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di

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Collisions of Fat Points

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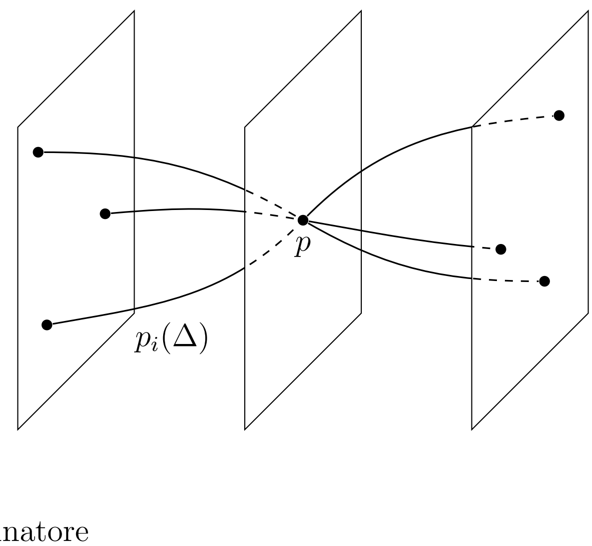

Il Coordinatore
del corso di Dottorato

Il Relatore

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Introduction

Flat limits are a fundamental tool in algebraic geometry and also one of the first examples on how schemes appear naturally from classical algebraic varieties. However such versatility also means that even questions that are easily formulated can be hard to answer. Such is the problem of computing the flat limit, for a complex parameter t approaching 0, of n fat points in $\mathbb{P}^2$ of given multiplicities $m_1, \ldots, m_n$ at $p_1(t), \ldots, p_n(t)$ when, for $t \to 0$, all the points $p_i(t)$ tend to the same point p.

A few elementary examples are known and some appear in textbooks. There are results of C. Ciliberto and R. Miranda in [6], where an answer is given under some generality hypothesis and when there are up to 5 points of the same multiplicity coming together, of L. Evain in [10], where an answer is given for up to four fat points of the same multiplicity colliding in a sequence, and of J. Roé in [14].

The collision problem is closely related to the problem of polynomial interpolation and problems of this kind have some importance in the applications (e.g. algebraic statistic). In fact, one of the main techniques used to work on polynomial interpolation consists of performing some kind of degeneration on the configuration of the points, moving them into a particularly good special position and then close with a semicontinuity argument. Finding a suitable position for this kind of argument is delicate, since it should be special enough to be treatable and at the same time general enough that the numbers involved do not "jump" on the special fiber. In this framework, the study of general collisions of fat points to a single point proposes itself as a useful tool to construct suitable degenerations. An example of an argument of this kind has recently be made by Evain in [10] to prove the Segre-Gimigliano-Harbourne-Hirshowitz conjecture (1.18) when the number of points is a square.

Our techinque consists in degenerating the linear system L_t given by the curves passing through the points $p_i(t)$ with the assigned multiplicities: in this setting the local equations of the curves in the limit system L_0 are the elements of the ideal defining the limit scheme. Since the problem is local around p, we consider $\mathbb{A}^2$ rather than the more commonly used $\mathbb{P}^2$ as our ambient space; with minimal effort, the core arguments can be adapted to any quasiprojective surface.

To find the linear system L_0 we extend the analysis of the *matching conditions*, introduced by Ciliberto and Miranda, which is based on a degeneration of the plane to the union of two surfaces V and W, where V is $\mathbb{A}^2$ blown up at the coaelscence point p and $W \cong \mathbb{P}^2$ is the exceptional divisor of the blowup at p of the total space of a trivial family of affine planes (cf. Figure 2.1). The technique presented here allows one to move the study of the limit scheme to the study of some linear system on W, which is of the same type as those appearing in the problem of polynomial interpolation.

Particular attention should be paid to any multiple fixed components that may appear in this linear system. Most efforts revolved around determining the matching conditions associated to the presence of these curves. The fact that these base curves have to be blown-up repeatedly until the linear system becomes nef brings the analysis to infinitesimal neighborhoods of higher order. This analysis was started in [6] when the fixed curves are either lines or conics in the projective plane W; this work continues that project extending the class of fixed curves whose associated matching conditions have been analyzed and applying the analysis to a wider number of cases.

