^{B}(\alpha_{i})$ ($\alpha_{i} \in \mathbb{C}$) and the number operator \hat{n}_{i}^{B} ($i = \{\chi, \chi_{\perp}\}$ stands for the polarization state). We define the qubit-CV correlator as:

$$\mathscr{C}(\boldsymbol{\alpha}_{\boldsymbol{\chi}},\boldsymbol{\chi};\boldsymbol{\phi}) = \langle \hat{\boldsymbol{\sigma}}^{A}(\boldsymbol{\phi}) \otimes \hat{\Pi}^{B}_{\boldsymbol{\chi},\boldsymbol{\chi}_{\perp}}(\boldsymbol{\alpha}_{\boldsymbol{\chi}},\boldsymbol{\chi}) \rangle, \tag{6.3}$$

which is evaluated on a general micro-macro state $\hat{\rho}_{AB}$. Starting from this correlator, we can define the following parameter \mathscr{B} by adopting the CHSH approach:

$$\mathscr{B} = \mathscr{C}(\alpha'_{\chi}, \chi'; \phi') + \mathscr{C}(\alpha'_{\chi}, \chi'; \phi) + \mathscr{C}(\alpha_{\chi}, \chi; \phi') - \mathscr{C}(\alpha_{\chi}, \chi; \phi).$$
(6.4)

This parameter can be exploited to derive an inequality which can take the role of a nonlocality or an entanglement witness depending on the detection apparatus chosen to measure the $\hat{\Pi}^B_{\chi,\chi_\perp}(\alpha_{\chi},\chi)$ operators.

The displaced parity operators $\hat{\Pi}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi)$ adopted on the multiphoton field for the present hybrid approach can be directly measured [DDS⁺08, LCGS10] by combining the input field with a coherent state in a low reflectivity beam-splitter, and by measuring the parity of the output field: Fig. 6.1 (a). However, such technique requires photon-counting detectors with very high efficiency, a condition extremely difficult to achieve with the present technology. When the $\hat{\Pi}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi)$ operators are directly measured, no assumptions are necessary on the detection apparatus. In this case, the outcome of the $\hat{\sigma}^{A}(\phi)$ and $\hat{\Pi}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi)$ measurements can only be ±1, and the use of a local hidden variable model imposes the bound $|\mathscr{B}^{\text{LHV}}| \leq 2$ [CHSH69] on the \mathscr{B} parameter. A violation of this bound on the measured state $\hat{\rho}_{AB}$ confutes all LHV theories.

A different strategy can be adopted to measure the displaced parity operators. This strategy is based on an indirect measurement of the average value of the displaced parity operators, which can be performed by exploiting the connection between $\langle \hat{\Pi}(\alpha) \rangle$ and the Wigner function of the state: Fig. 6.1 (b). Indeed, the average value of the measurement operator on state $\hat{\rho}_i^B$ of the multiphoton mode is related to the value of its Wigner function at α_i : $W_{\Phi}^B(\alpha_i) = (2/\pi) \text{Tr}[\hat{\Pi}_i^B(\alpha_i) \rho_i^B]$. The latter can be easily reconstructed using a homodyne tomographic apparatus. This indirect approach requires some assumptions on the detection apparatus, that is, an *a-priori* characterization of the displaced parity operators is not suitable for a genuine nonlocality test, but can be still adopted to develop an entanglement inequality. In this case, the average values of the outcomes of the $\hat{\sigma}^A(\phi)$ and $\hat{\Pi}_{\chi,\chi_{\perp}}^B(\alpha_{\chi},\chi)$ measurements is limited by $\langle \hat{\sigma}^A(\phi) \rangle \leq 1$ and $\langle \hat{\Pi}_{\chi,\chi_{\perp}}^B(\alpha_{\chi},\chi) \rangle \leq 1$. Hence, by using a standard CHSH argument it can be shown that for all separable states the bound $|\mathscr{B}^{\text{sep}}| \leq 2$ holds. A violation of this bound witnesses an entangled state. The measurement settings for the single-photon mode \mathbf{k}_A [multiphoton mode \mathbf{k}_B] are given by the measurement

polarizations (ϕ, ϕ') [measured polarizations (χ, χ') and the chosen phase-space points $(\alpha_{\chi}, \alpha'_{\chi})$]. This requires a standard polarization detection system for the microscopic mode and a homodyne detection system for the multiphoton one, as shown in Fig. 6.1 (b).

6.2.2 Hybrid entanglement witness in presence of detection losses

Losses are modeled by inserting a beam-splitter of transmittivity $\eta \in [0, 1]$ in the path of the modes at hand, "tapping" the corresponding signal [JPR09]. The choice $\eta = 1$ ($\eta = 0$) corresponds to a lossless (fully-lossy) process. To this end, the measurement performed on the π_{χ} polarization of the multiphoton part is replaced by the operator [LJJ10]:

$$\hat{O}_{\chi}^{B}(\alpha_{\chi};\eta) = \begin{cases} \frac{1}{\eta} \hat{\Pi}_{\chi}^{B}(\alpha_{\chi}) + \left(1 - \frac{1}{\eta}\right) \hat{\mathbb{1}}_{\chi}^{B} & \text{if } \eta \in (0.5,1], \\ 2\hat{\Pi}_{\chi}^{B}(\alpha_{\chi}) - \hat{\mathbb{1}}_{\chi}^{B} & \text{if } \eta \in (0,0.5]. \end{cases}$$

$$(6.5)$$

where η is the detection efficiency of the apparatus. Such definition of the measurement operator is performed in order to correct the detrimental effect of losses on the properties of the detected state. Let us consider a general state $|\Phi\rangle_{\chi}^{B}$ on spatial mode \mathbf{k}_{B} and polarization $\vec{\pi}_{\chi}$ (although we illustrate our argument using pure states of mode *B*, our arguments apply equally to mixed states). After losses occur, the state evolves into a density matrix $\hat{\rho}_{\Phi\chi}^{\eta B}$. The average value of $\hat{O}_{\chi}^{B}(\alpha_{\chi};\eta)$ on such density matrix gives [LJJ10]:

$$\langle \hat{O}_{\chi}^{B}(\boldsymbol{\alpha}_{\chi};\boldsymbol{\eta}) \rangle_{\boldsymbol{\eta}} = \begin{cases} \frac{\pi}{2\eta} W_{\Phi}^{\boldsymbol{\eta}B}(\boldsymbol{\alpha}_{\chi}) + \left(1 - \frac{1}{\eta}\right) & \text{if } \frac{1}{2} < \boldsymbol{\eta} \leq 1, \\ 2W_{\Phi}^{\boldsymbol{\eta}B}(\boldsymbol{\alpha}_{\chi}) - 1 & \text{if } 0 \leq \boldsymbol{\eta} \leq \frac{1}{2}. \end{cases}$$
(6.6)

Here, $W_{\Phi}^{\eta B}(\alpha_{\chi})$ is the Wigner function of the detected state, which is related to the Wigner function of the initial state before losses $|\Phi\rangle_{\chi}^{B}$ by the gaussian convolution:

$$W_{\Phi}^{\eta B}(X_{\chi}, P_{\chi}) = \frac{2}{\pi(1-\eta)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dX'_{\chi} dP'_{\chi} W_{\Phi}^{B}(X'_{\chi}, P'_{\chi}) e^{-2\left[\frac{(X_{\chi}-\sqrt{\eta}X'_{\chi})^{2}}{1-\eta} + \frac{(P_{\chi}-\sqrt{\eta}P'_{\chi})^{2}}{1-\eta}\right]}.$$
 (6.7)

The measured Wigner function given in Eq. (6.7) corresponds to the *s*-parametrized quasiprobability distribution W_{Φ}^{B} , (α_{χ}, s) , of $|\Phi\rangle_{\chi}^{B}$ with $s = -\frac{(1-\eta)}{\eta}$ [CG69b, CG69a]. Exploiting the properties of such distributions, it is straightforward to prove that [LHL⁺09, LJJ10]:

$$|\langle \hat{O}^{B}_{\chi}(\alpha_{\chi};\eta)\rangle_{\eta}| \leq 1, \tag{6.8}$$

for all values of η . We can then define the overall measurement performed on the multiphoton state as:

$$\hat{O}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta) = \hat{O}^{B}_{\chi}(\alpha_{\chi};\eta) \otimes \hat{\mathbb{1}}^{B}_{\chi_{\perp}}.$$
(6.9)

with average values bounded by $|\langle \hat{O}^B_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta)\rangle_{\eta}| \leq 1$. Hence, by introducing the micro-macro correlator:

$$\widetilde{\mathscr{E}}_{\eta}(\alpha_{\chi},\chi,\phi) = \langle \hat{\sigma}^{A}(\phi) \otimes \hat{O}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta) \rangle_{\eta}, \qquad (6.10)$$

we define the following witness parameter:

$$\mathscr{W}_{\eta} = \tilde{\mathscr{C}}_{\eta}(\alpha'_{\chi}, \chi', \phi') + \tilde{\mathscr{C}}_{\eta}(\alpha'_{\chi}, \chi', \phi) + \tilde{\mathscr{C}}_{\eta}(\alpha_{\chi}, \chi, \phi') - \tilde{\mathscr{C}}_{\eta}(\alpha_{\chi}, \chi, \phi).$$
(6.11)

In order to define the bounds on \mathscr{W}_{η} satisfied by separable states, we consider a generic micro-macro separable state described by the density matrix $\hat{\rho}^{\text{sep}} = \sum_{i} p_{i} \hat{\rho}_{i}^{A} \otimes \hat{\rho}_{i}^{B}$. After detection losses on the multiphoton mode \mathbf{k}_{B} , such state evolves into $\hat{\rho}^{\text{sep}} = \sum_{i} p_{i} \hat{\rho}_{i}^{A} \otimes \mathscr{L}_{\eta} [\hat{\rho}_{i}^{B}]$, which gives:

$$|\mathscr{W}_{\eta}^{\text{sep}}| = \left|\sum_{i} p_{i} \left(\langle \hat{A}^{\prime} \rangle^{i} \langle \hat{B}^{\prime} \rangle_{\eta}^{i} + \langle \hat{A}^{\prime} \rangle^{i} \langle \hat{B} \rangle_{\eta}^{i} + \langle \hat{A} \rangle^{i} \langle \hat{B}^{\prime} \rangle_{\eta}^{i} - \langle \hat{A} \rangle^{i} \langle \hat{B} \rangle_{\eta}^{i} \right) \right|$$
(6.12)

where:

$$\langle \hat{A} \rangle^{i} = \operatorname{Tr} \left[\hat{\sigma}^{A}(\phi) \hat{\rho}_{i}^{A} \right],$$

$$\langle \hat{A}' \rangle^{i} = \operatorname{Tr} \left[\hat{\sigma}^{A}(\phi') \hat{\rho}_{i}^{A} \right],$$

$$\langle \hat{B} \rangle^{i}_{\eta} = \operatorname{Tr} \left[\hat{O}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta) \mathscr{L}_{\eta}[\hat{\rho}_{i}^{B}] \right],$$

$$\langle \hat{B}' \rangle^{i}_{\eta} = \operatorname{Tr} \left[\hat{O}^{B}_{\chi',\chi'_{\perp}}(\alpha'_{\chi},\chi';\eta) \mathscr{L}_{\eta}[\hat{\rho}_{i}^{B}] \right].$$

$$(6.13)$$

As all these terms satisfy $|\langle \hat{X} \rangle^i| \leq 1$ with $\hat{X} = \{\hat{A}, \hat{A}', \hat{B}, \hat{B}'\}$, we get:

$$|\mathscr{W}_{\eta}^{\mathrm{sep}}| \le 2. \tag{6.14}$$

This shows that for any separable state undergoing a lossy process on mode \mathbf{k}_B the witness parameter \mathcal{W}_{η} is bound to satisfy Eq. (6.14). Violation of this inequality witnesses entanglement in the system. Such bound can be explained by considering that separable states do not violate CHSH inequalities, and local processes such as losses cannot increase their nonlocal character. It is important to notice that, in virtue of the assumption that the macrostate of mode \mathbf{k}_B undergoes losses η before (rather than *at*) detection, this entanglement witness reveals the presence of entanglement *without* any assumption on the micro-macro source [see Fig. 6.1]. On the other hand, the lossy mechanism can be shifted to occur just before measurement, thus modeling the effects of a non-ideal detector. For $\eta = 1$, \mathcal{W}_{η} coincides with the CHSH-based parameter \mathcal{B} in Eq. (6.4).

6.3 Hybrid entanglement tests in a microscopic-macroscopic system based on parametric amplification

The system under consideration is the optical source analyzed in the previous chapter in Sec. 5.2.1 and based on the process of optical parametric amplification of a single photon belonging to an entangled photon pair.

At variance with the previous chapter, the output state is then measured by means of a polarization analysis apparatus and by a homodyne detection acting on the the $\vec{\pi}_{\chi}$ polarization mode [see Fig. 6.1]. This apparatus implements the indirect measurement of the displaced parity operators, as discussed in Sec. 6.2.

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6.3.1 Application of the hybrid CHSH-based entanglement inequality

We begin analyzing the CHSH-based inequality (6.4) in the lossless case (η =1). The correlation operator evaluated on $|\Psi^-\rangle_{AB}$ takes the form (see App. C.1):

$$\mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi) = (1 - \mathscr{Z}) \cos[2(\chi - \phi)] e^{-\mathscr{Z}}, \qquad (6.15)$$

where $\mathscr{Z}=2(e^{-2g}\overline{X}_{\chi}^{2}+e^{2g}\overline{P}_{\chi}^{2})$ is a function of the rotated variables set:

$$\overline{X}_{\chi} = X_{\chi} \cos(\chi/2) - P_{\chi} \sin(\chi/2), \qquad (6.16)$$

$$\bar{P}_{\chi} = X_{\chi} \sin(\chi/2) + P_{\chi} \cos(\chi/2).$$
 (6.17)

Here, (X_{χ}, P_{χ}) are the field quadratures and $\alpha_{\chi} = X_{\chi} + \iota P_{\chi}$. The correlator in Eq. (6.15) is maximized at the origin of the phase space, where $\mathscr{C}(0, 0, \chi; \phi) = \cos[2(\chi - \phi)]$, which is independent of the gain of the amplifier g and the number of generated photons $\overline{n} = \sinh^2 g$. It has the same form as for a Bell-CHSH test performed on a polarization photonpair where spin-1/2 operators are measured. The CHSH-based parameter \mathscr{B} is then maximized by choosing the measurement settings for $(\phi, \phi, \chi, \chi')$ corresponding to such case, which ensures the maximum degree of violation of the inequality $\mathscr{B}=2\sqrt{2}$.



Figure 6.2: CHSH-based parameter \mathscr{B}_{η} as a function of the number of lost photons $(1 - \eta)\langle n \rangle$ for different values of the gain *g*. We show the local realistic boundary $\mathscr{B}^{\text{sep}}=2$.

We are now in a position to address the possibility to observe micro-macro entanglement under realistic experimental conditions. We thus analyze the effects of detection efficiency of the homodyne apparatus, used for the measurement of the generalized parity operator on the multiphoton mode \mathbf{k}_B , and its effect onto the qubit-CV correlator (see App. C.2 for a discussion on the other sources of experimental imperfections). By restricting our attention to the origin of the phase space, where maximum non-classical effects are achieved, we get $\mathscr{C}_{\eta}(0,0,\chi;\phi) = \cos[2(\chi-\phi)]\mathscr{L}(g,\eta)$, where:

$$\mathscr{L}(g,\eta) = \frac{\eta [1 + 2\overline{n}(1-\eta)]}{(1 + 4\eta(1-\eta)\overline{n})^{3/2}}$$
(6.18)

is a loss-function for the test. Hence, the maximum amount of violation is directly deter-

mined by the loss-function as $\mathscr{B}_{\eta} = \mathscr{BL}(g, \eta)$. In Fig. 6.2 we show the value of \mathscr{B}_{η} as a function of the average number of lost photons $(1-\eta)\langle n \rangle$, where $\langle n \rangle = 3\overline{n} + 1$ is the mean number of the generated photons after the amplification process. The CHSH-based inequality of Eq. (6.4) is satisfied when only a moderate number of photons is lost. A lower bound $\eta_{\text{lim}} = 1/\sqrt{2}$ for the detection efficiency can be found below which a violation is not observed anymore. On the other hand, at set values of η there is a minimum gain $g_{\rm lim}(\eta)$ above which the presented test cannot detect micro-macro entangled correlations. Such threshold value decreases with the reduction of the efficiency η . The behavior of \mathscr{B}_{η} in the (η, g) -plane is shown by the contour plot in Fig. 6.3 (a). In order to relate the violation of the CHSH-based inequality to intrinsically non-classical features enforced at the level of the macro-part of the state, Fig. 6.3 (b) reports the negativity of the Wigner function of an amplified single-photon state versus η and g [SVD⁺09]. We observe that the transition of \mathscr{B}_{η} to the region below the classical limit is directly linked to the decrease in the negativity of the Wigner function itself. Indeed the value of the micro-macro correlator \mathscr{C}_{η} is determined by the excursion of the Wigner function in $X_{\chi} = P_{\chi} = 0$, as a function of the polarization of the injected photon.

6.3.2 Application of the hybrid entanglement witness in presence of detection losses

We complement the analysis of the investigated micro-macro system by discussing the use of the entanglement witness described in Sec. 6.2.2. The evaluation of the correlation



Figure 6.3: (a) Contour plot of the shifted loss function $\mathscr{L}(g,\eta) - 2^{-1/2}$ as a function of the gain *g* and the detection efficiency η . (b) Contour plot of the negativity of the Wigner function of an amplified single-photon state [SVD⁺09] against *g* and η , evaluated in the origin of the phase space. In both panels the solid line divides the region of entanglement $(|\mathscr{B}_{\eta}|>2, \text{ above the line})$ from the one in which entanglement cannot be inferred $(|\mathscr{B}_{\eta}|\leq 2, \text{ below the line}).$





Figure 6.4: (a) W_{η} against the detection efficiency η and the nonlinear gain g. (b) Contour plot of the effective loss function $h(\eta) \mathcal{L}(\eta, g)$. Entanglement can be revealed in the region above the black line. (c) Summary of the results obtained from our tests. We identify three regions in the (η, g) space, depending on whether entanglement can be demonstrated with our techniques.

operator over state $|\Psi^-\rangle_{AB}$ after losses leads to:

$$\tilde{\mathscr{C}}_{\eta}(\alpha_{\chi},\chi,\phi) = h(\eta) \mathscr{C}_{\eta}(\alpha_{\chi},\chi;\phi), \tag{6.19}$$

where $h(\eta)=1/\eta$ $(h(\eta)=2)$ for $1/2 < \eta \le 1$ $(0 \le \eta \le 1/2)$. More details can be found in App. C.3. Therefore, the entanglement witness can be directly obtained from the CHSHbased parameter as $\mathcal{W}_{\eta}=h(\eta)\mathcal{B}_{\eta}$. In Fig. 6.4 (a) we report the dependence of \mathcal{W}_{η} as a function of η and g: for single-photon states (i.e. at g = 0), the correction of losses introduced by the factor $h(\eta)$ allows one to observe micro-micro entanglement up to $\eta \sim 0.35$. As the number of photons in the macro-state increases, the damping in the negativity of the Wigner function induced by losses scales more rapidly than η and the $h(\eta)$ -correcting term becomes less effective. Fig. 6.4 (b) shows the behavior of the effective overall lossfunction $h(\eta)\mathcal{L}(\eta,g)$, highlighting the thresholds in g and η above which entanglement is observed. We note that the non-monotonic behaviour obtained for the inefficiency parameter at $\eta = 0.5$ is a property of the witness itself. However, being Eq. (6.5) a witness for entanglement, no special meaning can be attached to the lack of violation of the separability condition $|\mathscr{W}_{\eta}^{sep}| \leq 2$.

6.4 Conclusion and perspectives

In this chapter we have proposed an experimentally oriented approach to detect entanglement in a micro-macro entangled state involving a single-photon and a multiphoton bipartite system. We have used a hybrid CHSH-based inequality and an entanglement witness whose use against such a class of states is effective. Furthermore, the CHSHbased inequality can be adopted as a genuine nonlocality test when a direct measurement of the displaced parity operators is performed on the multiphoton field.

As experimental benchmark, we considered the bipartite state obtained by amplification of a single photon belonging to an entangled singlet state. The approach adopted in this chapter does not require any supplementary assumption on the source. We showed that with this approach, the entanglement in absence of detection losses is present in any photon-number regime. In presence of losses, the entanglement can be efficiently demonstrated in the few photon-number regime by adopting an hybrid entanglement witness which includes an appropriate correction for detection losses

While our study spurs further interest in the identification of suitable tests in the high-loss and large-photon-number region, it paves the way to an experimentally feasible demonstration of entanglement properties in an interesting class of states lying at the very border between quantum and classical domains. As a further perspective, the system based on parametric amplification can lead to the investigation of entanglement in a bipartite multiphoton-multiphoton system [De 11, SHB⁺09]. This scenario will be discussed in the next chapter.

Chapter 7

Detection of nonlocality in multiphoton-multiphoton systems and the role of measurement resolution

Since the initial paper by Einstein, Podolsky and Rosen [EPR35] and the formulation of the Bell's theorem [Bel64], the violation of local realistic theories by quantum mechanics has been analyzed both theoretically and experimentally. The experimental violation of a Bell's inequality when the size of the system progressively becomes larger is still an open challenge. This is due to the necessity of increasing the measurement resolution when the size of the system increases, and to the detrimental effect of decoherence. In the previous chapters we showed that the process of parametric amplification provides a platform to investigate the presence of quantum properties in a micro-macro system which can be tuned to produce states with larger number of particles. Here we discuss an optical source to generate a bipartite system of two multiphoton fields. This source is based on the process of parametric down-conversion in the spontaneous emission regime. We investigate the possibility of observing nonlocality with dichotomic measurements, showing both theoretically and experimentally that such detection strategy does not possess the necessary resolution to witness quantum properties. Then, we consider the adoption of a high resolution continuous-variables detection scheme, and we show that detection losses become more detrimental as the number of photon is increased. The obtained results, reported in Refs. [VST⁺10b, VTCS⁺11], highlight the need of performing a high efficiency and high resolution measurement to observe the violation of a Bell's inequality in multiphoton systems.

7.1 Quantum nonlocality in multiphoton systems

Since the discussion about nonlocality started by Einstein, Podolsky and Rosen (EPR) in 1935 [EPR35], the possibility of observing quantum phenomena at a macroscopic level seems to be in conflict with the classical description of our everyday world. The main

problem for such observation arises from the difficulty of sufficiently isolating a quantum system from the environment [Zur03]. Starting from an earlier idea discussed by Peres [Per93] and others in Ref. [KB07] it has been discussed that the emergence of macro-scopic realism and classical physics in systems of increasing size arises due to the lack of measurement resolution. They focused on the limits of the quantum effects observability in macroscopic objects, showing that, for large systems, macrorealism arises under coarse-grained measurements. Therefore the measurement problem seems to be a key ingredient in the attempt of understanding the limits of the quantum behavior of physical systems and the quantum-to-classical transition question.

In this context, the possibility of obtaining macroscopic quantum systems in laboratory has raised the problem of investigating entanglement and nonlocality in systems in which single particles cannot be addressed singularly. As shown in Ref. [CPHZ02], the demonstration of nonlocality in a multiphoton state produced by a nondegenerate optical parametric amplifier would require the experimental application of parity operators. On the other hand, the estimation of a coarse grained quantity, through collective measurements as the ones proposed in Ref. $[PDS^+06]$, would miss the underlying quantum structure of the generated state, introducing elements of local realism even in presence of strong entanglement and in absence of decoherence. In Ref. [RMD02] Reid et al. analyzed the possibility of obtaining the violation of Bell's inequality by performing dichotomic measurements on multiparticle quantum states. More specifically, in analogy with the spin formalism, they proposed to compare the number of photons polarized "up" with the number of photons polarized "down" at the exit of the amplifier. The result of this comparison could be either (+1) or (-1). In such a way Reid et al. revealed a small violation of the multiparticle Bell's inequality even in presence of losses and quantum inefficiency of detectors. It is worth nothing that this violation presents a fast decreasing behavior as a function of the generated photons number.

As a possible approach to overcome such limitation continuous-variables measurements, exploiting homodyne detection, have been proposed. However, the generalization of Bell's inequalities to quantum systems with continuous-variables has represented for long time a challenging issue. According to Bell, the positivity of the Wigner function would have allowed to construct a local hidden-variable model simulating correlations for any observable defined as functions of phase-space points [Bel87]. However Banaszek and Wodkiewicz showed that, in spite of the positivity of the Wigner function, the EPR state exhibits a high degree of nonlocality [BW98]. This study has later been extended by Chen et al. [CPHZ02], who showed that a maximal violation of Bell's inequality can be obtained by measuring pseudo-spin operators over the state produced by the non-degenerate optical parametric amplifier (NOPA), when the nonlinear gain of the amplifier grows and the NOPA state tends to the original EPR one. The relation between the positivity of the Wigner function and the possibility of observing a violation of Bell's inequality has then been clarified by Rezven et al. [RMMJ05]. They showed that only "nondispersive" dinamical variables, i.e. measurements whose Wigner representation takes as possible values only the eigenvalues of the corresponding operator, can be considered good candidates for a local hidden-variables theory. The violation of a Bell's inequality is then not only dependent on the system's Wigner function but also on the nature of the dynamical variables measured upon it. Moreover, there is another motivation to perform continuous-variables measurement, since the high detection efficiency of homodyne detection could lead to closing the detection loophole [GPFcvC⁺04]. Within this context, hybrid measurements involving both discrete- and continuous- variables observables in order to demonstrate Bell's test violations have been recently addressed in Refs. [CBS⁺10, SVP⁺11].

In the present chapter, we investigate the multiphoton-multiphoton states generated by high gain spontaneous parametric down-conversion (Sec. 7.2). In Sec. 7.3 the possibility of observing quantum correlations in such a multiphoton systems through dichotomic measurement will be analyzed, by addressing two different schemes [VST+10b]. More specifically, we will investigate the persistence of nonlocality in an increasing size n/2spin singlet state by studying the change in the correlations as n increases, both in the ideal case and in presence of losses. At last, experimental observation of multiphoton correlations will be reported in Sec. 7.4. The results obtained enlighten that dichotomic fuzzy measurements lack of the necessary resolution to characterize such states and show the extreme difficulty to observe quantum nonlocality in this experimental configuration. We then propose in Sec. 7.5 a further step towards the understanding of the nonlocality problem in continuous-variables systems, by addressing the possibility of performing high efficiency homodyne measurements in order to perform a Bell's test [VTCS⁺11]. The exploited multiphoton state source can be considered a paradigmatic system, since it can be related to the continuous-variables EPR state with an additional degree of freedom: the polarization. We study the violation of the Bell's test in the form proposed by Banaszek and Wodkiewicz [BW98] based on the measurement of the Wigner function at specific points of the phase space. By correlating the value of the Wigner function at different points of the phase space, we study the possibility of violating the Bell's inequality either in absence or in presence of losses, and we relate the results with the value of the nonlinear gain of the amplifier, that is, the size of the measured state.

7.2 Multiphoton quantum states generated by high-gain spontaneous parametric down-conversion

The paradigmatic system over which we perform our analysis is the one obtained by SPDC in a type-II OPA [KMW⁺95, EKD⁺04] discussed in Sec. 2.1. The low gain regime of such a system has been experimentally realized and deeply studied in the past few years, leading to the violation of Bell's inequalities [KMW⁺95] and to the observation of polarization-entanglement up to 12 photons per branch [EKD⁺04, CDP⁺06]. However, no theoretical and experimental demonstration of entanglement and nonlocality has been

given in the multiphoton regime.

The interaction Hamiltonian of the multiphoton system pf Eq. (2.3) presents full rotational invariance, and can be exploited to generate multiphoton states of the form (see Sec. 2.1.3):

$$|\Psi^{-}\rangle = \frac{1}{C^2} \sum_{n=0}^{\infty} \Gamma^n \sqrt{n+1} ||\psi_n^{-}\rangle, \qquad (7.1)$$

with:

$$|\psi_{n}^{-}\rangle = \sum_{m=0}^{n} \frac{(-1)^{m}}{\sqrt{n+1}} |(n-m)_{\pi}, m_{\pi_{\perp}}\rangle_{1} |m_{\pi}, (n-m)_{\pi_{\perp}}\rangle_{2},$$
(7.2)

where $\Gamma = \tanh g$ and $C = \cosh g$; $g = \chi t$ is the nonlinear gain (NL) of the process. Hence, the output state can be written as the weighted coherent superposition of singlet spin- $\frac{n}{2}$ states $|\psi_n^-\rangle$. The mean number of generated photons per polarization per mode is related to the nonlinear gain g by the exponential relation $\overline{n} = \sinh^2 g$, the overall number of photons per pulse is then given by $\langle n \rangle = 4\overline{n}$; a maximum value of $g_{exp} = 3.5$, corresponding to $\langle n \rangle = 1080$ per pulse, has been achieved [VST⁺10b].

7.2.1 Wigner function of the generated multiphoton states

The Wigner function of the multiphoton state can be obtained following the method described in Chap. 4 (see also [SVD⁺09]). We consider the presence of a lossy channel with transmittivity η , simulated by the presence of a beam splitter. We assume that the channel efficiency η is equal for all modes. The Wigner function of the state $\hat{p}_n^- = \mathscr{L}_{\eta}[|\psi_n^-\rangle\langle\psi_n^-|]$, where \mathscr{L}_{η} is the map describing the channel, can then be written as:

$$W(\alpha_{H}, \alpha_{V}, \beta_{H}, \beta_{V}, g, \eta) = \mathscr{N} \exp\left\{-\overline{\varepsilon} \sum_{\pi=H, V} \left[|\alpha_{\pi}|^{2} + |\beta_{\pi}|^{2} \right] \right\} \times \\ \times \exp\left\{-\overline{\mu} \left[2\operatorname{Re}(\alpha_{V}\beta_{H}) - 2\operatorname{Re}(\alpha_{H}\beta_{V}) \right] \right\}.$$
(7.3)

Such expression can be recovered from the results of Sec. 4.2.2 replacing the real quadrature variables $\{X_{\mathbf{k}_i,\pi}, P_{\mathbf{k}_i,\pi}\}$, where $\mathbf{k}_i = \{\mathbf{k}_1, \mathbf{k}_2\}$, and $\pi = \{\vec{\pi}_H, \vec{\pi}_V\}$, with the corresponding complex variables $\alpha_{\pi} = X_{\mathbf{k}_1,\pi} + \imath P_{\mathbf{k}_1,\pi}$ and $\beta_{\pi} = X_{\mathbf{k}_2,\pi} + \imath P_{\mathbf{k}_2,\pi}$. Here, the $\{\alpha_{\pi}\}_{\pi=H,V}$ quadratures correspond to the spatial mode \mathbf{k}_1 , the $\{\beta_{\pi}\}_{\pi=H,V}$ quadratures correspond to the spatial mode \mathbf{k}_2 , and:

$$\overline{\varepsilon} = \frac{\varepsilon(1+2S^2) - \mu 2CS}{\varepsilon^2 - \mu^2}, \qquad (7.4)$$

$$\overline{\mu} = \frac{\varepsilon 2CS - \mu(1 + 2S^2)}{\varepsilon^2 - \mu^2}, \qquad (7.5)$$

$$\mathcal{N} = \frac{1}{\pi^4} \left(\frac{1}{\varepsilon^2 - \mu^2} \right)^2, \tag{7.6}$$

where:

$$\varepsilon = \frac{1}{2} \left[1 + 2(1 - \eta)S^2 \right],$$
 (7.7)

$$\mu = (1-\eta)CS. \tag{7.8}$$

The lossless case can then be recovered by setting $\eta = 1$. We observe that the fourmodes Wigner function of the multiphoton state produced by the OPA is positive as the one produced by the NOPA discussed in [BW98]. We will show that, in spite of such a positivity, it is possible to demonstrate the violation of a Bell's inequality by performing continuous-variables measurement upon the state.

7.3 Dichotomic measurements on multiphoton states

Several possible extensions of dichotomic measurements in the macroscopic regime have been discussed [RMD02, BBB⁺08], showing that CHSH-type inequalities can be exploited in order to perform nonlocality tests also in many-particle collective states. Here we analyze the possibility of applying the OF and the TD method, introduced in Sec. 5.2.3, to the detection of quantum correlations in singlet spin-n/2 states.



Figure 7.1: (a) Dichotomic detection apparatus for the multiphoton state under investigation. (b) O-Filtering technique representation in the bidimensional Fock-Space $\{n_{\pi}, m_{\pi_{\perp}}\}$. The (+1) and (-1) regions correspond to a difference in the detected photon numbers $|n_{\pi} - m_{\pi_{\perp}}| > k$. The (0) region corresponds to an inconclusive measurement. (c) Dichotomic threshold measurement representation in the bidimensional Fock-Space $\{n_{\pi}, m_{\pi_{\perp}}\}$. Only those pulses containing a sufficiently high photon number can be detected due to the threshold response of the apparatus. Then, a dichotomic assignment is performed on the measurement outcomes.

7.3.1 Bell's test based on dichotomic measurements with inconclusive outcomes

Let us begin by briefly summarizing the content of Bell inequalities for a set of dichotomic observables, by generalizing further the results already obtained by Reid *et al.* [RMD02]. Consider a quantum state $\hat{\rho}$ defined in the Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Define \hat{O}_a^i the positive operator acting on subspace \mathscr{H}_1 , and the probability of finding the value *i* after the measurement *a* given by Tr $[\hat{\rho}(\hat{O}_a^i \otimes \hat{I})]$. The same relation holds for the positive operator \hat{O}_b^j acting on subspace \mathscr{H}_2 . The existence of a LHV model would imply that the expectation values of the observables *a* and *b* are predetermined by the value of the parameter $\lambda: \{D_a(\lambda), D_{a'}(\lambda), D_b(\lambda), D_{b'}(\lambda)\}$, hence the product $a \cdot b$ is equal to $D_a(\lambda)D_b(\lambda)$. For a fixed value of λ the variables D_n with $n = \{a, b, a', b'\}$ take the values -1, 1 and satisfy the CHSH inequality:

$$D_{a}(\lambda)D_{b}(\lambda) + D_{a}(\lambda)D_{b'}(\lambda) + D_{a'}(\lambda)D_{b}(\lambda) - D_{a'}(\lambda)D_{b'}(\lambda) \leq 2.$$
(7.9)

The same inequality holds by integrating this equation on the space of the hidden variable (λ) :

$$\int_{\Omega} d\mathbb{P}(\lambda) D_{a}(\lambda) D_{b}(\lambda) + \int_{\Omega} d\mathbb{P}(\lambda) D_{a}(\lambda) D_{b'}(\lambda) + \int_{\Omega} d\mathbb{P}(\lambda) D_{a'}(\lambda) D_{b}(\lambda) - \int_{\Omega} d\mathbb{P}(\lambda) D_{a'}(\lambda) D_{b'}(\lambda) \le 2,$$
(7.10)

where $\mathbb{P}(\lambda)$ is the measure of the λ probability space. If there is a local hidden variables model for quantum measurement taking values [-1, +1], then the following inequality must be satisfied:

$$S_{CHSH} = E^{\rho}(a,b) + E^{\rho}(a,b') + E^{\rho}(a',b) - E^{\rho}(a',b') \le 2,$$
(7.11)

where $E^{\rho}(a,b) = \int_{\Omega} D_a(\lambda) D_b(\lambda) d\mathbb{P}(\lambda)$. The violation of (7.11) proves that a LHV variables model for the considered experiment is impossible.

We now discuss the feasibility of a Bell's inequality test when the OF and the TD detection methods are adopted. This analysis is motivated by the increase in the visibility obtained with this measurement operators with respect to the pure dichotomic case. Both strategies present the feature of having three possible outcomes $\{-1,1,0\}$, at variance with a genuine dichotomic measurement. In order to clarify the validity of a Bell test in presence of such kind of POVM's, let us consider the case in which at the A site a standard dichotomic measurement is performed, while at the B site a POVM measurement is carried out.

Consider the outcomes for which the Bob's results are different from 0. In this case the expectation value of the product of a and b is conditioned by the event: "outcome b different from zero". In a LHV model these conditional expectations are represented by:

$$E^{\rho}(a \cdot b) = \int_{\Omega'} D_a(\lambda) D_b(\lambda) d\mathbb{P}'(\lambda), \qquad (7.12)$$

where Ω' is the hidden variable probability sub-space for which, for any $D_{b'}(\lambda)$, is $D_b(\lambda) \neq 0$ and $d\mathbb{P}' = d\mathbb{P} / \int_{\Omega'} d\mathbb{P}$. Similarly:

$$E^{\rho}(a \cdot b') = \int_{\Omega''} D_a(\lambda) D_b(\lambda) d\mathbb{P}''(\lambda), \qquad (7.13)$$

where Ω'' is the hidden variable probability sub-space for which, for any $D_b(\lambda)$, is $D_{b'}(\lambda) \neq 0$ and $d\mathbb{P}'' = d\mathbb{P} / \int_{\Omega''} d\mathbb{P}$. Since for different random variables D_b and $D_{b'}$ these conditional expectation values can in principle refer to different subensembles Ω' and Ω'' of the original ensemble Ω , in general the equation (7.10) does not hold any more and the measured quantity, based on the detection of conditional values, is:

$$\int_{\Omega'} d\mathbb{P}'(\lambda) D_a(\lambda) D_b(\lambda) + \int_{\Omega'} d\mathbb{P}'(\lambda) D_{a'}(\lambda) D_b(\lambda) + \int_{\Omega''} d\mathbb{P}''(\lambda) D_a(\lambda) D_{b'}(\lambda) - \int_{\Omega''} d\mathbb{P}''(\lambda) D_{a'}(\lambda) D_{b'}(\lambda).$$
(7.14)

Let us consider the class of LHV models such that, for a fixed value of λ , simultaneously is: $D_b(\lambda) \neq 0, D_{b'}(\lambda) \neq 0$. In this case the inequality (7.10) still holds since it becomes:

$$\int_{\Omega^*} d\mathbb{P}^*(\lambda) D_a(\lambda) D_b(\lambda) + \int_{\Omega^*} d\mathbb{P}^*(\lambda) D_a(\lambda) D_{b'}(\lambda) + \int_{\Omega^*} d\mathbb{P}^*(\lambda) D_{a'}(\lambda) D_{b'}(\lambda) - \int_{\Omega^*} d\mathbb{P}^*(\lambda) D_{a'}(\lambda) D_{b'}(\lambda) \le 2,$$
(7.15)

where Ω^* is the hidden variable probability common subspace for which $D_b(\lambda) \neq 0$ and $D_{b'}(\lambda) \neq 0$. Let us now make a *fair sampling* assumption: (a) the probability of rejecting a measurement does not depend on the hidden parameter λ and on the measurement settings, i.e. $\Omega' = \Omega'' = \Omega^*$ [AK03]. In this case the experimentally observed quantity (7.14) will follow the LHV inequality (7.15), and its violation implies the non-locality of the considered system. Finally, this LHV model can be directly generalized to the case in which both A and B sites perform a POVM measurement by including also at Alice stage an inconclusive outcome (0) and conditioning the expectation value of the product of *a* and *b* to the additional event: "outcome *a* different from 0".

Let us now conclude by discussing how these considerations apply to the OF and the TD detection methods. In both cases, the conditions $\{D_b(\lambda) \neq 0\}$ and $\{D_{b'}(\lambda) \neq 0\}$ correspond to the event that the photons survive the action of the measurement device tuned along the directions **b** or **b'**, respectively. In the O-Filter case, the previous assumption (a) is motivated by the experimentally tested state-independency of the probability of a conclusive outcome. More precisely, (i) the POVM operation is independent of the input state; (ii) the POVM probability P_{conc} is independent on the selected measurement basis. Then, on these premises any sampling or filtering made on the particles by our system can be defined a fair sampling operation. In the TD case, the previous assumption is further legitimated by a third condition: (iii) the Hilbert subspace leading to a conclusive outcome is invariant under any rotation of the polarization basis. In other words, when an

event leads to a (± 1) outcome for a specific choice of the measurement basis, it would correspond to a conclusive outcome if measured in another basis.

These considerations permits to observe that, for the TD device the fair sampling condition can be assumed, while care should be taken when this condition is assumed for the OF device. Finally, the measurement improvement attained via the OF and the TD devices by the implementation of the POVM strategy is realized at the cost of a decrease of the total quantum efficiency η_{tot} and then of a corresponding enhancement of the detection loophole.

7.3.2 Theoretical results of the Bell's test in absence of losses

We begin our analysis on the macroscopic-macroscopic state by evaluating in this section the correlations existing between the two spatial modes of the spin- $\frac{n}{2}$ singlet states [Eq.(7.2)]. We use a pure dichotomic measurement scheme, where the (+1) and (-1) outcomes are assigned whether the difference in the number of photons with two orthogonal polarization is positive or negative. Finally, if the detected difference in the number of photons is 0, one of the (±1) outcomes is randomly assigned to the event with equal probability p = 1/2. We note that this choice is a subcase of the threshold detection and O-filtering methods introduced in the previous sections, corresponding to the values h = 0and k = 0.

The scheme for evaluating the correlations is sketched in Fig. 7.1. The two spatial modes of the $|\psi_n^-\rangle$ are analyzed with the dichotomic measurement apparatus here described. The polarization basis on mode \mathbf{k}_1 is fixed on $\{\vec{\pi}_+, \vec{\pi}_-\}$, while on mode \mathbf{k}_2 the analysis basis is varied over the Bloch sphere. In particular, due to the SU(2) symmetry of the emitted states, it is sufficient to consider only the linear polarizations case, defined by the rotation: $\vec{\pi}_{\theta} = \cos \theta \vec{\pi}_+ + \sin \theta \vec{\pi}_-$. The fringe patterns are then obtained by evaluating the coincidences between the outcomes of the two detection apparatus on modes \mathbf{k}_1 and \mathbf{k}_2 . More specifically, this measurement corresponds to the evaluation of the averages:

$$D_{|\psi_{n}^{-}\rangle}^{(\pm1,\pm1)}(\theta) = \langle \psi_{n}^{-} | \left(\hat{T}_{+,-}^{(\pm1)}(0) \right)_{A} \otimes \left(\hat{T}_{\theta,\theta_{\perp}}^{(\pm1)}(0) \right)_{B} | \psi_{n}^{-} \rangle =$$

$$= \langle \psi_{n}^{-} | \left(\hat{F}_{+,-}^{(\pm1)}(0) \right)_{A} \otimes \left(\hat{F}_{\theta,\theta_{\perp}}^{(\pm1)}(0) \right)_{B} | \psi_{n}^{-} \rangle,$$
(7.16)

where the singlet spin- $\frac{n}{2}$ states of Eq. (7.2) in the analyzed polarization basis reads:

$$|\psi_n^-\rangle = \sum_{m=0}^n \sum_{p=0}^n \varepsilon_{m,p}^n(\theta) |(n-m)+,m-\rangle_A |p\theta,(n-p)\theta_\perp\rangle_B,$$
(7.17)

where:

$$\boldsymbol{\varepsilon}_{m,p}^{n}(\boldsymbol{\theta}) = \sum_{q(m,p)} (-1)^{q} \boldsymbol{\alpha}_{\boldsymbol{\theta}}^{m+p-2q} \boldsymbol{\beta}_{\boldsymbol{\theta}}^{n-m-p+2q} \left[\binom{n-m}{p-q} \binom{n-p}{m-q} \binom{m}{q} \binom{p}{q} \right]^{\frac{1}{2}}, \quad (7.18)$$

with $\alpha_{\theta} = \cos \theta$, $\beta_{\theta} = \sin \theta$. The limits of the sum over q have an explicit dependence on the values of p and m and are not reported here. Finally, by direct application of the measurement operator, the interference fringe patterns are evaluated as:

$$D_{|\psi_{n}^{-}\rangle}^{(\pm 1,\pm 1)}(\theta) = \sum_{\{m,p\}} |\varepsilon_{m,p}^{n}(\theta)|^{2}.$$
(7.19)

The extension of the sums over m and p depends on the choice of the outcome on each spatial mode according to the definitions of Eqs. (5.9-5.11) and (5.12-5.14).

In Fig. 7.2 we report the results obtained for different values of the number of photons *n*. The simplest case [see Fig. 7.2 (a)], corresponding to a spin- $\frac{1}{2}$ state, presents the



Figure 7.2: Theoretical interference fringe-patterns for singlet spin- $\frac{n}{2}$ states. The polarization basis on mode \mathbf{k}_1 is kept fixed while on mode \mathbf{k}_2 the basis is varied to obtain the fringe pattern. Figures correspond to values of (a) n = 1, (b) n = 7, (c) n = 25 and (d) n = 51. The sinusoidal pattern of the spin- $\frac{1}{2}$ progressively transforms into a linear pattern. In all figures, blue solid lines correspond to the coincidences of both the (+1,+1) and (-1,-1) outcome configurations, while red dashed lines correspond to the (+1,-1) and (-1,+1) outcomes on the two spatial mode. Note that the maximum for each fringe is 0.5, which is the probability to obtain one of the two possible anti-correlated outcomes ($\mp 1, \pm 1$).

well-known sinusoidal pattern. The sinusoidal pattern is responsible for the violation of Bell's inequalities as no classical system can present this dependence on the phase θ . For progressively higher values of *n*, as shown in Fig. 7.2 (b-d), the fringe pattern changes its dependence from the phase from a sinusoidal to a linear form. The latter represents the typical response of a pair of classicaly anti-correlated spin-**J** systems, analyzed through a dichotomic "which emisphere" measurement [Red89], i.e. the measurement of the angular momentum sign. Such detection scheme is completely analogous to the dichotomic strategy analyzed in this section.

