Scuola Dottorale In Ingegneria Sezione di Ingegneria dell'Elettronica Biomedica, dell'Elettromagnetismo e delle Telecomunicazioni

## XIV cICLO

# Microwave Systems for Plasma Heating in Nuclear Fusion and for Buried Objects Detection 

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## Abstract

Nel presente lavoro di tesi vengono trattate approfonditamente alcune tematiche relative allo studio di sistemi alla microonde per applicazioni di rilevazione di oggetti sepolti ed accoppiamento di potenza elettromagnetica a plasmi fusionistici. In particolare, il lavoro sarà convenientemente presentato in due parti nelle quali si esporranno estensivamente sia l'impianto teorico associato allo stato dell'arte di ciascun argomento, sia i principali contributi di innovazione che costituiscono l'elemento originale del lavoro. Particolare enfasi sarà rivolta alla presentazione dei risultati, che verranno organizzati in modo da rendere il più possibile esaustiva l'analisi dei diversi scenari.
Nella prima parte sarà dapprima introdotto il problema della rilevazione e localizzazione di oggetti cilindrici sepolti, operando alcune ipotesi semplificative circa la struttura fisica dei mezzi considerati, in particolare si assumerà un modello di propagazione elettromagnetica ideale nel quale non sono tenute in considerazione le perdite dovute alla conducibilità non infinita nei conduttori e nulla nei dielettrici. Questi ultimi, inoltre, sono stati assunti lineari, isotropi, omogenei e non dispersivi, in modo da ricavare un modello elettromagnetico semplificato quantunque utile in molte applicazion pratiche. Gli scenari di riferimento, in questo caso, sono molteplici, a partire dalle indagini sottosuperficiali in terreni sabbiosi per attività di sminamento, fino alle prospezioni non invasive di strutture in cemento o lastricati stradali.
Le caratteristiche del problema di localizzazione possono essere associate con l'inversione dello scattering elettrognetico che è molto spesso trattato attraverso la "teoria dei problemi inversi". A differenza delle tecniche più utilizzate, come ad esempio la tomografia o l'approccio Ground Penetrating Radar, la tecnica che è stata sviluppata prevede un formalismo di tipo elettromagnetico per il problema diretto (che riguarda cioè la derivazione dei contributi di campo elettromagnetico generato dalle correnti indotte sugli oggetti e successivamente re-irradiato verso l'antenna), mentre affronta il problema dell'inversione attraverso una tecnica basata sulla stima delle direzioni di arrivo del segnale diffuso dagli oggetti sepolti.

Nel Capitolo 1, sarà quindi costruita l'impalcatura teorica che riguarda il modello del segnale e del rumore, e saranno brevemente introdotti i metodi di stima che sono stati implementati. in particolare, essi saranno divisi in tre gruppi a seconda delle loro caratteristiche. Alla fine del capitolo sarà invece descritta la tecnica statistica che è stata adottata per estrarre una stima della posizione degli oggetti. Nel Capitolo 2 verrà analizzata la localizzazione di un singolo cilindro, supposto dapprima perfettamente conduttore e successivamente dielettrico, assumendo diverse combinazioni tra l'indice di rifrazione del cilindro e quello del mezzo di sepoltura. In questo caso, verranno eseguite delle variazioni di alcuni parametri, come ad esempio il raggio dell'oggetto o la posizione rispetto all'array, per derivare alcune indicazioni circa la robustezza della procedura di stima. In secondo luogo sarà presentata una analisi legata alla possibilità di estrarre informazioni circa le dimensioni dell'oggetto sepolto.
Nel Capitolo 3, il procedimento verrà esteso al caso delle localizzazione di due cilindri, attraverso l'utilizzo di alcuni algoritmi di data mining, di cui ci siamo serviti per elaborare i dati ottenuti dalla stima delle direzioni di arrivo. Dopo un breve accenno sui costituenti fondamentali della "clustering analysis", saranno descritti gli algoritmi di nostro interesse. Successivamente, verrà dato un ampio spazio alla presentazione dei risultati all'interno di una casistica molto esauriente nella quale sono stati considerati inizialmente due cilindri conduttori variando l'indice di rifrazione del mezzo opite e ancora due cilindri dielettrici aventi differente indice di rifrazione rispetto al mezzo di sepoltura e rispetto a sè stessi.
Anche in questo caso la variazione dei parametri ha interessato sia la dimensione dei cilindri che la loro posizione, evidenziando un buona capacità di localizzazione in molti dei casi esaminati. In particolare, la procedura si dimostrata particolarmente adatta alla localizzazione di cavità cilindriche sia nel caso di un singolo oggetto che di coppie di oggetti. Alla fine del capitolo sarà offerta una ampia bibliografia sui diversi argomenti.
Nella seconda parte della tesi, ci occuperemo della progettazione di alcuni componenti a microonde per il riscaldamento di plasmi fusionistici. Il lavoro prodotto in questo campo, scaturisce dalla collaborazione pluriennale dell'Università di Roma Tre con ENEA-Fusmag all'interno dell'Associazione Euratom. In particolare, nel corso degli anni 2009, 2010 e 2011, si sono affrontate alcune problematiche relative alla progettazione di un sistema addizionale di riscaldamento a radiofrequenza per la risonanza ibrida inferiore, nell'ambito del progetto europeo denominato ITER (International Thermonuclear Reactor Experiment). La collaborazione ha rigurdato il task dell'European Fusion Development Agreement (EFDA) WP08-03-01 LH4IT
e si è conclusa con la pubblicazione di un Report tecnico sull'argomento (EU contribution to the ITER LHCD development plan).
Dopo una introduzione generale sulle macchine termonucleari a confinamento magnetico (Tokamak), verranno presentati i principali obbiettivi tecnologici che si è inteso raggiungere con il presente studio. Nel Capitolo 1 saranno fornite alcune basi teoriche riguardanti l'implementazioni di componenti a microonde di particolare interesse per questo tipo di applicazioni. Saranno considerati rispettivamente, un convertitore tra quattro guide retangolari in regime unimodale e una guida circolare in regime surmodato, una serie di filtri di modo ottenuti tramite opportune corrugazioni in guida rettangolare, ed infine alcune rilevanti tipologie di curve in guida rettangolare surmodata con delle interessanti propriet in grado di mitigare l'eccitazione di modi spuri dovuta alla presenza della discontinuit.
Lo studio di componenti in guida d'onda in regime surmodato è un argomento che difficilmente viene affrontato nella letteratura scientifica; anche gli strumenti teorici a disposizione del progettista sono di gran lunga trascurati rispetto ad altri settori dell'elettromagnetismo applicato. Appare quindi di particolare interesse aver conseguito un modesto know-how in questo tipo di applicazioni, non solo in fase di sintesi, attraverso l'utilizzo di CAD elettromagnetici, ma sopratutto nel momento di analisi, dove si è sviluppata una certa "sensibilità" nell'affrontare i numerosi problemi della propagazione elettromagnetica in regime multi-modale.
Nel Capitolo 2 sarà descritto in dettaglio il funzionamento di un componente innovativo in grado di convertire quattro guide a sezione rettangolare operanti nel loro modo fondamentale, in una guida circolare surmodata rispetto alla frequenza di utilizzo, operante nel solo modo circolare elettrico. Lo scopo di questo componente è quello di ridurre al minimo possibile l'attenuazione in guida e consentire la trasmissione di potenza in lunghi tratti di linea di trasmissione. Nello specifico, saranno descritte le fasi principali del progetto, lo studio prestazionale ed infine l'analisi delle non idealità. Nel Capitolo 3 sarà affrontato il complicato tema della progettazione di componenti in guide rettangolari funzionanti in regime surmodato, con particolare attenzione verso il filtraggio selettivo e il controllo dell'eccitazione di modi di ordine superiore. A questo proposito, è stata di particolare interesse la progettazione di curve in guida rettangolare in regime multi-modale in grado di limitare il più possibile l'accoppiamento tra il modo fondamentale e i modi di ordine superiore all'uscita della discontinuità, in modo da minimizzare lo scambio di potenza tra essi, conservando pressochè inalterata la trasmissione del modo fondamentale, come richiesto per questo tipo di applicazioni. Infine, anche per questa sezione, un'ampia bibliografia fornirà gli adeguati strumenti di approfondimento.

## Introduction

During my three-years of research activity, I have been dealt with two main topics: the localization of buried targets of cylindrical shape, and the design of microwave components for fusion engineering applications. Therefore, the present thesis is organized in two parts, each one developing exhaustively the various aspects connected with the arguments, with a particular emphasis laid on the outcomes. Moreover, each part of the manuscript will conclude with an extensive and selected bibliography that can be considered as the starting point for further developments.

The problem of reconstruction of buried objects by the use of electromagnetic survey, is generally referred as "microwave imaging" or "electromagnetic inverse problem". The principal technique implemented in this frame is the electromagnetic tomography, which is based on the Born Approximation (BA) and can be applied to approximate the integral equation resulting to the scattering inversion, assuming that the electromagnetic field scattered by the object is small if compared with the incident field. The scattered field is given as the "difference" between the total field and the unperturbed field which is the field reflected at the interface when the objects are absent, providing a function accounting for the dielectric contrast between the object and the surrounding medium. The unknowns of the problem are the spatial distributions of both the relative dielectric permittivity and the conductivity inside a fixed domain of investigation. Under BA, the relationship between the unknown contrast function and the scattered field data is provided by an the integral equation that, for cases in which the scattering amplitude can be directly measured (modulus and phase), then reduces to solving the Fredholm integral equation of the electric potential in terms of the scattering amplitude. Such a solution can also be calculated numerically by iterative procedures that in the most of cases, take a considerable computational effort.
On the other hand, Ground Penetrating Radar (GPR) is a well assessed diagnostic technique that finds many applications involving the detection and localization of objects buried in different media. The basic principle on which GPR is based on,
concerns the electromagnetic wave that is radiated from a transmitting antenna, traveling through the material at a velocity which is determined by the relative dielectric constant of the material. When the transmitted wave hits an object, it produces a scattered wave that move backward to the antenna. In particular, part of the energy carried by the incident wave is reflected by the surface of the encountered object and can be captured by a receiver. Electromagnetic waves travel at a specific velocity that is determined by the permittivity of the material. The relationship between the velocity of the wave and material properties is the fundamental basis for using GPR to investigate the subsurface. In fact, since the velocity depends on the dielectric properties of media, a signal passing through two different materials over the same distance will arrive at different times. The velocity is proportional to the inverse square root of the permittivity of the material, and therefore, the difference between the time of arriving (ToA) of the pulse reflected by the interface and the ToA of the wave traveling through the material and then scattered back to the surface, can provide useful information about the presence of buried objects on the track of the transmitted signal.
The recording of both pulses is called a "trace", which is such a time-history of the travel of a single pulse from the transmit antenna to the receive antenna, and includes all of its different travel paths. All the traces recorded during a period of time are collected together to form a radargram. The round-trip travel time is greater for deep objects than for shallow objects. Therefore, the ToA for the reflected wave recorded on each trace can be used to determine the depth of the buried object, if the velocity of the wave in the subsurface is known.
The widely use of GPR is due to the some advantage offered by the instrumentations that can be stated as the low cost and easiness of employ. Moreover, the flexibility of the GPR system is ensured by the adoption of antennas (mostly portable) working at different frequencies and that can be moved. Despite of this, the main drawback that affects the use of GPR consists in the difficult interpretation of the radargram in order to achieve clear and accurate information about the presence, location and dimension of the buried objects.
For what stated up to now, it is important to develop innovative techniques that could be able to deal with the non-invasive localization problem that, in addition, shows different features depending on the particular area of application. For example, in the typical GPR surveys the main difficulty is represented by the soil attenuation, so that the losses of the ground could affect seriously the localization, on the other hand, in civil engineering applications, it is important to see through dry concrete walls and the attenuation is suitably small. Another important application
is the remediation of unexploded mines, which consists on the survey of few centimeters deep on the ground, that very often is a dry soil. Differently happens for geophysical prospections where higher range capability is instead required.

In this study we present a procedure that could be conveniently implemented for low-attenuation environments and for low relative electric permittivities of the embedding media (up to ten). The most important advantage of this technique consists in its simplicity and low-computational implementation, while the evident drawbacks are related to the excessive approximation of the analytical model that force to carry on the study in order to consider more realistic scenarios. Anyway, the outcomes presented in this work, cannot be ignored encouraging further developments.
The approach that we follow is hybrid electromagnetic-statistical because we consider the direct electromagnetic scattering rigorously by means of a full-wave scattering solver implementing the Cylindrical Wave Approach in the spectral domain, while we derived a simplified model for the scattered signal reconstruction. In fact, by using an uniform linear array of sensors, we consider the total scattered field by a cylindrical infinitely long object, as a superimposition of narrowband signals each one having a different Direction-of-Arrival (DoA) corresponding to the direction of the maximum backscattered power at each antenna of the array. Furthermore, by triangulating all the DoAs we obtain a crossing pattern made of two different regions: one of them represented by a dense cloud of crossings, and the other region in which the number of crossings is small and they are quite sparse. Is intuitive to understand that the first region is the one having the higher probability to identify the object. So that, we adopt a simple an efficient statistical technique based on the Poisson distribution, that allow us to erase the second region and estimate the position of the center of the cylindrical cross section by averaging the co-ordinates of the first region crossings. We describe the signal and noise model in Chapter 1, together with many DoA estimation algorithms that we found in literature commonly referred to wireless communication systems. We also treat exhaustively the derivation of the statistical procedure based on the Poisson statistic and the sub-array processing adopted. In Chapter 2, we collect a great number of simulation results relevant to the single-object localization, considering both conducting and dielectric cylinders with several combinations of refractive indexes, embedded in different media. We provide a performance analysis by varying both the co-ordinates of the cylinder center in a two-dimensional reference system considering the horizontal displacement of the object along the array line as the abscissa, and the distance from the array axis as the ordinate. We also report a typical trend among the cylinder radius, its depth and dielectric permittivity, and a characteristic parameter of the array, that
could be generalized in order to estimate the object dimension. The outcomes of these simulations are published in [r1, r5] and presented in important national and international conferences [c1-c4] and [c10, c11], and we are also checking the possibility of using some Support Vector Machines methods for the DoA estimation [c12, c13].
The multiple-object localization is performed by using some algorithms used in Data Mining application an in particular for the clustering analysis. These methods are very effective in classifying homogeneous groups of data, by deriving a proximity function that iteratively updates the number of "clusters" found by the algorithms, together with their co-ordinates. For our scopes, the number of clusters is equal to the number of cylinders, and the same for cluster co-ordinates. We implemented several algorithms even though they return quite the same estimation, but not all of them can find arbitrary configurations of clusters, so that we use some algorithms to find the correct number of clusters and others to estimate the cluster co-ordinates. In Chapter 3 we describe these algorithms together with a brief introduction about the clustering analysis; successively we present many simulations of double-localization both for conducting and dielectric configuration. Also in these cases we provide performance analyses and derive some statements about them. The outcomes achieved for the multi-object localization are not yet published or presented elsewhere, and we take on a commitment to publish them as soon as possible.

The second part of the thesis is dedicated to the analysis and design of microwave components for the Lower Hybrid and Current Drive (LHCD) system relevant to the International Thermonuclear Experimental Reactor (ITER). The present work has been developed within the EURATOM-ENEA-"Roma Tre" University collaboration for the activities of the European Fusion Development Agreement (EFDA) task WP08-HCD-03-01 LH4IT, which leaded to the "EU Contribution to the ITER Lower Hybrid Current Drive Development Plan" of 2011.
It is well known that Tokamak machines required to implement additional heating systems that are mainly obtained by two typologies. The neutral-beam injection involves the introduction of high-energy (rapidly moving) atoms into the magnetically confined plasma. The atoms are ionized as they pass through the plasma and are trapped by the magnetic field. The high-energy ions then, by interacting with plasma, transfer part of their energy in repeated collisions, increasing the overall temperature. The other possibility to realize an effective plasma heating consists in the electromagnetic power injection, by means of three different frequencies corresponding to the resonances of plasma. The lower frequency system is the Ion Cyclotron Resonance Heating (ICRH) that, for ITER, covers the bandwidth from

40 to 55 MHz ; the intermediate frequency heating system is the Lower Hybrid, working at 5 GHz , and the upper frequency bound is represented by the Electron Cyclotron Resonance Heating (ECRH) that transfers electromagnetic power to the plasma at 170 GHz .
Anyway the ECRH is affected by serious technological problem, since high-power-steady-state sources at very high frequency are very hard to create. More feasible is the ICRH even if some problems occur limiting the efficiency, as for example the Debye shielding. Furthermore, even though both ECRH and ICRH can safely provide an efficient power injection of many MWs, these two system have many difficulties to allow the tokamak to operate continuously, by establishing a non-inductive current drive (NICD) in the toroidal co-ordinate. For these reason, the LH system is used; in fact, besides heating capability, it also provides (in theory) an efficient current drive.
Nevertheless, there are several technological problems that have to be taken into account in the LHCD transmission line design, and many microwave components must be properly optimized. After a brief introduction about the fundamentals of Tokamak machines and radiofrequency plasma heating, described in Chapter 1, we go into detail of the design of some important components. In Chapter 2 we introduce the concept of a mode converter able to transform four rectangular waveguide excited in their fundamental mode, into a circular waveguide which is oversized at the working frequency. To better realize the scope of such a component, one must consider that the engineering of heating system is stressed by strict requirements in terms of power transfer, and the overall system must be optimized to dissipate the lowest amount of power through the line. To accomplish with this constraint, the transmission line must be designed to minimize the attenuation losses cased by the large distance between generators and plasma vessel. At the same time, the waveguide used to carry the electromagnetic power must be properly dimensioned in terms of power handling. A good trade-off between these two main specifications, can be achieved by delivering high power with low attenuation, and it is possible only considering oversized waveguides. In fact, the first electric mode propagating in a oversized circular waveguide allows to have a low attenuated transmission (because this particular mode does not affect the waveguide walls) and at the same time the oversizing structure is adequate to carry many MWs power on.
After designing and optimizing the mode converter, we conclude for the unsuitability of such a component, since it is fairly reliable because poorly fault-tolerant. In fact, if all the input waveguides are fed uniformly, the behavior is really outperforming but, as one of them turns off or some malfunctioning occurs, the performances are
drastically compromised. The outcomes of this study are published on [r2, t1] and presented in important homeland and international conferences $[\mathrm{c} 5, \mathrm{c} 8]$.
Furthermore, we consider the opportunity to shorten the distance between klystrons and plasma vessel, that allow us to design an oversized rectangular waveguide system. This part of the study must consider to be quite significant since the oversized propagation relevant theory is very fair when not altogether absent in literature. In particular, in Chapter 2, we analyze and design mode filters that could attenuate the power content of higher order modes propagating in oversized regime, since the most of the power content must be carried on the fundamental mode, as required by the launcher specifications. On top of this, we also design innovative bends in oversized rectangular waveguide that show the important capability to have a low coupling between the fundamental mode and higher order modes, together with very good performances in terms of reflection and transmission coefficients. In Chapter 3 we introduce the theoretic framework of curved waveguide propagation and then we develop optimized design of several existent curved framework. Moreover, we synthesize an innovative curved profile (that we called Trapezoidal-Miter Bend) able to fulfill the specifications with the important improvement consisting in an appreciable reduction of the bending radius, that means an overall reduction of the curved path and thus, of the attenuation. The outcomes of these conceptual and technological achievements are published in [r2-r4, t2, t3] and presented in the occasion of significant conferences [c6-c9].
Other important contributions have been brought by the candidate to several work [r6-r10], even though not as principal author.

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[t3] S. Meschino, S. Ceccuzzi, F. Mirizzi, L. Pajewski, G. Schettini, "Bends in Rectangular Waveguide," in "EU Contribution to the ITER LHCD development plan," Report EFDA TASK: WP09-HCD-03-01 LH4IT.

## CONFERENCE PROCEEDINGS

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## Parte I

## Localization of Buried Objects

## Capitolo 1

## Signal model and localization procedure

In this part of the thesis, a new technique relevant to the microwave imaging of buried cylindrical objects is going to be proposed. Our purpose consists in developing a non-destructive detection method, useful in many technical applications, such as in remote sensing, geophysics and civil engineering. Many different technologies are currently used to this aim as, above all, Ground Penetrating Radar [1], Electromagnetic Tomography [2], Acoustic and Thermal Imaging [3, 4]. ], and more. Innovative methods are required to improve the detection capabilities in a big number of contexts, like for example the detection of sub-surface utilities (pipes, cables, etc.), the localization of damages in the road pavement, land mine reclamations, geophysical prospecting and so on.
The localization of objects in an unknown scenario is an inverse-scattering problem and it depends strongly on several components: the method adopted to solve the forward scattering problem or to collect experimental data, the characteristics of the incident field, the scatterer geometry, their number, and of course the electromagnetic properties of both the objects and the medium in which they are embedded. In particular, the procedure presented in this chapter assumes a half-space geometry in which air and ground are separated, by a planar interface. The two media are considered linear, isotropic, homogeneous and lossless. Moreover, cylindrical objects with axes parallel among them and to the interface are embedded in the ground. The whole structure has an infinite extension along the direction identified by the axes of the cylinders. The propagating vector of the incident field lies in the plane orthogonal to the interface and to cylinder axis. The problem is thus two-dimensional.
A sketch of the scenario is depicted in Fig. 1.1. We assume the incident field to
be $z$-polarized and uniform along $z$, that is $\mathbf{E}_{\text {inc }}(\mathbf{r})=E_{\text {inc } z}\left(\mathbf{r}_{\mathbf{t}}\right) \hat{\mathbf{z}}$ (transverse magnetic - TM). For symmetry reasons, both the scattered electric field and the total electric field turn out to be independent of $z$ and $z$ polarized, $\mathbf{E}_{\text {scat }}(\mathbf{r})=E_{\text {scat }, z}\left(\mathbf{r}_{\mathbf{t}}\right) \hat{\mathbf{z}}$ and $\mathbf{E}(\mathbf{r})=E_{z}\left(\mathbf{r}_{\mathbf{t}}\right) \hat{\mathbf{z}}$, where $\mathbf{r}$ is the position of the measurement point and $\mathbf{r}_{\mathbf{t}}$ is his transversal component, such that $\mathbf{r}=x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+z \hat{\mathbf{z}}=\mathbf{r}_{\mathbf{t}}+z \hat{\mathbf{z}}$.


Figure 1.1: Geometry of the problem.
As a general statement, all the inversion algorithms are associated with the method used to solve the forward scattering problem, which depends on the incident field, the physical-geometrical object configuration and the propagating media as well. We may assume that the whole structure shows a cylindrical symmetry, then the fields are obtained by the superposition of cylindrical wave functions (obtained by solving the Helmholtz equation with appropriate boundary conditions and imposing the Sommerfeld condition of radiation at infinity) centered at the axis of each cylinder.
The method of solving the forward scattering problem is the Cylindrical Wave Approach (CWA) solver which is a rigorous and fast spectral-domain method that solves the scattering problem of perfectly-conducting [5] or dielectric [6] cylindrical objects with circular cross-section, buried in a dielectric half-space or in a multilevel medium, as a finite-thickness slab [7]. The CWA may deal with both TM and TE polarization fields. It can be applied for arbitrary values of permittivity, size, and position of the targets. Since the method is implemented in the frequency domain, dispersive soils can be modeled (soils with frequency-dependent permittivity). The technique can be employed to study the scattering of an incident pulsed wave, with a rather general time-domain shape [8].
As shown in $[5,9]$, for perfectly-conducting cylinders, and in [6] for dielectric cylinders, obstacles of general shape can be simulated through the CWA with good results, by using a suitable set of small circular-section cylinders. In order to study more in depth the wire-grid modeling of perfectly-conducting cylinders, we here
consider the reference case of a circular cylinder buried in a linear, isotropic, homogeneous, dielectric, lossless half-space, and we solve the scattering problem exactly, or by simulating the target through a circular array of N wires, see Fig. 1.2. The cylinders are parallel to the $y$ axis and the structure is assumed to be infinite along $y$ direction. A monochromatic plane wave, with wave-vector $\mathbf{k}_{i}$ lying in the $x z$ plane, impinges normally from medium 0 (a vacuum, with permittivity $\epsilon_{0}$ ) on the planar interface with medium 1 (the ground, with permittivity $\epsilon_{1}$ ). A normalized reference frame $(O, \xi, \zeta)$ is introduced, with coordinates $\xi=k_{0} x$ and $\zeta=k_{0} z, k_{0}=2 \pi / \lambda_{0}$ being the vacuum wavenumber and $\lambda_{0}$ the vacuum wavelength. In such reference frame, the radius of the big cylinder is called $R$, its burial depth is $\chi$ and the wire radius is $\alpha$ (Fig. 1.2).
In the CWA, the solution to the scattering problem is carried out in terms of $V(\xi, \zeta)$, representing the $y$-component of the electric/magnetic field: $V=E_{y}(\xi, \zeta)$ for $\mathrm{TM}(y)$ polarization, and $V=H_{y}(\xi, \zeta)$ for $\mathrm{TE}(y)$ polarization.
In order to obtain a rigorous solution for $V(\xi, \zeta)$, the total field can be expressed as the superposition of a set of terms, produced by the interaction between the incident field, the interface and the cylinders: the incident plane wave, the reflected field (due to the reflection in medium 0 of the incident plane wave by the interface), the transmitted field (due to the transmission in medium 1 of the incident plane wave by the interface), the field scattered by the cylinders in medium 1, the scattered-reflected field (due to the reflection in medium 1 of the scattered field by the interface), the scattered-transmitted field (due to the transmission in medium 0 of the scattered field by the interface). The scattered field is expressed as the sum of the fields scattered by each buried obstacle. The field scattered by a cylinder is represented in terms of a superposition of cylindrical waves, by means of unknown coefficients, and use is made of the plane-wave spectrum of such waves to treat their reflection and transmission by the interface. The presence of the planar interface, in fact, is taken into account by means of its complex plane-wave reflection and transmission coefficients. To express the scattered-reflected and scattered-transmitted fields, the presence of the planar surface is taken into account by considering the reflection and trans- mission of each elementary plane wave constituting the Fourier spectrum of a cylindrical function. The scattered-reflected field is given by the sum of the fields scattered by each buried cylinder and reflected by the interface; suitable reflected cylindrical functions are introduced. The scattered-transmitted field is expressed as the sum of the fields scattered by each buried cylinder and transmitted to medium 0 ; transmitted cylindrical functions are defined.
Given the expressions of all the field terms, the boundary conditions on the cylinder
surfaces are imposed: due to the hypothesis of perfectly-conducting cylinders, the tangential component of the total electric field must vanish. A linear system for the unknown coefficients of the cylindrical-waves expansions is obtained. Once such system is solved, the $V(\xi, \zeta)$ field is determined in any point of the space and for both polarizations. All the other components of the electromagnetic field are derived by using Maxwell's equation.


Figure 1.2: Geometry of the scattering problem: a) a buried circular cylinder with radius $\mathrm{R} ; b$ ) simulation of the cylinder in $a$ ) by means of N smaller circular cylinders with radius $\alpha$.

### 1.1 Assumptions

The electromagnetic propagation model used to represent the fields is derived from Maxwell's equations. In particular, in a vacuum, far away from sources (charges and currents), they form the following system:

$$
\begin{gather*}
\nabla \cdot \mathbf{D}=0  \tag{1.1}\\
\nabla \cdot \mathbf{B}=0  \tag{1.2}\\
\nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}  \tag{1.3}\\
\nabla \times \mathbf{B}=\mu_{0} \epsilon_{0} \frac{\partial \mathbf{E}}{\partial t} \tag{1.4}
\end{gather*}
$$

where $\mathbf{B}$ is the magnetic induction $\left[\frac{W b}{m^{2}}\right], \mathbf{E}$ is the electric field $\left[\frac{V}{m}\right], \epsilon_{0}=8,84 \times 10^{-12}$ $\frac{F}{m}$ and $\mu_{0}=12,56 \times 10^{-7} \frac{H}{m}$ are the electric and magnetic permittivity, respectively. Properly combining the four equations above, we derive the following homogeneous
wave equation:

$$
\begin{align*}
& \nabla^{2} \mathbf{E}-\frac{1}{c^{2}} \frac{\partial \mathbf{E}^{2}}{\partial t^{2}}=0  \tag{1.5}\\
& \nabla^{2} \mathbf{B}-\frac{1}{c^{2}} \frac{\partial \mathbf{B}^{2}}{\partial t^{2}}=0 \tag{1.6}
\end{align*}
$$

where $c=\frac{1}{\sqrt{\mu_{0} \epsilon_{0}}} \approx 3 \times 10^{8} \frac{m}{s e c}$ is the propagation velocity in a vacuum. Eqs. (1.5) and (1.6) are both vectorial, but in many cases it is possible to introduce a scalar approximation and consider the vectors $\mathbf{B}$ and $\mathbf{E}$ as scalar functions $B(\mathbf{r}, t)$ and $E(\mathbf{r}, t)$ depending on position $\mathbf{r}$ and time [10]. So that, instead of Eqs. (1.5) and (1.6) we have the following scalar equations:

$$
\begin{align*}
& \nabla^{2} E(\mathbf{r}, t)-\frac{1}{c^{2}} \frac{\partial E(\mathbf{r}, t)^{2}}{\partial t^{2}}=0  \tag{1.7}\\
& \nabla^{2} B(\mathbf{r}, t)-\frac{1}{c^{2}} \frac{\partial B(\mathbf{r}, t)^{2}}{\partial t^{2}}=0 \tag{1.8}
\end{align*}
$$

Considering $E(\mathbf{r}, t)$ and assuming the same for $B(\mathbf{r}, t)$, every solution in the form $E(\mathbf{r}, t)=f\left(\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right)$ fulfills the scalar equation, Eq. (1.7), being $\boldsymbol{\alpha}=|\boldsymbol{\alpha}| \cdot \hat{\boldsymbol{\alpha}}$, con $|\boldsymbol{\alpha}|=\frac{1}{c}$.
The function $f\left(\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right)$ represents a traveling plane wave in $\hat{\boldsymbol{\alpha}}$ direction, having propagation velocity equal to $c$. A further assumption is made for the solutions of the Eq. (1.7), in particular, we consider only those that are consistent with the the narrowband condition.
A signal emitted from a narrowband transmitter can be expressed as in [?]:

$$
\begin{equation*}
E(\mathbf{0}, t)=s(t) e^{-j \omega t} \tag{1.9}
\end{equation*}
$$

where $\omega=2 \pi f_{0}$ and the signal bandwidth $B_{s}$ is lower the carrier frequency $f_{0}$, as sketched in Fig. 1.3.


Figure 1.3: Narrowband representation
Being $|\mathbf{r}| \ll \frac{c}{B_{s}},\left|\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right| \ll \frac{1}{B_{s}}$ we obtain:

$$
\begin{equation*}
E(\mathbf{r}, t)=s\left(t-\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right) e^{-j \omega\left(t-\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right)} \approx s(t) e^{-j \omega\left(t-\mathbf{r}^{T} \cdot \boldsymbol{\alpha}\right)} \tag{1.10}
\end{equation*}
$$

Moreover, considering $\mathbf{k}=\boldsymbol{\alpha} \omega$, Eq. (1.10) comes into the following equation:

$$
\begin{equation*}
E(\mathbf{r}, t) \approx s(t) e^{-j \omega\left(t-\mathbf{r}^{T} \cdot \mathbf{k}\right)} \tag{1.11}
\end{equation*}
$$

where $\mathbf{k}$ is the wave vector and $|\mathbf{k}|=k=\frac{\omega}{c}$ is the related wave number, noting that $\mathbf{k}$ and $\hat{\boldsymbol{\alpha}}$ have the same direction as the propagating vector.
In a Cartesian coordinate system subsists that:

$$
\begin{equation*}
\mathbf{k}=(\cos \theta, \sin \theta)^{T} \tag{1.12}
\end{equation*}
$$

where $\theta$ is the angle between $\mathbf{k}$ and the $x$ axis as shown in Fig. 1.4.
In the case of interest, an isotropic source emits a field that satisfies Eq. (1.11). This field is received by a sensor placed in $\mathbf{r}_{1}$. Once the time is fixed, the phase of Eq. (1.11) is constant if $\mathbf{r}^{T} \cdot \mathbf{k}=$ const, and, supposing the propagation direction and frequency to be constant too, when $\mathbf{r}=$ const. Therefore, all points lying on the spherical surface of radius $\mathbf{r}$ have uniform phases. This means that the isotropic source emits a spherical wave with amplitude inversely proportional to the distance from the source and phase-constant on spherical wavefronts. If we consider an


Figure 1.4: Parametric model
array of receiving antennas, it is necessary to introduce the far-field (or Fraunhofer) condition, which consists to impose a minimum distance between the receivers and the source in order to assume the impinging field as a plane wave. In particular, when this condition is satisfied, the spherical wavefront approximates a plane wave and then we can consider the array as a phase constant line for the impinging plane wave.
The far-field condition is often expressed like $|\mathbf{r}| \geq \frac{2 \mathcal{D}^{2}}{\lambda}$, where $\mathcal{D}$ is the width of the array ( $\mathcal{D}=d M$ where $d$ is the spacing among the $M$ antennas of the array) and $\lambda$ is the wavelength of the incoming signal.
With reference to an array of receiving antennas, in a Cartesian coordinate system, we represent the generic sensor as a receiving point with distinct coordinates. As
shown in Fig. 1.4, $\mathbf{r}_{m}=\left(x_{m}, y_{m}\right)^{T}$. Following the Eqs. (1.11) and (1.12), the field received by the $m$-th sensor and coming with the Direction of Arrival (DoA) $\theta$ is:

$$
\begin{equation*}
E(\mathbf{r}, t) \approx s(t) e^{j\left[\omega t-k\left(x_{m} \cos \theta+y_{m} \sin \theta\right)\right]} \tag{1.13}
\end{equation*}
$$

If we consider a constant frequency response at the $m t h$ sensor, like $g_{m}(\theta)$, in the whole signal bandwidth, the received field is proportional to the incident one at the point $\mathbf{r}_{m}$. Suppressing the harmonic time dependence, the output is:

$$
\begin{equation*}
x_{m}(t)=g_{m}(\theta) s(t) e^{-j k\left(x_{m} \cos \theta+y_{m} \sin \theta\right)}=a_{m}(\theta) s(t) \tag{1.14}
\end{equation*}
$$

Remembering Eq. (1.11), Eq. (1.14) is fulfilled when the array width is narrower than the bandwidth $(f / B)$. This is often referred as narrowband condition. In particular for a $M$-element array, the output signal, included noise, is:

$$
\begin{equation*}
\mathbf{x}(t)=\sum_{m=1}^{M} x_{m}(t)=\mathbf{a}(\theta) s(t)+\mathbf{n}(t) \tag{1.15}
\end{equation*}
$$

where $\mathbf{a}(\theta)$ is the steering vector and $\mathbf{n}(t)$ is the noise vector. So that, for a signal incoming from the direction $\theta$, we obtain:

$$
\mathbf{a}(\theta)=\left[\begin{array}{lll}
a_{1}(\theta) & \cdots & a_{M}(\theta) \tag{1.16}
\end{array}\right]^{T}
$$

The most common uniform array geometries are depicted in Fig. 1.5.


Figure 1.5: Array geometric configuration $a$ ) unifom linear array (ULA) e $b$ ) uniform circular array (UCA).

In the case of an uniform linear array (ULA) with spacing $d$, the receiving-antenna positional vector is $\mathbf{r}_{m}=\left[\begin{array}{ll}(m-1) d & 0\end{array}\right]^{T}$, and considering all the elements having the same directivity $g_{1}(\theta)=\cdots=g_{M}(\theta)=g(\theta)=1$, the steering vector becomes

$$
\mathbf{a}_{\mathbf{U L A}}(\theta)=\left[\begin{array}{lllll}
1 & e^{-j k d \cos \theta} & e^{-2 j k d \cos \theta} & \cdots & e^{-(M-1) j k d \cos \theta} \tag{1.17}
\end{array}\right]^{T}
$$

The Eq. (1.17) is a Vandermonde vector, and all the components of the vector, have a unitary norm with a constant phase shift. For a $L$-element uniform circular array (UCA) of radius $R=\lambda /[4 \sin (\pi / L)]$ the positional and steering vectors become instead

$$
\begin{gathered}
\mathbf{r}_{\mathrm{m}}=\left[\begin{array}{lll}
\frac{R \cos 2 \pi(l-1)}{L} & \frac{R \sin 2 \pi(l-1)}{L}
\end{array}\right]^{T} \\
\mathbf{a}_{U C A}=\left[\begin{array}{llll}
1 & e^{-j k R \cos (\theta-2 \pi / L)} & \cdots & e^{-j k R \cos (\theta-(l-1) \pi / L)}
\end{array}\right]^{T}
\end{gathered}
$$

where $\theta \in[-\pi, \pi], l=1, \cdots, L$ and the radius $R$ is chosen to maintain an interelement spacing of $d=\lambda / 2$. With no loss of generality we assume that the signal $s(t)$ has unit power. In this case the complete narrowband model for the received signal can be written as

$$
\begin{equation*}
\mathbf{x}(t)=\sqrt{S N R} s(t) \mathbf{a}(\theta)+\mathbf{n}(t) \tag{1.1.}
\end{equation*}
$$

being SNR the signal-to-noise ratio defined as

$$
\begin{equation*}
S N R=\frac{\mathbf{E}_{\mathbf{S}}{ }^{H} \cdot \mathbf{E}_{\mathbf{S}}}{M \sigma_{n}^{2}} \tag{1.19}
\end{equation*}
$$

where $\mathbf{E} \mathbf{S}$ is the scattered field received by each sensor and $\sigma_{n}$ is the noise variance. Let's consider now $D$ signal impinging on a $M$-elements array. The vector collecting the signal DoAs is $\theta_{1}, \ldots, \theta_{D}$, and the output is:

$$
\begin{equation*}
\mathbf{x}(t)=\sum_{d=1}^{D} \mathbf{a}\left(\theta_{d}\right) s_{d}(t)+\mathbf{n}(t) \tag{1.20}
\end{equation*}
$$

In a matrix form the Eq. (1.20) becomes:

$$
\begin{align*}
\mathbf{A}(\theta) & =\left[\begin{array}{lll}
\mathbf{a}\left(\theta_{1}\right) & \cdots & \mathbf{a}\left(\theta_{D}\right)
\end{array}\right]  \tag{1.21}\\
\mathbf{s}(t) & =\left[\begin{array}{lll}
\mathbf{s}_{1}(t) & \cdots & \mathbf{s}_{D}(t)
\end{array}\right] \tag{1.22}
\end{align*}
$$

where $\mathbf{A}(\theta)$ is the $[M \times D]$ steering matrix formed by all the steering vectors, $\mathbf{s}(t)$ is the signal vector; moreover $\mathbf{n}(t)$ is the noise vector.
Finally, the well known parametric model used in radar signal processing is obtained [12]:

$$
\begin{equation*}
\mathbf{x}(t)=\sqrt{S N R} \mathbf{A}(\theta) \cdot \mathbf{s}(t)+\mathbf{n}(t) \tag{1.23}
\end{equation*}
$$

All the methods that will be presented henceforth, require fulfilled the condition $D<M$, that is the number of incoming signals must be smaller than the number of sensors of the array. The output of the sensors will be properly pre-processed and sampled at time instants $k=1,2, \ldots, K$. The $x(t)$ can be seen as an aleatory process whose characteristics can be investigated through the statistics of first and second order of signals and noise.

### 1.2 Direction of Arrival Estimation

Many of the algorithms used to estimate the DoAs, are based on a proper manipulation of the array correlation matrix of the array. In Fig. 1.6. a $M$-element ULA with $M$ potential weights is used to receive $x_{m}(k)$ signals includes additive, zero mean, Gaussian noise. Time is represented by the $k$ th time sample. Thus, the array output $y$ can be given in the following form [13]:

$$
\begin{equation*}
y(k)=\mathbf{w}^{T} \cdot \mathbf{x}(k) \tag{1.24}
\end{equation*}
$$

Thus, each of the $D$-complex signals arrives at angles $\theta_{i}$ and is intercepted by the $M$ antenna elements. It is initially assumed that the arriving signals are monochromatic and the number of arriving signals is $D<M$. The arriving signals are time varying and thus our calculations are based upon time snapshots of the incoming signal. At this stage we consider the process to be stationary so that, the matrix of steering vectors is not changing with time and the corresponding arrival angles are fixed. Unless otherwise stated, the time dependence will be suppressed in the following equations. We define the $M \times M$ array correlation matrix $\mathbf{R}_{\mathbf{x x}}$ as:


Figure 1.6: Signal model for a $M$-element array with $D$ arriving signals.

$$
\begin{align*}
\mathbf{R}_{\mathbf{x x}} & =E\left[\mathbf{x} \cdot \mathbf{x}^{H}\right]=  \tag{1.25}\\
& =E\left[(\mathbf{A} \cdot \mathbf{s}+\mathbf{n}) \cdot(\mathbf{A} \cdot \mathbf{s}+\mathbf{n})^{H}\right]= \\
& =\mathbf{A} \cdot E\left[\mathbf{s} \cdot \mathbf{s}^{H}\right] \cdot \mathbf{A}^{H}+E\left[\mathbf{n} \cdot \mathbf{n}^{H}\right]= \\
& =\mathbf{A} \cdot \mathbf{R}_{\mathbf{s s}} \cdot \mathbf{A}^{H}+\mathbf{R}_{\mathbf{n n}}
\end{align*}
$$

Where the source and the noise correlation matrices are found by the expected value of the respective absolute values squared:
$\mathbf{R}_{\mathbf{s s}}=E|\mathbf{s}(t)|^{2} ;[D \times D]$ signal correlation matrix;
$\mathbf{R}_{\mathbf{n n}}=E|\mathbf{n}(t)|^{2}=\sigma_{n}^{2} \cdot \mathbf{I} ; \quad[M \times M]$.
If we do not know the exact statistics for the noise and signals, but we can assume that the process is ergodic, then we can approximate the correlation by use of a time-averaged correlation. In that case the correlation matrices are defined by

$$
\left\{\begin{array}{l}
\hat{\mathbf{R}}_{\mathbf{x x}} \approx \frac{1}{K} \sum_{t=1}^{K} \mathbf{x}(t) \cdot \mathbf{x}^{H}(t)  \tag{1.26}\\
\hat{\mathbf{R}}_{\mathrm{ss}} \approx \frac{1}{K} \sum_{t=1}^{K} \mathbf{s}(t) \cdot \mathbf{s}^{H}(t) \\
\hat{\mathbf{R}}_{\mathbf{n n}} \approx \frac{1}{K} \sum_{t=1}^{K} \mathbf{n}(t) \cdot \mathbf{n}^{H}(t)
\end{array}\right.
$$

When the signals are uncorrelated, $\mathbf{R}_{\text {ss }}$ obviously has to be a diagonal matrix because off-diagonal elements have no correlation. When the signals are partly correlated, $\mathbf{R}_{\text {ss }}$ is nonsingular. When the signals are coherent, $\mathbf{R}_{\text {ss }}$ becomes singular because the rows are linear combinations of each other. The matrix of steering vectors, $\mathbf{A}$ is an $M \times D$ matrix where all columns are different. Their structure is Vandermonde and hence the columns are independent [13].
The goal of DoA estimation techniques is to define a function that gives an indication of the angles of arrival based upon maxima vs. angle. This function is traditionally called the pseudospectrum $P(\theta)$ and the units can be in energy or in watts (or at times energy or watts squared). There are several potential approaches to defining the pseudospectrum and in particular, Stoica and Moses [14] and Van Trees [15] give an in-depth explanation of many of these possible approaches.
We classify the parameter estimation techniques into two main categories, namely Non-Parametric (or spectral-based) and Parametric. The Non-Parametric techniques form some spectrum-like function of the parameter(s) of interest, e.g., the DoA. The locations of the highest (separated) peaks of the function in question are recorded as the DoA estimates. In this first group, are included also the so-called Subspace Decomposition methods. Parametric techniques, on the other hand, require a simultaneous search for all parameters of interest and some statistical hypothesis about the process. The latter approach often results in more accurate estimates, albeit at the expense of an increased computational complexity. We will summarize some of the more popular pseudospectra solutions in the next section.

### 1.2.1 Non-Parametric Methods

Spectral-based methods which are discussed in this section, can be classified into beamforming techniques and subspace-based methods. Non-Parametric methods refer to the spectral estimation algorithms that use additional assumptions about the signal generation based on the determination of the periodogram obtained by DFT (Discrete Fourier Transform) of the sampled sequence. The first attempt to automatically localize signal sources using antenna arrays was through beamforming techniques.
The idea is to "steer" the array in one direction at a time and measure the output power. The steering locations which result in maximum power yield the DoA estimates. The array response is steered by forming a linear combination of the sensor outputs

$$
y(t)=\sum_{i=1}^{L} w_{i}^{*} \cdot x_{i}(t)=\mathbf{w}^{H} \cdot \mathbf{x}(t)
$$

Given samples $k=1, \cdots, K$ we obtain $K$ snapshots of $y,[y(1), \cdots, y(K)]$ and the Power Spectral Density (PSD) estimations expressed in Eq. (1.29). Different beamforming approaches correspond to different choices of the weighting vector w . For an excellent review of beamforming methods, we refer to [16].

$$
\begin{align*}
\hat{P}(\mathbf{w}) & =E\left\{\frac{1}{K} \sum_{k=1}^{K}|y(t)|^{2}\right\}=  \tag{1.27}\\
& =\frac{1}{K} \sum_{k=1}^{K} \mathbf{w}^{H} \cdot E\left\{\mathbf{x}(t) \mathbf{x}^{H}(t)\right\} \mathbf{w}= \\
& =\mathbf{w}^{H} \cdot \hat{\mathbf{R}}_{\mathbf{x x}} \cdot \mathbf{w}
\end{align*}
$$

So that, an estimation of the received power vs. the $\theta$ is possible by properly selecting the right-hand of the previous equation and notably by a particular choice of the weight vector $\mathbf{w}$.

## Conventional Beamforming

The conventional (or Bartlett) beamformer is a natural extension of classical Fourier-based spectral analysis $[17,18]$ to sensor array data. For an array of arbitrary geometry, this algorithm maximizes the power of the beamforming output for a given input signal. Suppose we wish to maximize the output power from a certain direction $\theta=\theta_{i}$.

The problem of maximizing the output power is then formulated as,

$$
\begin{align*}
\max _{w}\{P(\mathbf{w})\} & =\max _{w}\left\{E\left[\mathbf{w} \cdot \mathbf{x} \cdot \mathbf{x}^{H} \cdot \mathbf{w}\right]\right\}=  \tag{1.28}\\
& =\max _{w}\left\{\mathbf{w}^{H} \cdot E\left[\mathbf{x} \cdot \mathbf{x}^{H}\right] \cdot \mathbf{w}\right\}= \\
& =\max _{w}\left\{\mathbf{w}^{H} \cdot\left[\mathbf{a}(\theta) \cdot E\left[\mathbf{s} \cdot \mathbf{s}^{H}\right] \cdot \mathbf{a}^{H}(\theta)+E\left[\mathbf{n} \cdot \mathbf{n}^{H}\right]\right] \cdot \mathbf{w}\right\}= \\
& =\max _{w}\left\{\mathbf{w}^{H} \cdot\left[\mathbf{a}(\theta) \cdot E\left[|\mathbf{s}|^{2}\right] \cdot \mathbf{a}^{H}+E\left[|\mathbf{n}|^{2}\right]\right] \cdot \mathbf{w}\right\}= \\
& =\max _{w}\left\{E\left[|\mathbf{s}|^{2}\left|\mathbf{w}^{H} \cdot \mathbf{a}(\theta)\right|^{2}+\sigma_{n}^{2}|\mathbf{w}|^{2}\right]\right\}
\end{align*}
$$

where the assumption of spatially white noise is used. To obtain a non-trivial solution, the norm of $\mathbf{w}$ is constrained to $\left\|\mathbf{w}^{H} \cdot \mathbf{a}(\theta)\right\|=1$ when carrying out the above maximization. The resulting solution is the

$$
\begin{equation*}
\mathbf{w}=\frac{\mathbf{a}(\theta)}{\sqrt{\mathbf{a}^{H}(\theta) \cdot \mathbf{a}(\theta)}} \tag{1.29}
\end{equation*}
$$

The above weight vector can be interpreted as a spatial filter, which has been matched to the impinging signal. Intuitively, the array weighting equalizes the delays (and possibly attenuations) experienced by the signal on various sensors to maximally combine their respective contributions. Inserting the weighting vector into Eq. (1.29), the classical spatial spectrum is obtained

$$
\begin{equation*}
P_{B}(\theta)=\mathbf{a}^{H}(\theta) \cdot \mathbf{R}_{\mathbf{x x}} \cdot \mathbf{a}(\theta) \tag{1.30}
\end{equation*}
$$

The Bartlett DoA estimate is the spatial version of an averaged periodogram and is a beamforming DoA estimate. Under the conditions where s represents uncorrelated monochromatic signals and there is no system noise, Eq. (1.32) is equivalent to the following long-hand expression:

$$
\begin{equation*}
P_{B}(\theta)=\left|\sum_{i=1}^{D} \sum_{m=1}^{M} e^{j k d(m-1)\left(\cos \theta-\cos \theta_{i}\right)}\right| \tag{1.31}
\end{equation*}
$$

The periodogram is thus equivalent to the spatial finite Fourier transform of all arriving signals. This is also equivalent to adding all beamsteered array factors for each angle of arrival and finding the absolute value squared.
One of the limitations of the Bartlett approach to DoA estimation consists in the ability to resolve angles is limited by the array half-power beamwidth, as indicated in Fig. 1.7. An increase in resolution requires a larger array. For large array lengths with $d=\frac{\lambda}{2}$ spacing, the DoA resolution is approximately $\frac{1}{M}$. Thus, $\frac{1}{M}$ is the DoA resolution limit of a periodogram and in the case above is an indicator of the resolution of the Bartlett method. It should be noted that when two emitters are separated by an angle wider than the array resolution, they can be resolved but a bias is introduced. This bias cause the peaks to deviate from the true DoA. This bias asymptotically decreases as the array length increases.



