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Compactified Picard stacks over the moduli space of curves with marked points

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di

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Introduction

Statement of the problem

For any $d \in \mathbb{Z}$ and $g, n \geq 0$ such that $2g - 2 + n > 0$, denote by $\mathcal{P}ic_{d,g,n}$ the stack whose sections over a scheme S consist of flat and proper families $\pi : C \to S$ of smooth curves of genus g, with n distinct sections $s_i : S \to C$ and a line bundle L of relative degree d over C. Morphisms between two such objects are given by cartesian diagrams

$$\begin{array}{ccc} C & \xrightarrow{\beta_2} & C' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle s_{i'}} \\ S & \xrightarrow[\beta_1]{} & S' \end{array}$$

such that $s_{i'} \circ \beta_1 = \beta_2 \circ s_i$, $1 \leq i \leq n$, together with an isomorphism $\beta_3 : L \to \beta_2^*(L')$.

$\mathcal{P}ic_{d,g,n}$ is endowed with a natural forgetful map onto $\mathcal{M}_{g,n}$ and it is, of course, not complete.

In the present thesis we search for a compactification of $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$. By this we mean to construct an algebraic stack $\overline{\mathcal{P}}_{d,g,n}$ with a map $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$ with the following properties.

1. $\overline{\mathcal{P}}_{d,g,n}$ and $\Psi_{d,g,n}$ fit in the following diagram;

$$\begin{array}{ccc} \mathcal{P}ic_{d,g,n} & \hookrightarrow & \overline{\mathcal{P}}_{d,g,n} \\ \downarrow & & \downarrow{\scriptstyle \Psi_{d,g,n}} \\ \mathcal{M}_{g,n} & \hookrightarrow & \overline{\mathcal{M}}_{g,n} \end{array}$$

2. $\Psi_{d,g,n}$ is proper (or, ate least, universally closed);

3. $\overline{\mathcal{P}}_{d,g,n}$ has a geometrically meaningful modular description.

Note that in order to complete $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$ it is not enough to consider the stack of line bundles over families of n-pointed stable curves, since this is not complete as well. So, it is necessary to enlarge the category either admitting more general sheaves than line bundles or a bigger class of curves.

Motivation

The problem of compactifying the (generalized) jacobian of curves or of families of curves has been widely studied in the last decades, starting from the work of Igusa [I56] and of Mayer and Mumford [MM]. Since then, several solutions where found, specially in the case of irreducible curves (see [A04] for an overview and comparison results on these constructions). For irreducible curves a first answer was given by D'Souza in [DS79]. Later, Altman, Kleiman and others extended this work to families of irreducible curves with more general singularities than nodes (see [AK80] and also [Es01]). For reducible curves the situation is more intricate since one has to deal also with nontrivial combinatorial problems. A first solution, for a single reducible curve, is given by Oda and Seshadri in [OS79]. Then, in [C94], Caporaso constructs a compactification the universal Picard variety over $\overline{M}_g$ and later, in [P96], Pandharipande makes a more general construction that holds also for vector bundles of higher rank and coincides with Caporaso's in the case of line bundles. We also recall Simpson's general construction of moduli spaces of coherent sheaves on projective schemes in [Si94] and Schmitt's construction of algebraic stacks compactifying the universal moduli space of semistable vector bundles over smooth curves in [S04].

On the other hand, the construction of the moduli space of stable curves with marked points was done by Knudsen in [K83], following ideas of Mumford, with the scope of proving the projectivity of the moduli space of stable curves. Since then, $\overline{\mathcal{M}}_{g,n}$ itself became the subject of great interest, because of its rich geometry, and because of several applications. In particular, $\overline{\mathcal{M}}_{g,n}$ has a central role in Gromov-Witten theory and enumerative geometry. In fact, in part motivated by Witten's conjecture ([W91]), the study of the cohomology ring of $\overline{\mathcal{M}}_{g,n}$ attracted the attention of several algebraic geometers in the last decades and led to very important results.

We recall, for instance, Kontsevich's first proof of the Witten conjecture in ([K92]); the interaction between geometry and physics leading to the development of quantum cohomoly and Gromov-Witten theory (see e.g. [FP97]); the algebro-geometric inductive calculations on the cohomology ring of $\overline{\mathcal{M}}_{g,n}$ due to Arbarello and Cornalba in [AC98]; Faber's conjectures on the structure of the tautological ring of M_g and its pointed versions ([F99], [P02]), the ELSV formulas relating intersection formulas in $\overline{\mathcal{M}}_{g,n}$ with Hurwitz numbers ([ELSV1], [ELSV2]) and the recent proof by Faber, Shadrin and Zvonkine in [FSZ] of the generalized Witten conjecture ([W93]).

So, it is natural to search for a compactification of $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$ and to study its applications. Nevertheless, at least to our knowledge, there was no construction of compactified Picard varieties for curves with marked points until now.

Our interest in constructing such a space is also due to Goulden, Jackson and Vakil's "generalized ELSV formula"conjecturing a relation between the intersection theory of a $(4g-3+n)$-dimensional space and certain double Hurwitz numbers (see [GJV05] and [LV]). According to these authors, this space should be a suitable compactification of $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$ supporting particular families of classes satisfying certain properties. Unfortunately, we do not know yet if our space supports such classes, except for what they call ψ-classes, which turn out to be the pullback of the ψ-classes in $\overline{\mathcal{M}}_{g,n}$. It is certainly interesting to consider this as a future research problem.

Balanced Picard stacks over $\overline{\mathcal{M}}_g$

Let us start by considering the case $n = 0$ (and $g \geq 2$). As we already mentioned, the situation here is particularly fortunate since there are many constructions of compactified Picard varieties of stable curves.

In particular, in [C94], Caporaso addresses the problem of compactifying Pic_g^d over $\overline{M}_g$, where Pic_g^d denotes the "universal Picard variety of degree d"over M_g^0, parametrizing isomorphism classes of line bundles of degree d over automorphism-free nonsingular curves. We will now briefly describe this construction.

Fix an algebraically closed field k and consider the Hilbert scheme H of genus g curves defined over k embedded in $\mathbb{P}^r$ as nondegenerate curves of degree d, where $r = d - g$. There is a natural action of $PGL(r+1)$ in H corresponding to the choice of the coordinates used to embed the curves. For $d >> 0$, define

$$\overline{P}_{d,g} := H_d/PGL(r+1)$$

as the GIT-quotient of H_d, the locus of GIT-semistable points for this action (under a fixed suitable linearization). By results of [G82] and [C94], we know that, for $g \geq 2$, points in H_d correspond exactly to *quasistable* curves of genus g embedded by *balanced* line bundles (of degree d), where quasistable curves are semistable curves such that two exceptional components never meet, and balanced is a combinatorial condition on the multidegree of the line bundle on the curve (see Definition 2.1.1 below). In particular, given a quasistable curve X, there are only finitely many balanced multidegrees summing up to d.

By construction, $\overline{P}_{d,g}$ is endowed with a proper morphism ϕ_d onto $\overline{M}_g$ such that, for $g \geq 3$, $\phi_d^{-1}(M_g^0)$ is isomorphic to Pic_g^d. Moreover, given $[X] \in \overline{M}_g$, $\phi_d^{-1}(X)$ is a projective connected scheme with a finite number of components (that can not exceed a certain numerical invariant of the curve) and, if X has trivial automorphism group, $\phi_d^{-1}(X)$ is reduced and its smooth locus is isomorphic to the disjoint union of a finite number of copies of the jacobian of X, J_X.

Let $d >> 0$ such that $(d-g+1, 2g-2) = 1$. Then, the GIT-quotient yielding $\overline{P}_{d,g}$ is geometric (see [C94], Proposition 6.2) and the quotient stack associated to it, $[H_d/G]$, is a Deligne-Mumford stack with a strongly representable morphism onto $\overline{\mathcal{M}}_g$, where with G we will denote the group $PGL(r+1)$ (see [C05], 5.9). Moreover, it has the following modular description.

Consider the stack $\overline{\mathcal{P}}_{d,g}$ over SCH_k whose sections over a k-scheme S consist on families $\pi : X \to S$ of genus g quasistable curves over S endowed with a balanced line bundle of relative degree d over X and whose morphisms consist on cartesian diagrams of the curves plus an isomorphism between the line bundles (as in $\mathcal{P}ic_{d,g,n}$ above, ignoring the sections). There is a natural action of $\mathbb{G}_m$ on $\overline{\mathcal{P}}_{d,g}$ given by fiberwise scalar multiplication on the line bundles, leaving the curves fixed. Then, $[H_d/G]$ is isomorphic to the rigidification (in the sense of [ACV01]) of $\overline{\mathcal{P}}_{d,g}$ by the action of $B\mathbb{G}_m$ (see [C05] 5.10).

Let us now suppose that $(d-g+1, 2g-2) \neq 1$. Then, the quotient stack $[H_d/G]$ is not Deligne-Mumford and it is not possible to give a modular description of it using the same reasoning of Caporaso's in [C05], since it uses the existence of the analogue of Poincaré line bundles for families of stable curves, which does not exist in this general case. The main result of the first part of the present thesis consists exactly in showing that, if $d >> 0$, the quotient stack $[H_d/G]$ has the same modular description as in the case that $(d-g+1, 2g-2) = 1$. This follows from Theorem 2.3.1, where we show that $\overline{\mathcal{P}}_{d,g}$ is isomorphic to $[H_d/GL(r+1)]$, where $GL(r+1)$ acts by projection onto $G = PGL(r+1)$. This implies that $\overline{\mathcal{P}}_{d,g}$ is a smooth and irreducible Artin stack of dimension $4g-3$ endowed with a universally closed map onto $\overline{\mathcal{M}}_g$. Since, for d and d' such that $d = d' + m(2g-2)$ for some $m \in \mathbb{Z}$, $\overline{\mathcal{P}}_{d,g}$ is isomorphic to $\overline{\mathcal{P}}_{d',g}$, we get that the same statement holds in general for any $d \in \mathbb{Z}$ and for $g \geq 2$.

Then, using the universal property of the notion of rigidification, we give a modular description of the rigidification of $\overline{\mathcal{P}}_{d,g}$ along the action of $B\mathbb{G}_m$, $\overline{\mathcal{P}}_{d,g} /\!\!/ \mathbb{G}_m$, and we show that it is isomorphic to $[H_d/G]$ (see section 2.4 below). It also follows that $\overline{\mathcal{P}}_{d,g}$ is a $\mathbb{G}_m$-gerbe over $[H_d/G]$ (see 2.4.1).

In section 2.2 we consider the restriction of $[H_d/G]$ to the locus $\overline{\mathcal{M}}_g^d$ of d-general curves, i.e., the locus of genus g stable curves over which the GIT-quotient above is geometric. In Prop. 2.2.2 we show that this restriction, denoted by $[U_d/G]$, is a Deligne-Mumford stack and is endowed with a strongly representable map onto $\overline{\mathcal{M}}_g^d$. In particular, it gives a functorial way of getting a compactification of the relative degree d Picard variety for families of d-general curves (see Corollary 2.2.4 below). By this we mean that, given a family of stable curves $f : X \to S$, the base change of the moduli map $\mu_f : S \to \overline{\mathcal{M}}_g$ by $[H_d/G] \to \overline{\mathcal{M}}_g$ is a scheme, $\overline{P}_f^d$, yielding a compactification of the relative degree d Picard variety associated to f (see 1.1 for the Definition of the Picard variety associated to a morphism). This generalizes Proposition 5.9 of [C05] as observed in Remark 2.2.5. Instead, if the fibers of the family f are not d-general, the fiber product of the moduli map of f, $\mu_f : S \to \overline{\mathcal{M}}_g$, by the map $[H_d/G] \to \overline{\mathcal{M}}_g$ is not a scheme. Indeed, it is not even an algebraic space. However, it is canonically endowed with a proper map onto a scheme yielding a compactification of the relative degree d Picard variety associated to f (see Prop. 2.4.12).

In section 2.2.2 we give a combinatorial description of the locus in $\overline{M}_g$ of d-general curves, $\overline{M}_g^d$. For each d, $\overline{M}_g^d$ is an open subscheme of $\overline{M}_g$ containing all genus g irreducible curves and $\overline{M}_g^d = \overline{M}_g^{d'}$ if and only if $(d-g+1, 2g-2) = (d' - g + 1, 2g - 2)$. In Proposition 2.2.14 we also show that the $\overline{M}_g^d$'s yield a lattice of open subschemes of $\overline{M}_g$ parametrized by the (positive) divisors of $2g - 2$.

Let B be a smooth curve defined over an algebraically closed field k with function field K and X_K a smooth genus g curve over K whose regular minimal model over B is a family $f : X \to B$ of stable curves. Then, if $(d - g + 1, 2g - 2) = 1$, the smooth locus of the map $\overline{P}_f^d \to B$ is isomorphic to the Néron model of $\operatorname{Pic}^d X_k$ over B (see Theorem 6.1 of loc. cit.). We end chapter 2 by asking if, for any d, $[H_d/G]$, parametrizes Néron models of jacobians of smooth curves as it does if $(d - g + 1, 2g - 2) = 1$. In section 2.4.2 we show that the answer is no if $(d - g + 1, 2g - 2) \neq 1$, essentially because $[H_d/G]$ is not representable over $\overline{\mathcal{M}}_g$. We here focus on the case $d = g - 1$, which is particularly important. In fact for this degree all known compactified jacobians are canonically isomorphic and are endowed with a theta divisor which is Cartier and ample (see [A04]).

Compactified Picard stacks over $\mathcal{M}_{g,n}$

A consequence of what we have said so far is that, if we define $\overline{\mathcal{P}}_{d,g,0}$ to be equal to $\overline{\mathcal{P}}_{d,g}$, it gives an answer to our initial problem for $g \geq 2$ and $n = 0$. Chapter 3 is devoted to the case $n > 0$.

We start by introducing the notions of n-pointed quasistable curve and of balanced line bundles over these (see Definitions 3.1.3 and 3.1.4 below) and by noticing that, for $n = 0$ and $g \geq 2$, these coincide with the old notions. It turns out that, for $n > 0$, n-pointed quasistable curves are the ones we get by applying the stabilization morphism defined by Knudsen in [K83] (see 3.4.3 below) to $(n - 1)$-pointed quasistable curves endowed with an extra section without stability conditions. Moreover, balanced line bundles on n-pointed quasistable curves correspond to balanced line bundles on the quasistable curves obtained by forgetting the points and by contracting the rational components that get *quasidestabilized* without the points (see Lemma 3.1.10).

As a consequence, we also get a definition of n-pointed quasistable curves and of balanced line bundles over these for curves of genus 0 with at least 3 marked points and for curves of genus 1 with at least 1 marked point. It turns out that, for genus 0 curves, the notion of n-pointed quasistable coincides with the notion of n-pointed stable. Moreover, given an n-pointed stable curve X of genus 0, for each degree d, there is exactly one balanced multidegree summing up to d (see Remark 3.1.6).

We define $\overline{\mathcal{P}}_{d,g,n}$ to be the stack whose sections over a scheme S are given by families of genus g n-pointed quasistable curves over S endowed with a relative degree d balanced line bundle. Morphisms between two such sections are like in $\mathcal{P}ic_{d,g,n}$ above. We prove that $\overline{\mathcal{P}}_{d,g,n}$ is a smooth and irreducible algebraic (Artin) stack of dimension $4g-3+n$, endowed with a universally closed morphism onto $\overline{\mathcal{M}}_{g,n}$, giving a solution for our initial problem for all $g, n \geq 0$ such that $2g-2+n>0$ (see Theorem 3.2.2).

Our definitions imply that, for every integer d, $\overline{\mathcal{P}}_{d,0,3}$ is isomorphic to $\overline{\mathcal{M}}_{0,3} \times B\mathbb{G}_m$, that $\overline{\mathcal{P}}_{d,1,1}$ is isomorphic to $\overline{\mathcal{M}}_{1,1} \times B\mathbb{G}_m$ (see Propositions 3.2.7 and 3.2.10, respectively) and that Theorem 3.2.2 is true for $g \geq 2$ and $n=0$ (see Theorem 2.3.1).

Then, for $n>0$ and $2g-2+n>1$, we proceed by induction in the number of marked points n. Our construction goes along the lines of Knudsen's construction of $\overline{\mathcal{M}}_{g,n}$ in [K83], which consisted on showing that, for $n \geq 0$, $\overline{\mathcal{M}}_{g,n+1}$ is isomorphic to the universal family over $\overline{\mathcal{M}}_{g,n}$. In the same way, we show that there is an isomorphism between $\overline{\mathcal{P}}_{d,g,n+1}$ and the universal family over $\overline{\mathcal{P}}_{d,g,n}$, $\mathcal{Z}_{d,g,n}$ (see theorem 3.2.5), where $\mathcal{Z}_{d,g,n}$ is the stack whose sections over a k-scheme S consist on families of n-pointed quasistable curves over S, endowed with a balanced line bundle L and with an extra section. $\mathcal{Z}_{d,g,n}$ is naturally endowed with a universal line bundle $\mathcal{L}$, reason why it is universal over $\overline{\mathcal{P}}_{d,g,n}$ (see Proposition 3.2.3 below). The isomorphism between $\overline{\mathcal{P}}_{d,g,n+1}$ and $\mathcal{Z}_{d,g,n}$ is built explicitly and it generalizes Knudsen's notion of contraction and stabilization of n-pointed stable curves in this more general context of quasistable curves endowed with balanced line bundles. The main difference here is that we use the balanced line bundle itself tensored with the sections to contract and stabilize the curves, instead of the dualizing sheaf of the families used by Knudsen.

In order to make contraction work, we also need to prove some technical properties for line bundles over nodal curves. For instance, we show that a line bundle with sufficiently big multidegree is nonspecial, globally generated and normally generated (see Propositions 3.3.3 and 3.3.10). In particular, we get that, if L is a balanced line bundle of degree $d >> 0$ in an $(n+1)$-pointed quasistable curve X of genus $g \geq 0$ with $2g-2+n>0$, then

$L(p_1 + \cdots + p_n)$ is nonspecial, globally generated and normally generated, where $p_1, \ldots, p_n, p_{n+1}$ are the marked points of X (see Corollaries 3.3.6 and 3.3.12). We also get that the same holds for $(\omega_X(p_1 + \cdots + p_n + p_{n+1}))^m$, for $m \geq 3$, where ω_X denotes the dualizing sheaf of X (see Corollaries 3.3.5 and 3.3.11).

In section 3.5 we show that $\overline{\mathcal{P}}_{d,g,n}$ is endowed with a (forgetful) morphism $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$, given on objects by taking the stable model of the families and by forgetting the line bundle. We further study the fibres of $\Psi_{d,g,n}$.

Finally, in section 3.6, we study further properties of $\overline{\mathcal{P}}_{d,g,n}$. For instance, we show that if d and d' are such that $2g-2$ divides $d-d'$, then $\overline{\mathcal{P}}_{d,g,n}$ is isomorphic to $\overline{\mathcal{P}}_{d',g,n}$. We also study the map form $\overline{\mathcal{P}}_{d,g,n+1}$ to $\overline{\mathcal{P}}_{d,g,n}$ and its sections and we show that these yield Cartier divisors on $\overline{\mathcal{P}}_{d,g,n+1}$ with possibly interesting intersection properties.

Again, there is an action of $B\mathbb{G}_m$ on $\overline{\mathcal{P}}_{d,g,n}$ given by scalar product on the line bundles, leaving the curves and the sections fixed, so $\overline{\mathcal{P}}_{d,g,n}$ can never be Deligne-Mumford. By construction, for $n > 0$ and $2g-2+n > 1$, the rigidification of $\overline{\mathcal{P}}_{d,g,n}$ along this action of $\mathbb{G}_m$, denoted by $\overline{\mathcal{P}}_{d,g,n} \mathbin{/\!\!/} \mathbb{G}_m$, is Deligne-Mumford if and only if $\overline{\mathcal{P}}_{d,g,n-1}$ is and, in the same way, the natural map from $\overline{\mathcal{P}}_{d,g,n} \mathbin{/\!\!/} \mathbb{G}_m$ onto $\overline{\mathcal{M}}_{g,n}$, denote it again by $\Psi_{d,g,n}$, is proper and strongly representable if and only if $\Psi_{d,g,n-1}$ is. So, we get that, for any $n \geq 0$ and $g \geq 2$, $\overline{\mathcal{P}}_{d,g,n} \mathbin{/\!\!/} \mathbb{G}_m$ is a Deligne-Mumford stack proper and strongly representable over $\overline{\mathcal{M}}_{g,n}$ if and only if $(d-g+1, 2g-2) = 1$ (see Proposition 3.6.3). For $g = 0$ and 1 we get that, for any integer $d \in \mathbb{Z}$, $\overline{\mathcal{P}}_{d,0,n} \mathbin{/\!\!/} \mathbb{G}_m$ is isomorphic to $\overline{\mathcal{M}}_{0,n}$ for $n \geq 3$ and that $\overline{\mathcal{P}}_{d,1,n} \mathbin{/\!\!/} \mathbb{G}_m$ is isomorphic to $\overline{\mathcal{M}}_{1,n+1}$ for $n \geq 1$ (see Proposition 3.6.4).

We would also like to observe that another possible approach to the construction of $\overline{\mathcal{P}}_{d,g,n}$ would be to use Baldwin and Swinarski's GIT construction of the moduli space of stable maps and, in particular, of $\overline{M}_{g,n}$, in [BS08], and then proceed as Caporaso in [C94].

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Contents

Chapter 1

Preliminaries and notation

We will always consider schemes and algebraic stacks locally of finite type over an algebraically closed base field k. We will always indicate schemes with roman letters and stacks with calligraphic letters.

1.1 The relative Picard functor

Let X be an S-scheme with structural morphism $\pi : X \to S$. Given another S-scheme T, we will denote by $\pi_T : X_T \to T$ the base-change of π under the structural morphism $T \to S$.

$$\begin{array}{ccc} X_T := T \times_S X & \longrightarrow & X \\ {\scriptstyle \pi_T}\downarrow & & \downarrow{\scriptstyle \pi} \\ T & \longrightarrow & S \end{array}$$

Given a family of nodal curves f, we will denote by $\mathcal{P}ic_f$ the *relative Picard functor* associated to f and by $\mathcal{P}ic_f^d$ its subfunctor of line bundles of relative degree d. $\mathcal{P}ic_f$ is the fppf-sheaf associated to the functor $\mathcal{P} : \mathrm{SCH}_B \to \mathrm{Sets}$ which associates to a scheme T over B the set $\mathrm{Pic}(X_T)$. In particular, if the family f has a section, $\mathcal{P}ic_f(T) = \mathrm{Pic}(X_T)/\mathrm{Pic}(T)$ (see [BLR], chapter 8 for the general theory about the construction of the relative Picard functor).

Thanks to more general results of D. Mumford and A. Grothendieck in [M66] and [Gr], we know that $\mathcal{P}ic_f$ (and also $\mathcal{P}ic_f^d$) is representable by a scheme Pic_f, which is separated if all geometric fibers of f are irreducible (see also [BLR], 8.2, Theorems 1 and 2). Pic_g^d, the "universal degree d Picard variety", coarsely represents the degree d Picard functor for the universal family of (automorphism-free) nonsingular curves of genus g, $f_g : \mathcal{Z}_g \to M_g^0$. Furthermore, it was proved by Mestrano and Ramanan in [MR85] for $\mathrm{char}\, k = 0$ and later on by Caporaso in [C94] for any characteristic that Pic_g^d is a fine moduli space, that is, there exists a Poincaré line bundle over

$\mathrm{Pic}^d_g \times_{M^0_g} \mathcal{Z}_g$, if and only if the numerical condition $(d-g+1, 2g-2) = 1$ is satisfied.

1.2 Curves

Let S be a scheme. By a genus g curve X over S (or a family of curves over S) we mean a proper and flat morphism $X \to S$ whose geometric fibers are connected projective and reduced curves of genus g over k having at most nodes as singularities.

If we do not specify the base scheme S, by a curve X we will always mean a curve over k.

1.2.1 Line bundles on reducible curves

We will denote by ω_X the canonical or dualizing sheaf of X. For each proper subcurve Z of X (which we will always assume to be complete), denote by $Z' := \overline{X \setminus Z}$, by $k_Z := \sharp(Z \cap Z')$ and by g_Z its arithmetic genus. Recall that, if Z is connected, the adjunction formula gives

$$w_Z := \deg_Z \omega_X = 2g_Z - 2 + k_Z. \tag{1.1}$$

For $L \in \mathrm{Pic}\, X$ its *multidegree* is $\underline{\deg} L := (\deg_{Z_1} L, \dots, \deg_{Z_\gamma} L)$ and its (total) degree is $\deg L := \deg_{Z_1} L + \cdots + \deg_{Z_\gamma} L$, where $Z_1, \dots, Z_\gamma$ denote the irreducible components of X.