The transition with increasing *n* towards a classical response for the singlet spin- $\frac{n}{2}$ can be explained observing that this measurement lacks of the necessary resolution [CHRB09] to observe the peculiar quantum properties of these states. Their characterization would require a more sophisticated detection apparatus able to discriminate the value *m* of the spin projection, i.e. in our case the difference in the orthogonally polarized photon number, and not only its sign. An example of such measurement [Per93] is given by the parity operator $\hat{P}_{\pi,\pi_{\perp}} = \sum_{m=0}^{n} (-1)^{m} |(n-m)\pi, m\pi_{\perp}\rangle \langle (n-m)\pi, m\pi_{\perp}|$. The correlation between the two spatial modes of the singlet spin- $\frac{n}{2}$ states evaluated with this measurement operator leads to the following expression:

$$P_{|\psi_n^-\rangle}(\theta) = \langle \psi_n^- | \left(\hat{P}_{+,-} \right)_A \otimes \left(\hat{P}_{\theta,\theta_\perp} \right)_B | \psi_n^- \rangle = (-1)^n \frac{\sin\left[(n+1)\theta \right]}{(n+1)\sin\theta}.$$
(7.20)

This correlation function violates a CHSH inequality of an amount $S_{CHSH} = 2.481 > 2$ [Per93] even in the asymptotic limit of large number of particle $(n \rightarrow \infty)$. However, such scheme based on the parity operator requires a sharp photon-number measurement in order to discriminate with unitary efficiency among contiguous values of the spin projection.

As a further analysis, let us plot (Fig. 7.3) the function $D_{|\psi^n\rangle}^{(\pm 1,\pm 1)}(\theta)/L(\theta)$, which corresponds to the ratio between the interference fringe pattern of the macro-macro configuration and a linear function of θ . The choice of the curve $L(\theta)$ as a reference is motivated by the following consideration. The evaluation of the CHSH parameter in a system characterized by the linear response leads to the maximum value in a classical framework $S_{CHSH} = 2$. Hence, this function $L(\theta)$ can be considered as the boundary between the "classical" and the "quantum" regions, since it represents the response of two classical anti-correlated systems to this test. In Fig. 7.3, we note that the ratio $D_{|\psi_{-}^{n}\rangle}^{(\pm 1,\pm 1)}(\theta)/L(\theta)$ presents a number of intersections with the axis y = 1 (unitary ratio) proportional to the value of *n*. This depends on the explicit functional form of the interference fringe pattern of Eq. (7.19). Indeed, analyzing the explicit expression [Eq. (7.18)] of the coefficients $\varepsilon_{m,p}^{n}(\theta)$, we find a sum of terms $(\cos \theta)^{m+p-2q} (\sin \theta)^{n-m-p+2q}$, where the sum of the exponents is equal to the number of photons *n*. Hence, the fringe pattern $D_{|\psi_{i}^{n}\rangle}^{(\pm 1,\pm 1)}(\theta)$ [Eq. (7.19)] can be re-organized in a Fourier series expansion containing all the harmonics up to k = 2n. With increasing *n*, the difference between $D_{|\psi_{n}^{n}\rangle}^{(\pm 1,\pm 1)}(\theta)$ and the linear function $L(\theta)$ is progressively reduced, since more harmonics are present in the Fourier expansion which asymptotically reaches the expansion of $L(\theta)$.


Figure 7.3: Plot of the interference fringe pattern $D_{|\psi_n^-\rangle}^{(\pm 1,\pm 1)}(\theta)$ for singlet spin- $\frac{n}{2}$ states divided by a linear function $L(\theta)$ corresponding to the behaviour of two distinct classical macroscopic objects.

We are now in the position to address the possibility of violating a Bell's inequality with such a dichotomic measurement apparatus. In our case, the positive operators $\hat{O}_{a(b)}^{i}$ are given by the dichotomic measurement operators $\{\hat{T}_{\pi,\pi_{\perp}}^{(\pm 1)}(0), \hat{F}_{\pi,\pi_{\perp}}^{(\pm 1)}(0)\}$ set at zero threshold h = 0 and k = 0. In order to theoretically investigate the feasibility of a CHSH test on the spin- $\frac{n}{2}$ states, we evaluated the S_{CHSH} parameter in such system. The value of the S_{CHSH} has been numerically maximized over the measurement angles $\{\theta, \theta', \varphi, \varphi'\}$ of Alice's $[\mathbf{a}(\theta) \text{ or } \mathbf{a}'(\theta')]$ and Bob's $[\mathbf{b}(\varphi) \text{ or } \mathbf{b}'(\varphi')]$ polarization bases. In Fig. 7.4 we report the results obtained for different values of the number of photons, and hence the spin, of the analyzed state. We observe the decrease in the absolute value of $S_{CHSH}^{|\Psi_n^-\rangle}$ analogously to what reported in [RMD02, BBB+08] for an equivalent Bell's inequalities test. However, the asymptotic behavior for high *n* shows that the parameter S_{CHSH} never falls below the classical limit, but the amount of violation progressively becomes smaller and any decoherence process may forbid its experimental observation.

In conclusion, the increase in the number of photons renders the dichotomic mea-



Figure 7.4: Value of the CHSH parameter $S_{CHSH}^{|\psi_n^-\rangle}$ for singlet spin- $\frac{n}{2}$ states for an optimal choice of the angle settings and the dichotomic "majority-voting" measurement. We observe the progressive decrease in the amount of violation for an increasing value of the number of photons present in the state.

surement inefficient for the complete characterization of the state, and the decreased correlations approaches asymptotically classical ones. Furthermore, any small amount of decoherence practically cancels the possibility to observe nonlocal features in the n/2singlet state.

7.3.3 **Effect of losses**

To conclude this theoretical analysis we consider the possibility to exploit different dichotomic measurement schemes to observe quantum correlations in the spin-n/2 single states $|\psi_n^-\rangle$. More specifically, we consider the threshold detection or the O-filtering methods introduced in Sec. 5.2.3. In particular, we analyze how both the visibility and the form of the fringe pattern are modified exploiting this different measurement schemes. The main idea beyond this approach concerns the possibility of beating the losses effects on the macro-macro correlations, by using a more sophisticated measurement than a pure dichotomic one. The effect of losses has been evaluated by numerically calculating the action of the map that describes the lossy process, that is, $\mathscr{L}[\hat{\rho}] = \sum_k \gamma_k \hat{a}^k \hat{\rho} \hat{a}^{\dagger k} \gamma_k^{\dagger}$ where $\gamma_k = \frac{1}{\sqrt{k!}} (1 - \eta)^{k/2} \eta^{(\hat{a}^{\dagger} \hat{a})/2}$, on the distribution of the singlet spin-*n*/2 states. We first analyze the correlations obtained by the OF detection scheme, introduced in

Sec. 5.2.3. The fringe pattern can be calculated by evaluating the average:

$$F_{|\psi_n^-\rangle}^{(\pm 1,\pm 1)}(\theta,h) = \left\langle \left(\hat{F}_{+,-}^{(\pm 1)}(h)\right)_A \otimes \left(\hat{F}_{\theta,\theta_\perp}^{(\pm 1)}(h)\right)_B \right\rangle.$$
(7.21)

We performed a numerical simulation, in order to consider also the transmission over a lossy channel, with an analogous procedure to the one described in the previuos section. We report in Fig. 7.5 the fringe pattern obtained for the n = 51 singlet states for the lossless case and a channel efficiency $\eta = 0.3$. We note that, as the OF threshold k is increased, the tails of the fringe pattern are damped, while the form of the fringe around the peaks remains unchanged. Furthermore, both the minimum and the maximum of the fringes are lowered by this filtering procedure. To understand the advantage of this measurement scheme with respect to the pure dichotomic case, we analyze in Fig. 7.7 (a) the

trend of visibility of the fringe pattern as a function of the threshold. We note that, for increasing k, the visibility is increased by the filtering process. This advantage obtained by exploiting the OF measurement can be explained by the following considerations. In absence of losses, the visibility of the fringe pattern is always unitary, as the analyzed state presents perfect polarization anti-correlations. After the transmission over a lossy channel, the binomial statistics added to the photon number distribution is responsible for the partial cancellation of this property. More precisely, if the difference between n_{π} and $m_{\pi_{\perp}}$ on any of the two spatial mode is little, losses may invert the outcome of a dichotomic measurement, i.e. for example the (+1) outcome may be converted to the (-1)outcome if unbalanced losses occur in that specific event. Such a process can generate the occurrence in the joint measurement of a result with positive correlations, i.e. (+1,+1) or (-1,-1), where in the decoherence-free case only anti-correlations are present. Thus, the visibility of the fringe pattern can be reduced by the presence of losses. However, to invert the outcome of matrix elements with $n_{\pi} - m_{\pi_{\perp}} = q \gg 0$, a strongly unbalanced losses in a single shot for the two polarization modes must occur. This event has a decreasing probability as q becomes larger, and the visibility of the fringe pattern progressively returns unitary as the threshold k is increased.



Figure 7.5: Effect of the O-Filtering detection technique on the fringe pattern of a n = 51 singlet state. (a) Transmittivity $\eta = 1$ (no losses) and (b) transmittivity $\eta = 0.3$. As the threshold k is increased, the tails of the fringe pattern are rounded.

Let us now consider the second POVM dichotomic measurement under investigation, the threshold detection TD. The interference fringe pattern with this measurement scheme can be calculated as:

$$T_{|\psi_{n}\rangle}^{(\pm1,\pm1)}(\boldsymbol{\theta},k) = \left\langle \left(\hat{T}_{+,-}^{(\pm1)}(k)\right)_{A} \otimes \left(\hat{T}_{\boldsymbol{\theta},\boldsymbol{\theta}_{\perp}}^{(\pm1)}(k)\right)_{B} \right\rangle.$$
(7.22)

In this expression, as before, the average is evaluated over the density matrix of the state after the numerical simulation of the lossy channel. In Fig. 7.6 we report the form of the fringe pattern for n = 51 in the lossless case [Fig. 7.6 (a)] and for $\eta = 0.3$. In the

lossless case, the threshold detector is ineffective for h < n due to the fixed number of photons in the state. In the lossy case, as the threshold h is increased we note that the TD device is responsible for the progressive return of the fringe patterns to their original form in absence of losses, i.e. for high values of n an approximately linear form. This behaviour can be explained as follows. While the original singlet-state has a well definite number of photons, the lossy channel reduces the number of photons to an average of $\eta \langle n \rangle$, with Poissonian fluctuations. At the measurement stage the threshold h in the TD device neglects [Fig. 7.1 (c)] the sectors of the Fock-space corresponding to a low number of photons. As h approaches the value h = n, only the events in which the original singlet state travels undisturbed in the channel (with probability η^{2n}) are selected, thus restoring the original correlation. We then analyze the effects of this measurement scheme in the visibility of the fringe pattern in Fig. 7.7. We note that this quantity increases with a slower rate with respect to the OF apparatus. Differently from the O-filtering case, on each spatial mode the zones of the Fock space in which $n_{\pi} - m_{\pi_{\perp}}$ is small are not filtered out, and the increase in the visibility is then much slower with the threshold h. However, also with the TD apparatus the visibility reaches asymptotically the unitary value, since as said for a threshold h = n only the original singlet state, having unitary visibility, is detected.



Figure 7.6: Effect of the threshold detection technique on the fringe pattern for a n = 51 singlet state. (a) Transmittivity $\eta = 1$ (no losses) and (b) transmittivity $\eta = 0.3$. As the threshold *h* on the total photon-number is increased, the fringe patterns progressively return to have approximately a linear dependence from the phase θ , as for the original n = 51 singlet state. The values of the thresholds are indicated in the figure.

The analysis carried in this section shows that both the OF and the TD detection strategies can be used to enhance the fringe pattern visibility in lossy conditions for the singlet spin- $\frac{n}{2}$ states. A comparison between the two schemes shows a greater enhancement for the OF device. To conclude the discussion, we briefly analyze the advantages of the two POVM schemes presented here in terms of the achievable violation of the CHSH inequality $|S_{CHSH}| \leq 2$. In the OF case, we expect that the fast increase in the visibility may

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lead to an increase in the amount of violation with respect to the pure dichotomic measurement. However, as already discussed, care should be taken in the application of the OF system in a Bell's inequality due to the basis dependent filtering of the detected state performed by this measurement device. In the TD case the effect of the threshold *h* is the restoration of the original correlations present in the $|\Psi_n^-\rangle$ state before the lossy channel. This means that the value of the *S*_{CHSH} parameter reaches for h = n the maximum value $S_{CHSH}^{|\Psi_n^-\rangle}$, reported in Fig. 7.4, and the amount of achievable violation becomes practically negligible for large *n*.



Figure 7.7: Trend of the visibility for the singlet spin states for n = 80 and $\eta = 0.05$. The black straight curve corresponds to the TD detection scheme, while the red dashed line to the OF apparatus. In both cases, the success probability is calculated as the sum of the rate for the two conclusive outcomes (+1) and (-1).

7.4 Experimental high visibility correlations in high-gain spontaneous parametric down-conversion

We have generated a multiphoton state through an EPR source and we have performed dichotomic measurement via OF and TD upon it. Let us now describe the experimental setup shown in Fig. 7.8. The excitation source was a Ti:Sapphire Coherent Mira mode-locked laser amplified by a Ti:Sapphire regenerative RegA device operating with repetition rate 250 kHz. The output beam, frequency-doubled by second-harmonic generation, provided the OPA excitation field beam at the UV wave-length (wl) $\lambda = 397.5$ nm with power 600 mW on mode \mathbf{k}_P . The SPDC source was a BBO crystal cut for type-II phase-matching, working in a non-collinear configuration [KMW⁺95], in a high gain regime. The evaluated non linear gain is $g = 3.49 \pm 0.05$ corresponding to the generation of an average number of photons per mode of $\bar{n} \approx 270$ per pulse, corresponding to an overall average value of $\langle n \rangle \approx 540$ on each spatial mode. The multiphoton fields on modes \mathbf{k}_1 and \mathbf{k}_2 were filtered by 1.5 nm interferential filters, coupled by single mode fibers and then sent to the detection stage.

In order to characterize the source, we performed a set of preliminary measurements exploiting a SPCM detector on both spatial modes, deliberately attenuating the generated



Figure 7.8: Experimental setup for the generation and detection of a bipartite macroscopic field. The high laser pulse on mode \mathbf{k}_P excites a type-II EPR source in the high gain regime, i.e. g = 3.5. The two spatial mode \mathbf{k}_1 and \mathbf{k}_2 are spectrally and spatially selected by interference filters and single mode fibers. After fiber compensation *C*, the two modes are analyzed in polarization and detected by four photomultipliers ($PM_1, PM_1^*, PM_2, PM_2^*$). The signals are then analyzed electronically to perform either the threshold dichotomic detection described in the paper or the Orthogonality filtering detection technique. Finally, the coincidences between the measurement outcomes are recorded to obtain the desired interference fringe patterns.

field in order to have only few photons incident on the detector. First, we measured the nonlinear gain of the amplifier studying how the detected signal increases by varying the power of the incident pump beam on the crystal. In Fig. 7.9 we report the counts registered



Figure 7.9: (a) Experimental evaluation of the amplifier NL-gain: we report the counts of an SPCM detector on mode \mathbf{k}_1 versus the normalized UV power, defined as I_{in}/I_{max} . The red curve reproduces the best fit of the experimental data, the expected trend function is reported in [EKD⁺04].

on mode \mathbf{k}_1 by a SPCM detector as a function of the normalized UV power signal. As a further investigation on the multiphoton field features, we registered the coincidences between the signals on mode \mathbf{k}_1 and \mathbf{k}_2 , as a function of the phase φ , that represents the variation of the polarization analysis basis on Bob site, i.e. $\vec{\pi}_{\varphi} = \vec{\pi}_H + e^{i\varphi}\vec{\pi}_V$. Both fields are detected by two SPCM at Alice's and Bob's sites. Again, the signals were attenuated in order to have few photons incident on the detectors, in order to work in a linear response regime for the SPCM. As stressed in [EKD⁺04], the trend of visibility decreases as the gain increases, this is due to losses and to limited detectors photon-number resolution. The decrease of visibility below the theoretical asymptotic value of 33% is due to the multimodal operation of the amplifier. However, differently from what is reported in $[EKD^+04]$, we observe a value of visibility that remains above 15% as far as the NL-gain reaches the value of 3.5, while in $[EKD^+04]$ the visibility seems to fall below 15% for gain values higher than 2.

7.4.1 Non-collinear SPDC analyzed with the orthogonality filter

The multiphoton fields at Alice's and Bob's site are analyzed in polarization and detected by two photomultipliers (PMs), labeles as (PM_1, PM_1^*) and (PM_2, PM_2^*) respectively. This devices produce on each pulse a macroscopic output electronic current, whose amplitude is linearly proportional to the number of incident photons.



Figure 7.10: (a) Fringe patterns obtained by filtering on the difference of the signals. The main visibility is 0.67 ± 0.02 . Coincidences have been normalized to the product of the signals detected on each of the analyzed outcomes of the OF. (b) Trend of visibility versus OF counts. Black points: experimental data. Red solid line: theoretical model for the experimental results. Green solid line: theoretical model rescaled to take into account the multimode operation of the amplifier.

Let us fix the polarization analysis basis at Bob's site: the PMs provide the electronic signals (I_+^2, I_-^2) corresponding to the field intensity on the mode \mathbf{k}_2 associated with the π -components $\{\vec{\pi}_+, \vec{\pi}_-\}$, respectively. By the OF, shot by shot the difference signals $\pm (I_+^2 - I_-^2)$ are compared with a threshold $\xi k > 0$, where ξ is a constant describing the response of the photomultipliers. When the condition $(I_+^2 - I_-^2) > \xi k$ is satisfied, a TTL pulse L_2 is realized at one of the two output ports of OF. Likewise, when the condition $(I_-^2 - I_+^2) > \xi k$ is satisfied, a L_2^* TTL pulse is realized at other output port of OF. The PM output signals are discarded for $-\xi k < (I_+^2 - I_-^2) < \xi k$. The same measurement strategy is adopted at Alice's site, where the output TTL signals (L_1, L_1^*) are generated. The fringe

patterns are obtained by the following procedure: the analysis basis at Alice's site is kept fixed while the basis at Bob's site is varied through an adjustable phase delay given by a Babinet-Soleil compensator. Finally the coincidences between the TTL signals at Alice's and Bob's site are taken into account, namely $(L_1, L_1), (L_1, L_1^*), (L_1^*, L_2), (L_1^*, L_2^*)$. We report in Fig. 7.10 (a) the corresponding fringe patterns obtained in the $\{\vec{\pi}_+, \vec{\pi}_-\}$ basis, analogous results are observed in the $\{\vec{\pi}_R, \vec{\pi}_L\}$ and $\{\vec{\pi}_H, \vec{\pi}_V\}$ basis, due to the irrotational invariance of the generated multiphoton state. The threshold k in this case was set so that the measured count rate was ~ 500 Hz.

For sake of completeness we report the trend of visibility as a function of the OF counts in Fig. 7.10 (b). We observe an increase of visibility as the counts detected decrease. The highest visibility obtained is not enough to violate the CHSH inequality, due to the inefficiency of a dichotomic measurement performed on a multiphoton quantum state. However, in accordance with theoretical predictions, we observe that the OF technique allows to minimize losses effects.

7.4.2 Non-collinear SPDC analyzed with threshold detection

A further investigation on the macro-macro correlation has been carried out by performing another dichotomic measurement on the amplified states on modes \mathbf{k}_1 and \mathbf{k}_2 . The signals detected by the photomultipliers (PM_1, PM_1^*) and (PM_2, PM_2^*) enter into two threshold detectors, that perform the shot by shot measurement illustrated in Sec. 7.3 (b). Each TD



Figure 7.11: (a) Fringe patterns obtained by filtering on the sum of the signals through the threshold detection system. The main visibility is 0.49 ± 0.02 . (b) Visibility versus threshold detector counts. Black points: experimental data. Red solid line: theoretical model for the experimental results. Green solid line: theoretical model rescaled to take into account the multimode operation of the amplifier.

works as follows: the PMs electronic signals $(I_+^2, I_-^2) [(I_+^1, I_-^1)]$ corresponding to the field intensity on the mode \mathbf{k}_2 (\mathbf{k}_1), associated with the π -components { $\vec{\pi}_+, \vec{\pi}_-$ } respectively,

enter into the TD. The sum signals $\pm (I_+^2 + I_-^2) [\pm (I_+^1 + I_-^1)]$ are compared with a threshold $\xi h > 0$. When the conditions $(I_+^2 + I_-^2) > \xi h$ and $I_+^2 - I_-^2 > 0$ $[(I_+^1 + I_-^1) > \xi h$ and $(I_+^1 - I_-^1) > 0]$ are satisfied, a TTL pulse $J_2(J_1)$ is realized at one of the two output ports of TD. On the other hand when the conditions $(I_+^2 + I_-^2) > \xi h$ and $I_-^2 - I_+^2 > 0$ $[(I_+^1 + I_-^1) > \xi h$ and $(I_-^1 - I_+^1) > 0]$ are satisfied, a TTL pulse $J_2^*(J_1^*)$ is realized at the other output port of TD. Finally the coincidences between signals $(J_1, J_2), (J_1, J_2^*), (J_1^*, J_2), (J_1^*, J_2^*)$ are registered. The obtained fringe patterns corresponding to a count rate of $C \sim 400$ Hz are shown in Fig. 7.11 (a). Finally, a study on the obtained visibility as a function of the fraction of considered data has been carried out. We report in Fig. 7.11 (b) the trend of visibility versus TD counts.

7.5 Continuous-variables Bell's inequality

Starting from the results of previous section with dichotomic operators, we now address the problem of analyzing the investigated multiphoton state through an efficient measurement method in order to observe the violation of a Bell's inequality. We study the violation of the Bell's test in the form proposed by Banaszek and Wodkiewicz in Ref. [BW98] based on the measurement of the Wigner function at specific points of the phase space. By correlating the value of the Wigner function at different points of the phase-space, we study the possibility of violating the Bell's inequality either in absence or in presence of losses, and we relate the results with the value of the nonlinear gain of the amplifier, i.e. the size of the measured state.

7.5.1 Definition of the Bell's inequality for a CV measurement

Such inequality is based on the CHSH test proposed by Banaszek and Wodkiewicz in Ref. [BW98]. In that paper, they apply their test on the output state generated by a two-modes nondegenerate optical parametric amplifier. Their nonlocality proof starts from the definition of the displaced parity operators:

$$\hat{\Pi}(\alpha;\beta) = \hat{D}_1(\alpha)(-1)^{\hat{n}_1}\hat{D}_1^{\dagger}(\alpha) \otimes \hat{D}_2(\beta)(-1)^{\hat{n}_2}\hat{D}_2^{\dagger}(\beta),$$
(7.23)

where $\hat{D}_1(\alpha)$ and $\hat{D}_2(\beta)$ are displacement operators for the two spatial modes \mathbf{k}_1 and \mathbf{k}_2 , respectively. Such operators have been already exploited in Chap. 5 in the context of hybrid entanglement tests on microscopic-macroscopic systems. They can be directly measured by combining the input field with a coherent state in a low reflectivity beam-splitter, and by measuring the parity of the resulting field [Fig. 6.1 (a)]. Since a parity operator measurement gives a ± 1 result, it fits perfectly on CHSH inequality [CHSH69] and can be used to show nonlocality of the NOPA wave function. Using displacements in the phase-space, the correlation between the two parties can be written as:

$$E(\mathbf{a}; \mathbf{b}) = \Pi(\boldsymbol{\alpha}; \boldsymbol{\beta}). \tag{7.24}$$

where $\Pi(\alpha;\beta) = \langle \hat{\Pi}(\alpha;\beta) \rangle$ is the expectation value of the displaced parity operator. The nonlocality parameter can then be written as:

$$\mathscr{B} = \Pi(0;0) + \Pi(\sqrt{\mathscr{I}};0) + \Pi(0;-\sqrt{\mathscr{I}}) - \Pi(\sqrt{\mathscr{I}},-\sqrt{\mathscr{I}}), \qquad (7.25)$$

with \mathscr{I} positive. For local theories the inequality $-2 \le \mathscr{B} \le 2$ holds. In Ref. [BW98] it is shown that this inequality is violated by the NOPA state, even for large values of the squeezing parameter when the output state resemble closely the original EPR state.

As already discussed by Banaszek and Wodkiewicz in Ref. [BW98] and in the previous chapter, the average value of the displaced parity operators is related to the Wigner function of the state according to:

$$W(\alpha;\beta) = \frac{4}{\pi^2} \Pi(\alpha;\beta). \tag{7.26}$$

While the average value of $\hat{\Pi}(\alpha;\beta)$ can be extrapolated indirectly from the Wigner function of the state through homodyne measurements [Fig. 6.1 (b)], in order to perform a nonlocality test the direct measurement scheme has be adopted.



Figure 7.12: Conceptual scheme of the multiphoton source and of the detection apparatus for the measurement of the continuous-variables Bell's inequality. The generated state is measured in polarization and then detected by four apparata suitable for the measurement of the displaced parity operators.

In this section we generalize the inequality of Eq. (7.25) to the more general fourmodes multiphoton state produced by a type-II OPA, in which the correlations are present in two degrees of freedom, that is, the spatial and the polarization one. We then need to generalize Eq. (7.25) to the four dimensional case in which the Wigner function is expressed as a function of the complex variables $\alpha = (\alpha_H, \alpha_V)$ and $\beta = (\beta_H, \beta_V)$, the subscript H, V standing for the horizontal and vertical polarizations, respectively. The \mathscr{B} parameter can then be rewritten as a function of the average value of the four-modes displaced parity operators $\Pi(\alpha_H, \alpha_V; \beta_H, \beta_V)$. Hence, the violation results to be function of the nonlinear gain of the amplifier and of the displacement of the state in the eight-dimensional phase space. The Bell's inequality then reads:

$$\mathscr{B} = \Pi(\tilde{\alpha}_{H}, \tilde{\alpha}_{V}; \tilde{\beta}_{H}, \tilde{\beta}_{V}) + \Pi(\tilde{\alpha}_{H}, \tilde{\alpha}_{V}; \tilde{\beta}_{H}', \tilde{\beta}_{V}') + \Pi(\tilde{\alpha}_{H}', \tilde{\alpha}_{V}'; \tilde{\beta}_{H}, \tilde{\beta}_{V}) - \Pi(\tilde{\alpha}_{H}', \tilde{\alpha}_{V}'; \tilde{\beta}_{H}', \tilde{\beta}_{V}'),$$
(7.27)

where $\{\tilde{\alpha}_H, \tilde{\alpha}_V\}$, $\{\tilde{\alpha}'_H, \tilde{\alpha}'_V\}$, $\{\tilde{\beta}_H, \tilde{\beta}_V\}$ and $\{\tilde{\beta}'_H, \tilde{\beta}'_V\}$ are the measurement setting for the displacements in the phase-space. The detection apparatus necessary for this test requires standard polarization analysis on each spatial mode, followed by four apparatuses for the measurement of the displaced parity operators [Fig. 7.12].

7.5.2 Violation of the inequality in absence of losses

We first analyze the perfect case in which the multiphoton state is not affected by decoherence nor losses. In this case the Wigner function in Eq. (7.3) reads:

$$W_{0}(\alpha_{H}, \alpha_{V}, \beta_{H}, \beta_{V}, g) = \left(\frac{2}{\pi}\right)^{4} \exp\left\{-2(1+2S^{2})\sum_{\pi=H,V} \left[|\alpha_{\pi}|^{2}+|\beta_{\pi}|^{2}\right]\right\} \times \exp\left\{-2CS\left[2\operatorname{Re}(\alpha_{V}\beta_{H})-2\operatorname{Re}(\alpha_{H}\beta_{V})\right]\right\}.$$
(7.28)

The displaced parity can be written as:

$$\Pi_0(\alpha_H, \alpha_V, \beta_H, \beta_V, g) = \left(\frac{\pi}{2}\right)^4 W_0(\alpha_H, \alpha_V, \beta_H, \beta_V, g).$$
(7.29)

Now the Bell parameter can be written as a function of eight phase-space variables and the nonlinear gain:

$$\mathcal{B}_{0}(z_{1}, z_{2}, z_{3}, z_{4}, z_{5}, z_{6}, z_{7}, z_{8}, g) = \Pi_{0}(z_{1}, z_{2}, z_{3}, z_{4}, g) + \Pi_{0}(z_{1}, z_{2}, z_{7}, z_{8}, g) + \Pi_{0}(z_{5}, z_{6}, z_{3}, z_{4}, g) - \Pi_{0}(z_{5}, z_{6}, z_{7}, z_{8}, g),$$
(7.30)

where z_i , with i = 1, ..., 8, represents the displacements in the phase-space for the fixed $\{\pi_H, \pi_V\}$ polarization basis, being z_1, z_2 and z_5, z_6 the displacement relative to the k_1 mode, while z_3, z_4 and z_7, z_8 relative to k_2 spatial mode. By fixing the value of the nonlinear gain g, we have then maximized the value of B, for different values of g. We found numerically that for g = 2 the maximum violation is obtained for real displacements given by: $z_1 = -0.0241, z_2 = -0.0066, z_3 = -0.0066, z_4 = 0.0241, z_5 = 0.0725, z_6 = 0.0198, z_7 = 0.0198, z_8 = -0.0725$ and corresponds to a violation equal to:

$$\mathscr{B}_0^{\max} \simeq 2.32. \tag{7.31}$$

In Fig. 7.13 is reported the trend of the Bell's inequality violation as a function of the nonlinear gain. We observe that for low values of g we have small violation, since



Figure 7.13: Trend of the violation of Bell's test in absence of losses as a function of the nonlinear gain g of the amplifier.

gaussian states with no squeezing cannot violate this inequality. For $g \ge 1$ the amount of violation progressively saturates and reaches its maximum value equal to \mathscr{B}_0^{\max} in Eq. (7.31). We note that the points in which we can observe the maximal violation of the inequality depend on the nonlinear gain of the amplifier since it changes the squeezing of the generated state. Increasing the value of g we obtain displacement amplitudes closer to zero, a requirement which represents an experimental challenge.

7.5.3 Resilience of the violation in presence of losses

Let us consider now the case in which the state undergoes a decoherence process, simulated by the presence of beam-splitter of transmittivity η (Fig. 7.12). The losses contribution is taken into account by the parameter $R = 1 - \eta$, and the Wigner function in the lossy case is given by Eq. (7.3). The displaced parity on the phase-space is given by:

$$\Pi(\alpha_H, \alpha_V, \beta_H, \beta_V, g, \eta) = \left(\frac{\pi}{2}\right)^4 W_0(\alpha_H, \alpha_V, \beta_H, \beta_V, g, \eta),$$
(7.32)

and the violation turns out to be dependent on the losses parameter. Similarly to the perfect case we define a Bell parameter given by:

$$\mathcal{B}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, g, \eta) = \Pi(z_1, z_2, z_3, z_4, g, \eta) + \Pi(z_1, z_2, z_7, z_8, g, \eta) + \Pi(z_5, z_6, z_3, z_4, g, \eta) - \Pi(z_5, z_6, z_7, z_8, g, \eta),$$
(7.33)

and we maximize it with respect to z_i for fixed values of g and η . In Fig. 7.14 (a) we report the trend of violation as a function of η , for different values of the nonlinear gain. We observe that the amount of violation decreases rapidly as a function of η , and the maximum value of $\eta = \eta^*$ for which we cannot observe a violation strongly depends on g. Fig. 7.14 (b) reports the trend of η^* such that $B(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, g, \eta^*) = 2$ as a function of g; we observe that the value of η^* increases with increasing the size of the system, and for high value of g it becomes practically impossible to observe a violation even in the presence of a small amount of losses. We stress that while the increasing of nonlinear gain g produces a larger squeezing of the multiphoton state, the presence of



Figure 7.14: (a) Trend of violation of Bell's test as a function of the transmittivity η for different values of the nonlinear gain g. Red dashed curve: g = 0.01. Green solid curve: g = 0.5. Blue dotted curve: g = 1. Black dash-dotted curve: g = 1.5. Cyan dash-dot-dotted curve: g = 2. (b) Limiting value of $\eta = \eta^*$ for which $\mathscr{B} = 2$ as a function of nonlinear gain g. The white region identifies the range of parameters where nonlocality can be detected.

losses is responsible for the broadening of the Wigner function [DSTC10]. This interplay between the two effects determines the points in which we can see the maximal violation of the Bell's inequality.

7.6 Conclusions and perspectives

In this chapter we have reported a thorough analysis on the possibility of observing quantum correlation on a multiphoton quantum system by performing probabilistic dichotomic measurements and continuous-variables measurements. We have addressed a specific class of multiphoton states: the ones obtained by the high-gain optical parametric amplifier working in a noncollinear configuration.

In order to violate a Bell's inequality with dichotomic measurements, we have introduced two dichotomization processes, based on the O-Filtering procedure and on a threshold detection scheme similar to the naked eye discussed in Sec. 5.2.3. It has been demonstrated that these two detection schemes reduce to a simple dichotomic measurement when their characteristic thresholds are set to 0. We have discussed in terms of LHV models the feasibility of a CHSH test with the two probabilistic measurements presented in this chapter. We have shown that such dichotomic measurements when performed on n/2-spin states with increasing n, asymptotically permits in the ideal case to violate a CHSH Bell's inequality even for large n. The shape of correlation functions has been investigated, and we have shown that the sinusoidal correlation pattern, typical of an 1/2-spin state, tends asymptotically to a triangular form, proper to classical correlations. When losses and decoherence are introduced the visibility of the correlation pattern is lowered and its shape turns out to be sinusoidal. In presence of losses, the violation of CHSH Bell's inequality is not allowed by a dichotomic measurement and more complicated detection schemes are required. Finally, we have shown experimentally that the measurement performed by the probabilistic dichotomic schemes, the orthogonality-filter and the threshold detector, allow to obtain higher visibility of correlation functions, not enough to violate CHSH Bell's inequality, but effective to reduce losses and decoherence effects.

We have then theoretically addressed the problem of observing nonlocality by performing continuous-variables measurements upon the investigated system. We have generalized the Bell's test proposed by Ref. [BW98] for a NOPA state for an enlarged four-mode multiphoton state. We have then applied the nonlocality test by addressing both the lossless and the lossy case. In the lossless case, a maximum violation equal to $\mathscr{B}_{max} = 2.32$ can be reached for increasing size of the investigated system, while in presence of losses the amount of violation fastly decreases by increasing the nonlinear gain or the parameter of losses. This renders extremely difficult to observe experimentally the quantum features for a increasing size system, even if an efficient measurement is performed upon it.

In conclusion the obtained results could allow to reach a deeper understanding about the problem of observability of nonlocality by adopting continous-variables measurement over increasing size quantum states.

Chapter 8

Characterization of the single-photon addition process: nongaussianity, nonclassicality and process tomography

In continuous-variables quantum information nongaussianity is a relevant resource, which permits to perform several tasks forbidden when only gaussian resources are employed, such as error correction or entanglement distillation. Within this context, it is then necessary to identify and characterize suitable protocols for the generation of nongassian states. The process of parametric down-conversion combined with conditional detection can be adopted for this purpose to implement a relevant nongaussian process, that is, single-photon addition [ZVB04a]. In this chapter we characterize the amount of nongaussianity induced by this process on a set of input coherent states. We analyze in details how the experimental imperfections affect the nongaussianity of the output states. The results of this analysis are reported in Ref. [BSG⁺10]. Finally, by reconstructing the tensor of the process we explicitly address the role of experimental imperfections in a state-independent form.

8.1 Generation and characterization of continuous variables nongaussian resources: state of the art

Within the framework of quantum information with continuous-variables [Bv05], nonclassical states of the radiation field represent a resource and much attention has been devoted to their generation schemes, which usually involve nonlinear interactions in optically active media. In order to implement the necessary nonlinearities, the reduction postulate provides an alternative mechanism to achieve effective nonlinear dynamics. More specifically, when a measurement is performed on a portion of a composite entangled system, the other component is conditionally reduced according to the outcome of the measurement. The resulting dynamics may be highly nonlinear, and may produce quantum states that cannot be generated by currently achievable nonlinear processes. Conditional measurements have been exploited to engineer nonclassical states and, in particular, have been recently employed to obtain nongaussian states. The latter become a relevant resource in different quantum information tasks, such as quantum computer [RGM⁺03, LRH08], improving teleportation [OKW00, CRM02, OPB03], cloning [CKN⁺05], and quantum information storage [CLP07]. Several realisations of nongaussian states have been reported so far, in particular from squeezed light [LHA⁺01, WTBG04, ZVB04a, ZVB04b, OTBL⁺06, OTBG06, NNNH⁺06, ODTBG07, Kim08, OFTBG09], close-tothreshold parametric oscillators [DCL⁺05, DdLP⁺10], in optical cavities [DDS⁺08], and in superconducting circuits [HWA⁺08]. Nongaussian operations are also interesting for tasks as entanglement distillation [OJTBG07, TNNT⁺10], and noiseless amplification [FBB⁺10, FBB⁺11, XRL⁺10] which also are obtained in a conditional fashion, accepting only those events heralded by a measurement result.

However, care should be taken indentifying the nongaussianity with a nonclassical property. In principle, nongaussianity does not directly imply the nonclassical character of a state and, in turn, classical nongaussian state may be prepared, e.g. by phase-diffusion of coherent states or photon-subtraction on thermal states [AAB⁺10]. On the other hand, in the applications mentioned above it is the presence of both nongaussianity and non-classicality which allows for enhancement of performances. Therefore, de-gaussification protocols of interest for quantum information are those providing nongaussianity in conjunction with nonclassicality.

Recently, several techniques have been developed in order to fully characterize the evolution of the components of a physical process. State tomography [Leo98, JWKM01], process tomography [OPG⁺04, LKK⁺08], and detector tomography [LFCR⁺08] have been developed as a mathematical tool to reconstruct from the experimental data an unknown physical state, its evolution, or its measurement stage respectively. All of these must obey some constraints; for instance, a map acting on density matrices space corresponding to a physical process is normally completely positive. This amounts to say that it must send physical states into physical states regardless of observing the system by itself or as a part of a larger ensemble to which it is de-coupled [Kra83]. These maps usually preserve the norm of the state, but there exist notable exceptions: non-trace preserving operations arise whenever a measurement on the system is involved. The experimental investigation, as well as the mathematical framework, is relatively at an early stage. Indeed, quantum process tomography of non trace-preserving maps has been presently implemented only in a reduced two-qubit Hilbert space [KSW+05, BSS+10]. In particular, for the case of continuous-variable processes, as a further issue it is not clear whether the experimental imperfections would actually prevent the linearity of the process.

In this chapter we characterize a relevant process exploited as a protocol to generate nonclassical, nongaussian states, that is, single-photon addition [ZVB04a, BSG⁺10]. This is done by first describing the process in Sec. 8.2, then by providing a theoretical model in Sec. 8.3 to describe the experimental implementation, which is reported in Sec. 8.4. We quantify experimentally in Sec. 8.5 the amount of nongaussianity obtained by adding a photon to a coherent state [ZVB04a, ZPB07, PZKB07, ZPK⁺09]. Differently from previous investigations [SRW05, ZPB07, KVP⁺08, KVH⁺09, SVD⁺09], we can explicitly address the two aspects of nongaussianity and nonclassicality at once. Then, we derive the conditions under which such process can be described by a linear mapping. Under these conditions, we reconstruct in Sec. 8.6 the tensor of the process by exploiting some a-priori knowledge. This will constitute a stimulus to more investigations in the area and to develop more sophisticated analytic tools.

8.2 The single-photon addition process

The process under investigation is described mathematically by the action of the creation operator \hat{a}^{\dagger} on an input state $\hat{\rho}$. However, the map acting on the input state $\hat{a}^{\dagger}\hat{\rho}\hat{a}$ cannot be written in the Kraus representation and is not completely positive. Furthermore, for certain input states the trace of the output state is increased by the action of the \hat{a}^{\dagger} operator: $\text{Tr}[\hat{a}^{\dagger}\hat{\rho}\hat{a}] > 1$. Since $\text{Tr}[\hat{a}^{\dagger}\hat{\rho}\hat{a}]$ is the probability that the process occurs, the condition $\text{Tr}[\hat{a}^{\dagger}\hat{\rho}\hat{a}] > 1$ corresponds to an unphysical situation. Hence, no direct deterministic implementation of the process can be performed. However, the photon addition process can be implemented in an approximate and heralded fashion [ZVB04a].



Figure 8.1: Conceptual scheme of the implementation of the single-photon addition process. The input coherent state is injected into an optical parametric amplifier, and the detection of a single-photon in the idler mode heralds a successful run of the process. Finally, the output field is measured with an homodyne detection apparatus.

A conceptual scheme of the implementation of the photon addition process is shown in Fig. 8.1: an input coherent beam $|\alpha\rangle$ is injected on mode \mathbf{k}_A in an optical parametric amplifier working in a noncollinear, type-I configuration. In this device, a three-wave nonlinear interaction occurs between the pump beam, the signal beam on mode \mathbf{k}_A and a third mode on mode \mathbf{k}_B . In the strong pump limit, the action of the amplifier can be expressed as the application of the squeezing operator:

$$\hat{U}_{AB}^{r} = \exp\left\{r(\hat{a}_{A}^{\dagger}\hat{a}_{B}^{\dagger} - \hat{a}_{A}\hat{a}_{B})\right\},\tag{8.1}$$

to the input state $|\alpha\rangle_A|0\rangle_B$. Here, *r* is the squeezing parameter, which depends on the pump intensity and the crystal non-linear coefficients. Here, \hat{a}_A and \hat{a}_B are the field operators associated to the spatial modes \mathbf{k}_A and \mathbf{k}_B . Despite the strong pump, the OPA typically works in the weak gain regime, so that \hat{U}_{AB}^r can be expanded in series up to the first order:

$$\hat{U}_{AB}^{r} \simeq \hat{\mathbb{1}}_{AB} + r(\hat{a}_{A}^{\dagger} \hat{a}_{B}^{\dagger}) - r(\hat{a}_{A} \hat{a}_{B}).$$
(8.2)

According to the scheme of Fig. 8.1 an APD is inserted on mode \mathbf{k}_B . Since the mode \mathbf{k}_B was originally in the vacuum state, the only term which can give a contribution in \hat{U}_{AB}^r is the second one. Therefore, the detection of a single photon on mode \mathbf{k}_B heralds the addition of a single photon to the coherent state, transforming it in the ideal case into:

$$\frac{1}{\sqrt{1+|\boldsymbol{\alpha}|^2}}\hat{a}_A^{\dagger}|\boldsymbol{\alpha}\rangle_A,\tag{8.3}$$

8.3 Experimental implementation: the model

The theoretical model adopted (see Refs. [OTBL+06, ODTBG07, OJTBG07]) provides an accurate description of the experimental results, and it will be exploited to characterize the nongaussianity and the nonclassicality of the output states, as well as to perform the reconstruction of the quantum map of the process.



Figure 8.2: Modelization of the photon addition process' implementation. (i) Two mode squeezing on spatial modes \mathbf{k}_A and \mathbf{k}_B . (ii) Parasitic squeezing models the mode mismatch between the pump and the input field. (iii) Partial trace on vacuum injected spatial modes \mathbf{k}_C and \mathbf{k}_D . (iv) Detection of the single photon which heralds the photon addition process. (v) Partial trace on the single photon mode \mathbf{k}_B .

A block diagram of the model resembling the experimental apparatus is shown in Fig. 8.2. The input state $\hat{\rho}_A \otimes |0\rangle_B \langle 0|$ of the process is injected on mode \mathbf{k}_A in an OPA. The action of the OPA is described by the two-mode unitary squeezing operator of Eq. (8.1).

In the OPA there might occur a certain modal mismatch between the pump and the input: this results in a parasitic amplification that introduces excess noise on the two output modes. The process is modeled as a set of two nondegenerate OPAs, one per each mode \mathbf{k}_A and \mathbf{k}_B , driven at a weaker strength γr . The parasitic amplification process couples the two modes \mathbf{k}_A and \mathbf{k}_B with two other modes \mathbf{k}_C and \mathbf{k}_D , initially in the vacuum state. The complete description of the amplification process takes the form:

$$\hat{\rho}_{ABCD} = \hat{U}_{AC}^{\gamma r} \hat{U}_{BD}^{\gamma r} \hat{U}_{AB}^{r} (\hat{\rho}_{A} \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0|) (\hat{U}_{AB}^{r})^{\dagger} (\hat{U}_{BD}^{\gamma r})^{\dagger} (\hat{U}_{AC}^{\gamma r})^{\dagger}.$$
(8.4)

The detection of the idler beam is performed by an APD on mode \mathbf{k}_B , that cannot resolve photon number. In the limit of small detection efficiency, we can approximate the detection process as the application of the \hat{a}_B annihilation operator on the mode \mathbf{k}_B . This low efficiency approximation is valid in the case of the present experimental implementation, where the overall detection efficiency is less than $\mu \leq 10\%$, due to spatial filtering ($\leq 75\%$), spectral filtering ($\leq 30\%$), and limited efficiency of the avalanche photodiode ($\sim 55\%$). Accurate spatial and spectral filtering is performed so that the mode detected by the APD is matched with the input mode detected with the balanced homodyne apparatus. This mode matching is performed with a nonunitary efficiency ξ , leading to a correction of the output state of the form:

$$\boldsymbol{\xi}\hat{\boldsymbol{\rho}}_{A,\checkmark} + (1-\boldsymbol{\xi})\hat{\boldsymbol{\rho}}_{A,\mathbf{x}}.$$
(8.5)

Here, $\hat{\rho}_{A,\sqrt{2}} = \text{Tr}_B[\hat{a}_B \text{Tr}_{CD}[\hat{\rho}_{ABCD}]\hat{a}_B^{\dagger}]$ is the output state on the signal mode \mathbf{k}_A conditioned to a successful trigger count belonging to the correct mode, and $\hat{\rho}_{A,x} = \text{Tr}_{BCD}[\hat{\rho}_{ABCD}]$ is the output state heralded by a faulty trigger event. Note that the partial trace on the additional modes \mathbf{k}_C and \mathbf{k}_D is performed since they are not observed in the experiment.

As a last source of experimental imperfection, we need to include also the homodyne efficiency η . This can be modeled by inserting a beam-splitter of transmittivity η , attenuating and mixing the output field with a vacuum state, before an ideal homodyne detection apparatus. Such element has not been included in the overall scheme of Fig. 8.2 since the homodyne apparatus belongs to the detection stage of the output states and not to the process itself.