Using these methods we obtain a complete classification for the general limits of up to 9 points of the same multiplicity (except for the case of 8 points, for which we only give a partial answer), for up to 4 points of any multiplicities and for several more particular cases.

In most cases the limit schemes we find are complicated; they cannot be expressed as a set of points with multiplicities in some infinitesimal neighborhoods and for this reason they require more work to be used in applications as part of a degeneration argument. There are however a handful of cases where the limit schemes are as simple as a fat point counted with some greater multiplicity; these cases are the most promising for easy application but are unforunately uncommon for reasons that will be clear. A list of those cases can be found in the last chapter.

Contents

Chapter 1

General results

Troughout this work all the schemes we will mention will be noetherian schemes over the field of the complex numbers $\mathbb{C}$.

1.1 Nonreduced schemes

We will encounter mainly two kinds of nonreduced schemes: multiple points and their limits, which we will discuss in the next section, and varieties whose irreducible components are counted with some multiplicity, intended as follows:

Notation 1.1 (Subscheme counted with multiplicity n)**.** If Z is a reduced closed subscheme of a variety X corresponding to the ideal sheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$ then, for any positive integer n, by "the subscheme X counted with multiplicity n" we mean the closed subscheme of X defined by the ideal sheaf $\mathcal{I}_Z^n \subseteq \mathcal{O}_X$ and denote it by nZ. Similarly if X is reducible by saying that one of its irreducible components X_0 has multiplicity m means that X contains mX_0 (i.e. it has it as a subscheme) but not $(m+1)X_0$.

Remark 1.2: The notion of being counted with some multiplicity *does* depend on the ambient space. For instance, a double point (say, the origin) in a complex affine line will be a scheme isomorphic to Spec $\frac{\mathbb{C}[x]}{(x^2)}$ while a double point in a complex affine plane will be isomorphic to Spec $\frac{\mathbb{C}[x,y]}{(x^2,xy,y^2)}$.

The following proposition gives us a way to compute the cohomology spaces of line bundles on hypersurfaces counted with multiplicity.

Proposition 1.3. *Let Y be an integral scheme and X an integral effective Cartier divisor in Y. Let $\mathcal{N}^*_{X|Y}$ be the conormal sheaf of X in Y. Then for any integer $n \geq 2$ there is a short exact sequence of $\mathcal{O}_{nX}$-modules*

$$0 \to \left(\mathcal{N}^*_{X|Y}\right)^{\otimes n-1} \to \mathcal{O}_{nX} \to \mathcal{O}_{(n-1)X} \to 0$$

Proof. We already know that the second morphism exists and is surjective as it comes from the restriction of nX to $(n-1)X$; we need to determine its kernel. Let $\mathcal{I}_X$ be the ideal sheaf of X; we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}_{nX} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{nX} & \longrightarrow & 0 \\ & & \downarrow & & || & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_{(n-1)X} & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{(n-1)X} & \longrightarrow & 0 \end{array}$$

where last vertical morphism is the one whose kernel we are interested in. Since the central morphism is the identity and the left one is an injection the snake lemma says that

$$\ker\left(\mathcal{O}_{nX} \to \mathcal{O}_{(n-1)X}\right) \cong \frac{\mathcal{I}_{(n-1)X}}{\mathcal{I}_{nX}}.$$

Let $U = \mathrm{Spec}A$ be any open affine subscheme of Y and let f be a local equation of X. We have an isomorphism

$$\left(\frac{(f)}{(f^2)}\right)^{\otimes n-1} \cong \frac{(f^{n-1})}{(f^n)}$$

defined by sending the elements $a_1 f \otimes \cdots \otimes a_{n-1} f$ into $a_1 \ldots a_{n-1} f^{n-1}$; the morphism is obviously surjective and it is injective because A is an integral domain and f is prime. The morphisms we get from the different open sets glue together to an isomorphism of sheaves and we fint the short exact sequence we sought. ∎

1.2 Zero-dimensional schemes

Zero-dimensional schemes are our main object of interest; it is not our intention to provide a complete survey on this subject here but merely to report a few results for ease of reference. More details can be found in many textbooks; [9] in particular provides and discusses several explicit examples of nonreduced schemes.