Given $\underline{d} = (d_1, \dots, d_\gamma) \in \mathbb{Z}^\gamma$, we set $\mathrm{Pic}^{\underline{d}}\, X := \{L \in \mathrm{Pic}\, X : \underline{\deg} L = \underline{d}\}$ and $\mathrm{Pic}^d\, X := \{L \in \mathrm{Pic}\, X : \deg L = d\}$. We have that $\mathrm{Pic}^d\, X = \sum_{|\underline{d}|=d} \mathrm{Pic}^{\underline{d}}\, X$, where $|\underline{d}| = \sum_{i=1}^{\gamma} d_i$.

The *generalized jacobian* of X is

$$\mathrm{Pic}^{\underline{0}}\ \mathrm{X} = \{L \in \mathrm{Pic}\, X : \underline{\deg} L = (0, \dots, 0)\}.$$

1.2.2 Stable and semistable curves

A *stable* curve is a nodal (connected) curve of genus $g \geq 2$ with ample dualizing sheaf. We will denote by $\overline{M}_g$ (resp. $\overline{\mathcal{M}}_g$) the moduli scheme (resp. stack) of stable curves and by $\overline{M}_g^0 \subset \overline{M}_g$ the locus of curves with trivial automorphism group.

A *semistable* curve is a nodal connected curve of genus $g \geq 2$ whose dualizing sheaf has non-negative multidegree.

It is easy to see that a nodal curve X is stable (resp. semistable) if, for every smooth rational component E of X, $k_E \geq 3$ (resp. $k_E \geq 2$). If X is semistable, the smooth rational components E such that $k_E = 2$ are called *exceptional*.

A semistable curve is called *quasistable* if two exceptional components never meet.

The *stable model* of a semistable curve X is the stable curve obtained by contracting all the exceptional components of X.

A *family of stable* (resp. *semistable*, resp. *quasistable*) curves is a flat projective morphism $f : X \to B$ whose geometrical fibers are stable (resp. semistable, resp. quasistable) curves. A line bundle of degree d on such a family is a line bundle on X whose restriction to each geometric fiber has degree d.

1.2.3 n-pointed stable and semistable curves

An *n-pointed curve* is a connected, projective and reduced nodal curve X together with n distinct marked points $p_i \in X$ such that X is smooth at p_i, $1 \leq i \leq n$.

Suppose that g and n are such that $2g - 2 + n > 0$. Then, we will say that an n-pointed curve of genus g is *stable* (resp. *semistable*) if the number of points where a nonsingular rational component E of X meets the rest of X plus the number of points p_i on E is at least 3 (resp. 2).

Suppose that $(X; p_1, \dots, p_n)$ is an n-pointed curve. It is easy to see that, analogously to the case of curves without marked points, $(X; p_1 \dots, p_n)$ is stable (respectively semistable) if and only the dualizing sheaf of X tensored with the marked points, $\omega_X(p_1 + \cdots + p_n)$, is ample (has non-negative multidegree) (see for instance [HM]).

A *family of n-pointed stable* (resp. *semistable*) curves is a flat and proper morphism $\pi : X \to S$ together with n distinct sections $s_i : S \to X$ such that the geometric fibers X_s together with the points $s_i(s)$, $1 \leq i \leq n$, are n-pointed stable (resp. semistable) curves.

1.3 Algebraic stacks

Let $\mathcal{S}$ be a category endowed with a Grothendieck topology.

Roughly speaking, a stack is a category fibered in groupoids over $\mathcal{S}$ such that isomorphisms are a sheaf and every descent datum is effective.

We will not give any details about this definition since there are plenty of good references; see, for instance [FGA] or [V89]. For a short introduction to the subject see also [F] or [E00].

We will denote by SCH (resp SCH_k) the category of schemes (resp. schemes over k) endowed with the flat topology.

Remark 1.3.1. Given a scheme S, the category SCH/S of schemes over S is a stack: the objects are morphisms of schemes with target S and a morphism from $f : T \to S$ to $f' : T' \to S$ is a morphism of schemes $g : T \to T'$ such that $f' = f \circ g$; the projection functor sends the object $T \to S$ to T and a morphism g to itself.

1.3.1 Representability

Definition 1.3.2. *A stack $\mathcal{F}$ is said to be representable if it is isomorphic to the stack induced by a scheme (see Remark 1.3.1 above).*

We say that a morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ is ***representable*** *(resp.* ***strongly representable****) if, for any scheme Y with a morphism $Y \to \mathcal{G}$, the fiber product $\mathcal{F} \times_{\mathcal{G}} Y$ is an algebraic space (resp. a scheme).*

Note that morphisms of schemes are always strongly representable.

Example 1.3.3. The morphism $\phi : \mathcal{M}_{g,1} \to M_g$ forgetting the section is strongly representable. In fact, giving a map of Y to $\mathcal{M}_g$ is equivalent to give a curve $\pi : C \to Y$ in $\mathcal{M}_g(Y)$: the image of the identity morphism $Id_Y : Y \to Y$. Then, given any other morphism $g : Z \to Y$, its image under the map to $\mathcal{M}_g$ is necessarily $\pi^*(f)$. It follows that the fiber product $Y \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$ is isomorphic to C.

Definition 1.3.4. *A strongly representable morphism of stacks $f : \mathcal{F} \to \mathcal{G}$ has property* ***P*** *if for any map $S \to \mathcal{G}$, where S is a scheme, the morphism of schemes $\mathcal{F} \times_{\mathcal{G}} S \to S$ has property* ***P****.*

1.3.2 Algebraicity

In what concerns to algebraic stacks, we will always follow the definitions of [L-MB00].

Definition 1.3.5. *A stack $\mathcal{F}$ is algebraic in the sense of Artin (resp. Deligne-Mumford) if there exists a smooth (resp. étale) and surjective strongly representable morphism $S \to \mathcal{F}$, where S is (the stack associated to) a scheme. We will also say that $S \to \mathcal{F}$ is a presentation or a smooth (resp. étale) atlas of $\mathcal{F}$.*

Note that it makes sense to say that $S \to \mathcal{F}$ is algebraic in virtue of Definition 1.3.4 above.

1.3.3 Quotient stacks

Since almost all stacks that we will consider throughout this thesis are quotient stacks, we will say something more about these now.

Let G be an algebraic group acting on a scheme S on the left. Let $[S/G]$ be the following category fibered over SCH: its objects are principal homogeneous G-bundles with a G-equivariant morphism to S and morphisms are those pullback diagrams which are compatible with the morphism to S.

It turns out that, in general, $[S/G]$ is an algebraic Artin stack since there is a natural map

$$S \to [S/G]$$

which is a smooth and surjective presentation of $[S/G]$ of relative dimension $\dim G$.

Definition 1.3.6. *A quotient stack is a stack of the form* $[S/G]$*, for some* S *and* G *as above.*

If the stabilizers of the action of G on S are all finite and reduced (for instance, if it is a GIT-geometric action), it turns out that $[S/G]$ is indeed Deligne-Mumford (see for instance [V89], 7.17).

Example 1.3.7. The satcks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ are a quotient stacks for the action of $PGL(r+1)$ in suitable subschemes of a certain Hilbert scheme H (see for instance [E00] for an overview about this construction).

1.3.4 Coarse moduli spaces for algebraic stacks

Definition 1.3.8. *A coarse moduli space for a stack* $\mathcal{F}$ *is an algebraic space* F *together with a morphism* $\pi : \mathcal{F} \to F$ *satisfying the following properties:*

- *for any algebraically closed field* Ω*,* π *induces an isomorphism between the connected components of the groupoid* $\mathcal{F}(\operatorname{Spec}\Omega)$ *and* $F(\operatorname{Spec}\Omega)$*;*
- π *is universal for morphisms from* $\mathcal{F}$ *onto algebraic spaces.*

Example 1.3.9. GIT-geometric quotients by the action of an algebraic group in a scheme are coarse moduli spaces for the quotient stack associated to that action (see [V89] 2.1 and 2.11). So, for instance M_g and $\overline{M}_g$ are coarse moduli spaces for the stacks $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$, respectively.

Chapter 2

Balanced Picard stacks over $\overline{\mathcal{M}}_g$

In the present chapter we will try to give an answer to our initial problem for $n = 0$. So, we consider the stack $\mathcal{P}ic_{d,g}$, parametrizing families of nonsingular curves of genus g endowed with a line bundle of relative degree d over these families and we try to get a modular compactification of it over $\overline{\mathcal{M}}_g$.

As we already mentioned, this case is particularly fortunate since there are already several constructions of compactified jacobians for families of stable curves.

Our approach will be to consider Caporasos's construction, which is made by means of a GIT quotient, and try to give a modular description of the quotient stack associated to it (see 1.3.3 above).

We will start by giving an overview of the whole construction and by discussing some details associated to it. For example, we give, for every integer d, a geometrical description of the locus of genus g stable curves over which we get Deligne-Mumford stacks strongly representable over $\overline{\mathcal{M}}_g$. In this case, our stacks parametrize Néron models of jacobians of smooth curves in a sense that will be made precise (see 2.4.2 below). We also show that this point of view allows us to get compactified Picard varieties (of degree d) for families of stable curves in a functorial way. By this we mean that, given a family of stable curves $f : \mathcal{X} \to S$, the fiber product of our stacks by the moduli map $\mu_f : S \to \overline{\mathcal{M}}_g$ is either a compactification of the relative degree d Picard variety associated to f or has a canonical map onto it (see 1.1 for the definition of Picard variety associated to a morphism).

2.1 Balanced line bundles over semistable curves

Recall that Gieseker's construction of $\overline{M}_g$ consists of a GIT-quotient of the action of $SL(N)$, for some $N \in \mathbb{Z}$, on a Hilbert scheme where it is possible to embed all semistable curves of genus g (the "Hilbert point"of the curve) (see

[G82]). Gieseker shows that in this Hilbert scheme, in order for the Hilbert point of a curve to be GIT-semistable, it is necessary that the multidegree of the line bundle giving its projective realization satisfies an inequality, called the "Basic Inequality". Later, in [C94], Caporaso shows that this condition is also sufficient.

We will now give the definition of this inequality, extending the terminology introduced in [CCC04].

Definition 2.1.1. *Let X be a semistable curve of genus $g \geq 2$ and L a degree d line bundle on X.*

(i) We say that L (or its multidegree) is semibalanced if, for every connected proper subcurve Z of X the following ("Basic Inequality") holds

$$m_Z(d) := \frac{dw_Z}{2g-2} - \frac{k_Z}{2} \leq \deg_Z L \leq \frac{dw_Z}{2g-2} + \frac{k_Z}{2} := M_Z(d). \quad (2.1)$$

(ii) We say that L (or its multidegree) is balanced if it is semibalanced and if $\deg_E L = 1$ for every exceptional component E of X. The set of balanced line bundles of degree d of a curve X is denoted by B^d_X.

(iii) We say that L (or its multidegree) is stably balanced if it is balanced and if for each connected proper subcurve Z of X such that $\deg_Z L = m_Z(d)$, the complement of Z, Z', is a union of exceptional components.

The set of stably balanced line bundles of degree d on X will be denoted by $\tilde{B}^d_X$.

Remark 2.1.2. Balanced multidegrees are representatives for multidegree classes of line bundles on X up to twisters (that is, to elements in the degree class group of X, Δ_X, which is a combinatorial invariant of the curve defined in [C94]). More particularly, in [C05], Proposition 4.12, Caporaso shows that, if X is a quasistable curve, every multidegree class in Δ_X has a semibalanced representative and that a balanced multidegree is unique in its equivalence class if and only if it is stably balanced.

In [BMS1] and [BMS2] we have studied, together with Simone Busonero and Lidia Stoppino, several geometrical and topological properties of stable curves using exactly this invariant and its combinatorial properties. It is remarkable how combinatorics give interesting tools to the study of reducible nodal curves.

We now list some easy consequences of the previous definition.

Remark 2.1.3. (A) If a semistable curve X admits a balanced line bundle L, then X must be quasistable.

(B) To verify that a line bundle L is balanced it is enough to check that $\deg_Z L \geq m_Z(d)$, for each proper subcurve Z of X and that $\deg_E L = 1$ for each exceptional component E of X.

(C) If X is a stable curve, then a balanced line bundle L on X is stably balanced if and only if, for each proper connected subcurve Z of X, $\deg_Z L \neq m_Z(d)$.

(D) Let X be a stable curve consisting of two irreducible components, Z and Z', meeting in an arbitrary number of nodes. Then X admits a degree d line bundle which is balanced but not stably balanced if and only if $\frac{d-g+1}{2g-2} w_Z \in \mathbb{Z}$ (equivalently if $\frac{d-g+1}{2g-2} w_{Z'} \in \mathbb{Z}$).

(E) A line bundle is balanced (resp. stably balanced) if and only if $L \otimes \omega_X^{\otimes n}$ is balanced (resp. stably balanced), for $n \in \mathbb{Z}$. So, given integers d and d' such that $\exists n \in \mathbb{Z}$ with $d \pm d' = n(2g-2)$, there are natural isomorphisms $B_X^d \cong B_X^{d'}$ (and $\tilde{B}_X^d \cong \tilde{B}_X^{d'}$).

For (A) and (B) see [CE] Remark 3.3. (C) and (E) are immediate consequences of the definition. For (D) note that, given a balanced γ-uple $\underline{d} \in \mathbb{Z}^\gamma$ such that $|\underline{d}| = d$, there exists a (balanced degree d) line bundle L in X such that $\underline{\deg} L = \underline{d}$. Since $k_Z = w_Z - 2g_Z + 2$, we can write $m_Z(d)$ as $\frac{d-g+1}{2g-2} w_Z + g_Z - 1$, which is an integer by hypothesis. In the same way, $M_{Z'}(d) = d - m_Z(d)$ is an integer too, so $(m_Z(d), M_{Z'}(d))$ is a balanced multidegree which is not stably balanced.

2.1.1 Caporaso's construction

Let $\overline{P}_{d,g} \to \overline{M}_g$ be Caporaso's compactification of the universal Picard variety of degree d, $\mathrm{Pic}_g^d \to M_g^0$, constructed in [C94]. We will now recall some basic facts about this construction.

For $d >> 0$, and $g \geq 2$, $\overline{P}_{d,g}$ is the GIT-quotient

$$\pi_d : H_d \to H_d/PGL(r+1) =: \overline{P}_{d,g}$$

where $H_d := (\mathrm{Hilb}_{\mathbb{P}^r}^{dt-g+1})^{ss}$, the locus of GIT-semistable points in the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^r}^{dt-g+1}$, with $r = d-g$, which is naturally endowed with an action of $PGL(r+1)$ leaving H_d invariant. $\overline{P}_{d,g}$ naturally surjects onto $\overline{M}_g$ via a proper map $\phi_d : \overline{P}_{d,g} \to \overline{M}_g$ such that, for $g \geq 3$, $\phi_d^{-1}(\overline{M}_g^0)$ is isomorphic to Pic_g^d.

Remark 2.1.4. Note that, even if Caporaso's original results were stated for $g \geq 3$, the whole GIT construction holds also for $g = 2$. In fact, the universal Picard variety Pic_g^d over M_g^0 in [C94] exists only for $g \geq 3$ (since M_g^0 is empty otherwise), so it only makes sense to compactify it for $g \geq 3$. However, the description of the GIT-stable and semistable points of the Hilbert schemes where we embed the curves, works also for $g = 2$ as we can see by analyzing the proofs in [C94]. This description is all we need for our results.

For $[X] \in \overline{M}_g$, denote by $\overline{P}_{d,X}$ the inverse image of X by ϕ_d. $\overline{P}_{d,X}$ is a connected projective scheme having at most Δ_X irreducible components, all of dimension g. In addition, if X is automorphism-free, the smooth locus of $\overline{P}_{d,X}$ is isomorphic to the disjoint union of a finite number of copies of J_X.

Points in H_d correspond to nondegenerate quasistable curves in $\mathbb{P}^r$ embedded by a balanced line bundle.

Let $H_d^s \subseteq H_d$ be the locus of GIT-stable points. These correspond to nondegenerate quasistable curves in $\mathbb{P}^r$ embedded by a stably balanced line bundle of degree d.

Definition 2.1.5. *Let X be a semistable curve of arithmetic genus $g \geq 2$. We say that X is d-general if all degree d balanced line bundles on X are stably balanced. Otherwise, we will say that X is d-special.*

Denote by $U_d := (\phi_d \circ \pi_d)^{-1}(\overline{M}_g^d)$ the subset of H_d corresponding to d-general curves. U_d is an open subset of H_d where the GIT-quotient is geometric (i.e., all fibers are $PGL(r+1)$-orbits and all stabilizers are finite and reduced), invariant under the action of $PGL(r+1)$.

$U_d = H_d$ if and only if $(d-g+1, 2g-2) = 1$, so the GIT-quotient yielding $\overline{P}_{d,g}$ is geometric if and only if $(d-g+1, 2g-2) = 1$ (see Proposition 6.2 of loc. cit.).

2.2 Balanced Picard stacks on d-general curves

By reasons that will be clear in a moment (see Section 2.2.1 below), call $\overline{P}_{d,g}^{\text{Nér}}$ the GIT-quotient of U_d by $PGL(r+1)$, for $g \geq 2$.

For the time being let

$$G := PGL(r+1).$$

Let us now consider the quotient stack $[U_d/G]$.

Recall that, given a scheme S over k, a section of $[U_d/G]$ over S consists of a pair $(\phi : E \to S, \psi : E \to U_d)$ where ϕ is a G-principal bundle and ψ is a G-equivariant morphism. Arrows correspond to those pullback diagrams which are compatible with the morphism to U_d.

Let $\overline{\mathcal{M}}_g^d \subset \overline{\mathcal{M}}_g$ be the moduli stack of d-general stable curves. There is a natural map from $[U_d/G]$ to $\overline{\mathcal{M}}_g^d$, the restriction to d-general curves of the moduli stack of stable curves, $\overline{\mathcal{M}}_g$. In fact, the restriction to U_d of the stabilization morphism from H_d to $\overline{M}_g$ factors through $\overline{M}_g^d$ and, since U_d is invariant under the action of G, this yields a map from $[U_d/G]$ to $\overline{M}_g^d$.

We will start by proving the following general result about representability of morphisms of Deligne-Mumford stacks.

Lemma 2.2.1. *Let $f : \mathcal{F} \to \mathcal{G}$ be a representable morphism of Deligne-Mumford stacks admitting coarse moduli spaces F and G, respectively. Then, if the morphism induced by f in the coarse moduli spaces, $\pi : F \to G$, is strongly representable, also f is strongly representable.*

Proof. We must show that, given a scheme B with a morphism to $\mathcal{G}$, the fiber product of f with this morphism, $\mathcal{F}_B$, is a scheme.

$$\begin{array}{ccccc} \mathcal{F}_B & \longrightarrow & \mathcal{F} & \longrightarrow & F \\ \downarrow & & \downarrow f & & \downarrow \pi \\ B & \longrightarrow & \mathcal{G} & \longrightarrow & G \end{array}$$

Since f is representable, we know that $\mathcal{F}_B$ is an algebraic space, so to show that it is indeed a scheme it is enough to show that there is a projective morphism from $\mathcal{F}_B$ to a scheme (see [Vie94] 9.4). Consider the fiber product of the induced morphism from B to G with π, F_B. Since, by hypothesis, π is representable, F_B is a scheme and is endowed with a natural morphism to $\mathcal{F}_B$, $\rho : \mathcal{F}_B \to F_B$, the base change over B of the map from $\mathcal{F}$ to F. Since F is the coarse moduli space of $\mathcal{F}$, this map is proper (see [V89] 2.1), so also ρ is proper. Now, to show that ρ is projective it is enough to see that it has finite fibers, which follows from the fact that the stacks are Deligne-Mumford. □

Proposition 2.2.2. *The quotient stack $[U_d/G]$ is Deligne-Mumford for every $d \in \mathbb{Z}$ and for every $g \geq 2$ and its natural map onto $\overline{\mathcal{M}}_g^d$ is strongly representable.*

Proof. The fact that $[U_d/G]$ is Deligne-Mumford comes from the well known fact that a quotient stack is Deligne-Mumford if and only if the action of the group on the scheme is GIT-geometric, that is, if all stabilizers are finite and reduced. Since U_d is the locus of curves where balanced line bundles are necessarily stably balanced, the Hilbert point of a d-general curve is GIT-semistable if and only if it is GIT-stable, so the GIT-quotient of U_d by G is geometric.

The proof of the strong representability of the natural map from $[U_d/G]$ to $\overline{\mathcal{M}}_g^d$ consists on two steps: first we prove that it is representable and then we use it to prove strong representability.

To prove representability it is sufficient to see that given any section of our quotient stacks over the spectrum of an algebraically closed field k', the automorphism group of it injects into the automorphism group of its image in $\overline{\mathcal{M}}_g^d$ (see for example [AV02] 4.4.3). But a section of our quotient stack over an algebraically closed field consists of a map onto a orbit of the action of G in U_d. So, the automorphism group of that section is isomorphic to the stabilizer of the orbit. The image of our section consists of a stable curve X: the stable model of the projective curve associated to that orbit. As this must be d-general, it is GIT-stable and we can use [C94] section 8.2 to conclude that the stabilizer of the orbit injects into the automorphism group of X.

So, the map from $[U_d/G]$ to $\overline{\mathcal{M}}_g^d$ is representable. It follows now immediately that it is also strongly representable from Lemma 2.2.1 and the fact that the GIT-quotients yielding $\overline{P}_{d,g}^{\text{Nér}}$ and $\overline{M}_g$ are geometric (see Example 1.3.9). $\square$

Definition 2.2.3. *Let $f : X \to S$ be a family of stable curves. A compactification of the relative Picard variety of degree d associated to f is a projective S-scheme P whose fiber over closed points ξ of S corresponding to automorphism-free fibers X_ξ of f is isomorphic to $\phi_d^{-1}(X_\xi)$.*

The following is an immediate consequence of the previous Proposition.

Corollary 2.2.4. *The Deligne-Mumford stack $[U_d/G]$ gives a functorial way of getting compactifications of the relative Picard variety of degree d for families of d-general curves in the sense of Definition 2.2.3.*

Remark 2.2.5. Consider $[H_d/G]$, the quotient stack of the action of $G = PGL(r+1)$ in H_d. Then, if $(d-g+1, 2g-2) = 1$, $[H_d/G] = [U_d/G]$ and all we said in this section was already proved in [C05], section 5, for $[H_d/G]$. Note also that in loc. cit. $[H_d/G]$ is denoted by $\overline{\mathcal{P}}_{d,g}$.

2.2.1 Néron models of families of d-general curves

Recall that, given a DVR (discrete valuation ring) R with function field K and an abelian variety A_K over K, the Néron model of A_K, $N(A_K)$, is a smooth model of A_K over $B = \operatorname{Spec} R$ defined by the following universal property (cf. [BLR] Definition 1): for every smooth scheme Z over B with a map $u_K : Z_K \to A_Z$ of its generic fiber, there exists an unique extension of u_K to a B-morphism $u : Z \to N(A_K)$. Note that $N(A_K)$ may fail to be proper over B but it is always separated.

Let $f : X \to B$ be a family of stable curves with X nonsingular. Denote by X_k the closed fiber of the family and by X_K its generic fiber. The question is how to construct the Néron model of the Picard variety $\operatorname{Pic}^d X_K$ in a functorial way over $\overline{\mathcal{M}}_g$. Even if it is natural to look at the Picard scheme (of degree d) of the family, $\operatorname{Pic}^d_f \to B$, which is smooth and has generic fiber equal to $\operatorname{Pic}^d X_K$, it turns out to be non satisfactory since it fails to be separated over B if the closed fiber X_k of f is reducible.

Consider the quotient stack $[U_d^{st}/G]$, where U_d^{st} is the locus of points in U_d that parametrize d-general stable curves.

It is clear that the statement of Proposition 2.2.2 holds for $[U_d^{st}/G]$ since U_d^{st} is a G-invariant subscheme of U_d. So, given a family of d-general stable curves $f : X \to B$, the fiber product $[U_d^{st}/G] \times_{\overline{\mathcal{M}}_g^d} B$, where $B \to \overline{\mathcal{M}}_g$ is the moduli map associated to the family f, is a scheme over B, denoted by P_f^d.

$$\begin{array}{ccc} P_f^d & \longrightarrow & \mathcal{P}_{d,g}^{\text{Nér}} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \overline{\mathcal{M}}_g^d \end{array}$$

Suppose X is regular. Then, from [C05], Theorem 6.1, we get that $P_f^d \cong N(\operatorname{Pic}^d X_K)$.

2.2.2 Combinatorial description of d-general curves in $\overline{M}_g$

Recall the notions of d-general and d-special curve from Definition 2.1.5.