8.3.1 Wigner function of the output field

The Wigner function associated to the state above described reads [Fer11]:

$$W_{\mathscr{A}[|\alpha\rangle\langle\alpha|]}(X,P) = \frac{1}{\pi\sigma^2} \left(1 - \delta_{\alpha} - \zeta_{\alpha} + \delta_{\alpha} \frac{(x - \sqrt{2}\kappa\alpha)^2 + p^2}{\sigma^2} \right) e^{-\frac{(x - \sqrt{2}\sqrt{g_{\text{eff}}\alpha})^2}{\sigma^2} - \frac{p^2}{\sigma^2}},$$
(8.6)

where:

$$\delta_{\alpha} = \frac{\delta}{1 + \frac{h(g-1)}{hg-1}\alpha^2}, \qquad (8.7)$$

$$\zeta_{\alpha} = \frac{\delta_{\alpha} \sigma^2 \alpha^2}{2g_{\rm eff}}, \qquad (8.8)$$

$$\kappa = \frac{2\eta - 1}{2\sqrt{g_{\text{eff}}}},\tag{8.9}$$

$$g_{\rm eff} = \eta g h, \qquad (8.10)$$

and:

$$g = \cosh^2 r, \tag{8.11}$$

$$h = \cosh^2(\gamma r), \tag{8.12}$$

$$\delta = \frac{2\zeta \eta h^2 g(g-1)}{\sigma^2 (hg-1)}, \qquad (8.13)$$

$$\sigma = 2\eta (hg - 1) + 1. \tag{8.14}$$

8.4 Experimental implementation: results

The complete setup of the experimental apparatus is shown in Fig. 8.3. The output of a mode-locked Ti:Sapphire laser, operating at $\lambda = 850$ nm and generating a pulse train with duration 230 fs and repetition rate 800 kHz, is split in three parts. The first one attenuated through a set of neutral density filters acts as the input coherent state of the experiment, with $|\alpha|$ varying in the range [0, 1.5]. The second part acts as the local oscillator (LO) of the homodyne measurement apparatus. Finally, the third part doubled in frequency is the pump beam of the OPA at $\lambda_p = 425$ nm. The pump and the coherent seed are temporally superimposed through an adjustable delay line and injected into the optical parametric amplifier, implemented by a 100 μ m thick slab of potassium niobate. The output field of the vacuum injected mode \mathbf{k}_B is coupled into a single mode fiber, spectrally filtered by a diffraction grating followed by a slit, and detected by a single-photon APD. The results of the single-photon measurements on this mode \mathbf{k}_B heralds the reconstruction of the output on mode \mathbf{k}_A by homodyne quantum tomography. In order to obtain an accurate 50 : 50 splitting at the homodyne beam-splitter, polarization matching is optimized by a sequence of polarizing beam-splitters and a half waveplate.

The output states are reconstructed by a maximum likelihood algorithm [Lvo04] interpolating 800,000 data points arranged into 12 histograms each corresponding to a phase bin. In Fig. 8.4 (a) we report the fidelities between the reconstructed density matrices and the expected states calculated from the theoretical model. The used model parameters [see Eqs. (8.7-8.14)] are r = 0.105, $\gamma = 0.425$, $\xi = 0.96$ and $\eta = 0.71$. As it will detailed later



Figure 8.3: Layout of the experimental apparatus. An OPA is injected with a coherent state of variable amplitude $|\alpha|$ in the range [0,1.5]. Such OPA is driven in frequency-degenerate and non-collinear regime, so to generate an idler at the same wavelength $\lambda = 2\lambda_p$ as the coherent seed; this is then spatially filtered with a single-mode fiber, spectrally filtered by a diffraction grating and a slit. Finally, the idler is detected by an APD. The observation of the output conditioned by an APD count results in single-photon addition. The quantum state of the output is reconstructed by homodyne detection. Mode-matching with the local oscillator exploits polarization: the signal and the LO are first matched on a polarizing beam splitter, and then combined using a half-wave plate and a second PBS so to realize an accurate 50:50 intensity splitting.

on, these values have been obtained by fitting the curves of the nongaussianity obtained from the experimental data. The corresponding average fidelity is $F = 0.989 \pm 0.006$.

In Fig. 8.4 (b) we report the reconstructed Wigner functions obtained for three different values of $|\alpha|$. They are in good agreement with the expected ones reported in Fig. 8.4 (c).

8.5 Nongaussianity of the single-photon addition process

Here we address in details the nongaussianity of the output state generated by the process of single-photon addition. Furthermore, we complement this analysis by investigating if the nongaussianity generated through this process is accompanied by the presence of nonclassical properties. This has been performed by defining a witness of nonclassicality to be evaluated alongside the nongaussianity.

This section is organized as follows. In Sec. 8.5.1 we describe the adopted criteria for the nongaussianity and the nonclassicality respectively. Then, in Sec. 8.5.2 we perform a thorough analysis of the process by evaluating numerically these quantities exploiting

the theoretical model developed in Sec. 8.3. Finally, in Sec. 8.5.3 we calculate the nongaussianity and the nonclassicality of the process from the experimentally reconstructed output states of the photon addition process. This analysis demonstrates the presence of nongaussian features in the output states, alongside with a corresponding degree of nonclassicality.

8.5.1 Nongaussianity and Nonclassicality measures

The nongaussianity measure we adopted is $\delta[\hat{\rho}]$ proposed in Refs. [GPB08, GP10], and it is defined as the quantum relative entropy between the nongaussian state $\hat{\rho}$ and a reference gaussian one $\hat{\tau}$ having the same covariance matrix of $\hat{\rho}$. Given this choice of the reference gaussian state, we have that $\text{Tr}[\hat{\rho}\log\hat{\tau}] = \text{Tr}[\hat{\tau}\log\hat{\tau}]$, as $\log\hat{\tau}$ is a polynomial of order at most two in the canonical variables [GPB08, HSH99]. We thus find

$$\begin{split} \delta[\hat{\rho}] &= \mathscr{S}(\hat{\rho} \| \hat{\tau}) = \operatorname{Tr}[\hat{\rho}(\log \hat{\rho} - \log \hat{\tau})] \\ &= \mathscr{S}(\hat{\tau}) - \mathscr{S}(\hat{\rho}), \end{split}$$

that is, $\delta[\hat{\rho}]$ is simply equal to the difference between the von Neumann entropy of $\hat{\tau}$ and the von Neumann entropy of $\hat{\rho}$. In Ref. [GPB08] it has been shown that this measure is non zero only for nongaussian states. It is also additive under tensor product, invariant under unitary gaussian operations, and in general it does not increase under generic completely positive gaussian channels. This measure is somehow preferable to that based on the Hilbert-Schmidt distance [GPB07] in a quantum information context, since it is based on an information-related quantity.

As already discussed in Sec. 3.2.5, nongaussianity is a property which does not directly imply the presence of nonclassical features. Indeed, a mixture of classical states, such as a mixture of coherent states $|\alpha\rangle\langle\alpha|+|-\alpha\rangle\langle-\alpha|$, can also be strongly nongaussian. In order to distinguish whether the nongaussianity generated by the photon-addition process is nonclassical, and hence useful for the application in quantum information protocols, we consider as a nonclassicality witness a quantity $v[\hat{\rho}]$ related to the negativity of the Wigner function. This is normalized to a reference, which we choose to be a singlephoton state $W_1(x, p)$. This reference has been chosen since it has the lowest value within the class of states we consider. Furthermore, this choice is dictated by the need of a measure which does not depend on the convention for the quadratures, and which sets to unity the highest value of $v[\hat{\rho}]$ attainable in the class of states under investigation. The nonclassicality is then defined as

$$\nu[\hat{\rho}] = \frac{\min_{\{x,p\}} [W(x,p)]}{\min_{\{x,p\}} [W_1(x,p)]}.$$
(8.15)

While this does not constitute a measure, it acts as a witness for nonclassical states whenever $v[\hat{\rho}] > 0$ and it quantifies the amount of negativity of the state in the range $0 < v[\hat{\rho}] < 1$.



Figure 8.4: (a) Experimental results for the fidelities between the measured density matrices and the expected state, evaluated through the theoretical model of the experiment. (b) Experimental Wigner functions for increasing values of $|\alpha|$. (c) Corresponding expected Wigner functions.

8.5.2 Numerical analysis of the model

The theoretical model developed in Sec. 8.3 allows us to investigate the role played in the nongaussianity and in the nonclassicality of the process by the different parameters

involved in the experimental implementation.

As a first step, we must take into account the finite value of the squeezing parameter r of the OPA. Indeed, as the squeezing is increased, the expansion of the operator of Eq. (8.1) cannot be limited to first order. In Fig. 8.5 we plot $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ as a function of the coherent amplitude α , for different values of r. We observe that the two trends are very similar, suggesting that the nongaussianity induced by photon-addition is essentially a nonclassical signature and thus useful for quantum information processing. It can be also observed how both nongaussianity and nonclassicality decrease by increasing the squeezing parameter. This can be explained by observing that, as shown in Eq. (8.2), for low values of r, the squeezing operator adds only one photon on each arm, while by increasing r we have to consider also the possible addition of many photons. Since the emission of more than one photon cannot be discriminated from the single-photon term due to the lack of photon number resolution of the detectors, the signal will result to be in a mixture of several terms, thus decreasing the nongaussianity and nonclassicality of the output state. In the *ideal* limit of $r \rightarrow 0$ the nongaussianity of the state is exactly equal to the one of the ideal photon added coherent state in Eq. (8.3). However, since the squeezing parameter is reduced, the probability of detecting one photon on mode \mathbf{k}_{R} drops to zero. Hence, a compromise between the nongaussianity of the output states and the count rate of the successful events has to be chosen.



Figure 8.5: (a) Nongaussianity $\delta[\hat{\rho}]$ and (b) nonclassicality $v[\hat{\rho}]$ as a function of the amplitude $|\alpha|$ of the input coherent state for different values of the squeezing parameter *r* (dashed lines); from top to bottom $r = \{0.15, 0.30, 0.45\}$. The black solid line corresponds to the nongaussianity of the ideal photon added coherent state, that is to the limit $r \to 0$.

We can now address the role played by the other experimental imperfections. Since in the OPA a certain modal mismatch between the optical modes of the pump beam and of the input coherent state may occur, we need to consider the effect of this additional noise source modeled by the parasitic gain γr , where in our case $\gamma \sim 0.425$. Accurate spatial and spectral filtering is performed so that the mode detected by the APD is matched with the mode detected by the homodyne. However, this task can be accomplished only with

a nonunitary efficiency $\xi \sim 0.96$. Furthermore, we need to consider the effect of the limited efficiency η of the homodyne detection. The sources of this nonunitary efficiency are manifold, such as optical loss, nonunitary detector quantum efficiency, and nonunitary mode-matching between the local oscillator and the analyzed signal. The overall efficiency results to be of $\eta \sim 0.71$. In Fig. 8.6 we plot the value of the nongaussianity $\delta[\hat{\rho}]$ [Fig. 8.6 (a)] and of the nonclassicality $v[\hat{\rho}]$ [Fig. 8.6 (b)] at fixed values of the coherent state amplitude $\alpha = 0.5$ and of the squeezing parameter r = 0.15 as a function of the noise parameters γ , ξ and η . We observe as expected that $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ decrease monotonically with γ , while they increase monotonically with ξ and η . For the values that characterize the present experimental implementation, the homodyne efficiency η is the source of imperfection that affects in the most detrimental way the nongaussianity and the nonclassicality of our states.



Figure 8.6: (a) Nongaussianity $\delta[\hat{\rho}]$ and (b) nonclassicality $v[\hat{\rho}]$ as a function of the noise parameters of the experimental setup for fixed amplitude $|\alpha| = 0.5$ and squeezing parameter r = 0.15. Blue dot-dashed line: $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ as a function of γ . Green dotted line: $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ as a function of ξ . Red dashed line: $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ as a function of η . Lower grey solid line: $\delta[\hat{\rho}]$ and $v[\hat{\rho}]$ for squeezing parameter r = 0.15 and no imperfections. Upper black solid line: $\delta[\hat{\rho}]$ of the ideal photon-added coherent state.

8.5.3 Experimental data

Finally, in this section we describe the values of $\delta[\hat{\rho}]$ and of $v[\hat{\rho}]$ obtained by the experimental data. The results are shown Fig. 8.7. In Fig. 8.7 (a) we report the trend of the nongaussianity of the output states for different values of the coherent state amplitude $|\alpha|$. The values of parameters used for the curve, corresponding to the expected trend from the theoretical model, are obtained from a fit of the experimental data: r = 0.105, $\gamma = 0.425$, $\xi = 0.96$ and $\eta = 0.71$. The APD dark count rates can be neglected being ~ 10 counts/s over an overall rate ~ $1-4 \cdot 10^3$ counts/s thanks to a gated detection, triggered by the laser cavity dumping electronics. The agreement between the experimental data and the model



Figure 8.7: (a) Nongaussianity $\delta[\hat{\rho}]$ as a function of the amplitude $|\alpha|$ of the input coherent state. (b) Nonclassicality $v[\hat{\rho}]$ – related to the minimum value of the Wigner function of $\hat{\rho}$ – as a function of $|\alpha|$. The red points are the experimental values from the reconstructed matrices. The black dashed lines are obtained fitting the experimental data with the model including the main experimental imperfections. The parameters are: r = 0.105, $\gamma = 0.425$, $\xi = 0.96$ and $\eta = 0.71$.

is satisfactory, and we can observe, as expected, a decrease in the nongaussianity as the input intensity $|\alpha|$ increases. The effect of the single-photon addition is more relevant for quantum states with a small average number of photons, and becomes only a small perturbation for higher numbers.

In Fig. 8.7 (b) we report the trend of $v[\hat{\rho}]$ recovered from the experimental data as a function of the amplitude $|\alpha|$. The experimental results confirm that the two quantities, nongaussianity and nonclassicality, show a similar trend. Hence, the nongaussianity induced by this photon-addition operation is essentially of a nonclassical origin.

8.6 Quantum process tomography of the single-photon addition process

In this section we conclude the analysis of the single-photon addition process by considering the reconstruction of the relative quantum map. Since its dynamics is induced by a conditional evolution, the resulting map in general may not preserve the trace of the input state.

8.6.1 Quantum maps

In Sec. 1.2 we described the general formalism underlying the time evolution of a quantum system, which lead to the Kraus representation of Eqs. (1.11) and (1.12). However,

some exceptions arise when considering a conditional evolution. In this type of processes, the system evolves through a probabilistic device and a successful run is heralded by the detection of a certain trigger event on an ancillary mode [Fig. 8.8 (b)]. On one side, the overall process including both successes and failures needs to be physical. However, the conditional process may not preserve the trace of the output states since it involves a reduction of the wave-function due to the measurement process, leading to a non trace-preserving process. These processes are often used in order to approximate a non-unitary



Figure 8.8: (a) Trace-preserving quantum operation. The input state $\hat{\rho}$ is transformed by the quantum channel \mathscr{E} in the output state $\mathscr{E}[\hat{\rho}]$. The trace of the input state is preserved by the channel: $\operatorname{Tr}\{\mathscr{E}[\hat{\rho}]\} = 1$. (b) Heralded quantum operation. The input state $\hat{\rho}$ is transformed by the quantum channel \mathscr{F} in the output state $\mathscr{F}[\hat{\rho}]$ upon realization of a conditional event. The trace of the input state is in general not preserved by the channel: $\operatorname{Tr}\{\mathscr{F}[\hat{\rho}]\} \neq 1$.

linear operator \hat{C} . For instance, this is the case for the photon addition process under analysis in this chapter ($\hat{C}=\hat{a}^{\dagger}$). Its action on a pure state $|\alpha\rangle$ gives an output $\sqrt{\mathcal{N}(\alpha)}\hat{C}|\alpha\rangle$, where $\mathcal{N}(\alpha)$ is an additional normalization factor, which might present a complex dependence on the state. Therefore, even if the operator \hat{C} is linear, the linearity of the process is canceled when the normalization factor is included. Let us consider the action of the process \hat{C} on a linear superposition of two states $|\alpha\rangle$ and $|\beta\rangle$. In general, the linearity condition for the normalized output states does not hold, as

$$\sqrt{\mathcal{N}(\alpha)}\hat{C}|\alpha\rangle + \sqrt{\mathcal{N}(\beta)}\hat{C}|\beta\rangle \neq \sqrt{\mathcal{N}(\alpha+\beta)}\hat{C}(|\alpha\rangle+|\beta\rangle).$$
(8.16)

However, if we ignore the normalization, we can follow the same treatment as for ordinary maps. We can then introduce a definition for the tensor $\{\mathscr{F}_{l,k}^{n,m}\}$ similar to that of Eq.

Characterization of the single-photon addition process

(1.12):

$$\mathscr{F}_{l,k}^{n,m} = \langle l | \hat{C} | n \rangle \langle m | \hat{C}^{\dagger} | k \rangle, \qquad (8.17)$$

which allows us to predict the evolution of the state as

$$\left(\mathscr{F}[\hat{\rho}]\right)_{l,k} = \sum_{n,m} \mathscr{F}_{l,k}^{n,m} \rho_{n,m}.$$
(8.18)

Finally, in order to obtain a physical output state, the density operator $\mathscr{F}[\hat{\rho}]$ has to be normalized at the end of the calculation to obtain:

$$\left(\bar{\mathscr{F}}[\hat{\rho}]\right)_{l,k} = \mathscr{N}(\mathscr{F}, \hat{\rho}) \left(\mathscr{F}[\hat{\rho}]\right)_{l,k},\tag{8.19}$$

where in general the normalization $\mathcal{N}(\mathcal{F}, \hat{\rho})$ factor depends on both the process \mathcal{F} and the input state $\hat{\rho}$.

Such definition can then be extended to more general processes, with an important remark. Consider a heralded process \mathscr{F}_0 when conditioning can be faulty in a fraction $1 - \xi$ of the events. We can call \mathscr{F}_1 the correct process, and \mathscr{F}_2 the failure. The output state of the whole process will be a convex combination of $\hat{\rho}_1$ and $\hat{\rho}_2$, being $\hat{\rho}_{1,2}$ the output states of the correct and of the faulty processes. For this class of transformations, the output state cannot be written as the convex combination:

$$\mathscr{F}_0[\hat{\rho}] \neq \xi \mathscr{F}_1[\hat{\rho}] + (1 - \xi) \mathscr{F}_2[\hat{\rho}], \tag{8.20}$$

as it would for trace-preserving maps. Indeed, the normalization must be applied at each step of the process. The correct form of the normalized output state of the process then takes the form:

$$\left(\bar{\mathscr{F}}_{0}[\hat{\rho}]\right)_{l,k} = \xi \mathscr{N}(\mathscr{F}_{1},\hat{\rho})\left(\mathscr{F}_{1}[\hat{\rho}]\right)_{l,k} + (1-\xi)\mathscr{N}(\mathscr{F}_{2},\hat{\rho})\left(\mathscr{F}_{2}[\hat{\rho}]\right)_{l,k}.$$
(8.21)

8.6.2 Reconstruction of the single-photon addition process

In this section we apply the above considerations to reconstruct the map tensor $\mathscr{F}_{l,k}^{n,m}$ of the single-photon addition process ($\hat{C}=\hat{a}^{\dagger}$) considered throughout this chapter. As for the previous sections, the complete characterization of this process is performed by exploiting the theoretical model of Sec. 8.3. In App. D we show that the model for the amplification process is described by a linear map thus allowing the use of such formalism. The parameters obtained by fitting the nongaussianity of the experimental data, as described in Sec. 8.5.3: {r = 0.105, $\gamma = 0.425$, $\xi = 0.96$, $\eta = 0.71$ } are used to simulate the action of the process on the Fock basis vectors $|i\rangle\langle j|$. In this case, the density matrix of the input state has elements $\rho_{n,m} = \delta_{i,n}\delta_{j,m}$, and the output state reads:

$$\left(\mathscr{E}[|i\rangle\langle j|]\right)_{l,k} = \sum_{n,m} \mathscr{E}_{l,k}^{n,m} \delta_{i,n} \delta_{j,m} = \mathscr{E}_{l,k}^{i,j}.$$
(8.22)

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Figure 8.9: (a) Diagonal elements $\mathscr{F}_{k,k}^{m,m}$ of the ideal photon addition process. (b) Diagonal elements $\mathscr{F}_{k,k}^{m,m}$ for the case of a conditioned OPA driven at r=0.105. (c) Diagonal elements $\mathscr{F}_{k,k}^{m,m}$ with a parasitic gain $\gamma=0.425$ and very low gain (d) Diagonal elements $\mathscr{F}_{k,k}^{m,m}$ including both experimental imperfections. The tensor has been normalized to the larger element.

Finally, the elements $\mathscr{E}_{l,k}^{i,j}$ can be directly recovered from the elements $(\mathscr{E}[|i\rangle\langle j|])_{l,k}$ of the output states.

For the purpose of the reconstruction of the process' tensor $\mathscr{F}_{l,k}^{n,m}$, we need to consider the different sources of noise. Since we are interested in the characterization of the process itself, we can ignore the action of the noise introduced by the homodyne detection apparatus. Indeed, the latter can be considered as a part of the characterization stage. The two main sources of noise can be identified in the finite amount of squeezing r in the OPA, and in the imperfect matching between the pump and the signal mode γ . The values of these parameters in our experimental implementation were r=0.105, and $\gamma=0.425$ [BSG⁺10]. The third source of imperfection, that is, spurious events at the trigger stage D_0 due either to dark counts or clicks originating from non-matching modes [Figs. 8.1-8.2], can be neglected. This is due to the efficient spatial and temporal filtering on the trigger arms, leading to an high triggering efficiency in our implementation ($\xi > 0.95$). In case this source of noise could not be neglected, it is necessary to apply the considerations on the non-convexity of the previous section.

The results of the reconstruction are reported in Fig. 8.9. First, in Fig. 8.9 (a) we report the diagonal elements of the ideal process, obtained for $r \rightarrow 0$ and $\gamma = 0$. For higher values of the squeezing parameter [r = 0.105, Fig. 8.9 (b)], the gain is chosen to be sufficiently low so that two-pair events are not significant. On the other hand, the effect of the parasite gain presents the same relevance: in this case, the action of the parasitic noise consists in the presence of uncorrelated clicks at D_0 that leave the state unchanged. This corresponds to the diagonal terms in Fig. 8.9 (c), considered in the limit of extremely low gain $r \rightarrow 0$. The overall process is modeled in the presence of these two imperfections [Fig. 8.9 (d)].

These results show that the reconstruction of the process' tensor allows to obtain relevant information on the dynamics of the physical system, that may not be evident by analyzing a specific choice of the input state. In this example, the adoption of the quantum map formalism reveals to be particularly clear and useful since it allows us to discuss the behaviour of parasite processes, but also gives us a way of quantifying their effect in a way that does not depend on the particular input state.

8.7 Conclusions and perspectives

Heralded processes represent a relevant class of quantum evolution. Indeed, nonlinear dynamics can be obtained by exploiting the conditional evolution obtained by a measurement performed on a portion of the system. Such class of processes can be exploited to produce nongaussian states, which represent a useful resource for many quantum information tasks such as computation, teleportation, or cloning. In this chapter we characterized a relevant conditional process, that is, single-photon addition on coherent states, by explicitly addressing the generated nongaussianity and nonclassicality by means of two suitable criteria. With this analysis, we recognized that the nongaussianity induced by such process is nonclassical, thus being useful as a resource for several quantum information tasks.

Furthermore, to deepen our analysis of the photon-addition process, we reconstructed the tensor of the process by exploiting the experimental data and some a-priori knowledge. This reconstruction permits to address individually the effect of each experimental imperfection and to predict the action of the process on a general input state. Such results can represent a starting point for future investigations aimed at obtaining a general mathematical framework [LKK⁺08] for the description and the reconstruction of conditional processes, without the need of any a-priori knowledge.

Part III

Robust Quantum sensing with multiphoton states

Chapter 9

Enhancing resolution of single-photon phase estimation in lossy conditions by parametric amplification

Quantum sensing represents one of the possible fields where quantum mechanics permits to obtain increased performances with respect to classical strategies. In this scenario, the typical strategy to measure an optical phase consists in sending an optical probe on the system and in measuring the probe state after the interaction. The aim of these protocols is to obtain the maximum resolution with minimal disturbance upon the system to be measured. However, while quantum strategies turn out to be useful in increasing the achievable performances, quantum benefits are typically extremely fragile under the action of losses. In this chapter we propose and realize experimentally a strategy, based on the process of parametric amplification, to increase the resolution of phase estimation protocols performed with single photons. This strategy is motivated by the results reported in Chap. 4, where we showed that the multiphoton states generated by parametric amplification of single photons are robust with respect to losses. By amplifying the single-photon probe state after the interaction with the sample, we can preserve the information encoded on the phase from the action of detection losses. The results of this theoretical and experimental study are reported in Ref. [VST+10a], and will be extended in the next chapter for a coherent probe state.

9.1 Minimally invasing quantum sensing in a lossy scenario

The aim of quantum sensing is to develop methods to extract the maximum amount of information from a system with minimal disturbance upon it. In the case of optical interferometry, the parameter to be estimated is an optical phase shift introduced by a sample. Within this context, it has been shown that the possibility of exploting quantum resources

can increase the achievable precision going beyond the semiclassical regime of operation [GLM04, GLM06, Hel76]. For example, in phase estimation protocols this can be achieved by the use of the so-called N00N states. These are maximally entangled states which are quantum mechanical superpositions of just two terms, corresponding to all the available photons N placed either in the signal arm or in the reference arm. The use of NOON states can enhance the precision in phase estimation to 1/N, thus improving the scaling with the number of resources N with respect to the classical strategies [BKA⁺00, Dow08]. This approach can have wide applications for minimally invasive sensing methods. Imaging of biological samples and of an ancient artifact are examples of situations where it is clearly beneficial to use weak light probes in order to avoid damaging of the sample. In the quantum domain there is an even stronger motivation to employ minimally invasive measurements, since the back action of the measurement actually changes the state of the quantum system under investigation. When dealing with the practical implementation of quantum-enhanced phase estimation protocols, these approaches present some limitations. On one side, the experimental realization of protocols involving N00N states is still limited in the few photon regime [DCS01, WPA⁺04, MLS04, EHKB05, NOO⁺07, OH10, AAS10]. Nevertheless, these quantum states are extremely fragile under losses and decoherence. Furthermore, the sample whose phase shift is to be measured may at the same time introduce high attenuation. Since quantum-enhanced protocols for phase estimation exploit fragile quantum mechanical features, the impact of environmental effects can be much more deleterious than in semiclassical schemes, up to destroying completely quantum benefits [RK07, SC07]. Very recently, theoretical and experimental investigations of quantum states of light in this context has lead to the best possible precision in two-mode interferometry, even in presence of experimental imperfections [HWD08, DDDS⁺09b, MC09, DDDS⁺09a, KDDW⁺10, LHL⁺09]. However, the used quantum states present a complex quantum mechanical superposition form so that their implementation is still limited to the few photon regime and relies on schemes involving post-selection at the detection stage. Furthermore, they require an a priori knowledge of the amount of losses introduced by the sample, unavailable in most of the cases.

In the present chapter, starting from a review of single-photon phase estimation reported in Sec. 9.2 we propose in Sec. 9.3 a strategy to improve the performances in present of losses. This approach is based on the phase-covariant optical parametric amplifier described in Sec. 4.2.1 [DSV08, DSS09b, DSS09a]. As shown in Chaps. 5 and 6 the state outing the amplifier can be manipulated by exploiting a detection scheme which combines features of discrete- and continuous- variables. By performing the amplification process of the microscopic probe after the interaction with the sample we can preserve the probe single-photon state from the losses detrimental effect thus enhancing the performance of the phase measurement. The achievable improvement results to be proportional to the number of generated photons and depends on the optical amplifier gain, and is shown experimentally in Sec. 9.4. Furthermore, since this protocol involves weak single photons as probe states and the amplification process acts after the probe-sample interac-

tion, this approach can be adopted in a minimally invasive scenario, such as biological or artifact systems.

9.2 Quantum metrology with single-photon states

Let us consider a single-photon, path-encoded interferometric setup, whose conceptual scheme is shown in Fig. 9.1 (a). The phase shift ϕ is probed by sending into the interferometer *M* qubits, each one in the state $2^{-1/2}(|1\rangle_{\mathbf{k}_1} + |1\rangle_{\mathbf{k}_2})$ generated after the transmission of a single photon through a 50/50 beam-splitter. After the propagation, the medium introduces on the probe beam a phase ϕ , and each photon is found in the state:

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left(|1\rangle_{\mathbf{k}_1} + e^{i\phi} |1\rangle_{\mathbf{k}_2} \right). \tag{9.1}$$

The output state is recombined into a 50/50 beam-splitter and detected through singlephoton detectors placed at the output modes of the interferometer. The detection of one photon over the *N* experiments with quantum efficiency η_{tot} leads to a detected difference signal $I = I(D'_1) - I(D'_2)$ with corresponding fluctuations δI equals to:

$$I = \eta_{\text{tot}} M \cos \phi; \quad \delta I = (\eta_{\text{tot}} M)^{1/2}.$$
(9.2)

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According to phase estimation theory, the uncertainty on the phase ϕ can be evaluated as:

$$\delta\phi = \frac{\delta I}{\left|\frac{\partial I}{\partial\phi}\right|}.\tag{9.3}$$

The sensitivity of the measurement around $\phi = \pi/2$, defined as $S = \delta \phi^{-1}$, can hence be estimated as:

$$S_{1\text{phot}} = \sqrt{\eta_{\text{tot}}M}.$$
(9.4)

This precision on the estimation of the phase ϕ presents a classical scaling with the number of trials as \sqrt{M} . Furthermore the quantum efficiency is responsible for a decrease of $\sqrt{\eta_{\text{tot}}}$ in the sensitivity of the interferometer. This scheme is analogous to the case, analyzed hereafter and sketched in Fig. 9.1 (b), in which the two spatial modes \mathbf{k}_1 and \mathbf{k}_2 are replaced by the polarization modes $\vec{\pi}_H$ and $\vec{\pi}_V$.



Figure 9.1: (a) Interferometric scheme for single-photon phase estimation in the spatial degree of freedom. (b) Equivalent scheme in the polarization degree of freedom, where the sample introduces a relative phase shift between the $\vec{\pi}_H$ and the $\vec{\pi}_V$ polarizations.

9.3 Increased resilience to losses through optical parametric amplification

The conceptual scheme herein proposed is sketched in Fig. 9.2, where the two different encoding of the phase shift ϕ in the spatial degree of freedom [Fig. 9.2 (a)] and in the polarization degree of freedom [Fig. 9.2 (b)] are shown. In order to preserve the information on the phase encoded in the probe photon after the interaction with the system, we inject the $|\phi\rangle$ qubit into an optical parametric amplifier. In this scheme, the amount of losses can be divided in two contributions, corresponding respectively to the losses 1 - p induced by the interaction with the sample, which occur before the amplification process, and the losses $1 - \eta$ at the transmission and detection stage. The action of the *optimal phase-covariant quantum cloning*, is to broadcast the phase information into a large number of particle before the main losses $1 - \eta$ can be efficiently reduced, while our approach cannot compensate for losses 1 - p that occur before the amplification stage. Indeed, the effect of losses in the macroscopic field is no more the complete cancelation of the phase information, but only the reduction of the detected signal. The latter consideration represents the key of the increased resilience to losses of this scheme.

Let us now describe the theory of the amplifier-based protocol. From hereafter, we consider the polarization-encoded scheme of Fig. 9.2 (b). The input probe state is a single photon in the $\vec{\pi}_+ = 2^{-1/2} (\vec{\pi}_H + \vec{\pi}_V)$ polarization state:

$$|+\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle + |V\rangle \right). \tag{9.5}$$

This state can be prepared conditionally by means of an entangled state source through the process of spontaneous parametric down-conversion, as described in Sec. 2.1.3. The probe state $|+\rangle$ is then sent into the sample, which, we assume, introduces a birefringent phase shift ϕ between the $\vec{\pi}_H$ and the $\vec{\pi}_V$ polarization components. The information on the phase ϕ is then encoded in the polarization state $\vec{\pi}_{\phi} = 2^{-1/2} (\vec{\pi}_H + e^{i\phi} \vec{\pi}_V)$ of the single photon:

$$|\phi\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle + e^{i\phi} |V\rangle \right). \tag{9.6}$$


Figure 9.2: (a) Single-photon phase estimation with the insertion of an optical parametric amplifier after the interaction with the sample. (b) Equivalent scheme in the polarization degree of freedom, where the sample introduces a relative phase shift between the $\vec{\pi}_H$ and the $\vec{\pi}_V$ polarizations. Here, losses are divided in two contributions: 1 - p are the losses before the amplifier, $1 - \eta$ are the transmission and detection losses.

Before the propagation over the transmission channel and the detection stage, the probe state $|\phi\rangle$ is injected in the OPA for the amplification process. Consider the loss before the amplifier by a parameter 1 - p. The injected mixed state takes the form:

$$\hat{\rho}^{p}_{\phi} = p |\phi\rangle \langle \phi| + (1-p) |0\rangle \langle 0|.$$
(9.7)

The obtained macrostate after the OPA $\hat{\rho}_{\phi}^{p,g} = \hat{U}_{\text{OPA}} \hat{\rho}_{\phi}^{p} \hat{U}_{\text{OPA}}^{\dagger}$ is described by the density matrix:

$$\hat{\rho}_{\phi}^{p,g} = p |\Phi^{\phi}\rangle \langle \Phi^{\phi}| + (1-p) |\Phi^{0}\rangle \langle \Phi^{0}|.$$
(9.8)

where $|\Phi^{\phi}\rangle$ is the wave function of the amplified single photon $|\phi\rangle$ state, and $|\Phi^{0}\rangle$ is the wave function of the spontaneous emission field, as described in Secs. 2.2.3 and 4.2.1. After the action of detection and transmission losses $1 - \eta$, the density matrix of the state can then be written as:

$$\hat{\rho}_{\phi}^{p,g,\eta} = p \hat{\rho}_{\phi}^{g,\eta} + (1-p) \hat{\rho}_{0}^{g,\eta}, \qquad (9.9)$$

where $\hat{\rho}_{\phi}^{g,\eta} = \mathscr{L}\left[|\Phi^{\phi}\rangle\langle\Phi^{\phi}|\right]$ and $\hat{\rho}_{0}^{g,\eta} = \mathscr{L}\left[|\Phi^{0}\rangle\langle\Phi^{0}|\right]$. Details over the calculation and expressions of the coefficients of $\hat{\rho}_{\phi}^{g,\eta}$ and $\hat{\rho}_{0}^{g,\eta}$ are reported in App. A.1.2.

9.3.1 Quantum Fisher Information for single photon amplified states at high losses

The quantum Fisher information associated to the probe states $\hat{\rho}_{\phi}^{p,g,\eta}$ after the amplification process and the losses $1 - \eta$ in the high losses regime, where at most one single photon is detected $\eta \langle \hat{n}_{\pm} \rangle \ll 1$, can be evaluated by keeping only the vacuum and single

photon contributions in the expressions (A.4-A.7) and (A.10-A.13) for the coefficients of the probe state. Then, the density matrix $\hat{\rho}_{\phi}^{p,g,\eta}$ reduces to:

$$\begin{aligned} \hat{\rho}_{\phi}^{p,g,\eta} &= (\mathscr{N}^{p,g,\eta})^{-1} \left\{ \left[\frac{p(1-\eta)}{C^2} \frac{1}{1-\Gamma^2(1-\eta)^2} + (1-p) \right] |0\rangle \langle 0| + \right. \\ &+ \frac{\eta}{1-\Gamma^2(1-\eta)^2} \left[\frac{p}{C^2} \frac{1+2\Gamma^2(1-\eta)^2}{1-\Gamma^2(1-\eta)^2} + (1-p)(1-\eta)\Gamma^2 \right] |\phi\rangle \langle \phi| + \\ &+ \frac{\eta}{1-\Gamma^2(1-\eta)^2} \left[\frac{p}{C^2} \frac{(1-\eta)^2\Gamma^2}{1-\Gamma^2(1-\eta)^2} + (1-p)(1-\eta)\Gamma^2 \right] |\phi_{\perp}\rangle \langle \phi_{\perp}| \right\}, \end{aligned}$$
(9.10)

where $\mathcal{N}^{p,g,\eta}$ is the opportune normalization constant in order to ensure the normalization condition $\text{Tr}[\hat{\rho}_{\phi}^{p,g,\eta}] = 1$. The density matrix using $x = \Gamma(1-\eta)$ reads:

$$\hat{\rho}_{\phi}^{p,g,\eta} = \frac{1}{\mathcal{N}^{p,g,\eta}} \begin{pmatrix} \frac{p(1-\eta)}{C^2} \frac{1}{1-x^2} + (1-p) & 0 & 0\\ 0 & \frac{\eta}{1-x^2} \left[\frac{p}{C^2} \frac{1+2x^2}{1-x^2} + \frac{1-p}{1-\eta} x^2 \right] & 0\\ 0 & 0 & \frac{\eta}{1-x^2} \left[\frac{p}{C^2} \frac{x^2}{1-x^2} + \frac{1-p}{1-\eta} x^2 \right] \end{pmatrix}$$
(9.11)

In Fig. 9.3 we report the plots of the matrix representation for both single photon and amplified states after the transmission over the lossy channel in the high losses regime. The advantage of the amplified scheme is the persistence, after losses, of a non-negligible $|\phi\rangle\langle\phi|$ element in the relevant density matrix [Fig. 9.3 (b)]. Such term corresponds to the effective detection of a photon and the acquisition of information over ϕ . Furthermore, in the amplified case, we can observe the appearance of a $|\phi_{\perp}\rangle\langle\phi_{\perp}|$ contribution, which is due to the quantum cloning process performed by the amplifier.

Starting from the definition, the quantum Fisher Information relative to the density matrix $\hat{\rho}_{\phi}^{p,g,\eta}$ can be evaluated as Eq. (1.66) [Par09]. In our case, the calculation are simplified since the density matrix $\hat{\rho}_{\phi}^{p,g,\eta}$ [Eq. (9.11)] is already in diagonal form. By direct application of the method we obtain:

$$\partial_{\phi}\hat{\rho}_{\phi}^{p,g,\eta} = \frac{\iota(\rho_2 - \rho_3)}{2} \big[|\phi\rangle \langle \phi_{\perp}| + |\phi_{\perp}\rangle \langle \phi| \big], \qquad (9.12)$$

where ρ_2 and ρ_3 are the eigenvalues corresponding to the eigenvectors $|\phi\rangle$ and $|\phi_{\perp}\rangle$ respectively. The evaluation of the Fisher Information according to Eq. (1.66) leads to $H_{\phi}^{p,g,\eta} = \frac{(\rho_2 - \rho_3)^2}{\rho_2 + \rho_3}$. Substituting the expression of the eigenvalues we find:

$$H_{\phi}^{p,g,\eta} = \frac{\frac{p^2}{C^4} \eta \left[\frac{1+x^2}{1-x^2}\right]^2 \left\{\frac{p}{C^2}(1+3x^2) + \frac{1-p}{1-\eta}(1-x^2)2x^2\right\}^{-1}}{\left\{\frac{p}{C^2}\frac{1}{1-x^2}\left[1+\eta\frac{4x^2}{1-x^2}\right] + (1-p)\left[1+\frac{\eta}{1-\eta}\frac{2x^2}{1-x^2}\right]\right\}}.$$
(9.13)

In absence of the amplification stage, the quantum Fisher information reduces to:

$$H^{p,g=0,\eta}_{\phi} = p\eta, \qquad (9.14)$$

which corresponds to the quantum Fisher information of a single photon after the action of an overall quantum efficiency $\eta_{tot} = p\eta$. In absence of losses before the amplification



Figure 9.3: Evolution of the density matrix of the probe for single photon and single photon amplified states in high lossy conditions: $\eta = 10^{-5}$. (a) Density matrix of a single photon state after losses. (b) Density matrix of the amplified single photon state $\hat{\rho}_{\phi}^{p,g,\eta}$ for p = 1 and g = 5.2, corresponding to $\langle \hat{n}_{\phi} \rangle \approx 24600$ and to an average number of detected photons $\eta \langle \hat{n}_{\phi} \rangle \approx 0.25$.

stage, we obtain:

$$H^{p=1,g,\eta}_{\phi} = \frac{\eta}{1+\eta \frac{4x^2}{1-x^2}} \left[1 + \frac{4x^2}{(1-x^2)(1+3x^2)} \right].$$
(9.15)

Since this expression of the quantum Fisher information is valid in the limit $\eta \langle \hat{n}_{\pm} \rangle \ll 1$, we can further simplify the expression by expanding Eq. (9.13) in power series of η and by keeping only the first order term in η . In this regime and for a high gain g such that $\bar{n} = \sinh^2 g \gg 1$, the Fisher information reduces to the following expression:

$$H^{p,g,\eta}_{\phi} = \eta p \frac{2\overline{n}}{1+p^{-1}} \stackrel{p=1}{\to} \eta \overline{n}.$$
(9.16)

9.3.2 Quantum metrology with intensity measurements

Let us now consider a specific choice of the detection apparatus and the data processing strategy for exploiting the amplified state. The output field is analyzed by measuring the value of the difference in the number of photons present in the two orthogonal polarizations $\vec{\pi}_+$ and $\vec{\pi}_-$, corresponding to the operator $\hat{D} = \hat{n}_+ - \hat{n}_-$. Finally, the value of the phase ϕ is recoverd from the recorded signal $\langle \hat{D} \rangle$.

We can now proceed with the evaluation of the sensitivity $S_{\phi}^{p,g,\eta}$ achievable with the amplifier-based protocol and the chosen detection strategy. This can be done in the Heisenberg picture, exploiting the input-output relations for the joint amplifier-lossy channel system. The time evolution maps the field operators \hat{a}_{\pm}^{\dagger} into \hat{c}_{\pm}^{\dagger} :

$$\hat{c}_{\pm}^{\dagger} = \sqrt{\eta} \left(\hat{a}_{\pm}^{\dagger} C \pm \hat{a}_{\pm} S \right) + \iota \sqrt{1 - \eta} \, \hat{b}_{\pm}^{\dagger}, \qquad (9.17)$$

where \hat{b}_{\pm}^{\dagger} are the field operators of the vacuum modeling the action of the lossy channel. In order to perform the calculation over the complete density matrix, we need to analyze the injected and the spontaneous emission components of $\hat{\rho}_{\phi}^{p,g,\eta}$ separately. For the $\hat{\rho}_{\phi}^{g,\eta}$ state (i.e. the injected component), the average number of photons in $\vec{\pi}_{+}$ and $\vec{\pi}_{-}$ polarizations are:

$$\langle \hat{n}_{+} \rangle_{\hat{\rho}_{\phi}^{g,\eta}} = \eta \left[\bar{n} + (2\bar{n}+1)\cos^{2}\left(\frac{\phi}{2}\right) \right]; \quad \langle \hat{n}_{-} \rangle_{\hat{\rho}_{\phi}^{g,\eta}} = \eta \left[\bar{n} + (2\bar{n}+1)\sin^{2}\left(\frac{\phi}{2}\right) \right]. \tag{9.18}$$

where $\overline{n} = \sinh^2 g$. For the vacuum injected component $\hat{\rho}_0^{g,\eta}$ we have:

$$\langle \hat{n}_{+} \rangle_{\hat{\rho}_{0}^{g,\eta}} = \langle \hat{n}_{-} \rangle_{\hat{\rho}_{0}^{g,\eta}} = \eta \bar{n}.$$
 (9.19)

The average value of \hat{D} calculated over $\hat{\rho}_{\phi}^{p,g,\eta}$ [Eq. (9.9)] gives, by exploiting the results of Eqs. (9.18-9.19):

$$\langle \hat{D} \rangle = p \eta \left(2\bar{n} + 1 \right) \cos \phi.$$
 (9.20)

According to phase estimation theory, the uncertainty $\delta\phi$ obtained by recording the signal $I(\phi)$ can be calculated as: $\delta\phi = \delta I \left|\frac{\partial I}{\partial\phi}\right|^{-1}$. The fluctuations on $\langle \hat{D} \rangle$ are obtained by starting from the definition $\sigma^2 = \langle \hat{D}^2 \rangle - \langle \hat{D} \rangle^2$, where $\langle \hat{D}^2 \rangle = \langle \hat{n}_+^2 \rangle + \langle \hat{n}_-^2 \rangle - 2 \langle \hat{n}_+ \hat{n}_- \rangle$, thus giving:

$$\sigma^{2} = p \left[\eta^{2} \left(12\bar{n}^{2} + 8\bar{n} \right) + \eta \left(4\bar{n} + 1 \right) - p\eta^{2} \left(2\bar{n} + 1 \right)^{2} \cos^{2} \phi \right] + (1-p) \left[\eta^{2} \left(4\bar{n}^{2} + 2\bar{n} \right) + \eta 2\bar{n} \right].$$
(9.21)

Finally to evaluate the sensitivity on the phase ϕ , it is necessary to calculate the derivative of $\langle \hat{D} \rangle$ with respect to ϕ , which reads:

$$\frac{\partial \langle \hat{D} \rangle}{\partial \phi} = -p\eta (2\overline{n} + 1) \sin \phi. \qquad (9.22)$$

Accordingly [see Eq.(9.3)] for the uncertainty over the phase ϕ , the sensitivity of this measurement scheme reads:

$$S_{\phi}^{p,g,\eta} = p\eta \left(2\bar{n}+1\right) |\sin\phi| \left\{ p \left[\eta^2 \left(12\bar{n}^2+8\bar{n}\right) + \eta \left(4\bar{n}+1\right) - p\eta^2 \left(2\bar{n}+1\right)^2 \cos^2(\phi) \right] + \left(1-p\right) \left[\eta^2 \left(4\bar{n}^2+2\bar{n}\right) + \eta 2\bar{n} \right] \right\}^{\frac{1}{2}}.$$
(9.23)

The maximum of the sensitivity is reached in the inflection point of $\langle \hat{D} \rangle$, corresponding to $\phi = \pi/2$, leading to:

$$S_{\text{ampl}} = S_{\phi=\pi/2}^{p,g,\eta} = \frac{p(2\bar{n}+1)}{\{[p(8\bar{n}^2+6\bar{n})+4\bar{n}^2+2\bar{n}]+\eta^{-1}[p(2\bar{n}+1)+2\bar{n}]\}^{\frac{1}{2}}}.$$
 (9.24)

The ideal case with no losses before amplification (p = 1) gives the maximum achievable sensitivity:

$$S_{\text{ampl}}^{p=1} = \frac{\eta(2\bar{n}+1)}{\left[\eta(12\bar{n}^2 + 8\bar{n}) + 4\bar{n} + 1\right]^{1/2}}.$$
(9.25)

Since the maximum of the sensitivity is obtained for $\phi = \pi/2$, when the phase shift is unknown an adaptive protocol is obtained to maximize the performances of the scheme. In App. E.3 we show that it is sufficient to use a simple two-stage strategy in which we first find a rough estimate of the phase ϕ_{est} employing conventional phase estimation methods, and then we use it to tune the zero-reference so that our scheme operates at its optimal working point detailed above. We also show that the resources employed in the first stage of this adaptive strategy are asymptotically negligible with respect to the resources employed in the second high-resolution stage. We will thus neglect the first stage in the following analysis.