Figure 1.7: Bartlett pseudospectrum: a) $\theta_{1}=-10^{\circ}$ and $\theta_{2}=10^{\circ}$; b) $\theta_{1}=-5^{\circ} \mathrm{e}$ $\theta_{2}=5^{\circ}$ 。

## Capon Beamforming

In an attempt to alleviate the limitations of the above beamformer, such as its resolving power of two sources spaced closer than a beamwidth, researchers have proposed numerous modifications. A well-known method was proposed by Capon [19], and was later interpreted as a dual of the beamformer by Lacoss [20]. The optimization problem was posed as

$$
\mathbf{w}: \begin{cases}\mathbf{w}^{H} \cdot \mathbf{a}\left(\theta_{i}\right)=1 & \theta=\theta_{i}  \tag{1.32}\\ \min _{w} P(\mathbf{w}) & \theta \neq \theta_{i}\end{cases}
$$

where $P(w)$ is as defined in Eq. (1.29). Hence, Capon's beamformer (also known as the Minimum Variance Distortionless Response filter in the acoustics literature) attempts to minimize the power contributed by noise and any signals coming from other directions than $\theta_{i}$, while maintaining a fixed gain in the "look direction" $\theta_{i}$. It is also alternatively a maximum likelihood estimate of the power arriving from one direction while all other sources are considered as interference. Thus the goal is to maximize the signal-to-interference ratio (SIR) while passing the signal of interest undistorted in phase and amplitude. The source correlation matrix $\mathbf{R}_{\mathrm{ss}}$ is assumed to be diagonal. This maximized SIR is accomplished with a set of array weights $\mathbf{w}$ as shown in Fig. 1.6, where the array weights are given by

$$
\begin{equation*}
\mathbf{w}=\frac{\mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{a}(\theta)}{\mathbf{a}^{H}(\theta) \cdot \mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{a}(\theta)} \tag{1.33}
\end{equation*}
$$

Inserting the above weight into (2.6) leads to the following Capon ""spatial spectrum":

$$
\begin{equation*}
P_{C}(\theta)=\frac{1}{\mathbf{a}^{H}(\theta) \cdot \mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{a}(\theta)} \tag{1.34}
\end{equation*}
$$

It is easy to see why Capon's beamformer outperforms the classical one given in Eq. (1.32), as the former uses every available degree of freedom to concentrate the received energy along one direction, namely the bearing of interest. This is reflected by the constraint given in Eq. (1.34). The power minimization can also be interpreted as sacrificing some noise suppression capability for more focused "nulling" in the directions where there are other sources present. The spectral leakage from closely spaced sources is therefore reduced, though the resolution capability of the Capon beamformer is still dependent upon the array aperture and clearly on the $S N R$. A number of alternative methods for beamforming have been proposed, addressing various issues such as partial signal canceling due to signal coherence [21] and beam shaping and interference control [?, 23].
Looking at Fig. 1.8 It is clear that the Capon estimate has much greater resolution than the Bartlett estimate. In the case where the competing sources are highly correlated, the Capon resolution can actually become worse. The derivation of the Capon (ML) weights was conditioned upon considering that all other sources are interferers. If the multiple signals can be considered as multipath signals, with Rayleigh amplitude and uniform phase, then the uncorrelated condition is met and the Capon estimate will work. The advantage of the Bartlett and Capon estimation methods is that these are nonparametric solutions and one does not need an a priori knowledge of the specific statistical properties.


Figure 1.8: Capon pseudospectrum for $\theta_{1}=-5^{\circ}$ e $\theta_{2}=5^{\circ}$.

### 1.2.2 Subspace-Based Methods

Many spectral methods in the past, have implicitly called upon the spectral decomposition of a covariance matrix to carry out the analysis (e.g., Karhunen-

Lokve representation). One of the most significant contributions came about when the eigen-structure of the covariance matrix was explicitly invoked, and its intrinsic properties were directly used to provide a solution to an underlying estimation problem for a given observed process. Early approaches involving invariant subspaces of observed covariance matrices include principal component factor analysis [24] and errors-in-variables time series analysis. In the engineering literature, Pisarenko's work [25] in harmonic retrieval was among the first to be published. However, the tremendous interest in the subspace approach is mainly due to the introduction of the MUSIC (Multiple SIgnal Classification) algorithm [26, 27].
It is interesting to note that while earlier works were mostly derived in the context of time series analysis and later applied to the sensor array problem, MUSIC was indeed originally presented as a DoA estimator. It has later been successfully brought back to the spectral analysis with its later developments (see e.g. [28, 29]).
Since the correlation matrix $\mathbf{R}_{\mathrm{xx}}$ is an $M \times M$ Hermitian matrix, it is equal to its complex conjugate transpose such that $\mathbf{R}_{\mathrm{xx}}=\mathbf{R}_{\mathbf{x x}}{ }^{H}$. The array correlation matrix has $M$ eigenvalues $\left(\lambda_{1}, \lambda_{2}, \cdots, \lambda_{M}\right)$ along with $M$ associated eigenvectors $\mathbf{E}=\left[\begin{array}{llll}\mathbf{E}_{\mathbf{1}} & \mathbf{E}_{\mathbf{2}} & \cdots & \mathbf{e}_{M}\end{array}\right]^{T}$. If the eigenvalues are sorted from smallest to largest, we can divide the matrix $\mathbf{E}$ into two subspaces such that $\mathbf{E}=\left[\mathbf{E}_{\mathbf{S}} \mathbf{E}_{\mathbf{N}}\right]$ where $\mathbf{E}_{\mathbf{S}}$ is the signal subspace and $\mathbf{E}_{\mathbf{N}}$ is called the noise subspace. In particular, $\mathbf{E}$ and is composed of $D$ eigenvectors associated with signals and of $M-D$ eigenvectors associated with the noise. For uncorrelated noise, the eigenvalues are $\lambda_{1}=\lambda_{2}=\cdots=\lambda_{M-D}=\sigma_{n}^{2}$. The signal subspace $\mathbf{E}_{\mathbf{S}}$ is composed of $D$ eigenvectors associated with the arriving signals. The noise subspace is an $M \times(M-D)$ matrix, while the signal subspace is an $M \times D$ matrix.

The most important property of these subspaces is that, if signals and noise are uncorrelated, the signal subspace $\mathbf{E}_{\mathbf{S}}$ and the noise subspace $\mathbf{E}_{\mathbf{N}}$ are mutually orthogonal. Consequently, the scalar product between noise and signal in null, moreover, since the DoAs belong to the signal subspace, we can select the steering vector that nullify the scalar product with the noise subspace matrix. In particular, referring to Eq. (1.25), the eigenvalues of $\mathbf{R}_{\mathrm{xx}}$ are given by the sum of the eigenvalues of $\mathbf{A} \cdot \mathbf{R}_{\mathbf{x x}} \cdot \mathbf{A}^{H}$ and $\mathbf{R}_{\mathbf{n n}}$. Thus, if the signals are also uncorrelated among them, $\mathbf{R}_{\mathrm{ss}}=E\left[\mathbf{s} \cdot \mathbf{s}^{H}\right]$ is a $D \times D$ diagonal matrix, full rank, and with $D$ real eigenvalues. Moreover, the linear transformation $\mathbf{A} \cdot \mathbf{R}_{\mathbf{x x}} \cdot \mathbf{A}^{H}$ is a $M \times M$ full rank hermitian matrix, with $D$ eigenvalues different from zero. In particular, these eigenvalues are related to the signals.
Furthermore, supposing the noise as a determination of a Gaussian process having zero mean and variance $\sigma_{n}^{2}$, and assumed to be uncorrelated with signals, $\mathbf{R}_{\mathbf{n n}}$ is a
$M \times M$ diagonal matrix, in which the diagonal elements are equal among them and to the variance. In particular it admits $M$ equal eigenvalues $\left(q_{1}=q_{2}=\cdots=q_{M}=\sigma_{n}^{2}\right)$. So that, the eigenvalues of $\mathbf{R}_{\mathrm{xx}}$ are

$$
\lambda_{i}=\left\{\begin{array}{ll}
p_{i}+\sigma_{n}^{2} & i=1, \cdots, D  \tag{1.35}\\
\sigma_{n}^{2} & i=D+1, \cdots, M
\end{array} \quad \lambda_{i} \in \mathbb{R}\right.
$$

and, since $\mathbf{R}_{\mathbf{x x}}$ is hermitian, the associate eigenvectors are mutually orthogonal. Diagonalizing $\mathbf{R}_{\mathbf{x x}}$ the array correlation matrix we obtain a representation of the whole vectorial space in terms of eigenvalues and eigenvectors, as follows:

$$
\begin{equation*}
\mathbf{R}_{\mathbf{x x}}=\mathbf{A} \cdot \mathbf{R}_{\mathbf{x x}} \cdot \mathbf{A}^{H}+\mathbf{R}_{\mathbf{n n}}=\mathbf{A} \cdot \mathbf{R}_{\mathbf{x x}} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I}=\mathbf{U} \cdot \boldsymbol{\Lambda} \cdot \mathbf{U}^{H} \tag{1.36}
\end{equation*}
$$

where $\mathbf{U}$ is a unitary matrix which columns are the eigenvectors of $\mathbf{R}_{\mathbf{x x}}$, while $\boldsymbol{\Lambda}$ is a diagonal matrix in which the $M$ eigenvalues are sorted from smallest to largest $\left(\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{M} \geq 0\right)$.
It is important to point out that, every vector orthogonal to $\mathbf{A}$ is a $\mathbf{R}_{\mathrm{xx}}$ eigenvector related to the noise eigenvalue $\sigma_{n}^{2}$. This property comes directly from the uncorrelation among signals and noise.
As the noise eigenvalues are $M-D$, the noise subspace admits $M-D$ linearly independent vectors that form an orthonormal base. At the same time, the signal subspace (admitting $D$ eigenvalues larger than $\sigma_{n}^{2}$ ) is composed of $D$ linearly independent vectors. We can split the two eigenspaces with the relevant orthonormal basis:

$$
\begin{equation*}
\mathbf{R}_{\mathbf{x x}}=\mathbf{U}_{\mathbf{S}} \cdot \boldsymbol{\Lambda}_{\mathbf{S}} \cdot \mathbf{U}_{\mathbf{S}}{ }^{H}+\mathbf{U}_{\mathbf{N}} \cdot \boldsymbol{\Lambda}_{\mathbf{N}} \cdot \mathbf{U}_{\mathbf{N}}{ }^{H} \tag{1.37}
\end{equation*}
$$

where $\mathbf{U}=\left[\mathbf{U}_{\mathbf{S}} \mathbf{U}_{\mathbf{N}}\right]=\mathbf{E}=\left[\mathbf{E}_{\mathbf{S}} \mathbf{E}_{\mathbf{N}}\right] ; \boldsymbol{\Lambda}=\left[\boldsymbol{\Lambda}_{\mathbf{S}} \boldsymbol{\Lambda}_{\mathbf{N}}\right] ; \boldsymbol{\Lambda}_{\mathbf{N}}=\sigma_{n}^{2} \mathbf{I}$.

## Linear Prediction

The goal of the linear prediction method is to minimize the prediction error between the output of the mth sensor and the actual output [30, 31]. The goal is to find the weights that minimize the mean-squared prediction error; the solution for the array weights is given as

$$
\begin{equation*}
\mathbf{w}_{m}=\frac{\mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{u}_{m}}{\mathbf{u}_{m}^{T} \cdot \mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{u}_{m}} \tag{1.38}
\end{equation*}
$$

where $\mathbf{u}_{m}$ is the Cartesian basis vector which is the $m$ th column of the $M \times M$ identity matrix.

Upon substitution of these array weights into the calculation of the pseudospectrum, it can be shown that:

$$
\begin{equation*}
P_{L P_{m}}=\frac{\mathbf{u}_{m}^{T} \cdot \mathbf{R}_{\mathbf{x x}}^{-1} \cdot \mathbf{u}_{m}}{\left|\mathbf{u}_{m}^{T} \cdot \mathbf{R}_{\mathbf{x x}}{ }^{-1} \cdot \mathbf{a}(\theta)\right|^{2}} \tag{1.39}
\end{equation*}
$$

Although the choice made can dramatically affect the final resolution. If the array center element is chosen, the linear combination of the remaining sensor elements might provide a better estimate because the other array elements are spaced about the phase center of the array [30]. This would suggest that odd array lengths might provide better results than even arrays because the center element is precisely at the array phase center.
It is obvious that under these conditions, the linear predictive method provides superior performance over both the Bartlett estimate and the Capon estimate, as shown in Fig. 1.9. The efficacy of the performance is dependent on the array element chosen and the subsequent $\mathbf{u}_{m}$ vector. When one selects the arrival signals to have different amplitudes, the linear predictive spectral peaks reflect the relative strengths of the incoming signals. Thus, the linear predictive method not only provides DoA information but it also provides signal strength information. This linear prediction


Figure 1.9: Linear Prediction pseudospectrum for $\theta_{1}=-5^{\circ}$ e $\theta_{2}=5^{\circ}$.
technique is sometimes referred to as an autoregressive (AR) method [32]. It has been argued that the spectral peaks using linear prediction are proportional to the square of the signal power.

## Maximum Entropy

The Maximum Entropy method is attributed to Burg [33, 34]. A further explanation of the maximum entropy approach is given in [35]. The goal is to find a pseudospectrum that maximizes the entropy function subject to constraints. The details of the Burg derivation can be found in the references discussed previously. Considering Gaussian signals with a limited bandwidth, having a density power spectrum $S(f)$, the entropy function is defined as

$$
\begin{equation*}
H(S)=\int_{-f_{N}}^{f_{N}} \ln [S(f)] d f \tag{1.40}
\end{equation*}
$$

where $f_{N}$ is the Nyquist frequence.
Maximizing the entropy function for the pseudospectrum in the interval $[0,2 \pi]$ we obtain

$$
\begin{equation*}
H\left(P_{M E}\right)=\int_{0}^{2 \pi} \ln \left[P_{M E}(\theta)\right] d \theta \tag{1.41}
\end{equation*}
$$

where the correlation between the $i$ th and the $j$ th element of the array, $r_{i j}$, has to fulfill the following relation:

$$
\begin{equation*}
r_{i j}=\int_{0}^{2 \pi} P_{M E}(\theta) \cos \left(2 \pi \tau_{i j}(\theta)\right) d \theta \tag{1.42}
\end{equation*}
$$

being $\tau_{i j}(\theta)$ the time delay of the received signal coming from the DoA $\theta$ for the sensors $i$ and $j$.
The solution of this optimization problem can be found in the references previously cited [34].
The pseudospectrum $P_{M E}(\theta)$ is given by :

$$
\begin{equation*}
P_{M E_{j}}(\theta)=\frac{1}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{c}_{j} \cdot \mathbf{c}_{j}^{H} \mathbf{a}(\theta)\right|} \tag{1.43}
\end{equation*}
$$

where $\mathbf{c}_{j}$ is the $j$ th column of the inverse array correlation matrix $\mathbf{R}_{\mathbf{x x}}{ }^{-1}$. It should be noted that the Maximum Entropy method, when we select the central column from $\mathbf{R}_{\mathrm{xx}}{ }^{-1}$, gives the same pseudospectra as the linear predictive method. The choice of $\mathbf{c}_{j}$ can dramatically effect the resolution achieved. The center columns of the inverse array correlation matrix tend to give better results under the conditions assumed in this chapter (Fig. 1.10).

## Pisarenko Harmonic Decomposition

The Pisarenko Harmonic Decomposition (PHD) DoA estimate is named after the Russian mathematician who devised this Minimum Mean-Squared Error (MMSE)


Figure 1.10: Maximum Entropy pseudospectrum for $\theta_{1}=-5^{\circ}$ e $\theta_{2}=5^{\circ}$.
approach $[25,36]$. The goal is to minimize the mean-squared error of the array output under the constraint that the norm of the weight vector be equal to unity. The eigenvector that minimizes the mean-squared error corresponds to the smallest eigenvalue. The corresponding PHD pseudospectrum is given by:

$$
\begin{equation*}
P_{P H D_{j}}(\theta)=\frac{1}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{E}_{\mathbf{1}}\right|^{2}} \tag{1.44}
\end{equation*}
$$

where $\mathbf{E}_{\mathbf{1}}$ is the eigenvector associated with the smallest eigenvalue of $\mathbf{R}_{\mathbf{x x}}$. An example of the PHD DoA estimation is sketched in Fig. 1.11.


Figure 1.11: Pisarenko Harmonic Decomposition pseudospectrum for $\theta_{1}=-5^{\circ}$ and $\theta_{2}=5^{\circ}$.

Following this approach, other methods consider not only a single eigenvalue of
the signal subspace, but the entire signal or noise subspace. We obtain a class of methods based on the so called eigenvalues decomposition ([37, 38]).

## Minimum Norm

The Minimum-Norm method was developed by Reddi [39] and by Kumaresan and Tufts [40]. This method is also lucidly explained by Ermolaev and Gershman [41]. The Min-Norm method is only relevant for ULA. The algorithm optimizes the weight vector by solving the optimization problem

$$
\left\{\begin{array}{l}
\min _{\mathbf{w}} \mathbf{w}^{H} \cdot \mathbf{w}  \tag{1.45}\\
\mathbf{E}_{\mathbf{S}}{ }^{H} \cdot \mathbf{w}=0 \\
\mathbf{w}^{H} \cdot \mathbf{u}_{1}=1
\end{array}\right.
$$

where
$\mathbf{E}_{\mathbf{S}}$ is the signal subspace generated by the eigenvectors $\left[\begin{array}{lll}\mathbf{v}_{1} & \cdots & \mathbf{v}_{D}\end{array}\right]$;
$\mathbf{u}_{1}$ Cartesian basis vector (first column of the $M \times M$ identity matrix)
The solution to the optimization yields the Min-Norm pseudospectrum gives

$$
\begin{equation*}
P_{M N}(\theta)=\frac{\left(\mathbf{u}_{1}^{T} \cdot \mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}{ }^{H} \cdot \mathbf{u}_{1}\right)^{2}}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}{ }^{H} \cdot \mathbf{u}_{1}\right|^{2}} \tag{1.46}
\end{equation*}
$$

where $\mathbf{E}_{\mathbf{N}}$ is the noise subspace generated by the eigenvectors $\left[\begin{array}{lll}\mathbf{v}_{D+1} & \cdots & \mathbf{v}_{M}\end{array}\right]^{T}$. Since the numerator term in Eq. (1.40) is a constant, we can normalize the pseudospectrum as follows

$$
\begin{equation*}
P_{M N}(\theta)=\frac{1}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}{ }^{H} \cdot \mathbf{u}_{1}\right|^{2}} \tag{1.47}
\end{equation*}
$$

It should be noted by looking at Fig. 1.12, that the pseudospectrum from the Min-Norm method is almost identical to the PHD pseudospectrum. The Min-Norm method combines all noise eigenvectors whereas the PHD method only uses thefirst noise eigenvector.

## MUSIC

This approach was first posed by Schmidt [27] and is a popular high resolution eigenstructure method. MUSIC promises to provide unbiased estimates of the number of signals, the angles of arrival, and the strengths of the waveforms. MUSIC makes the assumption that the noise in each channel is uncorrelated making the noise correlation matrix diagonal. The incident signals may be somewhat correlated


Figure 1.12: Minimum Norm pseudospectrum $\theta_{1}=-5^{\circ}$ and $\theta_{2}=5^{\circ}$.
creating a non-diagonal signal correlation matrix. However, under high signal correlation the traditional MUSIC algorithm breaks down and other methods must be implemented to correct this weakness. These methods will be discussed later in this chapter.
One must know in advance the number of incoming signals or search the eigenvalues to determine the number of incoming signals. If the number of signals is $D$, the number of signal eigenvalues and eigenvectors is $D$, and the number of noise eigenvalues and eigenvectors is $M-D$ ( $M$ is the number of array elements). Because MUSIC exploits the noise eigenvector subspace, it is sometimes referred to as a subspace method. As before we calculate the array correlation matrix assuming uncorrelated noise with equal variances

$$
\begin{equation*}
\mathbf{R}_{\mathbf{x x}}=\mathbf{U}_{S} \cdot \boldsymbol{\Lambda}_{S} \cdot \mathbf{U}_{S}^{H}+\mathbf{U}_{N} \cdot \boldsymbol{\Lambda}_{N} \cdot \mathbf{U}_{N}^{H} \tag{1.48}
\end{equation*}
$$

where $\boldsymbol{\Lambda}_{N}=\sigma_{n}^{2} \mathbf{I} ; \boldsymbol{\Lambda}_{S}=\left(p_{i}+\sigma_{n}^{2}\right) \mathbf{I} ; \mathbf{U}_{N}=\left[\begin{array}{lll}\mathbf{v}_{D+1} & \cdots & \mathbf{v}_{M}\end{array}\right]^{T} ; \mathbf{U}_{S}=\left[\begin{array}{lll}\mathbf{v}_{1} & \cdots & \mathbf{v}_{D}\end{array}\right]^{T}$, being the eigenvalues $\mathbf{R}_{\mathbf{x x}}$ :

$$
\lambda_{i}= \begin{cases}p_{i}+\sigma_{n}^{2} & i=1, \cdots, D  \tag{1.49}\\ \sigma_{n}^{2} & i=D+1, \cdots, M\end{cases}
$$

For uncorrelated signals, the noise subspace eigenvectors are orthogonal to the array steering vectors at the angles of arrival

$$
\mathbf{v}_{i}^{H} \cdot \mathbf{v}_{j}=\delta_{i j}= \begin{cases}1 & i=j  \tag{1.50}\\ 0 & i \neq j\end{cases}
$$

so that

$$
\mathbf{R}_{\mathrm{xx}} \cdot{ }^{H} \mathbf{v}_{i}=\lambda_{i} \cdot{ }^{H} \mathbf{v}_{i}= \begin{cases}\left(p_{i}+\sigma_{n}^{2}\right) \cdot \mathbf{v}_{i} & i=1, \cdots, D  \tag{1.51}\\ \sigma_{n}^{2} \mathbf{v}_{i} & i=D+1, \cdots, M\end{cases}
$$

and

$$
\begin{equation*}
\left(\mathbf{R}_{\mathrm{xx}}-\sigma_{n}^{2} \mathbf{I}\right) \cdot \mathbf{v}_{i}=0 \tag{1.52}
\end{equation*}
$$

If we consider only the noise subspace for $i=D+1, \cdots, M$, we can decompose $\mathbf{R}_{\mathrm{xx}}$ following the Eq. (1.39) and obtaining

$$
\left(\mathbf{R}_{\mathrm{xx}}-\sigma_{n}^{2} \mathbf{I}\right) \cdot \mathbf{v}_{i}=\mathbf{A} \cdot \mathbf{R}_{\mathrm{ss}} \cdot \mathbf{A}^{H} \cdot \mathbf{v}_{i}
$$

The previous equation has solutions for $\mathbf{A} \cdot \mathbf{v}_{i}=0$. So that, the base generated by the noise eigenvalues $\mathbf{U}_{N}=\left[\begin{array}{lll}\mathbf{v}_{D+1} & \cdots & \mathbf{v}_{M}\end{array}\right]^{T}$ is orthogonal to the array steering vectors at the angles of arrival in $\mathbf{A}$. Because of this orthogonality condition, one can show that the Euclidean distance is

$$
\begin{equation*}
d^{2}=\mathbf{a}^{H}(\theta) \cdot \mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}{ }^{H} \cdot \mathbf{a}(\theta)=0 \tag{1.53}
\end{equation*}
$$

for each and every arrival angle $\theta_{1}, \theta_{2}, \cdots, \theta_{D}$. Placing this distance expression in a denominator creates sharp peaks at the angles of arrival. The MUSIC pseudospectrum is now given as

$$
\begin{equation*}
P_{M U S I C}(\theta)=\frac{1}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}{ }^{H} \cdot \mathbf{a}(\theta)\right|} \tag{1.54}
\end{equation*}
$$

where $\mathbf{E}_{\mathbf{N}}=\mathbf{U}_{\mathbf{N}}=\left[\begin{array}{lll}\mathbf{v}_{D+1} & \cdots & \mathbf{v}_{M}\end{array}\right]^{T}$.
It should be understood that in all examples discussed earlier, it was assumed that the array correlation matrix was of the form given in Eq. (1.35), that the noise variance for all elements was identical, and that the different signals were completely uncorrelated. In the case where the source correlation matrix is not diagonal, or the noise variances vary, the plots can change dramatically and the resolution will be worse. We can repeat Eq. (1.35) without assuming that we know the signal statistics and considering $\hat{\mathbf{R}}_{\mathbf{x x}}, \hat{\mathbf{R}}_{\mathrm{ss}}$ and $\hat{\mathbf{R}}_{\mathrm{nn}}$ instead of $\mathbf{R}_{\mathbf{x x}}, \mathbf{R}_{\mathrm{ss}}$ and $\mathbf{R}_{\mathbf{n n}}$, as discussed previously (Fig. 1.13).

### 1.2.3 Coherent Signals

Many of the direction-finding methods require the number of directional sources, and their performance is dependent on the perfect knowledge of this number [42]. In


Figure 1.13: MUSIC pseudospectrum for $\theta_{1}=-5^{\circ}$ and $\theta_{2}=5^{\circ}: a$ ) considering $\mathbf{R}_{\mathbf{x x}}$ and b) $\hat{\mathbf{R}}_{\mathbf{x x}}$.
particular, the interference canceling capability of the optimal beamformer discussed earlier, assumes that the signal and interference are uncorrelated. In our model, when multiple sources emit, the DoA estimation problem is related to the angle of arrival of multiple plane waves, impinging simultaneously on a ULA. Considering the beamforming techniques, they assume that the design of the optimal weights is based upon the condition that the signal is not correlated with the interference, but when two correlated plane waves occur, one of them is considered as correlated interference. The beamformer, minimizing the mean output power subject to lookdirection constraints, minimizes also the "interference" output power [43].
In a subspace view, the source covariance matrix $\mathbf{R}_{\mathrm{ss}}$ becomes rank deficient, and this results in a divergence of a signal eigenvector into the noise subspace. Therefore, it will be $\mathbf{E}_{N}^{H} \cdot \mathbf{a}(\theta) \neq 0$ for any $\theta$ and the spectrum may fail to produce peaks at the DoA locations. The ability to resolve correlated source, decreases dramatically for highly correlated signals [44].
In the simple case of two coherent sources, there is a fairly straightforward way to de-correlate the signals. The idea is to employ a Forward-Backward Averaging (FBA) [45], by means of a modified $M \times M$ correlation matrix $\mathbf{J}$, whose components are zero except for the ones on the anti-diagonal. Then for ULA case it holds that

$$
\begin{equation*}
\mathbf{J} \cdot \mathbf{a}^{H}(\theta)=e^{-j(M-1) k d \cos \theta} \mathbf{a}(\theta) \tag{1.55}
\end{equation*}
$$

The so-called backward array correlation matrix, takes the form

$$
\begin{equation*}
\mathbf{R}_{B}=\mathbf{J} \cdot \mathbf{R}_{\mathbf{x x}}{ }^{H} \cdot \mathbf{J}=\mathbf{A} \cdot \boldsymbol{\Phi}^{-j(M-1)} \cdot \mathbf{R}_{\mathbf{s s}} \cdot \boldsymbol{\Phi}^{-(M-1)} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I} \tag{1.56}
\end{equation*}
$$

where $\boldsymbol{\Phi}$ is a diagonal matrix with $e^{j k d(\delta-1) \cos \theta}, \delta=1, \ldots, D$ on the diagonal. By averaging the usual array correlation matrix and $\mathbf{R}_{B}$, we obtain the FB array cor-
relation matrix

$$
\begin{equation*}
\mathbf{R}_{F B}=\frac{1}{2}\left(\mathbf{R}_{\mathrm{xx}}+\mathbf{R}_{B}\right)=\mathbf{A} \cdot \tilde{\mathbf{R}}_{\mathrm{ss}} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I} \tag{1.57}
\end{equation*}
$$

where the new source correlation matrix is $\tilde{\mathbf{R}}_{\mathrm{ss}}=\left(\mathbf{R}_{\mathrm{ss}}+\boldsymbol{\Phi}^{-j(M-1)} \cdot \mathbf{R}_{\mathrm{ss}} \cdot \boldsymbol{\Phi}^{-(M-1)}\right) / 2$ and generally has full rank. The FB version of any correlation-based algorithm, simply consists of replacing $\mathbf{R}_{\mathbf{x x}}$ with $\mathbf{R}_{F B}$.
In more general scenarios where more than two coherent sources are present, FBA cannot restore the rank of the source correlation matrix on its own. A heuristic solution of this problem was first proposed in [21] for uniform linear array, and later formalized and extended in the so-called spatial smoothing technique [46, 47]. Anyway, in this thesis, we suppose to face the presence of two signals, postponing the treatment of more signals in further works.

### 1.2.4 Parametric Methods

While the non-parametric methods presented previously are computational attractive, they do not always yield sufficient accuracy. In particular for scenario involving highly correlated signals (that is for example the case of two scatterers very close each other or even overlapping) the performance of spectral-based methods may be insufficient. In these cases, an alternative is represented by the so-called parametric array processing methods, for which coherent signals impose no conceptual difficulties.
The most frequently used model-based approach is the Maximum Likelihood (ML) technique, that requires a statistical framework for the data generation process. Two different assumption about the emitter signals have led to corresponding ML approaches (deterministic and stochastic); furthermore subspace-based approximations are also derived. Parametric DoA estimation methods are generally computationally complex, but for ULA case, a less demanding class of methods are considered.

## Deterministic ML

While the background and receiver noise in the assumed data model can be thought as realizations of independent noise sources, the same is usually not possible for the emitter signals. Therefore, we can model the noise as a stationary Gaussian white random process whereas the signal waveforms are deterministic and unknown. Assuming spatially white and circularly symmetric noise [Appendice A], the second-
order moments take the form

$$
\begin{gather*}
E\left[\mathbf{n}(t) \cdot \mathbf{n}(s)^{H}\right]=\sigma_{n}^{2} \mathbf{I} \delta_{t, s}  \tag{1.58}\\
E\left[\mathbf{n}(t) \cdot \mathbf{n}(s)^{T}\right]=0 \tag{1.59}
\end{gather*}
$$

where $t, s$ are two different time samples and $\delta_{t, s}$ is the Kronecker delta function. As a consequence of these assumptions, the output vector $\mathbf{x}(t)$ is also circularly symmetric and temporally white Gaussian random process, with mean $\mathbf{A}(\theta) \cdot \mathbf{s}(t)$ and covariance matrix $\sigma_{n}^{2} \mathbf{I}$. The likelihood function is the Probability Density Function (PDF) of all observations given the unknown parameters. The PDF of the vector $\mathbf{x}(t)$ is then a complex $M$-variate Gaussian ( $M$ is the number of array sensors) :

$$
\begin{equation*}
\frac{1}{\left(\pi \sigma_{n}^{2}\right)^{M}} e^{\frac{-\|\mathbf{x}-\mathbf{A}(\theta) \cdot \mathbf{s}\|^{2}}{\sigma_{n}^{2}}} \tag{1.60}
\end{equation*}
$$

where $\|\cdot\|$ denotes the Euclidean norm, and the time dependence has been neglected for convenience. Since the realizations are independent, the likelihood function is

$$
\begin{equation*}
L_{D M L}\left(\theta, \mathbf{s}, \sigma_{n}^{2}\right)=\prod_{t=1}^{K}\left(\pi \sigma_{n}^{2}\right)^{-M} e^{\frac{-|\mathbf{x}(t)-\mathbf{A} \cdot \mathbf{s}(t)|^{2}}{\sigma_{n}^{2}}} \tag{1.61}
\end{equation*}
$$

It can be noted by Eq. (1.61) that the unknown parameters in the likelihood function are the $\operatorname{DoAs} \theta$, the signal waveform $\mathbf{s}$ and the noise variance $\sigma_{n}^{2}$. The ML estimates of these unknowns, consists in maximizing $L_{D M L}$ over the arguments $\left(\theta, \mathbf{s}, \sigma_{n}^{2}\right)$, or alternatively, minimizing the negative $\log$-likelihood function $-\log L_{D M L}$. Then, neglecting the constant terms, we get:

$$
\begin{equation*}
-\log L_{D M L}\left(\theta, \mathbf{s}, \sigma_{n}^{2}\right)=M \log \sigma_{n}^{2}+\left.\frac{1}{K \sigma_{n}^{2}} \sum_{t=1}^{K}\|\mathbf{x}(t)-\mathbf{A} \cdot \mathbf{s}(t)\|\right|^{2} \tag{1.62}
\end{equation*}
$$

Therefore, explicit minima with respect to $\sigma_{n}^{2}$ and $\mathbf{s}(t)$ are given by [48, 49]:

$$
\begin{gather*}
\sigma_{n}^{2}=\frac{1}{M} \operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{R}}\right\}  \tag{1.63}\\
\hat{s}(t)=\mathbf{A}^{\dagger} \cdot \mathbf{x}(t) \tag{1.64}
\end{gather*}
$$

where $\hat{\mathbf{R}}$ is the sample covariance matrix of Eq. (1.26), $\mathbf{A}^{\dagger}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$, and $\Pi_{\mathbf{A}}^{\perp}$ is the orthogonal projector onto the null-space of $\mathbf{A}^{H}$ :

$$
\begin{gather*}
\mathbf{A}^{\dagger}=\left(\mathbf{A}^{H} \cdot \mathbf{A}\right)^{-1} \cdot \mathbf{A}^{H}  \tag{1.65}\\
\Pi_{\mathbf{A}}=\mathbf{A} \cdot \mathbf{A}^{\dagger}  \tag{1.66}\\
\Pi_{\mathbf{A}}^{\perp}=\mathbf{I}-\Pi_{\mathbf{A}} \tag{1.67}
\end{gather*}
$$

Substituting Eqs. (1.63)and (1.64) into Eq. (1.61) we obtain the following minimization problem:

$$
\begin{equation*}
\hat{\theta}_{D M L}=\arg \left\{\min _{\theta} \operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{R}}\right\}\right\} \tag{1.68}
\end{equation*}
$$

The samples of $\mathbf{x}(t)$ are projected onto a model subspace orthogonal to all signal components, and the power $\frac{1}{K} \sum_{t=1}^{K}\left\|\Pi_{\mathbf{A}}^{\perp} \cdot \mathbf{x}(t)\right\|^{2}=\operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{R}}\right\}$ is estimated. Since only a finite number of samples is available, the energy is not exactly measured and $\hat{\theta}_{D M L}$ will deviate from the real $\theta$. However, if the scenario is stationary (as it is in our case), the error will converge to zero as the number of samples increase. This remains valid for correlated signals, although the accuracy is dependent upon signal correlation. Notice also that Eq. (1.68) reduces to Bartlett beamformer in the case of a single signal source.

## Stochastic ML

An alternative to the DML method is obtained by modeling the signal waveform as a Gaussian random Process and it is known as Stochastic Maximum Likelihood (SML) method.
A first important prerogative of this method is that it is applicable even if the data are not Gaussian. In fact, the accuracy of the signal parameter estimates can be shown to depend only on the second-order statistics (power and correlation) of the signal waveforms. So that, the Gaussian signal model is used only to the aim of modify the flexibility of the DML algorithm.
Let the signal waveforms be zero-mean and circularly symmetric

$$
\begin{gather*}
E\left[\mathbf{s}(t) \cdot \mathbf{s}(s)^{H}\right]=\mathbf{P} \delta_{t, s}  \tag{1.69}\\
E\left[\mathbf{s}(t) \cdot \mathbf{s}(s)^{T}\right]=0 \tag{1.70}
\end{gather*}
$$

leading the vector of observations $\mathbf{x}(t)$ to be a white, zero-mean and circularly symmetric Gaussian random vector with covariance matrix

$$
\begin{equation*}
\mathbf{R}=\mathbf{A}(\theta) \cdot \mathbf{P} \cdot \mathbf{A}^{H}(\theta)+\sigma_{n}^{2} \mathbf{I} \tag{1.71}
\end{equation*}
$$

The likelihood function in this case, depends on $\theta, \mathbf{P}$ and $\sigma_{n}^{2}$. The negative $\log$ likelihood, ignoring constant terms, is proportional to the estimated power

$$
\begin{equation*}
\frac{1}{K} \sum_{t=1}^{K}\left\|\Pi_{\mathbf{A}}^{\perp} \cdot \mathbf{x}(t)\right\|^{2}=\operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{R}}\right\} \tag{1.72}
\end{equation*}
$$

Even if Eq. (1.60) is a strongly non-linear function, for fixed $\theta$, the minimum with respect to $\sigma_{n}^{2}$ and $\mathbf{P}$ can be shown to be $[50,51]$

$$
\begin{gather*}
\hat{\sigma}_{n, S M L}^{2}(\theta)=\frac{1}{M-D} \operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp}\right\}  \tag{1.73}\\
\hat{P}_{S M L}(\theta)=\mathbf{A}^{\dagger} \cdot\left(\hat{\mathbf{R}}-\hat{\sigma}_{n, S M L}^{2}(\theta) \mathbf{I}\right) \cdot \mathbf{A}^{\dagger H} \tag{1.74}
\end{gather*}
$$

With replacing Eqs. (1.61) and (1.62) into Eq. (1.60), the following compact form is obtained

$$
\begin{equation*}
\hat{\theta}_{D M L}=\arg \left\{\min _{\theta} \log \left|\mathbf{A}(\theta) \cdot \hat{\mathbf{P}}_{\mathbf{S M L}} \cdot \mathbf{A}^{H}(\theta)+\hat{\sigma}_{n, S M L}^{2}(\theta) \mathbf{I}\right|\right\} \tag{1.75}
\end{equation*}
$$

The Eq. (1.62) is also highly non-linear function of its argument $\theta$. A Newton-type technique implementation of the numerical recipe is reported in [52] and an excellent statistical accuracy is reached. In literature, several comparisons between DML and SML are presented [53,54], and the SML method have been shown to have a better accuracy than DML in large sample cases and for a small number of sensor (together with a low SNR and highly correlated signals). In other words, for our scopes, these two methods appear to be equivalent.

## Subspace-Based Approximations

As noted previously, subspace-based methods offer significant performance improvements in comparison to conventional beamformers. The MUSIC method, for instance, shows the same accuracy of the DML method (with a large number of samples) when the signals are uncorrelated [55]. When source correlations occur, these methods suffer a resolution problem cause by a large bias in finite samples. In this paragraph we consider the most performing subspace-based methods for linear array (Root MUSIC and ESPRIT), and also other methods having the same accuracy as the ML methods [56, 57, 58, 59]. The computational cost of these Subspace Fitting methods are however, less than for the ML methods.

## Root MUSIC

The MUSIC algorithm in general can apply to any arbitrary array regardless of the position of the elements. Root-MUSIC implies that the MUSIC algorithm is reduced to finding roots of a polynomial as opposed to merely plotting the pseudospectrum or searching for peaks in the pseudospectrum. Barabell [35] simplified the MUSIC algorithm for the case where the antenna is a ULA. Recalling that the MUSIC pseudospectrum is given by Eq. (1.54), one can simplify the denominator
expression by defining the matrix $\mathbf{C}=\mathbf{E}_{\mathbf{N}} \cdot \mathbf{E}_{\mathbf{N}}^{\mathbf{H}}$ which is Hermitian. This leads to the root-MUSIC expression

$$
\begin{equation*}
P_{R M U}(\theta)=\frac{1}{\left|\mathbf{a}^{H}(\theta) \cdot \mathbf{C} \cdot \mathbf{a}(\theta)\right|} \tag{1.76}
\end{equation*}
$$

If we have an ULA, the $m$-th element of the array steering vector is given by

$$
\begin{equation*}
a_{m}(\theta)=e^{j k d(m-1) \cos \theta} \quad m=1,2, \ldots, M \tag{1.77}
\end{equation*}
$$

The denominator argument in Eq. (1.75) can be written as

$$
\begin{equation*}
\mathbf{a}^{H}(\theta) \cdot \mathbf{C} \cdot \mathbf{a}(\theta)=\sum_{m=1}^{M} \sum_{n=1}^{M} e^{-j k d(m-1) \sin \theta} C_{m n} e^{j k d(n-1) \sin \theta}=\sum_{l=-M+1}^{M-1} c_{l} e^{j k d l \sin \theta} \tag{1.78}
\end{equation*}
$$

where $c_{l}$ is the sum of the diagonal elements of $\mathbf{C}$ along the $l$-th diagonal such that

$$
c_{l}=\sum_{n-m=l} C_{m n}
$$

It should be noted that the matrix $\mathbf{C}$ has off-diagonal sums such that $c_{0}>\left|c_{l}\right|$ for $l \neq 0$. Thus the sum of off-diagonal elements is always less than the sum of the main diagonal elements. In addition, $c_{l}=c_{-l}^{*}$. We can simplify Eq. (1.76) to be in the form of a polynomial whose coefficients are $c_{l}$, thus

$$
\begin{equation*}
D(z)=\sum_{l=-M+1}^{M-1} c_{l} z^{l} \tag{1.79}
\end{equation*}
$$

where $z=e^{-j k d \sin \theta}$.
The roots of $D(z)$ that lie closest to the unit circle correspond to the poles of the MUSIC pseudospectrum (Fig. 1.14). Thus, this technique is called Root-MUSIC. The polynomial of Eq. (1.65) is of order $2(M-1)$ and thus has roots of $z_{1}, z_{2}, \ldots, z_{2(M-1)}$ . Each root can be complex and using polar notation can be written as

$$
\begin{equation*}
z_{i}=\left|z_{i}\right| e^{j \arg \left(z_{i}\right)} \quad 1=1,2, \ldots, 2(M-1) \tag{1.80}
\end{equation*}
$$

where $\arg \left(z_{i}\right)$ is the phase of angle $z_{i}$.
Exact zeros in $D(z)$ exist when the root magnitudes $\left|z_{i}\right|=1$. One can calculate the DoA by comparing $e^{j \arg \left(z_{i}\right)}$ to $e^{j k d \sin \theta_{i}}$ to get

$$
\begin{equation*}
\theta_{i}=\arccos \left(\frac{1}{k d} \arg \left\{\hat{z}_{i}\right\}\right) \tag{1.81}
\end{equation*}
$$

An example of the Root-MUSIC estimation algorithms is represented in Fig. 1.15. The roots found with Root-MUSIC clearly do not reflect exactly the actual location


Figure 1.14: Example of roots in Cartesian system coordinates.


Figure 1.15: MUSIC pseudospectrum and roots found with root-MUSIC for $\theta_{1}=$ $-4^{\circ}$ and $\theta_{2}=8^{\circ}$.
of the angles of arrival, but they indicate two angles of arrival. The roots themselves show the existence of an angle of arrival at near $8^{\circ}$ which is not obvious from the plot of the MUSIC pseudospectrum. The error in locating the correct root locations owes to the fact that the incoming signals are partially correlated, that we approximated the correlation matrix by time averaging, and that the $S N R$ is relatively low.

## ESPRIT

ESPRIT stands for Estimation of Signal Parameters via Rotational Invariance Techniques and was first proposed by Roy and Kailath [60] in 1989. Useful summaries of this technique are given by Liberti and Rappaport [61]. The goal of the

ESPRIT technique is to exploit the rotational invariance in the signal subspace which is created by two arrays with a translational invariance structure. ESPRIT inherently assumes narrowband signals so that one knows the translational phase relationship between the multiple arrays to be used. As MUSIC, ESPRIT assumes that there are $D<M$ narrow-band sources centered at the center frequency $f_{0}$. These signal sources are assumed to be of a sufficient range so that the incident propagating field is approximately planar. The sources can be either random or deterministic and the noise is assumed to be random with zero-mean. ESPRIT assumes multiple identical arrays called doublets. These can be separate arrays or can be composed of sub-arrays of one larger array. It is important that these arrays are displaced translationally but not rotationally. An example is shown in Fig. 1.15 where a four element linear array is decomposed in two identical three-element subarrays or in two doublets. These two sub-arrays are translationally displaced by the distance $d$. Let us label these arrays as $A_{1}$ and $A_{2}$. The signals induced on each


Figure 1.16: Doublet composed of two identical displaced arrays.
array are given by

$$
\left\{\begin{array}{l}
\mathbf{x}_{\mathbf{1}}(k)=\mathbf{A}_{\mathbf{1}} \cdot \mathbf{s}(k)+\mathbf{n}_{\mathbf{1}}(k)  \tag{1.82}\\
\mathbf{x}_{\mathbf{2}}(k)=\mathbf{A}_{\mathbf{2}} \cdot \mathbf{s}(k)+\mathbf{n}_{\mathbf{2}}(k)=\mathbf{A}_{\mathbf{2}} \cdot \boldsymbol{\Phi} \cdot \mathbf{s}(k)+\mathbf{n}_{\mathbf{2}}(k)
\end{array}\right.
$$

where $\boldsymbol{\Phi}=\operatorname{diag}\left\{e^{j k d \cos \theta_{1}}, e^{j k d \cos \theta_{2}}, \cdots, e^{j k d \cos \theta_{D}}\right\}$ is a $D \times D$ diagonal unitary matrix with phase shifts between the doublets for each DoA, and $\mathbf{A}_{i}$ is a Vandermonde matrix of steering vectors for sub-arrays $i=1,2$.
The complete received signal considering the contributions of both subarrays is given as

$$
\mathbf{x}(k)=\left[\begin{array}{l}
\mathbf{x}_{\mathbf{1}}(k)  \tag{1.83}\\
\mathbf{x}_{\mathbf{2}}(k)
\end{array}\right]=\left[\begin{array}{c}
\mathbf{A}_{\mathbf{1}} \\
\mathbf{A}_{\mathbf{1}} \cdot \boldsymbol{\Phi}
\end{array}\right] \cdot \mathbf{s}(k)+\left[\begin{array}{c}
\mathbf{n}_{\mathbf{1}}(k) \\
\mathbf{n}_{\mathbf{2}}(k)
\end{array}\right]
$$

It is important to point out that $\boldsymbol{\Phi}$ is isomorphic for a rotation in the real plane, therefore, after the transformation, it is still unitary and diagonal, and so it is rotationally invariant. For this reason $\boldsymbol{\Phi}$ is indicated as the rotational operator that
characterizes the method.
Consider what previously noticed about the eigenvalues decomposition, and since the signal subspace $\mathbf{E}_{\mathbf{S}}$ is the same of the subspace generated by $\mathbf{A}$, there must also exist a unique non-singular transformation matrix $\mathbf{T}$ such that such

$$
\begin{equation*}
\mathbf{E}_{\mathbf{S}}=\mathbf{A} \cdot \mathbf{T} \tag{1.84}
\end{equation*}
$$

so that,

$$
\mathbf{E}_{\mathbf{S}}=\left[\begin{array}{l}
\mathbf{E}_{1}  \tag{1.85}\\
\mathbf{E}_{2}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{A} \cdot \mathbf{T} \\
\mathbf{A} \cdot \Phi \cdot \mathbf{T}
\end{array}\right]
$$

where both $\mathbf{E}_{\mathbf{1}}$ and $\mathbf{E}_{\mathbf{2}}$ are $M \times D$ matrices whose columns are composed of the $D$ eigenvectors corresponding to the largest eigenvalues of $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$, respectively

$$
\left\{\begin{array}{l}
\mathbf{R}_{\mathbf{1 1}}=E\left[\mathbf{x}_{\mathbf{1}} \cdot \mathbf{x}_{\mathbf{1}}{ }^{H}\right]=\mathbf{A} \cdot \mathbf{R}_{\mathrm{ss}} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I} \\
\mathbf{R}_{\mathbf{2 2}}=E\left[\mathbf{x}_{\mathbf{2}} \cdot \mathbf{x}_{\mathbf{2}}{ }^{H}\right]=\mathbf{A} \cdot \boldsymbol{\Psi} \cdot \mathbf{R}_{\mathrm{ss}} \cdot \boldsymbol{\Psi}^{H} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I}
\end{array}\right.
$$

Since the arrays $A_{1}$ and $A_{2}$ are translationally related, the subspaces $\mathbf{E}_{\mathbf{1}}$ and $\mathbf{E}_{\mathbf{2}}$ are related by a unique non-singular transformation matrix $\boldsymbol{\Phi}$ such that

$$
\begin{equation*}
\mathbf{E}_{1}=\Psi \cdot \mathbf{E}_{2} \tag{1.86}
\end{equation*}
$$

Combining the previous Eqs. (1.85) and (1.86), and assuming that $\mathbf{A}$ is full-rank, we can derive the relationship

$$
\begin{equation*}
\mathbf{T} \cdot \Psi \cdot \mathbf{T}^{-1}=\Phi \tag{1.87}
\end{equation*}
$$

Thus, the eigenvalues of $\boldsymbol{\Phi}$ must be equal to the diagonal elements of $\boldsymbol{\Psi}$ such that $\lambda_{1}=e^{j k d \cos \theta_{1}}, \lambda_{2}=e^{j k d \cos \theta_{2}}, \cdots, \lambda_{D}=e^{j k d \cos \theta_{D}}$ and the columns of $\mathbf{T}$ must be the eigenvectors of $\boldsymbol{\Psi}$.
Now estimate the angles of arrival, given that $\lambda_{i}=\left|\lambda_{i}\right| e^{j \arg \left(\lambda_{i}\right)}$

$$
\begin{equation*}
\theta_{i}=\cos ^{-1}\left(\frac{\arg \left(\lambda_{i}\right)}{k d}\right) \quad i=1 \cdots, D \tag{1.88}
\end{equation*}
$$

If desired, one can estimate the matrix of steering vectors as

$$
\begin{equation*}
\mathbf{A}=\mathbf{E}_{\mathbf{S}} \cdot \mathbf{E}_{\Psi} \tag{1.89}
\end{equation*}
$$

where $\mathbf{E}_{\boldsymbol{\Psi}}=\mathbf{T}^{-1}$.