Following the notation of [CE], we will denote by Σ_g^d the locus in $\overline{M}_g$ of d-special curves. So, Σ_g^d consists of stable curves X of genus g such that $\tilde{B}_X^d \setminus B_X^d \neq \emptyset$ (see Definition 2.1.1). In particular, Σ_g^d is contained in the closed subset of $\overline{M}_g$ consisting of reducible curves. Let us also denote by $\overline{M}_g^d$ the locus of d-general genus g stable curves (so $\Sigma_g^d \cup \overline{M}_g^d = \overline{M}_g$, for all $d \in \mathbb{Z}$).

From [C94], Lemma 6.1, we know that $\overline{M}_g^d$ is the image under ϕ_d of U_d, so it is an open subset of $\overline{M}_g$.

Recall that a *vine curve* is a curve with two smooth irreducible components meeting in an arbitrary number of nodes. The closure in $\overline{M}_g$ of the vine curves of genus g is precisely the locus of reducible curves.

In Proposition 2.2.10, we give a geometric description of Σ_g^d.

Example 2.2.6. Let $d = 1$. From [CE], Prop. 3.15, we know that, if g is odd, Σ_g^1 is empty and that if g is even, Σ_g^1 is the closure in $\overline{M}_g$ of the locus of curves $X = C_1 \cup C_2$, with C_1 and C_2 smooth of the same genus and $\sharp(C_1 \cap C_2) = k$ odd.

Observe that, from the above example, we get that Σ_g^1 is the closure in $\overline{M}_g$ of the 1-special vine curves of genus g. In what follows we will see that this is always the case for any degree.

Lemma 2.2.7. *Let d be an integer greater or equal than 1. Then Σ_g^d is the closure in $\overline{M}_g$ of the locus of d-special vine curves.*

Proof. Let X be a genus g d-special curve. As X is stable, using Remark 2.1.3 (C), this means that there is a connected proper subcurve Z of X and a balanced line bundle L on X such that $\deg_Z L = m_Z(d)$.

So, let Z be a connected proper subcurve of X such that $\deg_Z L = m_Z(d)$ and such that w_Z is maximal among the subcurves satisfying this condition. The complementary curve of Z in X, Z' must be such that

$$\deg_{Z'} L = d - \deg_Z L = d - m_Z(d) = M_{Z'}(d).$$

Let us see that Z' is connected as well.

By contradiction, suppose $Z' = Z'_1 \cup \cdots \cup Z'_s$ is a union of connected components with $s > 1$. As $\deg_{Z'} L = M_{Z'}(d)$, also each one of its connected components Z'_i, $i = 1, \ldots, s$, must be such that $\deg_{Z'_i} L = M_{Z'_i}(d)$. In fact, suppose one of them, say Z'_j, is such that $\deg_{Z'_j} L < M_{Z'_j}(d)$. Then,

$$\deg_{Z'} L = \sum_{i=1}^{s} \deg_{Z'_i} L < \sum_{i=1}^{s} \frac{dw_{Z'_i}}{2g-2} + \frac{k_{Z'_i}}{2} = \frac{dw_{Z'}}{2g-2} + \frac{k_{Z'}}{2} = M_{Z'}(d)$$

leading us to a contradiction. Note that the sum of the $k_{Z'_i}$'s is $k_{Z'}$ because, being the Z'_i's the connected components of Z', they do not meet each other.

Now, let us consider $W := Z \cup Z'_1$. As $s > 1$, W is a connected proper subcurve of X with $w_W = w_Z + w_{Z'_1} > w_Z$. Indeed, as X is stable,

$$0 < w_Y < 2g - 2 \tag{2.2}$$

for every proper subcurve Y of X, since there are no exceptional components. Moreover,

$$\deg_W L = \deg_Z L + \deg_{Z'_1} L = \frac{dw_Z}{2g-2} - \frac{k_Z}{2} + \frac{dw_{Z'_1}}{2g-2} + \frac{k_{Z'_1}}{2} = \frac{dw_W}{2g-2} - \frac{k_W}{2}$$

because, being Z'_1 a connected component of Z', we have that

$$k_Z - k_{Z'_1} = k_Z - \sharp(Z \cap Z'_1) = k_W.$$

So, W is a connected proper subcurve of X with $\deg_W L = m_W(d)$ and with $w_W > w_Z$. This way, we achieved a contradiction by supposing that Z' is not connected.

As both Z and Z' are limits of smooth curves and Σ_g^d is closed in $\overline{M}_g$, then X lies in the closure in $\overline{M}_g$ of the locus of genus g d-special vine curves. $\square$

Given integers d and g, we will use the following notation to indicate greatest common divisor

$$G_d := (d - g + 1, 2g - 2).$$

>From [C94], we know that Σ_g^d is a proper closed subset of $\overline{M}_g$ and that $\Sigma_g^d = \emptyset$ if and only if $G_d = 1$ (see Prop. 6.2 of loc. cit.).

Remark 2.2.8. >From Lemma 2.2.7 and Remark 2.1.3(D) we conclude that a stable curve X is d-special if and only if there is a connected proper subcurve Z of X such that $\overline{X \setminus Z}$ is connected and $\frac{2g-2}{G_d}$ divides w_Z.

Remark 2.2.9. If $G_d = 2g-2$, which means that $d \equiv (g-1) \pmod{2g-2}$, an immediate consequence of the previous Remark is that all reducible curves are $(g-1)$-special. This is the opposite situation to the case $G_d = 1$.

>From Remark 2.2.8 we see that Σ_g^d depends only on G_d. This is evident in the following proposition, where we give a geometric description of Σ_g^d.

Proposition 2.2.10. *Let d be an integer greater or equal than 1. Then Σ_g^d is the closure in $\overline{M}_g$ of vine curves $X = C_1 \cup C_2$ such that*

$$\frac{2g-2}{G_d} \mid w_{C_1}.$$

More precisely, Σ_g^d is the closure in $\overline{M}_g$ of the following vine curves: given integers m and k with

$$1 \leq m < G_d$$

and

$$1 \leq k \leq \min\{\frac{2g-2}{G_d}m + 2, 2g - \frac{2g-2}{G_d}m\},\ k \equiv \frac{2g-2}{G_d}m \pmod 2,$$

then $X = C_1 \cup C_2$, with $\sharp(C_1 \cap C_2) = k$, and

- $g(C_1) = \frac{g-1}{G_d}m - \frac{k}{2} + 1$;
- $g(C_2) = g - \frac{g-1}{G_d} - \frac{k}{2}$.

Proof. The first part of the proposition is an immediate consequence of Lemma 2.2.7 and Remark 2.2.8.

Now, let X be a d-special genus g vine curve $X = C_1 \cup C_2$ with $\sharp(C_1 \cap C_2) = k$. >From Remark 2.2.8 we know that there exists an integer m such that

$$m\frac{2g-2}{G_d} = w_{C_1}$$

with $1 \leq m < G_d$ because, as X is a stable curve, $\frac{w_{C_1}}{2g-2}$ must be smaller than 1.

As $w_{C_1} = 2g(C_1) - 2 + k$, we get that $k \equiv \frac{2g-2}{G_d}m \pmod 2$ and that

$$g(C_1) = \frac{g-1}{G_d}m - \frac{k}{2} + 1.$$

Now, as $g = g(C_1) - g(C_2) + k - 1$, we get that

$$g(C_2) = g - \frac{g-1}{G_d} - \frac{k}{2}.$$

As $g(C_1)$ and $g(C_2)$ must be greater or equal than 0, we get, respectively, that

$$k \leq \frac{2g-2}{G_d}m + 2 \text{ and } k \leq 2g - \frac{2g-2}{G_d}m.$$

It is easy to see that if $g(C_1)$ or $g(C_2)$ are equal to 0 then $k \geq 3$. So, the vine curves we constructed are all stable. $\square$

Remark 2.2.11. Since by smoothing a vine curve in any of its nodes we get an irreducible curve, we see that the above set of "generators"of Σ_g^d is minimal in the sense that none of them lies in the closure of the others.

The dependence of Σ_g^d on G_d gets even more evident in the following proposition.

Proposition 2.2.12. *For every $d, d' \in \mathbb{Z}$, $G_d | G_{d'}$ if and only if $\Sigma_g^d \subset \Sigma_g^{d'}$.*

Proof. That $G_d | G_{d'}$ implies that $\Sigma_g^d \subset \Sigma_g^{d'}$ is immediate from Remark 2.2.8.

Now, suppose $\Sigma_g^d \subset \Sigma_g^{d'}$. If $G_d = 1$ then obviously $G_d | G_{d'}$. For $G_d \neq 1$ we will conclude by contradiction that $G_d | G_{d'}$. So, suppose $G_{d'} \nmid G_d$. Then, also $\frac{2g-2}{G_d'} \nmid \frac{2g-2}{G_d}$. We will show that there exists a stable curve X consisting of two smooth irreducible components C_1 and C_2 meeting in δ nodes ($\delta \geq 1$) which is d-special but not d'-special.

Take X such that $w_{C_1} = \frac{2g-2}{G_d}$. If such a curve exists and is stable then we are done because X will clearly be d-special and not d'-special. In fact, by construction, $\frac{2g-2}{G_{d'}}$ does not divide w_{C_1} and $\frac{2g-2}{G_{d'}}$ will not divide w_{C_2} too because $w_{C_2} = (2g-2) - \frac{2g-2}{G_d}$.

So, X must be such that

- $g(C_1) = \frac{g-1}{G_d} + 1 - \frac{\delta}{2}$
- $g(C_2) = g - \frac{g-1}{G_d} - \frac{\delta}{2}$
- $\delta \geq 1$ and $\delta \equiv \frac{2g-2}{G_d} \pmod 2$.

As $g(C_i)$ must be greater or equal than 0 and the curve X must be stable, we must check if such a construction is possible.

So, if $\frac{2g-2}{G_d} \equiv 1 \pmod 2$, take $\delta = 1$. Then we will have that $g(C_1) = \frac{g-1}{G_d} + \frac{1}{2}$ and $g(C_2) = g - \frac{g-1}{G_d} - \frac{1}{2}$, which are both greater than 1 because we are considering $G_d > 1$.

If $\frac{2g-2}{G_d} \equiv 0 \pmod 2$, take $\delta = 2$. Then we will have that $g(C_1) = \frac{g-1}{G_d}$ and $g(C_2) = g - \frac{g-1}{G_d} - 1$, again both greater than 1. We conclude that X is a stable curve. $\square$

The following is immediate.

Corollary 2.2.13. *For all d and d', $\Sigma_g^d = \Sigma_g^{d'}$ if and only if $G_d = G_{d'}$.*

For each positive divisor M of $2g-2$ there is an integer $d = M + g - 1$ such that $G_d = M$. So, for each such M, we can define

$$\Sigma_{g,M} := \Sigma_g^d \text{ and } \overline{M}_g^M = \overline{M}_g \setminus \Sigma_{g,M}.$$

For example, $\overline{M}_g^{2g-2}$ consists of irreducible curves of genus g and $\overline{M}_g^1 = \overline{M}_g$.

The following is now immediate.

Proposition 2.2.14. *The open subsets $\overline{M}_g^M$ associated to the positive divisors M of $2g-2$, form a lattice of open subschemes of $\overline{M}_g$ such that $\overline{M}_g^M \subset \overline{M}_g^{M'}$ if and only if $M'|M$.*

2.3 Modular description of Balanced Picard stacks over $\overline{\mathcal{M}}_g$

Suppose $g \geq 2$ and $(d-g+1, 2g-2) = 1$. Then $\overline{\mathcal{M}}_g^d = \overline{\mathcal{M}}_g$ and $[U_d/G] = [H_d/G]$ (see section 2.2.2). Moreover, from [C05], 5.10, we know that $[H_d/G]$ is the "rigidification"in the sense of [ACV01] (see section 2.4 below) of the category whose sections over a scheme S are pairs $(f : X \to S, L)$ where f is a family of quasistable curves of genus g and L is a balanced line bundle on X of relative degree d. Arrows between such pairs are given by cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

and an isomorphism $L \cong h^*L' \otimes f^*M$, for some $M \in \text{Pic } S$.

This description uses heavily the existence of Poincaré line bundles for families of quasistable curves, established in loc. cit., Lemma 5.5. However, this works only if $(d-g+1, 2g-2) = 1$.

In order to overcome this difficulty we will try to define the stack of line bundles of families of stable curves.

We will start by recalling the definition of "Picard stack associated to a morphism of schemes". Roughly speaking, a *Picard stack* is a stack together with an "addition"operation which is both associative and commutative. The theory of Picard stacks is developed by Deligne and Grothendieck on Section 1.4 of Exposé XVIII in [SGA4]. We will not include here the precise definition but we address the reader to [ibid.], [L-MB00] 14.4 and [BF], section 2.

Given a scheme X over S with structural morphism $f : X \to S$, the S-stack of (quasi-coherent) invertible $\mathcal{O}_X$-modules, $\mathcal{P}ic_{X/S}$, is a Picard stack: the one associated to the complex of length one

$$\tau_{\leq 0}(Rf_*\mathbb{G}_m[1]).$$

So, given an S-scheme T, $\mathcal{P}ic_{X/S}(T)$ is the groupoid whose objects are invertible $\mathcal{O}_{X_T}$-modules and whose morphisms are the isomorphisms between them (notation as in 1.1).

$\mathcal{P}ic_{X/S}$ fits in the exact sequence below, where, given an S-scheme T, $\mathrm{Pic}_{X/S}(T)$ is defined as $\mathrm{Pic}\, X_T/f_T^*(\mathrm{Pic}\, T)$ and $B\mathbb{G}_m(T)$ is the group of line bundles over T.

$$0 \to B\mathbb{G}_m \to \mathcal{P}ic_{X/S} \to \mathrm{Pic}_{X/S} \to 0$$

Now, let us consider the forgetful morphism of stacks $\pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$. The morphism π is strongly representable since, given a morphism Y with a map $h : Y \to \overline{\mathcal{M}}_g$, the fiber product $Y \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$ is isomorphic to the image of Id_Y under h, which is a family of stable curves of genus g, say $C \to Y$ (see Remark 1.3.3).

So, we define the category $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$ associated to π as follows. Given a scheme Y, morphisms from Y to $\overline{\mathcal{M}}_g$ correspond to families of stable curves over Y. So, the objects of $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}(Y)$ are given by pairs $(C \to Y, L)$ where $C \to Y$ is the family of stable curves of genus g associated to a map $Y \to \overline{\mathcal{M}}_g$ and L is a line bundle on $C \cong Y \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{M}}_{g,1}$. Morphisms between two such pairs are given by cartesian diagrams

$$\begin{array}{ccc} C & \xrightarrow{h} & C' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array} \tag{2.3}$$

together with an isomorphism $L \cong h^*L'$.

We will now concentrate on the following full subcategory of $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$ (and on a compactification of it).

Definition 2.3.1. *Let $\mathcal{P}_{d,g}$ (respectively $\overline{\mathcal{P}}_{d,g}$) be the category whose objects are pairs $(f : C \to Y, L)$ where f is a family of stable (respectively quasistable) curves of genus g and L a balanced line bundle of relative degree d over Y. Morphisms between two such pairs are defined as in $\mathcal{P}ic_{\overline{\mathcal{M}}_{g,1}/\overline{\mathcal{M}}_g}$.*

The aim of the present section is to show that both $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are algebraic (Artin) stacks. We will do it directly by showing that they are isomorphic to the quotient stacks we are about to define.

Recall from the section before that $GL(r+1)$ acts on H_d, the locus of GIT-semistable points in $\mathrm{Hilb}_{\mathbb{P}^r}^{dt-g+1}$, with $r = d - g$, by projecting onto

$PGL(r+1)$. Consider also the open subset of H_d parametrizing Hilbert points of stable curves and denote it by H_d^{st}. It is easy to see that H_d^{st} is a $GL(r+1)$-equivariant subset of H_d.

So, we can consider the quotient stacks $[H_d^{st}/GL(r+1)]$ and $[H_d/GL(r+1)]$. Given a scheme S, $[H_d^{st}/GL(r+1)](S)$ (respectively $[H_d/GL(r+1)](S)$) consists of $GL(r+1)$-principal bundles $\phi : E \to S$ with a $GL(r+1)$-equivariant morphism $\psi : E \to H_d$ (respectively $\psi : E \to H_d^{st}$). Morphisms are given by pullback diagrams which are compatible with the morphism to H_d (resp. H_d^{st}).

Theorem 2.3.2. *Let $d >> 0$ and $g \geq 2$. Then, $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are isomorphic, respectively, to the quotient stacks $[H_d^{st}/GL(r+1)]$ and $[H_d/GL(r+1)]$.*

Proof. Since the proof is the same for both cases, we will consider only the case of $[H_d/GL(r+1)]$.

We must show that, for every scheme $S \in \mathrm{SCH}_k$, the groupoids $\overline{\mathcal{P}}_{d,g}(S)$ and $[H_d/GL(r+1)](S)$ are equivalent.

Let $(f : X \to S, L)$ be a pair consisting of a family f of quasistable curves and a balanced line bundle L of relative degree d on X. We must produce a principal $GL(r+1)$-bundle E on S and a $GL(r+1)$-equivariant morphism $\psi : E \to H_d$. Since we can take d very large with respect to g (see Remark 2.1.3 (E)), we may assume that $f_*(L)$ is locally free of rank $r+1 = d-g+1$. Then, the frame bundle of $f_*(L)$ is a principal $GL(r+1)$-bundle: call it E. Now, to find the $GL(r+1)$-equivariant morphism to H_d, consider the family $X_E := X \times_S E$ polarized by L_E, the pullback of L to X_E. X_E is a family of quasistable curves of genus g and L_E is balanced and relatively very ample. By definition of frame bundle, $f_{E*}(L_E)$ is isomorphic to $\mathbb{C}^{(r+1)} \times E$, so that L_E gives an embedding over E of X_E in $\mathbb{P}^r \times E$. By the universal property of the Hilbert scheme H, this family determines a map $\psi : E \to H_d$. It follows immediately that ψ is a $GL(r+1)$-equivariant map.

Let us check that isomorphisms in $\overline{\mathcal{P}}_{d,g}(S)$ lead canonically to isomorphisms in $[H_d/GL(r+1)](S)$.

An isomorphism between two pairs $(f : X \to S, L)$ and $(f' : X' \to S, L')$ consists of an isomorphism $h : X \to X'$ over S and an isomorphism of line bundles $L \cong h^*L'$.

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & S & \end{array}$$

These determine an unique isomorphism between $f_*(L)$ and $f'_*(L')$ as follows
$$f_*(L) \cong f_*(h^*L') \cong f'_*(h_*(h^*L'))) \cong f'_*(L').$$

As taking the frame bundle gives an equivalence between the category of vector bundles of rank $r+1$ over S and the category of principal $GL(r+1)$-

bundles over S, the isomorphism $f_*(L) \cong f'_*(L')$ leads to an unique isomorphism between their frame bundles, call them E and E' respectively. This isomorphism must be compatible with the $GL(r+1)$-equivariant morphisms $\psi : E \to H_d$ and $\psi' : E' \to H_d$ because they are determined by the induced curves X_E and $X'_{E'}$ embedded in $\mathbb{P}^r$ by L_E and $L'_{E'}$.

Conversely, given a section $(\phi : E \to S, \psi : E \to H_d)$ of $[H_d/GL(r+1)]$ over S, let us construct a family of quasistable curves of genus g over S and a balanced line bundle of relative degree d on it.

Let C_d be the restriction to H_d of the universal family on $\mathrm{Hilb}_{\mathbb{P}^r}^{dt-g+1}$. The pullback of C_d by ψ gives a family C_E on E of quasistable curves of genus g and a balanced line bundle L_E on C_E which embeds C_E as a family of curves in $\mathbb{P}^r$. As ψ is $GL(r+1)$-invariant and ϕ is a $GL(r+1)$-bundle, the family C_E descends to a family C_S over S, where $C_S = C_E/GL(r+1)$. In fact, since C_E is flat over E and E is faithfully flat over S, C_S is flat over S too (see [EGA4], Proposition 2.5.1).

Now, since the action of $GL(r+1)$ on C_d is naturally linearized (see [C94], 1.4), also the action of $GL(r+1)$ on E can be linearized to an action on L_E, yielding descent data for L_E ([SGA1], Proposition 7.8). Moreover, L_E is relatively (very) ample so, using the fact that ϕ is a principal $GL(r+1)$-bundle, we conclude that L_E descends to a relatively very ample balanced line bundle on C_S, L_S (see proof of Proposition 7.1 in [GIT]).

It is straightforward to check that an isomorphism on $[H_d/GL(r+1)](S)$ leads to an unique isomorphism in $\overline{\mathcal{P}}_{d,g}(S)$. □

Remark 2.3.3. For a different proof that a section $(\phi : E \to S, \psi : E \to H_d)$ of $[H_d/GL(r+1)]$ over S leads to a family of quasistable curves of genus g over S and a balanced line bundle of relative degree d on it see the proof of Proposition 3.2.6 below.

We will call $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$, respectively, *balanced Picard stack* and *compactified balanced Picard stack.*

Remark 2.3.4. Since $\mathbb{G}_m$ is always included in the stabilizers at every point of the action of G both in H_d and in H_d^{st}, the quotient stacks above are never Deligne-Mumford. However, they are, of course Artin stacks with a smooth presentation given by the schemes H_d^{st} and H_d, respectively.

Notice also that, since the scheme H_d is nonsingular, irreducible and closed (see Lemmas 2.2 and 6.2 in [C94]), the algebraic stack $\overline{\mathcal{P}}_{d,g}$ is a smooth compactification of $\mathcal{P}_{d,g}$.

Moreover, combining the statement of Theorem 2.3.2 with the Remark 2.1.3 (E) above, we conclude that, for $g \geq 2$, $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$ are smooth and irreducible algebraic stacks for every $d \in \mathbb{Z}$.

Let

$$d\mathcal{P}_{d,g}$$

be the category over SCH_k whose sections over a scheme S, $d\mathcal{P}_{d,g}(S)$, consists of pairs $(f : X \to S, L)$, where f is a family of d-general quasistable curves of genus g and L is an S-flat balanced line bundle on X of relative degree d. Arrows between two such pairs are given by cartesian diagrams like in (2.3).

Using the same proof of Proposition 2.3.2 we conclude that $d\mathcal{P}_{d,g}$ is isomorphic to the quotient stack $[U_d/G]$.

2.4 Rigidified Balanced Picard stacks

In what follows we will relate $\mathcal{P}_{d,g}$ and $\overline{\mathcal{P}}_{d,g}$, respectively, with $[H_d^{st}/G]$ and $[H_d/G]$, where G denotes $PGL(r+1)$, using the notion of rigidification of a stack along a group scheme defined by Abramovich, Vistoli and Corti in [ACV01], 5.1 (recall that $H_d^{st} \subset H_d$ parametrizes embedded stable curves).

Note that each object $(f : X \to S, L)$ in $\overline{\mathcal{P}}_{d,g}$ have automorphisms given by scalar multiplication by an element of $\Gamma(X, \mathbb{G}_m)$ along the fiber of L. Since these automorphisms fix X, there is no hope that our stack $\overline{\mathcal{P}}_{d,g}$ can be representable over $\overline{\mathcal{M}}_g$ (see [AV02], 4.4.3). The rigidification procedure removes those automorphisms.

More precisely, the set up of rigidification consists of:

- a stack $\mathcal{G}$ over a base scheme $\mathbb{S}$;
- a finitely presented group scheme G over $\mathbb{S}$;
- for any object ξ of $\mathcal{G}$ over an $\mathbb{S}$-scheme S, an embedding

  $$i_\xi : G(S) \to \mathrm{Aut}_S(\xi)$$

  compatible with pullbacks.

Then the statement (Theorem 5.1.5 in [ACV01]) is that there exists a stack $\mathcal{G} /\!\!/ G$ and a morphism of stacks $\mathcal{G} \to \mathcal{G} /\!\!/ G$ over $\mathbb{S}$ satisfying the following conditions:

- For any object $\xi \in \mathcal{G}(S)$ with image $\eta \in \mathcal{G} /\!\!/ G(S)$, the set $G(S)$ lies in the kernel of $\mathrm{Aut}_S(\xi) \to \mathrm{Aut}_S(\eta)$;
- The morphism $\mathcal{G} \to \mathcal{G} /\!\!/ G$ above is universal for morphisms of stacks $\mathcal{G} \to \mathcal{F}$ satisfying condition (1) above;
- If S is the spectrum of an algebraically closed field, then in (1) above we have that $Aut_S(\eta) = \mathrm{Aut}_S(\xi)/G(S)$;
- A moduli space for $\mathcal{G}$ is also a moduli space for $\mathcal{G} /\!\!/ G$.

$\mathcal{G} /\!\!/ G$ is the *rigidification* of $\mathcal{G}$ along G.