The advantage of the amplification strategy can be evaluated by comparing this sensitivity with the one achievable without amplification. The latter corresponds to the case of single-photon phase estimation, analyzed in Sec. 9.2, with an overall transmission and detection efficiency $\eta_{\text{tot}} = \eta p$:

$$S_{1\text{phot}} = \sqrt{\eta p}.\tag{9.26}$$

We introduce the enhancement *E* as the merit figure for this analysis:

$$E = \left(\frac{S_{\text{ampl}}}{S_{1\text{phot}}}\right)^2. \tag{9.27}$$

E represents the reduction factor in the number of photons that would be sent onto the sample in order to obtain the same information on the phase ϕ , by exploiting the amplification strategy with respect to the single-photon probe scheme. In other words, this quantity represents the number of supplementary trials $\overline{N} = E$ necessary for the single-photon scheme to equal the sensitivity of the amplified one.

High losses regime

The high losses regime corresponds to the condition where, for the single-photon phase estimation protocol, most of the photons are not detected. In Fig. 9.4 we report the enhancement *E* as a function of the nonlinear gain *g* and of the efficiency η for a value of p = 0.5. We observe that a substantial enhancement of ~ 100 can be achieved with an efficiency $\eta \sim 10^{-3}$, showing the potential of our approach.



Figure 9.4: Enhancement of the amplified strategy as a function of the nonlinear gain g and of the efficiency η , analyzed in the high losses regime. The value of losses between the phase shifter and the amplifier is set as p = 0.5.

To further characterize the high losses regime, we investigate the limit where $\eta \langle n_{\pm} \rangle \ll$ 1, i.e. where at most only one photon is transmitted by the channel. In absence of losses before the amplification stage, the sensitivity reduces to:

$$S_{\text{ampl}}^{p=1} \to \sqrt{\eta \overline{n}},$$
 (9.28)

and the enhancement becomes:

$$E^{p=1} \to \overline{n}.\tag{9.29}$$

When losses 1 - p between the sample and the amplifier are introduced, the sensitivity reduces to:

$$S_{\text{ampl}} \rightarrow \sqrt{\eta p} \sqrt{\frac{2\overline{n}+1}{1+p^{-1}}},$$

$$(9.30)$$

and the enhancement to:

$$E = \left(\frac{S_{\text{ampl}}}{S_{1\text{phot}}}\right)^2 = \frac{2\overline{n}}{1+p^{-1}}.$$
(9.31)

We note that in this limit the enhancement does not depend any more on the efficiency of the detection stage but only on the gain g and from the losses 1 - p.

The squared sensitivity of this scheme, calculated in Eq. (9.24), reduces for $\eta \langle n_{\pm} \rangle \ll 1$ to $S^2 = \frac{1}{(\delta \phi)^2} \to H_{\phi}^{p,g,\eta}$ (i.e. it equals the quantum Fisher information). This means that, in the condition where at most a single photon is detected, the proposed measurement strategy is optimal. Measuring $\langle \hat{D} \rangle$ allows us to extract the maximum amount of information achievable with the probe states $\hat{\rho}_{\phi}^{p,g,\eta}$.

High gain limit

We conclude our analysis by investigating the high gain limit, where the number of photons generated by the amplifier satisfies the condition $\langle \hat{n} \rangle \gg 1$. For large values of the gain g the enhancement saturates to the value: $E_{\text{lim}} = \frac{p}{\eta(2p+1)}$. The trend of E_{lim} as a function of the efficiencies η and p is reported in the contour plot of Fig. 9.5-(a). According to this result, we can then identify a critical value of p above which the enhancement is greater than 1: $p_{\text{crit}} = \frac{\eta}{1-2\eta}$. As shown in Fig. 9.5-(b), the region where E > 1 corresponds to the condition where losses $1 - \eta$ at the detection and trasmission stage are greater than losses 1 - p before the amplifier. This analysis demonstrates that the amplifier can efficiently compensate for the loss of information that occurs after the OPA, while it cannot recover the amount of information lost before the amplification stage. In conclusion, the amplifier based strategy can lead to an effective enhancement when $\eta < \frac{p}{2p+1}$, while for high transmission and detection efficiency $\eta \ge 0.33$ no enhancement can be achieved by exploiting our amplification strategy.



Figure 9.5: (a) Contour plot of the enhancement *E* in the high gain limit as a function of the losses parameters 1 - p and $1 - \eta$ reported in a logarithmic scale. (b) Trend of the critical value p_{crit} of losses before the amplification for which E > 1 as a function of the detection efficiency.

9.3.3 Phase estimation through an orthogonality filter

In this section we investigate an alternative measurement strategy apt to obtain an enhancement in the resilience to losses of the interferometric estimation of an unknown phase. The presented method is based on a dichotomic threshold detection performed via the orthogonality filter device discussed in Sec. 5.2.3. This operation permits to increase the visibility of the amplified signal at the cost of a lower detection rate.

Due to the properties of the multiphoton states, the use of an OF would allow to extract information on the state from the shape of the photon-number distributions. The action of the O-Filter on any input density matrix can be described by the measurement operator $\hat{F}_{\pi,\pi_{\perp}}^{(\pm 1)}$ reported in Sec. 5.2.3. The interference fringe pattern can then be evaluated as:

$$I_{\rm OF}^{(\pm 1)}(\phi,k) = \left\langle \hat{F}_{\vec{\pi}_+,\vec{\pi}_-}^{(\pm 1)}(k) \right\rangle_{\hat{\rho}_{\phi}^{p,g,\eta}} = \operatorname{Tr}\left[\hat{\rho}_{\phi}^{p,g,\eta} \, \hat{F}_{\vec{\pi}_+,\vec{\pi}_-}^{(\pm 1)}(k) \right]. \tag{9.32}$$

Since the fringe pattern in presence of losses presents a cosinusoidal form with nonunitary visibility, we can write:

$$I_{\rm OF}^{(+1)}(\phi,k) = [I_{\rm max}(k) - I_{\rm min}(k)]\cos^2\left(\frac{\phi}{2}\right) + I_{\rm min}(k), \qquad (9.33)$$

$$I_{\rm OF}^{(-1)}(\varphi,k) = [I_{\rm max}(k) - I_{\rm min}(k)]\sin^2\left(\frac{\phi}{2}\right) + I_{\rm min}(k).$$
(9.34)

The maximum I_{max} and the minimum I_{min} of the fringes can be evaluated as:

$$I_{\max}(k) = \left\langle \hat{F}_{\vec{\pi}_{+},\vec{\pi}_{-}}^{(+1)}(k) \right\rangle_{\hat{\rho}_{+}^{p,g,\eta}}, \quad I_{\min}(k) = \left\langle \hat{F}_{\vec{\pi}_{+},\vec{\pi}_{-}}^{(-1)}(k) \right\rangle_{\hat{\rho}_{+}^{p,g,\eta}}.$$
(9.35)

where $\hat{\rho}^{p,g,\eta}_+$ corresponds to $\phi = 0$. In a phase estimation experiment, the value of an unknown phase shift ϕ is retrieved by measuring the signals $I^{(\pm 1)}(\phi, \bar{k})$ for a chosen value of \bar{k} , with sets the amount of filtering performed on the output state. Hence, the value of ϕ is obtained by applying an appropriate data processing on the experimental data, such as a Bayesian or a maximum-likelihood estimator [Hel76]. The visibility V(k) and the average signal $R_{\text{mean}}(k)$ of the fringe pattern are then defined by:

$$V(k) = \frac{I_{\max}(k) - I_{\min}(k)}{I_{\max}(k) + I_{\min}(k)}; \quad R_{\text{mean}}(k) = \frac{I_{\max}(k) + I_{\min}(k)}{2}.$$
 (9.36)

We report the trends of the visibilities [Fig. 9.6 (a)] and of the average signal [Fig. 9.6 (b)], for $\eta = 10^{-3}$ and different values of the non linear gain of the amplifier. We note that a visibility almost close to 1 can be obtained with a sufficient filtering threshold. As the gain is increased, the number of transmitted photons $\eta \langle \hat{n}_{\pm} \rangle$ becomes sufficient to detect all the *N* repeated trials. The action of the amplifier is then to compensate the effect of losses η by generating an high average number of photons. In the high lossy regime, at variance with the single-photon case, all pulses can be exploited to extract information about the phase ϕ . The action of the OF is then to select those events which can be discriminated with higher fidelity, leading to an increase in the visibility value. The latter operation is achieved at the cost of discarding a part of the data. This can be seen as an effective quantum efficiency of the scheme $\overline{\eta} = R_{\text{mean}}(k)$.

The sensitivity achievable with the OF-based strategy [see Eq. (9.3)] for the (+1) output signal of the OF reads:

$$S_{\rm OF}^{(+1)}(k) = \frac{1}{\delta\phi_{\rm OF}^{(+1)}(k)} = \frac{|\sin\phi (I_{\rm max}(k) - I_{\rm min}(k))|}{\left[(I_{\rm max}(k) - I_{\rm min}(k))\cos^2\frac{\phi}{2} + I_{\rm min}(k) \right]^{1/2}}.$$
(9.37)

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Figure 9.6: (a) Plot of the visibility as a function of the threshold k for fixed $\eta = 10^{-3}$ and different values of the gain. (b) Plot of the average detected signal, i.e. the filtering probability, as a function of the threshold k of the OF for $\eta = 10^{-3}$ and different values of the gain. The curves correspond to: g = 5 (black solid line), g = 4.5 (red dash-dotted line), g = 4 (blue dotted line), g = 3.5 (green dashed line).

The maximum of this quantity is obtained for $\phi = \frac{\pi}{2}$. An analogous result holds for the (-1) output signal of the OF. In conclusion, the sensitivity of the OF-based strategy in $\phi = \frac{\pi}{2}$ can be put in the form:

$$S_{\rm OF}(k) = \frac{1}{\delta\phi_{\rm OF}(k)} = V(k)\sqrt{R_{\rm mean}(k)}.$$
(9.38)

The average over *M* repeated experiments gives an improvement of \sqrt{M} : $S_{OF}^{M}(k) = S_{OF}(k)\sqrt{M}$. This expression shows that S_{OF} does not depend on the efficiency η , but only on the average filtered signal R_{mean} . The enhancement *E* [see Eq. (9.27)] in this case is:

$$E(k) = \left(\frac{S_{\rm OF}(k)}{S_{\rm 1phot}(k)}\right)^2 = V(k)^2 \frac{R_{\rm mean}(k)}{\eta}.$$
(9.39)

E is reported in Fig. 9.7 as a function of the threshold *k* of the O-Filter for an efficiency $\eta = 10^{-3}$. We note that a significant enhancement up to a value of ~ 200 can be achieved with the OF-based strategy. Indeed, the advantage of the QIOPA strategy is more evident in the high lossy regime. For low values of η , in the single-photon regime most of the pulses are not detected, thus degrading the quality of the estimation process. In the amplified case, the multiphoton field can survive the action of losses thus leading to a significant increase in the detected signal. Note that, since the maximum enhancement is achieved for $\phi = \pi/2$, in order to measure an unknown phase-shift an adaptive protocol is necessary [Nag88].



Figure 9.7: Trend of the enhancement *E* as a function of the threshold *k* for a detection efficiency $\eta = 10^{-3}$ for different values of the gain: g = 5 (black solid line), g = 4.5 (red dash-dotted line), g = 4 (blue dotted line), g = 3.5 (green dashed line).

9.4 Experimental implementation of the protocol

The above discussed protocol have been implemented in the high losses regime. As shown, in such highly detrimental condition, the amplifier-based strategy can lead to a significant enhancement in the performances of the phase estimation.

9.4.1 Experimental apparatus

The complete scheme is reported in Fig. 9.8. The laser system consists in a Ti:Sa modelocked MIRA900, pumped by a Verdi V5 Nd:Yag solid state laser. The output beam from the MIRA900 is injected into the Ti:Sa REGA9000 amplifier, pumped by a Verdi V18. The complete laser system allows to obtain a 1.5 W output beam at wavelength $\lambda = 795$ nm, that, after a second harmonic generation process, generates the pump beam at $\lambda_p = 397.5$ nm of power P = 750 mW. The pump beam is split between modes \mathbf{k}_p and \mathbf{k}'_{p} and sent to two nonlinear crystals C1 and C2. C1 acts as an entangled photon source (see Sec. 2.1.3). This source allows to conditionally prepare upon detection on mode \mathbf{k}_T a single photon state: $|+\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle)$ on mode \mathbf{k}_1 . This photon is then sent as a probe into the sample, which in our case consisted of a Babinet-Soleil compensator $[B(\phi)]$ which introduces a tunable birefringent phase shift between the $\vec{\pi}_H$ and the $\vec{\pi}_V$ polarization components. The information on the phase shift ϕ is then encoded in the polarization state $\vec{\pi}_{\phi} = 2^{-1/2} \left(\vec{\pi}_H + e^{i\phi} \vec{\pi}_V \right)$ of the single photon: $|\phi\rangle = \frac{1}{\sqrt{2}} \left(|H\rangle + e^{i\phi} |V\rangle \right)$. The probe state after the interaction is then superimposed spatially and temporally with the pump on mode \mathbf{k}'_{p} into the second crystal C2, working as an optical parametric amplifier in a collinear regime. Spatial and temporal matching between the two fields are obtained through an adjustable delay line (Z) and a dichroic mirror. The number of photons generated in the amplification process, as said, depends exponentially on the nonlinear gain of the amplifier g. The maximum value of g experimentally used is $g_{max} = 4.5$. Finally, after the amplification process the amplified field is filtered in frequency, coupled into a single mode fiber and sent to the detection stage where the information on the phase ϕ is retrieved.



Figure 9.8: Experimental setup of the QIOPA based interferometric scheme. The pump beam is split between modes \mathbf{k}_p and \mathbf{k}'_p . In the first BBO crystal (C1), the single photon probe in the $|+\rangle$ state on mode \mathbf{k}_1 is prepared upon detection on mode \mathbf{k}_T with a singlephoton SPCM-AQR14 detector (D_T) . Then, the phase ϕ is introduced through a Babinet-Soleil compensator $[B(\phi)]$ and the probe state $|\phi\rangle$ after the interaction is then injected in the QI-OPA and superimposed with the pump beam on mode \mathbf{k}'_p . (a) Measurement scheme based on detection of the average of the photon number difference $\hat{D} = \hat{n}_+ - \hat{n}_-$. (b) O-Filter based detection scheme.

9.4.2 Photon counting measurements

The phase ϕ , in the photon counting approach, is retrieved by measuring the photon number difference operator $\langle \hat{D} \rangle (\phi)$ [see Eq. (9.20)]. More specifically, after the amplification process the output field is filtered in frequency and coupled into a single-mode fiber. At the output of the fiber, after polarization compensation the field is attenuated up to the single-photon level, analyzed in polarization and detected by two single photon counting modules SPCM-AQR14, D_1 and D_1^* in Fig. 9.8 (a). The resulting signals, triggered by the click of detector $D_T(D_T^*)$ on mode \mathbf{k}_T , are hence subtracted and the difference in the number of photons \hat{D} is recorded as a function of the phase ϕ , varied by the Babinet-Soleil compensator $B(\phi)$ on the probe path. The fringe patterns, vs ϕ , recorted at the detector D_1 (D_1^*) are reported in Fig. 9.9 for the single-photon probe [Fig. 9.9 (a)] and the amplified state [Fig. 9.9 (b)]. In the first case the obtained visibility is $V_{1\text{phot}} = 0.45$, whose value differs from the expected unitary one since, in the employed high pump power regime, the first nonlinear crystal has a non-negligible probability to generate more than a single photon pair per pulse. This seed visibility value and the multimode operation of the OPA are also responsible for a reduction of the amplified state visibility up to $V_{\text{ampl}}(g_{\text{max}}) = 0.08$, and has been taken into account for analyzing the experimental results. The reduced visibility value in the amplified case is balanced by the increase of the detected signal with respect to the single photon one. The trend of the enhancement as a function of the nonlin-



Figure 9.9: Experimental fringe pattern for the single photon probe (a) and for the amplified beam (b).



Figure 9.10: Experimental results of the enhancement *E* versus the non-linear gain. Continuous line: theoretical prediction for the expected enhancement with $\eta = 3 \times 10^{-4}$, p = 0.15

ear gain of the amplifier is reported in Fig. 9.10 compared with the theoretical prediction for $\eta = 3 \times 10^{-4}$ and p = 0.15, leading to an experimental enhancement up to a factor ~ 200 .

9.4.3 Orthogonality-filter measurement

The OF based strategy has been experimentally implemented by adopting at the detection stage two photomultipliers, PM_1 and PM_1^* , as shown in Fig. 9.8 (b). The two intensity signals, proportional to the number of photons with two orthogonal polarization states, are compared shot-by-shot by an electronic discriminator that gives two TTL signals, which are taken into account in coincidence with the simultaneous click of the detector on trigger mode (D_T or D_T^*). By increasing the discrimination threshold of the OF we can achieve an higher visibility of the amplified field with respect to the one obtained through the SPCM based strategy. In Fig. 9.11 is reported the fringe pattern obtained with a fraction of detected signal equal to $R_{\text{mean}}(k) \sim 3.6 \times 10^{-4}$, with resulting visibility $V \sim 0.53$. By comparing the fringe pattern in Fig. 9.11 with the one in Fig. 9.9 (b), we observe that the amplified field visibility is increased by a factor \sim 7. The trend of



Figure 9.11: Experimental fringe pattern for a fraction of detected signal equal to $R_{\text{mean}}(k) \sim 3.6 \times 10^{-4}$, the obtained visibility is $V \sim 0.53$. The solid lines correspond to a sinusoidal fit of the experimental data.

visibility as a function of the percentage of detected pulses is reported in Fig. 9.12 (a), compared with the theoretical prediction for a value of losses p = 0.14 and detection efficiency $\eta = 0.005$. The effect of the orthogonality filter is an increase in the fringe



Figure 9.12: (a) Experimental trend of visibility as a function of the percentage of filtered signal. (b) Experimental results of the enhancement *E* versus the signal rate. (a-b) The continuous lines report the theoretical prediction for p = 0.14, $\eta = 0.005$.

pattern visibility when the filtering threshold becomes tighter. The value of the achieved detection efficiency is related with the amount of losses introduced by spectral filtering, spatial filtering, transmission losses in the optical fiber and detection efficiency of the photomultiplier. In Fig. 9.12 (b) we report the experimental enhancement *E* as a function of the signal rate R_{mean} . We observe that an enhancement up to a value ~ 16 can be achieved with the OF-based strategy in this losses regime.

9.5 Conclusions and perspectives

In the optical sensing context, the adoption of quantum resources can lead to a significant enhancement in estimating an unkwnown parameter, as for instance an optical phase. Hence, the ability to generate suitable quantum light probes and quantum detection strategies is a crucial prerequisite for the operation of any quantum sensor. However, the adoption of quantum resources renders such protocols extremely sensitive to losses, unavoidable in any experimental implementation, and in general to noise processes. Recently, much attention has been devoted to investigate which are the best possible strategies in a noisy environment. The optimal probes maximizing the sensitivity and the performance of the sensors can be theoretically determined, but the resulting quantum states are often very complicated, difficult to generate, extremely sensitive to losses and noise, and they require schemes involving some post-selection at the measurement stage.

In this chapter, we described the experimental implementation of a simple conceptual strategy, which involves single photons as probe states and that can be engineered with the existing quantum-optics technology. Such strategy is based on the adoption of an optical parametric amplifier placed after the probe-sample interaction and before the main losses occur. Our results show that a large sensitivity improvement can be achieved even if after the interaction of the probe with the sample the signal is affected by high losses. The enhancement is proportional to the number of photons generated by the amplification process, and it can be tuned by increasing the nonlinear gain of the sample. For this reason, the present strategy can be adopted when the sample to be measured results to be extremely fragile with respect to the intensity of the impinging signal. The adoption of the amplifier-based strategy allowed us to achieve an experimental enhancement up to a value ~ 200 in the high losses regime.

As a further perspective, this strategy can be adopted with different classes of probe states, such as for example coherent states. In the following chapter we describe a thorough theoretical analysis and the experimental implementation of the amplifier-based phase estimation protocol when coherent states of the electromagnetic field are adopted as the probe states.

Chapter 10

Achieving the quantum Cramer-Rao bound in coherent states phase estimation with noisy detectors by parametric amplification

In the previous chapter we proposed and realized experimentally a simple strategy based on the process of parametric amplification to increase the resolution of a single-photon phase estimation protocol in presence of detection losses. Here we extend the strategy analyzed in Chap. 9 by considering a coherent probe state. We show theoretically that the amplification-based strategy can achieve the Cramer-Rao bound of the lossless coherent probe state, thus efficiently protecting the information encoded on the phase. Finally, we perform the experimental implementation of the protocol in the high losses regime without needing any post-selection of the data, showing the advantage of the amplifierbased strategy in this detrimental condition. The results are reported in Ref. [SVL⁺11], and can lead to the application of the optical parametric amplifier to obtain quantum enhanced protocols in a lossy scenario.

10.1 Interferometry with noisy detection apparatuses

The proposed scheme employs a simple, conventional interferometric phase sensing stage that uses coherent-state probes [SVP⁺11]. These are amplified with an optical parametric amplifier after the interaction with the sample, but before the lossy detectors. No post-selection is employed to filter [GLM11, RPP⁺07] the output signal. The OPA transfers the properties of the injected state into a field with a larger number of particles, robust under losses and decoherence [SVD⁺09].

The present chapter is organized as follows. In Sec. 10.2 we derive the fundamental bounds of phase estimation protocols involving coherent probe states. Then, in Sec. 10.3 we describe and analyze theoretically the proposed scheme based on the process of optical parametric amplification. Finally, we consider a specific protocol based on the measurement of the photon-number difference, and we show that in a wide parameter range the proposed strategy permits to achieve asymptotically the quantum Cramer-Rao of the probe state with unitary detection efficiency. In Sec. 10.4 we present the experimental implementation of the amplifier-based coherent states protocol in the high losses regime.

10.2 Phase estimation with coherent states

The general scheme is shown in Fig.10.1, where a specific choice of the measurement apparatus has been performed. The input state is a coherent state with $\vec{\pi}_+$ polarization: $|\Psi^{\alpha}\rangle = |\alpha\rangle_+ \otimes |0\rangle_-$. The phase shift ϕ to be measured is introduced between the $\vec{\pi}_H$ and $\vec{\pi}_V$, transforming the probe state into: $|\Psi^{\alpha}_{\phi}\rangle = |\alpha e^{-i\phi/2}\cos(\phi/2)\rangle_+ \otimes |i\alpha e^{-i\phi/2}\sin(\phi/2)\rangle_-$. In all implementations, a certain amount of losses $1 - \xi$ is present in the interferometric stage. This effect can be included in the analysis by inserting a damping factor $\sqrt{\xi}$ in the coherent state amplitude of the output state. Finally the output state of the interferometer reads:

$$|\Psi_{\phi}^{\alpha,\xi}\rangle = |\alpha\sqrt{\xi}e^{-\iota\phi/2}\cos(\phi/2)\rangle_{+} \otimes |\iota\alpha\sqrt{\xi}e^{-\iota\phi/2}\sin(\phi/2)\rangle_{-}.$$
 (10.1)



Figure 10.1: Scheme of a phase estimation apparatus exploiting an input coherent probe states.

10.2.1 Quantum Cramer-Rao bound with coherent states

The calculation of the quantum Fisher information $H_{SQL}(\alpha, \xi)$ can be performed by writing the probe state $|\tilde{\Psi}_{\phi}^{\alpha,\xi}\rangle$ at the sample output in the $\{\vec{\pi}_H, \vec{\pi}_V\}$ polarization basis, which reads:

$$|\tilde{\Psi}^{\alpha,\xi}_{\phi}\rangle = e^{-\imath \hat{n}_V \phi} |\frac{\alpha \sqrt{\xi}}{\sqrt{2}}\rangle_H \otimes |\frac{\alpha \sqrt{\xi}}{\sqrt{2}}\rangle_V.$$
(10.2)

where $\hat{n}_V = \hat{a}_V^{\dagger} \hat{a}_V$ is the phase-shift generator for the $\vec{\pi}_V$ polarization mode. H_{SQL} can be evaluated as the variance of the phase-shift generator \hat{n}_V on the state $|\tilde{\Psi}_{\phi}^{\alpha,\xi}\rangle$ according to:

$$H_{\text{SQL}}(\alpha,\xi) = 4 \langle \tilde{\Psi}_{\phi}^{\alpha,\xi} | \delta^2 \hat{n}_V | \tilde{\Psi}_{\phi}^{\alpha,\xi} \rangle.$$
(10.3)

By explicitly performing the calculation we find:

$$H_{\rm SQL} = 2|\alpha|^2 \xi. \tag{10.4}$$

Such quantity represents the standard quantum limit, that is, the ultimate precision achievable by optimizing over all possible measurements and data processing strategies expressed by the quantum Cramer-Rao bound:

$$\delta^2 \phi_{\text{SQL}} \ge \frac{1}{MH_{\text{SQL}}}.$$
(10.5)

where M is the number of repeated experiments.

In the presence of detection losses, the quantum Fisher information can be evaluated with the same procedure and it reads:

$$H_{\rm SOL}^{\eta} = 2|\alpha|^2 \xi \eta. \tag{10.6}$$

10.2.2 Classical Fisher information with photon-counting measurements

The classical Fisher information for coherent states phase estimation and photon-counting measurements is calculated in the following way. We use the formulation for the probe state before the detection apparatus given in Eq. (10.1). The measurement operators describing photon-counting detection are the projectors over the Fock states:

$$\hat{\Pi}_{n^{(+)},n^{(-)}} = \hat{\Pi}_{n^{(+)}}^{(+)} \otimes \hat{\Pi}_{n^{(-)}}^{(-)},$$
(10.7)

where $\hat{\Pi}_{n^{(l)}}^{(l)} = |n^{(l)}\rangle_{ll} \langle n^{(l)}|$, with l = +, - labeling the optical mode. The probability distribution of the measurement outcomes can be evaluated as:

$$p(n^{(+)}, n^{(-)}|\phi) = \langle \Psi_{\phi}^{\alpha, \xi} | \hat{\Pi}_{n^{(+)}, n^{(-)}} | \Psi_{\phi}^{\alpha, \xi} \rangle.$$
(10.8)

Since the probe state and the measurement operator are separable with respect to the two optical modes, the probability distribution of the measurement outcomes factorizes according to:

$$p(n^{(+)}, n^{(-)} | \phi) = \prod_{l=+,-} p(n^{(l)} | \phi), \qquad (10.9)$$

where:

$$p(n^{(l)}|\phi) = {}_{l}\langle\beta_{l}|\hat{\Pi}_{n^{(l)}}^{(l)}|\beta_{l}\rangle_{l}.$$
(10.10)

The two distributions $p(n^{(l)}|\phi)$ can be evaluated separately, leading to:

$$p(n^{(+)}|\phi) = e^{-|\alpha|^2 \xi \cos^2(\phi/2)} \frac{(|\alpha|^2 \xi \cos^2(\phi/2))^n}{n!}, \qquad (10.11)$$

$$p(n^{(-)}|\phi) = e^{-|\alpha|^2 \xi \sin^2(\phi/2)} \frac{(|\alpha|^2 \xi \sin^2(\phi/2))^n}{n!}, \qquad (10.12)$$

and the two mode distribution reads:

$$p(n^{(+)}, n^{(-)}|\phi) = e^{-|\alpha|^2 \xi} \frac{(|\alpha|^2 \xi)^{n+m}}{n!m!} [\cos^2(\phi/2)]^n [\sin^2(\phi/2)]^m.$$
(10.13)

The classical Fisher information $I_{\rm coh}$ can be evaluated from its definition:

$$I_{\rm coh} = \sum_{n,m=0}^{\infty} \frac{[\partial_{\phi} p(n^{(1)}, n^{(2)} | \phi)]^2}{p(n^{(1)}, n^{(2)} | \phi)}.$$
 (10.14)

By explicitly evaluating the derivative and by replacing the obtained expressions in the definition (10.14) of the classical Fisher information we obtain:

$$I_{\rm coh} = |\alpha|^2 \xi. \tag{10.15}$$

We note that, by exploiting photon-counting measurements the quantum Fisher information cannot be saturated due to a constant factor 2. In presence of the detection losses η , the amplitude of the coherent state is rescaled by a factor $\sqrt{\eta}$ and the classical Fisher information reads:

$$I_{\rm coh}^{\eta} = |\alpha|^2 \xi \eta. \tag{10.16}$$

10.3 Phase estimation with noisy detectors by parametric amplification

In order to protect the probe state from the action of detection losses $1 - \eta$ that occur at the measurement stage, the amplifier-based strategy proceeds as follows. Before the amplification, a relative phase-shift of $\pi/2$ is inserted between the + and the – polarization components by means of a $\lambda/4$ birefringent waveplate with optical axis oriented at 45°, leading to:

$$|e^{-\iota\phi/2}\beta\cos(\phi/2)\rangle_{+}| - e^{-\iota\phi/2}\beta\sin(\phi/2)\rangle_{-}.$$
(10.17)

The resulting state is then injected in the optical parametric amplifier:

$$\hat{\mathscr{H}}_{\text{OPA}} = \imath \hbar \chi \left(\hat{a}_{H}^{\dagger} \hat{a}_{V}^{\dagger} \right) + \text{h.c.} = \imath \hbar \chi \left(\hat{a}_{+}^{\dagger 2} - \hat{a}_{-}^{\dagger 2} \right) / 2 + \text{h.c.}, \quad (10.18)$$

where $\hat{a}_{\pm} = (\hat{a}_H \pm \hat{a}_V)/\sqrt{2}$, and χ is the parameter that quantifies the strength of the interaction. It corresponds to a unitary operation:

$$\hat{U}_{\text{OPA}} = \exp[\tau(\hat{a}_{+}^{\dagger 2} - \hat{a}_{-}^{\dagger 2})/2 + \text{h.c.}], \qquad (10.19)$$



Figure 10.2: Scheme of the amplifier based protocol. The two red lines represent the polarization modes $\vec{\pi}_+$ and $\vec{\pi}_-$; \hat{U}_{ϕ} introduces the relative phase ϕ on the state; \mathscr{L}_{ξ} and \mathscr{L}_{η} are the loss transformations for the sample (which transforms the probe state to $|\Psi_{\phi}^{\beta}\rangle$) and detector loss respectively; the blue box is a $\lambda/4$ plate and the orange box represents the optical parametric amplifier; the black and the gray boxes represent the detection and data-processing. The quantum Fisher information H_{ampl} is achievable optimizing over the detection and the data-processing; the Fisher information I_{ampl} is achievable optimizing over the data-processing; the sensitivity S is what is achieved for a given detection and data-processing.

where $\tau = g e^{i\lambda} = \chi t$ is the amplifier gain (*t* being the interaction time). From Eq. (10.19), it is clear that the NOPA (a two-mode squeezer) is equivalent to two single-mode squeezers acting independently on the modes + and - with opposite phases, namely $\hat{U}_{\text{OPA}} = \hat{S}_{+}(-\tau) \otimes \hat{S}_{-}(\tau)$, where $\hat{S}_{l}(\tau) \equiv \exp[-\tau \hat{a}_{l}^{\dagger 2}/2 + \text{H.c.}]$, with l = +, -.

After the amplification, the state has evolved to $|\Psi_{\phi}^{\beta,g}\rangle = \hat{U}_{\text{OPA}}|\Psi_{\phi}^{\beta}\rangle$. The latter is detected by lossy detectors, parametrized by a quantum efficiency η . These are equivalent to perfect detectors preceded by a loss map \mathscr{L}_{η} . The action of this map on the state $|\Psi_{\phi}^{\beta,g}\rangle$ produces the mixed state:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} \equiv \mathscr{L}_{\eta}[|\Psi_{\phi}^{\beta,g}\rangle\langle\Psi_{\phi}^{\beta,g}|].$$
(10.20)

The output state is then detected to extract the available information on the phase ϕ by measuring the photon-number difference in the two orthogonal $\vec{\pi}_{\pm}$ polarization components.

10.3.1 Quantum Fisher information of the protocol

The explicit form of the density matrix of the probe state is:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \mathscr{L}_{\eta} \Big\{ \hat{S}_{+}(\tau_{+}) \hat{S}_{-}(\tau_{-}) \mathscr{L}_{\xi} \Big[\hat{D}_{+}(\alpha_{+}) \hat{D}_{-}(\alpha_{-}) |0\rangle \langle 0| \\ \hat{D}_{+}^{\dagger}(\alpha_{+}) \hat{D}_{-}^{\dagger}(\alpha_{-}) \Big] \hat{S}_{+}^{\dagger}(\tau_{+}) \hat{S}_{-}^{\dagger}(\tau_{-}) \Big\},$$
(10.21)

where $\hat{D}(\alpha)$ are the displacement operators. As shown in App. E.1, by exploiting some operatorial relations involving gaussian states the matrix $\hat{\rho}_{\phi}^{\beta,g,\eta}$ can be expressed as:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{D}_{+}(\tilde{\gamma}_{+})\hat{D}_{-}(\tilde{\gamma}_{-})\hat{S}_{+}(\tau_{+}^{\text{eff}})\hat{S}_{-}(\tau_{-}^{\text{eff}})\Big[\hat{\rho}_{+}^{th}(N_{\text{eff}})\otimes \hat{\rho}_{-}^{th}(N_{\text{eff}})\Big]\hat{S}_{+}^{\dagger}(\tau_{+}^{\text{eff}})\hat{S}_{-}^{\dagger}(\tau_{-}^{\text{eff}})\hat{D}_{+}^{\dagger}(\tilde{\gamma}_{+})\hat{D}_{-}^{\dagger}(\tilde{\gamma}_{-}).$$
(10.22)

Here, $\hat{\rho}_l^{\text{th}}$ are single-mode thermal states, while the expressions for the state parameters $\tilde{\gamma}_l$, N_l^{eff} and τ_l^{eff} can be found in App. E.1. The density matrix can be then separated in the two-single mode contributions $\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{\rho}_{\phi}^{(+)} \otimes \hat{\rho}_{\phi}^{(-)}$, where:

$$\hat{\boldsymbol{\rho}}_{\phi}^{(l)} = \hat{D}_{l}(\tilde{\boldsymbol{\gamma}}_{l})\hat{S}_{l}(\tau_{l}^{\text{eff}})\hat{\boldsymbol{\rho}}_{l}^{th}(N_{l}^{\text{eff}})\hat{S}_{l}^{\dagger}(\tau_{l}^{\text{eff}})\hat{D}_{l}^{\dagger}(\tilde{\boldsymbol{\gamma}}_{l}), \qquad (10.23)$$

with l = +, -. The quantum Fisher information can be evaluated starting from its definition of Eq. (1.66):

$$H(\alpha,\xi,\{g_l\},\{\lambda_l\},\eta) = \sum_{p,q=0}^{\infty} \frac{(\partial_{\phi} \rho_{p,q})^2}{\rho_{p,q}} + 2\sum_{i,j,m,n=0}^{\infty} \varepsilon_{i,j,m,n} |\langle \Psi_{i,j}| \partial_{\phi} \Psi_{m,n} \rangle|^2, \quad (10.24)$$

where g_l are the squeezing modulus for the two squeezers $l = +, -, \lambda_l$ are the squeezing phases, and:

$$\varepsilon_{i,j,m,n} = \frac{(\rho_{i,j} - \rho_{m,n})^2}{\rho_{i,j} + \rho_{m,n}}.$$
 (10.25)

Here $\rho_{p,q}$ and $|\Psi_{i,j}\rangle$ are respectively the eigenvalues and the eigenvectors of $\hat{\rho}_{\phi}^{\beta,g,\eta}$. These quantities can be obtained from the corresponding terms in the single mode $\hat{\rho}_{\phi}^{(l)}$ matrices, leading to:

$$\rho_{m,n} = \rho_m^{(+)} \rho_n^{(-)},$$
(10.26)

$$|\Psi_{m,n}\rangle = |\psi_m^{(+)}\rangle_+ \otimes |\psi_n^{(-)}\rangle_-, \qquad (10.27)$$

$$\rho_n^{(l)} = \frac{(N_l^{\text{eff}})^n}{(1+N_l^{\text{eff}})^{n+1}},$$
(10.28)

$$|\psi_n^{(l)}\rangle_l = \hat{D}_l(\tilde{\gamma}_l)\hat{S}_l(\tau_l^{\text{eff}})|n\rangle_l.$$
(10.29)

Finally, we obtain the following expression for *H*:

$$H(|\alpha|, \theta, \phi, \xi, g, \lambda, \eta) = \frac{2|\alpha|^2 \xi \eta}{\sqrt{1 + 4\eta (1 - \eta) \sinh^2 g}} \times \{\cosh[2(g - g_{\text{eff}})] - \cos(\lambda + 2\phi - 2\theta) \sinh[2(g - g_{\text{eff}})]\}.$$
(10.30)

All the details on the calculation are reported in App. E.2. The expression for *H* is maximized for $\cos(\lambda + 2\phi - 2\theta) = -1$. Here, λ and θ are respectively the optical phase of the

pump beam and of the coherent state. Eq. (10.30) means that the quantum Fisher information presents a ϕ dependence, and for each value of ϕ the maximum is achieved when the other parameters λ and θ are set according to $\phi + \lambda/2 - \theta = \pi/2$. The maximum value of the quantum Fisher information reads:

$$H_{\text{ampl}}(|\alpha|,\xi,g,\eta) = 2|\alpha|^2 \xi \eta \frac{e^{2(g-g_{\text{eff}})}}{\sqrt{1+4\eta(1-\eta)\sinh^2 g}}.$$
 (10.31)

Because of the dependence of H on ϕ , to achieve the maximum sensitivity H_{ampl} an adaptive strategy [Nag88] is necessary.

10.3.2 Classical Fisher information with photon-counting measurements

In this section we report the calculation leading to the classical Fisher information for the amplifier-based protocol, when the $\hat{\rho}_{\phi}^{\beta,g,\eta}$ is analyzed by means of photon-counting measurements. This quantity represents, according to the Cramer-Rao bound, the maximum precision achievable with the chosen probe state and detection apparatus, maximized over all possible data processing. For the amplifier-based protocol, the probability distribution $p(n^{(+)}, m^{(-)}|\phi)$ and the corresponding classical Fisher information from Eq. (10.14) can be separated in two independent single-mode contributions:

$$I_{\phi} = \sum_{l=+,-} I_{\phi}^{(l)}, \qquad (10.32)$$

where:

$$I_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{[\partial_{\phi} p(n^{(l)} | \phi)]^2}{p(n^{(l)} | \phi)}.$$
(10.33)

The starting point is the expression of Eq. (10.23) for the single-mode density matrix $\hat{\rho}_{\phi}^{(l)}$. From this expression, the photon-number probability distribution of $\hat{\rho}_{\phi}^{(l)}$ can be evaluted according to the procedure reported in App. E.4. The expression for $p(n^{(l)}|\phi) = \text{Tr}[\hat{\rho}_{\phi}^{(l)}\hat{\Pi}_{n^{(l)}}]$ takes the form:

$$p(n^{(l)}|\phi) = \frac{2(-1)^n}{1+2N_l^{\text{eff}}} e^{-2(\tilde{C}_{x_l}+\tilde{C}_{p_l})} \sum_{k=0}^n \sum_{j=0}^k \frac{2^k}{k!} \binom{n}{k} \binom{k}{j} \times \frac{U[-j, 1/2, -2\tilde{A}_{x_l}(\tilde{B}_{x_l})^2]U[-k+j, 1/2, -2\tilde{A}_{p_l}(\tilde{B}_{p_l})^2]}{(\tilde{A}_{x_l})^{j+1/2} (\tilde{A}_{p_l})^{k-j+1/2}}.$$
(10.34)

Here, U(a,b,z) are confluent hypergeometric functions, while the expressions for the coefficients $(\tilde{A}_{x_l}, \tilde{B}_{x_l}, \tilde{C}_{x_l})$ and $(\tilde{A}_{p_l}, \tilde{B}_{p_l}, \tilde{C}_{p_l})$ can be found in the App. E.4 in Eqs. (E.78-E.79). Finally, the classical Fisher information can be evaluated by calculating the derivative of the photon-number distribution according to the definition of Eqs. (10.32-10.33). The complete expression for the numerical evaluation of I_{ϕ} can be found in App. E.4.

10.3.3 Sensitivity with photon-number difference

Let us now fix the detection strategy apparatus, that is, the measurement of the photonnumber difference \hat{D} . In order to calculate the sensitivity *S* associated to this scheme, it is convenient to work in the Heisenberg picture. To this end, we need to consider the time evolution of the field operators due to the OPA and of the loss map \mathcal{L}_{η} . The latter is modeled by a beam-splitter of transmittivity η . By combining the resulting equations for the time evolution of the amplifier and of the beam-splitter we obtain the following expressions for the field operators at the detection stage:

$$\hat{c}_{+}^{\dagger} = \sqrt{\eta} \left(\hat{a}_{+}^{\dagger} C + e^{-\iota \lambda} \hat{a}_{+} S \right) - \iota \sqrt{1 - \eta} \hat{b}_{+}^{\dagger}, \qquad (10.35)$$

$$\hat{c}_{-}^{\dagger} = \sqrt{\eta} \left(\hat{a}_{-}^{\dagger} C - e^{-\iota \lambda} \hat{a}_{-} S \right) - \iota \sqrt{1 - \eta} \hat{b}_{-}^{\dagger}.$$
(10.36)

where \hat{b}_+ and \hat{b}_- are the annihilation operators for the second input port of the beamsplitter, $C = \cosh g$ and $S = \sinh g$. The chosen strategy is to measure the output photonnumber difference $\hat{D} = \hat{c}_+^{\dagger} \hat{c}_+ - \hat{c}_-^{\dagger} \hat{c}_-$ and to extrapolate the value of ϕ from it. By exploiting the expressions (10.35-10.36), the average of \hat{D} on the state $\hat{\rho}_{\phi}^{\beta,g,\eta}$ is:

$$\langle \hat{D} \rangle = \eta |\alpha|^2 \xi \left[\cos \phi (1 + 2\overline{n}) + \cos(\phi + \lambda - 2\theta) 2 \sqrt{\overline{n}(1 + \overline{n})} \right].$$
(10.37)

where $\overline{n} = \sinh^2 g$. To evaluate the resolution $\delta \phi$ on the estimated phase according to standard estimation theory, we need to calculate $\sigma^2(\langle \hat{D} \rangle) = \langle \hat{D}^2 \rangle - \langle \hat{D} \rangle^2$. By evaluating the average values $\langle (\hat{c}^{\dagger}_+ \hat{c}_+)^2 \rangle$ and $\langle (\hat{c}^{\dagger}_- \hat{c}_-)^2 \rangle$, we obtain:

$$\sigma^{2}(\langle \hat{D} \rangle) = \eta \left[a(\overline{n}, \eta) + \cos \phi \cos(\phi + \lambda - 2\theta) b(\overline{n}, \eta) \right], \tag{10.38}$$

where:

$$a(\overline{n},\eta) = 2\overline{n}(1+\eta+2\eta\overline{n})+|\alpha|^2\xi\left[1+2\overline{n}+\eta\overline{n}(6+8\overline{n})\right], \quad (10.39)$$

$$b(\overline{n},\eta) = 2\sqrt{\overline{n}(1+\overline{n})}|\alpha|^2\xi(1+\eta+4\eta\overline{n}).$$
(10.40)

We note that both the signal and the fluctuations depend on the difference between the phase θ of the coherent beam and the phase λ of the pump beam of the OPA. Finally, the resolution of this detection strategy can be evaluated according to standard estimation theory as:

$$\delta\phi = \frac{\sqrt{\sigma^2(\langle \hat{D} \rangle)}}{\left|\frac{\partial\langle D \rangle}{\partial\phi}\right|} = \frac{\sqrt{a(\overline{n},\eta) + \cos\phi\cos(\phi + \lambda - 2\theta)b(\overline{n},\eta)}}{|\alpha|^2\sqrt{\eta}\xi \left|\cos\phi(1 + 2\overline{n}) + \cos(\phi + \lambda - 2\theta)2\sqrt{\overline{n}(1 + \overline{n})}\right|}.$$
 (10.41)

The sensitivity is defined as $S_{\text{ampl}} = \delta \phi^{-1}$. Its optimal operating point is achieved for $\lambda - 2\theta = 0$ and for a value of the actual phase of $\phi = \pi/2$, corresponding to the steepest point of the signal $\langle \hat{D} \rangle$. The sensitivity of the scheme in this point is then:

$$S_{\max} = \frac{|\alpha|^2 \xi \sqrt{\eta} (1 + 2\overline{n} + 2\sqrt{\overline{n}(1 + \overline{n})})}{a^{1/2}(\overline{n}, \eta)}.$$
(10.42)

The fact that the sensitivity S_{ampl} depends on the parameter ϕ we want to estimate implies that the maximum sensitivity S_{max} can be achieved only by employing an adaptive strategy, where some initial measurements are performed to get an estimate of ϕ so that the apparatus can be employed in its optimal working point around $\phi = \pi/2$.