## Subspace Fitting Methods

Recall the structure of the eigen-decomposition of the array correlation matrix as in Eq. (1.36)

$$
\begin{equation*}
\mathbf{R}_{\mathbf{x x}}=\mathbf{A} \cdot \mathbf{R}_{\mathbf{s s}} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{I}=\mathbf{U}_{\mathbf{S}} \cdot \boldsymbol{\Lambda}_{\mathbf{S}} \cdot \mathbf{U}_{\mathbf{S}}{ }^{H}+\mathbf{U}_{\mathbf{N}} \cdot \boldsymbol{\Lambda}_{\mathbf{N}} \cdot \mathbf{U}_{\mathbf{N}}{ }^{H} \tag{1.90}
\end{equation*}
$$

As previously noted, the matrices $\mathbf{A}$ and $\mathbf{U}_{S}$ span the same range space whenever $\mathbf{R}_{\mathrm{ss}}$ has full rank. Generally, the number of signal eigenvectors in $\mathbf{U}_{S}$ equals the rank of $\mathbf{R}_{\text {ss }}$. This can be easily be seen by first expressing the identity up here as $\mathbf{I}=\mathbf{U}_{\mathbf{S}} \cdot \mathbf{U}_{\mathbf{S}}{ }^{H}+\mathbf{U}_{\mathbf{N}} \cdot \mathbf{U}_{\mathbf{N}}{ }^{H}$. Simplifying the $\sigma_{n}^{2} \mathbf{U}_{\mathbf{N}} \cdot \mathbf{U}_{\mathbf{N}}{ }^{H}$-term, we obtain

$$
\begin{equation*}
\mathbf{R}_{\mathbf{x x}}=\mathbf{A} \cdot \mathbf{R}_{\mathbf{s s}} \cdot \mathbf{A}^{H}+\sigma_{n}^{2} \mathbf{U}_{\mathbf{S}} \cdot \mathbf{U}_{\mathbf{S}}{ }^{H}=\mathbf{U}_{\mathbf{S}} \cdot \boldsymbol{\Lambda}_{\mathbf{S}} \cdot \mathbf{U}_{\mathbf{S}}{ }^{H} \tag{1.91}
\end{equation*}
$$

Post-multiplying on the right by $\mathbf{U}_{\mathbf{S}}$ (remember that $\mathbf{U}_{\mathbf{S}}{ }^{H} \cdot \mathbf{U}_{\mathbf{S}}=\mathbf{I}$ ) and re-arranging gives the relation

$$
\begin{equation*}
\mathbf{U}_{\mathbf{S}}=\mathbf{A} \cdot \mathbf{T} \tag{1.92}
\end{equation*}
$$

where $\mathbf{T}$ is the full-rank matrix

$$
\begin{equation*}
\mathbf{T}=\mathbf{R}_{\mathbf{s s}} \cdot \mathbf{A}^{H} \cdot \mathbf{U}_{\mathbf{S}}\left(\boldsymbol{\Lambda}_{\mathbf{S}}-\sigma_{n}^{2} \mathbf{I}\right)^{-1} \tag{1.93}
\end{equation*}
$$

The relation in Eq. (1.73) forms the basis for the Signal Subspace Fitting (SSF) approach. Since $\theta$ and $\mathbf{T}$ are unknown, it is possible to find them by estimating $\mathbf{U}_{S}$ and minimize a suitable distance measure between $\hat{\mathbf{U}}_{\mathbf{S}}$ and $\mathbf{A} \cdot \mathbf{T}$. For this purpose, the Frobenius norm is used, and the SSF estimate is obtained by solving the following non-linear optimization problem:

$$
\begin{equation*}
\{\hat{\theta}, \hat{\mathbf{T}}\}=\arg \min _{\theta, \mathbf{T}}\left\|\hat{\mathbf{U}}_{\mathbf{S}}-\mathbf{A} \cdot \mathbf{T}\right\|_{F}^{2} \tag{1.94}
\end{equation*}
$$

Similar to DML criterion, this is a separable non-linear least square problem [62]. The solution of the linear parameter $\mathbf{T}$ is

$$
\begin{equation*}
\hat{\mathbf{T}}=\mathbf{A}^{\dagger} \cdot \mathbf{U}_{\mathbf{S}} \tag{1.95}
\end{equation*}
$$

which, when substituting in Eq. (1.74), leads to the function

$$
\begin{equation*}
\hat{\theta}_{S S F}=\arg \left\{\min _{\theta} \operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{U}}_{\mathbf{S}} \cdot \hat{\boldsymbol{\Lambda}}_{\mathbf{S}} \cdot \hat{\mathbf{U}}_{\mathbf{S}}^{H}\right\}\right\} \tag{1.96}
\end{equation*}
$$

A variation in the SSF algorithm, can be performed since we consider the theory of the weighted least square [63], by introducing a weighting vector $\mathbf{W}$. In fact, the matrix $\boldsymbol{\Lambda}_{\mathbf{S}}$ contains the eigenvalues of the signal subspace that include also the noise
variance. So that, is intuitive to normalize the eigenvalue matrix in such a way to consider the actual signal eigenvalues:

$$
\begin{equation*}
\mathbf{W}_{\mathrm{opt}}=\left(\boldsymbol{\Lambda}_{\mathbf{S}}-\sigma_{n}^{2} \mathbf{I}\right)^{2} \boldsymbol{\Lambda}_{\mathbf{S}}{ }^{-1} \tag{1.97}
\end{equation*}
$$

Since $\mathbf{W}_{\text {opt }}$ depends on unknowns, we use instead estimated values

$$
\begin{equation*}
\hat{\mathbf{W}}_{\mathrm{opt}}=\left(\hat{\Lambda}_{\mathbf{S}}-\hat{\sigma}_{n}^{2} \mathbf{I}\right)^{2} \hat{\boldsymbol{\Lambda}}_{\mathbf{S}}^{-1} \tag{1.98}
\end{equation*}
$$

The estimator defined by the weights given into Eq. (1.78), is termed the Weighted Subspace Fitting (WSF) method, in the following equation

$$
\begin{equation*}
\hat{\theta}_{W S F}=\arg \left\{\min _{\theta} \operatorname{Tr}\left\{\Pi_{\mathbf{A}}^{\perp} \cdot \hat{\mathbf{U}}_{\mathbf{S}} \cdot \hat{\mathbf{W}}_{\mathbf{o p t}} \cdot \hat{\mathbf{U}}_{\mathbf{S}}^{H}\right\}\right\} \tag{1.99}
\end{equation*}
$$

It has been shown that WSF theoretically yields the same accuracy as the SML method, and a lower computational cost provided a fast method for computing the eigen-decomposition [52].

### 1.3 Statistical Sub-Array Processing

We consider a plane wave illuminating the region of interest assumed to be homogeneous, lossless and containing one or more targets located in the near field of the array of sensors. The ULA configuration suggests that high-resolution array processing methods could be used [64] for a near field detection. Such methods typically assume that the sources are infinitely far away so that the wavefront received on the array is supposed to be planar. For our problem, since the objects are located quite close to the receivers, this key assumption appears not valid. Moreover, a problem common to both the near field and far field array processing algorithms is that the number of incident wavefront and targets is not known a priori.
To deal with the non-planar nature of the wavefronts over the array, we partition the sensor array into subarrays, such that the scattered field can be considered locally planar at each subarray being the target in the far field of the subarray. Then, using these high resolution subarray processing (SAP) techniques, each subarray identifies a single direction of arrival (DoA) corresponding to the most dominant scatterer in the vicinity of that subarray. The localization of the objects in terms of their distances and bearings is achieved by triangulating the directions of arrival from all subarrays which turn into a crossing pattern of DoA intersections as sketched in Fig. 1.16 .

By examining a typical crossing structure, it is possible to notice that there are two distinct patterns where the crossings are either dense or sparse. In particular, a


Figure 1.17: DoAs $a$ ); b) Crossing Pattern.
dense crossing region indicates the object localization and is distinguished from a "background" region where the crossings are sparse. The problem of object detection is so reduced to the processing of the crossings obtained from our triangulation procedure.
For this purpose we introduce a simple and at the same time accurate stochastic model, developed by Hoaglin [65], describing the spatial distribution of DoA crossings. In fact, the crossings are inherently randomly distributed in the plane because of the presence of Gaussian noise, moreover this method is pretty consistent in terms of detection and false alarm rate (FAR). We consider two classes of crossings (dense and sparse) using two different spatial Poisson distributions [66]. The Poisson model in the target region has a large rate parameter while that one of the background region is considerably smaller. Based on these models, Şahin and Miller [67] developed a hypothesis test for the estimation of the rate parameters and the localization of dense crossing regions which indicate the presence of the target. We verify that the Poisson model is in fact rather accurate and reliable for our purpose.
The whole algorithm proceeds as follow:

1. Subarray processing: we partition the array of receivers so that the observed backscattered field is locally planar at each subarray. The DoAs are found using the DoA estimation algorithms described in the previous paragraph. The collected DoAs are then triangulated to obtain the crossing pattern.
2. Crossing analysis: the crossing pattern is modeled with two Poisson counting process, corresponding to target and background regions. After estimating the
required rate parameters, a hypothesis test is performed to determinate a set of ""window" regions corresponding to areas containing target.
3. Target extraction: the collected windows are aggregated in a number of disjoint groups. The total number of groups indicates the number of targets and by averaging the coordinates of the crossings of all the windows in each group, we can estimate the center of the corresponding target and thus the depth measurement.

### 1.3.1 SAP Technique and Statistical Analysis

The direction finding algorithms previously analyzed, assume plane wave incidences and determine the DoAs associated with each plane wave. For near field problems, both DoA and the distance of the source (in our case the scatterer) should be acquired. Here, a subarray processing technique requiring a one-dimensional DoA search for each subarray, is described. The idea behind this technique, is that one to reduce the aperture of the whole array partitioning it in a set of subarray that should be small enough to approximate the scattered field impinging upon the array to a locally planar one. Planar wavefronts are a consequence of the far field condition fulfillment, therefore, given the whole array span $\mathcal{D}$, we determine the distance that fulfill this condition that is $\frac{2 \mathcal{D}^{2}}{\lambda}$, where $\lambda$ is the wavelength in the medium of propagation. Any source located farther than this limit is in the far field of the array.
Now, consider that the subarray span is more smaller than $\mathcal{D}$ depending to the partition adopted. Choosing properly the number $S$ of subarrays, it is possible to locate objects in the near field region of the array of receivers.
At this stage, each subarray finds one DoA for the dominant planar wavefront in the total backscattered field. Once one DoA at each subarray is determined, all DoAs are triangulated to estimate the target locations.
In Fig. 1.16 the triangulation of the DoA is depicted together with the subarray partition. Inspecting Fig. 1.16b, we see two distinct regions where the density of the crossings are quite different: in the first region (background) the crossings are sparse, and in the second region target) the crossings are dense. By exploiting this difference, it is possible to isolate the target region. To this aim, we introduce a Poisson model for DoA crossings which has a large rate parameter in the target region and a small one in the background region.
Formally, for a given crossing pattern, we count the number of crossing $Y_{i}, i=$ $1,2, \ldots, N_{y}$ in a window of size $w_{x} \times w_{y}$, where $N_{y}$ is the total number of non-
overlapping windows ${ }^{1}$, while $w_{x}$ and $w_{y}$ are the width of the windows in $x$ and $y$ directions, respectively.
In order to ensure that $Y_{i}$ is Poisson distributed, we adopt the graphical technique described in [65]. In particular, for each count $k$ observed in $Y_{i}$ we plot $k$ versus $\left(\ln k!+\ln F_{k}\right)$ where $F_{k}=\sum_{j=1}^{N_{y}}\left[Y_{i}=k\right]$ is the number of data values $Y_{i}$ equal to $k$. If the fit to the Poisson model is verified, then the plot shoul form a straight line with slop approximately $\ln \lambda$ where $\lambda$ is the rate parameter of the distribution (Fig. 1.17). Examining the plot, it is possible to decompose the fitting curve in two


Figure 1.18: Example of fitness to Poisson model.
straight lines with negative and positive slope. The first line matches the smaller values of $k$ (such as $k \leq 2$ in Fig. 1.17), and the second line approximates the larger values of $k$ ( $k \geq 3$ in Fig. 1.17). It can be easily argued that these two regions correspond to background and target, which is expected to have a grater number of crossings compare with the background crossings number.
So that, we can identify $k_{b}$ and $k_{t}$ as the crossings count for background and target, with $k_{b}=0, \cdots, 2$ and $\left.k_{t}=3, \cdots, \infty\right)$. Moreover, $F_{k_{b}}=\sum_{j=1}^{N_{y}}\left[Y_{i}=k_{b}\right]$ and $F_{k_{b}}=\sum_{j=1}^{N_{y}}\left[Y_{i}=k_{t}\right]$ are the numbers of the windows that fulfill the background and target condition, respectively. Then, the rate parameter for the background and

[^0]target regions are given by their maximum likelihood estimates:
\[

$$
\begin{align*}
& \hat{\lambda}_{b}=\frac{1}{N_{b}} \sum_{k_{b}=0}^{2} k_{b} F_{k_{b}}  \tag{1.100}\\
& \hat{\lambda}_{t}=\frac{1}{N_{t}} \sum_{k_{t}=3}^{\infty} k_{t} F_{k_{t}} \tag{1.101}
\end{align*}
$$
\]

where $N_{b}=\sum_{k_{b}} F_{k_{b}}$ and $N_{t}=\sum_{k_{t}} F_{k_{t}}$.
The probability mass function in the background and target regions can be expressed as:

$$
\begin{align*}
& f_{X}\left(k \mid \text { Background, } \hat{\lambda}_{b}\right)=P\left(X=k \mid \text { Background }, \hat{\lambda}_{b}\right)=  \tag{1.102}\\
&=\frac{1}{k!} e^{-\hat{\lambda}_{b}}-\hat{\lambda}_{b}^{k} \\
& f_{X}\left(k \mid \text { Target }, \hat{\lambda}_{t}\right)=P\left(X=k \mid \text { Target }, \hat{\lambda}_{t}\right)=\frac{1}{k!} e^{-\hat{\lambda}_{t}}-\hat{\lambda}_{t}^{k} \tag{1.103}
\end{align*}
$$

To extract crossing clusters, we swept the region of interest with a test window of size $w_{x} \times w_{y}$. It is important that the area of the test is equal to the area of the non-overlapping windows used before to estimate the rate parameters. At each location of the test window (allowed at this stage to overlap), we count the number of crossings $T_{j}, j=1, \cdots, N_{\text {test }}$, where $N_{\text {test }}$ is the total number of the sweep windows in the region of interest. The number of overlapping windows $N_{\text {test }}$ defines the resolution of detection, and of course it has to be greater than $N_{y}$.
The hypothesis test allow us to determine whether the test window is over a background region or over a target region. The hypothesis test is formally defined as:

- $H_{0}$ : is Poisson distributed with a small rate parameter $\hat{\lambda}_{b}$,
- $H_{1}$ : is Poisson distributed with a small rate parameter $\hat{\lambda}_{t}$.

Based on this binary test, if $H_{0}$ is true, we decide that the window belongs to a background region, otherwise, if $H_{1}$ is true, we opt for a target window belonging. The generalized likelihood ratio for the hypothesis test is derived to the Eqs. (1.102) and (1.103) to be:

$$
\begin{equation*}
\Lambda\left(T_{j}\right)=\frac{f_{X}\left(T_{j} \mid H_{1}, \hat{\lambda}_{t}\right)}{f_{X}\left(T_{j} \mid H_{0}, \hat{\lambda}_{b}\right)} \tag{1.104}
\end{equation*}
$$

and the decision is made based on the following generalize likelihood ratio test:

$$
\begin{equation*}
\ln \left\{\Lambda\left(T_{j}\right)\right\}=T_{j} \lessgtr_{H_{0}}^{H_{1}} \mathcal{K} \tag{1.105}
\end{equation*}
$$

where the decision threshold $\mathcal{K}$ is found from a specific false alarm rate $P_{f a}$ using the following equation:

$$
\begin{equation*}
P_{f a}=\sum_{k=\mathcal{K}}^{\infty} f_{X}\left(k \mid H_{0}, \hat{\lambda}_{b}\right) \tag{1.106}
\end{equation*}
$$

that is the conditional probability occurring when a background window counts a number of crossings grater than $\mathcal{K}$. Following this formula, all windows which have $\mathcal{K}$ or more crossings in them will be declared as target locations. So that, the falsealarm probability $P_{f a}$ provides a degree of freedom, represented by the parameter $\mathcal{K}$, that is the number of crossings representative of a target window. Since we extract the cylinder center by averaging the coordinates of the crossings belonging to the target region, the choice of $\mathcal{K}$ affects the positional estimation.
Similarly the detection probability $P_{d}$ is as follow:

$$
\begin{equation*}
P_{d}=\sum_{k=\mathcal{K}}^{\infty} f_{X}\left(k \mid H_{1}, \hat{\lambda}_{t}\right) \tag{1.107}
\end{equation*}
$$

In Fig. 1.18 an example of the application of the stochastic Poisson model to a crossing pattern is presented. It is definitely clear to understand as the Poisson model provides a solid and reliable detection model. As we are going to explain in


Figure 1.19: Crossings pattern: $a)$ before and $b$ ) after the statistic processing.
the next chapters, the statistical filtering of the crossing pattern, allows to estimate the cylinder position by means of averaging the co-ordinates, reducing the estimation error due to background crossings.

## Chapter 2

## Detection of a Single Cylinder

In this chapter many simulation results relevant to a single cylinder detection and localization are presented and commented. On top on this, a flow chart of the adopted procedure is reported in Fig. 1.20, where the logic scheduling includes four main steps. The first one concerns the geometrical and physical concept of the simulations and in particular, the cylinder dimension, the array configuration, and the half-space electric properties. The second step consists in the forward scattering solution by means of the CWA solver, in order to obtain the electric field impinging to the array and coming from the scatterer in ground. Successively, the whole array is partitioned in a number of sub-arrays in order to fulfill the far-field condition at each sub-array for a shorter distance compared with the entire array. By using this sub-array processing technique it is possible to apply all the DoA algorithms described in the previous chapter at each sub-array and obtain a set of DoAs that can be triangulated to form the crossing pattern. The last step of the procedure is the statistical processing that will allow us to estimate the object position. In fact, depending on the configuration of both the array of antennas and the half-space configuration, the geometrical parameters of the problem change, as the far-field condition fulfillment, for instance.
Recalling the expression

$$
F F=2 \frac{\mathcal{D}^{2}}{\lambda}
$$

and assuming the array sensors perfectly adhering the ground (in such a way that we can neglect the refractive effects at the interface), we can obtain the following simple relation for our uniform linear sub-array geometry

$$
\begin{equation*}
F F=2 \frac{\mathcal{D}^{2}}{\lambda_{r}}=2 \frac{[(m-1) d]^{2}}{\lambda_{r}}=n \frac{\lambda_{0}}{2} \tag{2.1}
\end{equation*}
$$



Figure 2.1: Localization procedure flow-chart.
where $m$ is the number of elements for each sub-array, $d$ is the sub-array spacing, $\lambda_{r}$ is the wavelength in the lower half-space, and $n$ is the refractive index of the homogeneous medium. It is important to point out that Eq. (2.1) expresses the far-field distance associated to the sub-array but, when the linear dimension of the object is greater than $(m-1) d$, it lies in the near-field of the sub-array. Moreover, it has to be taken into account that the array spacing must be $d \leq \lambda / 2$, in order to avoid ambiguities in the direction finding process.

In the following paragraphs of this chapter, it will be clear that the accuracy of the proposed procedure is still effective even though the object cross-section exceed the $\lambda / 2$ limit, and thus we can certify an operative near-field detection capability of such a method. In a single cylinder configuration, we considered both a conductive and a dielectric cylinder, assuming ideal properties for both of them (infinite conductivity for the metallic cylinder and absence of losses for the dielectric one). At first, we performed the localization considering the cylinder in a vacuum, and subsequently, we analyzed the scenario with the cylinder in a dielectric half-space different from the one hosting the array. Our scope, consists in verifying the effectiveness of the DoA estimation methods (and consequently the precision of the statistical localization), in the simplest physical environment in which, a potential failure in the positional estimation, has to be ascribed to the bearing algorithm functioning.


Figure 2.2: DoA estimation by using an uniform linear array (ULA).

### 2.1 Simulation Settings

The localization procedure describe in Chapter 1, allow us to estimate the position of the center of the cylinder by averaging the co-ordinates, once fixed several simulation settings. The estimation error is defined as

$$
\begin{equation*}
e r r=\frac{\sqrt{(\Delta x)^{2}+(\Delta y)^{2}}}{a} \tag{2.2}
\end{equation*}
$$

where $\Delta x=\left|x_{c}-\hat{x}_{s}\right|$ and $\Delta y=\left|y_{c}-\hat{y}_{s}\right|$ are the difference between true center coordinates $\left(x_{0}, y_{0}\right)$ and estimated coordinates $\left(\hat{x}_{s}, \hat{y}_{s}\right)$, and $a$ is the radius of the cylinder. When err $=1$, the object estimated position is located in a point on the surface of the actual cylinder, so that we consider successfully detected an object whose estimated error is less than one.

The parameters that need to be fixed belong to both the physical-geometrical configuration of the presented scenario (the number and spacing of antennas, the number of sub-arrays, the distance between array and cylinder, the cross-section of the cylinder, the refractive index of both the dielectric cylinder and the ground, the position of the cylinder) and the computational settings of the localization procedure (the noise variance, the dimension of the filtering windows, and the false-alarm probability).
In particular, the dimension of the windows used for the statistical filtering of crossings, determines the resolution of the localization. We suppose to have square window with dimension $w_{x}=w_{y}=w$, and a total area of investigation to be a square of side $25 \lambda_{0}$, covering the whole length of the array of antennas. Considering
the window dimensions $w=\lambda_{0} / 8, \lambda_{0}, 2.5 \lambda_{0}$, and $5 \lambda_{0}$, the total number of windows is $40000,625,100$ and 25 , respectively.
We here consider the case of a conductive cylinder in a vacuum, positioned at a fixed distance of $3 \lambda_{0}$ from the array. Moreover, the cylinder is varying its radius in $\left[0.05 \lambda_{0}, 1.05 \lambda_{0}\right]$, with step size of $0.02 \lambda_{0}$, for a overall number of 50 different values. The array configuration consists in a 51 -sensors array partitioned into 17 sub-arrays, each one made of 3 -elements, equally spaced of $d=\lambda_{0} / 4$. As we can see by looking Fig. 2.3, when the window-size increases, also the smaller well-detected cylinder cross-section increases. On the other hand, the computational time clearly increases as the number of windows increases, that is when the window side is shorter. Since


Figure 2.3: Estimation error vs. cylinder normalized radius, for different values of the window side: $\left.\left.\left.a) w=\lambda_{0} / 8, b\right) w=\lambda_{0}, c\right) w=2.5 \lambda_{0}, d\right) w=5 \lambda_{0}$
the resolution acquired with $w=\lambda_{0} / 8$ and $w=\lambda_{0}$ (Fig. 2.3 $a$ and $b$ ) is quite similar around $\lambda_{0} / 4$, a reasonable choice is that to consider $w=\lambda$ providing a less computational complexity, in fact, with this choice we can reduce drastically the number of windows without paying much of resolution. Probably, an even better choice might be $w=\lambda / 2$ for a total number of 2500 windows.

It is also interesting to point out, that, as the resolution decreases, the error trend becomes smoother and almost linear. This kind of trend seems to depend on the cylinder dimension, in fact, when the object is too big compared with the filtering window, the statistical procedure erases the peripheral crossings corresponding to the edges of the cylinder. In Fig. 2.3a) this is particularly evident. When $a$ becomes greater then $0.6 \lambda_{0}$, the estimation error starts to increase. In the following Table 2.1, the setting of our simulations is resumed.

The overall number of the sensors is different only for ESPRIT algorithm because

| Number of sensors | 51 |
| :--- | :---: |
| Number of sub-arrays | 17 |
| Spacing $d$ | $\lambda_{0} / 4$ |
| Window size $w$ | $\lambda_{0}$ |
| Distance cylinder-array $h$ | $F F+a=n \frac{\lambda_{0}}{2}+a$ |
| Cylinder radius $a$ | $0.05 \lambda_{0} \leq a \leq 1.15 \lambda_{0}$ |
| Step size | $\frac{\lambda}{56}$ |
| Noise variance $\sigma_{n}^{2}$ | 0.25 |
| False-alarm probability | $10^{-6}$ |
| Estimation error | $\frac{\sqrt{\left(x_{c}-\hat{x}_{c}\right)^{2}+\left(y_{c}-\hat{y}_{c}\right)^{2}}}{a}$ |

Table 2.1: Simulations settings.
of geometrical considerations based on the implementation of this method. In fact, since the sub-array processing in this case, needs overlapping doublet-array, the number of sub-array (that is equal to the number of estimated DoAs), is higher for a 51 sensors array, introducing an additive computational cost. We take into account two implementations of this algorithm (indicated as ESPRIT-48 and ESPRIT-17); in the first one we assume a number of antennas equal to 51 with 48 sub-arrays estimating 48 DoAs, where the doublets are formed by 4 elements that sweep the array along its width. In the second case (ESPRIT-17) we reduce the number of sensors to the only 20 central one, partitioned into 17 sub-arrays estimating 17 DoAs. This last version often gives better results than the original one, especially when the object is placed centrally compared with the array axis. Other considerations about it, are going to be developed further in this chapter.

### 2.2 PEC Cylinder Localization

In this section we propose and analyze the localization problem connected with a perfectly conducting cylinder, with increasing radius, at first in a vacuum and then buried into a medium having refractive index $n=2$ or 3 . The simulation setting is defined in Table 2.1 and here briefly resumed. The radius $a$ of the cylinder is varying between $0.05 \lambda_{0}$ and $1.15 \lambda_{0}$ with a step-resolution of $0.02 \lambda_{0}$, at the same time, also the depth of the cylinder changes, following the rule $h=F F+a$. In other words, the cylinder is always placed at a range equal to the far-field distance calculated considering the only sub-array width, so that, when the cross-section of the cylinder becomes greater than $2 a=\lambda_{0} / 2$, the localization problem is to all intents and purposes, a near-field problem. As we can appreciate in this chapter, the proposed procedure seems to be effective also in these cases, providing a quite enough accurate positional estimation of the cylinder.
In the next Figs. 2.4-2.6 the estimation error vs. the normalized radius of the cylinder (with respect to the wavelength), is reported. We collect together nonparametric methods (Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, Pisarenko Harmonic Decomposition), MUSIC-root, MUSIC-ESPRIT, and parametric ML-based methods. The cylinder is initially placed in a vacuum and at the center of the array width, configuring a geometry with no half-spaces just to test the effectiveness of the DoA estimation methods in such a localization procedure.
In Fig. 2.4b)-2.6b), the region in which the estimation error is less than one is enlarged, and the smaller object size that can be precisely localized is plain to be $0.15 \lambda_{0}$.

It is clear that all the implemented algorithms work quite good, and with very similar results. In particular, the ESPRIT-17 seems to perform better than the others, showing the lowest estimation error. Moreover, as we expect from the theory of ML methods, in a single DoA case, the performances are equal among them and to non-parametric spectral-based methods (i.e. Bartlett). In all the algorithms is also possible to point out a dumped oscillating trend that is more mitigated in the ESPRIT-17 variation in Fig. 2.5b). This suggests that the peripheral sub-arrays influence the estimation for smaller radii, and gradually approach a linear trend as the radius increase. A possible explanation of such a behavior consists in the greater distance among the cylinder and the peripheral sub-arrays rather than the central ones, that is relevant to a lower field magnitude at the edges of the array, making the DoA estimation more doubtful. In fact, as the cross-section increases, the DoA intersections are more thickened around the actual center of the cylinder, mitigating


Figure 2.4: PEC cylinder localization in a vacuum, error vs. normalized radius $a / \lambda_{0}$ (Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD).


Figure 2.5: PEC cylinder localization in a vacuum, error vs. normalized radius $a / \lambda_{0}$ (MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48).


Figure 2.6: PEC cylinder localization in a vacuum, error vs. normalized radius $a / \lambda_{0}$ (WSF, DML, SML).
the effect of the peripheral sub-arrays estimation.
In the next Fig. 2.7, we fixed the cylinder cross-section to be a value that allows a good detection ( $a=0.5 \lambda_{0}$ ), varying the distance between object and array $h \in\left[0,5 \lambda_{0} ; 1,5 \lambda_{0}\right]$, normalized to the wavelength and with a resolution of $0.05 \lambda_{0}$. As we can see, all the methods improve the precision in estimating the DoA, moreover the non-parametric methods (both spectral-based and subspace-based) seems to show a lower gap between starting and ending distances rather than parametric methods (ML-based). Also in this case, ESPRIT-17 performs the lowest estimation error and the lowest error gap.
It is also interesting to check the procedure performances when the cylinder is placed close to one of the edges of the array. To this aim we simulate two different scenarios in which we change the horizontal coordinate of the cylinder center introducing an offset from $-5 \lambda_{0}$ to $10 \lambda_{0}$ with a step resolution of $0.25 \lambda_{0}$. The cylinder cross-section is fixed to $a=0.5 \lambda_{0}$ and the distance from the array line is the far-field range of Eq. 2.1.
In Fig. 2.7 the localization error vs. normalized distance $h / \lambda_{0}$ between the object surface and the array, is represented. As it is immediately clear just by a first look, the vertical displacement of the cylinder does not affect too much the localization procedure. To be precise, the estimation error decreases slowly as the distance increases.
In Fig. 2.8 the localization error trend vs. the horizontal offset is depicted. Also in this case, the procedure seems to be quite performing to peripheral detection (for the most of the DoA estimation methods), even if ESPRIT implementations are clearly affected by the horizontal position of the cylinder. In particular, since ESPRIT 17 makes use of the only central sub-arrays, it is quite obvious that it unfits the peripheral localization. Moreover, in this case ESPRIT 48 presents an unbalanced trend, reporting higher error values in the left side inly; this is probably due to an asymmetric localization capabilities for the first group of sub-arrays. For what concerns the other methods, the localization error increases (even if only slightly) at the edges of the range. This behavior is expected since, in peripheral localizations, part of the backscattered power is spread also in a outwards zone in which there are no sensors. Consequently, the amount of scattered power toward the array is slightly less than the central cylinder localization.
When the cylinder is embedded in a surrounding homogeneous medium, the problem scenario changes, introducing a half-space geometry. In particular, the array of sensors is supposed to adhere to the ground (being at a distance of $0.001 \lambda_{0}$ ), and then the interface effects could be reasonably neglected. Of course, as expressed

(b)

(c)

Figure 2.7: PEC cylinder localization in a vacuum, error vs. distance: a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.8: PEC cylinder localization in a vacuum, error vs. horizontal offset: $a$ ) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.
in Eq. 2.1, the sub-array far-field distance increases and also the minimum crosssection of the cylinder that is precisely localized slightly increase. In particular, this is clear by looking at Figs. 2.9-11, that the minimum dimension for a good localization is about $0.2 \lambda_{0}$ or greater (excepted for the ESPRIT-17).
As for a PEC cylinder in a vacuum, also in the half-spaces cases, an oscillating trend is obtained, almost for all the implemented methods; furthermore the estimation error vs. the distance $h$ is also oscillating, and differently from the previous case, the estimation error does not decrease as $h$ increases. In fact, by comparing Figs. 2.7, 2.12, and 2.16, is clear that in the vacuum case, together with an oscillating behavior of the estimation error, it is also affected by a decreasing trend inversely proportional to $h$. In the last case, occurring a refractive index of $n=3$, a further lack of accuracy has to be stresses. In particular, even though the procedure still perform a statistically precise localization, some of the implemented algorithms are to be considered less affordable. In Fig. 2.13, the loss of precision of Linear Prediction, Maximum Entropy and Pisarenko methods is quite evident, being their estimations more chaotic compared with the other methods that instead, still provide a clear trend similar to the case of $n=2$. Once again, ESPRIT-17 seems to be more robust showing almost the same behavior in both cases.


Figure 2.9: PEC cylinder localization in a homogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD).


Figure 2.10: PEC cylinder localization in a homogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48).


$$
a / \lambda_{0}
$$

(a)

(b)

Figure 2.11: PEC cylinder localization in ahomogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (WSF, DML, SML).

(b)

(c)

Figure 2.12: PEC cylinder localization in a homogeneous medium with $n=2$, error vs. distance: a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.13: PEC cylinder localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD).


Figure 2.14: PEC cylinder localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48).


Figure 2.15: PEC cylinder localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (WSF, DML, SML).


Figure 2.16: PEC cylinder localization in a homogeneous medium with $n=3$, error vs. distance: a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

### 2.3 Dielectric Cylinder Localization

In this section we simulate many configurations concerning a dielectric cylinder buried into a homogeneous medium. The refractive index of the cylinder is supposed to be $n_{c}=1,2,3,4,5$, while for the hosting medium it is $n_{m}=1,2,3$; also in these cases, we first assume a free-space localization just to derive some general statements about the algorithms and the overall functioning of our procedure.
The radius is assumed to be variable from $0.05 \lambda_{0}$ to $1.15 \lambda_{0}$, in 56 steps, with a resolution of $\lambda_{0} / 50$. The parameter $h$ varies with the radius, starting from the value that satisfies the far-field condition of the sub-array (Eq. 2.1), and following the rule $h=F F+a$. In Figs. 2.17-2.20 the estimation error for all the DoA estimation methods is plotted; in particular, it is possible to appreciate a general loss of precision in the localization of the cylinder, compared with the PEC cylinder case (see Figs. 2.4-2.6). Anyway, a detection capability is still performed for the simulated cases, showing error values concentrated under the localization threshold (indicated as a red line in figures) for radii larger than $\lambda_{0} / 2$.
Moreover, the first group of methods, summarized in Fig. 1.17a)-1.20a) and pertaining to spectral-based and subspace-based methods (except MUSIC), appear as less affordable in estimating objects having a lower cross-section. This tendency is clear by looking at the greater localization error in proximity of small radii. On the contrary, parametric methods seems to perform a better localization, providing an error at most twice than the threshold (Fig. 1.17c)-1.20c)) in the majority of the presented results.
Furthermore, as expected, as the refractive index of the cylinder increases, the localization trend improves, and also the minimum cross-section that can be detected with precision, gets smaller. In particular, by looking at Fig. 1.20b)-1.20c), is quite clear that the errors values fall down the threshold, for radii larger than $0.3 \lambda_{0}$, while, for the first group of methods (Fig. 1.20a)), the localization is well performed for bigger cross-sections. The general behavior of the algorithms, can be also deduced by looking at Figs. 2.21-2.22, where the mean value and the variance of the collected algorithms is shown. In particular, the non-parametric methods (indicated as MEAN 1 or VAR 1), provide the larger error trend, while the parametric ones, both subspace-based together with MUSIC, and ML-based (indicated as MEAN 2 or VAR 2 and MEAN 3 or VAR 3, respectively), are affected by an overall smaller error. In particular, for a refractive index of the cylinder equal to $n=3$, is well indicated that the estimation error is smaller than one for radii bigger than $\lambda_{0} / 2$, and when the index is fixed to $n=5$, the threshold roughly decrease to $0.3 \lambda_{0}$ (as the cylinder approaches the PEC case, the procedure improves their performances).

The performances in terms of the variance also indicate that for the groups indicated as VAR 1 and VAR 2, there are several peaks corresponding to extremely different estimations, while the parametric ML-based methods show to perform almost the same estimation in all the simulated cases. Moreover, it is clear that, for a refractive index of $n=5$, this tendency is very mitigated, and the localization become more affordable.
It is also interesting to underline, that the error trend is still oscillating (this is clear by looking at the mean and variance trends of Figs. 2.21-2.22). A plausible interpretation of such a behavior is that the field transmitted inside the cylinder at its first interface (air-to-medium) and the field reflected at the second interface (medium-to-air), corresponding to the cylinder edges, interfere among them and the peaks of the localization errors might arise from the constructive interference between the reflections from the two faces (upper and lower) of the dielectric objects. Moreover, as underlined by Van Bladel [68], the scattering cross-section of a dielectric cylinder immersed in a TM-polarized wave (electric field parallel to the axis) is affected by resonant peaks depending on the cross-section of the cylinder. This phenomenon is also evident in Figs. 2.17-2.19 and Figs. 2.21-2.22, where several sharpened peaks occur at different cross-sections.
When we consider a homogeneous medium surrounding the cylinder, different localization performances occur depending on the dielectric contrast. As a general statement, the scenario involving a half-space configuration is much more problematic that the previous PEC case. Anyway, in many cases, the procedure is still affordable to provide a good detection, even if the precision of the localization degrades. In fact, we have to consider not only the reflections and diffraction of the electromagnetic field in presence of a material body, but also the interference generates by the dielectric contrast of the two media. On top on this, the dielectric half-space introduces a drastically decreasing of the field received by the array, because a part of the field energy is also transmitted inside the cylinder, and thus, poor accuracy is to be expected if the electric field intensity inside the cylinder and the incident field intensity do not change much [69].
Generally speaking, we can underline as the first group of non-parametric methods (Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, Pisarenko), seem to be less robust in estimating dielectric object position. In fact, by looking at Figs. 2.27a)-2.29a), is quite clear that the overall enhancement in the localization process, due to the dielectric contrast increasing, appear to be quite mitigated if compared with sub-figures 2.27-2.29 b) and $c$ ), especially in reference to Linear Prediction, Maximum Entropy and Pisarenko algorithms.

In Figs. 2.23 and 2.24 the localization error vs. the horizontal position of the cylinder is represented, for $n_{c}=2$ and $n_{c}=5$, respectively. Also in this case, as happened for the perfectly-conducting case, the localization is quite precise; in particular, Pisarenko's method shows higher error values in comparison with the other methods, and the ESPRIT estimation performances are affected by the same problems described previously.
In Figs. 2.25 and 2.26, the error trend vs. vertical offsets is represented for $n_{c}=2$ and $n_{c}=5$, respectively. For a dielectric cylinder having a low refractive index, the estimation is affected by an oscillating behavior, but as the dielectric contrast increases, the effect of a vertical displacement of the cylinder is mitigated (see Fig. 2.26), and the procedure becomes more robust, notably for subspace-based and MLbased methods.
The same trend of the localization error, with reference to the dielectric contrast increase, is also revealed when the cylinder is embedded in a homogeneous medium, as depicted in Figs. 2.32 and 2.33 for the $n_{m}=2$ case, and in Figs. 2.39 and 2.40 for $n_{m}=3$.

(b)

(c)

Figure 2.17: Localization of a dielectric cylinder having $n_{c}=2$ in a vacuum, error vs. normalized radius $\left.a / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.18: Localization of a dielectric cylinder having $n_{c}=3$ in a vacuum, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(a)

(b)

(c)

Figure 2.19: Localization of a dielectric cylinder having $n_{c}=4$ in a vacuum, error vs. normalized radius $\left.a / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.20: Localization of a dielectric cylinder having $n_{c}=5$ in a vacuum, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(d)

Figure 2.21: Dielectric cylinder in a vacuum. Mean value vs. normalized radius $\left.\left.\left.\left.a / \lambda_{0}: a\right) n_{c}=2 ; b\right) n_{c}=3 ; c\right) n_{c}=4 ; d\right) n_{c}=5$.

(a)

(b)

(c)

(d)

Figure 2.22: Dielectric cylinder in a vacuum. Variance vs. normalized radius $a / \lambda_{0}$ : a) $n_{c}=2$; b) $n_{c}=3$; c) $n_{c}=4$;d) $n_{c}=5$.

(b)

(c)

Figure 2.23: Localization of a dielectric cylinder having $n_{c}=2$ in a vacuum, error vs. horizontal offset $\Delta \eta$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(b)

(c)

Figure 2.24: Localization of a dielectric cylinder having $n_{c}=5$ in a vacuum, error vs. horizontal offset $\Delta \eta$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(a)

(b)

(c)

Figure 2.25: Localization of a dielectric cylinder having $n_{c}=2$ in a vacuum, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(b)

(c)

Figure 2.26: Localization of a dielectric cylinder having $n_{c}=5$ in a vacuum, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.27: Localization of a dielectric cylinder having $n_{c}=3$ buried in a homogeneous medium with $n_{m}=2$, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.28: Localization of a dielectric cylinder having $n_{c}=4$ buried in a homogeneous medium with $n_{m}=2$, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(a)

(b)

(c)

Figure 2.29: Localization of a dielectric cylinder having $n_{c}=5$ buried in a homogeneous medium with $n_{m}=2$, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.30: Dielectric cylinder buried in a homogeneous medium with $n_{m}=2$.
Mean value vs. normalized radius $\left.a / \lambda_{0}: a\right) n_{c}=3$; b) $n_{c}=4$; c) $n_{c}=5$.

(a)

(b)

(c)

Figure 2.31: Dielectric cylinder buried in a homogeneous medium with $n_{m}=2$. Variance vs. normalized radius $\left.a / \lambda_{0}: a\right) n_{c}=3$;b) $n_{c}=4$; c) $n_{c}=5$.


Figure 2.32: Localization of a dielectric cylinder having $n_{c}=3$ in a homogeneous medium with $n_{m}=2$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(b)

(c)

Figure 2.33: Localization of a dielectric cylinder having $n_{c}=5$ in a homogeneous medium with $n_{m}=2$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

In particular, in Fig. 2.29, while two groups (MUSIC, root-MUSIC and ESPRIT, and ML-based parametric methods), show clearly a localization capability starting from a radius value equal to $\lambda_{0} / 5$, the group of non-parametric methods is still almost concentrated around the threshold, but some outlines still occur even for larger radii. This behavior is also stressed by looking at the statistical properties resumed in Figs. 2.30-2.31, when as the dielectric contrast increases, the localization become more precise and the three groups of methods differ from performances. In fact, in Fig. 2.30a), the average trends are very close and coincident in many configurations, but in the sub-figures 2.30b) and $c$ ), a result worsening of the first group of methods becomes clear. In addition of this, in Fig. 2.31, the variance behavior is considered to be significant to underline as the non-parametric methods confirm to have a great variability (in fact they are actually belonging to different kind of algorithms: spectral-based and subspace-based), instead of the other groups are more homogeneous, showing a slightly smoothed trend for radii larger than $\lambda_{0} / 4$. Moreover, in all the simulated cases, the third group of methods shown the same variance, indicating a total equivalence in estimating DoAs.
A similar comportment is stressed in Figs. 2.35-2.36, but in Fig. 2.34 a different trend is sketched: when the refractive index of the cylinder is smaller than the index of the surrounding medium, the localization becomes greatly effective, for many estimation algorithms. Also in this case, spectral-based together with Linear Prediction, Maximum Entropy, and P.H.D. methods, perform a worsen precision. By looking at Figs. 2.37-2.38, it is possible to point out that the mean and variance corresponding to this case, indicate that the localization is quite good and, on the other side, it get worse when the refractive index of the cylinder becomes $n_{c}=4$, getting better afterwards, when the dielectric contrast increases.
We can comment this behavior, observing that, when the refractive index of the cylinder is smaller than the index of the surrounding medium, the electric field propagating in the ground faces at first the interface with a less dense half-space, represented by the cylinder, and then, once transmitted inside the cylinder, a second interface with a denser medium. This combination of media is exactly opposite of what happened in all the other cases, and, moreover, it can be modeled as a cylindrical layered propagation problem, with the external layer that goes to infinity. In our opinion, such a problem is worthy to be studied intensively in future works. We can roughly presume that this combination of dielectric layers hallows an enhancement in the constructive interference between the electromagnetic field reflections by the two faces of the dielectric cylinder. This kind of trend is to the utmost clear, in the cavity cases, as we are going to see in the next paragraph.


Figure 2.34: Localization of a dielectric cylinder having $n_{c}=2$ buried in a homogeneous medium with $n_{m}=3$, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.35: Localization of a dielectric cylinder having $n_{c}=4$ buried in a homogeneous medium with $n_{m}=3$, error vs. normalized radius $a / \lambda_{0}: a$ ) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(b)

(c)

Figure 2.36: Localization of a dielectric cylinder having $n_{c}=5$ buried in a homogeneous medium with $n_{m}=3$, error vs. normalized radius $a / \lambda_{0}$ : a) Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.


Figure 2.37: Dielectric cylinder buried in a homogeneous medium with $n_{m}=3$.
Mean value vs. normalized radius $\left.\left.a / \lambda_{0}: a\right) n_{c}=2 ; b\right) n_{c}=4$; c) $n_{c}=5$.

(a)

(b)

(c)

Figure 2.38: Dielectric cylinder buried in a homogeneous medium with $n_{m}=3$. Variance vs. normalized radius $\left.a / \lambda_{0}: a\right) n_{c}=2$;b) $n_{c}=4$; c) $n_{c}=5$.


Figure 2.39: Localization of a dielectric cylinder having $n_{c}=2$ in a homogeneous medium with $n_{m}=3$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(a)

(b)

(c)

Figure 2.40: Localization of a dielectric cylinder having $n_{c}=5$ in a homogeneous medium with $n_{m}=3$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

### 2.3.1 Localization of Cavities

Several cases have been simulated, regarding a cylindrical cavity in a dielectric half-space. The target size has been varied; the burial depth has been changed together with the cylinder radius, following the rule $h=n_{m} \lambda_{0} / 2+a$. In Figs. 2.412.46, the estimation error is plotted vs. the normalized radius in sub-figures $a$ ), while in $b$ ) an enlargement of the region $\operatorname{err} \leq 1$ is reported; the hosting medium is assumed to have $n_{m}=2$ in Figs. 2.41-2.43 and $n=3$ in Figs. 2.44-2.46, respectively. The procedure is able to localize the center of the cavity with an estimation error lower than one in many cases with slightly different trends for the implemented algorithms. Non-parametric methods methods seem to be suitable for an accurate estimation of the target position, even if Linear Prediction, Pisarenko Harmonic Decomposition and Maximum Entropy are less robust when the permittivity of the ground is higher, as can be appreciated from Fig. 2.38. Subspace-based methods (both parametric and non-parametric) perform a precise localization of the cavity. Moreover, an oscillating trend is still present in Figs. 2.41-2.46. ML-based methods, are very effective in this case, as we can appreciate in Fig. 2.46, where all the estimated error values are lower the threshold.
By looking at Figs. 2.47-2.48, the mean error appears quite smoothed and below the unity and the variance indicates a homogeneity in the three groups of algorithms except in the case of Fig. 2.48b), where sharpened peaks occur, for the non-parametric group. As discussed before, this is caused by the lack of precision of some subspace-based algorithms (Linear Prediction, Pisarenko Harmonic Decomposition and Maximum Entropy) that deteriorate their performances as the refractive index of the surrounding medium increases.
In Figs. 2.49 and 2.50, the effect of the vertical offset on the localization error is represented. As previously stated, when the dielectric contrast decreases, the estimation is more affected by this issue, anyway in general, it does not damage the localization.
The localization procedure is very efficient for a cavity detection. In particular, by approximating the cylindrical wavefront to be locally planar, a possible explanation of this behavior can be found in the Snell's law: when $n_{c}<n_{r}$, the field that impinges at the interface and is transmitted inside the cylinder is limited by the total reflection angle, so that, with respect to the case of $n_{c}>n_{r}$, a smaller amount of energy enters the cylinder. Consequently, the field scattered by the object is mainly focused toward the array, making these cases better suitable to be treated with the procedure (together with the case of a conducting cylinder).

(a)

(b)

Figure 2.41: Cylindrical cavity localization in a homogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD).


Figure 2.42: Cylindrical cavity localization in a homogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48).


$$
a / \lambda_{0}
$$

(a)

(b)

Figure 2.43: Cylindrical cavity localization in a homogeneous medium with $n=2$, error vs. normalized radius $a / \lambda_{0}$ (WSF, DML, SML).


Figure 2.44: Cylindrical cavity localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD).

(a)


$$
a / \lambda_{0}
$$

(b)

Figure 2.45: Cylindrical cavity localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48).

When a lot of multiple internal reflections occurs, instead, also the contribution of interferences between the cylinder-to-ground interfaces is magnified, and the scattered field presents more radiation lobes forcing false DoA estimations. Obviously, also the mismatch due to the dielectric contrast between scatterer and ground, plays an important role.
As a further confirmation of our physical interpretation, in Fig. 2.51-2.52 we report two-dimensional plots of the magnitude of the total electric field, in the ground and inside the scatterer, as a function of the normalized coordinates $k_{0} x$ and $k_{0} z$. These plots have been calculated by using our codes implementing the CWA. In 2.51a), a cylindrical cavity is considered, in $2.51 b$ ), a dielectric cylinder with $n_{c}=5$ are represented for $n_{m}=2$, while in $2.52 a$ ), a dielectric cylinder with $n_{c}=5$, in 2.51b), a dielectric cylinder with $n_{c}=5$ and for $n_{m}=3$ are depicted. In the first case, the ground, the radius is $a=0.5 \lambda_{0}$ and the burial depth is $h=1.5 \lambda_{0}$ and in the second one the radius is $a=0.25 \lambda_{0}$ and the burial depth is $h=1.75 \lambda_{0}$.
The field values are codified through a gray scale ranging from black (lowest values) to white (highest values). The effects of the presence of the scatterer on the field pattern are in both cases very pronounced, however it is apparent that, when the permittivity of the cylinder is higher than the permittivity of the ground, the field outside the object is diffracted toward more direction and inevitably the localization procedure gives worst results.


Figure 2.46: Cylindrical cavity localization in a homogeneous medium with $n=3$, error vs. normalized radius $a / \lambda_{0}$ (WSF, DML, SML).


Figure 2.47: Cylindrical cavity localization in a homogeneous medium. Mean value vs. normalized radius $\left.\left.a / \lambda_{0}: a\right) n_{m}=2 ; b\right) n_{m}=3$.


Figure 2.48: Cylindrical cavity localization in a homogeneous medium. Variance vs. normalized radius $\left.\left.a / \lambda_{0}: a\right) n_{m}=2 ; b\right) n_{m}=3$.


Figure 2.49: Cylindrical cavity localization in a homogeneous medium with $n=2$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

(a)

(b)

(c)

Figure 2.50: Cylindrical cavity localization in a homogeneous medium with $n=3$, error vs. normalized distance $\left.h / \lambda_{0}: a\right)$ Bartlett, Capon, Linear Prediction, Max Entropy, Min Norm, PHD; b) MUSIC, root-MUSIC, ESPRIT-17, ESPRIT-48; c) WSF, DML, SML.

Finally, for the same cases considered in Fig. 2.51-2.52, we show in Fig. 2.53 the magnitude of the field scattered by the buried obstacle and transmitted in air, calculated along a line parallel to the interface $\left(k_{0} x=-0.0001\right)$ that covers the whole receiving array (i.e., it extends from $z=0$ to $z=50 d$ ). In the cavity case, the curve has a simpler behavior, therefore the field values measured by the receiving array are easier to be statistically interpreted.


Figure 2.51: Gray-scale map of the magnitude of the total electric field, in the ground and inside the cylinder; $n_{m}=2, a=0.5 \lambda_{0}, h=1.5 \lambda_{0}$. The scatterer in $a$ ) is a cavity, in $b$ ) is dielectric with $n_{c}=5$.


Figure 2.52: Gray-scale map of the magnitude of the total electric field, in the ground and inside the cylinder; $n_{m}=3, a=0.25 \lambda_{0}, h=1.75 \lambda_{0}$. The scatterer in a) $n_{c}=2$, in b) $n_{c}=5$.


Figure 2.53: Magnitude of the scattered-transmitted field impinging on the array.