By taking $\mathbb{S} = \operatorname{Spec} k$, $\mathcal{G} = \overline{\mathcal{P}}_{d,g}$ and $G = \mathbb{G}_m$ we see that our situation fits up in the setting above. It is easy to see that, since $GL(r+1)$ acts on H_d by projection onto $PGL(r+1)$, $[H_d/PGL(r+1)]$ is the rigidification of $[H_d/GL(r+1)] \cong \overline{\mathcal{P}}_{d,g}$ along $\mathbb{G}_m$: denote it by $\overline{\mathcal{P}}_{d,g} /\!\!/ \mathbb{G}_m$. Naturally, the same holds for $[H_d^{st}/PGL(r+1)]$ and $\mathcal{P}_{d,g} /\!\!/ \mathbb{G}_m$. The following is now immediate.

Proposition 2.4.1. *The quotient stacks $[H_d^{st}/G]$ and $[H_d/G]$ are isomorphic, respectively, to $\mathcal{P}_{d,g} /\!\!/ \mathbb{G}_m$ and to $\overline{\mathcal{P}}_{d,g} /\!\!/ \mathbb{G}_m$.*

Remark 2.4.2. The previous proposition holds, of course, also for the rigidification along $\mathbb{G}_m$ of $d\mathcal{P}_{d,g}$, $d\mathcal{P}_{d,g} /\!\!/ \mathbb{G}_m$, and $[U_d/G]$, which we have studied in section 2.2 (see the definition of $d\mathcal{P}_{d,g}$ in the end of section 2.3).

Recall from the beginning of section 2.3 that, if $(d-g+1, 2g-2) = 1$, $[H_d/G]$ has a modular description as the rigidification for the action of $B\mathbb{G}_m$ in a certain category.

In order to remove from $\overline{\mathcal{P}}_{d,g}$ the automorphisms given by the action of $B\mathbb{G}_m$, we first consider the auxiliar category $\overline{\mathcal{A}}_{d,g}$, whose objects are the same of $\overline{\mathcal{P}}_{d,g}$ but where morphisms between pairs $(C \to Y, L)$ and $(C' \to Y', L')$ are given by equivalence classes of morphisms in $\overline{\mathcal{P}}_{d,g}$ by the following relation. Given a cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{h} & C' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y' \end{array} \tag{2.4}$$

and isomorphisms $\phi : L \to h^*L'$ and $\psi : L \to h^*L'$, we say that ϕ is equivalent to ψ if there exists $\alpha \in \mathbb{G}_m$ such that $\underline{\alpha} \circ \psi = \phi$, where by $\underline{\alpha}$ we mean the morphism induced by α in L' (fiberwise multiplication by α).

There is an obvious morphism of $\overline{\mathcal{P}}_{d,g} \to \overline{\mathcal{A}}_{d,g}$ satisfying property (1) above and universal for morphisms of $\overline{\mathcal{P}}_{d,g}$ in categories satisfying it. However, it turns out that $\overline{\mathcal{A}}_{d,g}$ is not a stack. In fact, it is not even a prestack since, given an étale cover $\{\coprod_i Y_i \to Y\}$ of Y, the natural morphism $\overline{\mathcal{A}}_{d,g}(Y)$ to $\overline{\mathcal{A}}_{d,g}(\coprod_i Y_i \to Y)$, the category of effective descent data for this covering, is not fully faithful but just faithful.

Let us now consider the category $\overline{\mathcal{C}}_{d,g}$, with the following modular description. A section of $\overline{\mathcal{C}}_{d,g}$ over a scheme S is given by a pair $(f : X \to S, L)$, where f is a family of quasistable curves of genus g and L is a balanced line bundle on X of relative degree d. Arrows between two such pairs are given by cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

and equivalence classes of isomorphisms $L \cong h^*L' \otimes f^*M$, for some $M \in \text{Pic } S$, for the following relation. Isomorphisms $\phi : L \to h^*L' \otimes f^*M$ and $\psi : L \to h^*L' \otimes f^*N$ are equivalent if there exists an isomorphism $g : N \to M$ of line bundles on S such that the following diagram commutes.

$$\begin{array}{ccc} L & \xrightarrow{\phi} & h^*L' \otimes f^*M \\ & \searrow^{\psi} & \uparrow{\scriptstyle id \otimes f^*g} \\ & & h^*L' \otimes f^*N \end{array}$$

Straightforward computations show that $\overline{\mathcal{C}}_{d,g}$ is a prestack. Moreover, given the étale cover $(\coprod_i Y_i \to Y)$ of Y, we get that $\overline{\mathcal{A}}_{d,g}(\coprod_i Y_i \to Y)$ is isomorphic to $\overline{\mathcal{C}}_{d,g}(\coprod_i Y_i \to Y)$.

So, we conclude that the stackification of $\overline{\mathcal{C}}_{d,g}$ is the rigidification of $\overline{\mathcal{P}}_{d,g}$ by the action of $B\mathbb{G}_m$.

Proposition 2.4.3. *The stack $[H_d/G]$ (respectively $[H_d^{st}/G]$) is the stackification of the prestack whose sections over a scheme S are given by pairs $(f : X \to S, L)$, where f is a family of quasistable (respectively stable) curves of genus g and L is a balanced line bundle on X of relative degree d. Arrows between two such pairs are given by cartesian diagrams*

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ S & \longrightarrow & S' \end{array}$$

and an isomorphism $L \cong h^*L' \otimes f^*M$, for some $M \in \text{Pic } S$.

Remark 2.4.4. Let $d >> 0$. Then, as in the case $(d - g + 1, 2g - 2) = 1$, there is a canonical map from $[H_d/G]$ and $[H_d^{st}/G]$ to $\overline{P}_{d,g}$ and $P_{d,g}$, the GIT-quotients of H_d and H_d^{st}, respectively, by the action of $PGL(r + 1)$. If $(d - g + 1, 2g - 2) \neq 1$, these quotients are not geometric, which implies that these maps are universally closed but not separated. So, $\overline{P}_{d,g}$ and $P_{d,g}$ are not coarse moduli spaces for those stacks since the associated maps from the stacks onto them are not proper. However, at least if the base field has characteristic 0, we have that the GIT-quotients are *good moduli spaces* in the sense of Alper (see [A08]).

2.4.1 Gerbes

The content of this subsection is probably foreseeable to the experts.

Definition 2.4.5. *Let X be an S-space. A **gerbe** over X is an S-stack $\mathcal{G}$ endowed with a 1-morphism of S-stacks $A : \mathcal{G} \to X$, called the structural morphism, such that:*

1. *A is an epimorphism;*

2. *The diagonal* $\Delta : \mathcal{G} \to \mathcal{G} \times_X \mathcal{G}$ *is an epimorphism.*

Remark 2.4.6. 1. To say that A is an epimorphism means that given U in S and $x \in X(U)$ there exists U' in S, $x' \in \mathcal{G}(U')$ and $\eta : U' \to U$ surjective and flat such that $f(x') \cong \eta^*(x)$.

2. To say that Δ is an epimorphism means that given U in S and $x, x' \in X(U)$ such that $A(x) \cong A(x')$, $\exists U'$ in S and $\eta : U' \to U$ surjective and flat such that $\eta^*(x) \cong \eta^*(x')$.

Let H be a scheme endowed with a left action of $GL(m)$. Then we can form the quotient stack $[H/GL(m)]$. Recall that the objects of $[H/GL(m)]$ are principal homogeneous $GL(m)$-bundles with a $GL(m)$-equivariant morphism to H and the morphisms are pullback diagrams which are compatible with the morphism to H.

Suppose this action has the property of being constant along $\mathbb{G}_m$, meaning that, given $\alpha \in \mathbb{G}_m$, $\alpha h = h \ \forall h \in H$. This way we have naturally an induced action of $PGL(m)$ on H and, again, we can form the quotient stack $[H/PGL(m)]$. Of course, even if set theoretically, both quotients $H/GL(m)$ and $H/PGL(m)$ are the same, the quotient stacks are different. In fact, we have the following exact sequence of quotient stacks:

$$0 \to [*/\mathbb{G}_m] \to [H/GL(m)] \to [H/PGL(m)] \to 0 \tag{2.5}$$

where the map $[H/GL(m)] \to [H/PGL(m)]$, call it p, associates to an element $(E \to S, \phi : E \to H)$ in $[H/GL(m)]$ the $PGL(m)$-bundle $E/\mathbb{G}_m \to S$ with the $PGL(m)$-equivariant map $\phi^p : E/\mathbb{G}_m \to H$, which is well defined as ϕ is $GL(m)$-equivariant and the action of $GL(m)$ on H is $\mathbb{G}_m$-invariant.

Note that $[*/\mathbb{G}_m]$ is isomorphic to the classifying stack $B\mathbb{G}_m$.

The following lemma is an immediate consequence of a more general well-known result. We include it here by lack of a reference.

Lemma 2.4.7. *Under the above hypotheses, the quotient stack* $[H/GL(m)]$ *is a gerbe over* $[H/PGL(m)]$.

Proof. Let $(E \to S, \phi : E \to H)$ be an element in $[H/PGL(m)]$. As $E \to S$ is a $PGL(m)$-bundle, $\exists \eta : S' \to S$ surjective and flat such that $\eta^*(E)$ is isomorphic to the trivial $PGL(m)$-bundle $PGL(m) \times S'$. So, the image under p of the trivial $GL(m)$-bundle $GL(m) \times S' \to S'$ is, of course, isomorphic to $\eta^*(E) \to S'$. So, p is an epimorphism.

Now, suppose we have $(F \to T, \phi : F \to H)$ and $(F' \to T, \phi' : F' \to H)$ in $[H/GL(m)]$ which image under p in $[H/PGL(m)]$ are isomorphic. As both F and F' are $GL(m)$-bundles over T, we can find $\eta : T' \to T$, surjective and flat, such that both $\eta^*(F)$ and $\eta^*(F')$ are isomorphic to the trivial

$GL(m)$-bundle over T', $GL(m) \times T'$, with equivariant maps to H given by composition with ϕ and ϕ', respectively. To conclude, we must see that these maps are the same. But this follows from the fact that the image of ϕ and ϕ' image under p are isomorphic and the fact that they must be constant along $\mathbb{G}_m$. □

Remark 2.4.8. Once we have the exact sequence (2.5) above, we will say that $[H/GL(m)]$ is a $\mathbb{G}_m$-gerbe over $[H/PGL(m)]$.

The folowing is now immediate.

Corollary 2.4.9. *The compactified balanced Picard stack $\overline{\mathcal{P}}_{d,g}$, defined in section 2.3, is a $\mathbb{G}_m$-gerbe over $[H_d/G]$. Analogously, $\mathcal{P}_{d,g}$ is a $\mathbb{G}_m$-gerbe over $[H_d^{st}/G]$.*

2.4.2 Rigidified balanced Picard stacks and Néron models

Let us now consider the following question: does our balanced Picard stack $[H_d^{st}]$ parametrizes Néron models of families of stable curves for every d as $[U_d^{st}/G]$ does for families of d-general curves (see 2.2.1)?

Given a family of stable curves $f : X \to B = \operatorname{Spec} R$, we will denote by Q_f^d the base-change of the map $[H_d^{st}] \to \overline{\mathcal{M}}_g$ by the natural map $B \to \overline{\mathcal{M}}_g$: $B \times_{\overline{\mathcal{M}}_g} [H_d^{st}]$. Is Q_f^d isomorphic to $N(\operatorname{Pic}^d X_K)$?

The problem here is that the map $[H_d^{st}] \to \overline{\mathcal{M}}_g$ is not representable in general. In fact, if it were, $[H_d^{st}]$ would be a Deligne-Mumford stack. Indeed, it is easy to see that a stack with a representable map to a Deligne-Mumford stack is necessarily Deligne-Mumford. As we already mentioned, $[H_d^{st}]$ is not Deligne-Mumford in general since it is the quotient stack associated to a non-geometric GIT-quotient.

As a consequence of this, if the closed fiber of f is not d-general, then Q_f^d is not even equivalent to an algebraic space. In fact, from a common criterion for representability (see for example [AV02], 4.4.3 or the proof of Proposition 2.2.2), we know that Q_f^d would be equivalent to an algebraic space if and only if the automorphism group of every section of $[H_d^{st}]$ over k with image in $\overline{\mathcal{M}}_g$ isomorphic to $X_k \to k$ injects into the automorphism group of X_k. Since such a section corresponds to a map onto its orbit in H_d by the action of $PGL(r+1)$, the automorphism group of such a section is isomorphic to the stabilizer of that orbit. So, as X_k is not d-general, the stabilizer of its associated orbit in H_d is not finite, which implies that it cannot have an injective morphism to the automorphism group of X_k, which is, of course, finite.

The following example will clarify what we have just said.

Example 2.4.10. Let $d = g-1$ and $f : X \to B = \operatorname{Spec} R$ be a family of stable curves such that X is regular and X_k is a reducible curve consisting of two smooth components C_1 and C_2 of genus g_1 and g_2 respectively, meeting in one point (of course, $g = g_1 + g_2$). Then, the fiber over k of Q_f^d is a stack with a presentation given by a subscheme of the Hilbert scheme H_d, consisting of two connected components of dimension $r(r+2)+g$ (see [C94], Example 7.2). These correspond to projective realizations of X_k on $\mathbb{P}^r$ given by line bundles with the two possible balanced multidegrees on X_k: $(g_1, g_2 - 1)$ and $(g_1 - 1, g_2)$. By Proposition 5.1 of [C94], given a point h in one of these components, there is a point in $\overline{O_{PGL(r+1)}(h)}$ representing the quasistable curve with stable model X_k embedded by a line bundle of multidegree $(g_1 - 1, 1, g_2 - 1)$.

So, as a stack, the fiber of Q_f^d over k is reducible but the GIT quotient of the Hilbert scheme presenting it by the action of $PGL(r+1)$ is irreducible and isomorphic to the Jacobian of X_k. As a consequence, Q_f^d can never be isomorphic to the Néron model $N(\operatorname{Pic}^d X_K)$.

This is an example of a situation where the GIT-quotient identifies two components of the Hilbert scheme while in the quotient stack these two components remain separated.

Proposition 2.4.11. *Let $f : X \to B = \operatorname{Spec} R$ be a family of stable curves with X regular. Then, using the notation above, $Q_f^d \cong N \operatorname{Pic}^d X_K)$ if and only if and only if X_k is a d-general curve.*

2.4.3 Functoriality for non d-general curves

Let $f : X \to S$ be a family of stable curves. Denote by $\overline{P}_f^d$ the fiber product of $[H_d/G]$ by the moduli map of f, $\mu_f : S \to \overline{\mathcal{M}}_g$.

Recall that, if $(d - g + 1, 2g - 2) = 1$, $\overline{P}_f^d$ is a compactification of the relative degree d Picard variety associated to f in the sense of 2.2.3 (see Remark 2.2.5).

Let now $(d - g + 1, 2g - 2) \neq 1$. Then, since $[H_d/G]$ is not representable over $\overline{\mathcal{M}}_g$, we have just observed that the same cannot be true in general.

However, we have the following result.

Proposition 2.4.12. *Notation as before. Then $\overline{P}_f^d$ has a canonical proper map onto a compactification P of the relative degree d Picard variety associated to f.*

Proof. If all fibers of f are d-general, then from Corollary 2.2.4 it follows that $\overline{P}_f^d$ is a scheme and it gives a compactification of the relative degree d Picard variety associated to f.

Suppose now that not all fibers of f are d-general. Then, $\overline{P}_f^d$ is a stack with a presentation given by the subscheme H_d^f of H_d corresponding to the closure in H_d of the locus parametrizing curves isomorphic to the fibers of f. H_d^f is naturally invariant for the action of $PGL(r+1)$ on it.

Let P be the GIT-quotient of H_d^f by $PGL(r+1)$. P gives a compactification of the relative degree d Picard variety associated to f. The proper map $H_d^f \to P$ factorizes through $H_d^f \to \overline{P}_f^d$, the presentation map. In fact, even if P is not a coarse moduli space for $\overline{P}_f^d$, there is a canonical map from $\overline{P}_f^d$ onto P, which is universal for morphisms of $\overline{P}_f^d$ into schemes (see [V89] section 2). Now, since the map $H_d^f \to P$ is proper, then $\overline{P}_f^d \to P$ must be proper as well.

$\square$

Chapter 3

Compactifying the universal Picard stack over $\mathcal{M}_{g,n}$

The stacks $\overline{\mathcal{P}}_{d,g}$ defined in Chapter 2 give an answer to our initial problem for $n = 0$ and $g \geq 2$. In fact, they are algebraic stacks with a geometrically meaningful modular description and endowed with a universally closed map $\Psi_{d,g}$ onto $\overline{\mathcal{M}}_g$ such that $\Psi_{d,g}^{-1}(\mathcal{M}_g) = \mathcal{P}ic_{d,g,0}$. We will now try to generalize this construction to curves with marked points.

Our strategy is inspired in Knudsen's construction of $\overline{\mathcal{M}}_{g,n}$ in [K83], which is done by induction in the number of marked points n. The crucial point of this construction is the definition of contraction, which yields a morphism from $\overline{\mathcal{M}}_{g,n+1}$ onto $\overline{\mathcal{M}}_{g,n}$. Using the contraction morphism, Knudsen shows that, for $n \geq 0$ and $2g - 2 + n > 0$, $\overline{\mathcal{M}}_{g,n+1}$ is isomorphic to the universal family over $\overline{\mathcal{M}}_{g,n}$. As a consequence, it follows that the contraction morphism is representable, which implies that $\overline{\mathcal{M}}_{g,n+1}$ is algebraic if $\overline{\mathcal{M}}_{g,n}$ is.

After introducing the definitions of quasistable curve with marked points and of balanced line bundle over it, we define, for all $g, n \geq 0$ such that $2g - 2 + n > 0$, $\overline{\mathcal{P}}_{d,g,n}$ to be the stack whose sections over a scheme S are families of n-pointed quasistable curves endowed with balanced line bundles of relative degree d over these families (see Definition 3.2.1 below). Also in our case, the crucial point will be to generalize the notion of contraction in this more general context of n-pointed quasistable curves endowed with balanced line bundles and prove that this yields an isomorphism between $\overline{\mathcal{P}}_{d,g,n+1}$ and the universal family over $\overline{\mathcal{P}}_{d,g,n}$.

3.1 n-pointed quasistable curves and balanced line bundles

In the present section we will introduce the notions of quasistable curve and of balanced line bundle for curves with marked points. Our definitions are generalizations of the notions of quasistable and balanced for $n = 0$ and $g \geq 2$ introduced by Gieseker and Caporaso and that we dealt with in the previous chapter (see Definition 2.1.1 and 1.2.2).

As a consequence, we also get the notions of quasistable curves and balanced line bundles for $g = 0$ and $n \geq 3$ and for $g = 1$ and $n \geq 1$. Then, for $n > 0$ and $2g - 2 + n > 1$, n-pointed quasistable curves turn out to be the ones we get by applying the stabilization morphism defined by Knudsen in [K83] (see 3.4.3 below) to $(n-1)$-pointed quasistable curves endowed with an extra section without stability conditions. Moreover, balanced line bundles on n-pointed quasistable curves correspond to balanced line bundles on the quasistable curves obtained by forgetting the points and by contracting the rational components that get *quasidestabilized* without the points (see Lemma 3.1.10).

Recall that, according to Paragraph 1.2.3, n-pointed (semi)stable curves admit chains of smooth rational curves meeting the rest of the curve in one or two points. Since these will be very important in the whole discussion, we shall introduce the following notation for them.

Definition 3.1.1. *Let g and n be non-negative integers such that $2g-2+n > 0$ and let $(X; p_1, \ldots, p_n)$ be an n-pointed semistable curve of genus g.*

- *Let T be a proper subcurve of X with $g_T = 0$ and $k_T = 1$. Then T is a* ***rational tail*** *of X either if $g > 0$ or if $g = 0$ and if T contains at most one point among $\{p_1, p_2, p_3\}$;*
- *Let B be a proper subcurve of X with $g_B = 0$ and $k_B = 2$. Then B is a* ***rational bridge*** *of X either if $g > 1$ or if $g = 0$ and B does not contain any point among $\{p_1, p_2, p_3\}$ or if $g = 1$ and B does not contain p_1.*
- *A nonsingular rational component E such that the number of points where E meets the rest of X plus the number of marked points p_i on E is exactly 2 is called a* ***destabilizing*** *component. An* ***exceptional component*** *is a destabilizing component without marked points.*

We will also say that a rational bridge (resp. a rational tail) of an n-pointed semistable curve X is **maximal** if it is not contained in any other rational bridge (resp. rational tail) of X.

Remark 3.1.2. Note that the condition that $2g - 2 + n > 0$ implies that curves of genus $g = 0$ must have at least 3 marked points and that curves

Figure 3.1: Examples of 3-pointed semistable curves which are NOT quasistable

of genus $g = 1$ curves must have at least 1 marked point, so the previous definition makes sense.

Definition 3.1.3. *An n-**pointed quasistable** curve is an n-pointed semistable curve X such that*

1. *all destabilizing components are exceptional;*
2. *exceptional components can not be contained in rational tails;*
3. *each rational bridge contains at most one exceptional component.*

*A **family of n-pointed quasistable curves** is a proper and flat morphism with n distinct sections whose geometric fibers are n-pointed quasistable curves.*

See Figure 3.1 for examples of pointed semistable curves which are not quasistable.

Note that, in virtue of the previous definition, if X has genus $g = 0$, then X is quasistable if and only if X is stable. In fact, since X is rational, either it is irreducible or all proper subcurves of X that do not contain at least two points among $\{p_1, p_2, p_3\}$ are contained in a rational tail of X, so no exceptional components are allowed.

Suppose now that X is a 1-pointed quasistable curve of genus 1. Then X can be of 3 distinguished topological types, as we can see in Figure 3.2, where the numbers near the curves indicate the geometric genus of the respective components.

For $n > 1$, all n-pointed genus 1 curves can be obtained from these by attaching rational tails and rational bridges. So, all n-pointed genus 1 curves will have at most one maximal rational bridge which is not contained in any rational tail (recall that a rational component E intersecting the rest of the curve in two points and with $p_1 \in E$ is not considered to be a rational bridge) and, in particular, at most one exceptional component. The definition of balanced line bundles on n-pointed quasistable curves of genus 1 that we propose below is inspired by these facts.

Figure 3.2: 1-pointed quasistable curves of genus 1.

To each proper subcurve Z of X, denote by t_Z the number of rational tails meeting Z.

Let us now define balanced line bundles on pointed quasistable curves.

Definition 3.1.4. *Let X be an n-pointed quasistable curve of genus g with $2g-2+n>0$ and L a line bundle on X of degree d. We say that L (or its multidegree) is* ***balanced*** *if the following conditions hold:*

- *$deg_E L = 1$ for every exceptional component E of X;*
- *the degree of L on rational bridges can be either 0 or 1;*
- *if T is a rational tail of X, then $deg_T L = -1$;*
- *if $g \neq 1$ and Z is a proper subcurve of X which is not contained in any rational tail and in any rational bridge of X, then the degree of L on Z must satisfy the following inequality*

$$|deg_Z L - \frac{d(w_Z - t_Z)}{2g-2} - t_Z| \leq \frac{k_Z - t_Z - 2b_Z^L}{2} \tag{3.1}$$

  where b_Z^L denotes the number of rational bridges where the degree of L is zero meeting Z in two points.

- *if $g = 1$ and Z is a proper subcurve of X which is not contained in any rational tail and in any rational bridge of X, then $\deg_Z L$ must satisfy the following inequality*

$$|deg_Z L - d - t_Z| \leq \frac{k_Z - t_Z}{2}. \tag{3.2}$$

Note that, if $g \geq 2$ and $n = 0$, t_Z and b_Z^L are equal to 0 for all proper subcurves Z of X, and inequality (3.1) reduces to the “Basic Inequality”introduced by Gieseker in [G82]. In fact, for $n = 0$, Definition 3.1.4 coincides with the definition 2.1.1 of balanced multidegree for quasistable.

Notice also that if $g = 0$ and Z is an irreducible component of X which is not contained in any tail of X, we have that $k_Z = t_Z$. So, for rational curves, Definition 3.1.4 can be rewritten as follows.

Definition 3.1.5. *Let L be a line bundle of degree d on an n-pointed (quasi)-stable curve X of genus 0. We say that L is balanced if the following two conditions hold.*

1. $\deg_T L = -1$ *if T is a tail of X,*
2. *if Z is a proper subcurve of X which is not contained in any tail of X,* $\deg_Z L = d + k_Z$.

Remark 3.1.6. >From Lemma 3.1.9 below and the previous Definition it follows that if X is an n-pointed (quasi)stable curve of genus 0 then, for each degree $d \in \mathbb{Z}$, there is exactly one balanced multidegree summing up to d.