Definition 1.4. Let Z be a 0-dimensional scheme. The *degree* of Z, denoted by $\deg Z$, is the dimension of its ring of regular functions as a complex vector space.

Proposition 1.5. *Let $Z \subseteq \mathbb{P}^r$ be a 0-dimensional subscheme. The degree of Z, the Hilbert polynomial of Z and the number of conditions that passing through Z imposes to the hypersurfaces of large enough degree d are all equal.*

Proof. Let I_Z be the homogeneous ideal of $\mathbb{C}[X_0, \ldots, X_r]$ defining Z and let $(I_Z)_d$ be the homogeneous part of degree d. The number of conditions imposed to hypersurfaces of degree d by the passage through Z is the dimension of the quotient vector space

$$\frac{\mathbb{C}[X_0, \ldots, X_r]_d}{(I_Z)_d}$$

which is the value at d of the Hilbert function of Z. For d large enough, the values of the Hilbert function are given by the Hilbert polynomial of Z and the degree of the Hilbert polynomial is equal to the dimension of Z, which is 0.

The constant term (in this case the only term) of the Hilbert polynomial can be computed in general as $\chi(\mathcal{O}_Z)$. If R is the ring of regular functions on Z we have that $\chi(\mathcal{O}_Z) = h^0(Z, \mathcal{O}_Z) = \dim_{\mathbb{C}} R = \deg Z$. $\square$

Remark 1.6: This proposition means that the number of conditions imposed by Z is intrinsic; i.e. it does not depend on d (as long as it is large enough), on the immersion of Z in $\mathbb{P}^r$ or even on r itself.

Proposition 1.7. *Let X, Y be 0-dimensional schemes such that $X \subseteq Y$ and $\deg X = \deg Y$. Then $X = Y$.*

Proof. Let $Y = \mathrm{Spec} B$ for some $\mathbb{C}$-algebra B and let $I \subseteq B$ be the ideal defining X. We have the exact sequence of $\mathbb{C}$-vector spaces

$$0 \to I \to B \to \frac{B}{I} \to 0$$

and we know that the second and third term have the same dimension. This means that the dimension of I is 0 i.e. I is the zero ideal. Since I is the ideal defining the subscheme X we have that $X = Y$. $\square$

1.3 Degenerations

Definition 1.8. A 1-dimensional degeneration is a morphism $\pi : \mathcal{X} \to \Delta$ where Δ is a complex disk, $\mathcal{X}$ is a Cohen-Macaulay variety and π is proper and flat. For any $t \in \Delta$ we will denote the fiber of π over t by $\mathcal{X}_t$.

A 1-dimensional degeneration of varieties of dimension n is a 1-dimensional degeneration whose fibers have all dimension n i.e. $\mathcal{X}$ has dimension $n + 1$.

Definition 1.9. We will say that a 1-dimensional degeneration $\pi : \mathcal{X} \to \Delta$ with $\mathcal{X}$ smooth is *Global Normal Crossing (GNC)* if

1. the generic fiber $\mathcal{X}_t$ is smooth.
2. if $\mathcal{X}_{t_0}$ is a singular fiber, then for all $p \in \mathcal{X}_{t_0}$ there is an analytic open neighborhood U of p with local coordinates $(x_0, \ldots, x_n)$ such that $(\mathcal{X}_{t_0})_{\text{red}} \cap U$ has equation $x_0 \cdot \cdots \cdot x_k = 0$ for some $k \leq n$.
3. if $\mathcal{X}_{t_0}$ is a singular fiber then $\mathcal{X}_{t_0} = \sum_i n_i V_i$ where all the V_i are irreducible smooth Weil divisors.