### 2.4 Object-Size Estimation

In this section, we explore the possibility to estimate the diameter of a conducting or dielectric scatterer, cylindrically shaped and infinitely long, at first in a vacuum and also embedded in a dielectric half-space. In particular we consider the problem configuration as exposed extensively in the previous paragraphs of this chapter, in which an array of receivers is used to detect a buried cylinder of radius $a$ and distance $h$ from the array. It has been shown that, by partitioning the whole array in 3 -elements grouped sub-arrays, equally spaced by $d=\lambda_{0} / 4$ (where $\lambda_{0}$ is the wavelength of the impinging electromagnetic field in the upper vacuum half-space), it is possible to investigate the objects that are in the near-field of the whole array, giving also a rough approximation of their position.
The sub-array that shows the bigger eigenvalue of the sub-array correlation matrix (i.e. the $i$-th subarray), is the one overhanging the object, as one can clearly deduce by looking at Fig. 2.54. On top on this, is straightforward to notice the value of the maximum eigenvalue of the sub-array correlation matrices, is connected with the received backscattered power. Let start to note that the spectral norm of the correlation matrix $\mathbf{R}$ is defined as the maximum eigenvalue of $\mathbf{R}, \kappa_{m} a x$ [libro Haykin, Adaptive Filter Theory 3rd, chapter 4]:

$$
\begin{equation*}
\|\mathbf{R}\|_{S}=\kappa_{\max } \tag{2.3}
\end{equation*}
$$

Furthermore, considering a linear FIR filter as in Fig. 1.6 (and shown hereafter), represented by a wide-sense stationary signal $s(k)$ having correlation matrix $\mathbf{R}$, and a Gaussian noise $n(k)$, having zero mean and variance $\sigma_{n}^{2}$, the average power of the


Figure 2.54: Trend of the maximum eigenvalue of the sub-array correlation matrices for a cylinder $a$ ) placed at the center of the array or $b$ ) peripherally. The half-space containing the cylinder has a refractive index $n=2$, the radius of the cylinder is $a=\lambda_{0} / 4$ and the distance between the array and the cylinder is $h=F F+a$.
signal component of the filter output $y(k)$, is

$$
\begin{equation*}
P_{o}=\mathbf{w}^{H} \cdot \mathbf{R} \cdot \mathbf{w} \tag{2.4}
\end{equation*}
$$

where the elements of the vector $\mathbf{w}$ are the filter coefficients. The average power of the noise $\mathcal{N}_{0}$ is calculated by assuming the noise correlation matrix as $\sigma_{n}^{2} \mathbf{I}$, and applying the previous Eq. (2.6), we found

$$
\begin{equation*}
\mathcal{N}_{\mathrm{o}}=\sigma_{\mathrm{n}}^{2} \mathbf{w}^{\mathrm{H}} \cdot \mathbf{I} \cdot \mathbf{w}=\sigma_{\mathrm{n}}^{2} \mathbf{w}^{\mathrm{H}} \cdot \mathbf{w} \tag{2.5}
\end{equation*}
$$

and thus, the output signal-to-noise ratio is derived as

$$
\begin{equation*}
(S N R)_{o}=\frac{P_{o}}{\mathcal{N}_{o}}=\frac{\mathbf{w}^{H} \cdot \mathbf{R} \cdot \mathbf{w}}{\sigma_{n}^{2} \mathbf{w}^{H} \cdot \mathbf{w}} \tag{2.6}
\end{equation*}
$$

The optimum filter coefficient vector $\mathbf{w}$ that maximizes $(S N R)_{o}$, is subject to the constraint [libro Haykin]

$$
\mathbf{w}^{H} \cdot \mathbf{w}=1
$$

so that the maximum value of the output signal-to-noise ratio is given by

$$
\begin{equation*}
(S N R)_{o, \max }=\frac{\kappa_{\max }}{\sigma_{n}^{2}} \tag{2.7}
\end{equation*}
$$

We simulated many different geometrical configuration, changing the value of the distance between the object and the array in the range $h \in[F F+a ; 6 F F+a]$ with a step resolution of $\lambda_{0} / 20$ for a total number of 51 iterations. At the same time, we


Figure 2.55: Signal model for a $M$-element array with $D$ arriving signals.
vary we varying the cylinder cross-section in the range $a \in\left[0.05 \lambda_{0} ; 1.05 \lambda_{0}\right]$, with 51 iterations, for an overall matrix of $51 \times 51$ values in which each column represent the maximum eigenvalue among all the eigenvalues of the 17 sub-array correlation matrices, corresponding to a value of $a$ (varying with the columns) for all the distances $h$, that varies with the row.
In Fig. 2.56, the maximum eigenvalue is shown vs. normalized distance $h / \lambda_{0}$, for several values of radius $a$. In particular, it is possible to appreciate as in Fig. 2.56a) (concerning a conductive cylinder in a vacuum), the tendency is quite regular, therefore in our opinion in this case it might be possible to extrapolate the cylinder size. On the contrary, in Fig. 2.56b)-2.56c), the presence of a surrounding homogeneous medium, causes an oscillating behavior of the maximum eigenvalue, as noticed in the previous section for the estimation error. For a better readability of the pictures, only few tendencies are plotted, but all the other values have been calculated, too, and show to fit the general trend.
It is interesting to point out that, plotting the maximum eigenvalue vs. radius $a$, and varying the distance $h$ thus obtaining different curves, an almost typical trend can be stressed. In Fig. 2.57a)-2.57b), is clear that the maximum eigenvalue increases as the radius becomes larger and, at the same time, it decreases as the distance $h$ increases, with no oscillations; in Fig. 2.57c), indeed, the trend is oscillating. This behavior suggests the possibility to estimate the cylinder cross-section, based on the maximum eigenvalue among all the sub-arrays matrices.
For the dielectric case in a vacuum, in Figs. 2.58a) and $b$ ), the maximum eigenvalue vs. distance for several values of the radius $a$, is represented for two different refractive indexes ( $n_{c}=2$ and $n_{c}=3$ ). As in the PEC case, the trend is descent with the distance from the array.
In Fig. 2.59a) and $b$ ) the maximum eigenvalue vs. the normalized radius $a / \lambda_{0}$ for
different distances $h$ is represented for dielectric cylinders in a vacuum. What previously stated about interferences between the upper and a lower face of the cylinder, and about multiple internal reflections between these two faces, seems to be confirmed, as if the scatterer was a dielectric slab. It can be easily verified that the SNR peaks in Fig. 2.59b), are centered on radius values such that, in this simplified model, the multiple internal reflections are in phase at the upper face of the cylinder, and constructive interference occurs. Moreover, the peaks are higher for larger values of the cylinder radius, when these considerations, based on geometrical optics, are more valid. It might be expected that peaks of SNR should correspond to an easier detection (low values of the localization error), and that small values of the SNR should correspond to peaks of the localization error. However, the localization error (see Figs. 2.17-2.18) behavior shows a lower number of peaks, with respect to the SNR, and some peaks of the localization error (worse detection) are centered on the same radius values as the SNR peaks. Anyway, it might be taken into account that the SNR refers to only one sub-array and thus, it has not to be confused with the whole array SNR.
In Figs. 2.61-2.62, it is possible to appreciate as the SNR behavior becomes more regular if the cylinder has a refractive index lower that the embedding medium, for a fixed dielectric contrast. In particular, by looking at Fig. 2.60a) is quite evident the descendant trend of the maximum eigenvalue as the normalized distance increases, and in Fig. 2.61a), the peaks of the SNR introduced by the multiple reflections interference are stressed. In fact, by looking at Figs. 2.51a)-2.52a), it is possible to appreciate as, when $n_{c}<n_{m}$, the scattered field distribution is more regular and the SNR at each sub-array can be consider to have a similar trend of Fig. 2.61a). Otherwise, when $n_{c}>n_{m}$, the filed distribution depicted in Figs. 2.51b)-2.52b), is quite irregular and not predictable. This causes a lack of efficiency of the DoA signal model, based on the geometrical optic approximation, and also the whole SNR of the array cannot be deduced by the SNR of the sub-array overhanging the object. In Figs. 2.61-2.62, we report the SNR of the signal received by the sub-array, as a function of the cylinder radius normalized to the vacuum wavelength, for an air cavity embedded in a dielectric half-space with refractive index equal to 2 or 3 . Some peaks are present, centered on radius values such that the multiple internal reflections are in phase at the upper face of the cavity and constructive interference occurs.
With respect to the previous couple of figures, we now can see a smaller number of peaks because the permittivity of the air cavity is smaller than the permittivity of the previously considered dielectric cylinder, therefore the wavelength inside the
object is larger and, in the analyzed radius range, there is a smaller number of cases satisfying the resonance condition.
So that, it is not straightforward to explain the oscillations of the localization error through geometrical-optics considerations based on the geometrical and physical properties of the scenario. Unlike the SNR, the oscillations of the localization error are affected also from the statistical procedure applied to the DoA crossing pattern, for example the testing-window size and the decision threshold (false-alarm probability) play an important role.
In conclusion, in the cases when the SNR trend is more regular (PEC cylinder, cavity) it seems to be possible to estimate also the dimension of the buried object, by means of the maximum eigenvalue among all of the sub-array correlation matrices, that is directly linked to the maximum value of the SNR.


Figure 2.56: Maximum eigenvalue vs. distance $h$ and radius $a$ : a) conductive cylinder in a vacuum; $b$ ) conductive cylinder in a homogeneous medium with $n_{m}=2 ; c$ ) conductive cylinder in a homogeneous medium with $n_{m}=3$.


Figure 2.57: Maximum eigenvalue vs. radius $a$ and distance $h: a$ ) conductive cylinder in a vacuum; $b$ ) conductive cylinder in a homogeneous medium with $n_{m}=2 ; c$ ) conductive cylinder in a homogeneous medium with $n_{m}=3$.


Figure 2.58: Maximum eigenvalue vs. distance $h$ and radius $a$ : $a$ ) dielectric cylinder with $n_{c}=2$ in a vacuum; $b$ ) dielectric cylinder with $n_{c}=3$ in a vacuum.


Figure 2.59: Maximum eigenvalue vs. radius $a$ and distance $h$ : $a$ ) dielectric cylinder with $n_{c}=2$ in a vacuum; $b$ ) dielectric cylinder with $n_{c}=3$ in a vacuum.


Figure 2.60: Maximum eigenvalue vs. distance $h$ and radius $a: a$ ) cylindrical cavity in a homogeneous medium having $n_{m}=2 ; b$ ) dielectric cylinder with $n_{c}=3$ in a homogeneous medium with $n_{m}=2$.


Figure 2.61: Maximum eigenvalue vs. radius $a$ and distance $h$ : a) cylindrical cavity in a homogeneous medium having $n_{m}=2 ; b$ ) dielectric cylinder with $n_{c}=3$ in a homogeneous medium with $n_{m}=2$.


Figure 2.62: Maximum eigenvalue vs. distance $h$ and radius $a$ : a) cylindrical cavity in a homogeneous medium having $n_{m}=3 ; b$ ) dielectric cylinder with $n_{c}=2$ in a homogeneous medium with $n_{m}=3$.


Figure 2.63: Maximum eigenvalue vs. radius $a$ and distance $h$ : a) cylindrical cavity in a homogeneous medium having $n_{m}=3 ; b$ ) dielectric cylinder with $n_{c}=2$ in a homogeneous medium with $n_{m}=3$.

## Chapter 3

## Detection of Multiple Cylinders

In this chapter we focus on the localization of multiple cylinders, located in the same geometrical and physical scenario as considered before. It is essential for the algorithm to estimate the number of cylinders and their mutual distance, together with their distance from the array.
As we indicated in Chapter 1, the methods that we used to estimate the DoAs, have limitations for efficiency if the fields are correlated. To avoid this, we here consider the forward-backward (FB) representation of some algorithms that particularly suffer a loss of efficiency. In particular non-parametric methods (with exception of MU.SI.C.), have been implemented with their FB-form, resulting appropriate to our scopes.
Following the same procedure used for the single object localization case, we present the geometry of the multiple objects detection scenarios in Figure 3.1. The number of cylinders is fixed to be equal two for all the simulations. This assumption could be considered as a limitation of the procedure, but actually, as we point out further in this chapter, a number of objects grater than two can be treated just repeating the basic sub-array configuration, once determined the minimum horizontal distance among the objects. In fact the magnitude of the signal correlation depends, among other things, from the mutual distance of the cylinders, and consequently the DoA estimation capabilities of the algorithms.
In order to determine, not without ambiguity (just think of the case of cylinders of different sizes, overlapped at a different burial depth), both the number and the relative position of objects we use (and compare among them) several clustering algorithms, widely used in the field of image processing and remote sensing data [70]. These methods allow us, in a large number of cases and with sufficient accuracy, to properly detect the cylinders. We present in the next section a brief description of the clustering problem and algorithms, used for our purpose, and then we present
the numerical results of many simulations (relevant to both conductive and dielectric configuration) that allow us to extrapolate significant assumptions about the effectiveness of this technique and the opportunity of further developments concerning multiple cylinders localization.


Figure 3.1: Multiple-objects localization: two cylinders.

### 3.1 Clustering Analysis

In clustering, the goal is to understand the macroscopic structure and relationships among the objects by considering the ways in which they are similar and dissimilar. In many datasets, the distribution of objects with respect to some similarity relationship is not uniform, so that some of the objects resemble each other more closely than average. Such a subset is called a cluster. In a good clustering, objects from different clusters should resemble each other less than average. For any particular dataset, there are many ways to compare objects, so a clustering always implicitly contains some assumption about the meaning of similarity [71].
Clustering techniques can be divided into three kinds: those based on distances among objects in the geometrical sense described above (clusters are objects that are unusually close to each other); those based on density of objects (clusters are regions where objects are unusually common); or those based on probability distributions (clusters are sets of objects that fit an expected distribution well). These are called distance-based, density-based, and distribution-based clusterings, respectively.
Clustering techniques can also be distinguished by whether they carve up the objects into disjoint clusters at a single level (partitional clustering), or give a complete
hierarchical description of how objects are similar to each other (hierarchical clustering), using a dendrogram. As well, some clustering techniques need to be told how many clusters to look for, while others will try to infer how many are present. The simplest partitional clustering technique is $k$-means. Given a dataset considered as a set of points in $m$-dimensional space, a set of $k$ cluster centers are chosen at random. Each point in the dataset is allocated to the nearest cluster center. The centroid of each of these allocated sets of points is computed, and these centroids become the new cluster centers. The process is repeated until the cluster centers do not change. Because $k$ is a parameter to the algorithm, the number of clusters must be known or guessed beforehand.
An example in two dimensions is shown in Fig. 3.2. The crosses represent data points. If the cluster centers (circles) are placed as shown in Fig. 3.2a), then each object is allocated to its nearest cluster center. This relationship is shown by dashed lines. After this initial, random, allocation, each cluster center is moved to the centroid of the objects that belong to it, as shown in Fig. 3.2b).
The allocations of objects to new cluster centers is again shown by the dashed lines. It is clear that the allocation of objects to clusters will not change further, although the cluster centers will move slightly in subsequent rounds of the algorithm. The $k$ - means algorithm is simple and fast to compute. Typical density-based partitional clustering algorithms choose an object at random to be a potential cluster center and then examine its neighborhood. Objects that are sufficiently close are added to the cluster, and then their neighbors are considered, in turn. This process continues until no further points are close enough to be added. If enough points have been found, that is the potential cluster is large enough, then it becomes one of the clusters and its members are removed from further consideration. The process is repeated until no new clusters can be found. Some objects may not be allocated to any cluster because there are not enough other objects near them. We implemented to our purpose, the so-called Density-Based Scan Algorithm with Noise, or DBSCAN [72, 73].
A hierarchical clustering method [74] works by grouping data objects into a tree of clusters. Hierarchical clustering methods can be further classified into agglomerative and divisive hierarchical clustering, depending on whether the hierarchical decomposition is formed in a bottom-up or top-down fashion. In particular, the agglomerative hierarchical clustering consists in a bottom-up strategy starts by placing each object in its own cluster and then merges these atomic clusters into larger and larger clusters, until all of the objects are in a single cluster or until certain termination conditions are satisfied. Most hierarchical clustering methods belong to


Figure 3.2: $K$ - means algorithm, with objects denoted by crosses, and $k$ initial cluster centers denoted by circles. The dashed lines indicate which cluster center is closest to each object. a) Initialization of the algorithm, b) Second round of the algorithm, one object has moved from one cluster to another, and all objects are closer to their center than in the previous round.
this category. They differ only in their definition of intercluster similarity. On the other side, the divisive hierarchical clustering is a top-down strategy that does the reverse of agglomerative hierarchical clustering by starting with all objects in one cluster. It subdivides the cluster into smaller and smaller pieces, until each object forms a cluster on its own or until it satisfies certain termination conditions, such as a desired number of clusters is obtained or the distance between the two closest clusters is above a certain threshold distance.

In Fig. 3.3 an example of agglomerative hierarchical clustering is sketched. The group choice is based on a particular measure of similarity or dissimilarity, generally referred as proximity, that is computed for each data point and successively, from these value, a dendrogram (a kind of tree of proximities) is derived. Up to now, we generally refer to proximity as a measure of (dis)similarity among data, without


Figure 3.3: Hierarchical algorithm, with $a$ ) objects denoted by black dots, centroids denoted by red dots, and clusters denoted by circles; b) dendrogram.
any operative definition about it. The most common similarity criterion is distance; a good clustering method should produce cluster in which the intra-class distance is low and the inter-class distance is high. By using this kind of approach, the attribute of (dis)similarity is justified, and the classes of different data so grouped, can be considered as homogeneous clusters of data, based on this proximity criterion. The basic form of data that is taken into account in cluster analysis, is represented by a $m \times n$ multivariate matrix $\mathbf{X}$, whose elements are variables that constitute the data point coordinates in the space of parameters. The entry $x_{i j}$ in $\mathbf{X}$ presents the $j$-th variable of object $i$ :

$$
\mathbf{X}=\left[\begin{array}{cccc}
x_{11} & x_{12} & \cdots & x_{1 n}  \tag{3.1}\\
x_{21} & x_{22} & \cdots & x_{2 n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m 1} & x_{m 2} & \cdots & x_{m n}
\end{array}\right]
$$

The proximity measure used in clustering approaches is very similar to the Euclidean distance among the data point and the centroids of each cluster. The distance is defined as

$$
\begin{equation*}
d_{k, l}=\frac{1}{N} \sum_{j=1}^{N} \sum_{i=1}^{M}\left[x(k)_{i j}-c(l)_{i}\right]^{2} \tag{3.2}
\end{equation*}
$$

where $d_{k, l}$ represents the distance between the element of cluster $k$ and the centroid of cluster $l$, while $x(k)_{i j}$ is the value of the element in the $i$-th row and $j$-column of the matrix $\mathbf{X}$, and $c(l)_{i}$ is the value of the $i$-th row of the centroid vector of cluster number $l$.
There are many distance metrics that can be used as proximity measure, the most
commonly adopted are Manhattan, Minkowski, and Cranberra distances, summarized in the following Table 3.1, where $x(k)_{i}$ is the $k$-th variable value of the $p$ -

| Measure | Formula |
| :--- | :--- |
| Euclidean distance | $d_{i j}=\left\{\sum_{k=1}^{p}\left[x(k)_{i}-x(k)_{j}\right]^{2}\right\}^{\frac{1}{2}}$ |
| Manhattan distance | $d_{i j}=\sum_{k=1}^{p}\left\|x(k)_{i}-x(k)_{j}\right\|$ |
| Minkowski distance | $d_{i j}=\left\{\left\|\sum_{k=1}^{p}\left[x(k)_{i}-x(k)_{j}\right]^{r}\right\|\right\}^{\frac{1}{r}} \quad(r \geq 1)$ |
| Cranberra distance | $d_{i j}=\sum_{k=1}^{p} \frac{\left\|x(k)_{i}-x(k)_{j}\right\|}{\left[\left\|x(k)_{i}\right\|+\left\|x(k)_{j}\right\|\right]} \quad x(k)_{i}, x(k)_{j} \neq 0$ |

Table 3.1: Proximity measures.
dimensional observation for individual $i$ and $x(k)_{j}$ is the $k$-th variable value of the $p$-dimensional observation for individual $j$. An insight of the double cylinder localization is shown in Fig. 3.4 where, once applied the DoA estimation algorithms in order to find two crossing clouds, we perform a cluster analysis and extract the coordinates of the two cylinder centers that are, in practice, the coordinates of the cluster centroids.

### 3.1.1 K-means Algorithm

Clustering algorithms group a set of documents into subsets or clusters. The cluster algorithms goal is to create clusters that are coherent internally, but clearly different from each other. In other words, documents within a cluster should be as similar as possible; and documents in one cluster should be as dissimilar as possible from documents in other clusters.
Clustering is the most common form of unsupervised learning. No supervision means that there is no human expert who has assigned documents to classes. In clustering, it is the distribution and makeup of the data that will determine cluster membership. The procedure follows a simple and easy way to classify a given data set through a certain number of clusters (assume $k$ clusters) fixed a priori. The main idea is to define $k$ centroid, one for each cluster. These centroids should be placed in a cunning way because of different location causes different result. So, the better choice is to place them as much as possible far away from each other.
The next step is to take each point belonging to a given data set and associate it to the nearest centroid. When no point is pending, the first step is completed and an early group is done. At this point we need to re-calculate $k$ new centroids as barycenter of the clusters resulting from the previous step. After we have these $k$ new centroids, a new binding has to be done between the same data set points and

(a)

(b)

(c)

Figure 3.4: Double cylinder localization: $a$ ) DoAs estimation; $b$ ) crossing clouds; $c$ ) cluster analysis and estimation.
the nearest new centroid. A loop has been generated. As a result of this loop we may notice that the $k$ centroids change their location step by step until no more changes are done. In other words centroids do not move any more.
Finally, this algorithm aims at minimizing an objective function, in this case a squared error function. The objective function. The following description of $k$ means can be found in MacQueen [75]. Given $N$ documents in a $i$-dimensional vectorial space, $X=X_{1}, X_{2}, \cdots, X_{N}$, we partition the documents in $K$ group $P=$ $P_{1}, P_{1} \ldots, P_{K}$ satisfying the following properties:

- $\bigcup_{1}^{K} P_{i}=X$ : all the objects in a group must belong at least to one cluster;
- $\bigcap_{1}^{K} P_{i}=\emptyset:$ every object must belong to one and only one cluster;
- $\emptyset \subset A_{i} \subset X$ : it cannot exist a single cluster;
- $1<K<N$ : the number of clusters must be less than the number of documents.

We represent the partition using a matrix $\overline{\bar{U}} \in \mathbb{N}^{K \times N}$, where the generic element $u_{i j}=0$, indicates the belonging of the $j$ object to the $i$. Moreover, we assume $C=C_{1}, C_{2}, \cdots, C_{K}$ as $K$ centroids.
The objective function

$$
\begin{equation*}
V(U, C)=\sum_{i=1}^{K} \sum_{X_{j} \in P_{i}}^{N}\left\|X_{j}-C_{i}\right\|^{2} \tag{3.3}
\end{equation*}
$$

Finally, this algorithm aims at minimizing an objective function, as follow:

1. generate $U_{v}$ and $C_{v}$;
2. calculate $U_{n}$ minimizing $V\left(U, C_{v}\right)$;
3. calculate $C_{n}$ minimizing $V\left(U_{n}, C\right)$;
4. if the algorithm converges we stop, otherwise we repeat the procedure considering $U_{v}=U_{n}$ and $C_{v}=C_{n}$, and coming again to stage 2 .

Typically, the algorithm converges if

- no changes happen in the matrix $\overline{\bar{U}}$;
- the difference between the objective function at two different iterations is less than a fixed threshold.


### 3.1.2 Fuzzy C-means Algorithm

The binary character of partitions described so far may not always be a convincing representation of the structure of data [76]. Consider the set of two-dimensional patterns illustrated in Fig. 3.5. While we can easily detect three clusters, their character is different. The first one is quite compact, with highly concentrated
patterns. The other two exhibit completely different structures. They are far less condensed, with several patterns whose allocation to a given cluster may be far less certain. In fact, we may be tempted to allocate them to two clusters with varying degrees of membership. This simple and appealing idea forms a cornerstone of fuzzy sets-collections of elements with partial membership in several categories. These sit-


Figure 3.5: Three clusters with outliers of partial membership (pointed by arrows) in the clusters.
uations of partial membership occur quite often. Structures (clusters) may not be well separated for a variety of reasons. There may be noise or lack of discriminatory power of the feature space in which the patterns are represented. Some patterns could be genuine outliers.
The fuzzy algorithms are quite useful in solving certain kinds of clustering problems where a population $\mathbf{X}$ of $n$ objects, each represented by some vector of $s$ numerical features or measurements $x \in \mathbb{R}^{s}$, is to be decomposed into subpopulations (or clusters) of similar objects. The Fuzzy c-means (FCM) algorithms [77] use the set of feature vectors, along with some initial guess about the cluster substructure, to obtain a partitioning of the objects into fuzzy clusters, and as a by-product of the partitioning procedure, produce a prototypical feature vector representing each subpopulation. FCM is known to produce a reasonable partitioning of the original data in many cases. Let us assume that the partition matrix $\mathbf{U}=\left[u_{i k}\right]$ consisting of grades of membership distributed in the unit interval, by following a specific criterion of proximity. In particular let $c \geq 2$ be an integer; let $\mathbf{X}=\left\{x_{1}, \cdots, x_{n}\right\} \in \mathbb{R}^{s}$ be a finite data set containing at least $c<n$ distinct points; and let $\mathbb{R}^{c n}$ denote the set of all real $c \times n$ matrices. A non-degenerate fuzzy c-partition of $\mathbf{X}$ is conveniently
represented by the matrix $\mathbf{U}=\left[u_{i k}\right] \in \mathbb{R}^{c n}$, the entries of which satisfy

$$
\begin{gather*}
u_{i k} \in[0,1] \quad 1 \leq i \leq c ; 1 \leq k \leq n  \tag{3.4}\\
\sum_{i=1}^{c} u_{i k}=1 ; \quad 1 \leq k \leq n  \tag{3.5}\\
\sum_{k=1}^{n} u_{i k}>0 ; \quad 1 \leq i \leq c \tag{3.6}
\end{gather*}
$$

The set of all matrices in $\mathbb{R}^{c n}$ satisfying Eqs. (3.4-6) is denoted by $M_{f c n}$. A matrix $\mathbf{U} \in M_{f c n}$ can be used to describe the cluster structure of $\mathbf{X}$ by interpreting $u_{i k}$ as the grade of membership of $\mathbf{x}_{k}$ in cluster $i$; $u_{i k}=0.95$ represents a strong association of $\mathbf{x}_{k}$ to cluster $i$, while $u_{k}=0.01$ represents a very weak one. Note that $M_{c n}$, the subset of $M_{f c n}$ which contains only matrices with all $u_{i k}$ 's in $[0,1]$, is exactly the set of non-degenerate crisp (or conventional) c-partitions of $\mathbf{X}$. Other useful information about cluster substructure can be conveyed by identifying prototypes (or cluster centers) $\mathbf{v}=\left(\mathbf{v}_{1}, \cdots, \mathbf{v}_{c}\right)^{T} \in \mathbb{R}^{c s}$, where $\mathbf{v}_{i}$; is the prototype for class $i$, $1 \leq i \leq c, \mathbf{v}_{i} \in \mathbb{R}^{s}$. Good partitions $\mathbf{U}$ of $\mathbf{X}$ and representatives ( $\mathbf{v}_{i}$ for class $i$ ) may be defined by considering minimization of one of the family of c-means objective functionals $J_{m}:\left(M_{f c n} \times \mathbb{R}^{c s}\right) \rightarrow \mathbb{R}$ defined by

$$
\begin{equation*}
J_{m}(\mathbf{U}, \mathbf{v})=\sum_{k=1}^{n} \sum_{i=1}^{c}\left(u_{i k}\right)^{m}\left\|\mathbf{x}_{k}-\mathbf{v}_{i}\right\|^{2} \tag{3.7}
\end{equation*}
$$

where $l<m<\infty$ and $\|\cdot\|$ is any inner product induced norm on $\mathbb{R}^{s}$. This approach was first given for $m=2$ in Dunn [78] and then generalized to the above range of values of $m$ in Bezdek [79]. For $m>1$, Bezdek gave the following necessary conditions for a minimum $\left(\mathbf{U}^{*}, \mathbf{v}^{*}\right)$ of $J_{m}(\mathbf{U}, \mathbf{v})$ over $M_{f c n} \times \mathbb{R}^{c s}$ :

$$
\begin{equation*}
\left(\mathbf{v}^{*}\right)=\frac{\sum_{k=1}^{n}\left(u_{i k}^{*}\right)^{m} \mathbf{x}_{k}}{\sum_{k=1}^{n}\left(u_{i k}^{*}\right)^{m}} \quad \forall i \tag{3.8}
\end{equation*}
$$

and for each $k$ such that $d_{i k}^{*}=\| \mathbf{x}_{k}-\mathbf{v}_{i}^{*}>0 \forall i$, then

$$
\begin{equation*}
u_{i k}^{*}=\left[\sum_{j=1}^{c} a_{i j k}^{*}\right]^{-1} \forall i \tag{3.9}
\end{equation*}
$$

where

$$
a_{i j k}^{*}=\left[d_{i k}^{*} / d_{j k}^{*}\right]^{\frac{1}{(m-1)}}
$$

but if $k$ is such that $d_{i k}^{*}=0$ for some $i$, then $u_{i k}^{*} \forall i$ are any non-negative numbers satisfying

$$
\begin{equation*}
\sum_{i=1}^{c} u_{i k}^{*}=1 \quad u_{i k}^{*}=0, d_{i k}^{*} \neq 0 \tag{3.10}
\end{equation*}
$$

The FCM algorithms consists of iterations alternating among Eqs. (3.7) and (3.9). The process is started either with an initial guess for the partitioning $\mathbf{U}$ or an initial guess for the prototype vectors $\mathbf{v}$, and is continued until successive iterates of the partitioning matrix barely differ; that is, iteration stops with the first $\mathbf{U}^{r+l}$ such that $\left\|\mathbf{U}^{r+1}-\mathbf{U}^{r}\right\|<\epsilon$ where $\epsilon$ is a small positive number. The numerical convergence results which follow concern the behavior of the sequences $\left\{\mathbf{U}^{r}\right\}$ and $\left\{\mathbf{v}^{r}\right\}$, while the stochastic theory refers to how well minima of Eq. (3.7) actually represent the cluster substructure of a population under certain statistical assumptions.

### 3.1.3 DBSCAN Algorithm

The Density Based Spatial Clustering of Applications with Noise algorithm [72] can identify clusters in large spatial data sets by looking at the local density of database elements, using only one input parameter. Furthermore, the user gets a suggestion on which parameter value that would be suitable. Therefore, minimal knowledge of the domain is required.
In Density based notion, a cluster is a set of dense points. Cluster can grow in any direction that the density leads, thus density based algorithms can discover clusters of arbitrary shapes. This also provides a natural protection against noisy outliers. The key idea is that for each point of a cluster the neighborhood of a given radius has to contain at least a minimum number of points, i.e. the density in the neighborhood has to exceed some threshold. Before explaining how the algorithm works, we need son introducing definitions. In the rest of this section, we will rewrite almost entirely the original algorithm implementation reported in [72].
Definition 1: The $\epsilon$-neighborhoodof a point $p$, denoted by $N_{\epsilon}(p)$, is defined by $N_{\epsilon}(p)=\{q \in D \mid \operatorname{dist}(p, q) \leq \epsilon\}$.
A naive approach could require for each point in a cluster that there are at least a minimum number (MinPts) of points in an $\epsilon$-neighborhood of that point. However, this approach fails because there are two kinds of points in a cluster, points inside of the cluster (core points) and points on the border of the cluster (border points). In general, an $\epsilon$-neighborhood of a border point contains significantly less points than an $\epsilon$-neighborhood of a core point. Therefore, we would have to set the minimum number of points to a relatively low value in order to include all points belonging to the same cluster. This value, however, will not be characteristic for the respective cluster, particularly in the presence of noise. Therefore, we require that for every point $p$ in a cluster $C$ there is a point $q$ in $C$ so that $p$ is inside of the $\epsilon$-neighborhood of $q$ and $N_{\epsilon}(q)$ contains at least MinPts points. This definition is elaborated in the following.


Figure 3.6: a) Core points and border points and b) directly-density core points and border points.

Definition 2: A point $p$ is directly density-reachable from a point $q$ wrt. (i.e. with respect to) $\epsilon$ and MinPts if:

1. $p \in N_{\epsilon}(q)$ and
2. $\left|N_{\epsilon}(q)\right| \geq$ MinPts (core point condition).

Obviously, directly density-reachable is symmetric for pairs of core points. In general, however, it is not symmetric if one core point and one border point are involved. Fig. 3.6 shows the asymmetric case.
Definition 3: A point $p$ is density-reachable from a point $q$ wrt. $\epsilon$ and MinPts, if there is a chain of points $p_{1}, \cdots, p_{n} ; p_{1}=q, p_{n}=p$ such that $p_{i+1}$ is directly densityreachable from $p_{i}$.
Density-reachability is a canonical extension of direct density-reachability. This relation is transitive, but it is not symmetric. Fig. 3.7 depicts the relations of some sample points and, in particular, the asymmetric case. Although not symmetric in general, it is obvious that density-reachability is symmetric for core points. Two border points of the same cluster $C$ are possibly not density reachable from each other because the core point condition might not hold for both of them. However, there must be a core point in $C$ from which both border points of $C$ are densityreachable. Therefore, we introduce the notion of density-connectivity which covers this relation of border points.
Definition 4: A point $p$ is density-connected to a point $q$ wrt. $\epsilon$ and MinPts if there is a point $o$ such that both, $p$ and $q$ are density-reachable from $o$ wrt. $\epsilon$ and MinPts.

Density-connectivity is a symmetric relation. For density reachable points, the relation of density- connectivity is also reflexive (c.f. Fig. 3.7).
Now, we are able to define our density-based notion of a cluster. Intuitively, a cluster is defined to be a set of density-connected points which is maximal wrt. density-
reachability. Noise will be defined relative to a given set of clusters. Noise is simply the set of points in $D$ not belonging to any of its clusters.
Definition 5: Let $D$ be a database of points. A cluster $C$ wrt. $\epsilon$ and MinPts is a non-empty subset of $D$ satisfying the following conditions:

1. $\forall p, q$ : if $p \in C$ and $q$ is density-reachable from $p$ wrt. $\epsilon$ and MinPts, then $q \in C$.(Maximality)
2. $\forall p, q \in C: p$ is density-connected to $q$ wrt. $\epsilon$ and MinPts. (Connectivity)

Definition 6: Let $C_{1}, \cdots, C_{k}$ be the clusters of the database $D$ wrt. parameters $\epsilon$ and MinPts, $i=1, \cdots, k$. Then we define the noise as the set of points in the database $D$ not belonging to any cluster $C_{i}$, i.e. noise $=\left\{p \in D \mid \forall i: p \notin C_{i}\right\}$.
Note that a cluster $C$ wrt. $\epsilon$ and MinPts contains at least MinPts points because of the following reasons. Since $C$ contains at least one point $p, p$ must be densityconnected to itself via some point $o$ (which may be equal to $p$ ). Thus, at least o has to satisfy the core point condition and, consequently, the $\epsilon$-Neighborhood of o contains at least MinPts points. The following lemmas are important for validating the


Figure 3.7: a) Density-reachability and $b$ ) density-connectivity.
correctness of our clustering algorithm. Intuitively, they state the following. Given the parameters $\epsilon$ and MinPts, we can discover a cluster in a two-step approach. First, choose an arbitrary point from the database satisfying the core point condition as a seed. Second, retrieve all points that are density-reachable from the seed obtaining the cluster containing the seed.
Lemma 1: Let $p$ be a point in $D$ and $\left|N_{\epsilon}(p)\right| \geq$ MinPts. Then the set $O=\{o \mid$ $o \in D$ and $o$ is density-reachable from $p$ wrt. $\epsilon$ and MinPts $\}$ is a cluster wrt. $\epsilon$ and MinPts.
It is not obvious that a cluster $C$ wrt. $\epsilon$ and MinPts is uniquely determined by any of its core points. However, each point in $C$ is density-reachable from any of the core points of $C$ and, therefore, a cluster $C$ contains exactly the points which are
density-reachable from an arbitrary core point of $C$. Lemma 2: Let $C$ be a cluster wrt. $\epsilon$ and MinPts and let $p$ be any point in $C$ with $\left|N_{\epsilon}(p)\right| \geq$ MinPts. Then $C$ equals to the set $O=\{o \mid o$ is density-reachable from $p$ wrt. $\epsilon$ and MinPts $\}$. Ideally, we would have to know the appropriate parameters $\epsilon$ and MinPts of each cluster and at least one point from the respective cluster. Then, we could retrieve all points that are density-reachable from the given point using the correct parameters. But there is no easy way to get this information in advance for all clusters of the database. However, there is a simple and effective heuristic to determine the parameters $\epsilon$ and MinPts of the "thinnest", i.e. least dense, cluster in the database.
Therefore, DBSCAN uses global values for $\epsilon$ and MinPts, i.e. the same values for all clusters. The density parameters of the "thinnest" cluster are good candidates for these global parameter values specifying the lowest density which is not considered to be noise. To find a cluster, DBSCAN starts with an arbitrary point $p$ and retrieves all points density-reachable fromp wrt. $\epsilon$ and MinPts. If $p$ is a core point, this procedure yields a cluster wrt. $\epsilon$ and MinPts (see Lemma 2). If $p$ is a border point, no points are density-reachable from $p$ and DBSCAN visits the next point of the database.
Since we use global values for $\epsilon$ and MinPts, DBSCAN may merge two clusters according to definition 5 into one cluster, if two clusters of different density are "close" to each other. Let the distance between two sets of points $S_{1}$ and $S_{2}$ be defined as $\operatorname{dist}\left(S_{1}, S_{2}\right)=\min \left\{\operatorname{dist}(p, q) \mid p \in S_{1}, q \in S_{2}\right\}$. Then, two sets of points having at least the density of the thinnest cluster will be separated from each other only if the distance between the two sets is larger than $\epsilon$. Consequently, a recursive call of DBSCAN may be necessary for the detected clusters with a higher value for MinPts. This is, however, no disadvantage because the recursive application of DBSCAN yields an elegant and very efficient basic algorithm. Furthermore, the recursive clustering of the points of a cluster is only necessary under conditions that can be easily detected.

### 3.1.4 Hierarchical Clustering Algorithm

Hierarchical clustering solutions which are in the form of trees called dendrograms, are of great interest for a number of application domains. Hierarchical trees provide a view of the data at different levels of abstraction [80].
Hierarchical clustering solutions have been primarily obtained using agglomerative algorithms[81, 82], in which objects are initially assigned to their own cluster and then pairs of clusters are repeatedly merged until the whole tree is formed.
As we mentioned previously, hierarchical algorithms are subdivided into agglomer-
ative hierarchical algorithms and divisive hierarchical algorithms (see Fig. 3.8 [83]). Agglomerative hierarchical clustering starts with every single object in a single cluster. Then it repeats merging the closest pair of clusters according to some similarity criteria until all of the data are in one cluster. There are some disadvantages for agglomerative hierarchical clustering, such as (a) data points that have been incorrectly grouped at an early stage cannot be reallocated and (b) different similarity measures for measuring the similarity between clusters may lead to different results. If we treat agglomerative hierarchical clustering as a bottom-up clustering method, then divisive hierarchical clustering can be viewed as a top-down clustering method. Divisive hierarchical clustering starts with all objects in one cluster and repeats splitting large clusters into smaller pieces. Divisive hierarchical clustering has the same drawbacks as agglomerative hierarchical clustering.
Hierarchical algorithms can be expressed in terms of either graph theory or matrix algebra [84].


Figure 3.8: Agglomerative hierarchical clustering and divisive hierarchical clustering.

### 3.1.5 Data Spectroscopic Algorithm

In this paragraph, we provide only a briefly description of the Data Spectroscopic Algorithm (DS), recalling some fundamentals from the original work by Shi, Belkin and Yu [85].
This algorithm takes as input a data set $\mathbf{X}=\left\{x_{1}, \cdots, x_{n}\right\} \subset R^{d}$. From this data an $n \times n$ affinity matrix $\mathbf{K}=\left[K_{n}\right]_{i j}$ is constructed where each entry is a function of the Euclidean distance between two data points;

$$
\begin{align*}
\left(K_{n}\right)_{i j} & =\frac{K\left(x_{i}, x_{j}\right)}{n}  \tag{3.11}\\
K(x, y) & =\exp ^{-\frac{\|x-y\|^{2}}{2 \sigma^{2}}} \tag{3.12}
\end{align*}
$$

being $\mathbf{K}(x, y)$ the Gaussian kernel with variance $\sigma^{2}$. The eigenvalues $\lambda_{1}, \lambda_{2}, \cdots, \lambda_{n}$ and eigenvectors $\mathbf{e}_{1}, \mathbf{e}_{2}, \cdots, \mathbf{e}_{n}$ of the affinity matrix $\mathbf{K}_{n}$ are then calculated. The number of clusters is then estimated by the number of eigenvectors $\mathbf{e}_{j}$ which have no sign changes up to precision $\epsilon$. A vector $\mathbf{e}=\left(e_{1}, e_{2}, \cdots, e_{n}\right)$ has no sign changes up to precision $\epsilon$ if either $\forall i e_{i}>-\epsilon$ or $\forall i e_{i}<\epsilon$.
The most important feature of this algorithms is not properly the suitability with our scopes, rather than estimate the number of clusters (such as in our case the number of cylinders), to give as input to the other clustering methods. In fact, the DS Algorithm, is the only, among those we implemented, to have no a priori condition on the number of clusters.

### 3.1.6 Comparison of Clustering Algorithms

The analysis of clustering methods and the comparison among their performances, is a very discussed topic in literature since a long time [86, 75], and it is not possible to define a definitive performance comparison method; moreover this kind of problem is out of the topics of the present work. Anyway, during our simulations, we tried to extrapolate some general statements for what concerns the application of these method to our purpose. In particular, we did not find a drastic performance gap among the algorithms, observing that Data Spectroscopic methods performs definitely worse than the other, but, on the contrary, it is the only algorithm of those we implemented, that is able to find the number of clusters (an then the number of cylinders). On the other hand, we find the k-means algorithm performing better in many cases, so that we will refer to it in the following paragraphs, for what concerns the data analysis representation.
There are number of methods available for comparing partitions, but they are usually computational intensive; the most popular one is described as Hubert and Arabie's modified index [87]. For example, considering a two-cluster classification, all pairs of a dataset must be compared; if there are $n$ data points and then $\binom{n}{2}$ comparisons must be made, thus the method is an $O\left(n^{2}\right)$ computation [88].
The feature common to all clustering methods (except DS algorithm) is that the number of clusters is specified, then the algorithms only partition the data points into clusters. An effective full comparison among the algorithms, can be made by allowing the number of clusters to very into finite range of values. In our problem, we have a fixed number of clusters, even if for practical uses, the number of buried cylinder is often unknown a priori, so that we contemplate to extend this comparison in future works.
For a fixed number of clusters, any clustering algorithm assigns every data point
to a cluster. Comparing two methods, we must measure the so-called inter-rater agreement: different raters can disagree about measurement results from the same object by e.g. variations in the procedures of carrying out the experiment, interpreting the results and, subsequently, presenting them. All these stages may be affected by experimenter's bias, that is, a tendency to deviate towards what is expected by the rater. The standard measure of inter-rater agreement is Cohen's $\kappa$ statistic [89], that can be interpreted as an interclass correlation [90] measurement.
Cohen's $\kappa$ measure pairwise agreement among a set of raters, is expressed by

$$
\begin{equation*}
\kappa=\frac{P_{o}-P_{e}}{1-P_{e}} \tag{3.13}
\end{equation*}
$$

where $P_{o}$ is the observed probability of agreement between the two raters and $P_{e}$ is the expected probability of agreement under the assumption of independent rating by the two raters. Of course, $\kappa$ must be less or equal to 1 , and its lower bound depends on $P_{e}$, moreover the grater is $\kappa$, the fairer is the agreement between raters. So that, maximizing $\kappa$, the observed agreement probability is large and the clustering is performed quite the same by the two algorithms.
On top on this, when the number of points in dataset is very large ( $n \geq 100$ ), $\kappa_{\max }$ is approximately zero by the Law of Large Numbers. In particular, the rate at which the statistic approaches zero can be computer for two clusters case. As demonstrated in [88], the expected value of $\kappa_{\max }$, is equal to

$$
\begin{equation*}
E\left[\kappa_{\text {max }}\right]=2^{-\frac{n}{2}}\binom{n / 2}{n / 4} \approx(2 \sqrt{n \pi})^{-1} \quad \text { for large } n \tag{3.14}
\end{equation*}
$$

Since our datasets are typically grater than 300 units, Eq. (3.14) leads to $\kappa_{\max } \approx$ 0.0163 , that means that the observed probability of agreement for large datasets is actually very close to the expected probability of agreement, or in other words, for a large number of data points and two specified clusters, the partitioning of data points into the two clusters is quite the same for all the raters, and the number of outliers (non concordant partitions) is non influent. For such a region we consider assuming any peculiar significance, at least for the present work, a complete description of all the simulated cases associated with all the implemented clustering algorithms, since in our intention, we tried to demonstrate the possibility to face the localization problem as presented in Chapter 1, by using some statistical classification methods when a multiple target localization occurs. Anyway, in our simulations we noticed an outperforming of k -means compared with other methods in the most of cases. In fact, since the target we are trying to locate are cylinders, those algorithms that seek spherical shapes (i.e. k-means) typically do better than methods seeking cluster of arbitrary shape.

### 3.2 Simulation settings

In the following paragraphs the localization of two PEC or dielectric cylinders will be presented. In particular we consider two different scenarios: at first the cylinders are buried into a homogeneous medium having refractive index $n_{m}=2$ and successively the refractive index is fixed to $n_{m}=3$. The cylinders are supposed to have three different radii, that will be combined to analyze several combinations. The radius size is chosen to be $a_{1}=0.25 \lambda_{0}, a_{2}=0.5 \lambda_{0}$, or $a_{3}=\lambda_{0}$, and so there are six possible values for the quantity $\left(a_{i}-a_{j}\right)$, where $i=1, \cdots, 3$ and $j=1, \cdots, 3$. A cylinder axis has coordinates $(-10 \pi,-h)$ where $h=F F+a_{i}$ (being FF the far-field distance, see Eq. (2.1) of Chapter 2). The second cylinder has the same ordinate and abscissa varies in the range $[-5 \pi, 10 \pi]$ with a step resolution of $0.25 \pi$ that is, in $k_{0}$ units equals to $\lambda_{0} / 8\left(k_{0} \lambda_{0} / 8=\left(2 \pi / \lambda_{0}\right)\left(\lambda_{0} / 8\right)\right)$, for total 61 points. Normalizing with respect of $\pi$, we define a horizontal offset $\Delta \eta \in[-5,10]$.
With this strategy, we can make several assessments about the procedure performances vs. horizontal proximity of the cylinders and also vs. the reciprocal size. In fact, it is quite obvious that the larger object should be better located.
Subsequently, we provide a second set of simulations in which the ordinate of the second cylinder is varied while the position of the first cylinder is fixed. In these scenarios, the abscissa of both the cylinders does not vary, and it is equal to $\pm 5 \pi$. Thus, as the vertical co-ordinate depends on the far-field distance $F F$, we adopt a sweep in the range $\left[-F F-a_{i},-F F-5 a_{1}\right.$ ], with a resolution of $0.1 \pi$ (that is equal to $\lambda_{0} / 20$ in $k_{0}$ units), for total 51 points. So that, we define a vertical offset $\Delta \chi \in[0,5]$ for this kind of simulations, considering only a configuration of equal-size cylinders. In dielectric cylinders case, the geometrical combinations are repeated also for different combinations of the cylinder refractive indexes. In fact, once fixed the refractive index of the ground, we simulate two different scenarios, involving two main categories: two dielectric cylinders having the same and different refractive indexes. In both the cases we simulate all the radii combinations, varying $\Delta \eta$, and only the equal-radii pairs, for variations of $\Delta \chi$.
So that, for $n_{m}=2$, we take into account the cylinders refractive indexes $n_{c i}=1 ; 3$, where $i$ indicates the cylinders. We consider these cases very significant, since they represent a low-dielectric-contrast configuration, and also the presence of cavities. Our purpose is to check the localization procedure performances for dielectric objects but also for cavities, which was well-detected in single-object cases.
Furthermore, we present the outcomes considering the localization error defined in Eq. (2.4) for both the cylinders and for all the implemented methods. Actually, the implementation of ESPRIT is now changed; in fact, we provide an optimization in
the double-localization case of this method, choosing a 19 -element array partitioned in 16 doublets each made of 4 elements. The resulting number of estimated DoA is 16 , that is only one less the other methods. This variation is required by the clustering approach, since the number of crossings cannot be too high otherwise the implemented methods are not able to identify separate clusters of crossings. Nevertheless, this revised ESPRIT method, allow us to perform an equivalent analysis by reducing the overall number of sensors drastically.

### 3.3 Localization of Two PEC Cylinders

The double-PEC cylinders localization, with reference to a surrounding medium with refractive index $n_{m}=2$, shows to be quite effective in presence of a horizontal offset, as indicated in Figs. 3.9-14. In particular, in most of the cases, the localization error does not cross the threshold. By considering the following figures, some assessments can be pointed out; in particular, two outcomes are evident: first of all, when the double-objects configuration involves cylinders having different sizes, the larger is the object the better it is localized, and secondly the moving cylinder is better localized.
Not much can be exposed further about the first point, but it being understood that the proving of a such predictable behavior is anyway a result that could not be stated a priori. It is instead more significant the second issue; in fact, this unbalanced localization could be intuitive for the different sizes configurations, and too hastily ascribed to that, but it happens also in equal-size cases, and so more attention must be paid on it. A possible explanation is that the moving cylinder "attracts" more DoA that the fixed one. In fact, the fixed cylinder is placed at the edge of the array, limiting the number of sub-arrays that could be able to detect it. On the contrary, the moving cylinder can be detected not only by the overhanging sub-arrays but also by the peripheral one laying on the other edge.
Moreover, the correlation between signals has to be taken into account. In fact, when the two cylinders are close (within a distance of 3-4 $\lambda_{0}$ ), the fields scattered by the cylinders toward the group of sub-array overhanging the objects and in-between of them, is greatly correlated, inducing false DoA estimations. When the horizontal offset increases, the scattered fields are poorly correlated and the estimation algorithms perform a better estimation. We must notice that the localization error of the left column of Figs. 3.9-3.14, shows a "step" of the estimation error in correspondence with this distance; moreover the horizontal range distance at which the field is highly correlated increases as the refractive index of the hosting medium
increases (see Figs. 3.18-3.29).
This produces a larger amount of crossings identifying the moving object cluster and thus a better localization is achieved. In addition, only those sub-arrays overhanging the fixed cylinder, can receive the correlated field, because it is mainly focused in the in-between of the two cylinder, so that in Figs. 3.9-3.14 the fixed cylinder localization is less precise. Thus, it is more significant in terms of localization performances vs. $\Delta \eta$, the first fixed cylinder behavior, and in this context we can observe that the horizontal resolution capability of the procedure depends on the fixed cylinder size. In particular, in Fig. 3.9a)-3.9f) the localization error is under the threshold for $\Delta \eta \geq 4$, and going further the error becomes homogeneous as one could expect since the cylinder size is the same. The error levels of the all methods is quite the same (except P.H.D.), and all the most for parametric methods. It is also interesting that the resolution remains the same also in Figs. 3.10-3.11, confirming that it depends on the fixed cylinder size. In fact, in the right-side pictures, we can appreciate that the average error decreases as the moving cylinder increases, but the value of $\Delta \eta$ since which both the cylinders are correctly localized, does not change. In Fig. $3.12 a)-3.12 f$ ) the correct-double-localization occurs as a shorter horizontal offset is reached, and more precisely since $\Delta \eta=2 \pi$. Even though the overall method performances are improved, profiting by the radius increment, P.H.D. estimation is still the less precise seems to be not able to localize also medium-size objects. In Figs. 3.13-3.14, a further reduction of the localizing threshold is achieved, and when the cylinders have the larger dimensions, the localization is almost uniform along $\Delta \eta$. Anyway, a clear worse performing can be observed for the non-parametric methods and all the most for the P.H.D. one.
For what concerns the vertical sensitivity, an unexpected behavior has been observed. In fact, even if the starting configuration geometry is absolutely symmetric (in the first iteration the cylinders are placed at $(-5 \pi,-h)$ and $(5 \pi,-h)$, respectively), the localization error for the two cylinders is very different.


Figure 3.9: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.10: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$ and $a_{2}=0.5 \lambda_{0}$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.11: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$ and $a_{3}=\lambda_{0}$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.12: Localization of two PEC cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.13: Localization of two PEC cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$ and $a_{3}=\lambda_{0}$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.14: Localization of two PEC cylinders having radii equal to $a_{3}=\lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.15: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.