It follows also that the multidegree of a balanced line bundle on an n-pointed quasistable curve of genus 1 is uniquely determined except if it has rational bridges which are not contained in rational tails and no exceptional component.

Remark 3.1.7. In [C1] there is a general notion of Balanced line bundles for Binary curves, i. e., curves consisting of two nonsingular rational curves meeting in an arbitrary number of points. In particular, if the curves meet in two points, then the genus of the curve is equal to 1. We point out that our definition of Balanced line bundles for n-pointed quasistable curves of genus 1 is different from that one since ours takes into account the marked points of the curve and works just for curves with at least one marking.

Using the notation of 3.1.4, denote by

$$m_Z(d, L) := \frac{dw_Z + (3g - 3 - d)t_Z}{2g - 2} + b_Z^L - \frac{k_Z}{2}$$

and by

$$M_Z(d, L) := \frac{dw_Z + (g - 1 - d)t_Z}{2g - 2} - b_Z^L + \frac{k_Z}{2}.$$

Then, inequality (3.1) can be rewritten in the following way

$$m_Z(d, L) \leq \deg_Z L \leq M_Z(d, L)$$

Example 3.1.8. In figure 3.3 we can see an example of a 12-pointed quasistable curve X consisting of two components of genus bigger than 0, C

Figure 3.3: 12-pointed quasistable curve with assigned balanced multidegree in rational tails and rational bridges.

and D, intersecting each other in 1 point and other rational components belonging to rational tails or rational bridges. The numbers on the figure indicate the multidegrees of a balanced line bundle on rational tails and on rational bridges. They are uniquely determined with the exception of the rational bridge where there is no exceptional component. In this case, other possibilities would be either $(1, 0)$ or $(0, 1)$.

Consider $d = 0$. Then, from Inequality (3.1), we see that the only possibility for a balanced line bundle of degree 0 on X completing the multidegree of the figure is to assign to C degree 0 and to D degree -1. In fact, inequality (3.1) states that the degree of L on C can be either 0, 1 or 2, while on D it must be 0 or 1, so $(0, 1)$ is the only possible choice in order to the total degree sum up to 0. If, insted, we had chosen the degree in the rational bridge with no exceptional component to be 1, then L should have degree -1 on C. However, in this case inequality (3.1) would change to C: it would give $-1, 0, 1, 2, 3$ as possible degrees.

Consider now the case $d = g - 1$. Then, since $g = g_C + g_D + 2$, we can write $g - 1$ as $g_C + g_D + 1$. However, since the multidegrees assigned in the figure to rational tails and rational bridges sum up to -1, the sum of the degree of L on C with the degree of L on D must be $g_C + g_D + 2$. Inequality (3.1) asserts that the degree of L on C must be in between $g_C + 1$ and $g_C + 4$ while on D it must be g_D or $g_D + 1$. So, we have two possibilities: $(g_C + 2, g_D)$ and $(g_D + 1, g_D + 1)$. If, instead, we had chosen the degree on the rational bridge to be 1, then the sum of the degree of L on C with the degree of L on D should be $g_C + g_D + 1$. However, inequality (3.1) would change to C, giving $g_C, \dots, g_C + 5$ as the possible degrees on C. So, also in this case we would have two possibilities for the degrees of a balanced line bundle of total degree $g - 1$ on C and D: $(g_C, g_D + 1)$ and $(g_C + 1, g_D)$.

3.1.1 First properties

Lemma 3.1.9. *Let X be an n-pointed quasistable curve and suppose X admits a balanced line bundle L on X of degree d, for some $d \in \mathbb{Z}$. Then, if Z is a proper subcurve of X that is contained in a rational tail, then* $\deg_Z L = k_Z - 2$ *and if Z is contained in a rational bridge, then* $\deg_Z L$ *is either equal to $k_Z - 2$ or $k_Z - 1$.*

In particular, the multidegree of L on rational tails is unique and is independent of d.

Proof. Let us begin by showing that the multidegree of L on rational tails is uniquely determined. So, suppose T is a rational tail of X. If T is irreducible, then the multidegree of L on T is just the degree of L on T, which is necessarily -1.

Now, suppose T is reducible. Then there is exactly one irreducible component E of T meeting the rest of the curve (in exactly one point). We will call E the foot of the rational tail. E is a smooth rational curve meeting the rest of T in $k_E - 1$ points: denote by $E_1, \dots, E_{k_E-1}$ the irreducible components of T meeting E. Then, each E_i, $i = 1, \dots, k_E - 1$ is the foot of a rational tail contained in T. In fact, each one of these, if not irreducible, is attached to another rational chain that cannot intersect the rest of the curve since in that case T would contain cycles (which would force $p_a(T)$ to be bigger than 0). So, T is the union of E with $k_E - 1$ rational tails meeting E, and

$$-1 = \deg_T L = \deg_E L + \deg_{\overline{T \setminus E}} L = \deg{}_E L - (k_E - 1)$$

which implies that

$$\deg_E L = k_E - 2.$$

Note that we don't have to check if inequality (3.1) is satisfied since it does not apply for subcurves of X contained in rational tails.

Now, iterating the same procedure, it is clear that the degree of each irreducible component of T will be determined since T is the union of E with other $k_E - 1$ rational tails with foots $E_1, \dots, E_{k_E-1}$.

Now, consider a rational bridge B. Then, B meets the rest of the curve in two points, p_1 and p_2, and these are linked by a chain of (rational) irreducible components of B, $E_1, \dots, E_{l_B}$, each one meeting the previous and the next one, for $i = 2, \dots, l_B - 1$. Moreover, each E_i can have rational tails attached. Denote by $B_1, \dots, B_{l_B}$ respectively the proper subcurves of B consisting of E_i and the rational tails attached to it, for $i = 1, \dots, l_B$. So, $B = B_1 \cup \dots \cup B_{l_B}$ is the union of l_B rational bridges of length 1.

By definition, the degree of L in B can be either 0 or 1, and the same holds for each B_i, $i = 1, \dots, l_B$. If $\deg_{B_i} L = 0$, then, in order to the multidegree of L on B_i sum up to 0, $\deg_{E_i} L$ must be equal to the number of rational tails attached to it: $t_{E_i} = k_{E_i} - 2$. If, instead, the degree of L on

B_i is equal to one, then $\deg_{B_i} L$ must be equal to $t_{E_i} + 1 = k_{E_i} - 1$. Note that inequality (3.1) gives that

$$t_{E_i} - 1 \leq \deg_{E_i} L \leq t_{E_i} + 1$$

for $i = 1, \ldots, l_Z$, so either t_{E_i} or $t_{E_i} + 1$ are allowed. The multidegree of L on the rest of B_i is fixed since $\overline{B_i \setminus E_i}$ consists of rational tails (that, of course, cannot intersect each other).

Now, if B contains one exceptional component E among the E_i's, say E_j, the degree of B must be necessarily 1 (note that on each rational tail we can have at most one exceptional component by definition of pointed quasistable curve). In this case, we must have that $k_{E_j} = 2$, which implies that E_j has no rational tails attached, and the degree of L on it must be 1. Moreover, the degree of L on the other rational subcurves B_i, for $i \neq j$, must be 0.

If, instead, B does not contain any exceptional component, then we can choose the degree of L in B to be either 1 or 0. If we choose it to be 0, then the degree of L on each B_i must be 0, for $i = 1, \ldots, l_B$. If we choose it to be one, we can freely choose one of the B_i's where the degree of L is 1 and in all the others the degree of L must be 0. □

Lemma 3.1.10. *Let X be an n-pointed quasistable curve with assigned multidegree on rational bridges. Let X' be the quasistable curve obtained by contracting all rational tails and rational bridges with assigned degree zero and by forgetting the points. Then, for each degree d, the set of balanced multidegrees on X' summing up to d and the set of balanced multidegrees on X summing up to d with the given assigned multidegree on rational bridges are in bijective correspondence.*

Proof. Let L' be a balanced line bundle on X' with degree d. This means that, given a proper subcurve Z' of X', inequality 2.1 holds for Z', that is,

$$-\frac{k_{Z'}}{2} + \frac{dw_{Z'}}{2g-2} \leq \deg_{Z'} L' \leq \frac{dw_{Z'}}{2g-2} + \frac{k_{Z'}}{2} \tag{3.3}$$

and that the degree of L' on exceptional components is equal to 1.

Let C_i be an irreducible component of $X = C_1 \cup \cdots \cup C_\gamma$ such that C_i is not contained in any rational tail and in any rational bridge. Define the multidegree $\underline{d} = (d_1, \ldots, d_\gamma)$ on X by declaring that $d_i = \deg_{C_i'} L' + t_{C_i}$ where C_i' is the image of C_i on X'. Then we easily see that this defines a balanced multidegree on X (note that the multidegree of L on rational bridges is fixed by hypothesis). In fact, since $k_{C_i} = k_{C_i'} + t_{C_i} + 2r^L_{C_i}$ and $g_{C_i} = g_{C_i'} - b^L_{C_i}$, we have that

$$d_i = \deg_{C_i'} L + t_{C_i} \leq \frac{dw_{C_i'}}{2g-2} + \frac{k_{C_i'}}{2} + t_{C_i} =$$

$$= \frac{d(2g_{C_i'} - 2 + k_{C_i'})}{2g-2} + \frac{k_{C_i} - t_{C_i}}{2} - b^L_{C_i} + t_{C_i} =$$

$$= \frac{d(2g_{C_i} + 2b^L_{C_i} - 2 + k_{C_i} - t_{C_i} - 2b^L_{C_i})}{2g-2} + \frac{k_{C_i}}{2} + \frac{t_{C_i}}{2} - b^L_{C_i} =$$

$$= \frac{dw_{C_i}}{2g-2} + \frac{k_{C_i}}{2} - \frac{d}{2g-2} t_{C_i} + \frac{t_{C_i}}{2} - b^L_{C_i} =$$

$$= \frac{dw_{C_i}}{2g-2} + \frac{k_{C_i}}{2} + \frac{g-1-d}{2g-2} t_{C_i} - b^L_{C_i}$$

and also that

$$d_i \geq \frac{dw_{C'_i}}{2g-2} - \frac{k_{C'_i}}{2} + t_{C_i} =$$

$$= \frac{dw_{C_i}}{2g-2} - \frac{k_{C_i}}{2} + \frac{3g-3-d}{2g-2} t_{C_i} + b^L_{C_i}$$

so inequality (3.1) holds for C_i if and only if inequality 2.1 holds for C'_i. It is easy to see that this is true more generally for any proper subcurve Z of X not contained in any rational tail and in any rational bridge. □

3.2 Balanced Picard stacks over quasistable curves with marked points

We will now generalize Definition 2.3.1 of balanced Picard stacks to curves with marked curves.

Definition 3.2.1. *For any integer d and $g, n \geq 0$ with $2g-2+n > 0$, denote by $\overline{\mathcal{P}}_{d,g,n}$ the following category fibered in groupoids over the category of schemes over k. Objects over a k-scheme S are families $(\pi : X \to S, s_i : S \to X, L)$ of n-pointed quasistable curves over S and a balanced line bundle L on X of relative degree d.*

Morphisms between two such objects are given by cartesian diagrams

$$\begin{array}{ccc} X & \xrightarrow{\beta_2} & X' \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & {\scriptstyle \pi'}\downarrow\uparrow{\scriptstyle t_i} \\ S & \xrightarrow[\beta_1]{} & S' \end{array}$$

such that $t_i \circ \beta_1 = \beta_2 \circ s_i$, $1 \leq i \leq n$, together with an isomorphism $\beta_3 : L \to \beta_2^*(L')$.

We will refer to $\overline{\mathcal{P}}_{d,g,n}$ as the degree d ***Balanced Picard stack*** *for n-pointed quasistable curves of genus g.*

Note that $\overline{\mathcal{P}}_{d,g,n}$ contains $\mathcal{P}ic_{d,g,n}$ for all $n \geq 0$.

In what follows we will prove the following statement.

Theorem 3.2.2. *The degree d Balanced Picard stack $\overline{\mathcal{P}}_{d,g,n}$ is a smooth and irreducible algebraic (Artin) stack of dimension $4g-3+n$ endowed with a universally closed map onto $\overline{\mathcal{M}}_{g,n}$.*

Recall that, for $n = 0$ and $g \geq 2$, $\overline{\mathcal{P}}_{d,g,0}$ coincides with the stack $\overline{\mathcal{P}}_{d,g}$ defined in Chapter 2, so Theorem 3.2.2 holds in this case.

The cases $g = 0$ and $g = 1$ must be treated separately: we will show that $\overline{\mathcal{P}}_{d,0,3} \cong \overline{\mathcal{M}}_{0,3} \times B\mathbb{G}_m$ and $\overline{\mathcal{P}}_{d,1,1} \cong \overline{\mathcal{M}}_{1,2} \times B\mathbb{G}_m$ (see Propositions 3.2.7 and 3.2.10, respectively), so Theorem 3.2.2 clearly holds in this case.

Then, following Knudsen's construction of $\overline{\mathcal{M}}_{g,n}$ (see [K83]), we will show that Theorem 3.2.2 holds for all $d \in \mathbb{Z}$ and $g, n \geq 0$ such that $2g - 2 + n > 0$ using the following induction argument. We will prove that, for $n > 0$ with $2g - 2 + n > 1$, $\overline{\mathcal{P}}_{d,g,n+1}$ is isomorphic to the universal family over $\overline{\mathcal{P}}_{d,g,n}$.

By universal family over $\overline{\mathcal{P}}_{d,g,n}$ we mean an algebraic stack $\mathcal{Z}_{d,g,n}$ with a map onto $\overline{\mathcal{P}}_{d,g,n}$ admitting n-sections $\sigma^i_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \mathcal{Z}_{d,g,n}, i = 1, \ldots, n$ and endowed with an (universal) invertible sheaf $\mathcal{L}$ such that, given a family $f : C \to S, s_i : S \to C, i = 1, \ldots, n$ of n-pointed quasistable curves and a balanced line bundle L over C of relative degree d, the following diagram, commuting both in the upward and downward directions,

$$\begin{array}{ccc} C & \xrightarrow{\pi_2} & \mathcal{Z}_{d,g,n} \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle f} & & \downarrow\uparrow{\scriptstyle \sigma^i_{d,g,n}} \\ S & \xrightarrow[\mu_f]{} & \overline{\mathcal{P}}_{d,g,n} \end{array} \tag{3.4}$$

is cartesian and induces an isomorphism between $\pi_2^*(\mathcal{L})$ and L.

Let $\mathcal{Z}_{d,g,n}$ be the category whose sections over a scheme Y are families of n-pointed quasistable curves $X \to Y, t_i : Y \to X, i = 1, \ldots, n$ endowed with a balanced line bundle M of relative degree d and with an extra section $\Delta : Y \to X$. Morphisms in $\mathcal{Z}_{d,g,n}$ are like morphisms in $\overline{\mathcal{P}}_{d,g,n}$ compatible with the extra section. $\mathcal{Z}_{d,g,n}$ is an algebraic stack endowed with a forgetful morphism onto $\overline{\mathcal{P}}_{d,g,n}$ admitting n sections given by the diagonals $\delta_{1,n+1}, \ldots, \delta_{n,n+1}$.

It is easy to see that, given a family of n-pointed quasistable curves $f : C \to S, s_i : S \to C, i = 1, \ldots, n$ and a balanced line bundle L over C of relative degree d, diagram (3.4) is cartesian, where π_2 is defined by associating to the identity morphism $1_C : C \to C$ the fiber product of $f : C \to S$ with itself

$$\begin{array}{ccc} C \times_S C & \xrightarrow{p_2} & C \\ {\scriptstyle f^* s_i}\uparrow\downarrow{\scriptstyle p_1} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s_i} \\ C & \xrightarrow[f]{} & S \end{array} \tag{3.5}$$

endowed with an extra section $\Delta : C \to C \times_S C$ given by the diagonal and with the relative degree d line bundle $p_2^*(L)$. Given another object $h : Y \to C$ of C, $\pi_2(h)$ is defined to be the fiber product of h and p_1 defined in (3.5), naturally endowed with the $n+1$ pullback sections and with the pullback of $p_2^*(L)$.

The universal sheaf over $\mathcal{Z}_{d,g,n}$, $\mathcal{L}$, is defined by associating to each section $(X \to Y, t_i, M, \Delta)$ of $\mathcal{Z}_{d,g,n}$ over Y, the line bundle $\Delta^*(M)$ over Y. It is easy to see that this defines an invertible sheaf on $\mathcal{Z}_{d,g,n}$.

Now we easily check that $\mathcal{L}$ is the universal sheaf over $\mathcal{Z}_{d,g,n}$. Indeed, given an object $h : Y \to C$ on C, $\pi_2^*(\mathcal{L})(h) = \mathcal{L}(\pi_2(h)) \cong h^*(L)$, so it is isomorphic to the sheaf defined by L on C, considered as a stack.

We have just proved the following.

Proposition 3.2.3. *The algebraic stack $\mathcal{Z}_{d,g,n}$ defined above endowed with the invertible sheaf $\mathcal{L}$ is the universal family over $\overline{\mathcal{P}}_{d,g,n}$ for the moduli problem of n-pointed quasistable curves with a balanced degree d line bundle.*

Remark 3.2.4. >From propositions 3.2.7 and 3.2.10 we have that for $n \geq 3$, $\mathcal{Z}_{d,0,n} \cong \overline{\mathcal{M}}_{0,n+1} \times B\mathbb{G}_m$ and that for $n \geq 1$, $\mathcal{Z}_{d,1,n} \cong \overline{\mathcal{M}}_{1,n+2} \times B\mathbb{G}_m$.

Now, suppose we can show that, for all $n \geq 0$, there is a forgetful morphism $\Psi_{d,g,n}$ from $\overline{\mathcal{P}}_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$ such that the image under $\Psi_{d,g,n}$ of an n-pointed quasistable curve X over S endowed with a balanced degree d line bundle is the stable model of X over S forgetting the line bundle. These morphisms would yield commutative diagrams as follows, for all $n > 0$ such that $2g-2+n > 1$.

$$\begin{array}{ccccc} & & \overline{\mathcal{P}}_{d,g,n} & & \\ & \swarrow^{\Phi_{d,g,n}} & & \searrow^{\Psi_{d,g,n}} & \\ \overline{\mathcal{P}}_{d,g,n-1} & & & & \overline{\mathcal{M}}_{g,n} \\ & \searrow_{\Psi_{d,g,n-1}} & & \swarrow_{\Pi_{g,n}} & \\ & & \overline{\mathcal{M}}_{g,n-1} & & \end{array} \qquad (3.6)$$

Since $\Pi_{g,n}$ and $\Phi_{d,g,n}$ are the morphisms from the universal families over $\overline{\mathcal{P}}_{d,g,n-1}$ and $\overline{\mathcal{M}}_{g,n-1}$, respectively, it follows that $\Psi_{d,g,n}$ is universally closed (or proper) if and only if $\psi_{d,g,n-1}$ is. For $g \geq 2$ and $n = 0$, it follows from Chapter 2 (see Remark 2.4.4), that $\psi_{d,g,0}$ is universally closed, so we have that $\psi_{d,g,n}$ is universally closed for all $n \geq 0$ for all $n \geq 0$ and $g \geq 2$. For $g = 0$ and $g = 1$ the result follows immediately in virtue of Propositions 3.2.7 and 3.2.10 and Remark 3.2.4.

So, Theorem 3.2.2 will follow from the following statement, that we will prove in 3.4.3 bellow.

Theorem 3.2.5. *For all $d \in \mathbb{Z}$ and $n > 0$ with $2g-2+n > 1$, $\overline{\mathcal{P}}_{d,g,n+1}$ is isomorphic to the algebraic stack $\mathcal{Z}_{d,g,n}$.*

Recall that, for $g \geq 2$ and $n = 0$, our proof of Theorem 3.2.2 consisted on showing that $\overline{\mathcal{P}}_{d,g,0} = \overline{\mathcal{P}}_{d,g}$ is isomorphic to the quotient stack $[H_d/GL(r+1)]$ (see Theorem 2.3.2 above). The action of $GL(r+1)$ in H_d naturally lifts to an action in $\mathcal{Z}_d$, where $\mathcal{Z}_d$ is the restriction to H_d of the universal family over the Hilbert scheme. Using a similar proof we can show that $\mathcal{Z}_{d,g,1}$ is isomorphic to the quotient stack $[\mathcal{Z}_d/GL(r+1)]$. Nevertheless, we will now give a proof of this fact that, in one direction, is slightly different of the proof of Theorem 2.3.2 and that could be an alternative proof of it.

Proposition 3.2.6. *Let $d >> 0$. Then the stack $\overline{\mathcal{P}}_{d,g,1}$ is isomorphic to the quotient stack $[\mathcal{Z}_d/GL(r+1)]$.*

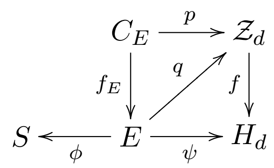
Proof. We must show that, For every scheme $S \in SCH_k$, the groupoids $\mathcal{Z}_{d,g,1}(S)$ and $[\mathcal{Z}_d/GL(r+1)](S)$ are equivalent. Let $(f : C \to S, s : S \to C, L)$ be a section of $\mathcal{Z}_{d,g,1}$ over S, i.e., a triple consisting of a family f of quasistable curves with a section s and a balanced line bundle L of relative degree d on C. Denote by G the group $GL(r+1)$. We must produce a principal $GL(r+1)$-bundle E on S and a G-equivariant morphism $q : E \to \mathcal{Z}_d$. We will proceed as in the proof of Theorem 2.3.2. Since we are considering d to be very large, we may assume that $f_*(L)$ is locally free of rank $r+1 = d-g+1$. Then, the frame bundle of $f_*L)$ is a principal $GL(r+1)$-bundle: call it E. Now, to find the G-equivariant morphism to $\mathcal{Z}_d$, consider the family $C_E := C \times_S E$ polarized by L_E, the pullback of L to C_E. C_E is a family of quasistable curves of genus g, endowed with a section s_E and L_E is balanced and relatively very ample. Moreover, the pullback of a morphism endowed with a section is naturally endowed with a section, call it s_E. By definition of frame bundle, $f_{E*}(L_E)$ is isomorphic to $\mathbb{C}^{(r+1)} \times E$, so that L_E gives an embedding over E of C_E in $\mathbb{P}^r \times E$. By the universal property of the Hilbert scheme H, this family determines a map $\psi : E \to H_d$, which is clearly G-equivariant. Furthermore, the following diagram is cartesian

$$\begin{array}{ccc} C_E & \xrightarrow{q} & \mathcal{Z}_d \\ {\scriptstyle s_E}\uparrow\downarrow{\scriptstyle f_E} & & \downarrow{\scriptstyle f} \\ C & \xrightarrow[\psi]{} & H_d \end{array} \tag{3.7}$$

Since q is naturally G-equivariant and s_E is G-equivariant by construction, qs_E is a G-equivariant morphism from E to $\mathcal{Z}_d$. This way, we got a section (E, qs_E) of $[\mathcal{Z}_d/G](S)$. It is easy to check that isomorphisms in $\mathcal{Z}_{d,g,1}(S)$ leads canonically to isomorphisms in $[\mathcal{Z}_d/G](S)$.

Conversely, given a section $(\phi : E \to S, q : E \to \mathcal{Z}_d)$ of $[\mathcal{Z}_d/G]$ over S, let us construct a family of quasistable curves of genus g over S with a section and a balanced line bundle of relative degree d on it. This part of the proof is different of the proof of Theorem 2.3.2.

The pullback of the identity morphism of $\mathcal{Z}_d$ by q gives a family C_E on E of quasistable curves of genus g and a balanced line bundle L_E on C_E which embeds C_E as a family of curves in $\mathbb{P}^r$. In fact, C_E is obtained pulling back $\mathcal{Z}_d \to H_d$ via ψ, where ψ is the composition of q with $\mathcal{Z}_d \to H_d$, which is naturally G-equivariant.



Moreover, by the universal property of the pullback, q induces a section $s_E : E \to C_E$.

As ψ is G-invariant and ϕ is a G-bundle, the family C_E descends to a family C_S over S, where $C_S = C_E/G$. We must check that both the section s_E and the balanced line bundle L_E also descend to C_S and that C_S is flat over S.

Let us see that C_S is a flat family by showing that C_E is locally G-equivariantly a product $C_W \times G$ for some W-flat family C_W for an open $W \subset S$.