Figure 10.3: (a) Theoretical framework for phase estimation protocols. (b) Plot of the ratio between the sensitivity square $(S_{\text{max}})^2$ and the quantum Fisher information H_{SQL} connected to the standard quantum limit (SQL) as a function of the nonlinear gain g of the amplifier and of the detection efficiency η , with $|\beta|^2 = 20$. Our scheme achieves the SQL for a wide range of parameters. (c) Plot of the ratio between the quantum Fisher information H_{ampl} from our scheme and H_{SQL} as a function of the nonlinear gain g of the amplifier and of the detection efficiency η , with $|\beta|^2 = 20$. (d) Comparison between the classical Fisher information I_{ampl} (points) and the sensitivity $(S_{\text{max}})^2$ (lines) for $|\beta|^2 = 9$.

10.3.4 Achieving the lossless quantum Cramer-Rao bound

Let us now discuss the theoretical results obtained in the previous sections for the amplifier based protocol. To gauge the efficiency of our method we start by noting that for $\bar{n} \gg (2\eta)^{-1}$ and $|\alpha| \gg 1/(\sqrt{2}\xi)$ the sensitivity (10.42) gives $S_{\text{max}} \simeq \sqrt{2}|\beta|$, so that in this regime the QCR bound $\delta^2 \phi \ge 1/(M2|\beta|^2)$ of the state $|\Psi_{\phi}^{\beta}\rangle$ (before the amplification and the detector loss) can be attained. The quantum Fisher information, through the QCR bound, measures the best precision achievable when optimizing over the possible detection and data processing strategies. To show that our choice to measure the photon-number difference \hat{D} can be optimal, note that for $\bar{n} \gg (8\eta)^{-1}$ and $|\beta|^2 \gg 1/2$ the ratio $S^2/H_{\text{ampl}} \rightarrow 1$, see Fig. 10.3 (b)-(c). In other words, increasing the amplifier gain, the effects of the detector loss can be asymptotically removed [DDS10]. In addition, our data processing can be optimal for even a wider range of parameters. In fact, the sensitivity *S* closely tracks the classical Fisher information I_{ampl} also for small values of \bar{n} , see Fig. 10.3 (d). The quantity I_{ampl} represents the maximum amount of information that can be extracted from the probe state using our choice of measurement, optimizing over all possible data-processing.



Figure 10.4: (a) Conventional (unamplified) coherent-state interferometry with sample and detection loss $1 - \xi$ and $1 - \eta$ respectively: it can achieve the SQL bound connected to the quantum Fisher information (QFI) H_{SQL}^{η} . (b-c) Comparison between our scheme and the conventional scheme of Fig. 10.4 (a): logarithmic plots of the ratio $(S_{ampl})^2/H_{SQL}$ as a function of η and g for $|\beta|^2 = 20$ (b), and as a function of $|\beta|$ for g = 3.4 (c).

Consider now, as in standard interferometry, the case with no amplification, where a coherent state is subject to both the sample and detector loss [Fig. 10.4 (a)]. Our method always outperforms it; this can be seen in Figs. 10.4 (b) and (c) where the enhancement E is plotted for different values of the gain and the detection efficiency. Recently, the optimal strategy in the presence of loss was derived [DDDS⁺09b, KDDW⁺10]. It employs the state that maximizes the quantum Fisher information in lossy conditions. Of course, while this strategy cannot be beaten if one could access the optimal measurement that attains the QCR bound, both this measurement and the creation of these states without using post-selection are beyond the reach of practical implementations, especially for states with large average photon-numbers. In contrast, our amplifier-based protocol uses readily

available input states and detection strategies, and can be implemented with the current technology.

10.4 Experimental implementation of the amplifier-based protocol in the high losses regime

The optical setup is reported in Fig. 9.10. To acquire the phase shift to be measured, the probe coherent state is prepared by attenuating, filtering in frequency and preparing in the $\vec{\pi}_+$ polarization state a part of the laser beam. Then, the probe state is injected into the sample, simulated by a Babinet-Soleil compensator, and spatially and temporally matched to the pump it is injected into the OPA. In this experimental realization the phases of the pump and of the coherent state are not stabilized: this will reduce the achievable enhancement by a fixed numerical factor of 4. In contrast to previous realizations of parametric amplification of coherent states [ZVB04a, ZVB05, BSG⁺10] which focused on the single-photon excitation regime, we could achieve a large value for the nonlinear gain, up to g = 3.3, corresponding to a number of generated photons per mode $\bar{n} \sim 180$ in spontaneous emission. In addition, our scheme is also able to exploit the polarization degree of freedom. As usual, the two output orthogonal polarizations are filtered in frequency and coupled into a single-mode fiber. Finally, they are detected by two avalanche photodiodes SPCM-AQR14 (D_1, D_1^*). Their count rates are then subtracted to obtain the value of $\langle \hat{D} \rangle$, and recorded as a function of the phase ϕ , introduced by the Babinet.

10.4.1 Theoretical model for the experiment in the high losses regime

In the described implementation, no phase stabilization is performed on the optical path of the pump beam, hence the phase varies randomly at each experimental run. To model such effect, an average on the phase λ with a uniform distribution $\mathscr{P}(\lambda) = \frac{1}{2\pi}$ must be performed on both the signal and the fluctuations. In this case, the average signals in the two polarizations $\vec{\pi}_+$ and $\vec{\pi}_-$ are given by:

$$\overline{\langle \hat{n}_+ \rangle} = \eta \left[\overline{n} + |\alpha|^2 \xi (1 + 2\overline{n}) \cos^2(\phi/2) \right], \qquad (10.43)$$

$$\overline{\langle \hat{n}_{-} \rangle} = \eta \left[\overline{n} + |\alpha|^2 \xi (1 + 2\overline{n}) \sin^2(\phi/2) \right].$$
(10.44)

Then the average on $\langle D \rangle$ given by Eq. (10.37) trasforms into:

$$\overline{\langle \hat{D} \rangle} = |\alpha|^2 \eta \xi \cos \phi (1 + 2\overline{n}). \tag{10.45}$$

In the high losses regime investigated throughout the paper, the number of the photons effectively impinging on the detector is smaller than one, since $\eta \langle n_{\pm} \rangle < 1$. In this regime, the single-photon counting process is described by a Poissonian statistics. Hence, the fluctuation on the difference signal can be evaluated as:

$$\sigma^{2}(\overline{\langle \hat{D} \rangle}) = \sigma^{2}(\overline{\langle \hat{n}_{+} \rangle}) + \sigma^{2}(\overline{\langle \hat{n}_{-} \rangle}) = \overline{\langle \hat{n}_{+} \rangle} + \overline{\langle \hat{n}_{-} \rangle}.$$
 (10.46)



Figure 10.5: Experimental setup for the practical implementation of the protocol. The output of the excitation source is doubled in frequency through a second harmonic generation process to generate the experiment pump beam on mode \mathbf{k}_p . The remainder of the 795 nm beam is then separated from the pump beam through a dichroic mirror, and is prepared in the coherent state $|\alpha\rangle_+$ on mode \mathbf{k}_1 by controlled attenuation, spectral filtering and polarizing optics. The coherent state probe then acquires the phase shift by interacting with the sample (in our case, a Babinet-Soleil compensator). Then, the coherent state probe is superimposed spatially and temporally with the pump beam through an adjustable delay line (Z) and is then injected into the OPA after the acquisition of the phase. Finally, the output field is filtered in frequency and coupled into a single-mode fiber. Then, the field is attenuated to simulate the action of losses and detected by two single-photon detectors (D_1, D_1^*) .

By explicitly substituting the expressions for $\overline{\langle \hat{n}_+ \rangle}$ and $\overline{\langle \hat{n}_- \rangle}$ we obtain the following expression for the sensitivity:

$$S_{\exp}^{\phi} = \frac{|\alpha|^2 \xi \sqrt{\eta} (1+2\overline{n})|\sin\phi|}{\sqrt{2\overline{n}+|\alpha|^2 \xi (1+2\overline{n})}}.$$
(10.47)

The optimal point is achieved for $\phi = \pi/2$, where the sensitivity is:

$$S_{\exp} = \frac{|\alpha|^2 \xi \sqrt{\eta} (1+2\overline{n})}{\sqrt{2\overline{n}+|\alpha|^2 \xi (1+2\overline{n})}}.$$
(10.48)

10.4.2 Enhancement in the high losses regime

The results of the experiment are reported in Fig. 10.6. Since the sample losses $1 - \xi$ act as a scaling factor on the coherent state amplitude as $\alpha \rightarrow \beta = \alpha \sqrt{\xi}$, we evaluated $|\alpha|$ by estimating the average number of photons $|\beta|^2$ after the interaction with the sample. In Figs. 10.6 (a) and (b) we report the fringe patterns obtained by measuring the single counts at the single photon detectors for the amplified case and the coherent state case,



Figure 10.6: Experimental results. (a) Fringe patterns for the amplified coherent state (g = 3.3) and (b) for the unamplified coherent state strategy for $|\beta|^2 \sim 22.8$.

respectively. An enhancement of ~ 200 in the counts rate for the former case is observed without significantly affecting the visibility of the fringe pattern [Fig. 10.7 (a)], leading to an increased phase resolution. We measured the enhancement E_{exp} achievable with our protocol with respect to the conventional unamplified interferometry, defined as the squared ratio between the measured sensitivities with and without the amplifier, see Fig. 10.7 (b). Our measurement shows a good agreement with the theoretical predictions. A significant enhancement up to a value of $E_{exp} = 186.3 \pm 9.3$ has been achieved. In the operating regime of our experimental implementation the sensitivity of S_{exp} scales as $\eta^{1/2}$ [see Fig. 10.7 (a), inset], as for a coherent state only. Hence, the observed enhancement is mainly due to the strong increase in the counts rate due to the amplification process.



Figure 10.7: Experimental results. (a) Fringe pattern visibility and (b) experimental enhancement E_{exp} as a function of the nonlinear gain g for $|\beta|^2 \sim 5.8$, $\eta \sim 1.46 \times 10^{-4}$ (experiment: black diamond points; theory: black solid line) and $|\beta|^2 \sim 22.8$, $\eta \sim 3.48 \times 10^{-5}$ (experiment: green star points; theory: green dashed line). Inset: experimental plot of the sensitivity as a function of the efficiency ratio η/η_0 , where η_0 is a reference efficiency.

10.5 Conclusion and perspectives

In this chapter we presented a strategy to perform phase estimation protocols in the presence of noisy detectors. This approach involves coherent states as input signals, and phase sensitive amplification after the interaction with the sample and before detection losses. The accuracy of the protocol can reach the performances of a lossless probe state even in presence of imperfect detectors.

We then presented the experimental implementation of this protocol in highly lossy scenario, showing the advantage of this technique even in a highly detrimental regime. We obtained an experimental enhancement up to a value $E_{exp} \sim 200$ with respect to the standard quantum limit in presence of the same amount of detection losses. Furthermore, at variance with many implementations involving the generation of quantum probe states [GLM11, RPP⁺07], the present implementation does not require any post-selection of the experimental data.

As a further perspective, the present strategy could be exploited in phase estimation protocols with noisy detectors involving different classes of probe states, including quantum resources such as squeezed light. This approach could lead to the possibility of achieving sub-shot noise performances in phase estimation protocols with noisy detectors.

Conclusions

The possibility of observing quantum properties, such as entanglement, in systems involving a large number of particles is still an open challenge. The main experimental difficulties to be addressed are the uncontrolled interaction with the environment, that is, decoherence, and the necessity of performing quantum measurements with the required resolution. When the size of the system progressively increases, the requirements in terms of isolation from the environment and of measurement resolution becomes progressively more demanding, thus rendering very difficult, if not impossible, to observe quantum properties in macroscopic systems. For these reasons, it becomes crucial to identify suitable platforms for the investigation of quantum features in multiparticle systems.

In parallel, decoherence represents one of the main challenges to be overcome when dealing with the application of quantum mechanical theory to quantum information processing. In all implementations, the action of decoherence is responsible for the loss of the benefits achievable with quantum mechanical resources. A relevant example is provided by quantum sensing, which aims to perform precision measurements of a physical parameter, such as an optical phase beyond any classical limit. In this context, the adoption of multiphoton, maximally entangled states can allow to reach the Heisenberg limit. However, this approach is extremely sensitive to the detrimental effect of losses, which rapidly deletes any quantum benefit. Hence, it becomes relevant to develop suitable strategies to perform phase estimation protocols even in lossy conditions.

In the present thesis, we analyzed the quantum states generated by the process of optical parametric amplification to perform fundamental tests of quantum mechanics and quantum sensing. In Chap. 4 we showed that the process of parametric amplification permits to broadcast the properties of a microscopic single-photon seed into an output state with a large number of photons. Then, we analyzed the effect of the action of a lossy channel in the output multiphoton states. By adopting different criteria based on both discreteand continuous-variables, we showed that the multiphoton states produced by parametric amplification present a significant resilience to losses, and that nonclassical properties can be observed in a large losses regime [DSS09b, DSS09a, SVD⁺09, DSSV10, SSD10].

Motivated by these results, in Chap. 5 we analyzed the possibility of applying the optical parametric amplifier in order to amplify a single-photon belonging to an entangled pair. In such a way, the initial entanglement between the two particles is broadcasted into a multiphoton state. We then investigated the possibility of detecting the entangle-

ment in realistic experimental conditions, that is, in presence of losses, by exploiting dichotomic measurements [SVSD10]. The entanglement in this case can be demonstrated by making a supplementary assumption on the optical source. Then, we considered the possibility of manipulating the amplified multiphoton states in order to increase the distinguishability of the output multiphoton states [VSSD10]. The practical impossibility of observing genuine entanglement in this microscopic-macroscopic configuration with dichotomic measurements suggests that high resolution measurements are necessary for this task. Hence, we considered in Chap. 6 the possibility of exploiting homodyne detection on the multiphoton part of the state [SVP+11]. We showed that genuine micromacro entanglement can be detected. These results on the investigated source of micromacro entanglement suggested several further studies on how to detect entanglement for multiphoton states [SHB⁺09, SSB⁺11, STS⁺11, RSS11]. More specifically, entanglement in the micro-macro system may be more difficult to be detected than in a genuine macroscopic-macroscopic source. Further investigations are necessary in this direction. The same approach, involving low and high resolution measurements, has been applied to a high gain spontaneous parametric down-conversion source, generating a bipartite system of two multiphoton states [VST⁺10b, VTCS⁺11]. We addressed the violation of a Bell's inequality in this macroscopic-macroscopic system. By exploiting low resolution dichotomic measurements, the violation of the Bell's inequality results to be practically impossible due to the coarse-grained nature of the detection apparatus. By exploiting homodyne detection, Bell's inequality can be violated. In all the above cases the required detection efficiency increases with the size of the system. In conclusion in order to observe quantum properties in multiphoton systems it is necessary to detect almost all the involved particles.

The process of parametric amplification can be applied in a different context to generate nonclassical and nongaussian continuous-variables states. In Chap. 8 we reported the experimental characterization of the single-photon addition process on coherent states. By explicitly addressing the nongaussianity and nonclassicality of the output states [BSG⁺10], and by reconstructing the map associated to the process, at variance with previous realizations [ZVB04a] we obtained a full insight on the properties of the system. Such process can find application in several continuous-variables quantum information process, as well as in the context of measurement-induced quantum operations.

The significant resilience to losses of the multiphoton states generated by optical parametric amplification can find application in several quantum information protocols. Among them, we consider the adoption of this device to perform phase estimation in lossy conditions. As a first step, we considered a minimally invasive scenario, when the phase estimation is performed by single-photon probes. By amplifying the probe state after the interaction with the sample, we can protect the information on the phase from the action of losses without altering the number of photon which effectively impinge onto the sample. This is demonstrated both theoretically and experimentally in Chap. 9, where we report of an experimental enhancement up to a factor 200 [VST⁺10a]. Finally, the same approach

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has been exploited with coherent state probes in Chap. 10. We explicitly addressed the optimality of the protocol, showing that the performances of the input coherent states can be retrieved by exploiting noisy measurements and the amplification strategy. Indeed, the quantum Fisher information of the amplified field in presence of noisy detectors can reach the quantum Cramer-Rao bound associated to the input coherent state analyzed with ideal detectors. The protocol has been implemented in the high losses regime, showing that a significant enhancement can be obtained also in this extremely detrimental regime $[SVL^+11]$.



Figure 10.8: Transition from the quantum dynamics of microscopic systems to the classical dynamics of macroscopic systems [Zur91].

In conclusion, in this thesis we analyzed several applications of a class of multiphoton states generated by optical parametric amplification. From the fundamental side, we considered different optical configurations in order to detect entanglement in a hybrid microscopic-macroscopic scenario and to detect nonlocality in a multiphoton-multiphoton system. The obtained results suggests that to reveal quantum properties in this system it is necessary to exploit high resolution measurements, since the detection of all the involved particles seems to be a crucial requirement. Further studies are necessary in order to identify suitable strategies that can be adopted to witness quantum features in systems of growing size. A starting point is provided by the continuous-variables approach, which can lead to the development of more sophisticated entanglement tests. In the quantum sensing context, we showed that the process of parametric amplification can find application in order to perform phase estimation experiments in noisy conditions. The obtained results can lead to protocols robust with respect to losses. As a future perspective, the application of the amplifier-based strategy to quantum probe state, such as squeezed light, can lead to the possibility of obtaining sub-shot-noise scaling in presence of lossy detectors.

Appendix A

Density matrix for the amplified single-photon states in presence of losses

In this appendix we report the derivation and the expressions of the density matrix of the amplified single photon states after the propagation over a lossy channel. The action of losses on a single mode state, identified by the wave vector \mathbf{k}_i and the polarization state $\vec{\pi}_i$, is defined by the following superoperator [DSEB04]:

$$\mathscr{L}_{\mathbf{k}_{i},\vec{\pi}}[\hat{\sigma}] = \sum_{p=0}^{\infty} (1-\eta_{i})^{p/2} \eta_{i}^{(\hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}}^{\dagger} \hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}})/2} \frac{\hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}}^{p}}{\sqrt{p!}} \hat{\sigma} \frac{\hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}}^{\dagger} \eta_{i}^{(\hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}}^{\dagger} \hat{a}_{\mathbf{k}_{i},\vec{\pi}_{i}})/2}}{\sqrt{p!}} (1-\eta_{i})^{p/2}.$$
 (A.1)

As a further assumption, we consider the losses to be independent from the polarization state $\vec{\pi}_i$.

In the following sections we calculate the density matrix coefficients for the two different amplifiers, that is, corresponding to the collinear and the noncollinear configurations when injected by a single photon state.

A.1 Density matrix of the phase-covariant amplified states after the propagation over a lossy channel

In this section we report the derivation of the density matrix coefficients for the collinear optical parametric amplifier. More specifically, in Sec. A.1.1 we focus our analysis on the field emitted in the spontaneous emission regime. Then, in Sec. A.1.2 we consider the field emitted by amplification of a single-photon with equatorial polarization. For this class of input state, the present amplifier acts as an optimal phase-covariant cloning machine. Finally, for sake of completeness we consider in Sec. A.1.3 the field emitted by amplification of a single-photon with linear $\{\vec{\pi}_H, \vec{\pi}_V\}$ polarization.

A.1.1 Density matrix for the spontaneous emission field

We begin by considering the spontaneous emission state $\hat{\rho}_{\eta}^{0}$. The starting point is the expression for the unperturbed state $|\Phi_{\text{OPA}}^{0}\rangle$:

$$|\Phi_{\text{OPA}}^{0}\rangle = \frac{1}{C} \sum_{i,j=0}^{\infty} \left(e^{-\iota\phi} \frac{\Gamma}{2} \right)^{i} \left(-e^{-\iota\phi} \frac{\Gamma}{2} \right)^{j} \frac{\sqrt{(2i)!} \sqrt{(2j)!}}{i! j!} |(2i)\phi,(2j)\phi_{\perp}\rangle, \quad (A.2)$$

where $C = \cosh g$ e $\Gamma = \tanh g$, with g gain of the amplifier, with the state written in the polarization basis $\{\vec{\pi}_{\phi}, \vec{\pi}_{\phi_{\perp}}\}$. The density matrix after the action of the lossy channel $\hat{\rho}_{\eta}^{0}$ is obtained by applying the map of Eq. (A.1) to the two-mode density matrix of the state $\hat{\rho}_{\text{OPA}}^{0} = |\Phi_{\text{OPA}}^{0}\rangle\langle\Phi_{\text{OPA}}^{0}|$ according to: $\hat{\rho}_{\eta}^{0} = (\mathscr{L}_{\phi} \otimes \mathscr{L}_{\phi_{\perp}})[\hat{\rho}_{\text{OPA}}^{0}]$. The output density matrix is the sum of four terms with different parities:

$$\begin{split} \hat{\rho}_{\eta}^{0} &= \sum_{i,j,k,q=0}^{\infty} \left\{ \left(\hat{\rho}_{\eta}^{0} \right)_{ijkq} | (2i+1)\phi, (2j)\phi_{\perp} \rangle \left\langle (2k+1)\phi, (2q)\phi_{\perp} \right| + \\ &+ \left(\hat{\rho}_{\eta}^{0} \right)_{ijkq} | (2i)\phi, (2j)\phi_{\perp} \rangle \left\langle (2k)\phi, (2q)\phi_{\perp} \right| + \\ &+ \left(\hat{\rho}_{\eta}^{0} \right)_{ijkq} | (2i+1)\phi, (2j+1)\phi_{\perp} \rangle \left\langle (2k+1)\phi, (2q+1)\phi_{\perp} \right| + \\ &+ \left(\hat{\rho}_{\eta}^{0} \right)_{ijkq} | (2i)\phi, (2j+1)\phi_{\perp} \rangle \left\langle (2k)\phi, (2q+1)\phi_{\perp} \right| \right\}. \end{split}$$
(A.3)

The density matrix coefficients exhibit an explicit dependence on the parity and their expression is reported hereafter. For i, j, k, q even we obtain:

$$\left(\hat{\rho}_{\eta}^{0}\right)_{ijkq} = \frac{1}{C^{2}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}} (\sqrt{\eta})^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i}{2}!\frac{j}{2}!\frac{k}{2}!\frac{q}{2}!}$$

$$\left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{1+\frac{i+j+k+q}{2}} {}_{2}F_{1}\left(-\frac{i}{2},-\frac{k}{2};\frac{1}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1}\left(-\frac{j}{2},-\frac{q}{2};\frac{1}{2};R^{2}\Gamma^{2}\right).$$

$$(A.4)$$

For i, k odd and j, q even, we obtain:

$$\left(\hat{\rho}_{\eta}^{0}\right)_{ijkq} = \frac{1}{C^{2}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}-1} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}} R\Gamma^{2} \left(\sqrt{\eta}\right)^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i-1}{2}!\frac{j}{2}!\frac{k-1}{2}!\frac{q}{2}!}$$

$$\left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{1+\frac{i+j+k+q}{2}} {}_{2}F_{1} \left(\frac{1-i}{2}, \frac{1-k}{2}; \frac{3}{2}; R^{2}\Gamma^{2}\right) {}_{2}F_{1} \left(-\frac{j}{2}, -\frac{q}{2}; \frac{1}{2}; R^{2}\Gamma^{2}\right).$$

$$(A.5)$$

For i, k even and j, q odd, we obtain:

$$\left(\hat{\rho}_{\eta}^{0}\right)_{ijkq} = \frac{1}{C^{2}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}-1} R\Gamma^{2} (\sqrt{\eta})^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i}{2}!\frac{j-1}{2}!\frac{k}{2}!\frac{q-1}{2}!}$$

$$\left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{1+\frac{i+j+k+q}{2}} {}_{2}F_{1} \left(-\frac{i}{2}, -\frac{k}{2};\frac{1}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1} \left(\frac{1-j}{2}, \frac{1-q}{2};\frac{3}{2};R^{2}\Gamma^{2}\right).$$

$$(A.6)$$

Density matrix of the phase-covariant amplified states after the propagation over a lossy channel215

Finally, for i, j, k, q odd we obtain:

$$\left(\hat{\rho}_{\eta}^{0}\right)_{ijkq} = \frac{1}{C^{2}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}-1} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}-1} R^{2} \Gamma^{4} \left(\sqrt{\eta}\right)^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i-1}{2}!\frac{j-1}{2}!\frac{k-1}{2}!\frac{q-1}{2}!} \left[\frac{1}{2}\right]^{1+\frac{i+j+k+q}{2}} {}_{2}F_{1}\left(\frac{1-i}{2},\frac{1-k}{2};\frac{3}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1}\left(\frac{1-j}{2},\frac{1-q}{2};\frac{3}{2};R^{2}\Gamma^{2}\right).$$

$$(A.7)$$

In these expressions, $R = 1 - \eta$ is the losses parameter, and $_2F_1(\alpha, \beta; \gamma; z)$ are hypergeometric functions [Sla66].

A.1.2 Density matrix for the amplified single-photon with equatorial polarization

Here we report the coefficients for the density matrix of the amplified single-photon states, when a single photon with equatorial polarization $\vec{\pi}_{\phi}$ is injected in the state $|1\phi, 0\phi_{\perp}\rangle$. The starting point is the expression of the unperturbed state $|\Phi_{\text{OPA}}^{\phi}\rangle$:

$$|\Phi_{\rm OPA}^{\phi}\rangle = \frac{1}{C^2} \sum_{i,j=0}^{\infty} \left(e^{-i\phi} \frac{\Gamma}{2} \right)^i \left(-e^{i\phi} \frac{\Gamma}{2} \right)^j \frac{\sqrt{(2i+1)!} \sqrt{(2j)!}}{i! \, j!} |(2i+1)\phi, (2j)\phi_{\perp}\rangle,$$
 (A.8)

where $C = \cosh g$ e $\Gamma = \tanh g$, with g gain of the amplifier, with the state written in the polarization basis $\{\vec{\pi}_{\phi}, \vec{\pi}_{\phi_{\perp}}\}$. We now apply the lossy channel map of Eq. (A.1) to the two-mode density matrix of the state $\hat{\rho}_{\text{OPA}}^{\phi} = |\Phi_{\text{OPA}}^{\phi}\rangle\langle\Phi_{\text{OPA}}^{\phi}|$ according to: $\hat{\rho}_{\eta}^{\phi} = (\mathscr{L}_{\phi} \otimes \mathscr{L}_{\phi_{\perp}})[\hat{\rho}_{\text{OPA}}^{\phi}]$. The output density matrix is the sum of four terms with different parities:

$$\begin{split} \hat{\rho}_{\eta}^{\phi} &= \sum_{i,j,k,q=0}^{\infty} \left\{ \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i+1)\phi, (2j)\phi_{\perp} \rangle \langle (2k+1)\phi, (2q)\phi_{\perp} | + \\ &+ \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i)\phi, (2j)\phi_{\perp} \rangle \langle (2k)\phi, (2q)\phi_{\perp} | + \\ &+ \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i+1)\phi, (2j+1)\phi_{\perp} \rangle \langle (2k+1)\phi, (2q+1)\phi_{\perp} | + \\ &+ \left(\hat{\rho}_{\eta}^{\phi} \right)_{ijkq} | (2i)\phi, (2j+1)\phi_{\perp} \rangle \langle (2k)\phi, (2q+1)\phi_{\perp} | \right\}. \end{split}$$
(A.9)

We now report the expressions of the parity-dependent density matrix coefficients. For i, j, k, q even we obtain:

$$\left(\hat{\rho}_{\eta}^{\phi} \right)_{i,j,k,q} = \frac{1}{C^4} \left(\frac{\Gamma}{2} e^{-\iota \phi} \right)^{\frac{i+k}{2}} \left(-\frac{\Gamma}{2} e^{-\iota \phi} \right)^{\frac{j+q}{2}} R(\sqrt{\eta})^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i}{2}!\frac{j}{2}!\frac{k}{2}!\frac{q}{2}!} (i+1)(k+1)$$

$$\left[\frac{1}{1-R^2\Gamma^2} \right]^{2+\frac{i+j+k+q}{2}} {}_2F_1\left(-\frac{i}{2}, -\frac{k}{2};\frac{3}{2};R^2\Gamma^2 \right) {}_2F_1\left(-\frac{j}{2}, -\frac{q}{2};\frac{1}{2};R^2\Gamma^2 \right).$$
(A.10)

For i, k odd and j, q even, we obtain:

$$\left(\hat{\rho}_{\eta}^{\phi}\right)_{i,j,k,q} = \frac{1}{C^{4}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}-1} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}} (\sqrt{\eta})^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i-1}{2}!\frac{j}{2}!\frac{k-1}{2}!\frac{q}{2}!}$$

$$\left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{2+\frac{i+j+k+q}{2}} {}_{2}F_{1}\left(-\frac{1+i}{2}, -\frac{1+k}{2};\frac{1}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1}\left(-\frac{j}{2}, -\frac{q}{2};\frac{1}{2};R^{2}\Gamma^{2}\right).$$

$$(A.11)$$

For i, k even and j, q odd, we obtain:

$$\begin{split} \left(\hat{\rho}_{\eta}^{\phi}\right)_{i,j,k,q} &= \frac{1}{C^{4}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}-1} R^{2} \Gamma^{2} \left(\sqrt{\eta}\right)^{i+j+k+q} (i+1)(k+1) \\ &\frac{\sqrt{i!j!k!q!}}{\frac{i}{2}!\frac{j-1}{2}!\frac{k}{2}!\frac{q-1}{2}!} \left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{2+\frac{i+j+k+q}{2}} {}_{2}F_{1}\left(-\frac{i}{2},-\frac{k}{2};\frac{3}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1}\left(\frac{1-j}{2},\frac{1-q}{2};\frac{3}{2};R^{2}\Gamma^{2}\right) \\ &(A.12) \end{split}$$

Finally, for i, j, k, q odd we obtain:

$$\left(\hat{\rho}_{\eta}^{\phi}\right)_{i,j,k,q} = \frac{1}{C^{4}} \left(\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{i+k}{2}-1} \left(-\frac{\Gamma}{2} e^{-\iota\phi}\right)^{\frac{j+q}{2}-1} R\Gamma^{2} \left(\sqrt{\eta}\right)^{i+j+k+q} \frac{\sqrt{i!j!k!q!}}{\frac{i-1}{2}!\frac{j-1}{2}!\frac{k-1}{2}!\frac{q-1}{2}!} \left[\frac{1}{1-R^{2}\Gamma^{2}}\right]^{2+\frac{i+j+k+q}{2}} {}_{2}F_{1}\left(-\frac{1+i}{2},-\frac{1+k}{2};\frac{1}{2};R^{2}\Gamma^{2}\right) {}_{2}F_{1}\left(\frac{1-j}{2},\frac{1-q}{2};\frac{3}{2};R^{2}\Gamma^{2}\right).$$
(A.13)

Again, in these expressions, $R = 1 - \eta$ is the losses parameter, and $_2F_1(\alpha, \beta; \gamma; z)$ are hypergeometric functions [Sla66].

A.1.3 Density matrix for the amplified single-photon with $\vec{\pi}_H, \vec{\pi}_V$ polarization

The procedure for the evaluation of the density matrix of the state after losses is the same applied in the previous sections. Let us analyze the $|\Phi_{OPA}^{H}\rangle$ state, the same results apply for the $|\Phi_{OPA}^{V}\rangle$ state by relabelling the optical modes. The density matrix after the amplification process reads:

$$\hat{\rho}_{\text{OPA}}^{H} = |\Phi_{\text{OPA}}^{H}\rangle \langle \Phi_{\text{OPA}}^{H}| = \frac{1}{C^4} \sum_{n,m=0}^{\infty} \Gamma^{n+m} \sqrt{n+1} \sqrt{m+1} |(n+1)H, nV\rangle \langle (m+1)H, mV|.$$
(A.14)

After the application of the lossy channel map: $\hat{\rho}_{\eta}^{H} = (\mathscr{L}_{H} \otimes \mathscr{L}_{V})[\hat{\rho}_{\text{OPA}}^{H}]$, we finally obtain:

$$\hat{\rho}_{\eta}^{H} = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} \sum_{k=0}^{\infty} \left(\sum_{p=0}^{\infty} \overline{\gamma}_{ijk;p} \right) |iH, jV\rangle \langle kH, (k+j-i)V| + \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} \left(\sum_{p=j+1-i}^{\infty} \overline{\gamma}_{ijk;p} \right) |iH, jV\rangle \langle kH, (k+j-i)V|,$$
(A.15)

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where the coefficients $\overline{\gamma}_{i\,ik;p}$ are:

$$\overline{\gamma}_{ijk;p} = \frac{\Gamma^{2p+i+k-2}}{C^4} \sqrt{p+i} \sqrt{p+k} \eta^{k+j} R^{2p+i-1-j} \left[\binom{p+i}{i} \binom{p+i-1}{j} \binom{p+k}{k} \binom{p+k-1}{k+j-1} \right]^{\frac{1}{2}}.$$
(A.16)

A.2 Density matrix of the universal amplified states after the propagation over a lossy channel

In this section we report the detailed calculation of the coefficient of the density matrix for the $|\Phi^{1\psi}\rangle$ states in presence of losses. We focus our attention on the $|1\psi\rangle_1$ case only, since the calculation for the complementary state $|1\psi_{\perp}\rangle_1$ is similar.

First we investigate the features of the interaction Hamiltonian. Due to the properties of $\hat{\mathscr{H}}_{SPDC}$, the time evolution operator in the interaction picture $\hat{U} = \exp(-i\hat{\mathscr{H}}_{SPDC}t/\hbar)$ can be decomposed as the product of two independent operators $\hat{U} = \hat{U}_{\mathscr{A}} \otimes \hat{U}_{\mathscr{A}'}$, acting on two different Hilbert spaces corresponding to the two sets of modes [PSS⁺03, DPS04]:

$$\mathscr{A} \equiv \{ (\mathbf{k}_1, \vec{\pi}_{\psi}), (\mathbf{k}_2, \vec{\pi}_{\psi_\perp}) \}; \quad \mathscr{A}' \equiv \{ (\mathbf{k}_1, \vec{\pi}_{\psi_\perp}), (\mathbf{k}_2, \vec{\pi}_{\psi}) \}.$$
(A.17)

The operators $\hat{U} = \hat{U}_{\mathscr{A}} \otimes \hat{U}_{\mathscr{A}'}$ take the form:

$$\hat{U}_{\mathscr{A}} = \exp\left[\chi t(\hat{a}_{1\psi}^{\dagger}\hat{a}_{2\psi_{\perp}}^{\dagger} - \hat{a}_{1\psi}\hat{a}_{2\psi_{\perp}})\right], \qquad (A.18)$$

$$\hat{U}_{\mathscr{A}'} = \exp\left[-\chi t (\hat{a}_{1\psi_{\perp}}^{\dagger} \hat{a}_{2\psi}^{\dagger} - \hat{a}_{1\psi_{\perp}} \hat{a}_{2\psi})\right].$$
(A.19)

In the case of a separable input state in the OPA $\hat{\rho} = \hat{\rho}_{\mathscr{A}} \otimes \hat{\rho}_{\mathscr{A}'}$, the amplified states can be written in a separable form:

$$\hat{\rho}(t) = \hat{U}\hat{\rho}\hat{U}^{\dagger} = \left(\hat{U}_{\mathscr{A}}\hat{\rho}_{\mathscr{A}}\hat{U}_{\mathscr{A}}^{\dagger}\right) \otimes \left(\hat{U}_{\mathscr{A}'}\hat{\rho}_{\mathscr{A}'}\hat{U}_{\mathscr{A}'}^{\dagger}\right).$$
(A.20)

This separability property will be exploited in the remaining part of the paper both for the calculation of the density matrix after losses and for the evaluation of the Bures distance.

A.2.1 Density matrix coefficients for the amplified states

In this section we derive the density matrix coefficients for the amplified states $|\Phi_{SPDC}^{1\psi(1\psi_{\perp})}\rangle$ after the transmission over a lossy channel. We focus our attention on the $|1\psi\rangle_1$ case only, since the calculation for the complementary state $|1\psi_{\perp}\rangle_1$ is similar. Due to the property of the universal amplifier analyzed previously, we analyze separately the two subspaces \mathscr{A} and \mathscr{A}' [Eq.(A.20)]. Since the time evolution operators $\hat{U}_{\mathscr{A}}$ and $\hat{U}_{\mathscr{A}'}$ are equivalent apart from a global phase factor (-1), the quantum states for the amplifier \mathscr{A}' can be

directly derived from the expressions obtained for amplifier \mathscr{A} . Only two relevant cases are considered: the vacuum injected state $\hat{U}_{\mathscr{A}}|0\rangle$ (spontaneous emission) and the singlephoton injected $\hat{U}_{\mathscr{A}}|1\psi\rangle_1$ state. This analysis can be performed separately for the two amplifiers since the separability feature also holds after the amplified state propagates over a lossy channel in both \mathbf{k}_1 and \mathbf{k}_2 spatial modes. This is a consequence of the properties of the lossy channel map, which being a "local" transformation, acts independently on each mode. The output state after losses reads:

$$\mathscr{L}[\hat{\boldsymbol{\rho}}(t)] = \mathscr{L}_{\mathscr{A}}\left[\hat{U}_{\mathscr{A}}\hat{\boldsymbol{\rho}}_{\mathscr{A}}\hat{U}_{\mathscr{A}}^{\dagger}\right] \otimes \mathscr{L}_{\mathscr{A}'}\left[\hat{U}_{\mathscr{A}'}\hat{\boldsymbol{\rho}}_{\mathscr{A}'}\hat{U}_{\mathscr{A}'}^{\dagger}\right].$$
(A.21)

Here $\mathscr{L}_{\mathscr{A}} = \mathscr{L}_{\mathbf{k}_1, \vec{\pi}_{\psi}} \otimes \mathscr{L}_{\mathbf{k}_2, \vec{\pi}_{\psi_\perp}}$, $\mathscr{L}_{\mathscr{A}'} = \mathscr{L}_{\mathbf{k}_1, \vec{\pi}_{\psi_\perp}} \otimes \mathscr{L}_{\mathbf{k}_2, \vec{\pi}_{\psi}}$ are the maps induced by losses for the two subspaces, where the single mode map $(\mathbf{k}_i, \vec{\pi})$ is given by Eq.(A.1) Again, the transmission efficiency of the channels η_i are assumed to be polarization independent. We then label η_1 and η_2 the two efficiencies for the two spatial modes.

We begin with the analysis of the spontaneous emission regime. The calculation proceeds as follows. Starting from the quantum state for the subsystem $\mathscr{A} \hat{U}_{\mathscr{A}} |0\rangle$, the output state after the transmission over the lossy channel is obtained by applying the lossy channel map (A.1) to the density matrix of the state $\hat{\rho}^0_{\mathscr{A}}$. The same procedure applies for the single photon amplified states, where the seed of the amplifier \mathscr{A} is the single photon state $|1\psi\rangle$. In this case, the input state in the lossy channel is $\hat{U}_{\mathscr{A}}$. By applying the lossy channel map over the density matrix $\hat{\rho}^{1\psi}_{\mathscr{A}}$ of the state, we find the desired output states. Details on the calculation and the complete expressions of the coefficients for the density matrices $\hat{\rho}^{1\psi}_{\mathscr{A}}(\eta_1, \eta_2)$ and $\hat{\rho}^0_{\mathscr{A}}(\eta_1, \eta_2)$ are reported below.

Let us emphasize that, due to analogy of the Hamiltonian of the two amplifier \mathscr{A} and \mathscr{A}' , the density matrices of the states $\hat{\rho}^0_{\mathscr{A}'}(\eta_1,\eta_2)$ and $\hat{\rho}^{1\psi_{\perp}}_{\mathscr{A}'}(\eta_1,\eta_2)$ for amplifier \mathscr{A}' can be directly derived from Eqs.(A.24-A.26) and (A.29-A.31) by substituting (Γ) with ($-\Gamma$) and by re-labelling the indexes describing the spatial and polarization modes.

We begin with the analysis of the spontaneous emission regime. The quantum state for the subsystem \mathscr{A} is given by:

$$\hat{U}_{\mathscr{A}}|0\rangle = \frac{1}{C}\sum_{n=0}^{\infty}\Gamma^{n}|n\psi\rangle_{1}\otimes|m\psi_{\perp}\rangle_{2}.$$
(A.22)

The output state after the transmission over the lossy channel is obtained by applying the lossy channel map (A.1) to the density matrix of the state $\hat{\rho}^{0}_{\mathscr{A}}$:

$$\hat{\boldsymbol{\rho}}^{0}_{\mathscr{A}}(\boldsymbol{\eta}_{1},\boldsymbol{\eta}_{2}) = \left(\mathscr{L}_{\mathbf{k}_{1},\vec{\pi}_{\psi}} \otimes \mathscr{L}_{\mathbf{k}_{2},\vec{\pi}_{\psi_{\perp}}}\right) \left[\hat{\boldsymbol{\rho}}^{0}_{\mathscr{A}}\right].$$
(A.23)

After direct application of the lossy channel map on the density matrix, the following

expression is obtained:

$$\hat{\rho}_{\mathscr{A}}^{0}(\eta_{1},\eta_{2}) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=i-j}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{0}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j)} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}| + \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} \sum_{k=0}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{0}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i
(A.24)$$

where the coefficients for $i \ge j$ and i < j are given by:

$$\left[\hat{\rho}_{\mathscr{A}}^{0}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j)} = \frac{1}{C^{2}} \Gamma^{i+k} \frac{\eta_{1}^{(i+k)/2} \eta_{2}^{(2j+k-i)/2} R_{2}^{i-j} \sqrt{i!k!}}{(i-j)! \sqrt{j!(j+k-i)!}}$$

$$\times _{2}F_{1} \left(1+i, 1+k, i+i-j; \Gamma^{2}R_{1}R_{2} \right),$$

$$\left[\hat{\rho}_{\mathscr{A}}^{0}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i

$$\times _{2}F_{1} \left(1+j, 1+j+k-i, 1+j-i; \Gamma^{2}R_{1}R_{2} \right),$$

$$(A.25)$$$$

where ${}_{2}F_{1}(a,b,c;z)$ is the hypergeometric function defined in Ref. [Sla66]. The same procedure has been applied to the stimulated case, where the seed of the amplifier \mathscr{A} is the single photon state $|1\psi\rangle$. In this case, the input state in the lossy channel has the following expression:

$$\hat{U}_{\mathscr{A}}|1\psi\rangle_{1} = \frac{1}{C^{2}}\sum_{n=0}^{\infty}\Gamma^{n}\sqrt{n+1}|(n+1)\psi\rangle_{1}\otimes|m\psi_{\perp}\rangle_{2}.$$
(A.27)

By applying the lossy channel map over the density matrix $\hat{\rho}_{\mathscr{A}}^{1\psi}$ of the state, we find:

$$\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_1,\eta_2) = \left(\mathscr{L}_{\mathbf{k}_1,\vec{\pi}_{\psi}} \otimes \mathscr{L}_{\mathbf{k}_2,\vec{\pi}_{\psi_{\perp}}}\right) \left[\hat{\rho}_{\mathscr{A}}^{1\psi}\right]. \tag{A.28}$$

The application of the map leads to the following expression for the density matrix:

$$\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) = \sum_{i=0}^{\infty} \sum_{j=0}^{i-1} \sum_{k=i-j}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j+1)} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}| + \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \sum_{k=0}^{\infty} \left[\hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) \right]_{ijk}^{(i\geq j)} |i\psi\rangle_{1} \langle k\psi| \otimes |j\psi_{\perp}\rangle_{2} \langle (j+k-i)\psi_{\perp}|,$$
(A.29)

where the coefficients for $i \ge j + 1$ and $i \le j$ are given by:

$$\begin{bmatrix} \hat{\rho}_{\mathscr{A}}^{1\psi}(\eta_{1},\eta_{2}) \end{bmatrix}_{ijk}^{(i\geq j)} = \frac{1}{C^{4}} \Gamma^{i+k-2} \frac{\eta_{1}^{(i+k)/2} \eta_{2}^{(2j+k-i)/2} R_{2}^{i-j-1} \sqrt{i!k!}}{(i-j-1)! \sqrt{j!(j+k-i)!}}$$

$$\times _{2}F_{1} \left(1+i,1+k,i-j;\Gamma^{2}R_{1}R_{2}\right),$$
(A.30)

$$\left[\hat{\rho}_{\mathscr{A}}^{1\psi}(T_1, T_2) \right]_{ijk}^{(i < j)} = \frac{1}{C^2} \Gamma^{i+k} \frac{T_1^{(i+k)/2} T_2^{(2j+k-i)/2} R_1^{j-i+1} \sqrt{j!(j+k-i)!}}{(j-i+1)! \sqrt{i!k!}}$$

$$\times (j+1)(j+k-i+1)_2 F_1 \left(2+j, 2+j+k-i, 2+j-i; \Gamma^2 R_1 R_2\right).$$
(A.31)

According to previous considerations, the density matrices of the states $\hat{\rho}^{0}_{\mathscr{A}'}(\eta_1,\eta_2)$ and $\hat{\rho}^{1\psi_{\perp}}_{\mathscr{A}'}(\eta_1,\eta_2)$ for amplifier \mathscr{A}' can be directly derived from Eqs.(A.24-A.26) and (A.29-A.31) by substituting (Γ) with ($-\Gamma$) and by re-labelling the indexes describing the spatial and polarization modes. Finally, the complete output state can be reconstructed as:

$$\hat{\rho}^{1\psi}(\eta_1,\eta_2) = \hat{\rho}^{1\psi}_{\mathscr{A}}(\eta_1,\eta_2) \otimes \hat{\rho}^0_{\mathscr{A}'}(\eta_1,\eta_2). \tag{A.32}$$

A.2.2 Density matrix coefficients on the reduced k₁ spatial mode for the amplified states

In this section we report the expression of the coefficients for the reduced density matrix on spatial mode \mathbf{k}_1 of the $|\Phi_{SPDC}^{1\psi}\rangle$ after the propagation over a lossy channel. Such result has been exploited in the calculation of the Bures distance, where the action of the O-Filter device has been analyzed. The starting point of the calculation is the expression (A.1) of the $|\Phi_{SPDC}^{1\psi}\rangle$. After the partial trace on mode \mathbf{k}_2 , the density matrix $\hat{\rho}_{\mathbf{k}_1}^{1\psi} = \text{Tr}_{\mathbf{k}_2} \left[|\Phi_{SPDC}^{1\psi}\rangle \langle \Phi_{SPDC}^{1\psi}| \right]$ reads:

$$\hat{\rho}_{\mathbf{k}_1}^{1\psi} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\Gamma^{2n+2m}}{C^6} (n+1) |(n+1)\psi\rangle_1 \langle (n+1)\psi| \otimes |m\psi_{\perp}\rangle_1 \langle m\psi_{\perp}|.$$
(A.33)

Finally, the application of the lossy channel map leads to the following density matrix:

$$\hat{\rho}_{\mathbf{k}_{1}}^{1\psi}(\boldsymbol{\eta}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[\rho_{\mathbf{k}_{1}}^{1\psi}(\boldsymbol{\eta}) \right]_{ij} |i\psi\rangle_{1} \langle i\psi| \otimes |j\psi_{\perp}\rangle_{1} \langle j\psi_{\perp}|, \qquad (A.34)$$

where the coefficients are given by:

$$\left[\rho_{\mathbf{k_1}}^{1\psi}\right]_{ij} = \frac{\Gamma^{2i+2j-2}}{C^6} \eta^{i+j} \left(i + \Gamma^2 R\right) \left(1 - \Gamma^2 R\right)^{-3-i-j}.$$
 (A.35)

Appendix B

Generalized micro-macro entanglement criterion for dichotomic operators

In this appendix we demonstrate the inequality of Eq.(5.21), which gives a generalized bound for an entanglement test in a micro-macro bipartite system and dichotomic measurements.