In fact, in Fig. 3.15a)-3.15f), on the left-side pictures (relevant to the fixed cylinder) the localization performs quite good, excepted for some DoA estimation methods (i.e. P.H.D. and root-MUSIC), while in the right-side the error is higher. This could be well explained if the higher-error region were localized toward the right size of the plot, corresponding to the zone in which the cylinder is more distant from the array, but on the contrary, the error trend seems to be oscillating quite uniformly in a definite range $\Delta \chi \in[2,4]$.
A possible interpretation of such a curious trend, can be associated not just to the vertical offset sensitivity of the procedure, but to the capabilities of the clustering algorithms to deal with symmetric clustering dataset. The same happens in Figs. 3.16-3.17 even if in these two groups of figures, an error decrease in the left-side pictures can be appreciated. Moreover, all the DoA methods appear more sensitive to the vertical variation, and the non-parametric methods become significantly less affordable. On the contrary both parametric methods and ESPRIT behave quite regularly even though affected by the ambiguity previously described. This is a good result, in particular for ESPRIT performance, because it could allow us to use a lower number of antenna elements.
When a medium having refractive index $n_{m}=3$ is considered, the localization suffers from a performance worsening. Moreover, we report only the equal-size cases, since the general behavior has been well-defined during the previous analysis. So that, in Figs. 3.18-3.20 a higher average estimation error is observed. In particular, even if the error is still decreasing as $\Delta \eta$ increases, it goes under the threshold, only for some DoA methods, and generally the localization precision is poor. In particular, a lack of accuracy can be noted for non-parametric methods, MUSIC, root MUSIC and sometimes ML-based methods (that in this case are differentiated), but ESPRIT seems to be more robust. This point suggests that the outer elements of the array could affect the localization as the refractive index of the burial medium increases. For what concerns the sensitivity to the vertical offset, a similar behavior of the previous $n_{m}=2$ case can be derived. In Figs. 3.21-3.23 can be appreciated how the ESPRIT method is outperforming if compared with the other methods.
Even though the localization in this case is not accurately provided, the procedure is still able to detect the objects. In fact, the average error for methods represented in Figs. 3.18-3.20, varies in the interval $\operatorname{err}_{i} \in[5,15]$, where $i=1,2$, and it is inversely proportional to the radius. Consequently, the average distance gap between the estimated co-ordinates of the object center and the real one is at most $5 \lambda_{0}$. Furthermore, it has to be pointed out, that the error committed on estimating the vertical co-ordinate (Figs. 3.21-3.23) is quite mitigated in this case (excepted for
some non-parametric methods and root MUSIC), suggesting a problematic identification of the cluster abscissas by the clustering methods. It could be interesting to repeat these simulations by using more efficient clustering methods, considering also the possibility to derive one ad-hoc algorithm.
Another damaging factor comes into play as the refractive index of the burial medium increases, and in particular it is relevant to the DoA estimation model. In fact, the simplified scenario which does not consider the effects due to the ground-air planar interface, is no more able to fit the propagation in a dense medium, and the geometrical optic approximation implicitly adopted, becomes improper. In particular, on one side, the electromagnetic-field wavelength in the medium hosting the cylinders becomes smaller when the refractive index is higher, so the object is electrically larger. On the other hand, as the refractive index of the homogeneous medium increases, the limit angle at the planar interface reduces the transmission from the medium to the array (since the transmission between a more dense medium and a less dense medium is interdicted by the limit angle towards the less dense medium). A more rigorous analysis of electromagnetic propagation in stratified media would be necessary.


Figure 3.16: Localization of two PEC cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.17: Localization of two PEC cylinders having radii equal to $a_{3}=\lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.18: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.19: Localization of two PEC cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.20: Localization of two PEC cylinders having radii equal to $a_{3}=\lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.21: Localization of two PEC cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.22: Localization of two PEC cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.23: Localization of two PEC cylinders having radii equal to $a_{3}=\lambda_{0}$, buried in a homogeneous medium with $n_{m}=3$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.

### 3.4 Localization of Two Dielectric Cylinders

In this section some results relevant to the localization of two dielectric cylinders embedded in a homogeneous medium having refractive index $n_{m}=2$ are presented. In particular we analyze different configurations in which the objects are supposed to have different size and the different refractive index. The first group of figures, refer to the case of two cylindrical cavities having 6 different combinations of radii, by partitioning the values $a_{1}, a_{2}$ and $a_{3}$ in any way, where the localization error is reported vs. $\Delta \eta$. In Figs. 3.24-3.26, the estimation of the standing cylinder is quite corrupted within a distance of about $4 \lambda_{0}$ between the cylinders. This is most of all evident when the cylinder is small (see the left column of Figs. 3.24-3.26) because the field received by the overhanging sub-arrays is more affected by the field scattered by the moving cylinder, since the correlation between the scattered signals is still present but less prominent than the PEC case. In fact, when the dimensions of the two cylinders are similar, the cause of a damaged estimation is due mainly to the effect of the correlated field, but in these cases (see Figs.3.27-3.29, left column) the localization error decreases.

Almost the same trend can be deduced by looking Figs. 3.30-3.35, when the presence of two dielectric cylinders having refractive index $n_{c}=3$ is simulated. The localization is less precise, because the permittivity of the objects is now larger than the permittivity of the medium, anyway the "step" behavior of the estimation error previously described is still clear. It is quite interesting, by looking at Figs. 3.33 and 3.35 , that medium-size objects are better localized than larger objects. In fact, the far-field region is fixed to be $n_{m} \lambda_{0} / 2$ by Eq. (2.1), and it is good since the dimension of the cylinders (or even one of them) are smaller than or equal to this quantity. Since the radii of the cylinders are normalized to $\lambda_{0}$, the electric dimension of the cylinders depends on the value of $n_{m}$, that fixes the threshold of a correct far-field distance for $a_{i} \leq 3 \lambda_{0} / 4$ (that is the case of Fig. 3.33). Instead, when $a_{i}=a_{3}$ the two objects are in the near-field of some sub-arrays, and the estimation is further worsened.
In Figs. 3.36-3.38, two cylinders with different refractive indexes are simulated. The standing cylinder is a cavity and the moving one has $n_{c}=3$. With this group of simulations, we intend to check the procedure in order to have information about the simultaneous localization of two different cylinders, once fixed the dielectric contrast. The cylinders have same dimensions $a_{i}=a_{1}, a_{2}, a_{3}$ for the figures, respectively. We can see hat the "step" in the localization error trend is still present, but out of the high-correlated field region,the procedure is quite affordable, even if the cavity is better localized, es observed in the previous chapter.


Figure 3.24: Localization of two dielectric cylindrical cavities having radii equal to $a_{1}=0.25 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.25: Localization of two dielectric cylindrical cavities having radii equal to $a_{1}=0.25 \lambda_{0}$ and $a_{2}=0.5 \lambda$, respectively, buried in a homogeneous medium with $n_{m}=$ 2; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.26: Localization of two dielectric cylindrical cavities radii having equal to $a_{1}=0.25 \lambda_{0}$ and $a_{3}=\lambda$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.27: Localization of two dielectric cylindrical cavities having radii equal to $a_{2}=0.5 \lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.28: Localization of two dielectric cylindrical cavities having radii equal to $a_{2}=0.5 \lambda_{0}$ and $a_{3}=\lambda$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.29: Localization of two dielectric cylindrical cavities having radii equal to $a_{3}=\lambda_{0}$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.30: Localization of two dielectric cylinders having radii equal to $a_{1}=$ $0.25 \lambda_{0}$, and refractive index $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.31: Localization of two dielectric cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$ and $a_{2}=0.5 \lambda$, respectively, and refractive index $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.32: Localization of two dielectric cylinders having radii equal to $a_{1}=0.25 \lambda_{0}$ and $a_{3}=\lambda$, respectively, and refractive index $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.33: Localization of two dielectric cylinders having radii equal to $a_{2}=0.5 \lambda$, and refractive indexes $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.34: Localization of two dielectric cylinders having radii equal to $a_{2}=0.5 \lambda_{0}$ and $a_{3}=\lambda$, respectively, and refractive indexes $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.35: Localization of two dielectric cylinders having radii equal to $a_{3}=$ $\lambda$, and refractive index $n_{c}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.36: Localization of two dielectric cylindrical objects having radii equal to $a_{1}=0.25 \lambda$, and refractive indexes $n_{c 1}=1$ and $n_{c 2}=3$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.37: Localization of two dielectric cylindrical objects having radii equal to $a_{2}=0.5 \lambda$, and refractive indexes $n_{c 1}=1$ and $n_{c 2}=3$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.38: Localization of two dielectric cylindrical objects having radii equal to $a_{3}=\lambda$, and refractive indexes $n_{c 1}=1$ and $n_{c 2}=3$, respectively, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. horizontal position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.

In the following Figs. 3.39-3.41, the localization error vs. distance $\Delta \chi$ of two equalsize cavities is simulated. In particular, the left column refers to the standing object, while the right one pertains the moving object. The horizontal distance between the center of the cavities is fixed. As it is clear, the estimation of the position of the standing cylinder is more precise, and the same happens when the objects are supposed to be dielectric cylinders with $n_{c}=3$ (see Figs. 3.42-3.44), even if in this case, the estimation performances are worse.
In conclusion, we extend the localization procedure described in the previous chapters, for a double-cylinder configurations, by using some clustering analysis algorithms. The simulation results, showed that the estimation of the position is quite effective in most of the presented cases. We varied both the size of the cylinders and its refractive index, performing the analysis for various horizontal distances and depths.
In particular, better performances are highlighted when perfectly conducting objects or cavities are simulated. Possible future works, could take into account mixed configuration involving both PEC and dielectric cylinders; moreover, also the number of the objects could be increased in order to test the clustering analysis algorithm performances.


Figure 3.39: Localization of two different dielectric cylindrical objects having radii equal to $a_{1}=0.25 \lambda$, and refractive indexes $n_{c 1}=n_{c 2}=1$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.40: Localization of two different dielectric cylindrical objects having radii equal to $a_{2}=0.5 \lambda$, and refractive indexes $n_{c 1}=n_{c 2}=1$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.41: Localization of two dielectric cylindrical objects having radii equal to $a_{3}=\lambda$, and refractive indexes $n_{c 1}=n_{c 2}=1$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.42: Localization of two different dielectric cylindrical objects having radii equal to $a_{1}=0.25 \lambda$, and refractive indexes $n_{c 1}=n_{c 2}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position:
a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.43: Localization of two dielectric cylindrical objects having radii equal to $a_{2}=0.5 \lambda$, and refractive indexes $n_{c 1}=n_{c 2}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; d) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; f) WSF, DML, SML center estimations of the moving cylinder.


Figure 3.44: Localization of two dielectric cylindrical objects having radii equal to $a_{3}=\lambda$, and refractive indexes $n_{c 1}=n_{c 2}=3$, buried in a homogeneous medium with $n_{m}=2$; estimation error for the two centers vs. vertical position: a) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the fixed cylinder; b) Bartlett, Capon, Linear Prediction, Maximum Entropy, Minimum Norm, P.H.D. center estimations of the moving cylinder; c) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the fixed cylinder; $d$ ) MU.SI.C., root MU.SI.C., ESPRIT center estimations of the moving cylinder; e) WSF, DML, SML center estimations of the fixed cylinder; $f$ ) WSF, DML, SML center estimations of the moving cylinder.

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## Part II

Design of Advanced Electromagnetic Components for Nuclear Fusion Applications

## Chapter 1

## ITER Lower-Hybrid Current-Drive System

Since many years, the research on thermonuclear fusion technology has become a challenging field of application and development for the design of new microwave components. The present work has been developed within the EURATOM-ENEA"Roma Tre" University collaboration for the activities of the European Fusion Development Agreement (EFDA) task WP08-HCD-03-01 (EU Contribution to the ITER Lower Hybrid Current Drive Development Plan), concerning the International Thermonuclear Experimental Reactor (ITER) project.
Before starting to explain the contents of our contribution in terms of analysis and design of microwave components for the Lower Hybrid and Current Drive (LHCD) system of ITER, a brief introduction about nuclear fusion technology and development is presented in this chapter, focusing on the additional radio-frequency (RF) heating sources, in particular at the lower hybrid resonance frequency [1]. Furthermore we describe in detail, the overall LH heating transmission line system, that has the job to carry several megawatt of RF power from the generators to plasma, exciting the resonance frequency of plasma with two important goals: first of all a massive heating of plasma, secondary but not less outstanding, electromagnetic power injection at this particular frequency can be used to realize a non-inductive current drive (NICD) to achieve a quasi steady-state regime. The power is coupled to plasma by using a particular launcher formed by an array of antennas (called Passive-Active Multi-junction or PAM [2,3]), that has the peculiarity to radiate a non-evanescent electromagnetic field only when facing plasma with specified characteristics; this device is the hard core of the whole system and it is engineered with respect of strict constrains in term of power efficiency and then, the transmission losses have to be imperatively minimized.

We have contributed to the ITER ambitious project, by analyzing and synthesizing some fundamental microwave components such as the Combiner-Splitter network, mode filters and bends in oversizes rectangular waveguide. The final scope is to maximize the power transmission from generators to the launcher reducing as much as possible the losses in terms of attenuation and spurious modes excitation, with respect to the system constraints.

### 1.1 Basics of Tokamak Machines

Under the proper conditions the low atomic number elements will react to convert mass to energy $\left(E=m c^{2}\right)$ via nuclear fusion. For example, the fusion of the hydrogen isotopes deuterium (D) and tritium ( T ) according to the reaction

$$
D+T \rightarrow^{4} \mathrm{He}+n \quad(Q=17.6 \mathrm{MeV})
$$

produces 17.6 MeV of energy. The fusion of $1 g$ of tritium together with $2 / 3 g$ of deuterium produces $1.6 \times 10^{5} \mathrm{kWh} h^{-1}$ of thermal energy [4]. As we can see by looking at Fig. 1.1, this kind of fusion reaction will provide much more energy that the uranium-iron fission reaction. In order for a fusion reaction to take place the two nuclei must have enough energy to overcome the repulsive Coulomb force acting between the nuclei and approach each other sufficiently close that the short-range attractive nuclear force becomes dominant. Thus, the fusion fuel must be heated to high temperatures. For the D-T reaction, the gas temperature must exceed 100 million degree Celsius before a significant fusion rate is feasible. An even higher gas temperature is required for the other fusion reactions. At such temperatures the gas exists as a macroscopically neutral collection of ions and unbound electrons which is called a plasma.
A fusion plasma cannot be maintained at thermonuclear temperatures if it is allowed to come in contact with the walls of the confinement chamber, because material eroded from the walls would quickly cool the plasma. Fortunately, magnetic fields can be used to confine a plasma within a chamber without contact with the wall. A charged particle moving in a magnetic field will experience a Lorentz force which is perpendicular to both the direction of particle motion and to the magnetic field direction. This force does not affect the component of particle motion in the magnetic field direction, but it causes acceleration at right angles to the particle direction in the plane perpendicular to the magnetic field direction, producing a circular particle motion in that plane. Thus, a particle in a magnetic field will move along the field and circle about it; that is, will spiral about the field line. The radius of the spiral,


Figure 1.1: Binding energy curve.
or gyroradius, is inversely proportional to the strength of the magnetic field, so that in a strong field charged particles move along magnetic field lines.
A tokamak (Toroidal Kamera Magnitnaya, invented by Sakharov and Tamm in the 1950s) is a toroidal chamber which uses a strong toroidal magnetic field, $B_{\phi}$, to contain a high temperature plasma within the torus. However, this field alone does not allow confinement of the plasma. In order to have an equilibrium in which the plasma pressure is balanced by the magnetic forces it is necessary also lo have a poloidal magnetic field in a tokamak this field is produced mainly by current in the plasma itself, this current flowing in the toroidal direction. The combination of the toroidal field $B_{\phi}$ and poloidal field $B_{\theta}$, gives rise to magnetic field lines which have a helical trajectory around the torus [5].
The toroidal field is produced by a set of toroidal field coils which encircle the plasma. The poloidal field is produced by an axial current which is induced by the transformer action of a set of primary poloidal field, or ohmic heating, coils, so the plasma can be viewed as the secondary winding of a transformer. This concept is illustrated in Fig. 1.2.
Since the plasma is an electrical conductor, the current flow through it, causes an ohmic (or resistive) heating; it is the same kind of heating that occurs in an electric light bulb, for instance. The heat generated depends on the resistance of the plasma and the current. But as the temperature of heated plasma rises, the resistance decreases and ohmic heating becomes less effective. It appears that the maximum plasma temperature attainable by ohmic heating in a tokamak is $20-30$ million degrees Celsius. To obtain still higher temperatures (20keV $\approx 230$ million degree Celsius), additional heating methods must be used.
Other heating systems are Neutral Beam Injection (NBI), consisting in neutral par-


Figure 1.2: Magnetic confinement in a Tokamak machine.
ticles (typically a hydrogen isotope such as deuterium) at high energy shot into the core of the plasma to transfer their energy to the plasma, raising the overall temperature, and RF heating, including Electron-Cyclotron Resonance heating (ECRH), Ion-Cyclotron Resonance heating (ICRH) and Lower-Hybrid Resonance heating (LHRH). These three additional RF systems, differs among them from the particular frequency used to excite the plasma at its resonances. In the following section we present some outlines of plasma wave theory, just to have some physical basics to understand the technological problem that have to be dealt with in nuclear fusion engineering. More detailed plasma physic features are out of the purpose of this work, but some exhaustive literature is reported in the Bibliography [6, 7].

### 1.1.1 Radio-Frequency Heating

Radio-frequency heating depends on the transfer of energy from electromagnetic waves generated by an external source to particles at suitable resonance frequencies. Resonance absorption of wave energy does not involve collisions and unlike ohmic heating, the process becomes more efficient with increasing temperature. A multi-species plasma in a magnetic field has several resonance frequencies capable of absorbing the energy of incident waves.
The fundamental properties of waves in plasma can be understood by considering low-frequency wave phenomena in a cold, uniform, uniformly magnetized, infinite, and homogeneous plasma which has an equilibrium solution characterized by an electric field $E_{0} \approx 0$ and a constant magnetic field $B_{0}$. The plasma is not isotropic, however, since the presence of a magnetic field provides one preferred direction. Without the magnetic field, the plasma may be represented by a simple dielectric
constant and the only wave solution is a simple electromagnetic wave that propagates above the plasma frequency, $\omega_{p e}$. The governing equations for perturbed solutions are Amperes and Faradays laws,

$$
\begin{align*}
& \nabla \times \mathbf{B}=\mu_{0} j+\frac{1}{c^{2}} \frac{\partial \mathbf{E}}{\partial t}  \tag{1.1}\\
& \nabla \times \mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}
\end{align*}
$$

together with the equation of motion for a particle of species $l=i, e$ and charge $q_{l}$ (being $i$ positive ions and $e$ electrons) in an electromagnetic field (Lorentz law):

$$
\begin{equation*}
m_{l} \frac{d v_{l}}{d t}=q_{l}\left(\mathbf{E}+\mathbf{v}_{l} \times \mathbf{B}\right) \tag{1.2}
\end{equation*}
$$

and the expression for the total current is

$$
\begin{equation*}
\mathbf{J}=\sum_{l} n_{l} q_{l} \mathbf{v}_{l} \tag{1.3}
\end{equation*}
$$

where the sum is over the species number $n_{l}$. Since the plasma has been presumed to be uniform and homogeneous in both space and time, we may Fourier transform these equations or what is equivalent, assume that

$$
\begin{align*}
\mathbf{E} & =\mathbf{E}_{\mathbf{1}} e^{j(\mathbf{k} \cdot \mathbf{r}-\omega t)} \\
\mathbf{B} & =\mathbf{B}_{\mathbf{0}}+\mathbf{B}_{\mathbf{1}} e^{j(\mathbf{k} \cdot \mathbf{r}-\omega t)}  \tag{1.4}\\
\mathbf{v} & =\mathbf{v}_{\mathbf{1}} e^{j(\mathbf{k} \cdot \mathbf{r}-\omega t)}
\end{align*}
$$

and $\mathbf{B}_{0}$ is the static magnetic field assumed to propagate in $z$-direction and $\left|\mathbf{B}_{1}\right| \ll$ $\left|\mathbf{B}_{0}\right|$. Now, by inserting the previous Eq. (1.4) in the equation of motion, Eq. (1.2), we obtain the linear form as

$$
\begin{equation*}
-j \omega m_{l} \mathbf{v}_{1 l}=q_{l}\left(\mathbf{E}_{1}+\mathbf{v}_{1 l} \times \mathbf{B}_{0}\right) \tag{1.5}
\end{equation*}
$$

where the second order terms have been neglected because we have assumed the waves having sufficiently low amplitude. Projecting the velocity onto the axes of a Cartesian system co-ordinates, we lead the following equations

$$
\begin{align*}
v_{x l} & =\frac{j q_{l}}{m_{l}\left(\omega^{2}-\omega_{c l}^{2}\right.}\left(\omega E_{x}+j \kappa_{l} \omega_{c l} E_{y}\right) \\
v_{y l} & =\frac{j q_{l}}{m_{l}\left(\omega^{2}-\omega_{c l}^{2}\right.}\left(\omega E_{y}-j \kappa_{l} \omega_{c l} E_{x}\right)  \tag{1.6}\\
v_{z l} & =\frac{j q_{l}}{m_{l} \omega} E_{z}
\end{align*}
$$

where $\kappa_{l}=q_{l} /\left|q_{l}\right|$ denote the sign of the charge for species $l$ and $\omega_{c l}=\left|q_{l}\right| B_{0} / m_{l}$ is the cyclotron frequency for species $l$. Introducing the rotating co-ordinates such that $\binom{v}{E}_{ \pm}=\binom{v}{E}_{x} \pm j\binom{v}{E}_{y}$, we may write both $v_{x}$ and $w_{y}$ of the previous Eq. (1.6) as

$$
\begin{equation*}
v_{ \pm}=\frac{j q_{l}}{m_{l}\left(\omega \mp \kappa_{l} \omega_{c l}\right)} E_{ \pm} \tag{1.7}
\end{equation*}
$$

Similarly, the current density of Eq. (1.3) may be written as

$$
\begin{align*}
J_{ \pm} & =j \epsilon_{0} \sum_{l} \frac{\omega_{p l}^{2}}{\left(\omega \mp \kappa_{l} \omega_{c l}\right)} E_{ \pm}  \tag{1.8}\\
J_{z} & =j \epsilon_{0} \sum_{l} \frac{\omega_{p l}^{2}}{\omega} E_{ \pm}
\end{align*}
$$

where

$$
\omega_{p l}^{2}=\frac{n_{l} q_{l}^{2}}{m_{l} \epsilon_{0}}
$$

is the plasma frequency for species $l$.
Combining the plasma current together with the displacement current into the Maxwell's equation in Eq. (1.1) for the magnetic field, we get

$$
\mathbf{k} \times \mathbf{B}=\mu_{0}\left(\mathbf{J}+\epsilon_{0}-j \omega \mathbf{E}\right)=-j \omega \epsilon_{0} \mathbf{K} \cdot \mathbf{E}
$$

where $\mathbf{k}$ is the wavevector whose amplitude is the index of refraction $\mathbf{n}=\mathbf{k} c / \omega$ and $\mathbf{K}$ is the dielectric tensor of plasma given by

$$
\mathbf{K}=\left(\begin{array}{ccc}
S & -i D & 0  \tag{1.9}\\
i D & S & 0 \\
0 & 0 & P
\end{array}\right)
$$

where the indicated elements are defined as

$$
\begin{align*}
S & =1-\sum_{l} \frac{\omega_{p l}^{2}}{\omega^{2}-\omega_{c l}^{2}} \\
D & =\sum_{l} \frac{\kappa_{l} \omega_{c l} \omega_{p l}^{2}}{\omega\left(\omega^{2}-\omega_{c l}^{2}\right)}  \tag{1.10}\\
P & =1-\sum_{j} \frac{\omega_{p l}}{\omega^{2}}
\end{align*}
$$

and the following algebraic parameters can be derived [8]

$$
\begin{align*}
& R=S+D=1-\sum_{l} \frac{\omega_{p l}^{2}}{\omega\left(\omega+\kappa_{l} \omega_{c l}\right)}  \tag{1.11}\\
& L=S-D=1-\sum_{l} \frac{\omega_{p l}^{2}}{\omega\left(\omega-\kappa_{l} \omega_{c l}\right)}
\end{align*}
$$

The Stix labels stand for the Sum, Difference, Right, Left, and Plasma terms, respectively.
The resulting wave equation

$$
\begin{equation*}
\mathbf{n} \times(\mathbf{n} \times \mathbf{E})+\mathbf{K} \cdot \mathbf{E}=0 \tag{1.12}
\end{equation*}
$$

gives the cold plasma dispersion relation that identifies the three principal radio frequency heating schemes. In a slab approximation it takes the form

$$
\begin{equation*}
A n^{4}-B n^{2}+C=0 \tag{1.13}
\end{equation*}
$$

where

$$
\begin{align*}
& A=S \sin ^{2} \theta+P \cos ^{2} \theta \\
& B=R L \sin ^{2} \theta+P S(1+\cos \theta)^{2}  \tag{1.14}\\
& C=P R L
\end{align*}
$$

and $\theta$ is the angle between $\mathbf{k}$ and the magnetic field $z$-directed. So that $x$ and $y$ are the radial and poloidal co-ordinates, respectively. The general solution of the dispersion relation can be conveniently written in terms of the refractive index $n$ and angle $\theta$ as follow

$$
\begin{gather*}
n^{2}=\frac{B \pm F}{2 A}  \tag{1.15}\\
\tan ^{2} \theta=-\frac{P\left(n^{2}-R\right)\left(n^{2}-L\right)}{\left(S n^{2}-R L\right)\left(n^{2}-P\right)}
\end{gather*}
$$

where $F^{2}=B^{2}-4 A C$.
The general condition for a resonance requires that $n^{2} \rightarrow \infty$ or, equivalently that $A \rightarrow 0$, that is

$$
\begin{equation*}
\tan ^{2} \theta=-\frac{P}{S} \quad \text { resonance condition } \tag{1.16}
\end{equation*}
$$

and the general cutoff condition, where $n=0$ is satisfied when

$$
\begin{equation*}
C=P R L=0 \quad \text { cutoff condition. } \tag{1.17}
\end{equation*}
$$

The so-called principal resonances occur at $\theta=0$ and $\theta=\pi / 2$, hence, for $\theta \rightarrow 0$, we require $S \rightarrow \infty$ (see Eq. (1.16), the resonance condition for $\theta=0$ ), since $P=0$ leads to cutoff condition. Since $S=1 / 2(R+L)$, we have two possibilities:

$$
\begin{array}{ll}
R \rightarrow \infty & \text { electron cyclotron resonance } \\
L \rightarrow \infty & \text { ion cyclotron resonance }
\end{array}
$$

So that, remembering that the cyclotron frequency is $\omega_{c l}=\left|q_{l}\right| B_{0} / m_{l}$ and $\kappa_{l}=q_{l} / q_{l}$, the term at the denominator of R and $\mathrm{L} \kappa_{l} \omega_{c l}=q_{l} B_{0} / m_{l}$ is positive defined for ions and negative for electrons. It is now clear that for a simple plasma of species $l=i, e$, the ion cyclotron resonance frequency happens when $\omega_{c i}=q_{i} B_{0} / m_{i}$ and the electron cyclotron resonance frequency when $\omega_{c e}=q_{e} B_{0} / m_{e}$, respectively. Since the electron mass lower than the ion mass (except that in partially ionized plasmas), the electron cyclotron frequency is higher than the ion cyclotron frequency ( $30-120 \mathrm{MHz}$ and $100-200 G H z$, respectively).
Between $\omega_{c i}$ and $\omega_{c e}$, it is possible to define the lower hybrid resonance. In particular, as $\theta \rightarrow \pi / 2, P / S \rightarrow \infty$ and, since $P \rightarrow \infty$ only for $\omega \rightarrow 0$ or $\omega_{p} \rightarrow \infty$, and both of them are nonsense solutions, we require that $S \rightarrow 0$. This resonances are called "hybrid resonance" because they generally need combinations of both $\omega_{c}=\omega_{c i}^{2}+\omega_{c e}^{2}$ and $\omega_{p}=\omega_{p i}^{2}+\omega_{p e}^{2}$, wich are extraordinary waves for perpendicular propagation. The dispersion relation for such a waves is given by

$$
\begin{equation*}
n_{X}^{2}=\frac{R L}{S}=\frac{\left[\left(\omega+\omega_{c i}\right)\left(\omega-\omega_{c e}\right)-\omega_{p}^{2}\right]\left[\left(\omega-\omega_{c i}\right)\left(\omega+\omega_{c e}\right)-\omega_{p}^{2}\right]}{\left(\omega^{2}-\omega_{c i}^{2}\right)\left(\omega^{2}-\omega_{c e}^{2}\right)+\omega_{p}^{2}\left(\omega_{c e} \omega_{c i}-\omega^{2}\right)} \tag{1.18}
\end{equation*}
$$

where the resonances are given by nullifying the denominator of the previous Eq. (1.18). The derivation of these frequencies is quite complicated and we remind to the references cited in Bibliography for a complete formulation. For our purpose it is important to consider the only lower hybrid resonance

$$
\begin{equation*}
\omega_{L H}^{2}=\omega_{c e} \omega_{c i}\left(\frac{\omega_{p e}^{2}+\omega_{c e} \omega_{c i}}{\omega_{p e}^{2}+\omega_{c e}^{2}}\right) \tag{1.19}
\end{equation*}
$$

so that, the lower hybrid resonance approaches the ion cyclotron resonance in the low density limit ( $\omega_{p e} \ll \omega_{c e}$ ), and the geometric mean $\sqrt{\omega_{c e} \omega_{c i}}$ in the high density limit $\left(\omega_{p e} \gg \omega_{c e}\right)$, so it in-between ion and electron cyclotron frequencies.
The two group of resonances of a cold plasma $(\theta=0, \pi / 2)$ induced different properties in resultant propagating waves. In particular, in the first case, the particle of specie $l$ is sped up the static magnetic filed $\mathbf{B}_{0}$ in the $z$-direction. Of course, this produces a particle velocity increasing and consequently an effective plasma heating. Anyway the ECRH is affected by serious technological problem, since high-power-steady-state sources at very high frequency are very hard to create. More feasible is the ICRH even if some problems occur limiting the efficiency $[9,10]$.
Another possibility could enhance the plasma temperature and also allow the tokamak to operate continuously (in theory). Such a condition could be achieved making $\mathbf{k} \| \mathbf{E}$, so that the electrons are sped up by a quasi-electrostatic field. The LH resonce fulfill this condition having $\theta \approx \pi / 2$, but actually a small component
of the wavevector parallel to the magneto-static field still exists. In high density plasma this reveals a component of $n_{\|}=c k_{\|} / \omega>1$, that establish a non-inductive current in accordance with Landau damping theory[7]. Of course this is only a poor explanation of the complex phenomena involving the excitation of a NICD system, anyway it help us to focus our attention on the LHCD System of ITER and the necessity to develop a Transmission Line system able to carry efficiently the power from generators to the launcher.

### 1.2 LHCD System Overview

The LHCD system is specialized for off-axis current drive and current profile control. It is designed to deliver to the ITER plasma a total power of 20 MW at 5 GHz.
In ITER, this is approximately equivalent to a driven current of $\sim 1.5 M A$ in the region $0.5<R / a<0.7$ where the distance $R$ is calculated from the center of the torus to the center of the tube having radius $a$.
The starting design of the whole LH system [11] is depicted in Fig. 1.3 and it was supposed to be composed of :

- a generator with 24 one MW klystron,
- a transmission line,
- one launcher.


Figure 1.3: Layout of LHCD System.

The launcher whose principle is made upon PAM (Passive Active Multi-junction) is firstly described. The wave launching structure is an array of four (two toroidal and two poloidal) radiating elements, located in the ITER vacuum vessel. The RF
power is transferred to the array by six circular transmission lines, following paths of unequal electrical length, from the power sources to the tokamak port. From an operational point of view, the LH system is composed of four identical subsystems (LH Subsystems), each consisting of:

- one PAM module;
- two sections of main transmission lines (MTL);
- six klystron power sources.

The physic principle beyond the PAM is based on the high-density plasma properties to have a density gradient if moving from the edges to the center. In particular, in the central region of the plasma bulk, the density is actually high and the electric field of an injected electromagnetic wave at the LH frequency is radially directed. Moreover, at the interface vacuum-plasma the density is much lower and the electric field result to be $z$-polarized and quasi-parallel to the magneto-static field $\mathbf{B}_{0}$ (as can be demonstrated by solving the wave equation, Eq. (1.12), for a low-edgedensity plasma over cutoff). So that, the polarization of the electric field at the edge must be $\mathbf{E}=\hat{\mathbf{z}} E_{z}$, and it can be fulfilled easily by using a rectangular wave guide excited in its fundamental electric mode. The other condition that has to be satisfied, involves the $n_{\|}$component that has to be grater than one. By defining $n_{\|}=c k_{\|} / \omega=c \Delta \Phi / \omega \Delta z$, (where $\Delta z$ is the short side of the rectangular waveguide), by fixing $\Delta \Phi=\approx \pi$ and respecting the fundamental condition $\Delta \Phi / \Delta z=2 \pi / \lambda$, we can design the rectangular waveguide dimension $\Delta z$ to realize the desired $n_{\|}$profile (Fig. 1.4).
Furthermore, the power coupling can be enhanced by alternating each waveguide with a closed-ended rectangular waveguide (this one called "passive" in opposition with the radiating "active" waveguide), re-radiating the field components reflected at the interface with the plasma. The single PAM module is then formed by 24 active and 25 passive waveguides; each module is placed in a matrix of $6 \times 8$ as indicated in Fig. 1.5. Each row is fed by a rectangular waveguide that is bifurcated to realize the row of active waveguides. The whole antenna is formed by four quadrants each made of one half of 3 adjacent modules equal to that represented in Fig. 1.5. The design of each quadrant consists in:

- $4 \times 3$ waveguides with E plane bends. The bending has been used in order to avoid to have the RF windows seeing the plasma in straight line;
- $4 \times 3 T E_{10}$ linear tapers,


Figure 1.4: PAM principle. The phase profile is obtained by means of a grid of rectangular waveguides excited in their fundamental $T E_{10}$ mode.

- $4 \times 3 T E_{10}$ to $T E_{30}$ mode converters. They are used to allow to divide the power into three in the vertical direction,
- $12 \times 3$ passive active multi-junctions modules.

For each block, the front face is made of an array of 12 horizontal rows of 24 active waveguides and 25 passive waveguides each.
More details about mode converters, linear tapers and RF windows can be found in [12], for our scopes it is important that in each quadrant, the $12 \times 3$ passive active multi-junction are collected in a $4 \times 3$ matrix (in which each element is fed by a $T E_{30}$ mode, that is converted in 3 waveguide excited in their fundamental $T E_{10}$ mode vertically) that turn into a $2 \times 3$ by using 3 dB couplers. A global view of the launcher is also represented in Fig. 1.6. In such a configuration every one of the 24 ports is fed by a rectangular waveguide.
The nominal RF power density in the PAM is around $33 \mathrm{MW} / \mathrm{m}^{2}$ (if only the active waveguides are taken into account) and it corresponds to an amplitude for the forward wave of $6 \mathrm{kV} / \mathrm{cm}$ at the grill mouth. The maximum power density for ITER is therefore well below the values achieved on present tokamaks.
The launched power is a function of the reflection at the grill mouth and is directly linked to the electric field at the grill. Power density up to $24 \mathrm{MW} / \mathrm{m}^{2}$ has been achieved on Tore Supra (Cadarache, France) for 75 sec corresponding to an electric field value of $5.4 \mathrm{kV} / \mathrm{cm}$ at the grill mouth [13]. During the same experimental campaign on Tore Supra, routine operation with an electric field between 3.8 $5.4 \mathrm{kV} / \mathrm{cm}$ was achieved for more than 4000 sec . These results tend to give a large safety margin for 5 GHz operation. A well established law shows indeed that the maximum tolerable electric field increases roughly linearly with the frequency. In the last ten years, depending on the klystrons number and their power supply


Figure 1.5: Launcher plug.
capabilities, the PAM configuration has been subjected to several modifications. At least two successive revised design were proposed [14]. Actually a high power $5 G H z$ klystrons provide maximum output power of 500 kW CW ; as a consequence 48 klystrons will be used to assure the requested $20 M W$ of LHCD coupled to the plasma. The reduced power of the klystron influences the MTL and also the PAM module configuration: it can be realized with only 24 active waveguides assembled in $6 \times 4$ waveguides as in Fig. 1.7. In order to transport the elevated RF power


Figure 1.6: Revised PAM module design.
from generators to the antenna, 6 Main Transmission Lines (MTLs) are used. In particular a single MTL carries $4 M W$ RF power combining the outputs of groups of 4 klystrons in a single circular waveguide, exciting the electric $T E_{01}$ mode, that is the lowest losses mode. In fact the attenuation through the $\sim 70 \mathrm{~m}$ path from klystrons to the launcher for a circular waveguide excited in its $T E_{01}$ mode, is about
$7 \times 10^{-3} \mathrm{~dB} / \mathrm{m}$ for total $0.49 \mathrm{~dB}(\approx 10 \%$ of the input power), instead of 1.75 dB $(\approx 30 \%)$ for rectangular waveguides. The MTL is briefly described by Fig. 1.8 and consists of 4 sections:


Figure 1.7: MTL scheme.

- the cryostat section (that we will not consider in the present work);
- the Splitting Network (SN) Section, running from the cryostat wall to the circular transmission line. Each MTL feeds 4 vacuum transmission lines (VTL) through its SN
- the Circular Transmission Line (CTL) Section, running from the splitting network to the LHCD klystron cabinet;
- the Recombining Network (RN) Section, located in the LHCD klystron cabinet, running from each individual power klystron to the CTL. Each CTL is fed by 4 klystrons through the RN.

The considered standard rectangular waveguide is the WR-229. The propagating mode is the $T E_{10}$; the waveguide sizes are $58.17 \mathrm{~mm} \times 29.08 \mathrm{~mm}$. The frequency at which the $T E_{20}$ mode propagates is 5.157 GHz , therefore the frequency margin is 157 MHz .
The waveguide is supposed made of copper and efficiently water cooled. Since the RF losses are high the power coming from 4 klystron is combined in order to feed a circular waveguide as indicated in Fig. 1.8.
The attenuation in a rectangular waveguide for the $T E_{10}$ mode [15] is given by

$$
\begin{equation*}
\alpha_{R W}=\frac{20}{\ln 10} \sqrt{\frac{\pi \epsilon f}{\sigma}} \frac{a+2 b\left(\lambda_{0} / \lambda_{c}\right)^{2}}{a b\left(\lambda_{0} / \lambda_{g}\right)} \tag{1.20}
\end{equation*}
$$

where in MKSA system $f$ is the frequency, $a \times b$ are the waveguides dimensions, $\lambda_{0}$ is the wavelength in free space, $\lambda_{c}$ is the cutoff wavelength, $\lambda_{g}$ is the wavelength in the waveguide, and $\sigma$ is the conductivity. The overall attenuation is then $0.025 \mathrm{~dB} / \mathrm{m}$, for a total of 1.75 dB for 70 m . The amplitude of the electric field $E$ for a forward power $P$ on matched load is

$$
\begin{equation*}
E=2 \sqrt{\frac{\lambda_{g} \eta}{\lambda_{0} a b} P} \tag{1.21}
\end{equation*}
$$

that is, for 1 MW of forward power, smaller that the breakdown amplitude ( $30 \mathrm{kV} / \mathrm{cm}$ which is the breakdown amplitude in dry air).

For the construction of the Combiner/Splitter component, the cross section of


Figure 1.8: Splitting/Combining Network: a) component design; b) rectangular to sector waveguide transition. labelfig:sottofigure
the circular waveguide is divided in four sector waveguides, with sector angle $\alpha=$ $\pi / 2$; each sector is then connected to a rectangular waveguide through a suitable transition. The device also converts the $T E_{10}$ mode of the rectangular waveguide in the $T E_{01}$ mode of the circular waveguide. The device has a double geometric symmetry with reference to the orthogonal partition axis, and then it is also used at the end of the MTL to split the main line into rectangular waveguides.
For a mode $T E_{m n}$, the attenuation in a circular waveguide of radius $a$ is given in MSKA system by

$$
\begin{equation*}
\alpha_{C W}=\frac{20}{a \ln 10} \sqrt{\frac{\pi \epsilon f}{\sigma}}\left[\frac{m^{2}}{\xi_{i}^{\prime 2}-m^{2}}+\left(\frac{\lambda_{0}}{\lambda_{c i}^{\prime \prime}}\right)^{2}\right] \frac{1}{\sqrt{1-\left(\frac{\lambda_{0}}{\lambda_{c i}^{\prime}}\right)^{2}}} \tag{1.22}
\end{equation*}
$$

where $\xi_{i}^{\prime}$ is the $n$-th root of the $m$-th derivative of the Bessel function and $\lambda_{c i}^{\prime \prime}=$ $2 \pi a / \xi_{i}^{\prime}$ is the cutoff wavelength. The azimuthal component of the electric field is
given by

$$
\begin{equation*}
E_{\phi}=V_{i}^{\prime \prime} \sqrt{\frac{\epsilon_{m}}{\pi}} \frac{\xi_{i}^{\prime}}{\sqrt{\xi_{i}^{\prime 2}-m^{2}}} \frac{J_{m}^{\prime}\left(\frac{\xi_{i}^{\prime} r}{a}\right)}{a J_{m}\left(\xi_{i}^{\prime}\right)} \cos (m \phi) \tag{1.23}
\end{equation*}
$$

and, for $m=0$ we have $\epsilon_{m}=1$, and the forward power $P$ is

$$
\begin{equation*}
P=\eta \sqrt{1-\left(\frac{\lambda_{0}}{\lambda_{c i}^{\prime \prime}}\right)^{2}}\left|V_{i}^{\prime \prime}\right|^{2} \tag{1.24}
\end{equation*}
$$

where $V I_{i}$ is the voltage of the propagating wave. For $m=0$ the maximum amplitude of the electric field can be derived by Eqs. (1.) and (1.) as follow

$$
\begin{equation*}
E_{\phi, \max }=\frac{\beta}{\alpha} \sqrt{\frac{P}{\eta \sqrt{1-\left(\frac{\lambda_{0}}{\lambda_{c i}^{\prime \prime}}\right)^{2}}}} \tag{1.25}
\end{equation*}
$$

with $\beta=0.8158$.
The minimum radius of this waveguide is limited by the constraint to propagate the $T E_{01}$ mode, and thus:

$$
\begin{equation*}
a_{\min }=\frac{\lambda_{c} \xi_{01}^{\prime}}{2 \pi}=36.59 \mathrm{~mm} \tag{1.26}
\end{equation*}
$$

where $\lambda_{c}=60 \mathrm{~mm}$ is the wavelength in vacuum at 5 GHz , that is imposed to be the cutoff frequency of the $T E_{01}$ mode for the converter and, at the same frequency $\xi_{01}^{\prime}=3.8317$. The maximum diameter is instead defined by the cutoff frequency of the next mode allow to propagating in the rectangular to section transformer, that is the $T E_{41}$ (as we demonstrate in the next chapter), so that the output of the combiner/splitter converter must be a circular cross-section waveguide having a radius in this range. On the other hand the attenuation in circular waveguide increases with decreasing radii, as indicated in Fig. 1.10. Thus, a circular waveguide taper must be designed to connect the output cross-section of the converter to the MTL.

$$
\begin{equation*}
a_{\max }=\frac{\lambda_{c} \xi_{41}^{\prime}}{2 \pi}=50.78 \mathrm{~mm} \tag{1.27}
\end{equation*}
$$

where $\xi_{41}^{\prime}=5.3175$. All components (splitting and recombining networks, bends) have, an efficiency lower than $100 \%$. In closed system it means that part of the RF power will be dissipated on spurious modes excitation. These modes, being partially or perfectly trapped modes, can cause undesirable resonance, which spoil the VSWR parameter of the transmission line and increase the electric fields in comparison with calculations predictions [16]. The distance between two neighbor resonance for a


Figure 1.9: Attenuation in circular waveguide.
transmission line of $l=70 m$ is not less than $\Delta f=c / 2 l \approx 2 \mathrm{MHz}$. In order to avoid these resonance, mode filters have to be applied. One of the classic mode filters is based on corrugated waveguides filled by absorber. Such kinds of mode filters are more efficient than the so-called resistive filters. The reason is that in a waveguide with frequent corrugation (impedance-type corrugation) the field do not penetrate inside the corrugation, where the absorber is placed. A space in-between the combs of the corrugation is actually a cutoff for modes with an electric field parallel to the corrugation. On the other hand, the fields of other non-symmetrical modes can penetrate in the corrugation and, thus, be absorbed.
In Chapters 2 we present the design of the combiner/splitter converter, concluding that this component is too critical for this application. Then, a revised transmission line concept is proposed, using 48 rectangular waveguides, oversized at 5 GHz . In Chapter 3, we design and optimize, for this oversized waveguide, filters and bends.

### 1.2.1 Rectangular-to-Circular-Sector Mode Converter

The basic theory of the sector waveguides is based on the following hypothesis:

- the radius of the sector waveguide must be constant along the direction of propagation,
- the angle of the sector must be less then $130^{\circ}$,
- the working frequency $f_{0}$ must be significantly greater than the cut-off frequency of the $T E_{01}$ mode in the sector waveguide.

If the three previous conditions are satisfied, the theory of the circular waveguides may be applied to the sector waveguides.
In particular the eigenfunctions of the sector waveguides, the functions that describe the field configuration in cylindrical co-ordinates $(\rho, \phi)$, may be straightforward derived from the corresponding eigenfunctions of the circular waveguides with radius $a$ :

$$
\begin{align*}
& T_{H}=\sqrt{\frac{z_{m}}{\left(\xi_{m n}^{\prime 2}-m^{2}\right) \pi}} \frac{J_{m}\left(\xi_{m n}^{\prime} \rho / a\right)}{J_{m}\left(\xi_{m n}^{\prime}\right)}\left\{\begin{array}{r}
\sin (m \phi) \\
\cos (m \phi)
\end{array} \quad T E_{m n}\right. \text { modes }  \tag{1.28}\\
& T_{E}=\frac{1}{\xi_{m n}} \sqrt{\frac{z_{m}}{\pi}} \frac{J_{m}\left(\xi_{m n} \rho / a\right)}{J_{m-1}\left(\xi_{m n}\right)}\left\{\begin{array}{l}
\sin (m \phi) \\
\cos (m \phi)
\end{array} \quad T M_{m n}\right. \text { modes }
\end{align*}
$$

where:

- $z_{m}$ is the Neumann number ( $z_{m}=1$ for $m=0$ and $z_{m}=0$ for $m \neq 0$ );
- $J_{m}(\xi)$ is the Bessel function of the first kind of order " $m$ ";
- $\xi_{m n}$ is the $n$-th zero of $J_{m}(\xi)$;
- $\xi_{m n}^{\prime}$ is the $n$-th zero of the function derivative $J_{m}^{\prime}(\xi)=d J_{m}(\xi) / d \xi$.

This is obtained by substitution in Eq. (1.28) of the angle $\pi$ with the sector halfangle $\alpha / 2$ and of the integer mode number $m$ with the fractional number:

$$
\begin{equation*}
\mu=\frac{2 \pi m}{\alpha} \tag{1.29}
\end{equation*}
$$

This last parameter changes the order of the Bessel function in Eq. (1.28), which becomes $J_{\mu}(\xi)$. The cutoff wavenumbers of the modes propagating in the sector waveguide are therefore:

$$
\begin{equation*}
k_{c \mu n}=\frac{\xi_{\mu n}^{\prime}}{a} \tag{1.30}
\end{equation*}
$$

depending on the sector angle $\alpha$.
So that, the modes that may propagate in a generic waveguide depend on the vacuum wavenumber:

$$
\begin{equation*}
k_{0}=2 \pi f_{0} \sqrt{\epsilon_{0} \mu_{0}} \tag{1.31}
\end{equation*}
$$

and on the cutoff wave number $k_{c}$ of the specific modes, that is on the geometric characteristics of the waveguide. At a given frequency only the modes with $k_{0}>k_{c}$ can propagate. For a circular waveguide, radius $a$, this means modes for which:

$$
\begin{equation*}
\xi_{m n}<2 \pi f_{0} a \sqrt{\epsilon_{0} \mu_{0}} \tag{1.32}
\end{equation*}
$$

The sector waveguide derived by splitting by four the circular waveguide, has a sector angle $\alpha=90^{\circ}$ (or $\alpha=\pi / 2$ ); according to Eq. (1.29) the fractional order of the Bessel functions to consider in Eq. (1.28) is $\mu=4 m$.
On the base of this last equivalence, modes in circular waveguide with $m=0$ correspond to propagating modes in the sector waveguides with $\mu=0$, and their cutoff wave numbers are not dependent on $\alpha$ and have the same numerical value of the corresponding modes in circular waveguide.
Furthermore, modes in circular waveguide with $m=1$, correspond to modes with $\mu=4$ in sector waveguide and circular modes with $m=2$ correspond to sector modes with $\mu=8$; modes with higher $\mu$ are in cutoff in the sector waveguide as in the circular one. From the above considerations follows that only modes with first index $\mu=4 m$ can propagate in the sector waveguide.
The sector waveguide has even geometric symmetry around the bisector of the angle $\alpha$. Therefore only TE and TM modes having even electric field symmetry around the same bisector may propagate, that is, with reference to the eigenfunctions in Eq. (1.28), TE modes with cosinusoidal dependence on $\phi$ and $T M$ modes with sinusoidal dependence on $\phi$. But for $\mu=0, \sin (\mu \phi)=0$ whatever is the value of $\phi$, therefore the $T M_{0 n}$ do not propagate in the sector waveguide. Between them the $T E_{01}$ mode is the fundamental one because it has the lowest cutoff wave number. Given two generic $T E$ modes $M$ and $N$, with eigenfunctions respectively $T_{M}$ and $T_{N}$, propagating in a waveguide with arbitrary cross section, the coupling coefficient between them is:

$$
\begin{equation*}
C_{M N}=\frac{k_{c N}^{2}}{k_{c M}^{2}-k_{c N}^{2}} \oint_{L} \tan \phi T_{N} \frac{\partial^{2} T_{M}}{\partial n_{s}^{2}} d l_{s} \tag{1.33}
\end{equation*}
$$

where $n_{s}$ and $l_{s}$ are the normal to the cross section surface and the tangential component to the cross-sectional contour, respectively, as in Fig. 1.11. The gradient of the wave function along the side-walls of the sector waveguide with $\phi= \pm \alpha / 2$ is:

$$
\frac{\partial T}{\partial n_{s}}=\frac{\partial T}{\rho \partial \phi}
$$

and for all the $T E_{0 n}$ modes the eigenfunction of the electric field is not dependent on the angular co-ordinate $\phi$, so that the previous derivative in is zero. Therefore the


Figure 1.10: Waveguide of arbitrary cross section.
coupling coefficient $C_{M N}$ in between modes $T E_{0 n}$ and whatever other $T E$ modes, are zero even if $\alpha$ is not constant along the propagation direction $z$. This means that, if the cross section of a sector waveguide with sector angle $\alpha=\pi / 2$ is not affected by significant deformations, then the sector waveguide will propagate only the $T E_{01}$ mode.

### 1.2.2 Mode Filters in Corrugated Waveguides

Generally speaking, classical mono-modal waveguides are not suitable to transmit high electromagnetic power across long paths, since they present high attenuation of the fundamental mode and low power handling capability. Such drawbacks can be overcome with the use of oversized waveguides that present the following advantages:

- broadband;
- high power handling capability;
- comparatively low value of attenuation;
- reduced production costs since they can be chosen among the standard waveguide series.

Due to the aforementioned features oversized waveguides are especially suitable for sub-millimeter waves and high power applications[17]. As a general rule circular waveguides operating in the $T E_{01}$ mode are used in order to further reduce transmission losses. Nevertheless, notwithstanding their higher losses, oversized rectangular waveguides, operating in the dominant $T E_{10}$ mode, are to be preferred. The latter ones present indeed the following advantages:

1. there is no need for mode converters from rectangular (assumed to be the standard output of the power sources) to circular waveguides;
2. potential transitions with other rectangular waveguides are realized by means of simple tapers or quarter wavelength transformers;
3. some devices can be designed more easily due to the simple electromagnetic field pattern;
4. the mode of interest presents a better stability (the $T E_{01}$ mode in circular waveguide is easily transformed into the $T M_{11}$ mode).