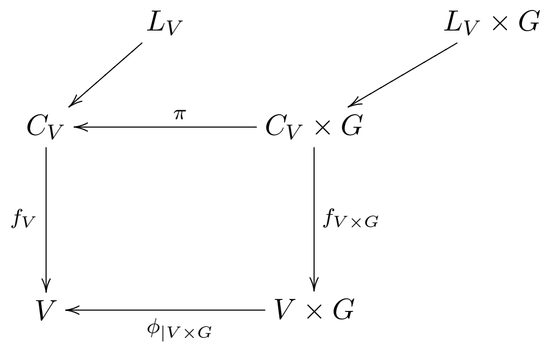
Since $G = GL(r+1)$ is special (see [SC]), the principal bundle E is trivial locally in the Zariski topology. So, let $V \subset S$ be an open subset of S such that $E_{|V} \cong V \times G$.

Let $\psi_0 : V \to H_d$ be defined as follows:

$$\psi_0(x) = \psi(x, 1_G)$$

for each $x \in V$. As ψ is G-invariant, $\psi_{|(V\times G)}(x, g) = \psi_0(x).g$, for every $x \in V$ and every $g \in G$ and, similarly, $q_{|(V\times G)}(x, g) = g.q(x, 1_G)$.

Let $f_V : C_V \to V$ be the family of quasistable curves of genus g over V induced by the morphism ψ_0 and L_V the balanced line bundle of relative degree d embedding C_V as a family of curves in $V \times \mathbb{P}^r$. Since L_V is relatively very ample, $f_{V*}(L_V)$ is locally free of rank $r+1$. Up to restricting to an open subset of V, we can assume $f_{V*}(L_V)$ is trivial. Let $f_{V\times G} : C_V \times G \to V \times G$ be the pullback family and $\pi^*(L_V) \cong L_V \times G$ the pullback of L_V to $C_V \times G$.



The frame bundle of $f_{V*}(L_V)$ is isomorphic to $V \times G$, which is isomorphic to $E_{|V}$. Furthermore, $f_{V\times G*}(\pi^*(L_V))$ is isomorphic to $\phi^*_{|V\times G}(f_{V*}(L_V))$ which is isomorphic to $V \times G \times \mathbb{C}^{r+1}$. So, $f_{V\times G*}(\pi^*(L_V))$ gives an embedding of $C_{V\times G}$ as a family of d-general quasistable curves in $V \times G \times \mathbb{P}^r$. By the universal property of the Hilbert scheme, such a family induces a G-equivariant morphism to H_d. By construction, this morphism must be equal to $\psi_{|(V\times G)}$.

We conclude that, locally, C_E is a G-equivariant product of a flat family C_V by G. In particular, we can apply Kempf's descent lemma which states that L_E descends to a line bundle over C_S if and only if, for every closed point $\xi \in E$ its stabilizer acts trivially on the fiber of L_E in ξ (see, for example, Theorem 2.3 of [DN]). From the local description of the family, we conclude that L_E descends to a line bundle L_S on C_S. Moreover, since $q = ps_E$ and by the local description of q, we get that also s_E is G-equivariant, so it descends to a section $s : S \to C_S$. So, $(C_S \to S, s : S \to C_S, L_S) \in \mathcal{Z}_{d,g,1}$.

It is straightforward to check that an isomorphism on $[\mathcal{Z}_d/G](S)$ leads to an unique isomorphism in $\mathcal{Z}_{d,g,1}(S)$. □

3.2.1 Balanced Picard stacks over genus 0 curves

Recall that the notions of n-pointed stable and quasistable curve coincide for curves of genus 0 (and $n \geq 3$) (see Remark 3.1.6 above).

In the present section we describe Balanced Picard stacks over (families of) n-pointed stable curves of genus 0. We will start by considering the case $n = 3$.

Let $(\pi : X \to S, s_i : S \to X) \in \overline{\mathcal{M}}_{0,3}$, $i = 1, \ldots, 3$. Then X is necessarily a trivial family: a stable rational curve with 3 distinguished marked points is necessarily smooth and has trivial automorphism group. Then, for any $d \in \mathbb{Z}$, $\mathcal{O}_{X/S}(d)$ is a line bundle of relative degree d over X and it is clearly balanced (since all fibers of the family are irreducible). Moreover, any other line bundle of relative degree d over X is isomorphic to it, the isomorphism being given by an element of $\mathbb{G}_m$.

So, we have proved the following result.

Proposition 3.2.7. *For any $d \in \mathbb{Z}$, $\overline{\mathcal{P}}_{d,0,3} \cong \overline{\mathcal{M}}_{0,3} \times B\mathbb{G}_m (\cong B\mathbb{G}_m)$.*

Let now $n > 3$. In view of theorem 3.2.5, consider the universal family over $\overline{\mathcal{P}}_{d,0,n-1}$, $\mathcal{Z}_{d,0,n-1}$. By applying an inductive argument based on the previous Proposition we have that $\mathcal{Z}_{d,0,n-1} \cong \overline{\mathcal{M}}_{0,n} \times B\mathbb{G}_m$. So, Theorem 3.2.2 will give the following result.

Proposition 3.2.8. *Let $d \in \mathbb{Z}$ and $n \geq 3$. Then $\overline{\mathcal{P}}_{d,0,n}$ is isomorphic to $\overline{\mathcal{M}}_{0,n} \times B\mathbb{G}_m$.*

3.2.2 Balanced Picard stacks over genus 1 curves

In order to describe Balanced Picard stacks over genus 1 curves, analogously to the case $g = 0$, we will start by considering $n = 1$. The general result will then follow from the induction process in the number of marked points that will be developed in section 3.4, yielding a proof of Theorem 3.2.2.

It is convenient to do a further assumption in this case: let us suppose that $d = 1$. In fact, in virtue of the next Lemma, this assumption is not a restriction at all.

Lemma 3.2.9. *Let d, d' be any integers. Then, $\overline{\mathcal{P}}_{d,1,1} \cong \overline{\mathcal{P}}_{d',1,1}$.*

Proof. It is enough to show that, for any $d \in \mathbb{Z}$, $\overline{\mathcal{P}}_{d,1,1} \cong \overline{\mathcal{P}}_{d+1,1,1}$.

Let $(\pi : X \to S, s : S \to X, L)$ be an 1-pointed quasistable curve over S of genus 1 endowed with a balanced line bundle L of relative degree d over X, i. e., an element of $\overline{\mathcal{P}}_{d,1,1}(S)$. Then, $(\pi : X \to S, s : S \to X, L(s))$ is an object of $\overline{\mathcal{P}}_{d+1,1,1}(S)$. In fact, since $n = 1$, the geometric fibers of π must be either irreducible genus 1 curves or curves consisting in two smooth rational curves meeting in two points (see Figure 3.2 above). To check that $L(s)$ is a balanced line bundle (of degree $d + 1$) over $(\pi : X \to S, s : S \to X)$, it is enough to see that, given a geometric fiber X_s of π, $L_{|X_s}$ is balanced, so only the later case when X_s is reducible matters. In this case, Definition 2.1.1 implies that the multidegree of L restricted to X_s is $(d-1, 1)$, where 1 is the degree on the exceptional component and $d-1$ is the degree on the rational component containing the marking. It follows immediately now that L is balanced (of degree d) if and only if $L(s)$ is balanced (of degree $d + 1$).

One checks immediately that this defines an equivalence of (fibered) categories and the result follows. □

Proposition 3.2.10. *For any integer d, we have that $\overline{\mathcal{P}}_{d,1,1} \cong \overline{\mathcal{M}}_{1,2} \times B\mathbb{G}_m$.*

Proof. >From Lemma 3.2.2 it is enough to consider the case $d = 1$.

Moreover, instead of showing directly that $\overline{\mathcal{M}}_{1,2} \times B\mathbb{G}_m$ is isomorphic to $\overline{\mathcal{P}}_{1,1,1}$, let us prove that $\mathcal{Z}_{1,1} \times B\mathbb{G}_m$ is isomorphic to $\overline{\mathcal{P}}_{1,1,1}$, where $\mathcal{Z}_{1,1}$ is the universal family over $\overline{\mathcal{M}}_{1,1}$.

Let $(\pi : X \to S, s : S \to X, \Delta : S \to X) \in \mathcal{Z}_{1,1}(S)$ (Δ is the extra section of π). Then, if Δ lies in the smooth locus of X, it is easy to see that $(\pi : X \to S, s : S \to X, \mathcal{O}_{X/S}(\Delta))$ is an element of $\overline{\mathcal{P}}_{1,1,1}(S)$. In fact, all geometric fibers of π must be irreducible curves, so $\mathcal{O}_{X/S}(\Delta)$ is certainly balanced. Otherwise, using an analogous procedure to the proof of Theorem 3.2.5 in 3.4.3, we will construct an element of $\overline{\mathcal{P}}_{1,1,1}$ out of this datum. Let $\mathcal{I}$ be the $\mathcal{O}_{X/S}$-ideal defining Δ and $\mathcal{K}$ the cokernel of the natural injective map

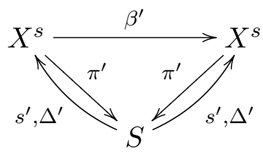
$$\mathcal{O}_{X/S} \to \mathcal{I}^{-1}.$$

Define

$$X^s := \mathbb{P}(\mathcal{K})$$

and consider the natural S-morphism $p : X^s \to X$. Then, X^s is a family of curves over S and it is not isomorphic to $X \to S$ if and only if Δ meets singular points of some geometric fibers of X over S. In this case, locally, X^s is the total transform of the blow up of X at that point with the reduced structure. Moreover, from Theorem 2.4 of [K83], the sections s and Δ have unique liftings to sections s' and Δ' of $X^s \to S$ compatible with the morphism p. So, in the geometric fibers where the curve has been blown up, Δ' must lie in a smooth point of the exceptional components of the blow up. So, it is easy to see that $(X^s \to S, s' : S \to X^s, \mathcal{O}_{X^s/S}(\Delta')) \in \overline{\mathcal{P}}_{1,1,1}(S)$. In fact, in the geometric fibers were the curve is reducible, Δ' must lie in the exceptional component of the blow up, so $\mathcal{O}_{X^s/S}(\Delta')$ restricted to those fibers has degree 1 on the exceptional components and degree 0 in the component containing the image of s', so it satisfies the conditions of Definition 2.1.1.

Let now β be an automorphism of $(\pi : X \to S, s : S \to X, \Delta : S \to X)$ and $\alpha \in B\mathbb{G}_m(S)$. β is an S-automorphism of $\pi : X \to S$ leaving the two sections fixed and α is just an element of $\mathbb{G}_m$. It is easy to see that β corresponds biunivocally to an automorphism β' of $(\pi^s : X^s \to S, s' : S \to X^s, \Delta' : S \to X^s)$ leaving s' and Δ' fixed. In fact, this follows from the fact that any automorphism of $\mathbb{P}^1$ fixing 3 distinct points is necessarily the identity.

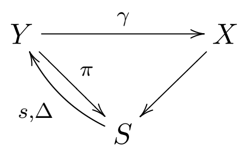


So, β' induces an automorphism of $(X^s \to S, s' : S \to X^s, \mathcal{O}_{X^s/S}(\Delta'))$, that is, an automorphism of $X^s \to S$ fixing s' and inducing an automorphism of $\mathcal{O}_{X^s/S}(\Delta')$. So, we associate to (β, α) the automorphism β' of $X^s \to S$ and the isomorphism $\alpha : \mathcal{O}_{X^S/S}(\Delta') \to \beta^*(\mathcal{O}_{X^s/S}(\Delta')) \cong \mathcal{O}_{X^s/S}(\Delta')$ given by fiberwise scalar multiplication by α.

Moreover, any other automorphism of $(X^s \to S, s' : S \to X^s, \mathcal{O}_{X^s/S}(\Delta'))$ must fix Δ' because the isomorphism class of $\mathcal{O}_{X^s/S}(\Delta')$ corresponds to the linear equivalence class of Δ', which is given just by Δ' since the geometric fibers of $X^s \to S$ are genus 1 curves. So, automorphisms of $(X^s \to S, s' : S \to X^s, \mathcal{O}_{X^s/S}(\Delta'))$ correspond to automorphisms of $(X^s \to S, s' : S \to X^s, \Delta' : S \to X^S)$ fixing the two sections and to an automorphism of $\mathcal{O}_{X^s/S}(\Delta')$ on itself, which is given by an element of $\mathbb{G}_m$.

So, we constructed a functor from $\mathcal{Z}_{1,1} \times B\mathbb{G}_m$ to $\overline{\mathcal{P}}_{1,1,1}$ which is full and faithful. In order to conclude that $\mathcal{Z}_{1,1} \times B\mathbb{G}_m \cong \overline{\mathcal{P}}_{1,1,1}$ it is enough to check that this functor is essentially surjective. Let $(\pi : Y \to S, s : S \to Y, L) \in \overline{\mathcal{P}}_{1,1,1}(S)$. Since L is a line bundle of degree 1 in a genus 1 curve, it is associated to an unique efective divisor (of degree 1): call it Δ. Of course, Δ can also be seen as a section of π. So, if all fibers of π are

1-pointed stable curves, it follows immediately that $(\pi : Y \to S, s : S \to Y, \Delta : S \to Y) \in \mathcal{Z}_{1,1}(S)$. Instead, if some of the geometric fibers of π have exceptional components, the idea is to blow down these components and endow this new curve with the extra section given by the image of Δ. The way to do it rigorously is standard: it is enough to define $X := \mathcal{P}roj(\oplus_{i\geq 1}\pi_*((\omega_{Y/S}(s))^{3i})) \to S$. In fact, the fibers of $\omega_{Y/S}(s)$ over S have degree 0 exactly in the exceptional components and positive degree in all the others, so the result is that $X \to S$ is isomorphic to $Y \to S$ everywhere except in the exceptional components, that get contracted to points in X (see section 3.5 for a rigorous proof). Moreover, there is an S-morphism $\gamma : Y \to X$ making the following diagram commute.



So, $X \to S$ endowed with the sections γs and $\gamma\Delta$ is an object of $\mathcal{Z}_{1,1}(S)$. It is easy to check that the above functor applied to $(X \to S, \gamma s, \gamma\Delta)$ yields an object of $\overline{\mathcal{P}}_{1,1,1}(S)$ which is isomorphic to $(\pi : Y \to S, s : S \to Y, L)$. □

Let now $n > 1$. In view of theorem 3.2.5, consider the universal family over $\overline{\mathcal{P}}_{d,1,n}$, $\mathcal{Z}_{d,0,n}$. By applying an inductive argument based on the previous Proposition we have that $\mathcal{Z}_{d,0,n} \cong \overline{\mathcal{M}}_{1,n+1} \times B\mathbb{G}_m$. So, Theorem 3.2.2 will give the following result.

Proposition 3.2.11. *Let $d \in \mathbb{Z}$ and $n \geq 1$. Then $\overline{\mathcal{P}}_{d,1,n}$ is isomorphic to $\overline{\mathcal{M}}_{1,n+1} \times B\mathbb{G}_m$.*

3.3 Properties of line bundles on reducible nodal curves

In this section we prove some technical properties of line bundles over (reducible) nodal curves that will be used later in the proof of Theorem 3.2.5.

3.3.1 Nonspecialty and global generation

Lemma 3.3.1. *Let X be a rational curve and L a line bundle on X such that, for each irreducible component Z of X, the degree of L on Z is smaller or equal to $k_Z - 2$, except possibly for one component Z_0, where the degree of L can be equal to $k_{Z_0} - 1$. Then, $H^0(X, L) = 0$.*

Proof. We will argue by induction on the number of irreducible components γ of X.

If $X = Z$ is irreducible, then $k_X = 0$ and $\deg_X L \le -1$, which clearly implies that $H^0(X, L) = 0$.

Now, suppose that X has $\gamma > 1$ irreducible components. Let Z_0 be an irreducible component of X such that $\deg_{Z_0} L - (k_{Z_0} - 2)$ is maximal among the irreducible components of X. In particular, if there is an irreducible component Z of X with $\deg_Z L = k_Z - 1$, then necessarily $Z_0 = Z$. Then, $\overline{X \setminus Z_0} = X_1 \cup \cdots \cup X_\delta$ is a disjoint union of (rational) subcurves of X and $(X_i, L_{|X_i})$ satisfy the hypothesis, for $i = 1, \ldots, \delta$. So, since each X_i has a number of irreducible components smaller than γ, we can apply the induction hypothesis on each one of them and conclude that $H^0(X_i, L_{|X_i}) = 0$ for $i = 1, \ldots, \delta$. So, a global section of L on X must be trivial along $\overline{X \setminus Z_0}$, and in particular it must be equal to zero in each one of the k_{Z_0} points where Z_0 meets the rest of X. Since $\deg_{Z_0} L \le k_Z - 1$, we conclude that all sections of L must be trivial also on Z. It follows that $H^0(X, L) = 0$. □

Corollary 3.3.2. *Let X be an n-pointed rational curve, with $n \ge 3$. Then, X is semistable if and only if $\omega_X(p_1 + \cdots + p_n)$ is globally generated, where $p_1, \ldots, p_n$ are the marked points of X.*

Proof. Let ω denote $\omega_X(p_1 + \ldots, p_n)$. We will start by showing that if X is semistable then ω is globally generated. So, for all $x \in X$, we must see that there are sections of ω that do not annulate in x.

Start by assuming that x is a nonsingular point of X. We must show that $h^0(X, \omega(-x)) < h^0(X, \omega)$.

By Riemann-Roch, we have that

$$h^0(\omega) = h^1(\omega) - 2 + n + 1 = h^0(\mathcal{O}_X(-p_1 - \cdots - p_n)) - 1 + n$$

and, since $(X; p_1, \ldots, p_n)$ is semistable, $(X, \mathcal{O}_X(-p_1 - \cdots - p_n))$ satisfies the hypothesis of Lemma 3.3.1, so $h^0(X, \mathcal{O}_X(-p_1 - \cdots - p_n)) = 0$. In fact, given an irreducible component Z of X, we have that $\deg_Z \mathcal{O}_X(-p_1 - \cdots - p_n) \le 0 \le k_Z - 2$ if Z is not a rational tail of X and for rational tails that $\deg_T \mathcal{O}_X(-p_1 - \cdots - p_n) \le -1 = k_Z - 2$.

So, again by Riemann-Roch, to show that $h^0(X, \omega(-x)) < h^0(X, \omega)$, we must show that $h^1(X, \omega(-x)) = h^0(X, \mathcal{O}(-p_1 - \cdots - p_n + x)) = 0$. But it is easy to see that, if $(X; p_1, \ldots, p_n)$ is semistable, then also $(X, \mathcal{O}(-p_1 - \cdots - p_n + x))$ satisfies the hypothesis of Lemma 3.3.1, which implies that $h^0(X, \mathcal{O}(-p_1 - \cdots - p_n + x)) = 0$.

Now, suppose that x is a singular point of X. To show that x is not a base point of X we must show that $h^0(X, \omega \otimes \mathcal{I}_x) < h^0(X, \omega)$. By contradiction, suppose these are equal. Let $\nu : Y \to X$ be the partial normalization of X at x. Then, if p and q denote the preimages of x under ν, we have that $h^0(Y, \nu^*\omega(-p - q)) = h^0(X, \omega)$. Since x is necessarily a disconnecting node of X, $Y = Y_1 \cup Y_2$ is the union of two rational curves. Arguing in the same way as before in Y_1 and Y_2 we easily see that $h^0(Y, \nu^*\omega(-p-q)) = n-2$, which is a contradiction, and we conclude.

Now, suppose that X is not semistable and let us see that ω is not globally generated. X being semistable means that there is a tail T of X without marked points; we will show that all $x \in T$ are base points for ω. >From what we have said so far it is enough to see that $h^0(X, \mathcal{O}_X(-p_1 - \cdots - p_n)) < h^0(X, \mathcal{O}(-p_1 - \cdots - p_n + x))$. Since $h^0(X, \mathcal{O}_X(-p_1 - \cdots - p_n)) = 0$, again by Lemma 3.3.1, it is enough to see that $\mathcal{O}_X(-p_1 - \cdots - p_n + x)$ has nontrivial sections. But this follows by observing that $\mathcal{O}_X(-p_1 - \cdots - p_n + x)_{|T}$ is a line bundle of degree one in a rational curve, so its space of sections has dimension 2. So, even if the node connecting T with the rest of X imposes one condition, there is a section of $\mathcal{O}_X(-p_1 - \cdots - p_n + x)$ that is nontrivial on T and we conclude.

□

Note that the statement of 3.3.2 does not hold in the case of curves with higher genus. In fact, if X is a nonsingular curve of genus greater or equal than one with one marked point p, $\omega_X(p)$ is not globally generated since p itself is a base point for $H^0(X, \omega_X(p))$.

Instead, for curves without marked points, the global generation of the dualizing sheaf is indeed related to the connectivity of the curve. In fact, from [BE91], we know that if X is a graph curve, i.e, a stable curve such that all irreducible components are rational curves, then $H^0(X, \omega_X)$ has no base points if and only if X has no disconnecting nodes (and analogously that ω_X is very ample if and only if X is 3-connected).

Lemma 3.3.3. *Let X be a nodal curve of genus g and $L \in \operatorname{Pic}^d X$. If $\deg_Z L \geq 2g_Z - 1$ for every connected subcurve $Z \subseteq X$, then $H^1(X, L) = 0$. Moreover, if strict inequality holds above for all $Z \subseteq X$, then L has no base points.*

To prove Lemma 3.3.3 we will use the following Lemma, which is Lemma 2.2.2 in [C2].

Lemma 3.3.4 (Caporaso,[C2])**.** *Let X be a nodal curve of genus g and $L \in \operatorname{Pic}^d X$. If, for every connected subcurve Z of X, $\deg_Z L \geq 2g_Z - 1$, then $h^0(X, L) = d - g + 1$.*

Proof (of Lemma 3.3.3). The first assertion follows immediately by Serre duality and by Lemma 3.3.4.

Now, assume that, for every $Z \subseteq X$, $\deg_Z L \geq 2g_Z$. We must show that L has no base points. Consider a closed k-rational point x in X. Suppose that x is a nonsingular point of X. We must show that

$$h^0(X, L(-x)) < h^0(X, L).$$

By our assumption on L, we can apply again Lemma 3.3.4 to $L(-x)$ to get that $h^0(X, L(-x)) = d - 1 - g + 1 = h^0(X, L) - 1$.

Suppose now that x is a nodal point of X. We must show that

$$H^0(X, L \otimes \mathcal{I}_x) \subsetneq H^0(X, L).$$

By contradiction, suppose these are equal. Then, if $\nu : Y \to X$ is the partial normalization of X at x, we get that

$$H^0(X, L) = H^0(Y, \nu^* L(-p - q)),$$

where p and q are the preimages of x by ν.

Suppose that x is not a disconnecting node for X. Then, it is easy to see that we can apply Lemma 3.3.4 to $(Y, \nu^* L(-p - q))$. Let $Z' \subseteq Y$ and $Z \subseteq X$ the subcurve of X such that $Z' = \nu^{-1}(Z)$. In fact, since x is not a disconnecting node for X, if Z' contains p and q, then $g_{Z'}{=}g_Z - 1$, so $\deg_{Z'} \nu^* L(-p - q) = \deg_Z L - 2 \geq 2g_Z - 2 = 2g_{Z'} - 1$. If Z contains only one among the points $\{p, q\}$, then $g_{Z'} = g_Z$ but $\deg_{Z'} \nu^* L(-p - q) = \deg_Z L - 1 \geq 2g_Z - 1 = 2g_{Z'} - 1$. Finally, if Z does not contain none of the points p and q, $g_{Z'} = g_Z$ and $\deg_{Z'} \nu^* L(-p - q) = \deg_Z L \geq 2g_Z$. So, we get that $h^0(Y, \nu^* L(-p - q)) = (d - 2) - (g - 1) + 1 = d - g$, leading us to a contradiction.

Suppose now that x is a disconnecting node for X. Then, Y is the union of two connected curves, Y_1 and Y_2, of genus g_1 and g_2, respectively, with $g_1 + g_2 = g$. Suppose that $p \in Y_1$ and $q \in Y_2$. Then,

$$h^0(Y, \nu^* L(-p - q)) = h^0(Y_1, \nu^*(L)_{|Y_1}(-p)) + h^0(Y_2, \nu^*(L)_{|Y_2}(-q)).$$

Also in this case, we can apply Lemma 3.3.4 to $(Y_i, h^0(Y_1, \nu^*(L)_{|Y_1}(-p)))$ and to $(Y_2, h^0(Y_1, \nu^*(L)_{|Y_2}(-q)))$. We get that

$$h^0(Y, \nu^* L(-p - q)) = (\deg_{Y_1}(\nu^* L) - g_1) + (\deg_{Y_2}(\nu^* L) - g_2) = d - g,$$

a contradiction. □

Corollary 3.3.5. *Let X be an n-pointed semistable curve of genus g with $2g - 2 + n > 0$ and let $M := \omega_X(p_1 + \cdots + p_n)$, where $p_1, \ldots, p_n$ are the n marked points of X. Then, for all $m \geq 2$, we have that*

1. *$H^1(X, M^m) = 0$;*
2. *M^m is globally generated.*

Proof. According to Lemma 3.3.3, it is enough to show that, given a subcurve Z of X, $\deg_Z M^m \geq 2g_Z$, for all $m \geq 2$. It is sufficient to prove the result for $m = 2$.