B.1 General treatment of dichotomic measurements

The density matrix of a separable state, composed by two subsystems A and B, can be written as:

$$\hat{\rho} = \sum_{i} p_i \left(\hat{\rho}_i^A \otimes \hat{\rho}_i^B \right). \tag{B.1}$$

We restrict our attention to the set of dichotomic measurements, i.e. (± 1) valued upon each subsystem \hat{O}_A and \hat{O}_B respectively. The average value of a generic measurement operator $\hat{O}^j = \hat{O}_A^j \otimes \hat{O}_B^j$ is given by $V^j = \text{Tr}(\hat{\rho}\hat{O}^j)$, where the superscript *j* refers to a specific choice of the operator \hat{O}^j . The average value of the *i*-*th* component of the decomposition of the density matrix reads:

$$v^{ij} = \operatorname{Tr}\left[\left(\hat{\rho}_{i}^{A} \otimes \hat{\rho}_{i}^{B}\right) \hat{O}^{j}\right] = \operatorname{Tr}_{A}\left(\hat{\rho}_{i}^{A} \hat{O}_{A}^{j}\right) \operatorname{Tr}_{B}\left(\hat{\rho}_{i}^{B} \hat{O}_{B}^{j}\right) = v_{A}^{ij} \cdot v_{B}^{ij}.$$
(B.2)

The average value V^j can then be reexpressed as:

$$V^{j} = \operatorname{Tr}\left(\sum_{i} p_{i}\left(\hat{\rho}_{i}^{A} \otimes \hat{\rho}_{i}^{B}\right) \hat{O}^{j}\right) = \sum_{i} p_{i} v^{ij}.$$
(B.3)

The following inequality holds:

$$\left|V^{j}\right| = \left|\sum_{i} p_{i} v^{ij}\right| \le \sum_{i} p_{i} \left|v^{ij}\right|,\tag{B.4}$$

since $p_i \ge 0$. The sum of the average value over three different operators \hat{O}^j , where $\{j = 1, ..., 3\}$, is given by the following expression:

$$\sum_{j=1}^{5} |V^{j}| \le \sum_{i} p_{i} \Big(|v^{i1}| + |v^{i2}| + |v^{i3}| \Big) \le \sum_{i} p_{i} \Big(|v^{i1}_{A}| + |v^{i2}_{A}| + |v^{i3}_{A}| \Big)$$
(B.5)

where, since $-1 \le v_B^{ij} \le +1$, the following inequality has been exploited:

$$\left|v^{ij}\right| = \left|v_A^{ij} \cdot v_B^{ij}\right| \le \left|v_A^{ij}\right|. \tag{B.6}$$

The latter can be always decomposed as: $\hat{\rho}_i^A = \sum_n q_n^i |\psi_n\rangle_{AA} \langle\psi_n|$, where the set $\{q_n^i\}$ of probabilities satisfied the normalization condition $\sum_n q_n = 1$. We can then derive the following inequality:

$$\sum_{j=1}^{3} \left| v_A^{ij} \right| = \sum_{j=1}^{3} \left| \operatorname{Tr} \left(\sum_n q_n^i | \psi_n \rangle_{AA} \langle \psi_n | \hat{O}_A^j \right) \right| \le \sum_{j=1}^{3} \sum_n q_n^i \left| \operatorname{Tr} \left(|\psi_n \rangle_{AA} \langle \psi_n | \hat{O}_A^j \right) \right|.$$
(B.7)

Substituting this result in Eq.(B.5), we obtain:

$$\sum_{j=1}^{3} |V^{j}| \leq \sum_{j=1}^{3} \sum_{i} p_{i} \sum_{n} q_{n}^{i} \left|_{A} \langle \psi_{n} | \hat{O}_{A}^{j} | \psi_{n} \rangle_{A} \right| \leq \sum_{i} p_{i} \sum_{n} q_{n}^{i} \max_{|\psi_{n}\rangle} \left(\sum_{j=1}^{3} \left|_{A} \langle \psi_{n} | \hat{O}_{A}^{j} | \psi_{n} \rangle_{A} \right| \right).$$
(B.8)

The insertion of the normalization condition for the coefficients $\{p_i\}$ and $\{q_n^i\}$ completes the proof. For all bipartite separable states, a dichotomic measurement on both sides obey the following inequality:

$$\sum_{j=1}^{3} \left| V^{j} \right| \leq \max_{|\psi\rangle} \sum_{j=1}^{3} \left| {}_{A} \langle \psi | \hat{O}_{A}^{j} | \psi \rangle_{A} \right|, \tag{B.9}$$

where the maximization is performed over all possible states of system A.

B.2 Specific Micro-Macro case

We now specialize the result of previous section in the microscopic-macroscopic states, that is, when system *A* is a single spin-1/2 particle. Let us make a specific choice for the measurement operators $\{\hat{O}_A^j\}_{j=1}^3$. For a single spin-1/2 particle, we choose the Pauli operators $\{\hat{\sigma}_A^j\}_{j=1}^3$. Hereafter, we remove the subscript *A* in all the equations for simplicity of notation. The entanglement criterium (B.9) for this choice of the system and operators then reads:

$$\sum_{j=1}^{3} \left| V^{j} \right| \le \max_{|\psi\rangle} \sum_{j=1}^{3} \left| \langle \psi | \hat{\sigma}^{j} | \psi \rangle \right| \le \sqrt{3}.$$
(B.10)

where the latter has been maximized over all possible choice of single particel states $|\psi\rangle = \alpha |+\rangle + \beta |-\rangle$.

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Appendix C

Evaluation of the correlators for the hybrid CHSH-based inequality and entanglement witness for the amplified entangled pair

In this appendix we report the full calculation for the correlators (6.15,6.18,6.19) of the hybrid entanglement tests performed in Chap. 5.

C.1 Correlator for the CHSH-based test in ideal conditions

In this section we report the full calculation of the correlator $\mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi)$ reported in the main letter. We begin with the two-mode correlation \hat{Q} , defined as:

$$\hat{Q}(\boldsymbol{\alpha}_{\boldsymbol{\chi}}, \boldsymbol{\alpha}_{\boldsymbol{\chi}_{\perp}}, \boldsymbol{\chi}; \boldsymbol{\phi}) = \hat{\boldsymbol{\sigma}}^{A}(\boldsymbol{\phi}) \otimes \left(\hat{\Pi}_{\boldsymbol{\chi}}^{B}(\boldsymbol{\alpha}_{\boldsymbol{\chi}}) \otimes \hat{\Pi}_{\boldsymbol{\chi}_{\perp}}^{B}(\boldsymbol{\alpha}_{\boldsymbol{\chi}_{\perp}}) \right).$$
(C.1)

This operator corresponds to the measurement of the generalized parity operator on both polarization modes $\{\vec{\pi}_{\chi}, \vec{\pi}_{\chi\perp}\}$ of the macro-part of our state. The average $\mathscr{Q}(\alpha_{\chi}, \alpha_{\chi\perp}, \chi; \phi) = {}_{AB} \langle \Psi^{-} | \hat{Q} | \Psi^{-} \rangle_{AB}$ is related to the correlator of the CHSH-based inequality by:

$$\mathscr{C}(\alpha_{\chi},\chi;\phi) = \frac{2}{\pi} \int d^2 \alpha_{\chi_{\perp}} \mathscr{Q}(\alpha_{\chi},\alpha_{\chi_{\perp}},\chi;\phi).$$
(C.2)

This expression holds by considering the closure relation $\frac{2}{\pi} \int d^2 \alpha_{\chi_{\perp}} \hat{\Pi}_{\chi_{\perp}}(\alpha_{\chi_{\perp}}) \equiv \mathbb{1}_{\chi_{\perp}}$, which in turns comes from the normalization of the Wigner function.

C.1.1 Two-mode correlator

We now calculate the two-mode correlator $\mathscr{Q}(\alpha_{\chi}, \alpha_{\chi_{\perp}}, \chi; \phi)$. Let us recall the expression of the micro-macro state under investigation:

$$|\Psi^{-}\rangle_{AB} = \frac{1}{\sqrt{2}} (|\phi\rangle_{A}|\Phi^{\phi}_{\perp}\rangle_{B} - |\phi_{\perp}\rangle_{A}|\Phi^{\phi}\rangle_{B}), \qquad (C.3)$$

where the state has been expressed in a generic equatorial polarization basis $\{\vec{\pi}_{\phi}, \vec{\pi}_{\phi_{\perp}}\}$. The value of $\mathscr{Q}(\alpha_{\chi}, \alpha_{\chi_{\perp}}, \chi; \phi)$ is obtained by exploiting the relation between the twomode $\hat{\Pi}^{B}_{\chi}(\alpha_{\chi}) \otimes \hat{\Pi}^{B}_{\chi_{\perp}}(\alpha_{\chi_{\perp}})$ operator and the two-mode Wigner function $_{B}\langle \Phi | \hat{\Pi}^{B}_{\chi}(\alpha_{\chi}) \otimes \hat{\Pi}^{B}_{\chi_{\perp}}(\alpha_{\chi_{\perp}}) | \Phi \rangle_{B} = \frac{\pi^{2}}{4} W_{\Phi}(\alpha_{\chi}, \alpha_{\chi_{\perp}})$. We get:

$$\mathscr{Q}(\boldsymbol{\alpha}_{\boldsymbol{\chi}}, \boldsymbol{\alpha}_{\boldsymbol{\chi}_{\perp}}, \boldsymbol{\chi}; \boldsymbol{\phi}) = \frac{\pi^2}{8} \left[W^B_{\boldsymbol{\phi}_{\perp}}(\boldsymbol{\alpha}_{\boldsymbol{\chi}}, \boldsymbol{\alpha}_{\boldsymbol{\chi}_{\perp}}) - W^B_{\boldsymbol{\phi}}(\boldsymbol{\alpha}_{\boldsymbol{\chi}}, \boldsymbol{\alpha}_{\boldsymbol{\chi}_{\perp}}) \right].$$
(C.4)

Here, $W_{\phi_{\perp}}^{B}$ and W_{ϕ}^{B} stand for the two-mode Wigner functions of an amplified $|\phi_{\perp}\rangle$ and $|\phi\rangle$ single-photon states respectively, evaluated at the rotated phase-space variables $\{\alpha_{\chi}, \alpha_{\chi_{\perp}}\}$. The correlator $\mathscr{Q}^{AB}(\alpha_{\chi}, \alpha_{\chi_{\perp}}, \chi; \phi)$ is then derived starting from the expression of the Wigner functions [SVD⁺09] [$S = \sinh g$ and $C = \cosh g$]:

$$W^{B}_{\phi_{\perp}}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) = \frac{4}{\pi^{2}} \left\{ 4 \left[|\alpha_{\phi_{\perp}}|^{2} (1 + 2S^{2}) + 2CS \operatorname{Re}(\alpha_{\phi_{\perp}}^{2} e^{i\phi}) \right] - 1 \right\} \times \\ \times e^{-2 \left[(|\alpha_{\phi_{\perp}}|^{2} + |\alpha_{\phi}|^{2})(1 + 2S^{2}) + 2CS\operatorname{Re}(\alpha_{\phi_{\perp}}^{2} e^{i\phi} - \alpha_{\phi}^{2} e^{i\phi}) \right]},$$

$$W^{B}_{\phi}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) = \frac{4}{\pi^{2}} \left\{ 4 \left[|\alpha_{\phi}|^{2} (1 + 2S^{2}) + 2CS \operatorname{Re}(\alpha_{\phi}^{2} e^{i\phi}) \right] - 1 \right\} \times \\ \times e^{-2 \left[(|\alpha_{\phi_{\perp}}|^{2} + |\alpha_{\phi}|^{2})(1 + 2S^{2}) + 2CS\operatorname{Re}(\alpha_{\phi_{\perp}}^{2} e^{i\phi} - \alpha_{\phi}^{2} e^{i\phi}) \right]},$$
(C.5)

by rotating the polarization of the phase-space variables $\{\alpha_{\phi}, \alpha_{\phi_{\perp}}\}$ as:

$$\alpha_{\phi} = e^{\iota(\chi - \phi)/2} [\alpha_{\chi} \cos(\chi - \phi) - \iota \alpha_{\chi_{\perp}} \sin(\chi - \phi)],$$

$$\alpha_{\phi_{\perp}} = e^{\iota(\chi - \phi)/2} [\alpha_{\chi_{\perp}} \cos(\chi - \phi) - \iota \alpha_{\chi} \sin(\chi - \phi)].$$
(C.6)

Finally, we replace the complex phase-space variables with the real quadrature variables $(X_{\chi}, P_{\chi}, X_{\chi_{\perp}}, P_{\chi_{\perp}})$ and finally obtain the full expression for $\mathscr{Q}(X_{\chi}, P_{\chi}, X_{\chi_{\perp}}, P_{\chi_{\perp}}, \chi; \phi)$. However, this is too lengthy and rather uninformative and will not be reported here.

C.1.2 Single-mode correlator

We now calculate the single mode correlator $\mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi)$. This choice of the measurement operator allows to capture the nonlocal features of the micro-macro state generated by amplification of an entangled photon pair. To evaluate this quantity we exploit Eq. (C.2):

$$\mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi) = \frac{2}{\pi} \int \int d\Omega \, \mathscr{Q}(X_{\chi}, P_{\chi}, X_{\chi_{\perp}}, P_{\chi_{\perp}}, \chi; \phi), \qquad (C.7)$$

where the integral in $d^2 \alpha_{\chi_{\perp}}$ has been replaced by the integral in the quadrature variables $d\Omega = dX_{\chi_{\perp}} dP_{\chi_{\perp}}$. After straightforward algebra, we obtain the following expression for the correlator:

$$\mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi) = \cos[2(\chi - \phi)]e^{-2(e^{-2g}\overline{X}_{\chi}^2 + e^{2g}\overline{P}_{\chi}^2)} \left[1 - 2(e^{-2g}\overline{X}_{\chi}^2 + e^{2g}\overline{P}_{\chi}^2)\right], \quad (C.8)$$

where $\{\overline{X}_{\chi}, \overline{P}_{\chi}\}$ define a set of rotated variables $\overline{X}_{\chi} = X_{\chi} \cos(\chi/2) - P_{\chi} \sin(\chi/2), \overline{P}_{\chi} = X_{\chi} \sin(\chi/2) + P_{\chi} \cos(\chi/2)$. The maximum of such correlation operator is obtained at the origin of the phase-space and reads $\mathscr{C}(0, 0, \chi; \phi) = \cos[2(\chi - \phi)]$.

C.2 Correlator for the CHSH-based test under detection losses and nonunitary injection efficiency

Here we report in details the calculation of the correlator $\mathscr{C}_{p,\eta}$, when detection losses and a nonunitary injection efficiency are taken into account. These two effects represent the two main issues for an experimental observation of entanglement in a micro-macro system.

The model for the effect of losses at the detection stage is performed by inserting a beam-splitter of transmittivity η along the transmission path of the field on mode \mathbf{k}_B . The other port of this beam-splitter is injected with a vacuum state, thus introducing vacuum-noise fluctuations in the system. Here we demonstrate that the correlator \mathscr{C}_{η} in presence of detection losses η can be evaluated as the convolution of the lossless correlator \mathscr{C} with a Gaussian function of the form:

$$\mathscr{C}_{\eta}(X_{\chi}, P_{\chi}, \chi; \phi) = \frac{2}{\pi(1-\eta)} \int \int dX'_{\chi} dP'_{\chi} C(X'_{\chi}, P'_{\chi}, \chi; \phi) e^{-2\left[\frac{(X_{\chi} - \sqrt{\eta}X'_{\chi})^{2}}{1-\eta} + \frac{(P_{\chi} - \sqrt{\eta}P'_{\chi})^{2}}{1-\eta}\right]}.$$
(C.9)

We begin by writing the density matrix $\hat{\rho}_{\eta}^{\Psi^{-}}$ of the micro-macro state after losses occur at the detection stage:

$$\hat{\rho}_{\eta}^{\Psi^{-}} = \frac{1}{2} \Big\{ |\phi\rangle_{A} \langle \phi| \otimes \mathscr{L} \big[|\Phi^{\phi_{\perp}}\rangle_{B} \langle \Phi^{\phi_{\perp}}| \big] |\phi_{\perp}\rangle_{A} \langle \phi_{\perp}| \otimes \mathscr{L} \big[|\Phi^{\phi}\rangle_{B} \langle \Phi^{\phi}| \big] \\ - |\phi\rangle_{A} \langle \phi_{\perp}| \otimes \mathscr{L} \big[|\Phi^{\phi_{\perp}}\rangle_{B} \langle \Phi^{\phi}| \big] - |\phi_{\perp}\rangle_{A} \langle \phi| \otimes \mathscr{L} \big[|\Phi^{\phi}\rangle_{B} \langle \Phi^{\phi_{\perp}}| \big] \Big\},$$
(C.10)

where $\mathscr{L}[\cdot]$ is the map that describes the action of detection losses. The evaluation of the correlation operator \mathscr{Q} on this density matrix leads to:

$$\mathscr{Q}_{\eta}(\alpha_{\chi}, \alpha_{\chi_{\perp}}, \chi; \phi) = \frac{\pi^2}{8} \left[W^B_{\eta, \phi_{\perp}}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) - W^B_{\eta, \phi}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) \right], \quad (C.11)$$

where $W^B_{\eta,\phi}$ and $W^B_{\eta,\phi_{\perp}}$ are the Wigner functions of the macrostates $|\Phi^{\phi}\rangle$ and $|\Phi^{\phi_{\perp}}\rangle$ after losses. The action of detection losses in the phase-space can be written in the form of a

Gaussian convolution [Leo93]:

$$W_{\eta}(X,P) = \int \int dX' dP' W(X,P) K_{\eta}(X,P,X',P'),$$
 (C.12)

where $K_{\eta}(X, P, X', P') = \frac{2}{\pi(1-\eta)} \exp\{-2\left[\frac{(X-\sqrt{\eta}X')^2}{1-\eta} + \frac{(P-\sqrt{\eta}P')^2}{1-\eta}\right]\}$. The correlator \mathscr{C}_{η} is obtained from \mathscr{Q}_{η} as:

$$\mathscr{C}_{\eta}(X_{\chi}, P_{\chi}, \chi; \phi) = \frac{2}{\pi} \int \int d\Omega \mathscr{Q}_{\eta}(X_{\chi}, P_{\chi}, X_{\chi_{\perp}}, P_{\chi_{\perp}}, \chi; \phi).$$
(C.13)

By writing explicitly the Wigner function after losses as a Gaussian convolution we obtain

$$\mathscr{C}_{\eta}(X_{\chi}, P_{\chi}, \chi; \phi) = \frac{2}{\pi} \int \int dX_{\chi_{\perp}} dP_{\chi_{\perp}} \mathscr{I}(X'_{\chi}, P'_{\chi}), \qquad (C.14)$$

where:

$$\mathscr{I}(X'_{\chi}, P'_{\chi}) = \int \int dX'_{\chi_{\perp}} dP'_{\chi_{\perp}} \mathscr{Q}(X'_{\chi}, P'_{\chi}, X'_{\chi_{\perp}}, P'_{\chi_{\perp}}, \chi; \phi) \times \int \int dX_{\chi_{\perp}} dP_{\chi_{\perp}} K_{\eta}(X_{\chi_{\perp}}, P_{\chi_{\perp}}, X'_{\chi_{\perp}}, P'_{\chi_{\perp}}).$$
(C.15)

By changing the integration variables as $X_{\chi_{\perp}} \rightarrow \tilde{X}_{\chi_{\perp}} = \frac{X_{\chi_{\perp}} - \sqrt{\eta} X'_{\chi_{\perp}}}{\sqrt{1-\eta}}$, $P_{\chi_{\perp}} \rightarrow \tilde{P}_{\chi_{\perp}} = \frac{P_{\chi_{\perp}} - \sqrt{\eta} P'_{\chi_{\perp}}}{\sqrt{1-\eta}}$ we have the explicit function:

$$\mathscr{I}(X'_{\chi},P'_{\chi}) = \frac{2|J|}{\pi(1-\eta)} \int \int d\tilde{X}_{\chi_{\perp}} d\tilde{P}_{\chi_{\perp}} e^{-2(\tilde{X}^{2}_{\chi_{\perp}}+\tilde{P}^{2}_{\chi_{\perp}})} \\ \times \int \int dX'_{\chi_{\perp}} dP'_{\chi_{\perp}} \mathscr{Q}(X'_{\chi},P'_{\chi},X'_{\chi_{\perp}},P'_{\chi_{\perp}},\chi;\phi),$$
(C.16)

where $|J| = 1 - \eta$. Eq. (C.9) is found by integrating over $d\tilde{X}_{\chi\perp}d\tilde{P}_{\chi\perp}$, using Eq. (C.2) to have $\mathscr{I}(X'_{\chi}, P'_{\chi}) = \mathscr{C}(X'_{\chi}, P'_{\chi}, \chi; \phi)$ and replacing this in Eq. (C.14).

We now proceed with the explicit calculation of Eq. (C.9). As a first step, we rotate the quadratures (X_{χ}, P_{χ}) and the integration variables (X'_{χ}, P'_{χ}) as:

$$\overline{\mathscr{X}}_{\chi} = \mathscr{X}_{\chi} \cos(\chi/2) - \mathscr{P}_{\chi} \sin(\chi/2),
\overline{\mathscr{P}}_{\chi} = \mathscr{X}_{\chi} \sin(\chi/2) + \mathscr{P}_{\chi} \cos(\chi/2),$$
(C.17)

with $\mathscr{X} = (X, X')$ and $\mathscr{P} = (P, P')$ and the convention that only primed (unprimed) variables are involved in the equations above. The correlator \mathscr{C}_{η} can be then expressed as a function of the rotated variables. After replacing the expression of K_{η} in the correlator \mathscr{C}_{η} , it is matter of some straightforward (although tedious) algebra to find that:

$$\mathscr{C}_{\eta}(\overline{X}_{\chi}, \overline{P}_{\chi}, \chi; \phi) = \frac{\cos[2(\chi - \phi)]e^{-2\left[\frac{\overline{X}_{\chi}^{2}}{\mathscr{M}} + \frac{\overline{P}_{\chi}^{2}}{\mathscr{N}}\right]}}{\sqrt{1 + 4\eta(1 - \eta)\overline{n}}} \times \left\{ 1 - \frac{(1 - \eta)(1 + 2\eta\overline{n})}{1 + 4\eta(1 - \eta)\overline{n}} - 2\eta \left[\frac{e^{2g}\overline{X}_{\chi}^{2}}{\mathscr{M}^{2}} + \frac{e^{-2g}\overline{P}_{\chi}^{2}}{\mathscr{N}^{2}} \right] \right\},$$
(C.18)

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with $\mathcal{M} = \eta e^{2g} + (1 - \eta)$ and $\mathcal{N} = \eta e^{-2g} + (1 - \eta)$. This expression is maximized at the origin of the phase space, reading:

$$\mathscr{C}_{\eta}(\overline{X}_{\chi}, \overline{P}_{\chi}, \chi; \phi) = \cos[2(\chi - \phi)] \mathscr{L}(\eta, g), \qquad (C.19)$$

where the loss function $\mathscr{L}(\eta, g)$ has the form:

$$\mathscr{L}(\eta,g) = \frac{\eta + 2\eta(1-\eta)\overline{n}}{(1+4\eta(1-\eta)\overline{n})^{3/2}}.$$
(C.20)

In typical experimental conditions, the injection of the single photon of the entangled pair $|\Psi^-\rangle_{AB}$ into the optical parametric amplifier, occurs with an efficiency p < 1 because of the imperfect matching between the optical modes of the amplifier and the singlephoton one. Such a non-ideality can be modeled by allowing for a probability p of correct single-photon injection and a complementary probability (1-p) that just vacuum state is injected in the amplifier and no correlations between the two output modes are set. This modifies the density matrix of the output modes as:

$$\hat{\rho}_{p}^{\psi^{-}} = p |\psi^{-}\rangle_{AB} \langle \psi^{-}| + (1-p) \frac{\hat{\mathbb{1}}_{A}}{2} \otimes |0\rangle_{B} \langle 0|, \qquad (C.21)$$

where $\hat{\mathbb{1}}_A = |H\rangle_A \langle H| + |V\rangle_A \langle V|$ is a completely mixed single-photon polarization state, and $|0\rangle_B \langle 0|$ is the vacuum state. The bipartite state after the amplification process then reads:

$$\hat{\rho}_{p}^{\Psi^{-}} = p|\Psi^{-}\rangle_{AB}\langle\Psi^{-}| + (1-p)\frac{\hat{\mathbb{1}}_{A}}{2}\otimes\left(\hat{U}_{OPA}|0\rangle_{B}\langle0|\hat{U}_{OPA}^{\dagger}\right).$$
(C.22)

We can now proceed with the calculation of $\hat{\mathscr{C}}(\alpha_{\chi}, \chi; \phi)$ as:

$$\mathscr{C}(\boldsymbol{\alpha}_{\boldsymbol{\chi}},\boldsymbol{\chi};\boldsymbol{\phi}) = p_{AB} \langle \Psi^{-} | \hat{\boldsymbol{\sigma}}^{A}(\boldsymbol{\phi}) \otimes \hat{\Pi}^{B}(\boldsymbol{\alpha}_{\boldsymbol{\chi}},\boldsymbol{\chi}) | \Psi^{-} \rangle_{AB} + (1-p) \mathrm{Tr} \bigg[\frac{\hat{\mathbb{1}}_{A}}{2} \otimes \left(\hat{U}_{OPA} | 0 \rangle_{B} \langle 0 | \hat{U}_{OPA}^{\dagger} \right) \hat{\boldsymbol{\sigma}}^{A}(\boldsymbol{\phi}) \otimes \hat{\Pi}^{B}(\boldsymbol{\alpha}_{\boldsymbol{\chi}},\boldsymbol{\chi}) \bigg].$$
(C.23)

As the second term factorizes (due to the lack of quantum correlations) and $\operatorname{Tr}\left[\frac{\mathbb{1}_{A}}{2}\hat{\sigma}^{A}(\phi)\right] = 0$, such contribution is null. Therefore, under non-ideal injection efficiency, the correlator is related to the ideal one according to $\mathscr{C}_{p}(X_{\chi}, P_{\chi}, \chi; \phi) = p \mathscr{C}(X_{\chi}, P_{\chi}, \chi; \phi)$. This result can be extended to the case of nonunitary detection efficiency, leading to:

$$\mathscr{C}_{\eta,p}(X_{\chi}, P_{\chi}, \chi; \phi) = p \, \mathscr{C}_{\eta}(X_{\chi}, P_{\chi}, \chi; \phi). \tag{C.24}$$

C.3 Correlator for the entanglement witness after detection losses and nonunitary injection efficiency

Here we sketch the steps needed for the calculation of the correlator $\mathscr{C}_{p,\eta}$ entering the entanglement test based on the witness operator of Eq. (6.11) under losses and non-ideal

photon injection. By using arguments similar to those put forward in the previous sections, we have:

$$\widetilde{\mathscr{C}}_{\eta}(\alpha_{\chi},\chi;\phi) = \frac{1}{2} \left\{ \operatorname{Tr} \left[\mathscr{L} \left[|\Phi^{\phi_{\perp}}\rangle_{B} \langle \Phi^{\phi_{\perp}}| \right] \widehat{O}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta) \right] - \operatorname{Tr} \left[\mathscr{L} \left[|\Phi^{\phi}\rangle_{B} \langle \Phi^{\phi}| \right] \widehat{O}^{B}_{\chi,\chi_{\perp}}(\alpha_{\chi},\chi;\eta) \right] \right\},$$
(C.25)

where $\mathscr{L}[\cdot]$ is the map describing the lossy process. We focus on the case $\eta \ge \frac{1}{2}$. By exploiting results that have been previously obtained here, we have:

$$\widetilde{\mathscr{C}}_{\eta}(\alpha_{\chi},\chi;\phi) = \frac{\pi}{4\eta} \int d^2 \alpha_{\chi_{\perp}} \left(W^{\eta}_{\phi_{\perp}}(\alpha_{\chi},\alpha_{\chi_{\perp}}) - W^{\eta}_{\phi}(\alpha_{\chi},\alpha_{\chi_{\perp}}) \right).$$
(C.26)

We now exploit the chain of relations:

$$\frac{\pi}{4} \int d^{2} \alpha_{\chi_{\perp}} \left(W^{\eta}_{\phi_{\perp}}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) - W^{\eta}_{\phi}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) \right) = \\
= \frac{2}{\pi} \int d^{2} \alpha_{\chi_{\perp}} \frac{\pi^{2}}{8} \left(W^{\eta}_{\phi_{\perp}}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) - W^{\eta}_{\phi}(\alpha_{\chi}, \alpha_{\chi_{\perp}}) \right) = \\
= \frac{2}{\pi} \int d^{2} \alpha_{\chi_{\perp}} \mathscr{Q}_{\eta}(\alpha_{\chi}, \alpha_{\chi_{\perp}}, \chi; \phi) = \mathscr{C}_{\eta}(\alpha_{\chi}, \chi; \phi),$$
(C.27)

so as to get $\tilde{\mathscr{C}}_{\eta}(\alpha_{\chi}, \chi; \phi; \eta) = \frac{1}{\eta} \mathscr{C}_{\eta}^{AB}(\alpha_{\chi}, \chi; \phi)$. With an analogous procedure, we obtain:

$$\tilde{\mathscr{C}}_{\eta}(\alpha_{\chi},\chi;\phi;\eta) = \begin{cases} \frac{1}{\eta} \mathscr{C}_{\eta}(\alpha_{\chi},\chi;\phi) & \text{if } \frac{1}{2} < \eta \leq 1, \\ 2 \mathscr{C}_{\eta}(\alpha_{\chi},\chi;\phi) & \text{if } \eta \leq \frac{1}{2}. \end{cases}$$
(C.28)

We can further generalize this result so as to take into account the effect of a nonunitary injection efficiency and finally get $\tilde{\mathscr{E}}_{\eta,p}(\alpha_{\chi},\chi;\phi;\eta) = p \,\tilde{\mathscr{E}}_{\eta}(\alpha_{\chi},\chi;\phi;\eta)$.

Appendix D

Linearity of the map describing the photon-addition process' implementation

In this section we demonstrate that the process of photon addition is described by a linear map, that is, possessing the following property:

$$\mathscr{A}\left[\sum_{n,m}\rho_{nm}|n\rangle\langle m|\right] = \sum_{n,m}\rho_{nm}\mathscr{A}\left[|n\rangle\langle m|\right].$$
(D.1)

To proceed with the proof, we now write a formal expression for the map that describes the process shown in Fig.8.2. By applying the sequence of operation (i)-(v) that represent the optical implementation of the photon-addition process, we obtain the following map:

$$\mathscr{A}[\hat{\rho}_{A}] = \operatorname{Tr}_{B} \left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD} \left[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) \left[(\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \left(\hat{\rho}_{A} \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{D} \langle 0| \right) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \right] (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) \left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \right].$$
(D.2)

We now proceed with the proof of the linearity of the process \mathscr{A} . Replacing the input density matrix $\hat{\rho}_A = \sum_{n,m} \rho_{nm} |n\rangle_A \langle m|$ we obtain:

$$\mathscr{A}[\hat{\rho}_{A}] = \mathscr{A}\left[\sum_{n,m} \rho_{nm} |n\rangle_{A} \langle m|\right] = \operatorname{Tr}_{B}\left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD}\left[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r})\right. \\ \left[(\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D})\left(\sum_{n,m} \rho_{nm} |n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0|\right)\right] (D.3) \\ \left.(\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D})\right](\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger})\right](\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger})\right].$$

By exploiting the linearity of the unitary squeezing process $\hat{U}_{AB}^r \otimes \hat{\mathbb{1}}_C \otimes \hat{\mathbb{1}}_D$:

$$\mathscr{A}[\hat{\rho}_{A}] = \operatorname{Tr}_{B} \left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD} \left[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) \left[\sum_{n,m} \rho_{nm} (\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \left(|n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0| \right) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \right] (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) (\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \right].$$
(D.4)

Linearity of the map describing the photon-addition process' implementation

By exploiting the linearity of the unitary parasitic squeezing processes $\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}$:

$$\mathscr{A}[\hat{\rho}_{A}] = \operatorname{Tr}_{B} \left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD} \left[\sum_{n,m} \rho_{nm} (\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) (\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \left(|n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0| \right) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) \right] (\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \right].$$
(D.5)

By exploiting the linearity of the partial trace operation Tr_{CD} :

$$\mathscr{A}[\hat{\rho}_{A}] = \operatorname{Tr}_{B} \left[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \left[\sum_{n,m} \rho_{nm} \operatorname{Tr}_{CD} \left[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) (\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \left(|n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0| \right) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) \right] \right] (\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \right].$$

$$(D.6)$$

By exploiting the linearity of the operation $\hat{\mathbb{1}}_A \otimes \hat{a}_B$, which is written in the form of a Kraus operator:

$$\mathscr{A}[\hat{\rho}_{A}] = \operatorname{Tr}_{B} \left[\sum_{n,m} \rho_{nm}(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD} \left[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) (\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \left(|n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0| \right) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) \right] (\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \right].$$
(D.7)

Finally by exploing the linearity of the partial trace operation Tr_B :

$$\mathscr{A}[\hat{\rho}_{A}] = \sum_{n,m} \rho_{nm} \operatorname{Tr}_{B} \Big[(\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}) \operatorname{Tr}_{CD} \Big[(\hat{U}_{AC}^{\gamma r} \otimes \hat{U}_{BD}^{\gamma r}) (\hat{U}_{AB}^{r} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) \Big(|n\rangle_{A} \langle m| \otimes |0\rangle_{B} \langle 0| \otimes |0\rangle_{C} \langle 0| \otimes |0\rangle_{D} \langle 0| \Big) (\hat{U}_{AB}^{r \dagger} \otimes \hat{\mathbb{1}}_{C} \otimes \hat{\mathbb{1}}_{D}) (\hat{U}_{AC}^{\gamma r \dagger} \otimes \hat{U}_{BD}^{\gamma r \dagger}) \Big] (\hat{\mathbb{1}}_{A} \otimes \hat{a}_{B}^{\dagger}) \Big] = \sum_{n,m} \rho_{nm} \mathscr{A} \Big[|n\rangle_{A} \langle m| \Big],$$
(D.8)

which concludes the proof:

$$\mathscr{A}[\hat{\rho}_A] = \mathscr{A}\Big[\sum_{n,m} \rho_{nm} |n\rangle_A \langle m|\Big] = \sum_{n,m} \rho_{nm} \mathscr{A}\Big[|n\rangle_A \langle m|\Big]. \tag{D.9}$$

Such result allows to express the action of the photon addition process in the form:

$$\left(\mathscr{A}[\hat{\rho}_{A}]\right)_{lk} =_{A} \langle l|\mathscr{A}[\hat{\rho}_{A}]|k\rangle_{A} = \sum_{n,m} \rho_{nmA} \langle l|\mathscr{A}\left[|n\rangle_{A}\langle m|\right]|k\rangle_{A} = \sum_{n,m} \rho_{nm} \mathscr{A}_{lk}^{nm}.$$
 (D.10)

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Appendix E

Quantum and Classical Fisher information for the amplifier-based phase estimation protocol with coherent states

In this appendix we elaborate on the material presented in Chaps. 9 and 10, giving more details on the derivation of the formulas presented there.

E.1 State evolution

In this section we calculate the explicit form of the output state $\hat{\rho}_{\phi}^{\beta,g,\eta}$ of our scheme, by exploiting some operatorial relations for Gaussian states. This will be useful to evaluate the quantum and classical Fisher informations in the following sections. The state impinging at the measurement stage after detection losses can be written in the form:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \mathscr{L}_{\eta} \Big\{ \hat{S}_{+}(\tau_{+}) \hat{S}_{-}(\tau_{-}) \mathscr{L}_{\xi} \Big[\hat{D}_{+}(\alpha_{+}) \hat{D}_{-}(\alpha_{-}) |0\rangle \langle 0| \\ \hat{D}_{+}^{\dagger}(\alpha_{+}) \hat{D}_{-}^{\dagger}(\alpha_{-}) \Big] \hat{S}_{+}^{\dagger}(\tau_{+}) \hat{S}_{-}^{\dagger}(\tau_{-}) \Big\},$$
(E.1)

where $\hat{D}_l(\alpha_l) = \exp(\alpha_l \hat{a}_l^{\dagger} - \alpha_l^* \hat{a}_l)$ is the displacement operator such that $\hat{D}(\alpha)|0\rangle = |\alpha\rangle$. The action of the lossy channel ξ and of the displacement operators can be interchanged as:

$$\mathscr{L}_{\xi} \Big[\hat{D}_{+}(\alpha_{+}) \hat{D}_{-}(\alpha_{-}) | 0 \rangle \langle 0 | \hat{D}_{+}^{\dagger}(\alpha_{+}) \hat{D}_{-}^{\dagger}(\alpha_{-}) \Big] = \hat{D}_{+}(\beta_{+}) \hat{D}_{-}(\beta_{-}) | 0 \rangle \langle 0 | \hat{D}_{+}^{\dagger}(\beta_{+}) \hat{D}_{-}^{\dagger}(\beta_{-}),$$
(E.2)

where $\beta_l = \sqrt{\xi} \alpha_l$. The output state then reads:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \mathscr{L}_{\eta} \Big\{ \hat{S}_{+}(\tau_{+}) \hat{S}_{-}(\tau_{-}) \hat{D}_{+}(\beta_{+}) \hat{D}_{-}(\beta_{-}) |0\rangle \langle 0| \hat{D}_{+}^{\dagger}(\beta_{+}) \hat{D}_{-}^{\dagger}(\beta_{-}) \hat{S}_{+}^{\dagger}(\tau_{+}) \hat{S}_{-}^{\dagger}(\tau_{-}) \Big\}.$$
(E.3)

The action of the squeezing operators and of the displacement operators can be now inverted according to:

$$\hat{D}(\alpha)\hat{S}(\tau) = \hat{S}(\tau)\hat{D}(\tilde{\alpha}_{+}), \qquad (E.4)$$

$$\hat{S}(\tau)\hat{D}(\alpha) = \hat{D}(\tilde{\alpha}_{-})\hat{S}(\tau), \qquad (E.5)$$

where $\tilde{\alpha}_{\pm} \equiv \alpha \cosh g \pm \alpha^* e^{i\lambda} \sinh g$. Using Eqs.(E.4-E.5) we can write:

$$\hat{S}_l(\tau_l)\hat{D}_l(\beta_l)|0\rangle = \hat{D}_l(\gamma_l)\hat{S}_l(\tau_l)|0\rangle, \qquad (E.6)$$

with $\gamma_l \equiv \beta_l \cosh \tau_l - \beta_l^* e^{i\lambda_l} \sinh \tau_l$. The output state can be then written as:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \mathscr{L}_{\eta} \Big\{ \hat{D}_{+}(\gamma_{+}) \hat{D}_{-}(\gamma_{-}) \hat{S}_{+}(\tau_{+}) \hat{S}_{-}(\tau_{-}) |0\rangle \langle 0| \hat{S}_{+}^{\dagger}(\tau_{+}) \hat{S}_{-}^{\dagger}(\tau_{-}) \hat{D}_{+}^{\dagger}(\gamma_{+}) \hat{D}_{-}^{\dagger}(\gamma_{-}) \Big\}.$$
(E.7)

By interchanging the action of the loss \mathscr{L}_{η} and of the displacement operators $\hat{D}_{l}(\gamma_{l})$, we obtain:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{D}_{+}(\tilde{\gamma}_{+})\hat{D}_{-}(\tilde{\gamma}_{-})\mathscr{L}_{\eta}\left\{\hat{S}_{+}(\tau_{+})\hat{S}_{-}(\tau_{-})|0\rangle\langle 0|\hat{S}_{+}^{\dagger}(\tau_{+})\hat{S}_{-}^{\dagger}(\tau_{-})\right\}\hat{D}_{+}^{\dagger}(\tilde{\gamma}_{+})\hat{D}_{-}^{\dagger}(\tilde{\gamma}_{-}),$$
(E.8)

where $\tilde{\gamma}_l = \sqrt{\eta} \gamma_l$. Finally, we exploit the following identity involving the action of \mathcal{L}_{η} on squeezed vacuum states [ACMTB09]:

$$\mathscr{L}_{\eta}\left[\hat{S}(\tau)|0\rangle\langle 0|\hat{S}^{\dagger}(\tau)\right] = \hat{S}^{\dagger}(\tau^{\text{eff}})\hat{\rho}^{th}(N^{\text{eff}})\hat{S}(\tau^{\text{eff}}), \tag{E.9}$$

where $\hat{\rho}^{th}$ is a thermal state. The effective modulus of the squeezing parameter τ^{eff} and the effective thermal noise N^{eff} take the form:

$$\tau^{\text{eff}} = \frac{1}{4} \log(\frac{P}{M}), \quad N^{\text{eff}} = \frac{-1 + \sqrt{PM}}{2},$$
 (E.10)

where:

$$P = \eta e^{2g} + 1 - \eta, \quad M = \eta e^{-2g} + 1 - \eta.$$
 (E.11)

We can express the output state after detection losses in the Gaussian form

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{D}_{+}(\tilde{\gamma}_{+})\hat{D}_{-}(\tilde{\gamma}_{-})\hat{S}_{+}(\tau_{+}^{\text{eff}})\hat{S}_{-}(\tau_{-}^{\text{eff}})\Big[\hat{\rho}_{+}^{th}(N_{\text{eff}})\otimes \hat{\rho}_{-}^{th}(N_{\text{eff}})\Big]\hat{S}_{+}^{\dagger}(\tau_{+}^{\text{eff}})\hat{S}_{-}^{\dagger}(\tau_{-}^{\text{eff}})\hat{D}_{+}^{\dagger}(\tilde{\gamma}_{+})\hat{D}_{-}^{\dagger}(\tilde{\gamma}_{-}).$$
(E.12)

E.1.1 Eigenvalues and Eigenvectors

From Eq. (E.12) one can calculate the spectrum of eigenvalues and eigenvectors of $\hat{\rho}_{\phi}^{\beta,g,\eta}$. As a first step, we observe that the density matrix of the state takes the form of a separable state $\hat{\rho}_{\phi}^{(+)} \otimes \hat{\rho}_{\phi}^{(-)}$, where:

$$\hat{\boldsymbol{\rho}}_{\phi}^{(l)} = \hat{D}_{l}(\tilde{\boldsymbol{\gamma}}_{l})\hat{S}_{l}(\tau_{l}^{\text{eff}})\hat{\boldsymbol{\rho}}_{l}^{lh}(N_{l}^{\text{eff}})\hat{S}_{l}^{\dagger}(\tau_{l}^{\text{eff}})\hat{D}_{l}^{\dagger}(\tilde{\boldsymbol{\gamma}}_{l}), \qquad (E.13)$$

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with l = +, -. Since the state for the two modes has the same Gaussian form, the joint spectrum can be obtained by analyzing directly the $\hat{\rho}_{\phi}^{(l)}$ single-mode state. By expanding the density matrix in the Fock basis we obtain:

$$\hat{\boldsymbol{\rho}}_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{(N_l^{\text{eff}})^n}{(1+N_l^{\text{eff}})^{n+1}} \hat{D}_l(\tilde{\boldsymbol{\gamma}}_l) \hat{S}_l(\tau_l^{\text{eff}}) |n\rangle_l \langle n| \hat{S}_l^{\dagger}(\tau_l^{\text{eff}}) \hat{D}_l^{\dagger}(\tilde{\boldsymbol{\gamma}}_l).$$
(E.14)

The eigenvalues and the eigenvectors of the state $\hat{\rho}_{\phi}^{(l)} = \sum_{n} \rho_{n}^{(l)} |\psi_{n}^{(l)}\rangle_{l} \langle \psi_{n}^{(l)}|$ are then respectively:

$$\rho_n^{(l)} = \frac{(N_l^{\text{eff}})^n}{(1+N_l^{\text{eff}})^{n+1}},$$
(E.15)

$$|\psi_n^{(l)}\rangle_l = \hat{D}_l(\tilde{\gamma}_l)\hat{S}_l(\tau_l^{\text{eff}})|n\rangle_l.$$
 (E.16)

Finally, the eigenvalues and the eigenvectors of the joint two-modes density matrix can be written as:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \sum_{m,n=0}^{\infty} \rho_{m,n} |\Psi_{m,n}\rangle_{HV} \langle \Psi_{m,n}|, \qquad (E.17)$$

$$\rho_{m,n} = \rho_m^{(H)} \rho_n^{(V)},$$
(E.18)

$$|\Psi_{m,n}\rangle_{HV} = |\psi_m^{(H)}\rangle_H \otimes |\psi_n^{(V)}\rangle_V.$$
 (E.19)

E.2 Quantum Fisher Information

In this section we describe the calculation of the quantum Fisher information (QFI) of the output state $\hat{\rho}_{\phi}^{\beta,g,\eta}$ of our scheme.