Nevertheless many spurious modes may be excited and propagate within an oversized system in proximity of discontinuities such as flanges, bends, tapers, etc. , giving rise to the so-called trapped mode resonances. Furthermore, the field pattern is not definite and varies along the waveguide if several modes are propagating simultaneously; so the transmission properties may deteriorate severely.
These unwanted effects may be restrained by inserting some mode filters or mode absorbers [18]. At the moment corrugated waveguides, with corrugations filled with an absorbing material, are the most suitable solution for high power applications since the resistive sheets filter can only handle low power [19]. Scientific literature related to this topic is scarce and totally lacking in guidelines useful to design such devices.
This work gives a contribution in this sense, mainly referring to filters based on rectangular waveguides. In particular it describes the relationship between the absorption of unwanted modes and the thickness of the absorbing material that fills the corrugations, deriving a dependence similar to a damped cosine. Such behavior is explained by means of an analytical model treating the corrugations as normal guiding structures where forward and backward waves propagate. A check for what will be analytically derived is given in the Chapter 3 showing the results of a series of full-wave simulations run on a desktop computer with a commercial software. Regardless of the waveguide cross section (rectangular, circular, etc.), corrugations are excited when they are transversally crossed by the field lines of any electric surface current. As a consequence the waveguide slots have to be designed so that the surface currents of the mode of interest cross them lengthwise, while the ones of the unwanted modes cut them perpendicularly. In this way the power carried by the unwanted modes only feeds the corrugations, generating inside them a set of modes that satisfies Maxwell equations and related boundary conditions.
As shown in Fig. 1.12, a single corrugation is defined by 4 dimensions: the height ( $b c$ ) and the width (ac) of its cross section and the thicknesses of the absorbing material
(hca) and of its empty region (hcv). The empty region is generally air-filled, but the vacuum will be considered instead of $\operatorname{air}\left(\epsilon_{r, a i r} \approx 1\right)$ to simplifies the mathematical notation. Inside geometries like the one depicted in Fig. 1.12, the electromagnetic


Figure 1.11: Waveguide of arbitrary cross section.
field is given by the superimposition of forward and backward waves given by the sum of the $N$ modes (both $T E$ and $T M$ ) above the cutoff frequency. In a general form it can be expressed as follows:

$$
\begin{align*}
\mathbf{E} & =\sum_{t=0}^{N} C_{i}^{+}\left(\mathbf{e}_{i}+\mathbf{e}_{z i}\right) e^{-k_{i} z}+C_{i}^{-}\left(\mathbf{e}_{i}-\mathbf{e}_{z i}\right) e^{k_{i} z}  \tag{1.34}\\
\mathbf{H} & =\sum_{t=0}^{N} C_{i}^{+}\left(\mathbf{h}_{i}+\mathbf{h}_{z i}\right) e^{-k_{i} z}+C_{i}^{-}\left(-\mathbf{h}_{i}+\mathbf{h}_{z i}\right) e^{k_{i} z}
\end{align*}
$$

being $\mathbf{h}_{z i}$ and $\mathbf{e}_{z i}$ the eigenvectors for respectively $T E$ and $T M$ modes [20]. The unknown wave amplitudes ( $C_{i}^{+}$and $C_{i}^{-}$) and the propagation constants $\left(k_{i}\right)$ change with the transmission medium, being vacuum or an absorbing material, but the expressions in Eq. (1.34) remain the same. From an electromagnetic point of view, absorbing materials are lossy dielectrics, thus, at the interface between the vacuum and the absorber, the following boundary conditions apply:

$$
\begin{align*}
& \hat{\mathbf{n}} \times\left.\left(\mathbf{H}_{a}-\mathbf{H}_{v}\right)\right|_{z=0}=0  \tag{1.35}\\
& \hat{\mathbf{n}} \times\left.\left(\mathbf{E}_{a}-\mathbf{E}_{v}\right)\right|_{z=0}=0
\end{align*}
$$

and

$$
\begin{align*}
& \left.\hat{\mathbf{n}} \cdot\left(\mu_{r} \mathbf{H}_{a}-\mathbf{H}_{v}\right)\right|_{z=0}=0  \tag{1.36}\\
& \left.\hat{\mathbf{n}} \cdot\left(\epsilon_{r} \mathbf{E}_{a}-\mathbf{E}_{v}\right)\right|_{z=0}=0
\end{align*}
$$

where the subscripts $v$ and $a$ respectively stand for vacuum and absorber. Only the equations in Eq. (1.35) are linearly independent because among the remaining ones, that refer to the longitudinal field vector, there are always a redundant equation and a null one. More precisely the equations of Eq. (1.36) are respectively null for $T M\left(\mathbf{H}_{z}=0\right)$ and $T E\left(\mathbf{E}_{z}=0\right)$ modes.
At the interface between the absorber and the perfect electric wall, the boundary conditions are as follows:

$$
\begin{align*}
& \hat{\mathbf{n}} \times\left.\mathbf{H}_{a}\right|_{z=h c a}=-\mathbf{J}_{s}  \tag{1.37}\\
& \hat{\mathbf{n}} \times\left.\mathbf{E}_{a}\right|_{z=h c a}=0
\end{align*}
$$

and

$$
\begin{align*}
& \left.\hat{\mathbf{n}} \cdot \mu_{r} \mathbf{H}_{a}\right|_{z=h c a}=0  \tag{1.38}\\
& \left.\hat{\mathbf{n}} \cdot \epsilon_{r} \mathbf{E}_{a}\right|_{z=h c a}=-\rho_{s}
\end{align*}
$$

For similar reasons, only the second equation of Eq.(1.38) can be used. To summarize, a set of three linearly independent equations, comprising all $N$ propagating modes, can be derived by the boundary conditions. Enforcing mode orthogonality, this set of equations reduces to $N$ linear systems with 3 equations and 4 unknowns that are the wave amplitudes of a single mode in the two propagation media $\left(C_{v}^{+}\right.$, $C_{v}^{-}, C_{a}^{+}$and $C_{a}^{-}$, where the subscript " $v$ " and " $a$ " stand for the vacuum and the absorber, respectively). Thus each system allows the calculation of the reflection coefficient at the corrugation input for one of the $N$ propagating modes.
The reflection coefficient of a corrugation is very important because it affects the performances of mode filters being directly related with their absorption capability. More precisely, since corrugations are lossy passive structures with only one port, the power that is not reflected is absorbed.

### 1.2.3 Maxwell's Equation for Curved Waveguide

The study of curves in cylindrical cross section waveguide requires at first, to find a convenient system of co-ordinates that allow us to transform the Maxwell's equations along the curvature. In particular, some basics of differential geometry are needed.
Differential geometry of curves is an area of geometry that involves the study of smooth curves in the plane and in Euclidean space using the methods of differential and integral calculus. In particular, the Frenet-Serret frame is a moving frame that can be used to develop an orthogonal system that describes the curve at a point
completely.
A first important classification of curves can be done, considering as regular a curve that can be described by a function which derivative is well defined and non zero on the interval on which the curve is defined. Many different curves typologies are known, but formally, the differential-geometric invariants are the so-called curvature and torsion of a curve. The fundamental theorem of curves asserts that the knowledge of these invariants completely determines the curve [21].
More details about the derivation of such a moving frame, can be found in the Appendix B of this work, together with the co-ordinate metrics $\left(h_{1}, h_{2}, h_{3}\right)$ for an arbitrary but constant cross section waveguide, that follow a specified longitudinal path. In particular, we use $u_{3}$ for the axial co-ordinates and $u_{1}, u_{2}$ for the transverse cross section co-ordinates. Since the waveguide has a constant cross section, $h_{1}$ and $h_{2}$ are the same occurring in a straight waveguide, and therefore independent of $u_{3}$; consequently $h_{3}$ is function of $u_{1}$ and $u_{2}$, and may be also dependent on $u_{3}$. In such a co-ordinate system, the Maxwell's equation can be written as:

$$
\begin{align*}
& \frac{\partial\left(h_{3} A_{3}^{E, H}\right)}{\partial u_{2}}-\frac{\partial\left(h_{2} A_{2}^{E, H}\right)}{\partial u_{3}}=\mp j \omega h_{2} h_{3} A_{1}^{B, D}  \tag{1.39}\\
& \frac{\partial\left(h_{1} A_{1}^{E, H}\right)}{\partial u_{3}}-\frac{\partial\left(h_{3} A_{3}^{E, H}\right)}{\partial u_{1}}=\mp j \omega h_{3} h_{1} A_{2}^{B, D} \\
& \frac{\partial\left(h_{2} A_{2}^{E, H}\right)}{\partial u_{1}}-\frac{\partial\left(h_{1} A_{1}^{E, H}\right)}{\partial u_{3}}=\mp j \omega h_{1} h_{2} A_{3}^{B, D}
\end{align*}
$$

where $A_{n}^{E, H}$ and $A_{n}^{B, D}$ represent equivalently electric field and magnetic intensity components, and magnetic field and electric displacement components, intending the signs in the right part of Eq. (1.39) associated with the electric (negative) and magnetic (positive) field, respectively.
Of course the relation between these component in free space are the straightforward equations $\mathbf{B}=\mu \mathbf{H}$ and $\mathbf{D}=\epsilon \mathbf{E}$, for the magnetic and electric field, respectively, where $\mu$ and $\epsilon$ are the permeability and permittivity in a linear, isotropic and homogeneous medium, respectively.
It is also convenient to use a bi-complex notation [22] to transform Eq. (1.39) into transverse and axial coordinates, and notably

$$
\begin{align*}
& A_{t}^{E, H}=A_{1}^{E, H}+i A_{2}^{E, H}  \tag{1.40}\\
& A^{E, H}=h_{3} A_{3}^{E, H}
\end{align*}
$$

where $A_{1}$ and $A_{2}$ are both real and directly related to the real imaginary part of the electric (magnetic) transverse field $E_{t}\left(H_{t}\right)$. In particular both $i$ and $j$ are imaginary constant such that $i^{2}=j^{2}=-1$, but $i j$ cannot be further reduced $(i j \neq-1)$. So
that, we can define [23] a differential operator $\mathcal{F}$, such that

$$
\begin{equation*}
\mathcal{F}=\frac{1}{h_{1}} \frac{\partial}{\partial u_{1}}[\cdot]+i \frac{1}{h_{2}} \frac{\partial}{\partial u_{2}}[\cdot] \tag{1.41}
\end{equation*}
$$

since $h_{1}$ and $h_{2}$ are independent of $u_{3}$, the first two equations in Eq. (1.36) can be combined together into the next equation

$$
\begin{equation*}
\mathcal{F} A^{E, H}=\frac{\partial A_{t}^{E, H}}{\partial u_{3}} \mp i j \omega h_{3} A_{t}^{B, D} \tag{1.42}
\end{equation*}
$$

To deal with the third equation of Eq. (1.36), another operator is required, and defined as

$$
\begin{equation*}
\overline{\mathcal{F}}=\frac{1}{h_{1} h_{2}}\left[\frac{\partial}{\partial u_{2}}\left(h_{1}[\cdot]\right)+i \frac{\partial}{\partial u_{1}}\left(h_{2}[\cdot]\right)\right] \tag{1.43}
\end{equation*}
$$

and the equation become

$$
\begin{equation*}
h_{3} \operatorname{Re}\left(\overline{\mathcal{F}} A_{t}^{E, H}\right)= \pm j \omega A_{B, D} \tag{1.44}
\end{equation*}
$$

with reference to the complex quantity $i$. It is possible to avoid the presence of $\operatorname{Re}[\cdot]$ in Eq. (1.39), by obtaining the associated imaginary parts $i h_{3} \operatorname{Im}[\cdot]$ and adding to Eq. (1.39). To this aim, we consider the double operator $\overline{\mathcal{F}} \mathcal{F}$ as follows

$$
\begin{equation*}
\overline{\mathcal{F}} \mathcal{F}=\frac{i}{h_{1} h_{2}}\left\{\frac{\partial}{\partial u_{1}}\left(\frac{h_{2}}{h_{1}} \frac{\partial}{\partial u_{1}}[\cdot]\right)+\frac{\partial}{\partial u_{2}}\left(\frac{h_{1}}{h_{2}} \frac{\partial}{\partial u_{2}}[\cdot]\right)\right\} \tag{1.45}
\end{equation*}
$$

resulting to an imaginary term if the operand is real. So that, we can apply this operator to $A^{E, H}$ and, since it is pure real, the result must be imaginary and so $\operatorname{Re}\left(A^{E, H}\right)=0$ must be superimposed. So that, considering Eq. (1.42), we get

$$
\begin{equation*}
\operatorname{Re}\left\{\overline{\mathcal{F}} \mathcal{F} A^{E, H}\right\}=\operatorname{Re}\left\{\overline{\mathcal{F}}\left[\frac{\partial A_{t}}{\partial u_{3}} \mp i j \omega h_{3} A_{t}^{B, D}\right]\right\}=0 \tag{1.46}
\end{equation*}
$$

Since $h_{1}$ and $h_{2}$ are independent of $u_{3}, \partial / \partial u_{3}$ commutes in the Eq. (1.46) out of the real part operator and, with reference to Eq. (1.44), we obtain

$$
\begin{equation*}
\mp \frac{\partial\left(j \omega A^{B, d}\right) / h_{3}}{\partial u_{3}} \pm \operatorname{Im}\left\{j \omega \overline{\mathcal{F}} h_{3} A^{B, D}\right\}=0 \tag{1.47}
\end{equation*}
$$

and, with some algebra, it is possible to lead to the searched expression

$$
\begin{equation*}
h_{3} \operatorname{Im}\left(\overline{\mathcal{F}} A_{t}^{E, H}\right)= \pm \partial\left(A^{E, H} / h_{3}\right) / \partial u_{3}-i \operatorname{Re}\left(A_{t}^{E, H} \mathcal{F}^{*} h_{3}\right) \tag{1.48}
\end{equation*}
$$

and, multiplying the previous Eq. (1.48) by $i$, it is possible to remove the real part operator into Eq. (1.44), leading to the expression

$$
\begin{equation*}
h_{3} \overline{\mathcal{F}} A_{t}^{E, H}= \pm j \omega A^{B, D}-i \partial\left(A^{E, H} / h_{3}\right) / \partial u_{3}-i \operatorname{Re}\left(A_{t}^{E, H} \mathcal{F}^{*} h_{3}\right) \tag{1.49}
\end{equation*}
$$

The previous Eq. (1.49) is actually representing two different equations, that is for the electric and magnetic field. It is possible to obtain a single compact form including both $E_{t}$ and $H_{t}$ terms, by defining a multiplier $N$ and combining in Eq. (1.42) the $E_{t}$ and $H_{t}$ contributes. To this aim it is convenient to split Eq. (1.48) in the next two equations:

$$
\begin{align*}
& \mathcal{F} E=\frac{\partial E_{t}}{\partial u_{3}}-i j \omega \mu h_{3} H_{t}  \tag{1.50}\\
& \mathcal{F} H=\frac{\partial H_{t}}{\partial u_{3}}+i j \omega \epsilon h_{3} E_{t} \tag{1.51}
\end{align*}
$$

in particular, if Eq. (1.50) is multiplied by $N$ and added to Eq. (1.49), it is easy to see that $E_{t}$ and $H_{t}$ occur in the two combinations $E_{t}+N H_{t}$ and $\epsilon N E_{t}-\mu H_{t}$. If we define $N$ as

$$
\begin{equation*}
N= \pm j \sqrt{\frac{\mu}{\epsilon}}= \pm N_{0} \tag{1.52}
\end{equation*}
$$

we obtain the same expression $E_{t} \pm N_{0} H_{t}$, apart from an initial factor $j \sqrt{\mu \epsilon}$. If we now define

$$
\begin{align*}
& C_{t^{ \pm}}=E_{t} \pm N_{0} H_{t}  \tag{1.53}\\
& C_{ \pm}=E \pm N_{0} H_{t}
\end{align*}
$$

that combined and elaborated with $\mathcal{F}$ operator lead to

$$
\begin{equation*}
\mathcal{F} C=\left[\frac{\partial}{\partial u_{3}}+i j \epsilon \omega h_{3} N\right] C_{t} \tag{1.54}
\end{equation*}
$$

where th ambiguity of sign $\pm$ depends on the sign chosen when defining $N$. By using Eq. (1.53) the two relations in Eq. (1.49) can be treated as one by using the following expression

$$
\begin{equation*}
h_{3} \overline{\mathcal{F}} C_{t}=-\left[j \omega \epsilon N h_{3}+i \frac{\partial}{\partial u_{3}}\right] \frac{C}{h_{3}}-i \operatorname{Re}\left(C_{t} \mathcal{F}^{*} h_{3}\right) \tag{1.55}
\end{equation*}
$$

where the only ambiguity consists in the sign of $N$.
To get a "wave equation" for the field in a general case when $u_{3}$ is allowed to vary arbitrarily, some complicated differential algebra is required, and this subject lie outside the present dissertation, anyway, when $u_{3}$ varies as $e^{-\gamma u_{3}}$ (where $\gamma$ is a constant), the propagation in the $u_{3}$ dimension is uniform and a useful expression can be derived [23] by taking

$$
\begin{equation*}
C_{t}=\frac{\mathcal{F}(C)}{-\gamma+i j \epsilon \omega h_{3} N} \tag{1.56}
\end{equation*}
$$

In particular, by applying the operator $\overline{\mathcal{F}}$ to Eq. 1.54 , we get $\overline{\mathcal{F} \mathcal{F} C}$, where further simplification can lead to the following wave equation:

$$
\begin{equation*}
\left\{\nabla_{t}^{2}+\frac{\partial}{\partial u_{3}}\left[\frac{1}{h_{3}} \frac{\partial}{\partial u_{3}} \frac{1}{h_{3}}+\omega^{2} \epsilon \mu\right\} C=\operatorname{Re}\left\{\frac{2 i j \epsilon \mu N h_{3}}{-\gamma+i j \epsilon \omega h_{3} N} \mathcal{F}(C) \mathcal{F}^{*}\left(\log h_{3}\right)\right\}\right. \tag{1.57}
\end{equation*}
$$

that can be solved for the transverse field component by dividing Eq. (1.55) by $j \omega \epsilon N h_{3}-i \gamma$ and thus eliminating the axial component $C$, to obtain

$$
\begin{equation*}
\mathcal{F}(C)=\mathcal{F}\left\{\frac{h_{3} \overline{\mathcal{F}}\left(C_{t}\right)+i \operatorname{Re}\left(C_{t} \mathcal{F}^{*} h_{3}\right)}{-j \epsilon \omega N+\frac{i \gamma}{h_{3}}}\right\}=\left(-\gamma+i j \epsilon \omega N h_{3}\right) C_{t} \tag{1.58}
\end{equation*}
$$

by solving Eq.(1.58) to find the transverse $C_{t}$ component and then extracting by simple algebra the various electric and magnetic terms, it is possible to apply Eq. (1.39) to find the electromagnetic field components at each point of the curvature. It should be noted that neither $\overline{\mathcal{F}} \mathcal{F}$ nor $\mathcal{F} \overline{\mathcal{F}}$ reduced to the transverse Laplacian $\nabla_{t}^{2}$, so that this method has the great convenience to obtain a tranverse wave equation parametric with the metric of the particular curve frame.

## Chapter 2

## Combiner/Splitter Design

The LHCD system for ITER has been examined in the "Detailed Design Description of the LHCD System" document [11], produced in the frame of the EFDAITER. That document gives the general description of the system, outlines its main components and makes the preliminary thermo-mechanical, electro-mechanical and nuclear analyses. In addition that document describes the operation modes, the assembly procedures and the maintenance of the system.
The report on the transmission line and, more in detail its appendix, studied the "combiner/splitter", that is a high power microwave device that connects the four independent rectangular waveguides, at the output of the four high power RF sources of a LHCD subsystem, to the single circular waveguide of the MTL analyzed in Chapter 1. This device also converts the fundamental $T E_{10}$ mode carried by the rectangular waveguides into the low losses, $T E_{01}$ mode launched into the circular one.
More details can be found in the following documents [24, 25, ?].
In particular the last two reports put in evidence the need of a more detailed analysis of the combiner/splitter in non-ideal condition of operations.
The present report concerns the final analysis of the combiner/splitter outlined in the previous report. The analysis been performed with the help of the ANSOFT computer code "High Frequency Structure Simulator" (HFSS ${ }^{\circledR}$ ), version 12, based on the Finite Elements Method (FEM).

### 2.1 The Combiner/Splitter Optimized Layout

In the analytic formulation reported in the previous chapter, was pointed out that, under a given set of conditions, in the rectangular to sector waveguide transition (Fig. 2.1), that is the basic component of the combiner/splitter, the fundamen-
tal $T E_{10}$ mode in the input rectangular waveguide can only be converted in the $T E_{01}$ mode in the output sector waveguide. Basing of what demonstrated before, the first higher spurious mode that could propagate in the sector waveguide, considering its geometrical symmetry around the bisector of its vertex angle, is the $T E_{41}$. To avoid any occasional, but possible, conversion to this spurious mode, as well to avoid any diffraction losses on the walls of the sector waveguides due to incoming spurious modes from the MTL, the radius of the sector waveguide was optimized [16].
Furthermore, we design the combiner device, by varying both the total length of the transition and the radius of the circular sector. In particular, the length of the device, $L_{\text {conv }}$ is supposed to vary in the range $[10 \mathrm{~cm}, 70 \mathrm{~cm}]$, with a step resolution of 1 cm . At the same time, the radius of the circular sector $R_{\text {conv }}$ is varied in the range specified by Eqs. (1.26) and (1.27), and here reported for readability: $R_{\text {conv }} \in[37 \mathrm{~mm}, 50 \mathrm{~mm}]$ (the cited values are rounded off for simulation convenience) with a resolution of 1 mm . The next analysis does not take into account losses due to the waveguide walls which are now considered as perfectly electric. According to


Figure 2.1: The rectangular-to-sector converter.
[16], a length of 500 mm , even if it does not correspond to the best result, represents a good compromise between the performance of the device ( $S_{11}=-42.1 \mathrm{~dB}$ ) and its dimensions. If considered necessary, better performances could be obtained with a longer device ( $S_{11}=-55 \mathrm{~dB}$ at $L_{\text {conv }}=700 \mathrm{~mm}$ ), but it must be taken into account that increasing the length of the devise also the attenuation will increase as the mechanical stress which the rectangular-to-circular sector transformers are subjected. The chosen dimensions that assure optimized performances to the transition are summarized in Table 2.1. A radius of $R=50 \mathrm{~mm}$ has been chosen for the sector waveguide because it is the best compromise between the constraints governing its design. In fact, not only this radius fixes the cut off frequency of the $T E_{41}$, both

| Design parameters | Dimensions |
| :--- | :---: |
| Input Rectangular Waveguide | WR-229 $\left(58.17 \times 29.08 \mathrm{~mm}^{2}\right)$ |
| Output Circular Waveguide | $R=50 \mathrm{~mm}$ |
| Sector angle | $\phi=90^{\circ}$ |
| Overall length | 500 mm |

Table 2.1: Rectangular-to-sector mode converters dimensions.
in the sector waveguide and in the circular one, at 5078 MHz , that is sufficiently far from the upper limit of the foreseen frequency bandwidth $(51, \mathrm{GHz} \pm 10 \mathrm{MHz})$, but it is also affordable for the power handling point of view.

### 2.2 Performance Analysis

A more detailed analysis of the behavior of the combiner/splitter in terms of power carried by the circular $T E_{01}$ mode, has been performed both in ideal and wrong conditions. In particular it has been derived an analytical method in which the component is modeled as a network with $N+M$ ports, as represented in Fig. 2.2, in which the inputs $(N)$ are the four rectangular $T E_{10}$ modes, while the outputs ( $M$ ) are the seven modes above cut off in the circular waveguide at $5 \mathrm{GHz}\left(T E_{11}, T M_{01}\right.$, $\left.T E_{21}, T E_{01}, T M_{11}, T E_{31}, T M_{21}\right)$. By assuming the modal orthogonality, it has been considered the following scattering representation of a microwave $(M+N) \times$ $(M+N)$ network, in which the last $M$ inputs and the first $N$ outputs as null values. Using the scattering parameters theory is possible to calculate the power content driven by each mode. In order to calculate the power carried by each propagating


Figure 2.2: Equivalent scattering model of the combiner/splitter network.
mode in the circular waveguide, we can refer to the following scattering parameters
representation:
that turns into a matrix form

$$
\begin{equation*}
\mathbf{V}_{i}^{\text {out }}=\mathbf{S}_{k, j} \cdot \mathbf{V}_{T E_{10}}^{i n, j} \text { where } i=1, \cdots, 7 ; k=5, \cdots, 11 \text { and } j=1, \cdots, 4 \tag{2.2}
\end{equation*}
$$

The total input power is then calculated as

$$
\begin{equation*}
P_{i n}=\frac{1}{2} \sum_{j=1}^{4}\left|V_{T E_{10}}^{i n, j}\right|^{2} \tag{2.3}
\end{equation*}
$$

while the output power of each mode is derived as

$$
\begin{equation*}
P_{i}^{\text {out }}=\frac{1}{2}\left|V_{i}^{\text {out }}\right|^{2} \tag{2.4}
\end{equation*}
$$

and the total output power is

$$
\begin{equation*}
P_{\text {out }}=\frac{1}{2} \sum_{i=1}^{7}\left|V_{i}^{\text {out }}\right|^{2} \tag{2.5}
\end{equation*}
$$

The percentage of power carried by each propagating mode can be written as

$$
\begin{equation*}
P_{[\%]}^{i}=\frac{100 P_{i}^{\text {out }}}{P^{\text {out }}} \tag{2.6}
\end{equation*}
$$

Actually part of the input power is reflected, so that also the power loss for reflection at the $i$-th input port for $i=1, \cdots, 4$ must be considered:

$$
\begin{equation*}
P_{i n, i}^{l o s s}=\frac{1}{2} \sum_{j=1}^{4}\left|S_{i, j} V_{T E_{10}}^{i n, j}\right|^{2} \tag{2.7}
\end{equation*}
$$

so that

$$
\begin{equation*}
P_{t o t}^{\text {loss }}=\sum_{i=1}^{4} P_{i n, i}^{\text {loss }} \tag{2.8}
\end{equation*}
$$

and the total input power must be necessarily $P_{\text {in }}=P_{\text {out }}+P_{\text {tot }}^{\text {loss }}$.

### 2.2.1 Ideal Working Conditions

Ideal conditions of work for the combiner means that the electric fields in its four rectangular waveguides, mode $T E_{10}$, have the same amplitude and the same phase. In these conditions the four input powers are combined in a pure $T E_{01}$ mode in the circular waveguide. If the splitting properties of the device are instead considered, in ideal conditions the circular waveguide carries a pure $T E_{01}$ mode and the RF power is roughly equally split in the four rectangular output waveguides of the device. These ideal conditions are visualized in Fig. 2.3, where the electric fields in the four circular sector waveguides, and their combination in the circular one, are shown. The


Figure 2.3: Rectangular to circular waveguide conversion.
most important outcomes of the previous analysis are summarized in the following Tables 2.2-2.3, and the derivation of the modal power content through the methods previously described, is presented in Table 2.4. In these tables the ports 1 to 4 corresponds to the four sector waveguides, while port 5 correspond to the circular one. The analytical model is well enough confirmed by the CAD simulations

| Parameters [dB] | Port 1 | Port 2 | Port 3 | Port 4 |
| :--- | :---: | :---: | :---: | :---: |
| Reflection at the sector input | -37.02 | -31.54 | -30.21 | -33.85 |
| Coupling efficiency | -6.034 | -6.02 | -6.028 | -6.038 |

Table 2.2: Combiner reflection and coupling @ 5 GHz .
(provided with Ansoft HFSS ${ }^{\circledR}$ ). In the following Fig. 2.4a)-2.4b), the reflection and transmission parameters, $\left|S_{11}\right|$ and $\left|S_{21}\right|$, are reported. In particular the overall considered bandwidth is from 4.9 GHz to 5.1 GHz with a step resolution of 1 MHz .

|  | Port $1\left[\mathrm{~dB} /{ }^{\circ}\right]$ | Port $2\left[\mathrm{~dB} /{ }^{\circ}\right]$ | Port $3\left[\mathrm{~dB} /{ }^{\circ}\right]$ | Port $4\left[\mathrm{~dB} /{ }^{\circ}\right]$ | Port $5\left[\mathrm{~dB} /{ }^{\circ}\right]$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Port 1 | $-14.99 /-55.47$ | $-12.76 / 134.03$ | $-11.51 / 139.81$ | $-12.72 /-46.26$ | $-6.02 /-55.18$ |
| Port 2 | $-12.76 / 134.03$ | $-15.07 /-55.68$ | $-12.77 /-45.96$ | $-11.53 / 139.91$ | $-6.02 /-55.11$ |
| Port 3 | $-11.51 / 139.81$ | $-12.77 /-45.96$ | $-14.99 /-55.59$ | $-12.73 / 133.63$ | $-6.03 / 124.76$ |
| Port 4 | $-12.72 /-46.26$ | $-11.53 / 139.91$ | $-12.73 / 133.63$ | $-14.94 /-55.35$ | $-6.02 / 124.72$ |
| Port 5 | $-6.02 /-55.18$ | $-6.02 /-55.11$ | $-6.03 / 124.76$ | $-6.02 / 124.72$ | $-33.27 /-67.59$ |

Table 2.3: Combiner/splitter conversion efficiency: amplitude and phase @ 5 GHz .

| Output modes | output power [\%] |
| :---: | :---: |
| $T E_{11}$ | 0.0069 |
| $T M_{01}$ | 0.1752 |
| $T E_{21}$ | 0.0014 |
| $T E_{01}$ | 99.8088 |
| $T M_{11}$ | 0.0037 |
| $T E_{31}$ | 0.0012 |
| $T M_{21}$ | 0.0028 |

Table 2.4: Modal power content @ 5 GHz .

The frequency sweep is about 200 MHz and it is much wider than the operating bandwidth, but we adopt a precautionary range in the possibility of further changing. In the sub-figure $a$ ), the $\left|S_{11}\right|$ level for all the input ports is suitably under -15 dB in the frequency interval $5 \mathrm{GHz} \pm 10 \mathrm{MHz}$, and in the same range, the $\left|S_{21}\right|$ parameter (sub-figure $b$ )) is quite smooth around -6 dB . A decrease of $\left|S_{11}\right|$ down to $-30 \mathrm{~dB} /-$ 37 dB , occurs at 5.025 GHz , and then increases again up to -20 dB . Such a $\left|S_{11}\right|$ level is extremely desirable, unfortunately at the same frequencies several notches affect the $\left|S_{21}\right|$ parameter, causing an unbalanced transmission among the input converters. To avoid this possibility, we design the component in order to have these oscillations out of the required bandwidth, and provide a transmission stability and uniformity. The total power loss, obtained from Eq. (2.8) is the $0.47 \%$ of the total input power. Considering a total input power of 4 MW , the total reflection losses are about $18,8 \mathrm{~kW}$. Even if the values of the $\left|S_{11}\right|$ parameter for the 4 input waveguide are not exactly the same, it can be roughly assumed that this power is equally divided between the 4 inputs, so that the reflected power at each input port is about 4.7 kW . The effective transmitted power is about 3.9735 MW , that is the $99.33 \%$ of the total input power. As explained in Chapter 2 , at 5 GHz only the circular $T E_{01}$ mode can be re-converted to the $T E_{10}$ mode in the rectangular waveguides and propagate backward trough the combiner. Thus, the power carried by the other modes remains


Figure 2.4: Splitting/combining network: a) reflection; b) transmission.
trapped into the circular waveguide and must be dissipated. In particular, in the ideal case, the total amount of trapped power is only $0.19 \%$ of the total input power, that is about 7.6 kW , and corrugations filled of absorbing material can be designed to solve this problem.

### 2.2.2 Fault-Tolerance Analysis

As pointed out previously, the wrong conditions can be due to either amplitude or phase anomalies, or to a combination of the two. Both the two anomalies have been considered in the following analysis.
In this section, basing on the principle described in Fig. 2.2, the quantitative analysis
of the most probable amplitude anomalies on the input wave ports is presented. In particular, 4 different cases are considered. As it will be shown, it is sufficient to consider only the case in which the anomaly affects just a single wave port to suitably describe the performances of the device for the actual purpose. The four amplitude anomalies that have been considered are the following:

- one out of the four rectangular wave ports injects only the $70 \%$ of the nominal 1 MW RF power;
- one out of the four rectangular wave ports injects only the $50 \%$ of the nominal 1 MW RF power;
- one out of the four rectangular wave ports injects only the $30 \%$ of the nominal 1 MW RF power;
- one out of the four rectangular wave ports does not inject RF power at all.

The simulation outcomes are summarized in the next Table 2.5. In Fig. 2.5a) the

|  | $100 \%$ | $70 \%$ | $50 \%$ | $30 \%$ | $0 \%$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $T E_{11}$ | 0.0069 | 0.2815 | 0.9815 | 2.0721 | 4.5692 |
| $T M_{01}$ | 0.1752 | 0.1717 | 0.1747 | 0.1783 | 0.1852 |
| $T E_{21}$ | 0.0014 | 0.027 | 0.08 | 0.168 | 0.3682 |
| $\mathbf{T E}_{\mathbf{0 1}}$ | $\mathbf{- 9 9 . 8 0 8 8}$ | $\mathbf{- 9 8 . 4 3 7 9}$ | $\mathbf{9 5 . 7 2 1}$ | $\mathbf{9 1 . 2 5 8 8}$ | $\mathbf{8 1 . 1 1}$ |
| $T M_{11}$ | 0.0037 | 0.521 | 1.5172 | 3.1429 | 6.8273 |
| $T E_{31}$ | 0.0012 | 0.4218 | 1.3974 | 2.9259 | 6.4088 |
| $T M_{21}$ | 0.0028 | 0.13 | 0.1361 | 0.264 | 0.5613 |

Table 2.5: Combiner/splitter amplitude anomalies @ 5 GHz : mode contents [\%] vs. input power [\%].
variation (in \%) of the RF power associated to the $T E_{01}$ mode as function of the wave port power reduction (fourth row of the previous Table 2.5) is reported. It must be taken into account that the percentage is referred to the total output power. If we consider the power reflection at each step we can understand the effective output power percentage compared with the actual input power injection (see Table 2.6). The effective output power associated to the $T E_{01}$ mode vs. the effective total input power (last row in Table 2.6) is plotted in Fig. 2.5b). As indicated in [24] the circular output waveguide has to carry on a minimum RF power content of $95 \%$ associated to the $T E_{01}$ (that is $5 \%$ of power deficit). So that, up to $50 \%$ of power reduction in one out of the four rectangular ports can be tolerated. Moreover, in

|  | $100 \%$ | $70 \%$ | $50 \%$ | $30 \%$ | $0 \%$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Total input power [MW] | 4 | 3.7 | 3.5 | 3.3 | 3 |
| Power reflection [\%] | 0.47 | 0.5475 | 0.7702 | 1.1249 | 1.9045 |
| Effective overall output power [MW] | 3.9812 | 3.6781 | 3.4692 | 3.255 | 2.9238 |
| Effective $T E_{01}$ power conversion [MW] | 3.9735 | 3.62 | 3.32 | 2.97 | 2.3715 |
| $T E_{01}$ vs. input power | 99.3375 | 97.838 | 94.85 | 90 | 79.05 |

Table 2.6: Combiner/splitter performances @ 5 GHz : performances vs. amplitude anomalies [\%].
this case about $4.38 \%$ of the total input power, that is about 125 kW , is trapped into the circular waveguide and must be properly dissipated to avoid possible damages in this waveguide. To complete the analysis of the combiner/splitter in non ideal conditions, the influence of the phase variation in one out of the four rectangular waveguides with reference to the remaining three, has been evaluated. The results of this study, for a phase variation in the range $0^{\circ}-180^{\circ}$, are summarized in Table 2.7. From this table it can be pointed out that the spurious $T E_{11}$ and $T M_{11}$ modal contents increase as the $T E_{01}$ mode content decreases, in other words, by increasing the phase difference there is a significant transfer of power from the $T E_{01}$ modes to the other two modes. The row related to the $T E_{01}$ mode, is more detailed in Fig. 2.6 .

|  | $0^{\circ}$ | $30^{\circ}$ | $60^{\circ}$ | $90^{\circ}$ | $120^{\circ}$ | $150^{\circ}$ | $180^{\circ}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T E_{11}[\%]$ | 0.007 | 1 | 3.61 | 7.5 | 11.85 | 15.36 | 16.63 |
| TE $_{\mathbf{0 1}}[\%]$ | $\mathbf{9 9 . 8}$ | $\mathbf{9 6 . 2}$ | $\mathbf{8 6 . 0 2}$ | $\mathbf{7 0 . 5}$ | $\mathbf{5 2 . 7 7}$ | $\mathbf{3 8 . 1 4}$ | $\mathbf{3 2 . 4 3}$ |
| $T M_{11}[\%]$ | 0.004 | 2.34 | 9.24 | 19.8 | 31.9 | 42 | 46 |

Table 2.7: Combiner/splitter modal content [\%] @ 5 GHz vs. phase anomalies [\%].

(b)

Figure 2.5: $T E_{01}$ modal content [\%]: a) compared with the total output power and b) compared with the effective input power.


Figure 2.6: $T E_{01}$ power content vs. phase anomalies.

### 2.3 Conclusive Remarks

The study of Combiner/Splitter has shown a good behavior in terms of performances of $\left|S_{11}\right|$ and coupling in ideal working conditions when all the four input waveguides are fed uniformly in amplitude and phase. In these conditions it has been calculated that the total amount of power injected in the circular waveguide at one side of the combiner/splitter, is about the $99 \%$ compared to the sum of the input powers of the four rectangular waveguides WR-229 standard at the other side. Furthermore, in this case, the coupling between the modes that propagate in the circular waveguide, at the frequency of 5 GHz , are quite acceptable in terms of power losses.
Anyway, in wrong working conditions, the performances of the device are affected by a steady decrease of efficiency. In particular, in the simple case in which only one rectangular waveguide suffers a reduction of power of $50 \%$ (without any phase anomalies), the total power losses for the lone $T E_{01}$ mode, with reference to the total input power, are about $5 \%$. This means that is enough a simple damage like the described ones, to force a substantial failure of the component and this could provoke the suppression of an entire main transmission line.
On the other hand, a phase variation $\Delta \phi$ in one of the four rectangular waveguides greater than $35^{\circ}$ determines a potential dangerous situation for the MTL in circular waveguide, due to the generation of spurious modes. In fact, even if it were possible a phase control of the four klystrons, the correct amplitude uniformity of all the input sectors cannot be assured. In addition, if one of the four klystrons is turned off as consequence of an internal or of a whichever external failure, the remaining three klystrons must be suddenly turned off to avoid severe damages to the MTL. This highly probable event eventually determines an overall power reduction of more than $16 \%$.
Taking into account the total number of klystrons in operation in the LHCD sytem (24/1 MW or 48/0.5 MW klystrons) the probability that one or more of them can be in a faulty condition during a RF pulse is very high, so that the availability of the whole 24 MW power of the LHCD system could be compromised.
In conclusion, the present and exhaustive analysis definitely confirms the results of the previous analyses. Therefore the use of this combiner/splitter to gather the RF power generated by four klystrons into a single circular waveguide, mode $T E_{01}$, is positively discouraged.
For what stated before, 48 MTLs in circular waveguide, mode $T E_{01}$, with direct coupling between rectangular waveguides, mode $T E_{10}$, and circular ones, mode $T E_{01}$, would seem the only available solution.

Anyway, a possible alternative can be considered. In fact, since the distance of the klystrons from the launcher influences the length, the path and hence the configuration of the MTL, by reducing this distance (determining new location of the RF generators for instance), oversized rectangular waveguides, transmitting the fundamental $T E_{10}$ mode, can be profitably used to build the MTL. Moreover integration of 48 rectangular waveguides through the port cell door could be possible. By the way, the absence of critical mode converters from rectangular waveguides, mode $T E_{10}$, to circular ones, mode $T E_{01}$, would further enhance the overall efficiency of the MTL when rectangular waveguides are used. In addition the criticality of the not fundamental circular TE01 mode in an oversized circular waveguide is avoided by launching the fundamental TE10 mode in a rectangular waveguide.
To limit the transmission losses to less than $5 \%$ (we consider this percentage as a threshold for an effective power transfer. If we consider that up to 24 MW of available power, the launcher has to inject 20 MW to the vacuum vessel, so that about 4 MW can be dissipated through the whole LHCD system. A realistic estimation of the power amount concerning the transmission line subsystem of $30 \%$ (1,2 MW that is exactly the $5 \%$ of the total power) seems to be convincing), a preliminary analysis about rectangular oversized waveguide performances must be provided. Moreover, the accidental conversion between the fundamental $T E_{10}$ mode and higher-order modes (by the presence of discontinuities such as bends) that can be excited and can propagate in oversized waveguide, has to be taken into account.
In particular, to deal with these spurious modes, adequate mode filters, able to work at high RF power, have been investigated and a preliminary optimization of these components has been carried out. The analysis of the bends and of the mode filters will be finalized once the real length and path of the MTL have been assessed.

## Chapter 3

## Oversized Rectangular Waveguide Components

As stated in the previous chapter, the combiner/splitter device is a good-performing component in normal working condition, showing in this case a great effectiveness in converting the four rectangular modes into an electric circular one. Nevertheless, it is poorly reliable when faults occur, even if with some differences between amplitudes and phase diseases, and it can be classified as not-reliable for high power applications.
This technical hitch induced to take into account the possibility of a fully oversized rectangular waveguide transmission line. In fact, since the transmission losses along the MTL are substantially reduced in proportion to the line length, it is possible to limit the power losses by shortening the distance between klystrons lodging and the launcher. This option suggests the possibility of using oversized rectangular waveguides to realize the MTL of the LHCD system. Moreover if the MTL is realized with rectangular waveguides, there is no need for mode converters, but only tapers to connect the different cross-sections are necessary. Therefore the transmission losses and the mode conversion efficiency of the mode converters must not be considered in the overall MTL transmission losses [11].
In conclusion the slightly higher transmission losses of the rectangular waveguides can be compensated by the absence of rectangular to circular mode converters and of their conversion efficiency. On top on this, the rectangular waveguides will transmit directly the fundamental $T E_{10}$ mode, reducing the complexity of a MTL based on circular waveguide transmitting a non fundamental $T E_{01}$ mode.
On the other hand, the realization of the MTL in waveguides of oversized rectangular cross-section, allow to propagate several higher-order modes that diminish the transmission efficiency of the fundamental $T E_{10}$ mode, dissipating some of the total

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power among the other propagating modes. This will produce a decrease of power injection in the fundamental mode, compared with the expected 20 MW , together with some problems of reflection toward the line [27].

To deal with this problem, it is mandatory the design of efficient mode filters in rectangular waveguide, that allow to partially attenuate the unwanted modes. Moreover, spurious modes can also be generated by any accidental discontinuity on the way. Therefore, it is very important to design some "controlled discontinuities", such as bends on both the $E$ and $H$ geometrical planes.

### 3.1 Preliminary Analysis

In Table 3.1, some standard rectangular waveguides that are above cut-off at 5 GHz , with their computed power capability at that frequency for the chosen propagation mode at different electric fields and VSWR, are roughly summarized. The power capability has been computed according to the following operative relation [28]:

$$
\begin{equation*}
\frac{P}{E_{\max }^{2}}=6.63 \times 10^{-1} a b \frac{\lambda_{0}}{\lambda_{g}} \tag{3.1}
\end{equation*}
$$

where $E_{\text {max }}$ (in $\mathrm{kV} / \mathrm{cm}$ ) is the max allowable electric field in the waveguide, while $a \times b$ (in $\mathrm{cm}^{2} 2$ ) is the surface of the cross-section of the waveguide; $\lambda_{0}$ and $\lambda_{g}$ are the wavelength in vacuum and the wavelength in the waveguide, respectively. The maximum allowable power $P_{\max }$ is instead obtained by waveguides specification guide. This peak power is obtained considering an electric strength of $30 \mathrm{kV} / \mathrm{cm}$ (ideal dry air) in the nominal frequency bandwidth of each waveguide.
The value indicated in Table 3.1, refer to a maximum electric $E_{\max }=6 \mathrm{kV} / \mathrm{cm}$ and a $V S W R=1.2$, that is about a reflection coefficient of $|\Gamma|=0.0909$. From

| Waveguide | Size $(a \times b)[\mathrm{mm}]$ | $\mathrm{P}[\mathrm{kW}]$ | $P_{\max }[\mathrm{kW}]$ | Bandwidth $\Delta f[\mathrm{GHz}]$ |
| :--- | :---: | :---: | :---: | :---: |
| WR-187 | $47.5 \times 22.15$ | 164 | 1040 | $3.94-5.99$ |
| WR-229 | $58.17 \times 29.11$ | 291 | 1600 | $3.22-4.90$ |
| WR-284 | $74.14 \times 34.04$ | 448 | 2430 | $2.60-3.95$ |
| WR-340 | $86.36 \times 43.18$ | 701 | 3800 | $2.17-3.00$ |
| WR-430 | $109.22 \times 54.61$ | 1150 | 5900 | $1.72-2.61$ |
| WR-510 | $129.54 \times 64.77$ | 1978 | 9200 | $1.16-2.20$ |
| WR-650 | $165.10 \times 82.55$ | 3195 | 15000 | $1.12-1.70$ |

Table 3.1: Power capability of standard rectangular waveguide @ 5 GHz .

| Waveguide | $\alpha_{R W}[\mathrm{~dB} / \mathrm{m}]$ | $f_{c}^{T E_{10}}[\mathrm{GHz}]$ | $f_{c}^{T E_{20}}[\mathrm{GHz}]$ | No. modes <br> $@ 5 \mathrm{GHz}$ |
| :--- | :---: | :---: | :---: | :---: |
| WR-187 | $3.4 \times 10^{-2}$ | 3.16 | 6.31 | 1 |
| WR-229 | $2.16 \times 10^{-2}$ | 2.58 | 5.16 | 1 |
| WR-284 | $1.6 \times 10^{-2}$ | 2.08 | 4.16 | 7 |
| WR-340 | $1.15 \times 10^{-2}$ | 1.74 | 3.47 | 8 |
| WR-430 | $8.7 \times 10^{-3}$ | 1.38 | 2.75 | 10 |
| WR-510 | $7.1 \times 10^{-3}$ | 1.15 | 2.31 | 13 |
| WR-650 | $5.4 \times 10^{-3}$ | 0.91 | 1.82 | 18 |

Table 3.2: Attenuation of standard rectangular waveguide @ 5 GHz .
this table it is evident that the WR-284 could carry the 500 kW of a single klystron if adequately pressurized and if the power reflection coming back from the load is limited to a $V S W R \leq 1.2$.
An additional parameter to consider for the choice of a waveguide is its specific attenuation $[\mathrm{dB} / \mathrm{m}]$ at a given frequency, as can be calculated with Eq. (1.20), here reported for convenience:

$$
\alpha_{R W}=\frac{20}{\ln 10} \sqrt{\frac{\pi \epsilon f}{\sigma}} \frac{a+2 b\left(\lambda_{0} / \lambda_{c}\right)^{2}}{a b\left(\lambda_{0} / \lambda_{g}\right)}
$$

The attenuation value, computed @ 5 GHz for the considered standard waveguides, is reported in Table 3.2, together with the cut-off value and the overall number of propagating modes. By analyzing this table it is evident that, to limit the transmission losses, the WR-284 can be used for shortened transmission lines. In fact the specific attenuation corresponds to a power losses of about $0.36 \% \mathrm{~m}^{-1}$. Anyway the power handling with an inner electric field limited to $6 \mathrm{kV} / \mathrm{cm}$ and a VSWR $=$ 1.2 is much lower than the specifications. The same happens for the WR-340, that can carry an average power of about 700 kW . This solution cannot be taken into account for two main reasons: at first, the overall power attenuation is lower than the $5 \%$ of the transmitted power for a transmission line length of about 14 m that is considered too short for the actual distance between generators plant and launcher; moreover, the transmitted power of the fundamental $T E_{10}$ is reduced because of the presence of higher order modes. Even though it were possible to attenuate the power content on these modes, the resulting power on the fundamental mode remain the same.

The power capability of the WR-430, with an inner electric field limited to $6 \mathrm{kV} / \mathrm{cm}$ and a VSWR $=1.2$ is about 1150 kW . This power capability can be increased up

| Mode | $\lambda_{c}[\mathrm{~mm}]$ | $\lambda_{g}[\mathrm{~mm}]$ | $f_{c}[\mathrm{GHz}]$ |
| :--- | :---: | :---: | :---: |
| $T E_{10}$ | 218.44 | 62.40 | 1.373 |
| $T E_{01} / T E_{20}$ | 109.22 | 71.81 | 2.747 |
| $T E_{11} / T M_{11}$ | 97.69 | 76.03 | 3.071 |
| $T E_{21} / T M_{21}$ | 77.23 | 95.30 | 3.884 |
| $T E_{30}$ | 72.81 | 105.90 | 4.120 |
| $T E_{31} / T M_{31}$ | 60.58 | 433.04 | 4.952 |

Table 3.3: WR-430: modes propagating @ 5 GHz .
to 3200 kW if the acceptable maximum electric field in the waveguide is brought to $10 \mathrm{kV} / \mathrm{cm}$ by pressurizing it by dry air or by SF6. Thus the standard rectangular waveguide WR-430 can be effectively used to transmit even more than the 500 kW generated by a single klystron in the $T E_{10}$ fundamental mode.
The WR-430 is oversized at 5 GHz , thus it can propagate high order modes other than the fundamental. These modes ( $T E_{m n}$ and/or $T M_{m n}$ ) can be easily obtained from the following relation:

$$
\begin{equation*}
\lambda_{c}=\frac{2}{\sqrt{\left(\frac{m}{a}\right)^{2}+\left(\frac{n}{b}\right)^{2}}} \tag{3.2}
\end{equation*}
$$

being $a$ and $b$ the waveguide largest and smallest rectangular sizes, respectively. All the modes having $\lambda_{c}>\lambda_{0}$ at 5 GHz can propagate inside the WR-430. These modes are listed in Table 3.3; for each of them also the cut-off frequency and the wavelength in waveguide are given
According to the that, the first mode in cut-off at 5 GHz is the $T E_{40}$. On the base of what explained, the path of the transmission line must be accurately designed. In particular, the total length has to be defined with respect to the attenuation (Table 3.3). For the WR-430 it is about $0.2 \% \mathrm{~m}^{-1}$, allowing to have a $25-30 \mathrm{~m}$ line, for an overall power losses lower than the $5-6 \%$ of the transmitted power.
Furthermore, mode filters and waveguide bends have to be properly designed in order to mitigate the higher modes power content, and the accidental generation of spurious modes, due to discontinuities along the line. The MTL path design has to consider not only the best choice in terms of curve typologies, but also the proper allocation of all the waveguides, considering the possibility of both horizontal and vertical curvatures (in other words the waveguide bend must be designed for a curve on both the $E$ and $H$ plane).
In addition, ideally, each MTL should provide almost the same power to the launcher in order to guarantee the uniformity of the PAM module radiation. This aspect
needs to be more detailed and discussed, and the problem of an nonuniform PAM excitation must be deeply studied.
It can be noted that, for a distance of $30 m$, the WR- 430 standard, has an attenuation of 0.261 dB , equal to the $6 \%$ of transmission losses. If it were possible to reduce the distance, other dimensions could be considered for the waveguides, with the advantage to reduce the number of propagating unwanted modes. Considering for example an hypothetic distance of $10 m$, the WR-229 can be used with an acceptable maximum electric field in the waveguide brought to $10 \mathrm{kV} / \mathrm{cm}$ (by pressurizing it by dry air or by SF6). Such solution not only will allow an unimodal propagation, but also a maximum power handling of 1.6 MW , and a total attenuation of $4.9 \%$. Of course this would be the optimal solution for an high power unimodal transmission.