Let Z be a subcurve of X. Then,

$$\deg_Z \omega_X = 2g_Z - 2 + k_Z$$

and

$$\deg_Z(M^2) \geq 4g_Z - 4 + 2k_Z = (2g_Z) + (2g_Z - 4 + 2k_Z).$$

So, if both g_Z and k_Z are bigger than zero or if one of them is bigger than two, we are done. We must threat the remaining cases separately.

Start by supposing that $g_Z = 0$ and $k_Z \leq 1$. Then either $k_Z = 0$ and $Z = X$ has at least three marked points since $2g - 2 + n > 0$ or $k_Z = 1$ and Z has at least one marked point of X. In both cases we have that $\deg_Z(M^2) \geq 2(-2 + 2) = 0 = 2g_Z$.

It remains to control the case when $g_Z = 1$ and $k_Z = 0$. Then $X = Z$ has genus one and since $2g - 2 + n > 0$, we have that $n > 0$ which implies that it has at least one marked point. So, $\deg_Z(M)^2 \geq 2(0 + 1) = 2 = 2g_Z$. □

Corollary 3.3.6. *Let $d >> 0$, $n > 0$ and X an n-pointed quasistable curve of genus g with $2g - 2 + n > 0$ endowed with a balanced line bundle L of degree d. Denote by $p_1, \dots, p_n$ the n marked points of X and let M be the line bundle $L(p_1 + \dots + p_{n-1}) \otimes (\omega_X(p_1 + \dots + p_{n-1}))^{-k}$, for any $k \leq 1$. Then, we have that, for all $m \geq 1$,*

1. *$H^1(X, M^m) = 0$;*
2. *M^m is globally generated.*

Proof. Again, accordingly to Lemma 3.3.3, the result follows if we prove that, for every subcurve Z of X, $\deg_Z M^m \geq 2g_Z$. It is enough to prove the result for $m = 1$.

Let Z be a subcurve of X which is not contained in any rational tail or in any rational bridge of X. Since $\deg_Z \omega_Z(p_1 + \dots + p_{n-1}) \geq 0$, we have that $\deg_Z M \geq \deg_Z L \otimes \omega_Z^{-1}$. Then, if $g = 0$, $\deg_Z M \geq d + t_Z - (k_Z - 2)$ and if $g = 1$, $\deg_Z M \geq d + t_Z - \frac{k_Z - t_Z}{2} - k_Z$ (see Definitions 2.1.1 and 3.1.5). In both cases, since we are considering $d >> 0$, clearly $\deg_Z M \geq 2g_Z$.

Now, suppose $g \geq 2$. By definition of balanced (see 2.1.1 above), we have that

$$\deg_Z L \geq \frac{d}{2g-2}(w_Z - t_Z) + i_{X,Z} = \frac{d}{2g-2}(2g_Z - 2 + k_Z - t_Z) + i_{X,Z},$$

where $i_{X,Z}$ is independent of d. If Z is rational, $k_Z - t_Z \geq 3$, so $\deg_Z L >> 0$ if $d >> 0$. In fact, if $k_Z = t_Z$, X would be rational; by the other hand, if $k_Z - t_Z$ is 1 or 2 and $g \geq 2$, Z should be contained in a rational tail or in a rational bridge of X, respectively, which cannot be the case by our assumption on Z. Suppose now that $g_Z = 1$. Then, $k_Z - t_Z \geq 1$, since otherwise X would have genus 1. So, also in this case, $\deg_Z L >> 0$. Finally, if $g_Z \geq 2$, it follows immediately that $\deg_Z L >> 0$. The same holds for $\deg_Z M$, which is asymptotically equal to $\deg_Z L$ since $d >> 0$.

Suppose now that Z is contained in a rational tail or in a rational bridge of X. Then, from Lemma 3.1.9, we have that $\deg_Z L \geq k_Z - 2$, so $\deg_Z M \geq (k_Z - 2) - (k_Z - 2) = 0 = 2g_Z$ and we conclude. □

3.3.2 Normal generation

Recall the following definition.

Definition 3.3.7. *A coherent sheaf $\mathcal{F}$ on a scheme X is said to be **normally generated** if, for all $m \geq 1$, the canonical map*

$$H^0(X, \mathcal{F})^m \to H^0(X, \mathcal{F}^m)$$

is surjective.

Note that if we take $\mathcal{F}$ to be an ample line bundle L on X, then if L is normally generated it is, indeed, very ample (see [M70], section 1). In this case, saying that L is normally generated is equivalent to say that the embedding of X via L on $\mathbb{P}^N$, for $N = h^0(X, L) - 1$, is projectively normal.

Normal generation of line bundles on curves has been widely studied. For instance, we have the following theorem of Mumford:

Theorem 3.3.8 (Mumford, [M70], Theorem 6)**.** *Let X be a nonsingular irreducible curve of genus g. Then, any line bundle of degree $d \geq 2g + 1$ is normally generated.*

Mumford's proof of Theorem 3.3.8 is based on the following Lemma.

Lemma 3.3.9 (Generalized Lemma of Castelnuovo, [M70], Theorem 2)**.** *Let N be a globally generated invertible sheaf on a complete scheme X of finite type over k and F a coherent sheaf on X such that*

$$H^i(X, F \otimes N^{-i}) = 0 \text{ for } i \geq 1.$$

Then,

1. $H^i(X, F \otimes N^j) = 0$ *for* $i + j \geq 0$, $i \geq 1$.
2. *the natural map*

$$H^0(X, F \otimes N^i) \otimes H^0(X, N) \to H^0(X, F \otimes N^{i+1})$$

is surjective for $i \geq 0$.

The proof of the following statement uses mainly the arguments of Knudsen's proof of Theorem 1.8 in [K83].

Proposition 3.3.10. *Let X be an n-pointed semistable curve of genus g with $2g-2+n>0$ and $L \in \operatorname{Pic}^d X$. If, for every subcurve Z of X, $\deg_Z L \geq 2g_Z$, then $L \otimes \omega_X(p_1+\cdots+p_n)$ is normally generated, where $p_1 \ldots, p_n$ are the marked points of X.*

Proof. Let D denote the divisor $p_1+\cdots+p_n$. Let Z be a subcurve of X. Since the multidegree of $\omega(D)$ is non-negative, $\deg_Z L \otimes \omega(D) \geq \deg_Z L \geq 2g_Z$, so both statements of Lemma 3.3.3 hold also for $L \otimes \omega(D)$. So, we can apply the generalized Lemma of Castelnuovo with $F=(L \otimes \omega_X(D))^m$ and $N=L \otimes \omega_X(D)$, for any $m>1$, and get that the natural map

$$H^0(X,(L\otimes\omega_X(D))^m)\otimes H^0(X,L\otimes\omega_X(D))\to H^0(X,(L\otimes\omega_X(D))^{m+1})$$

is surjective. So, to prove that $L \otimes \omega_X(D)$ is normally generated, it remains to show that the map

$$H^0(X,L\otimes\omega_X(D))\otimes H^0(X,L\otimes\omega_X(D))\xrightarrow{\alpha} H^0(X,(L\otimes\omega_X(D))^2)$$

is surjective.

Start by assuming that X has no disconnecting nodes. Then, if $g=0$, X is necessarily nonsingular and $n \geq 3$, so $\deg L \otimes \omega_X(D) \geq 2g+1$ and the result follows from Theorem 3.3.8.

Now, assume $g \geq 1$ and consider the following commutative diagram

$$\begin{array}{ccc}
\Gamma(L\otimes\omega_X(D))\otimes\Gamma(\omega_X)\otimes\Gamma(L(D)) & \longrightarrow & \Gamma(L\otimes\omega_X(D))\otimes\Gamma(L\otimes\omega_X(D)) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\
\Gamma(L\otimes\omega_X^2(D))\otimes\Gamma(L(D)) & \xrightarrow{\quad\gamma\quad} & \Gamma((L\otimes\omega_X(D))^2)
\end{array}$$

where $\Gamma(-)$ indicates $H^0(X,-)$.

If $g=1$, then X is either nonsingular or it is a ring of $\mathbb{P}^1$'s. In both cases ω_X is isomorphic to $\mathcal{O}_X$, so it is globally generated. Instead, if $g \geq 2$, from the proof of Theorem 1.8 in Knudsen we have that, since X has no disconnecting nodes, ω_X is globally generated too.

Moreover, from Lemma 3.3.3 applied to $L(D)$, we get that $H^1((L \otimes \omega_X(D)) \otimes \omega_X^{-1}) = H^1(L(D)) = 0$. So, we can apply the Generalized Lemma of Castelnuovo with $F = L \otimes \omega_X(D)$ and $N = \omega_X$ to conclude that β is surjective.

Now, since X has no disconnecting nodes, it cannot have rational tails. So, we can see X as a semistable curve without marked points and apply Corollary 3.3.5 to X and ω_X and get that $H^1((L\otimes\omega_X^2(D))\otimes(L(D))^{-1}) = H^1(\omega_X^2) = 0$. Since $L(D)$ is globally generated, again by Lemma 3.3.3, we can apply the Generalized Lemma of Castelnuovo with $F = L \otimes \omega_X^2(D)$ and $N = L(D)$ to conclude that also γ is surjective.

Since the above diagram is commutative, it follows that also α is surjective and we conclude.

Now, to show that α is surjective in general, let us argue by induction in the number of disconnecting nodes of X.

Let x be a disconnecting node of X and X_1 and X_2 the subcurves of X such that $\{x\} = X_1 \cap X_2$.

The surjectivity of α follows if we can prove the following two statements.

1. The image of α contains a section $s \in H^0(X, (L \otimes \omega_X(D)^2))$ such that $s(x) \neq 0$;

2. The image of α contains $H^0(X, (L \otimes \omega_X(D))^2 \otimes \mathcal{I}_x))$.

The first statement follows immediately from the fact that $L \otimes \omega_X(D)$ is globally generated (once more by 3.3.3).

Let M denote $L \otimes \omega_X(D)$. To prove (2) let us consider $\sigma \in H^0(X, M^2 \otimes \mathcal{I}_x)$. Then, $\sigma = \sigma_1 + \sigma_2$, with

$$\sigma_1 \in H^0(X, M^2 \otimes \mathcal{I}_{X_1}) \cong H^0(X_2, (M^2 \otimes \mathcal{I}_{X_1})_{|X_2}) \cong H^0(X_2, (M^2)_{|X_2} \otimes \mathcal{I}_x),$$

$$\sigma_2 \in H^0(X, M^2 \otimes \mathcal{I}_{X_2}) \cong H^0(X_1, (M^2 \otimes \mathcal{I}_{X_2})_{|X_1}) \cong H^0(X_1, (M^2)_{|X_1} \otimes \mathcal{I}_x).$$

By induction hypothesis, σ_1 is in the image of

$$H^0(X_2, M_{|X_2}) \otimes H^0(X_2, M_{|X_2}) \to H^0(X_2, (M^2)_{|X_2})$$

and σ_2 in the image of

$$H^0(X_1, M_{|X_1}) \otimes H^0(X_1, M_{|X_1}) \to H^0(X_1, (M^2)_{|X_1})$$

with both σ_1 and σ_2 vanishing on x.

Write σ_1 as $\sum_{l=1}^r u_l \otimes v_l$, with u_l and v_l in $H^0(X_2, M_{|X_2})$, for $l = 1, \dots, r$. Let $\nu : Y \to X$ be the partial normalization of X in x and p and q be the preimages of x on X_1 and X_2, respectively, via ν. Since $M_{|X_1}$ is globally generated, there is $s \in H^0(X_1, M_{|X_1})$ with $s(p) \neq 0$. Then there are constants a_l and b_l for $l = 1, \dots, r$ and $i = 1, \dots, k$ such that

$$a_l s(p) = u_l(q) \text{ and } b_l s(p) = v_l(q). \tag{3.8}$$

Define the sections $\bar{u}_l$ and $\bar{v}_l$ as u_l (resp. v_l) on X_2 and as $a_l s$ (resp. $b_l s$) on X_1, for $l = 1, \dots, r$. By (3.8), these are global sections of M and

$$\sum_{l=1}^{r} \bar{u}_l \otimes \bar{v}_l$$

maps to σ_1. In fact,

$$\sigma_1(x) = \sum_{l=1}^{r} u_l(q) \otimes v_l(q) = \sum_{l=1}^{r} (a_l s(p) \otimes b_l s(p)) = (\sum_{l=1}^{r} a_l b_l) s(p) \otimes s(p)$$

and, by hypothesis, $\sigma_1(x) = 0$ and $s(p) \neq 0$. This implies that $\sum_l^r a_l b_l = 0$, so $(\sum_{l=1}^r \bar{u}_l \otimes \bar{v}_l)_{|X_1} = 0$. We conclude that σ_1 is in the image of α.

In the same way, we would get that also σ_2 is in the image of α, so (2) holds and we are done. □

The next result follows from the proof of Theorem 1.8 in [K83], however we include it here since we shall use it in the following slightly more general form.

Corollary 3.3.11. *Let X be an n-pointed semistable curve of genus g such that $2g-2+n>0$ and let $p_1 \ldots, p_n$ be the marked points of X. Then, for $m \geq 3$, $(\omega_X(p_1+\cdots+p_n))^m$ is normally generated.*

Proof. Is an immediate consequence of Proposition 3.3.10 and the proof of Corollary 3.3.5. □

Corollary 3.3.12. *Let X be an n-pointed quasistable curve of genus g such that $2g-2+n>0$ and L a balanced line bundle on X of degree $d >> 0$. Then, if $n>0$, $L(p_1+\cdots+p_{n-1})$ is normally generated.*

Proof. Since X is an n-pointed quasistable curve, it is easy to see that X, endowed with the first $n-1$ marked points $(p_1,\ldots,p_{n-1})$, is an $(n-1)$-pointed semistable curve. Moreover, by the proof of Corollary 3.3.6, we can apply Proposition 3.3.10 to $L(p_1+\cdots+p_{n-1})\otimes\omega_X^{-1}(p_1+\cdots+p_{n-1})$. The result follows immediately now.

□

Corollary 3.3.13. *Let $d>>0$, $n>0$ and X an n-pointed quasistable curve of genus g, with $2g-2+n>0$, endowed with a balanced line bundle L. Let M denote the line bundle $L(p_1+\cdots+p_n)$, where $p_1,\ldots,p_n$ are the marked points of X. We have:*

1. *M is normally generated;*
2. *M is very ample.*

Proof. Statement (1) follows from the proof of the previous Corollary, which obviously works for $M=L(p_1+\cdots+p_n)$ as well.

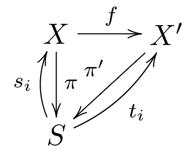
To show (2) it is enough to observe that M is ample since its degree on each irreducible component of X is positive. Since M is also normally generated, it follows that M is indeed very ample (see [M70], section 1). □

3.4 The contraction functor

The following definition generalizes the notion of contraction introduced by Knudsen in [K83] to the more general case of pointed quasistable curves endowed with balanced line bundles.

Definition 3.4.1. *Let $2g-2+n>0$ and $(\pi: X\rightarrow S, s_i: S\rightarrow X, L)$ be an $(n+1)$-pointed curve of genus g endowed with a line bundle of relative degree d. A contraction of X is an S-morphism from X into an n-pointed curve $(\pi': X'\rightarrow S, t_i: S\rightarrow X', L')$ endowed with a line bundle of relative degree d, L', and with an extra section $\Delta: S\rightarrow X'$ such that*

1. *for* $i = \ldots, n$*, the diagram*



commutes both in the upward and downward directions,

2. $\Delta = f s_{n+1}$,

3. f *induces an isomorphism between* $L(s_1 + \cdots + s_n)$ *and* $f^* L'(t_1 + \cdots + t_n)$,

4. *the morphism induced by* f *in the geometric fibers* X_s *is either an isomorphism or there is an irreducible rational component* $E \subset X_s$ *such that* $s_{n+1}(s) \in E$ *which is contracted by* f *into a closed point* $x \in X'_s$ *and*
$$f_s : X_s \setminus E \to X'_s \setminus \{x\}$$
is an isomorphism.

3.4.1 Properties of contractions

Proposition 3.4.2. *Let* $S = \operatorname{Spec} k$ *and* $f : X \to X'$ *a contraction from an* $(n+1)$*-pointed curve* $(X; p_1, \ldots, p_{n+1})$ *endowed with a degree* d *line bundle* L *into an* n*-pointed curve* $(X'; q_1, \ldots, q_n)$*, endowed with a degree* d *line bundle* L' *and with an extra point* r*. Then, if* $(X; p_, \ldots, p_{n+1})$ *is quasistable, also* $(X'; q_1, \ldots, q_n)$ *is quasistable and, in this case,* L *is balanced if and only if* L' *is balanced.*

Proof. Clearly, the assertion follows trivially if no irreducible component of X gets contracted by f. So, assume that there is an irreducible component E of X that gets contracted by f. Then, necessarily, $p_{n+1} \in E$, so no exceptional component of X gets contracted. Moreover, the condition that $f^* L'(q_1 + \cdots + q_n) \cong L(p_1 + \cdots + p_n)$ implies that $L(p_1 + \cdots + p_n)$ is trivial on the fibers of f, so it must have degree 0 on E. Now, we have only two possibilities: either $f(E) = \{r\}$ is a smooth point of X' or it is nodal.

Start by considering the case when r is smooth. Since $f(E) = \{r\}$, we must have that $k_E = 1$, i. e., E is a rational tail of X. So, if X is quasistable, E must contain exactly another special point p_i, for some $i = 1, \ldots, n$ and $r = q_i$. Let F' be the irreducible component of X' containing r and F the correspondent irreducible component of X (recall that f establishes an isomorphism between F and F' away from r). If $g_F > 0$, then it is clear that also X' is quasistable. Instead, if F is rational, even if $k_{F'} = k_F - 1$, F' has one more marked point than F. So, X' has the same destabilizing and

Figure 3.4: Contractions of quasistable pointed curves over k and balanced degree d line bundles.

exceptional components as X. In fact, if X is quasistable, it cannot be an exceptional component of X because F is would be contained in a rational tail of X. It follows that, if $(X; p_1, \ldots, p_{n+1})$ is quasistable, $(X'; q_1, \ldots, q_n)$ is a quasistable too.

Let us now check that, if we are contracting a rational tail of a quasistable curve, L is balanced if and only if L' is balanced. From the definition of contraction, we get that the multidegree of $L(p_1+\cdots+p_n)$ in the irreducible components of X that are not contracted must agree with the multidegree of $L'(q_1 + \cdots + q_n)$ in their images by f. In our case, this implies that the multidegree of L' on the irreducible components of X' coincides with the multidegree of L on the corresponding irreducible components of X, except on F', where we must have that

$$\deg_{F'} L' = \deg_F L - 1.$$

So, given a proper subcurve Z' of X', if Z' does not contain r, the balanced condition will be satisfied by L on Z if and only if it is satisfied by L' on Z' since $m_{Z'}(d, L') = m_Z(d, L)$, $M_{Z'}(d, L') = M_Z(d, L)$ and $\deg_{Z'}(d, L') = \deg_Z(d, L)$. Now, suppose $r \in Z'$ and let Z be the preimage of Z' by f. Then, $k_{Z'} = k_Z - 1$, $w_{Z'} = w_Z - 1$, $b_Z^L = b_{Z'}^{L'}$ and $t_{Z'} = t_Z - 1$, which implies that

$$m_{Z'}(d,L) = m_Z(d,L) - 1$$

and

$$M_{Z'}(d,L') = M_Z(d,L) - 1.$$

Since also $\deg_{F'} L' = \deg_F L - 1$, we conclude that, if L is balanced, then L' is balanced too. Now, to conclude that the fact that L' is balanced implies that also L is balanced we have to further observe that the degree of L on E is forced to be equal to -1 since E contains 2 special points and that the inequality (2.1) is verified on $\overline{X \setminus E}$ (that does not correspond to any proper subcurve of X'), which follows since $m_{\overline{X\setminus E}}(d,L) = M_{\overline{X\setminus E}}(d,L) = d+1 = \deg_{\overline{X\setminus E}} L$.

Now, suppose r is a nodal point of X. Then, p_{n+1} is the only marked point of X in E (otherwise, the condition that $f(p_i) = q_i$ for $i = 1, \dots, n$ would imply that one of these q_i's should coincide with r, which is nodal, and $(X', q_1, \dots, q_n)$ would not be a pointed curve). So, if $(X; p_1, \dots, p_{n+1})$ is quasistable, we must have that $k_E = 2$, i. e., E is a rational bridge of X (note that if $g = 1$ then necessarily $n > 0$, so $p_1 \notin E$). We must do a further distinction here. Suppose first that E intersects just one irreducible component of X: call it F and F' its associated irreducible component on X'. Now, if $X = E \cup F$, and if F is rational, X' is an irreducible genus 1 curve, which is clearly quasistable. If, instead, $g_F > 0$ or if $k_F \geq 3$, we see that all destabilizing and exceptional components of X' correspond to destabilizing and exceptional components of X and are contained in the same type of rational chains.

If, instead, E intersects two distinct irreducible components of X, it is easy to see that, also in this case, all destabilizing and exceptional components of X' correspond to destabilizing and exceptional components of X and are contained in the same type of rational chains. So, $(X'; q_1, \dots, q_n)$ will be quasistable if $(X; p_1, \dots, p_{n+1})$ is.

Now, since p_{n+1} is the only marked point in E, all irreducible components of X' have the same marked points as the corresponding irreducible components of X, so $f^*L'(q_1 + \cdots + q_n) \cong L(p_1 + \cdots + p_n)$ implies that the multidegree of L on the irreducible components of X' coincides to the multidegree of L on the corresponding irreducible components of X and that the degree of L on E is zero. Let Z' be a proper subcurve of X' and Z the corresponding proper subcurve of X. If Z does not intersect E or if it intersects E in a single point, then it is immediate to see that inequality (2.1) holds for L and Z if and only if it holds for L' and Z'. If, instead, Z intersects E in two points, then $g(Z') = g(Z) + 1$, $t_{Z'} = t_Z$, $b^L_{Z'} = b^L_Z - 1$ and $k_{Z'} = k_Z - 2$, so, we get that

$$m_{Z'}(d,L) = m_Z(d,L)$$

and

$$M_{Z'}(d, L') = M_Z(d, L).$$

Since also $\deg_{Z'} L' = \deg_Z L$, we conclude that if we are contracting a rational bridge, if L is balanced, L' will be balanced too. Now, to conclude that the fact that L' is balanced implies that also L is balanced we have to further observe that, by definition of contraction, the degree of L on E is forced to be 0 and that the inequality (2.1) is verified on $\overline{X \setminus E}$ (that does not correspond to any proper subcurve of X'), which is true since $m_{\overline{X\setminus E}}(d, L) = M_{\overline{X\setminus E}}(d, L) = d = \deg_{\overline{X\setminus E}} L$.

□

The following lemma is Corollary 1.5 of [K83].

Lemma 3.4.3. *Let X and Y be S-schemes and $f : X \to Y$ a proper S-morphism, whose fibers are at most one-dimensional. Let $\mathcal{F}$ be a coherent sheaf on X, flat over S such that $H^1(f^{-1}(y)\mathcal{F} \otimes_{\mathcal{O}_Y} k(y)) = (0)$ for each closed point $y \in Y$. Then $f_*\mathcal{F}$ is S-flat, $R^1 f_*\mathcal{F} = 0$ and, given any morphism $T \to S$, there is a canonical isomorphism*

$$f_*\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T \cong (f \times 1)_*(\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{O}_T).$$

If, moreover, $\mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$ is globally generated we have also that the canonical map $f^* f_*\mathcal{F} \to \mathcal{F}$ is surjective.

Corollary 3.4.4. *Let $(\pi : X \to S, s_i : S \to X, L)$ be an $(n+1)$-pointed quasistable curve endowed with a balanced line bundle of degree $d >> 0$. Let M be either the line bundle $L(s_1 + \cdots + s_n)$ or $(\omega_X(s_1 + \cdots + s_n))^3$. Then, for all $m \geq 1$, we have that*

1. *$\pi_*(M^m)$ is S-flat;*
2. *$R^1(\pi_*(M^m)) = 0$;*
3. *For all $i \geq 1$, the natural map*

   $$\alpha_i : \pi_* M^i \otimes \pi_* M \to \pi_* M^{i+1}$$

   is surjective;
4. *$\pi^*\pi_* M^m \to M^m$ is surjective.*

Proof. (1) (2) and (4) follow immediately from Corollaries 3.3.6 with $k = 0$ and 3.3.5, which assert that we can apply Lemma 3.4.3 to π and M, in both cases.