The QFI for a generic mixed state $\hat{\sigma} = \sum_{m} \sigma_{m} |\zeta_{m}\rangle \langle \zeta_{m}|$ can be evaluated as [Par09]:

$$H_{\phi} = \sum_{p} \frac{(\partial_{\phi} \sigma_{p})^{2}}{\sigma_{p}} + 2 \sum_{n,m} \varepsilon_{n,m} |\langle \zeta_{m} | \partial_{\phi} \zeta_{n} \rangle|^{2}.$$
(E.20)

Here σ_m and $|\zeta_m\rangle$ are respectively the eigenvalues and the eigenvectors of the density matrix, and $\varepsilon_{n,m} = (\sigma_n - \sigma_m)^2/(\sigma_n + \sigma_m)$. In the case of the output density matrix $\hat{\rho}_{\phi}^{\beta,g,\eta}$ of the amplifier-based protocol the eigenvalues and the eigenvectors are parametrized by the indices (n,m), and the QFI is:

$$H(\alpha,\xi,\{g_l\},\{\lambda_l\},\eta) = \sum_{p,q=0}^{\infty} \frac{(\partial_{\phi} \rho_{p,q})^2}{\rho_{p,q}} + 2\sum_{i,j,m,n=0}^{\infty} \varepsilon_{i,j,m,n} |\langle \Psi_{i,j} | \partial_{\phi} \Psi_{m,n} \rangle|^2, \quad (E.21)$$

where:

$$\varepsilon_{i,j,m,n} = \frac{(\rho_{i,j} - \rho_{m,n})^2}{\rho_{i,j} + \rho_{m,n}}.$$
(E.22)

We observe that, for the density matrix $\hat{\rho}_{\phi}^{\beta,g,\eta}$, the eigenvalues $\rho_{m,n}$ (E.17) are independent on the phase ϕ , and hence the first term in Eq.(E.21) vanishes. In order to calculate the second term, it is necessary to evaluate the following quantity: $|\langle \Psi_{i,j} | \partial_{\phi} \Psi_{m,n} \rangle|^2$. Such term can be written as:

$$\begin{split} \langle \Psi_{i,j} | \partial_{\phi} \Psi_{m,n} \rangle &= \langle \Psi_{i,j} | \partial_{\phi} \left(| \psi_m^{(1)} \rangle_1 \otimes | \psi_n^{(2)} \rangle_2 \right) = \\ &= \langle \Psi_{i,j} | \left(| \partial_{\phi} \psi_m^{(1)} \rangle_1 \otimes | \psi_n^{(2)} \rangle_2 + | \psi_m^{(1)} \rangle_1 \otimes | \partial_{\phi} \psi_n^{(2)} \rangle_2 \right) = \\ &= {}_1 \langle \psi_i^{(1)} | \partial_{\phi} \psi_m^{(1)} \rangle_1 \delta_{j,n} + \delta_{i,m\,2} \langle \partial_{\phi} \psi_i^{(2)} | \psi_m^{(2)} \rangle_2. \end{split}$$
(E.23)

Since the eigenvectors for the two-modes present an analogous form, it is necessary to evaluate only the term $_{l}\langle \psi_{i}^{(l)}|\partial_{\phi}\psi_{m}^{(l)}\rangle_{l}$. Let us focus on the $|\partial_{\phi}\psi_{m}^{(l)}\rangle_{l}$ state vector. Since the dependence on ϕ of the state is included only in the displacement operator $\hat{D}_l(\tilde{\gamma}_l)$, we can write:

$$|\partial_{\phi} \psi_m^{(l)}\rangle_l = \left[\partial_{\phi} \hat{D}_l(\tilde{\gamma}_l)\right] \hat{S}_l(\tau_l^{\text{eff}}) |m\rangle_l. \tag{E.24}$$

The latter can be evaluated by differentiating the displacement operator written in normallyordered form:

$$\partial_{\phi} \left[\hat{D}_{l}(\tilde{\gamma}_{l}) \right] = \partial_{\phi} \left[e^{-\frac{1}{2} |\tilde{\gamma}_{l}|^{2}} e^{\tilde{\gamma}_{l} a_{l}^{\dagger}} e^{-\tilde{\gamma}_{l}^{*} a_{l}} \right].$$
(E.25)

By differentiating the three exponential with respect to ϕ , and by exploiting the following commutation relation:

$$\left[\hat{a}_{l}, e^{\tilde{\gamma}\hat{a}_{l}^{\dagger}}\right] = \tilde{\gamma}_{l} e^{\tilde{\gamma}\hat{a}_{l}^{\dagger}}, \qquad (E.26)$$

the derivative of $\hat{D}_l(\tilde{\gamma}_l)$ reads:

$$\partial_{\phi} \left[\hat{D}_{l}(\tilde{\eta}) \right] = \left[C_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)} + \hat{F}_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)}(\hat{a}_{l},\hat{a}_{l}^{\dagger}) \right] \hat{D}_{l}(\tilde{\eta}).$$
(E.27)

The scalar $C_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}^{(l)}$ and the operator $\hat{F}_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}^{(l)}(\hat{a}_l,\hat{a}_l^{\dagger})$ are respectively:

$$C^{(l)}_{\alpha,\xi,g_l,\lambda_l,\eta,\phi} = \frac{1}{2} \big[\tilde{\gamma}_l(\partial_\phi \tilde{\gamma}_l^*) - (\partial_\phi \tilde{\gamma}_l) \tilde{\gamma}_l^* \big], \qquad (E.28)$$

$$\hat{F}^{(l)}_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}(\hat{a}_l,\hat{a}_l^{\dagger}) = (\partial_{\phi}\tilde{\gamma}_l)\hat{a}_l^{\dagger} - (\partial_{\phi}\tilde{\gamma}_l^*)\hat{a}_l.$$
(E.29)

By replacing the latter expressions in Eq.(E.24), the scalar product $_{l}\langle \psi_{i}^{(l)}|\partial_{\phi}\psi_{m}^{(l)}\rangle_{l}$ can be evaluated as:

$${}_{l}\langle\psi_{i}^{(l)}|\partial_{\phi}\psi_{m}^{(l)}\rangle_{l} = {}_{l}\langle i|\hat{S}_{l}^{\dagger}(\tau_{l}^{\text{eff}})\hat{D}_{l}^{\dagger}(\tilde{\gamma}_{l}) \left[C_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)} + \hat{F}_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)}(\hat{a}_{l},\hat{a}_{l}^{\dagger})\right]\hat{D}_{l}(\tilde{\gamma}_{l})\hat{S}_{l}(\tau_{l}^{\text{eff}})|m\rangle_{l}$$
(E.30)

Such average value can be evaluated by exploiting the operatorial identities:

 $\hat{S}^{\dagger}(\tau)\hat{a}\hat{S}(\tau) \ = \ \hat{a}\cosh g - \hat{a}^{\dagger}e^{\imath\lambda} \sinh g,$ (E.31)

$$\hat{S}^{\dagger}(\tau)\hat{a}^{\dagger}\hat{S}(\tau) = \hat{a}^{\dagger}\cosh g - \hat{a}e^{-\imath\lambda}\sinh g, \qquad (E.32)$$

$$\hat{D}^{\dagger}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \qquad (E.33)$$

$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^{*}.$$
(E.34)
$$\hat{D}^{\dagger}(\alpha)\hat{a}^{\dagger}\hat{D}(\alpha) = \hat{a}^{\dagger} + \alpha^{*}.$$

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We obtain:

$${}_{l}\langle\psi_{i}^{(l)}|\partial_{\phi}\psi_{m}^{(l)}\rangle_{l} = \delta_{i,m}A_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)} - \delta_{i,m-1}\sqrt{m}B_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)*} + \delta_{i,m+1}\sqrt{m+1}B_{\alpha,\xi,g_{l},\lambda_{l},\eta,\phi}^{(l)},$$
(E.35)

where the $A_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}^{(l)}$ and $B_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}^{(l)}$ quantities are defined as:

$$A^{(l)}_{\alpha,\xi,g_l,\lambda_l,\eta,\phi} = \frac{1}{2} \big[(\partial_{\phi} \tilde{\gamma}_l) \tilde{\gamma}_l^* - \tilde{\gamma}_l (\partial_{\phi} \tilde{\gamma}_l^*) \big], \tag{E.36}$$

$$B_{\alpha,\xi,g_l,\lambda_l,\eta,\phi}^{(l)} = \cosh g_l^{\text{eff}}(\partial_\phi \tilde{\gamma}_l) - e^{i\lambda_l} \sinh g_l^{\text{eff}}(\partial_\phi \tilde{\gamma}_l^*).$$
(E.37)

Note that the $\varepsilon_{i,j,m,n}$ coefficients present the following symmetries:

$$\varepsilon_{m,n,m,n} = 0, \quad \varepsilon_{i,j,m,n} = \varepsilon_{m,j,i,n}, \quad \varepsilon_{i,j,m,n} = \varepsilon_{i,n,m,j}.$$
 (E.38)

By inserting Eqs. (E.23)-(E.35) in Eq.(E.21) and by exploiting the symmetries of the $\varepsilon_{i,j,m,n}$ coefficients we obtain:

$$H(\alpha, \xi, \{g_l\}, \{\lambda_l\}, \eta) = 4 \sum_{m,n=0}^{\infty} \left[|B_{\alpha,\xi,g_l,\lambda_l,\eta}^{(1)}|^2(m+1) \times \varepsilon_{m+1,n,m,n} + |B_{\alpha,\xi,g_l,\lambda_l,\eta}^{(2)}|^2(n+1)\varepsilon_{m,n,m,n+1} \right].$$
(E.39)

The QFI $H_{\text{ampl}}(\alpha, \theta, \phi, \xi, g, \lambda, \eta)$ of the scheme is obtained by replacing $g_+ \to -g$ and $g_- \to -g$. This choice of the parameters is equivalent to the case described in the main paper (with $g_+ \to -g$, $g_- \to g$ and the additional $\pi/2$ phase shift in the probe state) leading to the same expression for the QFI. We finally obtain:

$$H(\alpha, \theta, \phi, \xi, g, \lambda, \eta) = \frac{2|\alpha|^2 \xi \eta}{\sqrt{1 + 4\eta (1 - \eta) \sinh^2 g}} \times$$

$$\{ \cosh[2(g - g_{\text{eff}})] - \cos(\lambda + 2\phi - 2\theta) \sinh[2(g - g_{\text{eff}})] \}.$$
(E.40)

The optimal condition corresponds to the case $\cos(\lambda + 2\phi - 2\theta) = -1$, where the QFI is:

$$H_{\rm ampl}(|\alpha|,\xi,g,\eta) = 2|\alpha|^2 \xi \eta \frac{e^{2(g-g_{\rm eff})}}{\sqrt{1+4\eta(1-\eta)\sinh^2 g}}.$$
 (E.41)

Again, the dependence of the QFI *H* of (E.40) on the parameter ϕ to be estimated implies that to achieve its maximum H_{ampl} , an adaptive strategy (see Sec. E.3) is necessary.

E.3 Adaptive protocol

In this section we detail a simple two-stage adaptive scheme, where first a rough estimate of the parameter ϕ is found, and then this estimate is employed in a second high-resolution stage of the protocol.

Let ϕ be the parameter we want to estimate (the phase) and assume that it is encoded in two different families of states, i.e. the family $\{\hat{\rho}_{\phi}\}_{\phi}$ and the family $\{\hat{\sigma}_{\phi}\}_{\phi}$. For example, the first family can be identified with the states of the system at the output of the interferometer when no amplification is used. The second family instead is identified as the the state at the output of the interferometer when the amplifier is active and where we have set the phase reference in such a way that the apparatus gives optimal performances for $\phi = 0$. In what follows we will consider a two stage estimation strategy in which *i*) first we perform M_1 measurements on the state $\hat{\rho}_{\phi}$ of the first family to get a preliminary estimation of ϕ , and then *ii*) we perform M_2 measurement on the state $\hat{\sigma}_{\phi}$ of the second family to improve our estimation (of course in the second stage we are facilitated by the fact that we have already acquired some info on ϕ).

Let then $\vec{x} = (x_1, x_2, \cdots)$ the data extracted from the first set of measurement and $\phi_{ext}^{(M_1)}(\vec{x})$ the estimation function we use to get the preliminary estimation of ϕ . Using the quantum Cramer-Rao (QCR) bound we have:

$$\delta^2 \phi_1 = \sum_{\vec{x}} P_1(\vec{x}) [\phi - \phi_{ext}^{(M_1)}(\vec{x})]^2 \ge \frac{1}{M_1 H_1(\phi)} , \qquad (E.42)$$

where $P_1(\vec{x})$ are the probability of getting the outcomes \vec{x} when measuring $\hat{\rho}_{\phi}^{\otimes M_1}$ and $H_1(\phi)$ is the quantum Fisher info associated with the family $\{\hat{\rho}_{\phi}\}_{\phi}$. For the sake of simplicity we assume that $x_{ext}^{(M_1)}(\vec{x})$ is unbiased, i.e.:

$$\sum_{\vec{x}} P_1(\vec{x}) \left[\phi - \phi_{ext}^{(M_1)}(\vec{x}) \right] = 0 , \qquad (E.43)$$

(generalization to the general case are possible).

In the second stage of the estimation we use the family $\{\hat{\sigma}_{\phi}\}_{\phi}$ where we modify the way the phase is mapped by rescaling it by $\phi_{ext}^{(M_1)}(\vec{x})$. This is possible for instance by changing the initial phase reference which effectively shifts the unknown phase ϕ to $\chi = \phi - \phi_{ext}^{(M_1)}(\vec{x})$: this is the new parameter we wish to recover. In the second stage, we perform measurements on $\hat{\sigma}_{\chi}^{\otimes M_2}$ obtaining the data $\vec{y} = (y_1, y_2, \cdots)$. We determine χ via the estimator $\chi_{est}^{(M_2)}(\vec{y})$ which again we assume to be unbiased, i.e.:

$$\sum_{\vec{y}} P_2(\vec{y}) \left[\chi - \chi_{ext}^{(M_2)}(\vec{y}) \right] = 0 , \qquad (E.44)$$

(here $P_2(\vec{y})$ is the probability of getting the outcomes \vec{y} when measuring $\hat{\sigma}_{\chi}^{\otimes M_2}$). The whole process can be described hence by introducing a joint estimator function:

$$\tilde{\phi}_{est}^{(M_1,M_2)}(\vec{x},\vec{y}) = \phi_{ext}^{(M_1)}(\vec{x}) + \chi_{est}^{(M_2)}(\vec{y}) .$$
(E.45)

characterized by a probability distribution $P_1(\vec{x})P_2(\vec{y})$ and which (by construction) is unbiased, i.e.:

$$\sum_{\vec{x},\vec{y}} P_1(\vec{x}) P_2(\vec{y}) \; \tilde{\phi}_{est}^{(M_1,M_2)}(\vec{x},\vec{y}) = \phi \; . \tag{E.46}$$

Adaptive protocol

Let us now compute the variance of the error associated with such estimator. Formally this is given by:

$$\begin{split} \delta^2 \tilde{\phi} &= \sum_{\vec{x},\vec{y}} P_1(\vec{x}) P_2(\vec{y}) \left[\phi - \tilde{\phi}_{ext}^{(M_1,M_2)}(\vec{x},\vec{y}) \right]^2 = \sum_{\vec{x}} P_1(\vec{x}) \left[\sum_{\vec{y}} P_2(\vec{y}) \left[\phi - \tilde{\phi}_{ext}^{(M_1,M_2)}(\vec{x},\vec{y}) \right]^2 \right] \\ &= \sum_{\vec{x}} P_1(\vec{x}) \left[\sum_{\vec{y}} P_2(\vec{y}) \left[\phi - \phi_{ext}^{(M_1)}(\vec{x}) - \chi_{est}^{(M_2)}(\vec{y}) \right]^2 \right] \\ &= \sum_{\vec{x}} P_1(\vec{x}) \left[\sum_{\vec{y}} P_2(\vec{y}) \left[\chi - \chi_{est}^{(M_2)}(\vec{y}) \right]^2 \right] \\ &\geqslant \sum_{\vec{x}} P_1(\vec{x}) \frac{1}{M_2 H_2(\chi)} = \sum_{\vec{x}} P_1(\vec{x}) \frac{1}{M_2 H_2(\phi - \phi_{ext}^{(M_1)}(\vec{x}))} \,, \end{split}$$

where we used the QCR bound on the estimation of χ and where $H_2(\chi)$ is the quantum Fisher info of the state $\hat{\sigma}(\chi)$. The above expression can now approximated by using the fact that for sufficiently large M_1 , $\phi_{ext}^{(M_1)}(\vec{x}) \simeq \phi$, i.e. $\chi \simeq 0$. This allows us to expand $H_2(\chi)$ around $\chi = 0$, i.e.:

$$H_2(\phi - \phi_{ext}^{(M_1)}(\vec{x})) \simeq H_2(0) + (\phi - \phi_{ext}^{(M_1)}(\vec{x})) H_2'(0) + (\phi - \phi_{ext}^{(M_1)}(\vec{x}))^2 H_2''(0)/2 , \quad (E.47)$$

which yields:

$$\begin{split} \delta^{2}\tilde{\phi} &\simeq \frac{1}{M_{2}}\sum_{\vec{x}}P_{1}(\vec{x}) 1/\left[H_{2}(0) + \left(\phi - \phi_{ext}^{(M_{1})}(\vec{x})\right)H_{2}'(0) + \left(\phi - \phi_{ext}^{(M_{1})}(\vec{x})\right)^{2}H_{2}''(0)/2\right] \\ &\simeq \frac{1}{M_{2}H_{2}(0)}\sum_{\vec{x}}P_{1}(\vec{x})\left[1 - \left(\phi - \phi_{ext}^{(M_{1})}(\vec{x})\right)\frac{H_{2}'(0)}{H_{2}(0)} - \left(\phi - \phi_{ext}^{(M_{1})}(\vec{x})\right)^{2}\frac{H_{2}''(0)}{2H_{2}(0)} \right. \\ &+ \left.\left(\phi - \phi_{ext}^{(M_{1})}(\vec{x})\right)^{2}\left[\frac{H_{2}'(0)}{H_{2}(0)}\right]^{2}\right] = \frac{1}{M_{2}H_{2}(0)}\left[1 - \delta^{2}\phi_{1}\left(\frac{H_{2}''(0)}{2H_{2}(0)} - \left[\frac{H_{2}'(0)}{H_{2}(0)}\right]^{2}\right] + \delta^{2}\phi_{1}^{2}\phi_$$

where we used Eq. (E.43) and the definition of $\delta^2 \phi_1$. Suppose now that $H_2(\chi)$ achieves its maximum for $\chi = 0$ (this is what happens thanks to our new choice of reference). This implies that $H'_2(0) = 0$ and $H''_2(0) \leq 0$. Therefore we get:

$$\delta^{2}\tilde{\phi} \geq \frac{1}{M_{2}H_{2}(0)} \left[1 + \delta^{2}\phi_{1} \frac{|H_{2}''(0)|}{2H_{2}(0)} \right] \geq \frac{1}{M_{2}H_{2}(0)} \left[1 + \frac{|H_{2}''(0)|}{2M_{1}H_{1}(\phi)H_{2}(0)} \right], \quad (E.49)$$

where in the last inequality we used the QCR bound (E.42). Defining $M = M_1 + M_2$ the total number of measurements, we can write:

$$\delta^2 \tilde{\phi} \ge \frac{1}{(1-p)MH_2(0)} \left[1 + \frac{|H_2''(0)|}{2pMH_1(\phi)H_2(0)} \right], \tag{E.50}$$

with $p = M_1/M$ begin the fraction of measurement we employ in the first step of the protocol. This equation provides the corrections to the accuracy we get when we adopt the adaptive strategy.

Observation I: It is worth comparing the above bound with the accuracy one could get if instead of performing the preliminary step one could have used all M copies to perform only the estimation on the states $\hat{\sigma}_{\phi}$. In this case the resulting accuracy would be $1/(MH_2(\phi))$. Do we gain something by going true the adaptive result? A positive answer would require:

$$\frac{1}{(1-p)MH_2(0)} \left[1 + \frac{|H_2''(0)|}{2pMH_1(\phi)H_2(0)} \right] \leqslant \frac{1}{MH_2(\phi)} , \qquad (E.51)$$

which can be cast as:

$$\frac{p+A}{p(1-p)} \leqslant B , \qquad (E.52)$$

with $B = H_2(0)/H_2(\phi)$ and $A = \frac{|H_2''(0)|}{2MH_1(\phi)H_2(0)}$. Since by assumption $B \ge 1$ and $A \ge 0$, one can easily verify that there are value of p which allows one to obtain Eq. (E.51) if B is sufficiently large.

Observation II: For fixed M we can optimize the right-hand-side of Eq. (E.50) with respect to p. This yields:

$$p_{opt} = \sqrt{A^2 + A} - A , \qquad (E.53)$$

(notice that this is and increasing function of A which is always positive and smaller than 1/2 – the latter being the asymptotic value reached for A >> 1). Consequently we can write:

$$\begin{split} \delta^2 \tilde{\phi} & \geqslant \quad \frac{1}{(1-p)MH_2(0)} \left[1 + \frac{|H_2''(0)|}{2pMH_1(\phi)H_2(0)} \right] = \frac{1}{(1-p)MH_2(0)} \left[1 + \frac{A}{p} \right] \\ & \geqslant \quad \frac{1}{MH_2(0)} \frac{\sqrt{A^2 + A}}{(\sqrt{A^2 + A} - A)(1 + A - \sqrt{A^2 + A})} \,. \end{split}$$

Now, for $M \gg 1$ we have that $A \rightarrow 0$. Therefore we can write:

$$\delta^{2}\tilde{\phi} \geq \frac{1}{MH_{2}(0)} \left[1 + 2\sqrt{A}\right] = \frac{1}{MH_{2}(0)} \left[1 + \sqrt{\frac{2|H_{2}''(0)|}{MH_{1}(\phi)H_{2}(0)}}\right].$$
 (E.54)

This implies that the resources M_1 employed in the first stage of the protocol can be neglected, and the precision asymptotically approaches the QCR of the second stage: the term with the square root in (E.54) is asymptotically negligible.

E.4 Classical Fisher information for the photon-counting measurement

In this section we describe the calculation for the classical Fisher information associated with our scheme when photon-counting measurements are performed. The output state of the protocol is described by the density matrix $\hat{\rho}_{\phi}^{\beta,g,\eta}$, while the measurement operators that describe photon-counting detectors are the projectors over Fock states:

$$\hat{\Pi}_{n^{(+)},n^{(-)}} = \hat{\Pi}_{n^{(+)}}^{(+)} \otimes \hat{\Pi}_{n^{(-)}}^{(-)}, \tag{E.55}$$

where $\Pi_{n^{(l)}}^{(l)} = |n^{(l)}\rangle_{ll} \langle n^{(l)}|$, with l = +, - labeling the optical mode. The probability distribution of the measurement outcomes can be evaluated as:

$$p(n^{(+)}, n^{(-)}|\phi) = \operatorname{Tr}[\hat{\rho}_{\phi}^{\beta, g, \eta} \hat{\Pi}_{n^{(+)}, n^{(-)}}].$$
(E.56)

The classical Fisher information associated to the probability distributions of the measurement outcomes is given by the following expression [Par09]:

$$I_{\phi} = \sum_{n,m=0}^{\infty} \frac{[\partial_{\phi} p(n^{(+)}, n^{(-)} | \phi)]^2}{p(n^{(+)}, n^{(-)} | \phi)}.$$
(E.57)

For the amplifier-based protocol, the probability distribution $p(n^{(+)}, n^{(-)}|\phi)$ can be separated in two independent single-mode contributions as:

$$p(n^{(+)}, n^{(-)}|\phi) = \prod_{l=+,-} p(n^{(l)}|\phi).$$
(E.58)

Here, $\hat{\rho}^{(l)}$ are the single-mode density matrices for modes l = +, - and:

$$p(n^{(l)}|\phi) = \text{Tr}[\rho^{(l)}\Pi_{n^{(l)}}^{(l)}].$$
(E.59)

In this case, the classical Fisher information can be separated in two single-mode contributions:

$$I_{\phi} = \sum_{l=+,-} I_{\phi}^{(l)}, \tag{E.60}$$

where:

$$I_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{[\partial_{\phi} p(n^{(l)} | \phi)]^2}{p(n^{(l)} | \phi)}.$$
 (E.61)

E.4.1 Photon-number distribution of the amplified coherent states

We begin by calculating the photon-number distribution of the amplified coherent states. The density matrix of the output state before the measurement stage is given by:

$$\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{D}_{+}(\tilde{\gamma}_{+})\hat{D}_{-}(\tilde{\gamma}_{-})\hat{S}_{+}(\tau_{+}^{\text{eff}})\hat{S}_{-}(\tau_{-}^{\text{eff}})\Big[\hat{\rho}_{+}^{th}(N_{\text{eff}})\otimes \hat{\rho}_{-}^{th}(N_{\text{eff}})\Big]\hat{S}_{+}^{\dagger}(\tau_{+}^{\text{eff}})\hat{S}_{-}^{\dagger}(\tau_{-}^{\text{eff}})\hat{D}_{+}^{\dagger}(\tilde{\gamma}_{+})\hat{D}_{-}^{\dagger}(\tilde{\gamma}_{-}).$$
(E.62)

To evaluate the photon-number distribution, we exploit the following identity between the elements of the density matrix expressed in the Fock basis $\hat{\rho} = \sum_{n,m=0}^{\infty} \rho_{n,m} |n\rangle \langle m|$ and the Wigner function of a general single-mode state $\hat{\rho}$:

$$\rho_{n,m} = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp W_{\hat{\rho}}(x,p) W_{n,m}(x,p), \qquad (E.63)$$

where $W_{n,m}(x,p)$ is the Wigner function associated to the operator $|n\rangle\langle m|$. Here, the (x,p) operators are defined according to $\Delta^2 x \Delta^2 p \ge 1/16$. The corresponding photonnumber distribution can be recovered from the diagonal elements $\rho_{n,n}$, by exploiting the expression of the Wigner function of a Fock state:

$$W_{n,n}(x,p) = \frac{2}{\pi} (-1)^n L_n[4(x^2 + p^2)] e^{-2(x^2 + p^2)}.$$
 (E.64)

Since the density matrix of the state $\hat{\rho}_{\phi}^{\beta,g,\eta} = \hat{\rho}_{\phi}^{(+)} \otimes \hat{\rho}_{\phi}^{(-)}$ is separable between the two modes, we can evaluate the distributions for the two components $\hat{\rho}_{\phi}^{(l)}$ separately. The first step is the evaluation of the Wigner function for the single-mode density matrix:

$$\hat{\boldsymbol{\rho}}_{\phi}^{(l)} = \hat{D}_{l}(\tilde{\boldsymbol{\gamma}}_{l})\hat{S}_{l}(\tau_{l}^{\text{eff}})\hat{\boldsymbol{\rho}}_{l}^{th}(N_{l}^{\text{eff}})\hat{S}_{l}^{\dagger}(\tau_{l}^{\text{eff}})\hat{D}_{l}^{\dagger}(\tilde{\boldsymbol{\gamma}}_{l}).$$
(E.65)

The Wigner function for this state takes the form of a Gaussian distribution of the form:

$$W_{\hat{\rho}_{\phi}^{(l)}}(x_{l},p_{l}) = \frac{2}{\pi} \frac{1}{1+2N_{l}^{\text{eff}}} e^{-\frac{2}{1+2N_{l}^{\text{eff}}}[2(x_{l}-x_{l}^{0})(p_{l}-p_{l}^{0})\sigma_{l}^{xp}]} e^{-\frac{2}{1+2N_{l}^{\text{eff}}}[(x_{l}-x_{l}^{0})^{2}\sigma_{l}^{xx}+(p_{l}-p_{l}^{0})^{2}\sigma_{l}^{pp}]},$$
(E.66)

where the first order and the second order moments are, respectively:

$$x_l^0 = \operatorname{Re}[\tilde{\gamma}_l], \quad p_l^0 = \operatorname{Im}[\tilde{\gamma}_l], \quad (E.67)$$

and:

$$\sigma_l^{xx} = \cosh(2g_l^{\text{eff}}) + \cos\lambda_l \sinh(2g_l^{\text{eff}}), \qquad (E.68)$$

$$\sigma_l^{pp} = \cosh(2g_l^{\text{eff}}) - \cos\lambda_l \sinh(2g_l^{\text{eff}}), \qquad (E.69)$$

$$\sigma_l^{xp} = \sin \lambda_l \sinh(2g_l^{\text{eff}}). \tag{E.70}$$

Here, g_l^{eff} and λ_l are respectively the absolute values and the phase of the squeezing parameters τ_l^{eff} . We can now proceed with the calculation of the single-mode photon-number distribution $p(n^{(l)}|\phi)$, which can be evaluated from the integral:

$$p(n^{(l)}|\phi) = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_l dp_l W_{\hat{\rho}_{\phi}^{(l)}}(x_l, p_l) W_{n,m}(x_l, p_l).$$
(E.71)

We first begin by performing the following rotation on the quadrature variables $(x_l, p_l) \rightarrow (x'_l, p'_l)$ of the $W_{\hat{p}_{\phi}^{(l)}}(x, p)$ function:

$$x'_l = x_l \cos \psi_l + p_l \sin \psi_l, \qquad p'_l = -x_l \sin \psi_l + p_l \cos \psi_l; \quad (E.72)$$

$$x_{l}^{\prime 0} = x_{l}^{\prime 0} \cos \psi_{l} + p_{l}^{\prime 0} \sin \psi_{l}, \qquad p_{l}^{\prime 0} = -x_{l}^{\prime 0} \sin \psi_{l} + p_{l}^{\prime 0} \cos \psi_{l}, \qquad (E.73)$$

where $\psi_l = \lambda_l/2$. The Wigner function in this rotated quadrature set is:

$$W_{\hat{\rho}_{\phi}^{(l)}}(x_{l}',p_{l}') = \frac{2}{\pi} \frac{1}{1+2N_{l}^{\text{eff}}} e^{-\frac{2}{1+2N_{l}^{\text{eff}}}[(x_{l}'-x_{l}'^{0})^{2}e^{2g_{l}^{\text{eff}}}]} e^{-\frac{2}{1+2N_{l}^{\text{eff}}}[(p_{l}'-p_{l}'^{0})^{2}e^{-2g_{l}^{\text{eff}}}]}.$$
 (E.74)

The same rotation is performed on the $W_{n,n}(x_l, p_l)$, which presents radial symmetry and hence its form is not affected by the rotation according to:

$$W_{n,n}(x'_l, p'_l) = \frac{2}{\pi} (-1)^n L_n \{ 4[(x'_l)^2 + (p'_l)^2] \} e^{-2[(x'_l)^2 + (p'_l)^2]}.$$
 (E.75)

We can then proceed with the evaluation of the integral (E.71). By performing the basis rotation $(x_l, p_l) \rightarrow (x'_l, p'_l)$ in the integration variable we obtain:

$$p(n^{(l)}|\phi) = \pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'_l dp'_l W_{\hat{\rho}_{\phi}^{(l)}}(x'_l, p'_l) W_{n,m}(x'_l, p'_l).$$
(E.76)

By expanding the Laguerre polynomials of the $W_{n,n}(x'_l, p'_l)$ function we obtain:

$$p(n^{(l)}|\phi) = \frac{4(-1)^n}{\pi(1+2N_l^{\text{eff}})} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \sum_{j=0}^k \frac{(-4)^k}{k!} \binom{k}{j}$$

$$\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx'_l dp'_l(x'_l)^j (p'_l)^{k-j} e^{-2[(x'_l)^2 + (p'_l)^2]} e^{-\frac{2}{1+2N_l^{\text{eff}}}[(x'_l - x'_l)^2 e^{2g_l^{\text{eff}}} + (p'_l - p'_l)^2 e^{-2g_l^{\text{eff}}}]}.$$
(E.77)

The integrals in dx'_l and dp'_l can be evaluated separately. We now define the following auxiliary functions:

$$\tilde{A}_{x_{l}} = 1 + \frac{e^{-2g_{l}^{\text{eff}}}}{1 + 2N_{l}^{\text{eff}}}, \quad \tilde{B}_{x_{l}} = \frac{x_{l}^{\prime 0}e^{-2g_{l}^{\text{eff}}}}{1 + 2N_{l}^{\text{eff}} + e^{-2g_{l}^{\text{eff}}}}, \quad \tilde{C}_{x_{l}} = \frac{(x_{l}^{\prime 0})^{2}e^{-2g_{l}^{\text{eff}}}}{1 + 2N_{l}^{\text{eff}} + e^{-2g_{l}^{\text{eff}}}}, \quad \tilde{C}_{x_{l}} = \frac{(x_{l}^{\prime 0})^{2}e^{-2g_{l}^{\text{eff}}}}{1 + 2N_{l}^{\text{eff}} + e^{-2g_{l}^{\text{eff}}}}, \quad \tilde{C}_{p_{l}} = \frac{(x_{l}^{\prime 0})^{2}e^{2g_{l}^{\text{eff}}}}{1 + 2N_{l}^{\text{eff}} + e^{2g_{l}^{\text{eff}}}}. \quad (E.79)$$

where the \tilde{B} and the \tilde{C} terms depend on the phase ϕ . Finally, by exploiting the definition of the confluent hypergeometric functions U(a,b;z), the single-mode photon number distribution can be written as:

$$p(n^{(l)}|\phi) = \frac{2(-1)^n}{1+2N_l^{\text{eff}}} e^{-2(\tilde{C}_{x_l}+\tilde{C}_{p_l})} \sum_{k=0}^n \sum_{j=0}^k \frac{2^k}{k!} \binom{n}{k} \binom{k}{j} \times \frac{U[-j, 1/2, -2\tilde{A}_{x_l}(\tilde{B}_{x_l})^2]U[-k+j, 1/2, -2\tilde{A}_{p_l}(\tilde{B}_{p_l})^2]}{(\tilde{A}_{x_l})^{j+1/2}(\tilde{A}_{p_l})^{k-j+1/2}}.$$
(E.80)

E.4.2 Derivative of the photon-number distribution and classical Fisher information

In order to evaluate the classical Fisher information according to Eqs.(E.60-E.61), we now need to evaluate the derivative of the photon-number distribution $p(n^{(l)}|\phi)$. The latter can be written in the following form:

$$p(n^{(l)}|\phi) = \sum_{k=0}^{n} \sum_{j=0}^{k} \omega_{n,kj} e^{-2(\tilde{C}_{x_l} + \tilde{C}_{p_l})} \frac{U[-j, 1/2, -2\tilde{A}_{x_l}(\tilde{B}_{x_l})^2]U[-k+j, 1/2, -2\tilde{A}_{p_l}(\tilde{B}_{p_l})^2]}{(\tilde{A}_{x_l})^{j+1/2}(\tilde{A}_{p_l})^{k-j+1/2}}.$$
(E.81)

Here, $\omega_{n,kj}$ includes all the coefficients independent from the phase ϕ . The derivative of the photon-number distribution $p(n^{(l)}|\phi)$ can then be written as the sum of three terms:

$$\partial_{\phi} p(n^{(l)} | \phi) = \sum_{i=1}^{3} D p_i(n^{(l)} | \phi).$$
 (E.82)

The term $Dp_1(n^{(l)}|\phi)$ presents the derivative of the exponential $e^{-2(\tilde{C}_{x_l}+\tilde{C}_{p_l})}$, leading to:

$$Dp_1(n^{(l)}|\phi) = (-2)\partial_{\phi}(\tilde{C}_{x_l} + \tilde{C}_{p_l})p(n^{(l)}|\phi).$$
(E.83)

The terms $Dp_2(n^{(l)}|\phi)$ and $Dp_3(n^{(l)}|\phi)$ exploit the following relation involving the derivatives of the confluent hypergeometric functions:

$$\partial_{\phi} U[a,b,f(\phi)] = -aU[a+1,b+1,f(\phi)]\partial_{\phi} f(\phi). \tag{E.84}$$

The remaining two terms can the be written as:

$$Dp_{2}(n^{(l)}|\phi) = \frac{2(-1)^{n}}{1+2N_{l}^{\text{eff}}}e^{-2(\tilde{C}_{x_{H}}+\tilde{C}_{p_{H}})}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\frac{2^{k}}{k!}\binom{k}{j}$$

$$\times \frac{U[1-j,3/2,-2\tilde{A}_{x_{l}}(\tilde{B}_{x_{l}})^{2}]U[-k+j,1/2,-2\tilde{A}_{p_{l}}(\tilde{B}_{p_{l}})^{2}]}{(\tilde{A}_{x_{l}})^{j+1/2}(\tilde{A}_{p_{l}})^{k-j+1/2}}j(-4)\tilde{A}_{x_{l}}\tilde{B}_{x_{l}}(\partial_{\phi}\tilde{B}_{x_{l}}),$$
(E.85)

and:

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$$Dp_{3}(n^{(l)}|\phi) = \frac{2(-1)^{n}}{1+2N_{l}^{\text{eff}}}e^{-2(\tilde{C}_{x_{H}}+\tilde{C}_{p_{H}})}\sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\frac{2^{k}}{k!}\binom{k}{j} \times \frac{U[-j,1/2,-2\tilde{A}_{x_{l}}(\tilde{B}_{x_{l}})^{2}]U[1-k+j,3/2,-2\tilde{A}_{p_{l}}(\tilde{B}_{p_{l}})^{2}]}{(\tilde{A}_{x_{l}})^{j+1/2}(\tilde{A}_{p_{l}})^{k-j+1/2}}(k-j)(-4)\tilde{A}_{p_{l}}\tilde{B}_{p_{l}}(\partial_{\phi}\tilde{B}_{p_{l}}).$$
(E.86)

Finally, the classical Fisher information can be evaluated according to:

$$I_{\phi} = \sum_{l=+,-} I_{\phi}^{(l)}.$$
 (E.87)

where:

$$I_{\phi}^{(l)} = \sum_{n=0}^{\infty} \frac{(\sum_{i=1}^{3} Dp_i(n^{(l)} | \phi))^2}{p(n^{(l)} | \phi)}.$$
 (E.88)

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Polarization preserving ultra fast optical shutter for quantum information processing

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Abstract: We present the realization of a ultra fast shutter for optical fields, which allows to preserve a generic polarization state, based on a self-stabilized interferometer. It exhibits high (or low) transmittivity when turned on (or inactive), while the fidelity of the polarization state is high. The shutter is realized through two beam displacing prisms and a longitudinal Pockels cell. This can represent a useful tool for controlling light-atom interfaces in quantum information processing.

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Amplification of polarization NOON states

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We analyze the quantum states obtained by optical parametric amplification of polarization NOON states. First we study, theoretically and experimentally, the amplification of a two-photon state by a collinear quantum injected optical parametric amplifier (QIOPA). We compare the stimulated emission regime with the spontaneous one, studied by Sciarrino *et al.* [Phys. Rev. A **77**, 012324 (2008)]. As a second step, we show that the collinear amplifier cannot be successfully used for amplifying *N*-photon states with N>2, and we propose to adopt a different scheme, based on a noncollinear QIOPA. We show that the state obtained by the latter amplification process preserves the λ/N feature and exhibits a high resilience to losses. Furthermore, measurement of part of the output state can be adopted to increase the pattern visibility. © 2009 Optical Society of America

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1. INTRODUCTION

In the past few years it has been proposed to exploit quantum effects to provide resolution enhancement in imaging procedures. Among the numerous problems that are currently studied under the general name of "quantum imaging," investigations concerning the quantum limits of optical resolution have a special importance, as they may lead to new concepts in microscopy and optical data storage. Such so-called superresolution techniques, studied for a long time at the classical level with a view to beating the Rayleigh limit of resolution, were recently revisited at the quantum level [1,2]. It was shown that it is possible to improve the performance of superresolution techniques by use of nonclassical light [3,4]. This approach, named "quantum lithography," may lead in the future to innovative microscopy techniques, to recording image features that are much smaller than the wavelength of the light, or to improving optical storage capacity beyond the wavelength limit. In such a framework, path entangled NOON $|\psi^N\rangle_{AB} = (1/\sqrt{2})(|N\rangle_A|0\rangle_B + |0\rangle_A|N\rangle_B)$ have states been adopted to increase the resolution in quantum interferometry. Indeed, in such states a single-mode phase shift ϕ induces a relative shift between the two components equal to $N\phi$ [5]. This feature leads to sub-Rayleigh resolution scaling as $\lambda/2N$, where λ is the wavelength of the field [6] [Fig. 1(a)]. Analogously, multiphoton polarization entangled states can be exploited to carry out quantum lithography by adopting the scheme reported in Fig. 1(b), which converts polarization-entanglement into pathentanglement. The theoretical and experimental study of photonic NOON states [7–9] has led to the experimental generation of two-, three-, and four-photon states by postselection [10-13] and to the conditional generation of a state with N=2 [14]. Very recently schemes for the generation of path entangled NOON states with high value of fidelity and arbitrary N have been proposed [8,15,16]. However, until now, the low number of photons generated has strongly limited the potential applications to quantum lithography and quantum metrology. Moreover a NOON state, like any superposition of macroscopic states, is supersensitive to losses: for a N-photon state a fractional loss 1/N would destroy the quantum effect responsible for the phase resolution improvement [17].

A natural approach to increase the number of photons and to minimize the effect of losses is to exploit a high optical parametric process. Recently the output radiation of an unseeded optical parametric amplifier (OPA) was exploited to demonstrate the typical $\lambda/4$ feature with a large number of photons [18] [Fig. 2(a)]. Even if the achieved visibility is equal to 20%, this value is sufficient for applications in lithography and imaging [16]. In such a framework it has been proposed to exploit stimulated parametric processes to improve the visibility and obtain higher signal values [19]. This process, also known as quantum injected optical parametric amplification, has found some important applications in the context of quantum information [20,21]. Let us stress that high resolution and intense light fields can also be obtained in a classical framework [22,23]. In that case the improved resolution relies on the nonlinear response of the recording medium rather than on the quantum features of the adopted light field.

In the present paper we investigate the task of the amplification of photonic NOON states by two different devices, both based on a quantum injected optical parametric amplifier (QIOPA). First, in Section 2, we review how a sub-Rayleigh $\lambda/2N$ resolution can be obtained by an interferometric device acting on a NOON state. Then, in Section 3 we study both theoretically and experimentally the amplification of a two-photon state by a collinear

Decoherence, environment-induced superselection, and classicality of a macroscopic quantum superposition generated by quantum cloning

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The high resilience to decoherence shown by a recently discovered macroscopic quantum superposition (MQS) generated by a quantum-injected optical parametric amplifier and involving a number of photons in excess of 5×10^4 motivates the present theoretical and numerical investigation. The results are analyzed in comparison with the properties of the MQS based on $|\alpha\rangle$ and N-photon maximally entangled states (NOON), in the perspective of the comprehensive theory of the subject by Zurek. In that perspective the concepts of "pointer state" and "environment-induced superselection" are applied to the new scheme.

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I. INTRODUCTION

The short handwritten note by Einstein on the back of a greetings card sent to Born on the first of January 1954 may be taken as the conceptual framework of the present work: "If φ_1 and φ_2 are two solutions of the same Schrödinger equation, $\varphi = \varphi_1 + \varphi_2$ is another solution of the same equation equally able to represent a possible situation. If however we are dealing with a 'macrosystem' and φ_1 and φ_2 are 'narrow' with respect to the macrocoordinates in the vast majority of cases φ cannot be 'narrow'. Narrowness respect to the macrocoordinates [i.e., macro-localization] is a property not only independent of the principles of quantum mechanics but also incompatible with them" [1].

As we can see since the early decades of quantum mechanics the counterintuitive properties associated with the superposition state of macroscopic objects and the problem concerning the "classicality" of quantum macrostates were the object of an intense debate epitomized in 1935 by the celebrated "Schrödinger's cat paradox" [2,3]. In particular, the actual feasibility of such quantum object has always been tied to the alleged infinitely short persistence of its quantum coherence, i.e., of its overwhelmingly rapid "decoherence." In modern times the latter property, establishing a rapid merging of the quantum rules of microscopic systems into classical dynamics, has been interpreted as a consequence of the entanglement between the macroscopic quantum system with the environment [4,5]. By tracing over the environmental variables in the final calculations, generally the pure quantum state decays irreversibly toward a probabilistic classical mixture [6]. Recently, the general interest in decoherence has been renewed in the framework of quantum information theory where it plays a fundamental detrimental role since it conflicts with the experimental realization of the quantum computer or of any quantum device bearing any relevant complexity [7]. In this respect a large experimental effort has been devoted recently to the implementation of macroscopic (i.e., many-particle) quantum superposition states (MQSs), adopting photons, atoms, and electrons in superconducting devices. Particular attention has been devoted to the realization of the MQS involving "coherent states" of light, which exhibits interesting and elegant Wigner function representations [8]. The most notable results of this experimental effort have been reached with atoms interacting with microwave fields trapped inside a cavity [9,10] or for freely propagating fields [11]. However, in spite of the long-lasting efforts spent in these endeavors, in these realizations the MQS has always proved to be so fragile that even the loss of a single particle was found to be able to spoil any possibility of a direct observation of its quantum properties. Precisely on the basis of these negative results in many scientific communities (and also within some influential editorial teams) grew the opinion that Schrödinger's cat is indeed an ill-defined and then avoidable concept since it fundamentally lacks any directly observable property [6].