### 3.2 Mode Filters

Scientific literature related to this topic is quite scarce and totally lacking in guidelines useful to design such devices; this work aims at giving a contribution in this sense. In detail, the dependence of mode filter performance on various geometrical parameters has been derived with analytical and numerical approaches. Subsequently the strength and distribution of the electric field inside the corrugations has been studied by means of a 3D full-wave Finite Element Method. The problems related with degenerate modes are finally examined and a structure where all propagating modes have different phase velocity has been analyzed. The mode filters design is based on the study of corrugations. Unwanted modes in oversized rectangular waveguides are generally divided into two classes:

- Class 1 - comprises $T E_{m n}$ and $T M_{m n}$ modes with $n \neq 0$;
- Class 2 - comprises $T E_{m 0}$ modes with $m \neq 1$ and can be further divided in modes with $m$ even and modes with $m$ odd.

The structure depicted in Fig. 3.1 can attenuate most of the spurious modal content of oversized rectangular waveguides. More precisely, modes of class 1 are attenuated by means of transversal corrugations in the lateral walls, while modes of class 2 with $m$ even are attenuated by means of longitudinal corrugations in the middle of the horizontal walls (corrugations like these also affect $T E_{0 n}$ modes). With reference to the rest, i.e. modes of class 2 with $m$ odd, they can not be attenuated by filters based on corrugated waveguides without affecting the fundamental mode. Inside both transversal and longitudinal corrugations, making use of the notation previously adopted (Chapter 1), modes are defined by the following eigenvectors for


Figure 3.1: Mode filters for rectangular waveguide.
$T E_{m n}$ modes, with $m=0,1,2, \ldots$ and $n=0,1,2, \ldots$, and for $T M_{m n}$ modes, with $m=1,2,3, \ldots$ and $n=1,2,3, \ldots$ :

$$
\begin{align*}
& \mathbf{h}_{z}=\cos \left(\frac{m \pi x}{a c}\right) \cos \left(\frac{n \pi y}{b c}\right) \hat{\mathbf{z}}  \tag{3.3}\\
& \mathbf{e}_{z}=\sin \left(\frac{m \pi x}{a c}\right) \sin \left(\frac{n \pi y}{b c}\right) \hat{\mathbf{z}} \tag{3.4}
\end{align*}
$$

where $(b c)$ and ( $a c$ ) are the height and the width of the corrugation, respectively. Once the complete expression of the electromagnetic field has been derived from Eq. (3.3) and (3.4), it has to be inserted in the boundary conditions. Subsequently, enforcing mode orthogonality, a linear system like the following one is obtained for each mode:

$$
\left\{\begin{array}{l}
C_{v}^{+}+C_{v}^{-}=C_{a}^{+}+C_{a}^{-}  \tag{3.5}\\
k_{z v}\left(C_{v}^{+}-C_{v}^{-}\right)=k_{z a}\left(C_{a}^{+}-C_{a}^{-}\right) \\
C_{a}^{+}=-C_{a}^{-} e^{2 k_{z a} h c a}
\end{array}\right.
$$

From Eq. (3.5) the derivation of reflection and transmission coefficient at the corrugation input is straightforward:

$$
\begin{equation*}
\Gamma_{i n}=\frac{C_{v}^{+}}{C_{v}^{-}}=1-\frac{2 k_{z a}}{k_{z a}+k_{z v} \tanh \left(k_{z a} h c a\right)} \tag{3.6}
\end{equation*}
$$

Eq. (3.6) has been plotted, as a function of the absorber thickness hca, in Fig. 3.2a) for the $T E_{10}$ mode at 5 GHz . The curves have been obtained considering corrugations with $b c=10 \mathrm{~mm}$ and several widths $a c$; Silicon Carbide ( SiC ) has been used as absorbing material.

The refection coefficient has a profile like a damped cosine superimposed on a constant value equal to -3 dB . In order to minimize it, corrugations have to be filled with about 4 mm of Silicon Carbide; $h c a>20 \mathrm{~mm}$ can be alternatively chosen in order to avoid that the absorption depends on SiC thickness. A similar behavior is obtained for the remaining propagating modes. Fig. 3.2b) shows the reflection coefficients of the three propagating modes in a corrugation with $b c=10 \mathrm{~mm}$ and $a c=100 \mathrm{~mm}$ at a working frequency of 5 GHz . Two structures, like those reported


Figure 3.2: Reflection coefficient at the input of a corrugation at 5 GHz and partially filled of $\left.\operatorname{SiC}\left(\rho=3.22 g / \mathrm{cm}^{3}, \epsilon_{r}=13.5, \tan \delta=0.44\right): a\right)$ of the $T E_{10}$ mode with $b c=10 \mathrm{~mm}$ and several widths $a c$, and $b$ ) of three propagating modes with $b c=10$ $\mathrm{mm}, a c=100 \mathrm{~mm}$.
in Fig. 3.3, have been simulated by using any 3D-FEM solver: they only have either lateral or longitudinal corrugations and mainly attenuate, respectively, modes of class 1 and class 2. For sake of clarity, the correspondence with previously-defined corrugation parameters has been reported in the same figure. Both filters are based on the standard waveguide WR-430 at the frequency of 5 GHz . Under these working conditions ten modes can propagate, as reported in Table 3.3. Unwanted modes are given by four couples of degenerate modes $\left(T E_{01} / T E_{20}, T E_{11} / T M_{11}, T E_{21} / T M_{21}\right.$, and $T E_{31} / T M_{31}$ ) plus the $T E_{30}$ mode. The latter one can not be absorbed by filters based on corrugations (in fact it has partially the same current distribution of the fundamental one, and any filtering of this mode will implicate also a filtering of the $T E_{10}$ ) so, if it would be accidentally generated along the path of the waveguide, a different filtering approach should be adopted. The outcomes of the parametric analysis are reported in Fig. 3.4a)-3.4b) for filters with lateral and longitudinal corrugations, respectively. Such results confirm the behavior predicted by the model: the absorption of unwanted modes oscillates before approaching to a constant value


Figure 3.3: Filters with either lateral $a$ ) or longitudinal b) corrugations.
as the absorber thickness varies. It has a mirror profile with reference to the reflection coefficients of Fig. 3.4. As previously said, this happens because the less power corrugations reflect, the more power the filter absorbs. The reflection coefficients for other modes are depicted in Fig. 3.3. The optimum absorber thickness given by the parametric analysis is a few tenths of millimeters smaller than the prediction of the model. In any case the model gives a good starting value for the research of the exact optimum thickness. A complete parametric analysis performed with HFSS ${ }^{\circledR}$, or similar software, takes a considerable amount of time if run on a general purpose computer. On the other side Eq. (3.6) can be plotted in a very short time.
With reference to the remaining corrugation dimensions, $h c v$ and $a c$, their effects are complex, reciprocally related and depend on the corrugation location. In any case two general rules can be extrapolated from the full-wave simulations. On one hand, as the depth of the corrugation empty region increases, also the coupling between spurious modes increases; in addition the corrugation length has a more influential effect on the performance of the mode filter. This phenomenon sets an upper bound to $h c v$, above which it becomes difficult to evaluate the filtering performance and to find clear dependences on the geometrical parameters. On the other hand, a relatively small value of $h c v$ too small (less than 4 mm for the present case) negatively affects the transmission properties of the fundamental mode. The aforementioned dependences have been used as guidelines for the preliminary design of mode filters. In particular some simulations using simultaneously both lateral and longitudinal corrugation has been developed; the filter parameters are resumed in he following Table 3.4, where the values are intended to be negative and expressed in dB.


Figure 3.4: Absorption performances at 5 GHz in a WR430 of: a) $T E_{11}$ and $T M_{11}$ modes with 20 lateral corrugations having $d c=10 \mathrm{~mm}, a c=42 \mathrm{~mm}, b c=7 \mathrm{~mm}$, and $h c v=4 \mathrm{~mm}$, and $b) T E_{20}$ and $T E_{01}$ modes with 2 longitudinal corrugations having $a c=100 \mathrm{~mm}, b c=10 \mathrm{~mm}$ and $h c v=20 \mathrm{~mm}$.

A 100 mm long device, comprising both lateral and longitudinal corrugations has been designed. Its performances are reported in Table 3.5, where the highest coupling coefficients have been highlighted in bold, while the elements on the main diagonal, represent transmission coefficients. Finally reflection coefficients are shown in Fig. 3.5 and their values at 5 GHz in Table 3.6. As can be appreciated by examining Table 3.4, the $T E_{10}$ mode and the $T E_{30}$ mode are essentially non attenuated ( $0.46 \%$ and $0.69 \%$, respectively), but the transmission of higher order modes are differently reduced. As expected, since degenerate modes have the same phase velocity, they are affected by different absorptions, that is with reference of their current distribution.
The performances shown in Table 3.4, confirm essentially good filtering capabilities, so that the corrugation method can be further developed and optimized once the definitive MTL path will be assessed. The corrugation reflection coefficient shown

|  | $a c[\mathrm{~mm}]$ | $b c[\mathrm{~mm}]$ | $h c v[\mathrm{~mm}]$ | $h c a[\mathrm{~mm}]$ | $d c[\mathrm{~mm}]$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Lateral | 42 | 7 | 4 | 4 | 3 |
| Longitudinal | 100 | 16 | 25 | 4 | none |

Table 3.4: Corrugation dimensions.

| IN $\backslash$ <br> OUT | $T E_{10}$ | $T E_{01}$ | $T E_{20}$ | $T E_{11}$ | $T M_{11}$ | $T E_{21}$ | $T M_{21}$ | $T E_{30}$ | $T E_{31}$ | $T M_{31}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T E_{10}$ | 0.02 | 82.38 | 67.69 | 73.71 | 65.62 | 76.22 | 84.15 | 25.61 | 54.5 | 67.99 |
| $T E_{01}$ | 78.13 | 6.63 | 75.08 | 78.96 | 63.97 | $\mathbf{1 1 . 7 3}$ | $\mathbf{1 0}$ | 74.22 | 56.82 | 62.45 |
| $T E_{20}$ | 64.97 | 65.85 | $\mathbf{4 . 0 1}$ | 70.49 | 65.11 | 80.28 | 70.98 | 79.34 | 57.36 | 72.30 |
| $T E_{11}$ | 83.80 | 81.88 | 77.22 | 6.46 | $\mathbf{1 0 . 6 9}$ | 77.59 | 67.82 | 63.28 | 28.35 | 17.26 |
| $T M_{11}$ | 67.49 | 67.65 | 96.36 | $\mathbf{1 0 . 6 9}$ | 1.81 | 71.47 | 64.21 | 68.63 | 25.32 | 25.15 |
| $T E_{21}$ | 70.94 | $\mathbf{1 1 . 7 1}$ | 62.33 | 63.34 | 63.37 | 20.99 | $\mathbf{1 3 . 3}$ | 68.5 | 87.03 | 80.64 |
| $T M_{21}$ | 80.09 | $\mathbf{1 0 . 0 1}$ | 83.32 | 65.9 | 63.79 | $\mathbf{1 3 . 3 1}$ | 5.48 | 76.26 | 61.53 | 68.47 |
| $T E_{30}$ | 25.61 | 73.47 | 66.89 | 62.11 | 61.61 | 70.18 | 75.19 | 0.03 | 50.2 | 68.93 |
| $T E_{31}$ | 59.59 | 61.97 | 54.44 | 28.79 | 25.27 | 50.99 | 59.88 | 56.4 | 1.93 | 21.68 |
| $T M_{31}$ | 95.84 | 67.26 | 72.25 | 17.26 | 25.21 | 73.41 | 63.37 | 63.06 | 21.66 | 16.58 |

Table 3.5: Coupling matrix among the propagating modes for a filter of 100 mm .


Figure 3.5: Reflection coefficients for several modes at the input of a corrugation with $b c=10 \mathrm{~mm}$ at 5 GHz and partially filled of $\mathrm{SiC}\left(\rho=3.22 \mathrm{~g} / \mathrm{cm}^{3}, \epsilon_{r}=13.5\right.$, $\tan \delta=0.44)$.
that the fundamental mode is almost totally confined inside the rectangular waveguide without any significant contact with corrugations. Anyway, the reflection of higher order modes, with reference to $T E_{01}, T E_{20}, T E_{21}, T E_{31}$, and $T M_{31}$, has to be enhanced to rich at least -20 dB .

| $T E_{10}$ | $T E_{01}$ | $T E_{20}$ | $T E_{11}$ | $T M_{11}$ | $T E_{21}$ | $T M_{21}$ | $T E_{30}$ | $T E_{31}$ | $T M_{31}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| -42.6 | -18.8 | -9.2 | -24.1 | -22.5 | -13.9 | -21.7 | -32.6 | -5.2 | -4.9 |

Table 3.6: Reflection coefficients [dB] @ 5 GHz .

Taking the previous design as reference, the study of mode filters has been focused on three main activities:

1. Power capability: the strength and distribution of the electric field inside the corrugations has been studied.
2. Parametric analysis: the dependence of mode filter performances on various geometrical parameters have been analyzed by means of HFSS ${ }^{\odot}$ simulations.
3. Degenerate modes: new structures where all propagating modes have different phase velocity have been investigated.

If the amplitude of the unwanted modes is too high, mode filters could be subject to electric discharges or excessive heating of the absorber. To properly design mode filters, the highest expected level of the unwanted modes must be known. Assuming that the output of the power sources is a pure $T E_{10}$ mode, the power carried by spurious modes, excited across the transmission line by discontinuities such as bends, has to be estimated. Up to now two cases have been studied with reference to the breakdown relevant parameters, just to have an idea about their order of magnitude. The first one consists in a WR-430, working at 5 GHz , with 20 lateral corrugations having $d c=10 \mathrm{~mm}, a c=42 \mathrm{~mm}, b c=7 \mathrm{~mm}$ and $h c v=4 \mathrm{~mm}$; the second one is a WR-430, working again at 5 GHz with 2 longitudinal corrugations having $a c=100 \mathrm{~mm}, b c=10 \mathrm{~mm}$ and $h c v=20 \mathrm{~mm}$, and they are represented in Figs. 3.6. The maximum electric field inside the corrugations has been calculated when each propagating unwanted mode has a power of 2 kW and a phase that maximizes the absorption. The results are about $7.3 \mathrm{kV} / \mathrm{cm}$ for the first case and about $2.3 \mathrm{kV} / \mathrm{cm}$ for the second one.
With reference to the first item, data have been collected in order to produce curves showing the behavior of reflection, transmission, absorption and conversion of different modes as a function of the corrugation dimensions. A preliminary iteration of the parametric analysis has given the outcomes reported in Table 3.6. These values

(a)

(b)

Figure 3.6: Average electric field between the closest walls of $a$ ) longitudinal corrugations and $b$ ) lateral corrugations.

|  | $a c[\mathrm{~mm}]$ | $b c[\mathrm{~mm}]$ | $h c v[\mathrm{~mm}]$ |
| :--- | :---: | :---: | :---: |
| Lateral | 34.4 | 7 | 4.6 |
| Longitudinal | 100 | 16 | 10 |

Table 3.7: Corrugation revised dimensions.
have been considered a good compromise between wanted and unwanted losses that take place in mode filters. Corrugation width (ac) mainly affects unwanted mode absorption and fundamental mode transmission; its value is fixed by the admissible power loss of $T E_{10}$ mode. Filters based on corrugated waveguides present the following drawback: total absorption is not given by the sum of absorptions calculated when single modes excite the waveguide. Depending on their phase, modes may interfere destructively just at the input of the corrugation and do not excite it. When modes have different phase velocity, the relative phase between them changes during the propagation; accordingly, if the mode filters are long enough, spurious modes will interfere both constructively and destructively at the corrugation inputs. Nevertheless, in case of degenerate modes, the relative phase remains unchanged through the filter. This is shown in Fig. 3.7 for a filter with a single longitudinal corrugation on the top wall: the magnitude of the component of the surface current exciting the corrugation has been plotted as an overlay for different excitations of the $T E_{01}$ and $T E_{20}$ modes. A phase shift between degenerate modes can be introduced


Figure 3.7: Overlays of surface current along $y$ direction for different excitations exciting the corrugation for different excitation cases involving the $T E_{01}$ and $T E_{20}$ modes.

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by slightly deforming the shape of the inner walls of the waveguide, as for example cutting arc-shaped longitudinal grooves in the central part of some walls [19]. A groove is defined by three dimensions: length $\left(L_{g}\right)$, width $(w g)$ and height $(h g)$, as indicated in Fig. 3.8. The effect of two lateral grooves with $b g=29 \mathrm{~mm}, h g=11$ mm and $a c=550 \mathrm{~mm}$ is shown in Fig. 3.9. Here the last two patterns of Fig. 3.7 can be recognized respectively at the input and at the output of the waveguide with grooves. Moreover, Fig. 3.9 shows how the relative phase shift between some degenerate modes varies in different waveguide cross-sections. In order to reduce the reflection coefficient at the transition with standard waveguides, the grooves must be tapered. Grooved waveguides can solve the problems related to degenerate modes; nevertheless, their integration with corrugations has to be deeply investigated.


Figure 3.8: Groove on the top wall of a waveguide.


Figure 3.9: Change in the relative phase between the $T E_{01}$ and $T E_{20}$ modes ( 2 W ) across 700 mm long standard $a$ ) and grooved $b$ ) WR-430. The last one has 2 lateral tapered grooves with $b g=29 \mathrm{~mm}, h g=11 \mathrm{~mm}$ and $a c=550 \mathrm{~mm}$.

### 3.3 Bends

The problem of the propagation of the electromagnetic field in curved waveguides of rectangular cross section is comprehensively examined in [23] (only for unimodal propagation regime). From a practical point of view, several solutions can be taken into account. In this context, the principal specifications that characterize the design are:

- minimize the reflection of the fundamental $T E_{10}$ mode ( $\left|S_{11}\right|$ parameter $)$;
- maximize the transmission of the fundamental $T E_{10}$ mode ( $S_{12}$ parameter);
- minimize the coupling between the $T E_{10}$ mode and other spurious modes that propagate at $5 \mathrm{GHz}\left(S_{1 j}, j=3, \cdots, 10\right)$.

Uniform bends in rectangular waveguides, are frequently used components in many microwave subsystems both a wide range of applications. Several contributions can be found in the technical literature concerning bends [29]. In particular, Rice [30] obtains two approximate formulas for the reflection coefficient of both the H and E plane bends with large radius of curvature.
Cochran [31] presents results of the propagation constants of bends as function of several parameters, by means of a combination of Bessel functions. Bates [32] analyzes the junction between straight and curved waveguides with a method based on an integral equation formulation; Lewin [23] derives approximate solutions through a perturbation analysis. Accatino [33] applied mode-matching technique to analyze both H and E plane bends, using an ad-hoc solution of the characteristic equation (which involves Bessel functions), allowing him to bypass the ill-conditioning of the problem for the propagation constants in the curved region. Weisshaar [34] presented an accurate method based on mode-matching technique where the Helmholtz equation in the curved region is transformed into an eigenvalue problem. Most of the above publications are based on mode matching technique [35] and give scattering parameter representations.
In addition to the analysis of uniform waveguide bends, we consider different bending geometry based on the mitered bends layout [36], where the bend profile is not circular but linear, configuring a cascade of junction of straight waveguide frames. In this case, we also develop an original component consisting in approximating a circular trajectory by means of a polygonal structure made of several straight waveguide segments. We call this kind of bend as "Trapezoidal Mitered Bend" which is described further in this chapter. The analysis outcomes shown a good behavior of this component in comparison with the uniform circular case, performing quite the
same but with an significant reduction of the equivalent bending radius and then of the overall length of the the curved waveguide framework.
For nuclear fusion relevant applications, a preliminary study of bends in oversized rectangular waveguide has been also presented in Nantisa and Tantawi work [37]. Recently, modern CAD tools for complex waveguide systems have been developed which are based on admittance or impedance multi-mode network representations (for instance [38, 39]).

### 3.3.1 Simple Circular Bends

For uniform circular-shape bends, we can use a notation to expand the transverse electric and magnetic field in the curved region [29] as an infinite series of modes. In particular, by looking at Fig. 3. 10, and considering a radius of curvature equal to $R_{b e n d}$, the transverse component of the electromagnetic field can be written by using the associated eigenvectors:

$$
\begin{align*}
\mathbf{e}_{t}^{(c)} & =\sum_{m=1}^{+\infty} V_{m}^{(c)} \mathbf{e}_{m}^{(c)}  \tag{3.7}\\
\mathbf{h}_{t}^{(c)} & =\frac{1}{1+\frac{x}{R_{\text {bend }}}} \sum_{m=1}^{+\infty} I_{m}^{(c)} \mathbf{h}_{m}^{(c)}
\end{align*}
$$

where the reference co-ordinate system is as specified in Fig. 3.11, and the subscripts $(c)$ and $(s)$ denote the curved and straight part, respectively. So that, the propagation in the curved region can be expanded into an infinite series of parallel waveguide eigenvectors as follows:

$$
\begin{align*}
& \mathbf{e}_{m}^{(c)}=\sum_{n=1}^{+\infty} d_{n}^{(m)} \mathbf{e}_{n}^{(s)}  \tag{3.8}\\
& \mathbf{h}_{m}^{(c)}=-\sum_{n=1}^{+\infty} d_{n}^{(m)} \mathbf{h}_{n}^{(s)}
\end{align*}
$$

where

$$
\begin{align*}
& \mathbf{e}_{n}^{(s)}=-\sqrt{\frac{2}{a}} \sin \left[\frac{n \pi}{a}\left(x+\frac{a}{2}\right)\right] \hat{\mathbf{y}}  \tag{3.9}\\
& \mathbf{h}_{n}^{(s)}=-\sqrt{\frac{2}{a}} \sin \left[\frac{n \pi}{a}\left(x+\frac{a}{2}\right)\right] \hat{\mathbf{x}}
\end{align*}
$$

where $n=1,2, \cdots$. The series in Eq. (3.8) are then inserted in the Helmholtz equation obtained for the curved region [?], obtaining a matrix of eigenvalues, and finally the linear system is solved by computing numerically the coefficients of the


Figure 3.10: Uniform bend on $a$ ) E-plane and b) H-plane.


Figure 3.11: Geometry of the waveguide bend.
expansion $d_{m}^{(m)}$ [29]. Once the curved waveguide modes are obtained, the junction between the straight and curved frameworks must be analyzed. Several techniques can used to this aim as indicated in [40, 41] and also a FEM solver can be used. We develop a parametric analysis, varying the radius of curvature of the bend and finding the optimized value for a WR- 430 waveguide at 5 GHz .
Since the only characterizing parameter of a simple circular-trajectory bend, is the bending radius $R_{\text {bend }}$ (once the waveguide dimensions are fixed), the design is not much flexible. The point of interest is the behavior of the device vs. the bending radius, in terms of scattering parameters: $\left|S_{11}\right|$ and $\left|S_{21}\right|$, and coupling between the fundamental $T E_{10}$ mode and all the other modes over cut-off.
Considering at first an E-plane circularly curved rectangular waveguide, we vary the radius $R_{\text {bend }}$ in the range $\left[10^{2}, 10^{3}\right] \mathrm{mm}$, with a step resolution of 5 mm , for total 180 points. The simulation has been carried out, and at the beginning we find the
value of $R_{\text {bend }}$ that simultaneously maximizes the transmission and minimizes the reflection back-going to the input. Once we obtain the optimized value of the bending radius $R_{\text {bend }}^{\text {opt }}$, we perform a frequency sweep simulation for a 20 MHz bandwidth around the central frequency of 5 GHz , that is $f \in[4.99,5.01] \mathrm{GHz}$. Finally, we check the coupling among the propagating modes.
In our analysis, basing on what presented in the previous Section 3.2, about mode filtering, we suppose to have at the input of the waveguide framework only the propagation of the fundamental $T E_{10}$ mode (or even that the higher order modes are strongly attenuated, for instance under -30 dB ). In order to face the transitory effect of the conversion of modes at the discontinuity, we extend the straight part of the waveguide framework $L_{r e f}$ up to 500 mm . A simple geometric layout of the curved waveguide cross section, is sketched in Fig. 3.13. In Fig. 3.14, the $\left|S_{11}\right|$ reflection

|  | w | Nominal frequency | 5 GHz |
| :---: | :---: | :---: | :---: |
|  | $\mathrm{R}_{\text {bend }}$ | Waveguide width | 54.61 mm |
|  |  | Input waveguide length $L_{-}$ref | 500 mm |
|  |  | Bend radius $\mathrm{R}_{\text {bend }}$ | $100-1000 \mathrm{~mm}$ |

Figure 3.12: E-plane curved rectangualr waveguide design paremeters.
parameter is plotted vs. the bending radius $R_{\text {bend }}$. In particular there are several negative peaks of the $\left|S_{11}\right|$ parameter (expressed with markers in the same figure), for instance for $R_{\text {bend }}=215,375,415,425,465,500 \mathrm{~mm}$. Apparently, one could be inclined to choose the most negative value ( $R_{\text {bend }}=465 \mathrm{~mm}$ ), anyway we have to consider that, in high power-transfer application, the specification on the reflection coefficient (once a suitable value of about -30 dB or less, is achieved) is subordinate to the corresponding transmission efficiency. In fact, viewing the next Fig. 3.14, it is clear that the higher power transmission occurs for $R_{\text {bend }}=415 \mathrm{~mm}$. On top of this, the $\left|S_{11}\right|$ parameter at this value is sufficiently small and less than -40 dB . For such a reason, the further optimization will be developed for $R_{b e n d}=415 \mathrm{~mm}$. At this value the transmission parameter is equal to -0.0025 dB that is the $99.94 \%$ of transmission efficiency, while for $R_{\text {bend }}=465 \mathrm{~mm}$ the transmission is $99.84 \%$. With this choice we gain the $0.1 \%$ of transmission that is about 500 W for each waveguide (for a total amount of 48 MTL it turn into 24 kW more or less). On the other hand we consider less significant to decrease the reflection from -44.85 dB for $R_{\text {bend }}=415 \mathrm{~mm}$, to -63.75 dB for $R_{\text {bend }}=465 \mathrm{~mm}$. Furthermore, with a 50 mm


Figure 3.13: H-plane curved rectangular waveguide: $\left|S_{11}\right|$ vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.


Figure 3.14: H-plane curved rectangular waveguide: $\left|S_{21}\right|$ vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.
shorter framework we may save about $4.35 \times 10^{-4} \mathrm{~dB}$ of attenuation.
Moreover, also the coupling parameters suggest this value showing a negative peak just for $R_{\text {bend }}=415 \mathrm{~mm}$, proving an effective stand-alone transmission of the the fundamental $T E_{10}$ mode. In Fig. 3.15, the most of the propagating modes are significantly under the -60 dB threshold, and they are not excited in practice. Only the degenerate modes pair $T E_{11}-T M_{11}$ are weakly allowed to propagate, for a power content lower than $-40 \mathrm{~dB}(0,001 \%)$. On the contrary, for $R_{\text {bend }}=465 \mathrm{~mm}$ we have a $T E_{11}$ content around -40 dB , but also a $T M_{11}$ mode excitation around -33 dB . Nevertheless, we must take into account that the power injected in the waveguide is very high and this percentage is equivalent to several tens of Watts.
For what exposed, we perform a frequency sweep of the curved waveguide for the optimized value of the bending radius. In particular we vary the frequency in the range $[4.99,5.01] \mathrm{GHz}$ with a step resolution of 1 MHz for total 20 points. This analysis is useful to make some assessments about the behavior of the component in the
bandwidth. As exposed in Chapter 1, the LHCD system is a narrowband system of less than 20 MHz of bandwidth, so that it is desirable a quite uniform trend in this range. In the following Figs. 3.16-3.18, the $\left|S_{11}\right|,\left|S_{21}\right|$, and coupling parameters are


Figure 3.15: H-plane curved rectangular waveguide: coupling vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.
shown. In particular, a linear trend of the parameters vs. frequency can be noted. Anyway, the dynamic is quite restricted in all the cases: the $\left|S_{11}\right|$ parameter is affected by a range between the bandwidth outers that is lower than 1 dB , decreasing as the frequency increases. The $\left|S_{21}\right|$ parameter shows an opposite trend and, in the worse case it is equal to $-0,004 \mathrm{~dB}$ that is about the $99.9 \%$ of transmission.


Figure 3.16: E-plane curved rectangular waveguide: $\left|S_{11}\right|$ at the optimized value of $R_{\text {bend }}=415 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.

Also the coupling between $T E_{10}$ and higher order modes increases moderately as the frequency increases. In this case the gap is lower than 3 dB for both the $T E_{11}$ and $T M_{11}$ mode. This value is quite significant in our opinion, because it means that the coupling varies with a factor of two in the bandwidth. The coupling between


Figure 3.17: E-plane curved rectangular waveguide: $\left|S_{21}\right|$ at the optimized value of $R_{\text {bend }}=415 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.


Figure 3.18: E-plane curved rectangular waveguide: coupling at the optimized value of $R_{\text {bend }}=415 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.
the fundamental mode and the other propagating modes at 5 GHz is reported in the following Table 3.8. As stated before, all the values are suitably adequate for our scopes.
For H-plane waveguide bends, the same analysis has been provided, and the obtained results confirm a potential use of such bends for high-power fusion engineering applications. The H-plane curved waveguide framework design is sketched in the next Fig. 3.19. Also in this case we note several potential optimized value of the bending radius. In particular, in Fig. 3.20 six values are indicated: $R_{\text {bend }}=450,540,630,660,675 \mathrm{~mm}$. Also in this case, as happened for the E-plane band optimization, the transmission coefficient $\left|S_{21}\right|$ performances give the final assessment: in Fig. 3.21 is clearly suggested the choice of $R_{\text {bend }}=630 \mathrm{~mm}$, and in this case, we have no ambiguity since the nearest value is exactly two orders magnitude

| Modes @ 5 GHz | Coupling [dB] |
| :--- | :---: |
| $T E_{10}$ | -0.0025 |
| $T E_{01}$ | -73.9 |
| $T E_{20}$ | -75.13 |
| $T E_{11}$ | -54.53 |
| $T M_{11}$ | -47.97 |
| $T E_{21}$ | -79.41 |
| $T M_{21}$ | -75 |
| $T E_{30}$ | -70.55 |
| $T E_{31}$ | -77.16 |
| $T M_{31}$ | -66.87 |

Table 3.8: Coupling matrix for an uniform E-plane bend.

| $\mathbf{R}_{\text {bend }}$ | Nominal frequency | 5 GHz |
| :--- | :--- | :---: |
|  | Waveguideheight | 109.22 mm |
|  | Input waveguide length L_ref | 500 mm |
|  | Bendradius R_bend | $100-1000 \mathrm{~mm}$ |

Figure 3.19: H-plane curved rectangualr waveguide design paremeters.
lower. A following frequency sweep analysis has been done considering a 20 MHz bandwidth around 5 GHz , and the outcomes confirm the component behavior. In particular, in this case the transmission coefficient printed in Fig. 3.24, presents a maximum for the central frequency of 5 GHz where is more than $99.98 \%$.
The gap between the bandwidth outers for the reflection coefficient is less than 1 dB and for transmission is less than $5 \times 10^{-4} \mathrm{~dB}$. Also the coupling is quite good around the chosen optimized value. In particular it is lower than -40 dB almost on the entire bandwidth (with the only exception of the $T E_{01}-T E_{20}$ at the lowest frequency) as depicted in Fig. 3.25.
Finally the other modes coupling values are presented in Table 3.9. Indeed, in Table 3.10 the optimized parameters for uniform E and H plane bends in oversized rectangular waveguide WR-430, are summarized. For what assessed, the E and H plane bends seem to be almost equivalent in terms of performances. In the H-plane bend have a higher transmission and a lower reflection, but it is longer and thus the overall attenuation is a little bit more. In particular the length gap between the


Figure 3.20: H-plane curved rectangular waveguide: $\left|S_{11}\right|$ vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.


Figure 3.21: H-plane curved rectangular waveguide: $\left|S_{21}\right|$ vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.


Figure 3.22: H-plane curved rectangular waveguide: coupling vs. $R_{\text {bend }} @ 5 \mathrm{GHz}$.


Figure 3.23: H-plane curved rectangular waveguide: $\left|S_{11}\right|$ at the optimized value of $R_{\text {bend }}=630 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.


Figure 3.24: H-plane curved rectangular waveguide: $\left|S_{21}\right|$ at the optimized value of $R_{\text {bend }}=630 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.


Figure 3.25: H-plane curved rectangular waveguide: coupling at the optimized value of $R_{\text {bend }}=630 \mathrm{~mm}$ for $f \in[4.99,5.01] \mathrm{GHz}$.
waveguide bends E and H is about 215 mm , corresponding to an additive attenuation of $4.3 \times 10^{-4} \mathrm{~dB}$, so that the transmission coefficient must be reduced of that quantity, resulting to be about $1.4 \times 10^{-3}$ that is quite similar compared with the $\left|S_{21}\right|$ of E-plane bend. Moreover, the coupling for the H-plane bend is a little worse but this not seems to be significant.

| Modes @ 5 GHz | Coupling [dB] |
| :--- | :---: |
| $T E_{10}$ | -0.0006 |
| $T E_{01}$ | -46.3 |
| $T E_{20}$ | -70.1 |
| $T E_{11}$ | -71.37 |
| $T M_{11}$ | -68.1 |
| $T E_{21}$ | -68.85 |
| $T M_{21}$ | -68.67 |
| $T E_{30}$ | -53.15 |
| $T E_{31}$ | -85.96 |
| $T M_{31}$ | -77.33 |

Table 3.9: Coupling matrix for an uniform H-plane bend.

| Spec. | E-plane | H-plane |
| :--- | :---: | :---: |
| $R_{\text {Bend }}[\mathrm{mm}]$ | 415 | 630 |
| $\left.\left\|S_{11}\right\| \mathrm{dB}\right]$ | -44.85 | -47.6 |
| $\left\|S_{21}\right\|[\mathrm{dB}]$ | $-2.5 \times 10^{-3}$ | $-6 \times 10^{-4}$ |
| coupling $[\mathrm{dB}]$ | $<-40 \mathrm{~dB}$ | $\leq-40 \mathrm{~dB}$ |

Table 3.10: Uniform bends design and performances.

### 3.3.2 Mitered Bends

A mitered bend consists in two parts of rectangular waveguide cross section connected as in Fig. 3.26. The theoretical fundamentals of the electromagnetic propagation in such a connected waveguide is extensively exposed for a mono-modal configuration in the exhaustive work of Cornet, Dussaux, and Chandezon [42], in which this two-dimensional problem is considered as two different problems according to the direction of incident electric field in relation to the bend. Both these

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problems ( E and H plane cases) are geometrically similar, but mathematically different. In an H-plane bend, the TE modes are generated, but in an E-plane bend, the longitudinal-section electric (LSE) modes are generated.
To solve this dual problem the Maxwell's equations are re-written in a tensorial


Figure 3.26: Mitered bend on $a$ ) E-plane and b) H-plane.
co-ordinate system and thus, the electric and magnetic components are expanded on trigonometric series that satisfy the boundary conditions on the perfectly conducting walls. The boundary conditions solution, allow us to determine the scattering matrix of this elementary structure.
Maxwell's equations in covariant form are solved in the oblique co-ordinate system $\left(\Sigma_{o}\right)$, which is obtained by the Cartesian system $\left(\Sigma_{c}\right)$ with the following transformation:

$$
u=x \quad v=y \quad w=z-x \tan (\phi)
$$

where $\phi$ is the angle between the cross section plane and the longitudinal vector $\hat{\mathbf{z}}$. The Jacobian matrices of the transformation between the two systems are

$$
A_{\Sigma_{o}}^{\Sigma_{c}}=\left[\begin{array}{ccc}
1 & 0 & -\tan (\phi)  \tag{3.10}\\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \quad A_{\Sigma_{c}}^{\Sigma_{o}}=\left[\begin{array}{ccc}
1 & 0 & \tan (\phi) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

In particular, In the case of H -plane bend, the incident $T E_{10}$ mode couples with the $T E_{m 0}$ mode only. The mode coupling in rectangular discontinuities can be demonstrated to be independent from the $b$ side of the waveguide, thus the covariant component can be derived [42]:

$$
\left\{\begin{array}{l}
\partial_{w} E_{v}=i k \eta H_{u}  \tag{3.11}\\
\partial_{w} \eta H_{u}=\frac{1}{1+\tan (\phi)}\left(i k-\frac{1}{i k} \partial_{u} \partial_{w}\right) E_{v}+\frac{2 \tan (\phi)}{1+\tan ^{2}(\phi)} \partial_{u} \eta H_{u}
\end{array}\right.
$$

where $\eta=\sqrt{\mu_{0} / \epsilon_{0}}$ is the impedance of a vacuum.
In the case of E-plane bend, the $T E_{10}$ mode generates the $L S E_{m 1}$ modes composed of
the superposition of $T E_{m 1}$ and $T M_{m 1}$ modes. In this case the transverse components are

$$
\left\{\begin{array}{l}
\partial_{w} \eta H_{v}=\frac{k^{2}-\beta^{2}}{i k\left[1+\operatorname{ta}^{2}(\phi)\right]} E_{u} \frac{\tan (\phi)}{1+\tan ^{2}(\phi)} \partial_{u} \eta H_{v}  \tag{3.12}\\
\partial_{w} E_{u}=\partial_{u}\left[\frac{i k}{\left(k^{2}-\beta^{2}\right)\left[1+\tan ^{2}(\phi)\right]} \partial_{u} \eta H_{v}+\frac{\tan (\phi)}{1+\tan ^{2}(\phi)} E_{u}\right]-i k \eta H_{v}
\end{array}\right.
$$

where $\beta=\pi / b$.
The resolution of differential systems of Eqs. (3.11) and (3.12) can be performed by means of the method of moments or FEM, resulting in an eigenvalue problem for the determination of the electric and magnetic transverse component. Finally, the scattering matrix of the whole structure can be obtained.
The analysis of mitered band both on E and H plane, do not show a good behavior for our scopes. In particular, with reference of Fig. 3.27, the E-plane mitered bend is firstly described.
The design analysis consists in varying the only design parameter $L$ (once fixed the waveguide cross section) in the range $[0.6,54.6] \mathrm{mm}$ with a step resolution of 0.5 mm . The next Fig. 3.28 shows the $\left|S_{11}\right|$ coefficient vs. $L$. where:

$$
\left\{\begin{array}{l}
d=\sqrt{2}(w-L)  \tag{3.13}\\
c=\sqrt{2} \frac{1+w}{2}
\end{array}\right.
$$

As it can be noticed, the $\left|S_{11}\right|$ value is quite high at its peak, around -15.28 dB for $L=27.1 \mathrm{~mm}$. This means that the reflection is the $29.6 \%$ of the injected power. This amount of reflected power is then not safely supported from klystrons which may be seriously damaged.
Moreover, the transmission is very poor, around -1.6 dB , that is the $69.1 \%$. For this reason this kind of structure cannot be taken into account for the transmission line. The coupling among the $T E_{10}$ and higher order modes, is specified in the next Table 3.11.
It can be noted that the coupling between the $T E_{10}$ mode and both of degenerating modes $T E_{11} / T M_{11}$ is quite high ( $2.75 \%$ for $T E_{11}$ and $17.3 \%$ for $T M_{11}$ ), as expected by the LSE theory.
For H-plane mitered bend, the layout is specified in Fig. 3.30. The design parameter $L$ has been varied in the range $[0.6,108.6] \mathrm{mm}$ with a step resolution of 0.5 mm . In this case the outcomes are quite better compared with the E-plane case. The reason of this improvement can be found in the LSE theory. In fact, in E-plane case LSE modes are generated by coupling with $T E_{10}$ mode, determining a power conversion among them. In H-plane case, these mode are not excited, and the $T E_{10}$ mode coupling, involves the $T E_{m 0}$ modes only. As reported in Figs. 3.31-3.32, the

| Modes @ 5 GHz | coupling [dB] |
| :--- | :---: |
| $T E_{01}$ | -83.38 |
| $T E_{20}$ | -76.89 |
| $T E_{11}$ | $\mathbf{- 1 5 . 6 1}$ |
| $T M_{11}$ | $-\mathbf{7 . 6 2}$ |
| $T E_{21}$ | -82.23 |
| $T M_{21}$ | -76.33 |
| $T E_{30}$ | -65.92 |
| $T E_{31}$ | -64.73 |
| $T M_{31}$ | -67 |

Table 3.11: E-plane mitered bend: coupling.
$\left|S_{11}\right|$ parameter in suitably under -40 dB for $L=6.1 \mathrm{~mm}$, but in spite of this, the transmission coefficient is still to low, around $-1 \mathrm{~dB}(79.4 \%)$. In conclusion, also for the H-plane case, this kind of solution must be avoided. The coupling for H-plane mitered bend is reported in Table 3.12.
Also in this case, the theoretic coupling is confirmed involving the degenerating

| Modes @ 5 GHz | coupling [dB] |
| :--- | :---: |
| $T E_{01}$ | $-\mathbf{8 . 5 8}$ |
| $T E_{20}$ | $\mathbf{- 1 9 . 5 2}$ |
| $T E_{11}$ | -48.29 |
| $T M_{11}$ | -77.67 |
| $T E_{21}$ | -80.49 |
| $T M_{21}$ | -81.2 |
| $T E_{30}$ | -83.77 |
| $T E_{31}$ | -83 |
| $T M_{31}$ | -89.2 |

Table 3.12: E-plane mitered bend coupling.
modes $T E_{10} / T E_{02}$ (the $13.9 \%$ for $T E_{20}$ mode and the $1.1 \%$ for $T E_{01}$ ). Moreover, the magnitude of $T E_{30}$ mode is now reduced, if comparing with the circular H-plane bend.
As a first general consideration, it seems that mitered solutions mitigate the coupling with $T E_{30}$ and, a modified version of this kind of curve, could achieved good performances in terms of return loss and transmission efficiency and simultaneously reduce the length of the curved framework.

|  | Nominal frequency | 5 GHz |
| :---: | :---: | :---: |
|  | Waveguide width w | 54.61 mm |
|  | Input waveguide length $\mathrm{L}_{\text {ref }}$ | 500 mm |
| W | L | $0.6-54.6 \mathrm{~mm}$ |

Figure 3.27: E-plane mitered bend design parameters.


Figure 3.28: E-plane mitered bend: $\left|S_{11}\right|$ vs. $L @ 5 \mathrm{GHz}$.


Figure 3.29: E-plane mitered bend: $\left|S_{21}\right|$ vs. $L @ 5 \mathrm{GHz}$.

|  | Nominal frequency | 5 GHz |
| :--- | :--- | :---: |
|  | Waveguide width W | 109.22 mm |
| Input waveguide length $\mathrm{L}_{\mathrm{ref}}$ | 500 mm |  |
| L | L | $0.6-108.6 \mathrm{~mm}$ |

Figure 3.30: E-plane mitered bend design parameters.


Figure 3.31: H-plane mitered bend: $\left|S_{11}\right|$ vs. $L$ @ 5 GHz .


Figure 3.32: H-plane mitered bend: $\left|S_{21}\right|$ vs. $L @ 5 \mathrm{GHz}$.

### 3.3.3 Trapezoidal Mitered Bend

As previously demonstrated, the mitered bend solution is not suitable for our purpose, both for the high reflection coefficient but above all, for the extremely poor transmission. At the same time this kind of structure has the advantage to be more compact than the circular trajectory bends, with a relevant reduction of the attenuation. By the way, the circular bends are quite good performing for our purposes, also with a view to the mode filters use. So that, considering a mitered bend approach, a better design flexibility can be achieved by using a modular structure, formed by $N$ trapezoidal elements (as described in Fig. 3.27). In this case, when $N$ is larger, the trapezoidal curve approximates a circular trajectory, indeed when $N$ is low (i.e. $N=2,3$ ) the structure can be considered as a modified version of the mitered bend. Thus, the design of this kind of bends, consists into the optimization of the mitered bend parameters for each one of the $N$ elements and also the number of elements becomes a design parameter. In this section the design of such a compo-


Figure 3.33: Trapezoidal mitered bend layout.
nent is considered. In particular the study involves several different types of curves with 2, 3, 4 and 5 elements. The two-element trapezoidal bends (Fig. 3.28), are obtained The two trapezoidal elements are obtained starting from the square with side $w \prime$ divided into two triangles isosceles with angle of $45^{\circ}$ (dividing the $90^{\circ}$ angle up on the right, into the number of trapezoidal elements), choosing arbitrarily the length $d=100 \mathrm{~mm}$ corresponding to the distance between the center of the reference system and the transverse sections of the waveguides. A parametric analysis for $0^{\circ} \leq \alpha \leq 45^{\circ}$ has been done; the value $\alpha=35^{\circ}$ seems to be the most suitable, anyway, the $\left|S_{21}\right|$ (absolute values) which has a maximum for $\alpha=25^{\circ}$. In other words the best value for the reflection coefficient does not coincide with the best value for the transmission. The performances of this curved framework are specified in the next Tables 3.13-3.14 for the E-plane and H-plane configurations, respectively. In particular, for $\alpha=35^{\circ}, S_{11}=-20.21 \mathrm{~dB}$ and $S_{21}=-1.55 \mathrm{~dB}$ (that is a transmission efficiency of about $70 \%$ ), while for $\alpha=25^{\circ}, S_{11}=-14.72 \mathrm{~dB}$ and $S_{21}=-0.31 \mathrm{~dB}$ (a

|  |  |  | Design parameters | E-plane | H-plane |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | 1_ref | 500 mm |  |
|  |  |  | w | 54.61 mm | 109.22 mm |
|  |  |  | $\mathrm{w}^{\prime}$ |  | $\mathrm{m}+\mathrm{w} / 2$ |
|  |  |  | $\alpha$ |  | $0^{\circ}$ to $45^{\circ}$ |
|  |  |  | $l$ | w/ | $\alpha+\cos \alpha)$ |
|  |  |  | $l^{\prime}$ | ( $\mathrm{w}^{\prime}-\mathrm{w}$ | in $\alpha+\cos \alpha)$ |

Figure 3.34: Two-element trapezoidal mitered bend design parameters.
transmission efficiency more than $93 \%$ ). In the H-plane configuration, this kind of

| Modes @ 5 GHz | $\alpha=25^{\circ}$ | $\alpha=30^{\circ}$ | $\alpha=35^{\circ}$ |
| :--- | :---: | :---: | :---: |
| $T E_{10}$ | $-\mathbf{0 . 3 1} \mathrm{dB}$ | $-\mathbf{0 . 4 7} \mathrm{dB}$ | $\mathbf{- 1 . 5 5} \mathrm{dB}$ |
| $T E_{01}$ | -95.54 dB | -78.68 dB | -91.82 dB |
| $T E_{20}$ | -84.63 dB | -87.41 dB | -9.22 dB |
| $T E_{11}$ | $\mathbf{- 2 8 . 6} \mathrm{~dB}$ | $\mathbf{- 2 1 . 5 6} \mathrm{~dB}$ | $\mathbf{- 1 4 . 2 5} \mathrm{~dB}$ |
| $T M_{11}$ | $\mathbf{- 2 0 . 6 1} \mathrm{~dB}$ | $\mathbf{- 1 3 . 5 9} \mathrm{~dB}$ | $\mathbf{- 6 . 6 7} \mathrm{~dB}$ |
| $T E_{21}$ | 81.29 dB | -84 dB | -102.85 dB |
| $T M_{21}$ | -74 dB | -80.92 dB | -82.14 dB |
| $T E_{30}$ | -83.57 dB | -85.51 dB | -78.14 dB |
| $T E_{31}$ | -87.36 dB | -92.34 dB | -72.6 dB |
| $T M_{31}$ | -82.3 dB | -84 dB | -75.28 dB |

Table 3.13: E-plane two-element trapezoidal mitered bend performances.
structure showed better performances accordingly with the mitered bend theory. In fact, looking at Table 3.14, for $\alpha=30^{\circ}$ the return loss ( $S_{11}=-37.91 \mathrm{~dB}$ ) and the transmission efficiency ( $S_{21}=-0.13 \mathrm{~dB}$ such as more than $97 \%$ of transmission) are optimized at the same time. Moreover, the $\left|S_{11}\right|$ and $\left|S_{21}\right|$ values seems to be not so bad as it was in the E-plane case ( $\left|S_{11}\right|$ around -40 dB and transmission efficiency greater than $95 \%$ ).
It is clear that the coupling affects only $T E_{01}$ and $T E_{30}$ modes. Furthermore, if the transverse section of the waveguide were reduced to WR-340, the $T E_{30}$ mode does not propagate and the only mode to filter would be the $T E_{01}$. This is a very favorable condition because this mode has orthogonal distribution of currents compared with the fundamental $T E_{10}$ mode and thus it can be filtered without attenuate the fundamental mode. Let consider the next configuration, concerning the 3 -elements

| Modes @ 5 GHz | $\alpha=30^{\circ}$ |
| :--- | :---: |
| $T E_{10}$ | -0.13 dB |
| $T E_{01}$ | -65.79 dB |
| $T E_{20}$ | $\mathbf{- 2 2 . 5 1} \mathrm{~dB}$ |
| $T E_{11}$ | -88.79 dB |
| $T M_{11}$ | -91.86 dB |
| $T E_{21}$ | -81.67 dB |
| $T M_{21}$ | -78.69 dB |
| $T E_{30}$ | $\mathbf{- 1 7 . 5 7} \mathrm{~dB}$ |
| $T E_{31}$ | -94.11 dB |
| $T M_{31}$ | -86.39 dB |

Table 3.14: H-plane two-element trapezoidal mitered bend performances.
$90^{\circ}$ trapezoidal mitered bends both on E and H planes. The design parameters are summarize in the table hereafter.
Compared with the previous two-elements design, the square of side $w^{\prime}$ is now divided into three parts, each one forming an angle of $30^{\circ}$ at the vertex up on the right side. As before, the choice of the slope of the side $l$ (angle $\alpha$ ) is the design parameters. Also in this case, the length of $d$ can be considered as an additive design parameter.
On E-plane, it is possible to verify a substantial bad performance in terms of return loss. The best value is $S_{11}=-21.42 \mathrm{~dB}$ (about the $0.72 \%$ of reflected power) corresponding to $\alpha=35^{\circ}$; this trend is confirmed from by the $\left|S_{21}\right|$ parameter vs. $\alpha$.
It can be noted that the value of $\alpha$ minimizing $\left|S_{11}\right|$, does not contextually maximize the transmission efficiency. In fact it is maximized for $\alpha=20^{\circ}\left(S_{21}=-0.25 \mathrm{~dB}\right.$ such as a transmission efficiency greater than $94 \%$ ) where the $\left|S_{11}\right|$ parameter is -13.851 dB (more than $4 \%$ of reflected power). On the other hand when the return loss is minimized, the transmission efficiency, $S_{21}=-2.08 \mathrm{~dB}$ (about $62 \%$ ), is too low. A balance between the necessity of a low $\left|S_{11}\right|$ and a high $\left|S_{21}\right|$ can be considered involving the values $\alpha=25^{\circ}, 30^{\circ}$ and resumed here after in Table 3.15. Of

| Parameters [dB] @ 5 GHz | $\alpha=25^{\circ}$ | $\alpha=30^{\circ}$ |
| :---: | :---: | :---: |
| $\left\|S_{11}\right\|$ | $-15.22(3 \%)$ | $-17.43(1.8 \%)$ |
| $\left\|S_{21}\right\|$ | $-0.48(89.5 \%)$ | $-1.04(78.7 \%)$ |

Table 3.15: E-plane three-element trapezoidal mitered bend reflection and transmission parameters.

|  | Design parameters | E-plane | H-plane |
| :---: | :---: | :---: | :---: |
|  | 1 ref | 500 mm |  |
|  | w | 54.61 mm | 109.22 mm |
|  | $\mathrm{w}^{\prime}$ | $100 \mathrm{~mm}+\mathrm{w} / 2$ |  |
|  | $\alpha$ | from $0^{\circ}$ to $45^{\circ}$ |  |
| $l$ | $2(\sin \alpha+\sqrt{3} \cos \alpha)$ |  |  |
| m | $\frac{\sqrt{3}\left(\mathrm{w}^{\prime}-\mathrm{w}\right)}{(\sin \alpha+\sqrt{3} \cos \alpha)}$ |  |  |
| $l^{\prime}$ | $\sqrt{2}\left\{\mathrm{w}^{\prime}-\frac{\mathrm{w}^{\prime} \sqrt{3}(\cos \alpha+\sin \alpha)}{2(\sin \alpha+\sqrt{3} \cos \alpha)}\right\}$ |  |  |
| $m^{\prime}$ | $\sqrt{2}\left\{\left(\mathrm{w}^{\prime}-\mathrm{w}\right)-\frac{\left(\mathrm{w}^{\prime}-\mathrm{w}\right) \sqrt{3}(\cos \alpha+\sin \alpha)}{2(\sin \alpha+\sqrt{3} \cos \alpha)}\right\}$ |  |  |

Figure 3.35: Three-element trapezoidal mitered bend design parameters.
course, the optimization has to be done taking into account the couplings between the fundamental $T E_{10}$ mode and spurious modes as described summarized in the following Table 3.16. By comparing the previous results, the best optimization seems

| Modes @ 5 GHz | $\alpha=25^{\circ}$ | $\alpha=30^{\circ}$ |
| :--- | :---: | :---: |
| $T E_{10}$ | -0.48 dB | $-\mathbf{1 . 0 4} \mathrm{dB}$ |
| $T E_{01}$ | -83.82 dB | -81 dB |
| $T E_{20}$ | -86.84 dB | -85 dB |
| $T E_{11}$ | $\mathbf{- 2 1 . 5 3} \mathrm{~dB}$ | $\mathbf{- 1 6 . 3 9 \mathrm { dB }}$ |
| $T M_{11}$ | $-\mathbf{1 3 . 5 6 3} \mathrm{dB}$ | .$- \mathbf{4 2} \mathrm{dB}$ |
| $T E_{21}$ | -81.46 dB | -91.49 dB |
| $T M_{21}$ | -96.53 dB | -81.82 dB |
| $T E_{30}$ | -78.77 dB | -71.97 dB |
| $T E_{31}$ | -78.35 dB | -70.81 dB |
| $T M_{31}$ | -87.89 dB | -81.92 dB |

Table 3.16: E-plane three-element trapezoidal mitered bend coupling.
to be $\alpha=25^{\circ}$ even if the return loss is greater. As general statement, it is possible to affirm that the E-plane 3 -elements $90^{\circ}$ trapezoidal bend, does not present good
performances so that further analyses are avoided.
On H-plane the situation is much better (as happen also in mitered bends cases); in this case it is possible to select three different values for $\alpha$ corresponding to low $\left|S_{11}\right|$ and acceptable $\left|S_{21}\right|$ parameters. The evidences are resumed in Table 3.17.
The reflection parameter is around -40 dB in all of these cases, the transmission

| Parameters [dB] @ 5 GHz | $\alpha=20^{\circ}$ | $\alpha=25^{\circ}$ | $\alpha=35^{\circ}$ |
| :---: | :---: | :---: | :---: |
| $\left\|S_{11}\right\|$ | $-37.51(0.017 \%)$ | $-17.43(0.0043 \%)$ | $-38.89(0.013 \%)$ |
| $\left\|S_{21}\right\|$ | $-0.2(95.5 \%)$ | $-0.44(90.36 \%)$ | $-0.19(95.72 \%)$ |

Table 3.17: H-plane three-element trapezoidal mitered bend reflection and transmission parameters.
efficiency corresponding to $\alpha=25^{\circ}$ seems to be too low and thus this value can be excluded from the optimization. The next Table 3.18 confirms that the coupling values at $\alpha=25^{\circ}$ are not so good compared with the other $\alpha$ values. In particular $\alpha=20^{\circ}$ seems to be the best balance considering very small coupling from the fundamental $T E_{10}$ mode and the pair of degenerating modes $T E_{01} / T E_{20}$. The higher coupling occurs for the $T E_{30}$ mode (about $3.36 \%$ of power) that is hard to filter as explained in the next section, anyway if it were possible to consider the WR-340 waveguide, the $T E_{30}$ mode would not propagate. On the contrary, in this present configuration using WR-430, it is mandatory the presence of mode filters for $T E_{01}$ and $T E_{30}$ modes.

| Modes @ 5 GHz | $\alpha=20^{\circ}$ | $\alpha=25^{\circ}$ | $\alpha=30^{\circ}$ |
| :--- | :---: | :---: | :---: |
| $T E_{10}$ | -0.2 dB | $\mathbf{- 0 . 4 4} \mathrm{~dB}$ | -0.19 |
| $T E_{01}$ | $-\mathbf{2 0} \mathrm{dB}$ | $-\mathbf{3 2 . 8 5} \mathrm{dB}$ | $\mathbf{- 1 9 . 3 2 \mathrm { dB }}$ |
| $T E_{20}$ | -41 dB | $\mathbf{- 1 8 . 1 1 ~ d B}$ | $\mathbf{. 2 7 . 4 1} \mathrm{~dB}$ |
| $T E_{11}$ | -72.81 dB | -75.84 dB | -100.71 dB |
| $T M_{11}$ | -75.54 dB | -80.66 dB | -96.31 dB |
| $T E_{21}$ | -78.46 dB | -82.34 dB | $95,1 \mathrm{~dB}$ |
| $T M_{21}$ | -81.5 dB | -82 dB | 86.48 dB |
| $T E_{30}$ | $\mathbf{- 1 4 . 7 3} \mathrm{~dB}$ | $\mathbf{- 1 1 . 0 1} \mathrm{~dB}$ | $\mathbf{- 1 6 . 9 5} \mathrm{~dB}$ |
| $T E_{31}$ | -85.81 dB | -80.39 dB | $\mathbf{- 9 4 . 5 8} \mathrm{~dB}$ |
| $T M_{31}$ | $-\mathbf{7 9 . 6 8 \mathrm { dB }}$ | -86.27 dB | -101.62 dB |

Table 3.18: H-plane three-element trapezoidal mitered bend coupling.