Let us now show that (3) holds. From Propositions 3.3.12 and 3.3.11, the statement holds if $S = \operatorname{Spec} k$. Since M satisfies the hypothesis of Lemma 3.4.3, the formation of π_* commutes with base change. So, α_i is surjective at every geometric point of S and we use Nakayama's Lemma to conclude that α_i is surjective. □

We now show that Knudsen's main lemma also holds for quasistable pointed curves and balanced line bundles of high degree.

Lemma 3.4.5. *Let $d >> 0$ and consider a contraction $f : X \to X'$ as in Definition 3.4.1. Denote by M and M', respectively, the line bundles $L(s_1 + \cdots + s_n)$ and $L'(t_1 + \cdots + t_n)$. Then, for all $m \geq 1$, we have that*

1. *$f^*(M')^m \cong M^m$ and $(M')^m \cong f_*(M^m)$;*
2. *$R^1 f_*(M^m) = 0$;*
3. *$R^i \pi_*(M^m) \cong R^i \pi'_*(M'^m)$ for $i \geq 0$.*

Proof. That $f^*(M')^m$ is isomorphic to M^m comes from our definition of contraction morphism. So, also $f_* f^*(M'^m)$ is isomorphic to $f_*(M^m)$. So, composing this with the canonical map from M'^m into $f_* f^*(M'^m)$, we get a map

$$M'^m \to f_*(M^m).$$

Since the fibers of f are at most smooth rational curves and M is trivial on them, also M^m is trivial on the fibers of f, so we can apply Lemma 3.4.3 to it. Since the previous morphism is an isomorphism on the geometric fibers of f and $f_*(M^m)$ is flat over S, we conclude that it is an isomorphism over S.

That $R^1 f_*(M^m) = 0$ follows directly from Lemma 3.4.3 while (3) follows from (1) and the Leray spectral sequence, which is degenerate by (2). □

3.4.2 Construction of the contraction functor

>From now on, consider $d >> 0$. Using the contraction morphism defined above, we will try to define a natural transformation from $\overline{\mathcal{P}}_{d,g,n+1}$ to $\mathcal{Z}_{d,g,n}$. Let $(\pi : X \to S, s_i : S \to X, L)$ be an $(n+1)$-pointed quasistable curve with a balanced line bundle L of relative degree d. For $i \geq 0$, define

$$\mathcal{S}_i := \pi_*(L(s_1 + \cdots + s_n)^{\otimes i})$$

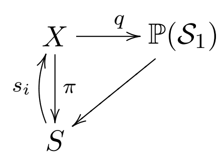
Since we are considering $d >> 0$, then, by Corollary 3.4.4, $R^1(\mathcal{S}_i) = 0$, so $\mathcal{S}_i$ is locally free of rank $h^0(L(s_1 + \cdots + s_n)^{\otimes i}) = i(d+n) - g + 1$, for $i \geq 1$, . Consider

$$\mathbb{P}(\mathcal{S}_1) \to S.$$

Again by Corollary 3.4.4, the natural map

$$\pi^*(\pi_* L(s_1 + \cdots + s_n)) \to L(s_1 + \cdots + s_n)$$

is surjective, so we get a natural S-morphism



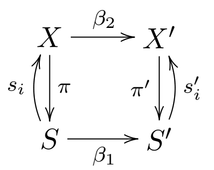
Define $Y := q(X)$, $N := \mathcal{O}_{\mathbb{P}(\mathcal{S}_1)}(1)_{|Y}$, and, by abuse of notation, call q the (surjective) S-morphism from X to Y. N is an invertible sheaf over Y and $q^*N \cong L(s_1 + \cdots + s_n)$.

Moreover, by Corollary 3.4.4 (3), we have that

$$Y \cong \mathcal{P}roj(\oplus_{i\geq 0}\mathcal{S}_i).$$

So, since all $\mathcal{S}_i$ are flat over S (again by Corollary 3.4.4), also Y is flat over S, so it is a projective curve over S of genus g (since the only possible contractions are of rational components).

So, if we endow $\pi_c : Y \to S$ with the sections $t_i := qs_i$, for $1 \leq i \leq n$, the extra section $\Delta := qs_{n+1}$ and $L^c := N(-t_1 - \cdots - t_n)$ as above, we easily conclude that $q : X \to Y$ is a contraction. Now, consider a morphism



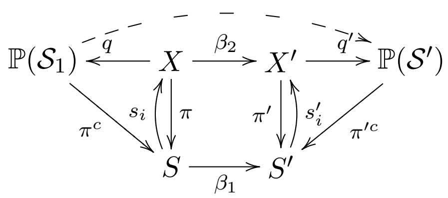
(3.9)

of $(n+1)$-pointed quasistable curves with balanced line bundles L and L' of relative degree d in $\overline{\mathcal{P}}_{d,g,n}$ and let us see that $(\beta_1, \beta_2, \beta_3)$, where β_3 is the isomorphism between L and β_2^*L', induces in a canonical way a morphism in $\mathcal{Z}_{d,g,n}$ between the contracted curves.

Define $\mathcal{S}' := \pi'_*L'(s'_1 + \cdots + s'_n)$. Recall that, to give an S'-morphism from $\mathbb{P}(\mathcal{S}_1)$ to $\mathbb{P}(\mathcal{S}')$ is equivalent to give a line bundle M on $\mathbb{P}(\mathcal{S}_1)$ and a surjection

$$(\beta_1\pi^c)^*(\pi'_*(L'(s'_1 + \cdots + s'_n))) \to M$$

where by π^c we denote the natural morphism $\mathbb{P}(\mathcal{S}_1) \to S$.



Since we are considering $d >> 0$, for $s' \in S'$, $h^0((\pi')^{-1}(s'), L'(s'_1 + \cdots + s'_n)_{|(\pi')^{-1}(s')})$ is constant and equal to $d + n - g + 1$. So, we can apply the theorem of cohomology and base change to conclude that there is a natural isomorphism

$$\beta_1^*\pi'_*L'(s_1 + \cdots + s'_n) \cong \pi_*\beta_2^*L'(s'_1 + \cdots + s'_n).$$

Now, since diagram 3.9 is cartesian and commutes un the upward direction two, the isomorphism $\beta_3 : L \to \beta_2^*L'$ induces

$$L(s_s + \cdots + s_n) \cong \beta_2^*(L'(s'_1 + \cdots + s'_n)),$$

yielding a natural isomorphism

$$\pi^{c*}\beta_1^*(\pi'_*(L'(s'_1+\cdots+s'_n)))\cong\pi^{c*}(\pi_*(L(s_1+\cdots+s_n))).$$

Composing this with the natural surjection

$$\pi^{c*}(\pi_*L(s_s+\cdots+s_n))\to\mathcal{O}_{\mathbb{P}(\mathcal{S}_1)}(1),$$

we conclude that there is a canonical surjection

$$\pi^{c*}\beta_1^*(\pi'_*(L'(s'_1+\cdots+s'_n)))\to\mathcal{O}_{\mathbb{P}(\mathcal{S}_1)}(1)$$

defining a natural S'-morphism from $\mathbb{P}(\mathcal{S}_1)\to\mathbb{P}(\mathcal{S}')$. This morphism naturally determines a morphism from X^c to X'^c, where X'^c is the image of X' in $\mathbb{P}(\mathcal{S}')$ via q', inducing a natural isomorphism between L^c and the pullback of L'^c, which is defined analogously to L^c by restricting $\mathcal{O}_{\mathbb{P}(\mathcal{S}')}(1)$ to X'^c and tensorizing with minus the sections of π'^c. The fact that all these morphisms are canonical implies that this construction is compatible with the composition of morphisms, defining a natural transformation. We have just proved the following proposition.

Proposition 3.4.6. *There is a natural transformation c from $\overline{\mathcal{P}}_{d,g,n+1}$ to $\mathcal{Z}_{d,g,n}$ given on objects by the contraction morphism defined in 3.4.1.*

3.4.3 Proof of the main Theorem

We can now prove our main Theorem.

Proof. (of Theorem 3.2.5) We must show that the contraction functor is an equivalence of categories, i. e., it is fully faithful and essentially surjective on objects. The fact that it is full is immediate. We can also conclude easily that it is faithful from the fact that a morphism of $\mathbb{P}^1$ fixing 3 distinct points is necessarily the identity. In fact, contraction morphisms induce isomorphisms on the geometric fibers away from contracted components and the contracted components have at least 3 special points and it is enough to use flatness to conclude.

In order to show that c is essentially surjective on objects we will use Knudsen's stabilization morphism (see [K83], Def. 2.3) and check that it works also for pointed quasistable curves with balanced line bundles.

So, let $\pi:X\to S$ be an pointed quasistable curve, with n sections $s_1,\dots,s_n$, an extra section Δ and a balanced line bundle L on X, of relative degree d. Let $\mathcal{I}$ be the $\mathcal{O}_X$-ideal defining Δ. Define the sheaf $\mathcal{K}$ on X via the exact sequence

$$0\to\mathcal{O}_X\xrightarrow{\delta}\mathcal{I}^{-1}\oplus\mathcal{O}_X(s_1+\cdots+s_n)\to\mathcal{K}\to 0$$

where δ is the diagonal morphism, $\delta(t)=(t,t)$.

Figure 3.5: Stabilization of pointed quasistable curves with balanced degree d line bundles.

Define

$$X^s := \mathbb{P}(\mathcal{K}).$$

and let $p : X^s \to X$ be the natural morphism from X^s to X. Theorem 2.4 of [K83] asserts that, in the case that X is a pointed stable curve, the sections $s_1 \dots, s_n$ and Δ have unique liftings $s'_1, \dots, s'_{n+1}$ to X^s making $X^s \to S$ an $(n+1)$-pointed stable curve and $p : X^s \to X$ a contraction. One checks easily that the same construction holds also if X is a quasistable pointed curve instead of a stable one. In fact, the assertion is local on S, the problem being the points where Δ meets non-smooth points of the fiber or other sections since in the other points X^s is isomorphic to X. In the case where Δ meets a non-smooth point of a geometric fiber, locally X^s is the total transform of the blow-up of X at that point with the reduced structure and s'_{n+1} is a smooth point of the exceptional component. In the case where Δ coincides with another section s_i in a geometric fiber X_s of X, then, locally, on X^s is the total transform of the blow-up of X at $s_i(s)$, again with the reduced structure, and s'_i and s'_{n+1} are two distinct smooth points of the exceptional component.

Let $L^s := p^*(L(s_1 + \cdots + s_n))(-s'_1 - \cdots - s'_n)$. Then the multidegree of $L^s(s'_1 + \cdots + s'_n)$ on a geometric fiber X^s_s coincides with the multidegree of $L(s_1 + \cdots + s_n)$ in the irreducible components of X^s_s that correspond to irreducible components of X and, in the possibly new rational components, the degree is 0. So, L^s is balanced of relative degree d.

To conclude, we must check that $c(X^s)$ is isomorphic to X. By definition, $c(X^s)$ is given by the image of X^s on $\mathbb{P}(\pi^s_*(L^s(s'_1 + \cdots + s'_n)))$. Consider the line bundle $L(s_1 + \cdots + s_n)$ on X. By Corollary 3.4.4, there is a natural surjection

$$\pi^*\pi_*(L(s_1 + \cdots + s_n)) \to L(s_1 + \cdots + s_n).$$

But, since $\pi_*(L(s_1+\cdots+s_n))$ is naturally isomorphic to $\pi^s_*p^*L(s_1+\cdots+s_n)$, we get a natural surjection

$$\pi^*(\pi^s_*L^s(s'_1 + \cdots + s'_n)) \to L(s_1 + \cdots + s_n)$$

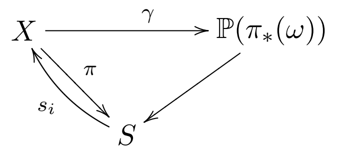
so, equivalently, a morphism f from X to $\mathbb{P}(L^s(s'_1+\cdots+s'_n))$. Since $p^*(L(s_1+\cdots + s_n)) = L^s(s'_1 + \cdots + s'_n)$ induces the natural morphism $q : X^s \to$

$\mathbb{P}(\pi^s_*(s'_1 + \cdots + s'_n))$, whose image is $c(X^s)$, naturally the image of f is $c(X^s)$. It is easy to check that f is an isomorphism on the geometric fibers, so, by flatness, we conclude that f gives an S-isomorphism between X and $c(X^s)$ as pointed quasistable curves and determines an isomorphism between the respective balanced degree d line bundles. $\square$

3.5 The forgetful morphism from $\overline{\mathcal{P}}_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$

Now, for each $n > 0$, we will try to construct a morphism $\Psi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ fitting in diagram (3.6) above.

Let $(\pi : X \to S, s_i : S \to X)$, $i = 1, \ldots, n$ be an n-pointed quasistable curve over S. Denote by ω the line bundle $(\omega_{X/S}(s_1 + \cdots + s_n))^3$. Then, by Corollary 3.4.4, $R^1(\pi_*\omega) = 0$, so it is locally free and there is an S-morphism $\gamma : X \to \mathbb{P}(\pi_*\omega)$ making the following diagram commute.



(3.10)

The restriction of γ to any fiber X_s of π maps X_s to its stable model in $\mathbb{P}(\omega)$, which is naturally endowed with the sections γs_i, for $i = 1, \ldots, n$. This follows from the fact that ω is very ample on the stable components of each fiber, whereas it has degree 0 on the exceptional components. Moreover, $\gamma(X)$ is flat over S. In fact, from Corollary 3.4.4, for any $i \geq 1$, the natural map

$$\pi_*\omega^i \otimes \pi_*\omega \longrightarrow \pi_*\omega^{i+1}$$

is surjective. It follows that $\gamma(X) \cong \mathcal{P}roj(\oplus_{i\geq 0}\pi_*(\omega^i))$, which is flat over S because each $\pi_*(\omega^i)$ is S-flat, again by Corollary 3.4.4.

Let us check that this yields a surjective morphism $\Psi_{d,g,n}$ from $\overline{\mathcal{P}}_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$ fitting in diagram (3.6) and making it commutative.

Let $(\pi : X \to S, s_i : S \to X, L)$ be an $(n+1)$-pointed quasistable curve of genus g endowed with a balanced line bundle L of relative degree d over X. It is immediate to check that, restricting ourselves to the geometric fibers of π, the diagram is commutative since in both directions we get the n-pointed curve which is the stable model of the initial one, after forgetting the last point. Now, since all families are flat over S, we conclude that the diagram is commutative.

The surjectivity of $\Psi_{d,g,n}$ follows from the fact that $\Psi_{d,g,0}$ is surjective (see [C05], Proposition 4.12) and from the commutativity of the diagram because $\Phi_{d,g,n}$ and $\Pi_{g,n}$ are the universal morphisms onto $\overline{\mathcal{P}}_{d,g,n-1}$ and $\overline{\mathcal{M}}_{g,n}$, respectively.

Moreover, the fibers of $\Psi_{d,g,n}$ over a pointed curve $X' \in \overline{\mathcal{M}}_{g,n}$ are the quasistable pointed curves X with stable model X' endowed with balanced degree d line bundles.

3.6 Further properties

Let X be an n-pointed quasistable curve over k. By applying the contraction morphism we get an $(n-1)$-pointed quasistable curve with an extra section. If we forget about this extra section and we iterate the contraction procedure n times, at the end we get a quasistable curve with no marked points, call it X_0. Denote by f this morphism from X to X_0.

Let ω_{X_0} be the dualizing sheaf of X_0. For each proper subcurve Z_0 of X_0, the degree of ω_{X_0} in Z_0 is $w_{Z_0} = 2g_{Z_0} - 2 + k_{Z_0}$. In particular, it has degree 0 on exceptional components of X_0. Consider now the pullback of ω_{X_0} via f, $f^*(\omega_{X_0})$. This is a line bundle on X having degree 0 on rational bridges and on rational tails; moreover, given a proper subcurve Z of X whose image under f is a proper subcurve Z_0 of X_0, $f^*(\omega_{X_0})$ has degree $w_{Z_0} = w_Z - t_Z$ on Z.

So, a line bundle L of degree d on X with given balanced multidegree on rational tails and rational bridges of X is balanced on X if and only if $L \otimes f^*(\omega_{X_0})$ is balanced on X of degree $d + (2g-2)$ and with the same multidegree on rational tails and rational bridges. In fact, for each proper subcurve Z of X which is not contained in rational tails or rational bridges, we have that

$$\deg_Z(L \otimes f^*(\omega_{X_0})) = \deg_Z L + w_Z - t_Z \leq$$

$$\leq \frac{dw_Z}{2g-2} + \frac{g-1-d}{2g-2}t_Z - b_Z^L + \frac{k_Z}{2} + w_Z - t_Z =$$

$$= \frac{(d+2g-2)w_Z}{2g-2} + \frac{g-1-(d+2g-2)}{2g-2}t_Z - b_Z^L + \frac{k_Z}{2}$$

and similarly that

$$\deg_Z(L \otimes f^*(\omega_{X_0})) \geq \frac{dw_Z}{2g-2} + \frac{3g-3-d}{2g-2}t_Z - b_Z^L - \frac{k_Z}{2} + w_Z - t_Z =$$

$$= \frac{(d+2g-2)w_Z}{2g-2} + \frac{3g-3-(d+2g-2)}{2g-2}t_Z - b_Z^L - \frac{k_Z}{2},$$

so $(L \otimes f^*(\omega_{X_0}))_{|Z}$ satisfies inequality (3.1) if and only if $L_{|Z}$ does.

In conclusion, we have the following result.

Proposition 3.6.1. *Let d and d' be integers such that there exists an $m \in \mathbb{Z}$ such that $d' = d + m(2g-2)$. Then, $\overline{\mathcal{P}}_{d,g,n}$ and $\overline{\mathcal{P}}_{d',g,n}$ are isomorphic.*

Proof. We must show that there is an equivalence of categories between $\overline{\mathcal{P}}_{d,g,n}$ and $\overline{\mathcal{P}}_{d',g,n}$. So, let $(\pi : X \to S, s_i : S \to X, L)$, $i = 1, \ldots, n$ be an object of $\overline{\mathcal{P}}_{d,g,n}$. Consider its image under $\Phi_{d,g,0} \circ \Phi_{d,g,1} \circ \cdots \circ \Phi_{d,g,n}$ and denote it by $(\pi_0 : X_0 \to S, L_0)$. According to 3.4.2, there is an S-morphism $q_0 : X \to X_0$. Then, by what we have seen above, $L' := L \otimes q_0^*(\omega^m_{X_0/S})$ is a balanced line bundle of relative degree d' over X, so $(\pi : X \to S, s_i : S \to X, L') \in \overline{\mathcal{P}}_{d',g,n}$.

It is easy to check that this defines an equivalence between $\overline{\mathcal{P}}_{d,g,n}$ and $\overline{\mathcal{P}}_{d',g,n}$. □

Proposition 3.6.2. *For all $n > 0$, there are forgetful morphisms $\Phi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n+1} \to \overline{\mathcal{P}}_{d,g,n}$ endowed with n sections $\sigma^1_{d,g,n}, \ldots, \sigma^n_{d,g,n}$ yielding Cartier divisors $\Delta^i_{d,g,n+1}$, $i = 1, \ldots, n$ such that $\sigma^i_{d,g,n}$ gives an isomorphism between $\overline{\mathcal{P}}_{d,g,n}$ and $\Delta^i_{d,g,n+1}$.*

Proof. The statement is true if we consider $\mathcal{Z}_{d,g,n}$ instead of $\overline{\mathcal{P}}_{d,g,n+1}$ (the sections are given by the diagonals $\delta_{i,n+1}$, for $i = 1, \ldots, n$, as we observed in section 3.2). In virtue of Theorem 3.2.5 the result follows if we define $\sigma^i_{d,g,n}$ as c^{-1} composed with $\delta_{i,n+1}$ for $i = 1 \ldots, n$. □

3.6.1 Rigidified balanced Picard stacks over quasistable curves with marked points

Analogously to the case $g \geq 2$ and $n = 0$, each object $(\pi : X \to S, s_i : S \to X, L)$, $i = 1, \ldots, n$, in $\overline{\mathcal{P}}_{d,g,n}$ has automorphisms given by scalar multiplication by an element of $\Gamma(X, \mathbb{G}_m)$ along the fibers of L leaving the curves fixed. In other words, there is an action of $B\mathbb{G}_m$ on $\overline{\mathcal{P}}_{d,g,n}$ which is invariant on the fibers of $\Psi_{d,g,n}$. So, there is no hope $\overline{\mathcal{P}}_{d,g,n}$ can be representable over $\overline{\mathcal{M}}_{g,n}$ (see [AV02], 4.4.3). Recall that the rigidification procedure, defined in [ACV01] (see section 2.4 above), fits exactly on our set up and produces an algebraic stack with those automorphisms removed.

Denote by $\overline{\mathcal{P}}_{d,g,n} /\!\!/ \mathbb{G}_m$ the rigidification of $\overline{\mathcal{P}}_{d,g,n}$ along the action of $B\mathbb{G}_m$. Exactly because the action of $B\mathbb{G}_m$ on $\overline{\mathcal{P}}_{d,g,n}$ leaves $\overline{\mathcal{M}}_{g,n}$ invariant, the morphism $\Psi_{d,g,n}$ descends to a morphism from $\overline{\mathcal{P}}_{d,g,n} /\!\!/ \mathbb{G}_m$ onto $\overline{\mathcal{M}}_{g,n}$, which we will denote again by $\Psi_{d,g,n}$, making the following diagram commutative.

$$\begin{array}{ccccc} & & \overline{\mathcal{P}}_{d,g,n} /\!\!/ \mathbb{G}_m & & \\ & \swarrow^{\Phi_{d,g,n}} & & \searrow^{\Psi_{d,g,n}} & \\ \overline{\mathcal{P}}_{d,g,n-1} /\!\!/ \mathbb{G}_m & & & & \overline{\mathcal{M}}_{g,n} \\ & \searrow_{\Psi_{d,g,n-1}} & & \swarrow_{\Pi_{g,n}} & \\ & & \overline{\mathcal{M}}_{g,n-1} & & \end{array} \tag{3.11}$$

So, the same argument we used to show that $\Psi_{d,g,n}$ is universally closed for all $n > 0$ with $2g - 2 + n > 1$ if and only if $\Psi_{d,g,0}$ is universally closed holds also in this case. Moreover, since, for $g \geq 2$ and $n = 0$, we have that $\Psi_{d,g,0} : [H_d/G] \to \overline{\mathcal{M}}_g$ is proper and strongly representable if and only if $(d - g + 1, 2g - 2) = 1$, we have that the same statement holds in general for every $n \geq 0$.

Proposition 3.6.3. *Let $g \geq 2$, $n \geq 0$ and $d \in \mathbb{Z}$. Then $\overline{\mathcal{P}}_{d,g,n} /\!\!/ \mathbb{G}_m$ is a Deligne-Mumford stack (of dimension $4g - 3 + n$) with a proper and strongly representable morphism onto $\overline{\mathcal{M}}_{g,n}$ if and only if $(d - g + 1, 2g - 2) = 1$.*

For curves of genus 0 and 1, propositions 3.2.8 and 3.2.11 immediately give the following result.

Proposition 3.6.4. *If $g = 0$ and $n \geq 3$, $\overline{\mathcal{P}}_{d,0,n} /\!\!/ \mathbb{G}_m \cong \overline{\mathcal{M}}_{0,n}$ and if $g = 1$ and $n \geq 1$, $\overline{\mathcal{P}}_{d,1,n} /\!\!/ \mathbb{G}_m \cong \overline{\mathcal{M}}_{1,n+1}$. In particular, for any integer d, $\overline{\mathcal{P}}_{d,g,n}$ is Deligne-Mumford and $\Psi_{d,g,n}$ is proper and strongly representable for $g = 0, 1$ with $2g - 2 + n > 0$.*

Remark 3.6.5. Let $d >> 0$, $g \geq 2$ and $n = 0$. Then, there is a canonical map from $\overline{\mathcal{P}}_{d,g,0} /\!\!/ \mathbb{G}_m$ to $\overline{P}_{d,g}$ (see 2.4.4 above). At least if the base field has characteristic 0, we have that $\overline{P}_{d,g}$ is a *good moduli space* for $\overline{\mathcal{P}}_{d,g,0}$ in the sense of Alper (see 2.4.4) (if $(d - g + 1, 2g - 2) = 1$ it is indeed a coarse moduli space). It would be certainly interesting to investigate if it is possible to construct good moduli spaces for $\overline{\mathcal{P}}_{d,g,n}$ in the general case, for example by investigating if our stacks are quotients stacks in general and then applying Theorem 13.6 of [A08].

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