In spite of these conclusions, very recently a new kind of MQS involving a number of particles N in excess of 5×10^4 has been realized, allowing the direct observation of entanglement between a microscopic and a macroscopic photonic state and showing a very high resilience to decoherence by coupling with environment [12]. Precisely, the MQS was generated by a quantum-injected optical parametric amplifier (QI-OPA) seeded by a single photon belonging to an Einstein-Podolsky-Rosen (EPR) entangled pair. We emphasize here that the reported QI-OPA can be considered for the present purpose as a paradigmatic system consisting of the simplest realizable "optimal phase-covariant quantumcloning machine" [13,14]. Indeed, precisely the process of "quantum cloning" was there responsible for the transfer of the entanglement and the superposition properties of a pure single-particle qubit into a multiparticle MQS. In other words, the QI-OPA encoded "optimally" into a macrostate the information associated with the input microstate, a seed qubit [15–19]. By this device, which includes an orthogonality filter [O filter (OF)] for enhanced state discrimination, the microstate-macrostate nonseparability was successfully tested and the micro-macro violation of the Bell's inequalities for spin-1 excitations was attained [12,20]. In view of this peculiar, striking behavior, we felt that a careful analysis of the decoherence of this novel MQS device was necessary. The present approach to decoherence will be cast within the useful framework developed in the past by Zurek [21]. Ac-

Anomalous Lack of Decoherence of the Macroscopic Quantum Superpositions Based on Phase-Covariant Quantum Cloning

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We show that all macroscopic quantum superpositions (MQS) based on phase-covariant quantum cloning are characterized by an anomalous high resilence to the decoherence processes. The analysis supports the results of recent MQS experiments and leads to conceive a useful conjecture regarding the realization of complex decoherence-free structures for quantum information, such as the quantum computer.

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Since the early decades of the last century the counterintuitive properties associated with the superposition state of macroscopic objects and the problem concerning the "classicality" of macroscopic states were the object of an intense debate epitomized in 1935 by the celebrated Schrödinger's paradox [1,2]. However, the actual feasibility of any macroscopic quantum superposition (MQS) adopting photons, atoms, and electrons in SQUIDS [3-6] was always found to be challenged by the very short persistence of its quantum coherence, i.e., by its overwhelmingly fast "decoherence." The latter property was interpreted as a consequence of the entanglement between the macroscopic system with the environment [7-9]. Recently, decoherence has received renewed attention in the framework of quantum information where it plays a detrimental role since it conflicts with the realization of any device bearing any relevant complexity, e.g., a quantum computer [10]. In particular, effort was aimed at the implementation of MQS involving coherent states of light, which exhibit elegant Wigner function representations [11]. Nevertheless, in all previous realizations the MQS was found so fragile that even the loss of a single particle spoiled any direct observation of its quantum properties.

The present work considers in general a novel type of MQS, one that is based on the amplification, i.e., or "quantum cloning", of a "microscopic" quantum state, e.g., a single-particle qubit: $|\phi\rangle = 2^{-(1/2)} (|\phi_1\rangle + e^{i\varphi} |\phi_2\rangle).$ Formally, the amplification is provided by a unitary cloning transformation \hat{U} , i.e., a quantum map which, applied to the microscopic state leads to the MQS macroscopic state: $|\Phi^{\phi}\rangle = \hat{U}|\phi\rangle$. In general, the amplification can be provided by a laser amplifier or by any quantum-injected nonlinear (NL) optical parametric amplification (OPA) process directly seeded by $|\phi\rangle$. The *atom laser* is also a candidate for this process, and then, within the exciting matter-wave context our model can open far reaching fields of novel scientific and technological endeavour. It can be shown that \hat{U} can be "information preserving", albeit slightly noisy, and able to transfer in the macroscopic domain the quantum superposition character of the single-particle qubit [12-14]. Furthermore, unlike most of the other MQS schemes, \hat{U} , being a fixed intrinsic dynamical property of the amplifier is not affected in principle by events of scattering of particles out of the system, i.e., by loss, a process which is generally the dominant source of decoherence. For the same reasons \hat{U} is largely insensitive to temperature effects. Let us investigate this interesting process by the quantum-injected OPA, often referred to as QI-OPA. By this device, indeed a high-gain phase ϕ -covariant cloning machine seeded by an entangled EPR photon pair, a macroscopic state consisting of a large number of photons $\mathcal{N} \approx 10^5$ was generated [14-18]. A sketchy draft of the apparatus is shown in the left part of Fig. 1. A polarization ($\vec{\pi}$) entangled couple of single photons (A, B) is parametrically generated by a standard Einstein-Podolsky-Rosen-Bohm (EPR) configuration in a NL crystal of BBO (beta-barium-borate) cut for type II phase matching and excited by a low intensity ultraviolet (UV) laser beam [12]. One of the photons, say A with state $|\phi^{\perp}\rangle_A$, measured by a detector (Det), provides the trigger signal for the overall experiment. The photon Bwith state $|\phi\rangle_B$ nonlocally correlated to A, is injected, via a dichroic mirror (DM) in another BBO NL-crystal excited



FIG. 1 (color online). Amplified states of the collinear optical parametric amplifier injected by a single-photon qubit generated in a type-II EPR source. Beam-splitter (BS) loss model for the amplified macroscopic states generated by the collinear QI-OPA.

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Wigner-function theory and decoherence of the quantum-injected optical parametric amplifier

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Recent experimental results demonstrated the generation of a macroscopic quantum superposition (MQS), involving a number of photons in excess of 5×10^4 , which showed a high resilience to losses. In order to perform a complete analysis on the effects of decoherence on these multiphoton fields, obtained through the quantum injected optical parametric amplifier, we investigate theoretically the evolution of the Wigner functions associated to these states in lossy conditions. Recognizing the presence of negative regions in the *W* representation as an evidence of nonclassicality, we focus our analysis on this feature. A close comparison with the MQS based on coherent $|\alpha\rangle$ states allows us to identify differences and analogies.

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I. INTRODUCTION

In the last decades the physical implementation of macroscopic quantum superpositions (MQSs) involving a large number of particles has attracted a great deal of attention. Indeed it was generally understood that the experimental realization of a MQS is very difficult and in several instances practically impossible owing to the extremely short persistence of quantum coherence, i.e., of the extremely rapid decoherence due to the entanglement established between the macroscopic system and the environment [1-4]. Formally, the irreversible decay toward a probabilistic classical mixture is implied theoretically by the tracing operation of the overall MQS state over the environmental variables [5,6]. In the framework of quantum information different schemes based on optical systems have been undertaken to generate and to detect the MQS condition. A cavity-QED scheme based on the interaction between Rydberg atoms and a high-Q cavity has lead to the indirect observation of macroscopic quantum superposition (Schrödinger cat) states and of their temporal evolutions. In this case the microwave MQS field stored in the cavity can be addressed indirectly by injecting in the cavity, in a controlled way, resonant or nonresonant atoms as ad hoc "measurement mouses" [7,8]. A different approach able to generate freely propagating beams adopts photonsubtracted squeezed states; experimental implementations of quantum states with an average number of photons of around four have been reported both in the pulsed and continuous wave regimes [9-12]. These states exhibit non-Gaussian characteristics and open new perspectives for quantum computing based on continuous-variable systems, entanglement distillation protocols [13,14], and loophole free tests of Bell's inequality.

In the last few years a novel "quantum injected" optical parametric amplification (QI-OPA) process has been realized in order to establish the entanglement between a singlephoton and a multiphoton state given by an average of many thousands of photons, a Schrödinger cat involving a "macroscopic field." Precisely, in a high-gain QI-OPA "phase-covariant" cloning machine the multiphoton fields were generated by an optical amplifier system bearing a high nonlinear (NL) gain g and seeded by a single photon belonging to an Einstein-Podolski-Rosen (EPR) entangled pair [15–19].

While a first theoretical insight on the dynamical features of the QI-OPA macrostates and a thorough experimental characterization of the quantum correlations were recently reported [20,21], a complete quantum phase-space analysis able to recognize the persistence of the QI-OPA properties in a decohering environment is still lacking [22,23]. Among the different representations of quantum states in the continuousvariable space [24], the Wigner quasiprobability representation has been widely exploited as an evidence of nonclassical properties, such as squeezing [25] and EPR nonlocality [26]. In particular, the presence of negative quasiprobability regions has been considered as a consequence of the quantum superposition of distinct physical states [27].

In the present paper we investigate the Wigner functions associated to multiphoton states generated by optical parametric amplification of microscopic single-photon states. We focus our interest on the effects of decoherence on the macrostates and on the emergence of the "classical" regime in the amplification of initially pure quantum states. The Wigner functions of these QI-OPA generated states in presence of losses are analyzed in comparison with the paradigmatic example of the superposition of coherent, Glauber's states, $|\alpha\rangle$.

The paper is structured as follows. In Sec. II, we introduce the conceptual scheme and describe the evolution of the system both in the Heisenberg and Schrödinger pictures. Section III is devoted to the calculation of the Wigner function of the QI-OPA amplified field. We first consider a singlemode amplifier, which is analogous to the case of photonsubtracted squeezed vacuum. Then we derive a compact expression of the Wigner function in the case of a two-mode amplifier in the "collinear" case, i.e., for common k vectors of the amplified output fields. In Sec. IV, we introduce, for the collinear case, a decoherence model apt to simulate the decohering losses affecting the evolution of the macrostate density matrix. This evolution is then compared to the case of the coherent $|\alpha\rangle$ MQS. Section V is devoted to a brief review of the features of coherent state superpositions (CSSs). Hence in Sec. VI we derive an explicit analytic ex-

Quantum-to-classical transition via fuzzy measurements on high-gain spontaneous parametric down-conversion

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We consider the high-gain spontaneous parametric down-conversion in a noncollinear geometry as a paradigmatic scenario to investigate the quantum-to-classical transition by increasing the pump power, that is, the average number of generated photons. The possibility of observing quantum correlations in such a macroscopic quantum system through dichotomic measurement will be analyzed by addressing two different measurement schemes, based on different dichotomization processes. More specifically, we will investigate the persistence of nonlocality in an increasing size $\frac{n}{2}$ -spin singlet state by studying the change in the correlations form as *n* increases, both in the ideal case and in presence of losses. We observe a fast decrease in the amount of Bell's inequality violation for increasing system size. This theoretical analysis is supported by the experimental observation of macro-macro correlations with an average number of photons of about 10³. Our results shed light on the practical extreme difficulty of observing nonlocality by performing such a dichotomic fuzzy measurement.

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I. INTRODUCTION

For a long time the investigation of entanglement and nonlocality has been limited to quantum systems of small size [1]. Theoretical and experimental works on Bell's inequalities have been devoted to the study of single-particle states, in which dichotomic measurements have been performed [2]. Nonlocality tests have been achieved with single-photon states, produced by parametric down conversion, by detecting polarization correlations [3–5]. More recently, the violation of Bell's inequality has been shown with a larger number of photons: on Greenberger-Horne-Zeilinger (GHZ) [6] and cluster states [7] up to four photons.

On the other hand, the possibility of observing quantum phenomena at a macroscopic level seems to be in conflict with the classical description of our everyday world knowledge. The main problem for such observation arises from the experimental difficulty of sufficiently isolating a quantum system from its environment, that is, from the decoherence process [8]. An alterative approach to explain the quantum-to-classical transition, conceptually different from the decoherence program, has been given, very recently, by Kofler and Brukner, along the idea earlier discussed by Bell, Peres [9], and others. These authors have given a description of the emergence of macroscopic realism and classical physics in systems of increasing size within quantum theory [10]. They focused on the limits of the quantum effects observability in macroscopic objects, showing that, for large systems, macrorealism arises under coarse-grained measurements. More specifically, they demonstrated that, while the evolution of a large spin cannot be described classically when sharp measurements are performed, a fuzzy measurement on a large-spin system would induce the emergence of the Newtonian time evolution from a full quantum description of the spin state. However, some counterexamples to such a modelization have been found later by the same authors: some nonclassical Hamiltonians violate macrorealism despite coarse-grained measurements [11]. One example is given by the time-dependent Schrödinger catlike superposition, which can violate macrorealism by adopting a suitable "which emisphere" measurement. Therefore the measurement problem seems to be a key ingredient in the attempt to understand the limits of the quantum behavior of physical systems and the quantum-to-classical transition question. As a further step, Kofler, Buric, and Brukner also demonstrated [12] that macrorealism does not imply a continuous spatiotemporal evolution. Indeed, they showed that the same Schrödinger catlike nonclassical Hamiltonian, in contact with a dephasing environment, no longer violates a Leggett-Garg inequality, while it still presents a nonclassical time evolution. In a recent paper Jeong et al. [13] contributed to the investigation about the possibility of observing the quantum features of a system when fuzzy measurement are performed on it, finding that extremely coarse-grained measurements can still be useful to reveal the quantum world where local realism fails.

In this context, the possibility of obtaining macroscopic quantum systems in the laboratory has raised the problem of investigating entanglement and nonlocality in systems in which single particles cannot be addressed singularly. As shown in Ref. [14], the demonstration of nonlocality in a multiphoton state produced by a nondegenerate optical parametric amplifier would require the experimental application of parity operators. On the other hand, the estimation of a coarse-grained quantity, through collective measurements as the ones proposed in Ref. [15], would miss the underlying quantum structure of the generated state, introducing elements of local realism even in the presence of strong entanglement and in the absence of decoherence. The theoretical investigation on a multiphoton system, obtained via parametric down-conversion, has been also carried out by Reid et al. [16]. They analyzed the possibility of obtaining the violation of Bell's inequality by performing dichotomic measurement on the multiparticle quantum state. More specifically, in analogy with the spin

Generation of Highly Resilient to Decoherence Macroscopic Quantum Superpositions via Phase-covariant Quantum Cloning

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Abstract In this paper we analyze the resilience to decoherence of the Macroscopic Quantum Superpositions (MQS) generated by optimal phase-covariant quantum cloning according to two coherence criteria, both based on the concept of Bures distance in Hilbert spaces. We show that all MQS generated by this system are characterized by a high resilience to decoherence processes. This analysis is supported by the results of recent MQS experiments of $N = 3.5 \times 10^4$ particles.

Keywords Macroscopic quantum superposition · Decoherence · Phase-covariant cloning

1 Introduction

The short handwritten note by Einstein on the back of a greetings card sent to Max Born on the first of January 1954 may be taken as the conceptual framework of the present work: "if φ_1 and φ_2 are two solutions of the same Schrödinger equation, $\varphi = \varphi_1 + \varphi_2$ is another solution of the same equation equally able to represent a possible situation. If however we are dealing with a "macrosystem" and φ_1 and φ_2

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Enhanced Resolution of Lossy Interferometry by Coherent Amplification of Single Photons

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In the quantum sensing context most of the efforts to design novel quantum techniques of sensing have been constrained to idealized, noise-free scenarios, in which effects of environmental disturbances could be neglected. In this work, we propose to exploit optical parametric amplification to boost interferometry sensitivity in the presence of losses in a minimally invasive scenario. By performing the amplification process on the microscopic probe after the interaction with the sample, we can beat the losses' detrimental effect on the phase measurement which affects the single-photon state after its interaction with the sample, and thus improve the achievable sensitivity.

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The aim of quantum sensing is to develop methods to extract the maximum amount of information from a system with a minimal disturbance on it. Indeed, the possibility of performing precision measurements by adopting quantum resources can increase the achievable precision going beyond the semiclassical regime of operation [1-3]. In the case of interferometry, this can be achieved by the use of the so-called NOON states, which are quantum mechanical superpositions of just two terms, corresponding to all the available photons N placed in either the signal arm or the reference arm. The use of N00N states can enhance the precision in phase estimation to 1/N, thus improving the scaling of the achievable precision with respect to the employed resources [4,5]. This approach can have wide applications for minimally invasive sensing methods in order to extract the maximum amount of information from a system with minimal disturbance. The experimental realization of protocols involving N00N states containing up to 4 photons have been realized in the past few years [6-10]. Other approaches [11,12] have focused on exploiting coherent and squeezed light to generated fields which approximate the features of NOON states. Nevertheless, these quantum states turn out to be extremely fragile under losses and decoherence [13], unavoidable in experimental implementations. A sample, whose phase shift is to be measured, may at the same time introduce high attenuation. Since quantum-enhanced modes of operations exploit fragile quantum mechanical features, the impact of environmental effects can be much more deleterious than in semiclassical schemes, destroying completely quantum benefits [14,15]. This scenario puts the beating of realistic, noisy environments as the main challenge in developing quantum sensing. Very recently, the theoretical and experimental investigations of quantum states of light resilient to losses have attracted much attention, leading to the best possible precision in optical two-mode interferometry, even in the presence of experimental imperfections [16–21].

In this work, we adopt a hybrid approach based on a high gain optical parametric amplifier operating for any polarization state in order to transfer quantum properties of different microscopic quantum states in the macroscopic regime [22,23]. By performing the amplification process of the microscopic probe after the interaction with the sample, we can beat the losses' detrimental effect on the phase measurement which affects the single-photon state after the sample. Our approach may be adopted in a minimally invasive scenario where a fragile sample, such as biological or artifacts systems, requires as few photons as possible impinging on it in order to prevent damages. The action of the amplifier, i.e., the process of optimal phase covariant quantum cloning, is to broadcast the phase information codified in a single photon into a large number of particles. Such multiphoton states have been shown to exhibit a high resilience to losses [24-26] and can be manipulated by exploiting a detection scheme which combines features of discrete and continuous variables. The effect of losses on the macroscopic field consists in the reduction of the detected signal and not in the complete cancellation of the phase information as would happen in the singlephoton probe case, thus improving the achievable sensitivity. This improvement does not consist in a scaling factor but turns out to be a constant factor in the sensitivity depending on the optical amplifier gain. Hence, the sensitivity still scales as \sqrt{N} , where N is the number of photons impinging on the sample, but the effect of the amplification process is to reduce the detrimental effect of losses by a factor proportional to the number of generated photons.

Let us review the adoption of single photons in order to evaluate the unknown phase φ , Fig. 1(a). The phase φ introduced in the path k_2 is probed by sending to the sample N input photons, each one in the state

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Resilience to decoherence of the macroscopic quantum superpositions generated by universally covariant optimal quantum cloning

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We show that the quantum states generated by universal optimal quantum cloning of a single photon represent a universal set of quantum superpositions resilient to decoherence. We adopt the Bures distance as a tool to investigate the persistence of quantum coherence of these quantum states. According to this analysis, the process of universal cloning realizes a class of quantum superpositions that exhibits a covariance property in lossy configuration over the complete set of polarization states in the Bloch sphere.

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I. INTRODUCTION

The observation of quantum properties in "macroscopic" systems has been at the focus of a long-lasting endavor to investigate the transition from the world of microscopic single-particle systems to the macroscopic "classical" domain. Uncontrolled interaction with the environment [1], that is, decoherence, is responsible for the progressive cancellation of the quantum properties and largely limits their experimental observation. To overcome this problem, several classes of quantum states possessing peculiar features, such as entanglement or negative Wigner distributions, have been investigated. Among them, the quantum superposition of "coherent" Glauber α states [2] and the Greenberger-Horne-Zeilinger (GHZ) states [3] have been found to be extremely fragile under decoherence, since the loss of only one particle is responsible for the cancellation of the quantum coherence present in the system.

About 10 years ago it was proposed to exploit the process of quantum cloning to generate a different class of multiphoton states (Fig. 1) [4,5]. This method recently led to the successful experimental realization of the macroscopic quantum superposition (MQS) of a large number of particles, $N \approx 5 \times 10^4$ [6,7]. The entanglement test reported in Ref. [7] for the collinear amplification regime was recently discussed by Sekatski *et al.* [8] Later, a recent paper [9] reported a detailed theoretical analysis of the adopted entanglement criteria and showed that a substantial degree of entanglement was indeed present in the micro-macro system dealt with in [7].

The persistence of quantum coherence in MQS states realized by phase-covariant cloning, that is, limited to a *one-dimensional* subspace of the entire Bloch sphere of the macroqubit, was analyzed on the basis of two criteria based on the definition of "distance" in the Hilbert space [10,11]. It was found that this limited physical system shows a high resilience to decoherence at variance with coherent $|\alpha\rangle$ state MQS. The feature of phase-covariance symmetry mostly consists of the relative simplicity of the required "collinear" structure and of the high efficiency of quantum-injected optical parametric amplification (QI-OPA). This one amplifies equally well the single-photon polarization states $|\phi\rangle$ belonging to the equatorial plane of the Bloch sphere of the injected microqubit [5,6].

Given this circumstance, the question arose whether there exists a physical system that exhibits the property of resilience to decoherence in a larger Hilbert space or, better, in the full space available to the generated macrostate. The present paper addresses this question. The "universal quantum cloning machine" realized in its "optimal" MQS mode nondegenerate configuration possesses the required property: resilience to decoherence is realized in the full Hilbert space spanned by the output macrostate [4,12–14].

In this paper, we report a theoretical analysis of the resilience to decoherence of quantum states generated by universal quantum cloning of a single-photon qubit. The basic tools of this investigation are provided by the two coherence criteria defined in Refs. [10] and [11]. There, the Bures distance [15–17]

$$\mathcal{D}(\hat{\rho},\hat{\sigma}) = \sqrt{1 - \sqrt{\mathcal{F}(\hat{\rho},\hat{\sigma})}},\tag{1}$$

where \mathcal{F} is a quantum "fidelity," has been adopted as a measure to quantify (I) the "distinguishability" between two quantum states $\{|\phi_1\rangle, |\phi_2\rangle\}$ and (II) the *degree of coherence*, that is, the MQS visibility, of their quantum superpositions $|\phi^{\pm}\rangle =$ $2^{-1/2}(|\phi_1\rangle \pm |\phi_2\rangle)$. These criteria were chosen according to the following considerations. (I) The distinguishability, that is, the degree of orthogonality, represents the maximum discrimination power among two quantum states available within a measurement. (II) The related visibility between the superposition $|\phi^+\rangle$ and the superposition $|\phi^-\rangle$ depends exclusively on the relative phase of the component states $|\phi_1\rangle$ and $|\phi_2\rangle$. Consider two orthogonal superpositions $|\phi^{\pm}\rangle$: $\mathcal{D}(|\phi^+\rangle, |\phi^-\rangle) = 1$. In the presence of decoherence the relative phase between $|\phi_1\rangle$ and $|\phi_2\rangle$ progressively randomizes and the superpositions $|\phi^+\rangle$ and $|\phi^-\rangle$ approach an identical fully mixed state leading to $\mathcal{D}(|\phi^+\rangle, |\phi^-\rangle) = 0$. The physical interpretation of $\mathcal{D}(|\phi^+\rangle, |\phi^-\rangle)$ as the visibility of a superposition $|\phi^{\pm}\rangle$ is legitimate insofar as the component states of the corresponding superpositions, $|\phi_1\rangle$ and $|\phi_2\rangle$ may be defined, at least approximately, as pointer states or einselected states [1]. Within the set of eigenstates characterizing any quantum

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Entanglement criteria for microscopic-macroscopic systems

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We discuss the conclusions that can be drawn on a recent experimental micro-macro entanglement test [De Martini, Sciarrino, and Vitelli, Phys. Rev. Lett. 100, 253601 (2008)]. The system under investigation is generated through optical parametric amplification of one photon belonging to an entangled pair. The adopted entanglement criterion makes it possible to infer the presence of entanglement before losses that occur on the macrostate under a specific assumption. In particular, an *a priori* knowledge of the system that generates the micro-macro pair is necessary to exclude a class of separable states that can reproduce the obtained experimental results. Finally, we discuss the feasibility of a micro-macro "genuine" entanglement test on the analyzed system by considering different strategies, which show that in principle a fraction ε , proportional to the number of photons that survive the lossy process, of the original entanglement persists in any loss regime.

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I. INTRODUCTION

The observation of quantum phenomena, such as quantum entanglement [1], has been mainly limited to systems of only few particles. One of the main open challenges for an experimental test in systems of large size is the construction of suitable criteria for the detection of entanglement in bipartite macroscopic systems. Much effort has been devoted in the past few years in this direction. Some of them, such as the partial-transpose criterion developed by Peres in Ref. [2], require the tomographic reconstruction of the density matrix, which for a system of a large number of particles becomes highly demanding from an experimental point of view. In order to avoid the necessity of the complete reconstruction of the state, a class of tests where only few local measurements are performed has been introduced under the name of "entanglement witness" [3]. For bipartite systems of a large number of particles, this approach has been further investigated considering the possibility of exploiting collective measurements on the multiparticle state. Within this context, Duan et al. proposed a general criterion in Ref. [4] based on continuous variable [5] observables. This general criterion was subsequently applied to the quantum extension of the Stokes parameters [6,7] to obtain an entanglement bound for such kinds of variables [8]. Other approaches have been developed based on spin variables [9] or pseudo-Pauli operators [10]. An experimental application of this criteria based on collective spin measurements has been performed in a bipartite system of two gas samples [11]. However, an experimental realization of most of these criteria in the quantum optical domain requires photon-number-resolving detectors with unitary efficiency, which is beyond the current technology. A feasible approach for the analysis of multiphoton fields has been developed in

the past few years and is based on the deliberate attenuation of the analyzed system up to the single-photon level. In this way, standard single-photon techniques and criteria can be used to investigate the properties of the field. The verification of the entanglement in the high-loss regime is evidence of the presence of entanglement before the attenuation, since no entanglement can be generated by local operations. Such an approach has been exploited in [12,13] to demonstrate the presence of entanglement in a high-gain spontaneous parametric downconversion (SPDC) source of up to 12 photons. An analogous conclusion has been theoretically obtained in Ref. [14] on the same system by exploiting symmetry considerations of the source. The attenuation method has been also applied to a different system, making it possible to obtain an experimental proof of the presence of entanglement between a single-photon state and a multiphoton state generated through the process of optical parametric amplification in a universal cloning configuration of up to 12 photons [15].

In this article we discuss recent experimental results of a micro-macro entanglement test [16], where the system under investigation is realized through the process of optical parametric amplification [17,18] of an entangled photon pair. The exploited entanglement criterion is an extension of the spin-based single-particle criterion of Ref. [12]. Such an extension requires a supplementary assumption which will be clarified in the remaining part of this article. In Sec. II we briefly review the properties of the micro-macro system realized in Ref. [16]. Then in Sec. III we discuss in details the performed entanglement test. In particular, we focus on the conditions adopted in order to justify the exploited entanglement criterion. Finally, in Sec. IV we perform a theoretical analysis of the micro-macro system based on the parametric amplification of an entangled pair. Several approaches for the verification of the entanglement property of the system will be addressed, showing that a substantial fraction ε of the original entanglement survives even in high-loss condition.

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Measurement-induced quantum operations on multiphoton states

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We investigate how multiphoton quantum states obtained through optical parametric amplification can be manipulated by performing a measurement on a small portion of the output light field. We study in detail how the macroqubit features are modified by varying the amount of extracted information and the strategy adopted at the final measurement stage. At last the obtained results are employed to investigate the possibility of performing a microscopic nonlocality test free from auxiliary assumptions.

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I. INTRODUCTION

The possibility of performing quantum operations in order to tailor quantum states of light on demand has been widely investigated in the last few years. Several fields of research have been found to benefit from the capability of generating fields possessing the desired quantum properties. Nonclassical states of light, such as sub-Poissonian light [1], squeezed light [2,3], or the quantum superposition of coherent states [4,5], have been generated in a conditional fashion. In this context, continuous-variable (CV) quantum information represents one of the most promising fields where conditional and measurement-induced non-Gaussian operations can find application. To this end, quantum interactions can be induced by exploiting linear optics, detection processes, and ancillary states [6]. For example, the process of coherent photon subtraction has been exploited to increase the entanglement present in Gaussian states [7,8] and to engineer quantum operations in traveling light beams [9]. Finally, very recently, conditional operations lead to the realization of different schemes for the implementation of the probabilistic noiseless amplifier [10-12], which can find interesting applications within the context of quantum phase estimation [13].

Strictly related to the engineering of quantum states of light for applications to quantum information, there is the problem of beating the decoherence due to losses which affect quantum resources interacting with an external environment. In the last few years a large investigation effort has been devoted to the decoherence process and the robustness of increasing size quantum fields, realized by nonlinear optical methods [14–17]. Recently, quantum phenomena generated in the microscopic world and then transferred to the macroscopic one via parametric amplification have been experimentally investigated. In Ref. [16] it has been reported the realization of a resilient to decoherence multiphoton quantum superposition (MQS) [18] involving a large number of photons and obtained by parametric amplification of a single photon belonging to a microscopic entangled pair: $|\psi^{-}\rangle = \frac{1}{\sqrt{2}}(|H\rangle_{A}|V\rangle_{B} - |V\rangle_{A}|H\rangle_{B})$, where A, B refer to spatial modes $\mathbf{k}_{A}, \mathbf{k}_{B}$ and the kets refer to single-photon polarization states $\vec{\pi}$ ($\pi = H, V$). This process

In this paper we consider several strategies for the realization of measurement-induced quantum operations on these multiphoton states, generated through the process of optical parametric amplification. We investigate theoretically how the measurement strategies, applied on a part of the multiphoton state before the final identification measurement, affect the distinguishability of orthogonal macroqubits. Such measurements based on the discrimination of multiphoton probability distributions combine features of both continuous and discrete variables techniques. The interest in improving the capability of identifying the state generated by the quantum injected optical parametric amplifier (QIOPA) system mainly relies on two motivations: the first one concerns the development of a discrimination method able to increase the transmission fidelity of the state after the propagation over a lossy channel, and hence to overcome the imperfections related to the practical implementation. Such increased discrimination capability in lossy conditions could find applications within the quantum communication context. The second reason concerns the scenario in which an appropriate preselection of the macroqubits could be adopted to demonstrate the microscopic-macroscopic nonlocality, free from the auxiliary assumptions requested if the filtering procedure was applied at the final measurement stage.

In previous papers [15,16] a probabilistic discrimination method, the orthogonality filter (OF), was introduced and successfully applied to an entanglement test in a microscopicmacroscopic bipartite system. The application of the OF strategy, acting at the measurement stage, is indeed not suitable for the demonstration of loophole-free microscopic-macroscopic nonlocality because of the presence of inconclusive results [20]. These correspond to the selection of different subensembles of data, depending on the choice of the measurement

has been realized through a nonlinear crystal pumped by an ultraviolet (UV) high power beam acting as a parametric amplifier on the single entangled injected photon (i.e., the qubit $|\phi\rangle_B$ on spatial mode **k**_B). In virtue of the unitarity of the optical parametric amplifier (OPA), the generated state was found to keep the same superposition character and the interfering properties of the injected qubit [14,15,19] and, by exploiting the amplification process, the single-photon qubit has been converted into a macroqubit involving a large number of photons.

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Non-Gaussianity of quantum states: An experimental test on single-photon-added coherent states

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Non-Gaussian states and processes are useful resources in quantum information with continuous variables. An experimentally accessible criterion has been proposed to measure the degree of non-Gaussianity of quantum states based on the conditional entropy of the state with a Gaussian reference. Here we adopt such a criterion to characterize an important class of nonclassical states: single-photon-added coherent states. Our studies demonstrate the reliability and sensitivity of this measure and use it to quantify how detrimental is the role of experimental imperfections in our implementation.

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I. INTRODUCTION AND DEFINITIONS

Quantum information offers a different viewpoint on fundamental aspects of quantum mechanics: it aims to assess and exploit the quantum properties of a physical system as a resource for a different, and hopefully more efficient, treatment of information. Indeed, within the framework of quantum information with continuous variables [1], nonclassical states of the radiation field represent a resource, and much attention has been devoted to their generation schemes, which usually involve nonlinear interaction in optically active media.

On the other hand, the reduction postulate provides an alternative mechanism to achieve effective nonlinear dynamics; if a measurement is performed on a portion of a composite entangled system, the other component is conditionally reduced according to the outcome of the measurement. The resulting dynamics may be highly nonlinear, and may produce quantum states that cannot be generated by currently achievable nonlinear processes. Conditional measurements have been exploited to engineer nonclassical states and, in particular, have been recently employed to obtain non-Gaussian states.

While Gaussian states, defined as states with a Gaussian Wigner function, are known to provide useful resources for tasks such as teleportation [2,3], cloning [4-6], or dense coding [7-9], there is an ongoing effort to study which protocols are allowed by non-Gaussian resources. The most notable example is certainly their use for an optical quantum computer [10,11], alongside their employment for improving teleportation [12–14], cloning [15], and storage [16]. Several implementations of non-Gaussian states have been reported so far, in particular from squeezed light [17-25], close-tothreshold parametric oscillators [26,27] in optical cavities [28], and in superconducting circuits [29]. Non-Gaussian operations are also interesting for tasks such as entanglement distillation [30,31] and noiseless amplification [32,33], which are also obtained in a conditional fashion, accepting only those events heralded by a measurement result.

In principle, non-Gaussianity is not directly related to the nonclassical character of a quantum state and, in turn, a classical non-Gaussian state may be prepared (e.g., by phase-diffusion of coherent states or photon subtraction on thermal states [34]). On the other hand, in the applications mentioned above it is the presence of both non-Gaussianity and nonclassicality which allows for enhancement of performances. Therefore, de-Gaussification protocols of interest for quantum information are those providing non-Gaussianity in conjunction with nonclassicality.

In this work we address the conditional dynamics induced by the so-called photon addition as a protocol to generate nonclassical non-Gaussian states. We quantify experimentally the amount of non-Gaussianity obtained by adding a photon to a coherent state [19,35–37]. Differently from previous investigations [35,38–41], we can explicitly address the two aspects of non-Gaussianity and nonclassicality at once. For the former, we adopt the non-Gaussianity measure $\delta[\varrho]$ proposed in [42,43], which is defined as the quantum relative entropy between the quantum state ϱ itself and a reference Gaussian state τ having the same covariance matrix as ϱ . Given this choice of reference Gaussian state, we have $\text{Tr}[\varrho \ln \tau] =$ $\text{Tr}[\tau \ln \tau]$, as $\ln \tau$ is a polynomial of order at most two in the canonical variables [42,44]. We thus find

$$\delta[\varrho] = S(\varrho \| \tau) = \operatorname{Tr} [\varrho(\ln \varrho - \ln \tau)]$$

= $S(\tau) - S(\varrho);$ (1)

that is, $\delta[\varrho]$ is simply equal to the difference between the von Neumann entropy of τ and the von Neumann entropy of ϱ . In Ref. [42] it has been shown that this measure is nonzero only for non-Gaussian states. It is also additive under tensor product, invariant under unitary Gaussian operations, and in general it does not increase under generic completely positive Gaussian channels. This measure is somehow preferable to that based on the Hilbert-Schmidt distance [45] in a quantum information context, since it is based on an information-related quantity. We note, however, that a mixture (e.g., doubly peaked) of classical states can also be strongly non-Gaussian. We therefore adopt an additional "nonclassicality" criterion.

Several measures of nonclassicality have been proposed in literature [46–49]; for our purposes we consider as a witness a quantity $\nu[\varrho]$ related to the negativity of the Wigner function. This is normalized to a reference, which we choose to be

Simulation of noise-assisted transport via optical cavity networks

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Recently, the presence of noise has been found to play a key role in assisting the transport of energy and information in complex quantum networks and even in biomolecular systems. Here we propose an experimentally realizable optical network scheme for the demonstration of the basic mechanisms underlying noise-assisted transport. The proposed system consists of a network of coupled quantum-optical cavities, injected with a single photon, whose transmission efficiency can be measured. Introducing dephasing in the photon path, this system exhibits a characteristic enhancement of the transport efficiency that can be observed with presently available technology.

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I. INTRODUCTION

The presence of noise in quantum transmission networks is generally considered to be deleterious for the efficient transfer of energy or classical or quantum information encoded in quantum states. Quantum networks, used for the transmission, are unavoidably interacting with an external noisy environment, and this interaction significantly affects the quantum coherence of the system evolution. It is indeed commonly accepted that the presence of decoherence [1] is responsible for the undesired and uncontrolled transfer of information from the system to the environment, which in turn reduces the coherence in quantum systems. However, recently noise has been found to play a positive role in creating quantum coherence and entanglement [2,3]. Motivated by fascinating experiments showing the presence of quantum beating in photosynthetic systems [4-6], subsequent theoretical work pointed to the idea that the remarkable efficiency of the excitation energy transfer in light-harvesting complexes during photosynthesis benefits from the presence of environmental noise [7,8]. Indeed, the intricate interplay between dephasing and quantum coherence as well as the entanglement behavior during the noise-assisted transport dynamics have been elucidated in more detail in Refs. [9–12]. Perhaps even more surprisingly, the dephasing was recently found to assist the transfer of classical and quantum information in communication complex quantum networks [13].

Recently, quantum optical systems have been exploited as a promising platform to simulate quantum processes [14–16]. For example, several implementations of systems simulating quantum random walks have been reported with linear optical resonators [17,18], linear optical elements [19,20], fiber networks [21], and optical waveguides [22–25]. Motivated by these results, here we propose a quantum optical scheme to investigate the noise-assisted excitation transfer process through a set of coupled optical cavities. We discuss a four-site optical network and derive the set of relevant parameters that

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rule the time evolution of the system. A detailed numerical simulation of this dynamics, when one cavity is injected with a single photon, is performed employing realistic experimental parameters, showing that the presence of a suitable dephasing process in each site of the network allows for a characteristic increase of the excitation transfer efficiency. Furthermore, we consider aspects such as phase stabilization of the cavities and the implementation of dephasing, which are necessary to observe a clear enhancement of the photon transfer rate from one cavity to an external detector, mimicking the so-called reaction center of the light-harvesting complexes. Finally, we investigate how entanglement degrades during the time evolution of the optical network.

The paper is organized as follows: In Sec. II we define the model that describes the dynamics of the four-site optical network analyzed in this paper, including the master equation for the two relevant noise processes. Then in Sec. III we perform a detailed derivation of a realistic set of parameters for the system. In Sec. IV we report the results of a numerical simulation of the dynamics of the network. Finally, the conclusions and final remarks are presented in Sec. V.

II. MODEL OF THE NETWORK

In this section we describe in detail the model underlying the dynamics of the proposed network of optical cavities. A schematic view of this system in relation to the light-harvesting complexes is shown in Fig. 1. Starting from the Hamiltonian describing noninteracting cavities, one has

$$\hat{\mathcal{H}}_{\rm cav} = \sum_{i} \hbar \omega \hat{a}_{i}^{\dagger} \hat{a}_{i}, \qquad (1)$$

where \hat{a}_i and \hat{a}_i^{\dagger} are the usual bosonic field operators, which annihilate and create a photon in cavity *i*, and ω is the resonance frequency, which we assume for simplicity to be equal for all cavities. The transfer of photons between the optical cavities is described by the following Hamiltonian term:

$$\hat{\mathcal{H}}_{\text{int}} = \sum_{(i,j)} \hbar g_{ij} (\hat{a}_i^{\dagger} \hat{a}_j + \hat{a}_i \hat{a}_j^{\dagger}), \qquad (2)$$

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Hybrid methods for witnessing entanglement in a microscopic-macroscopic system

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We propose a hybrid approach to the experimental assessment of the genuine quantum features of a general system consisting of microscopic and macroscopic parts. We infer entanglement by combining dichotomic measurements on a bidimensional system and phase-space inference through the Wigner distribution associated with the macroscopic component of the state. As a benchmark, we investigate the feasibility of our proposal in a bipartite-entangled state composed of a single-photon and a multiphoton field. Our analysis shows that, under ideal conditions, maximal violation of a Clauser-Horne-Shimony-Holt-based inequality is achievable regardless of the number of photons in the macroscopic part of the state. The difficulty in observing entanglement when losses and detection inefficiency are included can be overcome by using a hybrid entanglement witness that allows efficient correction for losses in the few-photon regime.

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I. INTRODUCTION

An open challenge for fundamental quantum physics is to affirm the quantum nature of a system that puts together a microscopic part and a mesoscopic one. This hybrid scenario can emerge in completely different experimental platforms ranging from individual spin systems interacting with multimode cavity fields, such as transmon qubits in coplanar transmission-line resonators [1,2], to ionic impurities embedded in ultracold atomic samples, such as the systems considered in some recent experiments reported in [3,4]. Another possible physical approach exploits a massive tiny mirror interacting optomechanically with a single photon within a Michelson interferometer [5–9]. This endeavor could contribute to challenge the observability of quantum features at the macroscopic level, which is one of the most fascinating open problems in quantum physics. The difficulties inherent in such a quest are manifold, and they are related on the one hand to the unavoidable interaction of the system with the surrounding environment [10–13]. On the other hand, one faces the debated problem of achieving a measurement precision sufficient to observe quantum effects at such macroscales [14,15]. In this context, it has been proven experimentally that a dichotomic measurement performed upon a multiphotonentangled state is not sufficient to catch quantumness [16]. The accuracy of the measurement is crucial for the observation of quantum features and should be put on the same footing as the use of proper entanglement and nonlocality criteria for macroscopic quantum systems [15,17-21].

To successfully tackle the manipulation and characterization of hybrid systems the following question is still open: How can we ascertain the nonclassical nature of a multipartite state that, *per se*, does not meet the criteria for quantumness that have been designed for system components of equal dimensionality? Our work provides a quantitative answer to this broad question. We introduce an investigative platform that can be built up without the necessity for information on the state itself, and this supports the general validity and broad applicability of our results.

We introduce a hybrid method to demonstrate experimentally the truly quantum mechanical features of a general microscopic-macroscopic system beyond any assumption on its state and without the necessity of any a priori state knowledge. We infer the entanglement properties by means of a hybrid approach that combines dichotomic measurements on a bidimensional system and phase-space inferences through the Wigner distribution associated with the macroscopic component of the state. Here, through the use of a hybrid entanglement test, we identify a valuable tool for our goals. While the microscopic part of the state is measured using spin-1/2 projection operators, the macroscopic counterpart undergoes phase-space measurements based on the properties of its Wigner function [17]. At variance with previous proposals [17,22], the approach presented in this paper is tailored to fully exploit the polarization-spin degree of freedom on both the microscopic and the macroscopic subsystems. We analyze the effects of losses on a Clauser-Horne-Shimony-Holt-like (CHSH-like) inequality test [23] and show that maximum violation is achieved when losses are absent, regardless of the size of the macroscopic part of the state. This is not the case under nonideal conditions. However, we show how losses can be efficiently taken into account to infer entanglement of our multiphoton state.

As a paradigmatic microscopic-macroscopic system (MMS), we investigate the state obtained from a fully microscopic-entangled system through an amplification process [24,25]. Such a system has been further considered recently as a benchmark to perform nonlocality tests with human-eye threshold detectors [26] or as a platform for absolute radiometry [27]. At variance with respect to Refs. [24] and [25], our approach does not require any assumption on the system under investigation and hence represents a genuine entanglement test.

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Continuous-variable nonlocality test performed over a multiphoton quantum state

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We propose to exploit a continuous-variable measurement, based on displacing the input field at different points of the phase space, over a multiphoton state produced by a high-gain optical parametric amplifier. We show that by correlating the different values of the displaced parity operators obtained from the two separated parties, it is possible to violate a Bell's inequality and thus demonstrate the nonlocality of the overall state. The robustness of the results against two independent sources of error, loss and dephasing, is also discussed.

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I. INTRODUCTION

The discussion of nonlocality started by Einstein, Podolsky, and Rosen (EPR) in 1935 [1] has yielded a definition of entanglement as the most characteristic feature of quantum mechanics given by Erwin Schrödinger [2] up to the formulation of the Bell's inequality in order to test the nonlocal character of quantum states [3]. Following Bohm's scheme [4], the EPR correlations have been analyzed by addressing singlet pairs of two-level systems but not the two-particle continuous wave function introduced by EPR in their argument about the completeness of quantum mechanics. Theoretical and experimental studies of quantum nonlocality and entanglement have then been carried out on discrete systems [5–8], and the generalization of Bell's inequalities to quantum systems with continuous variables has represented a challenging issue for a long time.

Initially, it was believed that the possibility of observing the violation of Bell's inequality by addressing position and momentum over the EPR state was prevented by the non-negativity of its Wigner function. Indeed, according to Bell, the positivity of the Wigner function would have allowed the construction of a local-hidden-variable model simulating correlations for any observable defined as a function of phase-space points [9]. However, Banaszek and Wodkiewicz showed that in spite of the positivity of the Wigner function, the EPR state exhibits a high degree of nonlocality [10]. This study was later extended by Chen et al. [11], who showed that a maximal violation of Bell's inequality can be obtained by measuring pseudospin operators over the state produced by a nondegenerate optical parametric amplifier (NOPA) when the nonlinear gain of the amplifier grows and the NOPA state tends to the original EPR one. The relation between the positivity of the Wigner function and the possibility of observing a violation of Bell's inequality has then been clarified by Rezven et al. [12]; they focused their attention on the explicit assumptions that are made in a Bell's test and that involve the nature of the dynamical variables measured in order to violate a Bell's inequality. Reference [12] shows that only "nondispersive" dynamical variables, i.e., variables

whose representatives as functions of hidden variables take as possible values the eigenvalues a_n such that $|a_n| \leq 1$, can be considered good candidates for a local-hidden-variable theory. The violation of a Bell's inequality is then not only dependent on the system's Wigner function but also on the nature of the measured dynamical variables.

From an experimental point of view, the demonstration of Bell's inequality involving the measurement of discrete degrees of freedom requires the introduction of either the locality or the detection loophole [13]. The adoption of atomic systems allows one to close the detection loophole but not the locality one [14], and conversely, light can be sent at large distances but the inefficiency of detectors and the presence of losses along the communication channel prevent the possibility of closing the detection loophole. A path toward a Bell's test on bipartite multiphoton systems could involve the adoption of homodyne measurements, which can be performed with very high detection efficiency [15]. Recently, hybrid measurements involving both discrete and continuous-variable observables in order to demonstrate Bell's test violations have been addressed in Refs. [16] and [17]. The discussion of nonlocality in continuous-variable systems is then still an open problem in which the adoption of feasible measurements in reliable systems turns out to be the key requirement.

We propose a further step toward the understanding of the nonlocality problem in continuous-variable systems by addressing the possibility of performing continuous-variable measurements for a multiphoton system in order to observe a Bell's test violation. The exploited multiphoton-state source can be considered a paradigmatic system since it is based on an optical parametric amplifier, similar to the one analyzed by Banaszek and Wodkiewicz in Ref. [10] [in which the multiphoton state generated by a nondegenerate optical parametric amplifier was placed in relation with the continuousvariable EPR state], but with an additional degree of freedom: polarization. Recently, the quantum correlations present in the multiphoton state obtained by the high-gain, spontaneous parametric down-conversion process that cannot be read by a fuzzy measurement performed on it have been analyzed [18]. Even if in principle the nonlocal nature of the state could be observed for any value of the nonlinear gain of the amplifier,

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