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In the four-elements design the square of side $w \prime$ is divided into four parts and each one forms an angle of $22^{\circ} .5 \prime$ at the vertex up on the right side, while in the five-elements case the angle is of $18^{\circ}$. Compared to the previous two and three elements frameworks, now there is an additional design parameters such as the angle $\alpha^{\circ} \leq \sigma \leq 45^{\circ}$. The optimization has been performed by varying all the design parameters ( $R_{b e n d}, \alpha$ and $\sigma$ ) simultaneously. Furthermore, for each value of $R_{b e n d}$ a contour plot has been derived for both $\left|S_{11}\right|$ and $\left|S_{21}\right|$, in order to extract some informations to provide a final refinement.
Also in this case the length $d=100 \mathrm{~mm}$, while the angular parameters $\alpha$ and $\sigma$ were varied. As general statement it is possible to expect a trend close to the circular bends as the number of trapezoidal elements increase.


Figure 3.36: Four-element trapezoidal mitered bend design parameters.
The best results are obtained for the five-element configuration both on E and H plane. For the E-plane case, we perform an optimization around the couple of values $\alpha=10^{\circ}$ and $\sigma=15^{\circ}$ for $R_{\text {bend }}=150 \mathrm{~mm}$, toward the values $\alpha=5^{\circ}$ and $\sigma=20^{\circ}$, as suggested by contour plots of Fig. 3.38, in the intervals $\alpha \in\left[0^{\circ}, 15^{\circ}\right]$ and


Figure 3.37: Five-element trapezoidal mitered bend design parameters.
$\sigma \in\left[10^{\circ}, 30^{\circ}\right]$, respectively, with $1^{\circ}$ step refinement, finding the best performance for $\alpha=7^{\circ}$ e $\sigma=19^{\circ}\left(\left|S_{11}\right|=-75.96 \mathrm{~dB}\right.$ and $\left|S_{21}\right|=-210^{-4} \mathrm{~dB}$, as reported in Fig $3.39 a$ ) and 3.39b). Coupling values are suitably lower than -40 dB , as reported in Table. 3.19. As can be noticed, the specifications in terms of reflection, transmission and coupling are fulfilled, but in this case the bending radius is reduced than in the circular case. This is a quite important outcome because the overall attenuation is further minimized. For the H-plane case, we perform an optimization around the


Figure 3.38: Contour plot optimization for five-element E-plane trapezoidal bend: a) $\left|S_{11}\right|$ and $\left.b\right) \log \left|S_{21}\right|$.

| Modes @ 5 GHz | $\alpha=7^{\circ}$ and $\sigma=19^{\circ}$ |
| :--- | :---: |
| $T E_{01}$ | -75.56 dB |
| $T E_{20}$ | -83.25 dB |
| $T E_{11}$ | -52.93 dB |
| $T M_{11}$ | -45.17 dB |
| $T E_{21}$ | -89.7 dB |
| $T M_{21}$ | -72.38 dB |
| $T E_{30}$ | -72.65 dB |
| $T E_{31}$ | -79.38 dB |
| $T M_{31}$ | -78.60 dB |

Table 3.19: E-plane five-element trapezoidal mitered bend coupling.
couple of values $\alpha=25^{\circ}$ and $\sigma=10^{\circ}$ for $R_{\text {bend }}=350 \mathrm{~mm}$, toward the values $\alpha=20^{\circ}$ and $\sigma=5^{\circ}$, as suggested by contour plots of Fig. 3.46, in the intervals $\alpha \in\left[20^{\circ}, 30^{\circ}\right]$ and $\sigma \in\left[5^{\circ}, 15^{\circ}\right]$, respectively, with $1^{\circ}$ step refinement, finding the best performance for $\alpha=22^{\circ}$ e $\sigma=9^{\circ}\left(\left|S_{11}\right|=-64.82 \mathrm{~dB}\right.$ and $\left|S_{21}\right|=-610^{-4} \mathrm{~dB}$, as reported in Fig

(a)

(b)

Figure 3.39: Five-element E-plane trapezoidal bend vs. frequency: a) $\left|S_{11}\right|$, b) $\left|S_{21}\right|$, and $c$ ) coupling.
$3.41 a)$ and $3.41 b$ ). Coupling values are, also in this case suitably lower than -40 dB , as reported in Table. 3.20. As can be noticed, the bending radius is reduced than in the circular case, minimizing the overall attenuation losses.
A comparison between the circular shape curve and the trapezoidal bend is summarized in Table 3.23. The advantage of the trapezoidal structure is quite clear, since not only provides quite the same performances for the $\left|S_{11}\right|,\left|S_{21}\right|$ and coupling parameters as the circular shape component, but it reduces the bending radius of the curvature, providing a more compact device that minimizes the attenuation. Of course, the manufacturing of such a component is more complicated and it requires a great precision.


Figure 3.40: Contour plot optimization for five-element H-plane trapezoidal bend: a) $\left|S_{11}\right|$ and b) $\log \left|S_{21}\right|$.

(a)

(b)

Figure 3.41: Five-element H-plane trapezoidal bend vs. frequency: a) $\left.\left|S_{11}\right|, b\right)\left|S_{21}\right|$, and $c$ ) coupling.

| Modes @ 5 GHz | $\alpha=22^{\circ}$ and $\sigma=9^{\circ}$ |
| :--- | :---: |
| $T E_{01}$ | -41.9 dB |
| $T E_{20}$ | -83.79 dB |
| $T E_{11}$ | -63 dB |
| $T M_{11}$ | -61.36 dB |
| $T E_{21}$ | -67.6 dB |
| $T M_{21}$ | -68.97 dB |
| $T E_{30}$ | -45.61 dB |
| $T E_{31}$ | -71.45 dB |
| $T M_{31}$ | -73.58 dB |

Table 3.20: H-plane five-element trapezoidal mitered bend coupling.

| Specifications | Circular Bend | Trapezoidal Bend |
| :--- | :---: | :---: |
| $\left\|S_{11}\right\| \mathrm{dB}$ | -44.84 | -75.96 |
| $\left\|S_{21}\right\| \mathrm{dB}$ | $-2.5 \times 10^{-4}$ | $-2 \times 10^{-4}$ |
|  | $T E_{11}$ | $T E_{11}$ |
| coupling dB | -54.85 | -52.93 |
|  | $T M_{11}$ | $T M_{11}$ |
|  | -46.71 | -45.17 |
| $R_{\text {bend }} \mathrm{mm}$ | 415 | 150 |

Table 3.21: E-plane. Five-element trapezoidal mitered bend vs. circular bend.

| Specifications | Circular Bend | Trapezoidal Bend |
| :--- | :---: | :---: |
| $\left\|S_{11}\right\| \mathrm{dB}$ | -49.16 | -64.82 |
| $\left\|S_{21}\right\| \mathrm{dB}$ | $-8 \times 10^{-4}$ | $-6 \times 10^{-4}$ |
|  | $T E_{01}-T E_{20}$ | $T E_{01}$ |
| coupling dB | -41.2 | -41.9 |
|  | $T M_{30}$ | $T E_{30}$ |
|  | -45 | -45.61 |
| $R_{\text {bend }} \mathrm{mm}$ | 630 | 350 |

Table 3.22: H-plane. Five-element trapezoidal mitered bend vs. circular bend.

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## Appendix A

## Gaussian Random Variables in Noise

A standard Gaussian random variable $w$ takes values over the real line and has the probability density function [91]

$$
\begin{equation*}
f(w)=\frac{1}{s q r t 2 \pi} e^{\left(-\frac{w^{2}}{2}\right)}, \quad w \in \mathcal{R} \tag{A.1}
\end{equation*}
$$

The mean of $w$ is zero and the variance is 1 . A (general) Gaussian random variable $x$ is of the form

$$
\begin{equation*}
x=\sigma w+\mu \tag{A.2}
\end{equation*}
$$

The mean of $x$ is $\mu$ and the variance is equal to $\sigma^{2}$. The random variable $x$ is a one-to-one function of $w$ and thus the probability density function follows from (A.1) as

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{2 \pi \sigma^{2}}} e^{\left(-\frac{(x-\mu)^{2}}{2 \sigma^{2}}\right)}, \quad w \in \mathcal{R} \tag{A.3}
\end{equation*}
$$

Since the random variable is completely characterized by its mean and variance, we denote $x$ by $\mathcal{N}\left(\mu, \sigma^{2}\right)$. In particular, the standard Gaussian random variable is denoted by $\mathcal{N}(0,1)$.
An important property of Gaussianity is that it is preserved by linear transformations: linear combinations of independent Gaussian random variables are still Gaussian. If $x_{1}, \cdots, x_{n}$ are independent and $x_{i} \approx \mathcal{N}\left(\mu_{\mathrm{i}}, \sigma_{\mathrm{i}}^{2}\right)$, then

$$
\begin{equation*}
\sum_{i=1}^{n} c_{1} x_{i} \approx \mathcal{N}\left(\sum_{\mathrm{i}=\mathrm{i}}^{\mathrm{n}} \mathrm{c}_{\mathrm{i}} \mu_{\mathrm{i}}, \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{c}_{\mathrm{i}}^{2} \sigma_{\mathrm{i}}^{2}\right) \tag{A.4}
\end{equation*}
$$

A standard Gaussian random vector $\mathbf{w}$ is a collection of $n$ independent and identically distributed (i.i.d.) standard Gaussian random variables $w_{1}, \cdots, w_{n}$. The
vector $\mathbf{w}=\left(w_{1}, \cdots, w_{n}\right)^{T}$ takes values in the vector space $\mathcal{R}^{n}$. The probability density function of $\mathbf{w}$ follows from (A.1):

$$
\begin{equation*}
f(\mathbf{w})=\frac{1}{(\sqrt{2 \pi})^{n}} e^{\left(-\frac{\|\mathbf{w}\|^{2}}{2}\right)}, \quad \mathbf{w} \in \mathcal{R}^{\mathbf{n}} \tag{A.5}
\end{equation*}
$$

Here $\|\mathbf{w}\|=\sqrt{\sum_{i=1}^{n} w_{i}^{2}}$, is the Euclidean distance from the origin to $\mathbf{w}=\left(w_{1}, \cdots, w_{n}\right)^{T}$. Note that the density depends only on the magnitude of the argument. Since an orthogonal transformation $\mathbf{O}$ (i.e., $\mathbf{O}^{T} \cdot \mathbf{O}=\mathbf{O} \cdot \mathbf{O}^{T}=\mathbf{I}$ ) preserves the magnitude of a vector, we can immediately conclude that if $\mathbf{w}$ is standard Gaussian, then $\mathbf{O} \cdot \mathbf{w}$ is also standard Gaussian.
What this result says is that $\mathbf{w}$ has the same distribution in any orthonormal basis. Geometrically, the distribution of w is invariant to rotations and reflections and hence $\mathbf{w}$ does not prefer any specific direction. Fig. A. 1 illustrates this isotropic


Figure A.1: The isobars, i.e., level sets for the density $f(\mathbf{w})$ of the standard Gaussian random vector, are circles for $n=2$.
behavior of the density of the standard Gaussian random vector w. Another conclusion from (A.4) comes from observing that the rows of matrix $\mathbf{O}$ are orthonormal: the projections of the standard Gaussian random vector in orthogonal directions are independent. How is the squared magnitude $\|\mathbf{w}\|^{2}$ distributed? The squared magnitude is equal to the sum of the square of $n$ i.i.d. zero-mean Gaussian random variables. In the literature this sum is called a $\chi$-squared random variable with $n$ degrees of freedom and denoted by $\chi_{n}^{2}$. With $n=2$, the squared magnitude has density

$$
\begin{equation*}
f(a)=\frac{1}{2} e^{-\frac{a}{2}}, \quad a \geq 0 \tag{A.6}
\end{equation*}
$$

and is said to be exponentially distributed.
Gaussian random vectors are defined as linear transformations of a standard Gaussian random vector plus a constant vector, a natural generalization of the scalar case
(cf. (A.2)):

$$
\begin{equation*}
\mathbf{x}=\mathbf{A} \cdot \mathbf{w}+\mu \tag{A.7}
\end{equation*}
$$

Here $\mathbf{A}$ is a matrix representing a linear transformation from $\mathcal{R}^{n}$ to $\mathcal{R}^{n}$ and $\mu$ is a fixed vector in $\mathcal{R}^{n}$. Several implications follow:

1. A standard Gaussian random vector is also Gaussian (with $\mathbf{A}=\mathbf{I}$ and $\mu=0$ ).
2. For any $\mathbf{c}$, a vector in $\mathcal{R}^{n}$, the random variable

$$
\begin{equation*}
\mathbf{c}^{T} \cdot \mathbf{x} \approx \mathcal{N}\left(\mathbf{c}^{T} \cdot \mu, \mathbf{c}^{T} \cdot \mathbf{A} \cdot \mathbf{A}^{T} \mathbf{c}\right) \tag{A.8}
\end{equation*}
$$

this follows directly from (A.4). Thus any linear combination of the elements of a Gaussian random vector is a Gaussian random variable. More generally, any linear transformation of a Gaussian random vector is also Gaussian.
3. If $\mathbf{A}$ is invertible, then the probability density function of x follows directly from (A.5) and (A.6):

$$
\begin{equation*}
\left.f(\mathbf{x})=\frac{1}{(\sqrt{2 \pi})^{n} \sqrt{\operatorname{det}\left(\mathbf{A} \cdot \mathbf{A}^{T}\right)}} e^{\left(-\frac{1}{2}\right.}(\mathbf{x}-\mu)^{T}\left(\mathbf{A} \cdot \mathbf{A}^{T}\right)^{-1}(\mathbf{x}-\mu)\right), \quad \mathbf{x} \in \mathcal{R}^{\backslash} \tag{A.9}
\end{equation*}
$$

The isobars of this density are ellipses; the circles of the standard Gaussian vectors being rotated and scaled by $\mathbf{A}$ (Fig. A.2). The matrix $\mathbf{A} \cdot \mathbf{A}^{T}$ replaces $\sigma^{2}$ in the scalar Gaussian random variable (cf. (A.3)) and is equal to the covariance matrix of $\mathbf{x}$ :

$$
\begin{equation*}
\mathbf{K}=E\left[(\mathbf{x}-\mu)(\mathbf{x}-\mu)^{T}\right]=\mathbf{A} \cdot \mathbf{A}^{T} \tag{A.10}
\end{equation*}
$$

For invertible $\mathbf{A}$, the Gaussian random vector is completely characterized by its mean vector $\mu$ and its covariance matrix $\mathbf{K}=\mathbf{A} \cdot \mathbf{A}^{T}$, which is a symmetric and non-negative definite matrix. We make a few inferences from this observation:

- Even though the Gaussian random vector is defined via the matrix $\mathbf{A}$, only the covariance matrix $\mathbf{K}=\mathbf{A} \cdot \mathbf{A}^{T}$ is used to characterize the density of $\mathbf{x}$. Is this surprising? Consider two matrices $\mathbf{A}$ and $\mathbf{A} \cdot \mathbf{O}$ used to define two Gaussian random vectors as in (A.6). WhenO is orthogonal, the covariance matrices of both these random vectors are the same, equal to $\mathbf{A} \cdot \mathbf{A}^{T}$; so the two random vectors must be distributed identically. We can see this directly using our earlier observation that $\mathbf{O} \cdot \mathbf{w}$ has the same distribution as $\mathbf{w}$ and thus $\mathbf{A} \cdot \mathbf{O} \cdot \mathbf{w}$ has the same distribution as $\mathbf{A} \cdot \mathbf{w}$.


Figure A.2: The isobars of a general Gaussian random vector are ellipses. They corresponds to level sets $\left\{\mathbf{x}:\left\|\mathbf{A}^{-1}(\mathbf{x}-\mu)\right\|^{2}=c\right\}$ for constants c .

- A Gaussian random vector is composed of independent Gaussian random variables exactly when the covariance matrix $\mathbf{K}$ is diagonal, i.e., the component random variables are uncorrelated. Such a random vector is also called a white Gaussian random vector.
- When the covariance matrix $\mathbf{K}$ is equal to identity, i.e., the component random variables are uncorrelated and have the same unit variance, then the Gaussian random vector reduces to the standard Gaussian random vector.
- Now suppose that $\mathbf{A}$ is not invertible. Then $\mathbf{A} \cdot \mathbf{w}$ maps the standard Gaussian random vector $\mathbf{w}$ into a subspace of dimension less than $n$, and the density of $\mathbf{A} \cdot \mathbf{w}$ is equal to zero outside that subspace and impulsive inside. This means that some components of $\mathbf{A} \cdot \mathbf{w}$ can be expressed as linear combinations of the others. To avoid messy notation, we can focus only on those components of $\mathbf{A} \cdot \mathbf{w}$ that are linearly independent and represent them as a lower dimensional vector $\hat{\mathbf{x}}$, and represent the other components of $\mathbf{A} \cdot \mathbf{w}$ as (deterministic) linear combinations of the components of $\hat{\mathbf{x}}$. By this stratagem, we can always take the covariance $\mathbf{K}$ to be invertible.

So far we have considered real random vectors. Complex random vectors are of the form $\mathbf{x}=\mathbf{x}_{R}+j \mathbf{x}_{I}$ where $\mathbf{x}_{R}$ and $\mathbf{x}_{I}$ are real random vectors. Complex Gaussian random vectors are ones in which $\left[\mathbf{x}_{R}, \mathbf{x}_{I}\right]^{T}$ is a real Gaussian random vector. The distribution is completely specified by the mean and covariance matrix of the real vector $\left[\mathbf{x}_{R}, \mathbf{x}_{I}\right]^{T}$. The same information is contained in the mean $\mu$, the covariance
matrix $\mathbf{K}$, and the pseudo-covariance matrix $\mathbf{J}$ of the complex vector $\mathbf{x}$, where:

$$
\begin{gather*}
\mu=E[\mathbf{x}]  \tag{A.11}\\
\mathbf{K}=E\left[(\mathbf{x}-\mathbf{m u})(\mathbf{x}-\mathbf{m u})^{*}\right]  \tag{A.12}\\
\mathbf{J}=E\left[(\mathbf{x}-\mathbf{m u})(\mathbf{x}-\mathbf{m u})^{T}\right] \tag{A.13}
\end{gather*}
$$

Here, $\mathbf{A}^{*}$ is the transpose of the matrix $\mathbf{A}$ with each element replaced by its complex conjugate, and $\mathbf{A}^{T}$ is just the transpose of $\mathbf{A}$. Note that in general the covariance matrix $\mathbf{K}$ of the complex random vector $\mathbf{x}$ by itself is not enough to specify the full second-order statistics of $\mathbf{x}$. Indeed, since $\mathbf{K}$ is Hermitian, i.e., $\mathbf{K}=\mathbf{K}^{*}$, the diagonal elements are real and the elements in the lower and upper triangles are complex conjugates of each other. Hence it is specified by $n^{2}$ real parameters, where $n$ is the (complex) dimension of $\mathbf{x}$. On the other hand, the full second-order statistics of $\mathbf{x}$ are specified by the $n(2 n+1)$ real parameters in the symmetric $2 n \times 2 n$ covariance matrix of $\left[\mathrm{x}_{R}, \mathbf{x}_{I}\right]^{T}$.
In wireless communication we are almost exclusively interested in complex random vectors that have the circular symmetry property, that means that $\mathbf{x}$ is circular symmetric if $e^{j \theta} \mathbf{x}$ has the same distribution of $\mathbf{x}$ for any $\theta$.
For a circular symmetric complex random vector $\mathbf{x}$,

$$
\begin{equation*}
E[\mathbf{x}]=E\left[e^{j \theta} \mathbf{x}\right]=e^{j \theta} E[\mathbf{x}] \tag{A.14}
\end{equation*}
$$

for any $\theta$; hence the mean $\mu=0$. Moreover

$$
\begin{equation*}
E\left[\mathbf{x} \cdot \mathbf{x}^{T}\right]=E\left[e^{j \theta} \mathbf{x}\left(e^{j \theta} \mathbf{x}\right)^{T}\right]=e^{2 j \theta} E\left[\mathbf{x} \cdot \mathbf{x}^{T}\right] \tag{A.15}
\end{equation*}
$$

for any $\theta$; hence the pseudo-covariance matrix $\mathbf{J}$ is also zero. Thus, the covariance matrix $\mathbf{K}$ fully specifies the first- and second-order statistics of a circular symmetric random vector. And if the complex random vector is also Gaussian, $\mathbf{K}$ in fact specifies its entire statistics. A circular symmetric Gaussian random vector with covariance matrix $\mathbf{K}$ is denoted as $\mathcal{C N}(0, \mathbf{K})$. Some special cases:

1. A complex Gaussian random variable $w=w_{R}+j w_{I}$ with i.i.d. zero-mean Gaussian real and imaginary components is circular symmetric. The circular symmetry of $w$ is in fact a restatement of the rotational invariance of the real Gaussian random vector $\left[w_{R}, w_{I}\right]^{T}$ already observed (cf. (A.8)). In fact, a circular symmetric Gaussian random variable must have i.i.d. zero-mean real and imaginary components. The statistics are fully specified by the variance $\sigma^{2}=E\left[|w|^{2}\right]$, and the complex random variable is denoted as $\mathcal{C N}\left(0, \sigma^{2}\right)$. (Note that, in contrast, the statistics of a general complex Gaussian random variable
are specified by five real parameters: the means and the variances of the real and imaginary components and their correlation.) The phase of $w$ is uniform over the range $[0,2 \pi]$ and independent of the magnitude $\|w\|$, which has a density given by

$$
\begin{equation*}
f(r)=\frac{r}{\sigma^{2}} e^{-\frac{r^{2}}{2 \sigma^{2}}}, \quad r \geq 0 \tag{A.16}
\end{equation*}
$$

and is known as a Rayleigh random variable. The square of the magnitude, i.e., $w_{1}^{2}+w_{2}^{2} 2$, is $\chi_{2}^{2}$, i.e., exponentially distributed, cf. (A.6). A random variable distributed as $\mathcal{C N}(0,1)$ is said to be standard, with the real and imaginary parts each having variance 0.5 .
2. A collection of n i.i.d. $\mathcal{C N}(0,1)$ random variables forms a standard circular symmetric Gaussian random vector w and is denoted by $\mathcal{C N}(0,1)$. The density function of $\mathbf{w}$ can be explicitly written as, following from (A.5),

$$
\begin{equation*}
f(\mathbf{w})=\frac{1}{\pi} e^{-\|\mathbf{w}\|^{2}}, \quad \mathbf{w} \in \mathcal{C}^{n} \tag{A.17}
\end{equation*}
$$

As in the case of a real Gaussian random vector $\mathcal{N}(0,1)$ (cf. (A.8)), we have the property that $\mathbf{U} \cdot \mathbf{w}$ has the same distribution as $\mathbf{w}$ for any complex orthogonal matrix $\mathbf{U}$ (such a matrix is called a unitary matrix and is characterized by the property $\mathbf{U} \cdot \mathbf{U}=\mathbf{I})$. This property is the complex extension of the isotropic property of the real standard Gaussian random vector. Note the distinction between the circular symmetry and the isotropic properties: the latter is in general much stronger than the former except that they coincide when w is scalar. The square of the magnitude of $\mathbf{w}$, as in the real case, is a $\chi_{2 n}^{2}$ variable.
3. If w is $\mathcal{N}(0,1)$ and $\mathbf{A}$ is a complex matrix, then $\mathbf{x}=\mathbf{A} \cdot \mathbf{w}$ is also circular symmetric Gaussian, with covariance matrix $\mathbf{K}=\mathbf{A} \cdot \mathbf{A}^{*}$, i.e., $\mathcal{N}(0, \mathbf{K})$. Conversely, any circular symmetric Gaussian random vector with covariance matrix $\mathbf{K}$ can be written as a linearly transformed version of a standard circular symmetric random vector. If $\mathbf{A}$ is invertible, the density function of $\mathbf{x}$ can be explicitly calculated via (A.15), as in (A.8),

$$
\begin{equation*}
f(\mathbf{x})=\frac{1}{\pi^{n} \operatorname{det} \mathbf{K}} e^{-\left(\mathbf{x}^{*} \cdot \mathbf{K}^{-1} \cdot \mathbf{x}\right)}, \quad \mathbf{x} \in \mathcal{C}^{n} \tag{A.18}
\end{equation*}
$$

When $\mathbf{A}$ is not invertible, the earlier discussion for real random vectors applies here as well: we focus only on the linearly independent components of $\mathbf{x}$, and treat the other components as deterministic linear combinations of these. This allows us to work with a compact notation.

## Appendix B

## Orthogonal System for Curved Structure

Solutions of Maxwell's equation in source-free, isotropic media

$$
\begin{equation*}
\nabla \times \mathbf{H}=j \omega \epsilon \mathbf{E} \nabla \times \mathbf{E}=-j \omega \mu \mathbf{H} \tag{B.1}
\end{equation*}
$$

can be obtained only in a few orthogonal co-ordinate systems [23]; perturbations of such systems normally introduce non-orthogonal features that further complicate the solutions. A means whereby a suitable orthogonal system can be obtained from a general one, has been developed by Tang [43] and it is relevant to applications in which waveguides are twisted or curved in space.
Fig. B. 1 shows a smooth curve in space described by a position vector $\mathbf{R}(s)$ which


Figure B.1: Serret-Frenet frame in curved space..
is a function of the arc length $s$ measured from an arbitrary point. At any point we can define a unit vector $\mathbf{a}_{\mathbf{t}}$ in the direction of the tangent at that point; moreover,
we derive the relation $\mathbf{R}(s)+\mathbf{a}_{\mathbf{t}} d s=\mathbf{R}(s+d s)$, from which we can derive that

$$
\begin{equation*}
\mathbf{a}_{\mathbf{t}}=\frac{d}{d s} \mathbf{R} \tag{B.2}
\end{equation*}
$$

In going from $s$ to $s+d s$, this tangential unit vector rotates by $d \theta$, defining a plane in which the principal normal to the curve $\mathbf{a}_{\mathbf{n}}$ is in relation with the vector $\mathbf{a}_{\mathbf{t}}$ by the following equation

$$
\begin{equation*}
\mathbf{a}_{\mathbf{t}}(s+d s)=\mathbf{a}_{\mathbf{t}}(s)+\mathbf{a}_{\mathbf{n}} d \theta \tag{B.3}
\end{equation*}
$$

and then,

$$
\begin{equation*}
\mathbf{a}_{\mathbf{n}}=\frac{d}{d s} \mathbf{a}_{\mathbf{t}}\left(\frac{d s}{d \theta}\right) \tag{B.4}
\end{equation*}
$$

where the term $d s / d \theta=1 / \chi$, being $\chi$ the curvature; so that

$$
\begin{equation*}
\mathbf{a}_{\mathbf{n}}=\mathbf{a}_{\mathrm{t}}^{\prime} / \chi \tag{B.5}
\end{equation*}
$$

(the prime representing the differentiation with respect to $s$ ).
Since $\mathbf{a}_{\mathbf{t}}{ }^{\prime}=\mathbf{R}^{\prime \prime}$, Eq. (B.4) gives the relation

$$
\begin{equation*}
\chi^{2}=\mathbf{R}^{\prime \prime} \cdot \mathbf{R}^{\prime \prime} \tag{B.6}
\end{equation*}
$$

We now define the binormal unit vector $\mathbf{a}_{\mathbf{b}}$ given by the direction of the right-handed rotation from the tangent to the normal. Hence, $\mathbf{a}_{b}=\mathbf{a}_{\mathbf{t}} \times \mathbf{a}_{\mathbf{n}}$ or equivalently

$$
\begin{equation*}
\mathbf{a}_{\mathrm{b}}=\mathbf{R}^{\prime} \times \frac{\mathbf{R} / \prime}{\chi} \tag{B.7}
\end{equation*}
$$

We can also relate $\mathbf{a}_{\mathbf{b}}$ with another important parameter that distinguishes the curved frame. As the point $s$ moves to $s+d s$, the normal vector $\mathbf{a}_{\mathbf{n}}(s)$ becomes $\mathbf{a}_{\mathbf{n}}(s+d s)$, and the difference $\mathbf{a}_{\mathbf{n}}(s+d s)-\mathbf{a}_{\mathbf{n}}(s)=\mathbf{a}_{\mathbf{n}}{ }^{\prime} d s$ is perpendicular to $\mathbf{a}_{\mathbf{n}}$ (being $\mathbf{a}_{\mathbf{n}}{ }^{\prime}$ the slope of the tangent to the unit circle having radii $\mathbf{a}_{\mathbf{n}}(s)$ ). Hence, $\mathbf{a}_{\mathbf{n}}{ }^{\prime}$ can be expressed in terms of multiples of the vectors $\mathbf{a}_{\mathbf{b}}$ and $\mathbf{a}_{\mathbf{t}}$, being both of them perpendicular to $\mathbf{a}_{\mathbf{n}}$ :

$$
\begin{equation*}
\mathbf{a}_{\mathbf{n}}^{\prime}=A \mathbf{a}_{\mathbf{b}}+B \mathbf{a}_{\mathbf{t}} \tag{B.8}
\end{equation*}
$$

and, by the definition of torsion, the coefficient $A=\tau$.
Substituting into Eq. (B.7), from Eqs. (B.3), (B.4) and (B.6), gives

$$
\begin{equation*}
\frac{\mathbf{R}^{\prime \prime \prime}}{\chi}-\frac{\mathbf{R}^{\prime \prime} \chi^{\prime}}{\chi^{2}}=A \mathbf{R}^{\prime} \times \frac{\mathbf{R}^{\prime \prime}}{\chi}+B \mathbf{R}^{\prime} \tag{B.9}
\end{equation*}
$$

Now, thus $\mathbf{R}^{\prime} \times \mathbf{R}^{\prime \prime} / \chi=\mathbf{a}_{\mathbf{b}}$ is perpendicular to $\mathbf{a}_{\mathbf{t}}=\mathbf{R}^{\prime}$ and $\mathbf{a}_{\mathbf{n}}=\mathbf{R}^{\prime \prime} / \chi$, the scalar product of Eq. (B.8) with $\mathbf{R}^{\prime} \times \mathbf{R}^{\prime \prime}$, solved for the torsion parameter, gives

$$
\begin{equation*}
A=\tau=\left(\mathbf{R}^{\prime} \times \mathbf{R}^{\prime \prime}\right) \cdot \frac{\mathbf{R}^{\prime \prime \prime}}{\left(\mathbf{R}^{\prime \prime} \cdot \mathbf{R}^{\prime \prime}\right)} \tag{B.10}
\end{equation*}
$$

This relation returns the torsion in terms of derivatives of the position vector. From $\mathbf{a}_{\mathbf{t}} \cdot \mathbf{a}_{\mathbf{n}}=0$, we have $\mathbf{R}^{\prime} \cdot \mathbf{R}^{\prime \prime}=0$, and differentiating gives $\mathbf{R}^{\prime \prime} \cdot \mathbf{R}^{\prime \prime}=-\mathbf{R}^{\prime} \cdot \mathbf{R}^{\prime \prime \prime}$. Taking the scalar product of Eq. (B.8) with $\mathbf{a}_{\mathbf{t}}=\mathbf{R}^{\prime}$ accordingly gives $B=\mathbf{R}^{\prime} \cdot \mathbf{R}^{\prime \prime \prime} / \chi=$ $-\left(\mathbf{R}^{\prime \prime} \cdot \mathbf{R}^{\prime \prime}\right) / \chi=-\chi$, from Eq. (B.5). Substituting in Eq. (B.7), we obtain

$$
\begin{equation*}
\mathbf{a}_{\mathbf{b}}=\frac{\left(\mathbf{a}_{\mathbf{n}}^{\prime}+\chi \mathbf{a}_{\mathbf{t}}\right)}{\tau} \tag{B.11}
\end{equation*}
$$

Differentiating the relation $\mathbf{a}_{\mathbf{b}}=\mathbf{a}_{\mathbf{t}} \times \mathbf{a}_{\mathbf{n}}$ gives $\mathbf{a}_{\mathrm{b}}{ }^{\prime}=\mathbf{a}_{\mathrm{t}}{ }^{\prime} \times \mathbf{a}_{\mathbf{n}}+\mathbf{a}_{\mathbf{t}} \times \mathbf{a}_{\mathbf{n}}{ }^{\prime}$. From Eq. (B.4) $\mathbf{a}_{\mathbf{t}}{ }^{\prime}=\chi \mathbf{a}_{\mathbf{n}}$ so $\mathbf{a}_{\mathbf{t}}{ }^{\prime} \times \mathbf{a}_{\mathbf{n}}=0$; and from Eq. (B.10), $\mathbf{a}_{\mathbf{t}} \times \mathbf{a}_{\mathbf{n}}{ }^{\prime}=\tau \mathbf{a}_{\mathbf{t}} \times \mathbf{a}_{\mathbf{b}}=-\tau \mathbf{a}_{\mathbf{n}}$. Hence

$$
\begin{equation*}
\mathbf{a}_{\mathbf{b}}^{\prime}=-\tau \mathbf{a}_{\mathbf{n}} \tag{B.12}
\end{equation*}
$$

The three mutually orthogonal vector $\mathbf{a}_{\mathbf{t}}, \mathbf{a}_{\mathbf{b}}$ and $\mathbf{a}_{\mathbf{n}}$ define a right-handed orthogonal co-ordinate system in any point of the curve, and is known as Serret-Frenet frame. As the point $s$ moves along the curve the frame rotates; if the curve is a plane curve the rotation is about the axis $\mathbf{a}_{\mathbf{b}}$; the effect of the torsion is to provide additional rotation about $\mathbf{a}_{\mathbf{t}}$.
A vector which measures the rate of rotation of the frame is the Darboux vector $\delta$, defined by

$$
\begin{equation*}
\delta=\tau \mathbf{a}_{\mathbf{t}}+\chi \mathbf{a}_{\mathbf{b}} \tag{B.13}
\end{equation*}
$$

and Eqs. (B.4), (B.7) and (B.11), can be put in the form

$$
\begin{align*}
& \mathbf{a}_{\mathbf{t}}^{\prime}=\chi \mathbf{a}_{\mathbf{n}}=\delta \times \mathbf{a}_{\mathbf{t}} \\
& \mathbf{a}_{\mathbf{n}}^{\prime}=-\chi \mathbf{a}_{\mathbf{t}}+\tau \mathbf{a}_{\mathbf{b}}=\delta \times \mathbf{a}_{\mathbf{n}}  \tag{B.14}\\
& \mathbf{a}_{\mathbf{b}}^{\prime}=-+\tau \mathbf{a}_{\mathbf{n}}=\delta \times \mathbf{a}_{\mathbf{b}}
\end{align*}
$$

From Eq. (B.13) the Darboux vector can be interpreted as an angular velocity vector describing the rate of the rotation of the trihedral $\left(\mathbf{a}_{\mathbf{t}}, \mathbf{a}_{\mathbf{b}}, \mathbf{a}_{\mathbf{n}}\right)$ as it moves along the curve.
The Serret-Frenet frame can be used as the basis for setting up an orthogonal coordinate system in the neighborhood of the curve. In fact, even if it constitutes an
orthogonal frame on the curve, for points off the curve the effect of the torsion is to destroy the orthogonality. Apart from the case of a plane curve the Serret-Frenet frame does not provide an othogonal co-ordinate system [23].
On top on this, an orthogonal frame can be derived by producing an alternative frame with zero rotation around $\mathbf{a}_{\mathbf{t}}$ (i.e. a frame with rotation rate $-\tau$ with respect to the Serret-Frenet one). Denoting the new frame with the unit vectors $\mathbf{a}_{\nu}, \mathbf{a}_{\beta}, \mathbf{a}_{\mathbf{t}}$, and let the angle between $\mathbf{a}_{\mathbf{n}}$ and $\mathbf{a}_{\mathbf{t}}$ be $\phi$, a function of $s$ such that

$$
\begin{equation*}
\frac{d}{d s} \phi=-\tau(s) \tag{B.15}
\end{equation*}
$$

Then for the frame $\mathbf{a}_{\nu}, \mathbf{a}_{\beta}, \mathbf{a}_{\mathbf{t}}$ the Darboux vector becomes, since there is zero net torsion,

$$
\begin{equation*}
\delta_{\mathbf{0}}=\chi \mathbf{a}_{\mathbf{b}}=\chi\left(\mathbf{a}_{\nu} \sin \phi+\mathbf{a}_{\beta} \cos \phi\right) \tag{B.16}
\end{equation*}
$$

Since $n=\nu \cos \phi-\beta \sin \phi$ is the co-ordinate distance in the direction of $\mathbf{a}_{\mathbf{n}}$, the expression of the line element in the $(\nu, \beta, s)$ system, being $\mathbf{r}$ the position vector of an arbitrary point P in the vicinity of the curve, reduces to

$$
\begin{equation*}
(d r)^{2}=(d \nu)^{2}+(d \beta)^{2}+[1-\chi(\nu \cos \phi-\beta \sin \phi)]^{2}(d s)^{2} \tag{B.17}
\end{equation*}
$$

This form identifies the metric coefficient as

$$
\begin{equation*}
h_{\nu}=h_{\beta}=1, \quad h_{s}=[1-\chi(\nu \cos \phi-\beta \sin \phi)] \tag{B.18}
\end{equation*}
$$

Accordingly to that, Maxwell's equation in the $(\nu, \beta, s)$ frame can be constructed. In the case of a rectangular waveguide whose axis follow the curve $\mathbf{R}(s)$, and the cross-section is defined by the relation $\nu= \pm a / 2$ and $\beta= \pm b / 2$, where $a$ and $b$ are the waveguide transverse dimensions.
For shapes of cross-section other than rectangular, consider the transformation

$$
\begin{equation*}
\nu=\nu(u, v), \quad \beta=\beta(u, v) \tag{B.19}
\end{equation*}
$$

to orthogonal curvilinear co-ordinate $u, v$. The orthogonality condition on $u$ and $v$ takes the form

$$
\begin{equation*}
\frac{\partial \nu}{\partial u} \frac{\partial \nu}{\partial v}+\frac{\partial \beta}{\partial u} \frac{\partial \beta}{\partial v}=0 \tag{B.20}
\end{equation*}
$$

The metric coefficients in the new co-ordinates system are

$$
\begin{equation*}
\left.\left.h_{u}=\left[\left(\frac{\partial \nu}{\partial u}\right)^{2}+\left(\frac{\partial \beta}{\partial u}\right)^{2}\right]^{( } 1 / 2\right) h_{v}=\left[\left(\frac{\partial \nu}{\partial v}\right)^{2}+\left(\frac{\partial \beta}{\partial v}\right)^{2}\right]^{( } 1 / 2\right) \tag{B.21}
\end{equation*}
$$

with $h_{s}$ as in Eq. (B.18).
For example, for a curved circular pipe, the polar co-ordinates $\left(\rho_{0}, \psi\right)$ transformation is given by $\rho_{0}=\nu^{2}+\beta^{2}$ and

$$
\begin{equation*}
h_{\rho}=1, \quad h_{\phi}=\rho, \quad h_{s}=1-\chi \rho \cos (\phi+\psi) . \tag{B.22}
\end{equation*}
$$

## Conclusions

In the first part of this thesis we describe a hybrid electromagnetic and statistic procedure for the localization of buried objects (both perfectly-conducting and dielectric). In particular, we adopt a signal model for the field scattered by the object, and received by an array of sensors, based on the narrowband signal representation. After that, we partition the whole array in a certain number of sub-arrays in order to locally validate the plane-wave approximation, and successively we apply at each sub-array several Direction of Arrival estimation algorithms (non-parametric, subspace based, maximum likelihood based). By triangulating all the DoAs we obtain a dataset made by the intersection co-ordinates, which has been processed by using a statistical technique based on the Poisson distribution. In fact, the investigation domain can be geometrically partitioned by means of fixed-size standing windows, each one containing a number of crossings. It can be demonstrated that the random variable represented by the number of windows which have on the inside $k$ crossings (iterating $k$ on the number of crossings), can be represented by a Poisson-based statistical distribution. In particular, the are two different probability functions, for the target region windows (in which the number of crossings is high and the rateparameter of the distribution in larger) and for the background windows (in which the crossings are sparse and the rate-parameter of the distribution is smaller).
Moreover, by using a sliding windows, at each step it is possible to verify a binary test and detecting the windows belonging to the target region. By removing the background region crossings from the investigation domain, it is possible to estimate the object position by averaging the co-ordinates.
We simulated many cases relevant to a perfectly conducting cylinder and a dielectric one. Moreover, we calculated the localization error varying the cylinder radius, its horizontal position, the distance from the array and the refractive index of the ground. For the dielectric case, we also varied the refractive index of the cylinder. The results achieved for a single-object localization are quite precise in many cases, and specially for a conductive cylinder or a cavity.
Moreover, we derived some interesting trends of the maximum output SNR versus
the cylinder size and distance from the array. In some cases (i.e. conductive cylinders and cavities) it is possible to cross-check the values of the localization error and the SNR level, in order to estimate also the dimension of the cylindrical object.
We also take into account the multiple object localization, developing a clustering analysis method to process the crossing pattern. In particular, we apply several clustering algorithms, generally used to derive homogeneous groups of data in a dataset, to our scopes. The idea is to identify the cylinder positions by considering them as the centroids of a group of co-ordinates that minimize the proximity function (that in this case is consists in a measure of the geometrical distance).
Also in this case we simulate many configurations concerning a double cylinder localization, considering the scatterers perfectly conducting or dielectric. The localization error value is reported for both the cylinders, varying the geometrical parameters (dimension, distance, horizontal offset with respect to the array center) and the permittivity of both the cylinders (in dielectric cases) and ground. The outcomes show a detection capability of the procedure, specially for conductive cylinders and cavities.
In the second part of this thesis, we dealt with the design of microwave components for the transmission line of the LHCD system of ITER. The first component that we considered is a mode converter taking as inputs four rectangular waveguide excited in the fundamental mode (representing the generator outputs), and collecting them into a circular oversized waveguide, in order to transfer more power in a single line. The utility of such a component was supposed to be to minimize the attenuation losses of the line, in fact the circular $T E_{01}$ mode is less attenuated because it does not affect the waveguide walls. We design the combiner by using a commercial FEM solver, and then we optimize the dimension of the component in order to minimize the reflection toward the generators and at the same time, maximize the transmission efficiency of the desired mode. In ideal working conditions, when the input waveguides are fed uniformly in amplitude and phase, the combiner performances are good and the the efficiency is more than $99 \%$.
Problems occur when the input waveguides are unbalanced in amplitude and/or in phase. In such a scenario, the combiner is poorly fault-tolerant; in fact all the other modes are allowed to propagate and the efficiency progressively decreases. At the same time the reflection of power toward the generators increases. Since the event of a power reduction, affecting one generator or a phase shift among the generators of the line is not so unusual, the component cannot be used for this high power application, because a possible malfunctioning will cause the suppression of the line and consequently a lack of power for the entire system.

The unsuitability of the combiner for the transmission line achievement, suggested to consider the design of an oversized-rectangular-waveguide system. In fact, due to reduction of the distance between generators and plasma vessel, the attenuation losses along the whole transmission line decreased, making possible the use of rectangular waveguides. Unfortunately, the reduction of the attenuation losses was not so strong to allow the realization of an unimodal system, therefore we worked on the optimization of a transmission line with oversized cross-section, in order to fulfill the power transmission requirements. In fact, in such a oversized structure, several higher order mode are allowed to propagate, while the most of the power transfer must be carried on the fundamental $T E_{10}$ mode. So that it is mandatory to design efficient mode filters in order to reduce the power content on higher order modes. In particular, designed innovative filters by means of corrugations (both longitudinal and transversal) partially filled with absorbing materials, considering the power absorption as function of the absorber thickness. With this strategy, a dependence similar to a damped cosine between the absorption performance and the thickness of the material that fills the corrugations, can be derived. Such behavior has been explained by means of an analytical model treating the corrugations as normal guiding structures where forward and backward waves propagate. This model is applied to filters for rectangular waveguides and has been simulated by means of full-wave FEM solver.
Supposing to have mode filters attenuating higher order and spurious modes, the presence of discontinuity on the line (i.e., bend) can further excite unwanted modes. Therefore, on one hand mode filters attenuate the propagation of these modes while, on the other hand, the presence of bends will excite them. So, it is important to properly design the bends to limit the excitation of spurious modes facilitating the mode filtering. Several bend profiles have been analyzed both on E and H planes, with a commercial FEM code: circular bend, mitered bend, and trapezoidal elements bend.
The principal specifications that characterize the bend design consist in the minimization of the refection of the fundamental mode ( $\left|S_{11}\right|$ parameter ) and at the same time the maximization of the transmission efficiency of the $T E_{10}$ mode ( $\left|S_{12}\right|$ parameter). Another important requirement is to minimize the coupling between the fundamental mode and other spurious modes that can propagate at the working frequency in the oversized rectangular waveguide.
The optimization of the simple circular bends (on E and H plane), lead to design efficient structures in terms of specifications fulfillment, but the optimized bending radius was still too long introducing several dBs of attenuation at each bend. In or-
der to reduce the bending radius, we designed a bend structure present in literature as mitered bend. This component resulted to be very poor in terms of transmission efficiency and thus it was rejected. We have found an excellent solution by considering the polygonal approximation of the circular bends by means of trapezoidal elements. With these structure we have obtained the same performances of the circular bend for what concerns the transmission efficiency, the reflection and coupling with spurious modes (as one could expect since it performs an approximation of the circular shape), with a significant reduction of the bending radius.

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Simone Meschino


[^0]:    ${ }^{1}$ The windows must be non-overlapping to guarantee the independence of random variable $Y_{i}$.