Proposition 1.10 (Triple Point Formula). *Let $\pi : \mathcal{X} \to \Delta$ be a GNC 1-dimensional degeneration. Let X be a singular fiber of π and let X_1 and X_2 be two irreducible components of X_{red} that appear with multiplicities m_1 and m_2 respectively. Let $E = X_1 \cap X_2$ and let D_E be the divisor on E of the triple points of X_{red} along E. Then*

$$\mathcal{N}_{E|X_1}^{\otimes m_2} \otimes \mathcal{N}_{E|X_2}^{\otimes m_1} \otimes \mathcal{O}_E(D_E) \cong \mathcal{O}_E$$

and, if π is a degeneration of surfaces, passing to the degrees,

$$m_2(E^2)_{X_1} + m_1(E^2)_{X_2} + \deg(D_E) = 0$$

Proof. X is a Cartier divisor on $\mathcal{X}$ and it is the fiber of a map over the complex disk, so

$$\mathcal{O}_X(X) \cong \mathcal{O}_X$$

which, restricted to E, gives

$$\mathcal{O}_E(X) \cong \mathcal{O}_E.$$

Let $X = \sum_i m_i X_i$; we have that

$$\begin{aligned} \mathcal{O}_E(m_1X_1) \otimes \mathcal{O}_E(m_2X_2) \otimes \mathcal{O}_E\left(\sum_{i\geq 3} m_iX_i\right) &\cong \mathcal{O}_E \\ \mathcal{O}_E(X_1)^{\otimes m_1} \otimes \mathcal{O}_E(X_2)^{\otimes m_2} \otimes \mathcal{O}_E\left(\sum_{i\geq 3} m_iX_i\right) &\cong \mathcal{O}_E \qquad (1.1) \end{aligned}$$

We have that

$$\mathcal{N}_{E|X_1} \cong \mathcal{O}_{X_1}(E) \otimes \mathcal{O}_E \cong \mathcal{O}_{X_1}(X_2) \otimes \mathcal{O}_E \cong \mathcal{O}_E(X_2)$$

and similarly $\mathcal{O}_E(X_1) \cong \mathcal{N}_{E|X_2}$. Moreover since X_{red} is a normal crossing surface no more than two components can meet along E and any component other than X_1 and X_2 that meets E must do so transversally; since the X_i's are smooth the only triple points of X are those where three components meet, meaning that $D_E = \sum_{i\geq 3} X_i \cap E$. We can then obtain the result by substituting these three sheaves in (1.1). ☐

1.4 Polynomial interpolation

Given n points $p_1, \dots, p_n$ in $\mathbb{A}^r$ one could ask to find a polynomial of a given degree d whose value and all its higher order derivatives up to a certain order m_i at the points p_i match an assigned set of values. This problem is called the problem of *polynomial interpolation* and is still far from being solved in general.

To reduce the difficulty of the problem, people reduce the problem to asking that at each point p_i the polynomial and all its derivatives up m_i vanish. In this form the problem can be formulated for projective spaces and can be rephrased geometrically as to describe the linear system of the hypersurfaces of degree d having multiplicity at least m_i at each point p_i and, in particular, find its dimension.

In dimension one there are the Hermite interpolation formulas, which give a complete and general answer, but in higher dimensions the situation is much more complicated. The main issue is the fact that the conditions imposed to the polynomial may not be independent.

In this work we will only be concerned with the two-dimensional case. A further common reduction that we will make in the following is to assume that the points $p_1, \dots, p_n$ are in general position.

Notation 1.11. By $L_d(m_1, \dots, m_n)$ we will indicate a linear system on $\mathbb{P}^2$ of all the plane curves of degree d having multiplicities $m_1, \dots, m_n$ respectively at some assigned points $p_1, \dots, p_n$ in general position. If some multiplicities $m_i, \dots, m_{i+k}$ are all equal to some value m we might write m^k in their place.

For the scope of this work, the problem of polynomial interpolation will be to determine the dimension of $L_d(m_1, \dots, m_n)$. The *virtual dimension* of such a system is defined as

$$\operatorname{vdim} L_d(m_1, \dots, m_n) := \frac{d(d+3)}{2} - \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2}$$

that is the dimension of the linear system of the curves of degree d minus the total number of equations expressing the conditions imposed to them. If the degree d is too small this number can get very negative, so we define the *expected dimension* of the system as

$$\operatorname{expdim} L_d(m_1, \dots, m_n) := \max\{\operatorname{vdim} L_d(m_1, \dots, m_n); -1\}$$

where dimension -1 indicates that the linear system is empty.

In general

$$\dim L \geq \operatorname{expdim} L \geq \operatorname{vdim} L.$$

The first inequality is an equality when either the system is empty or all the conditions imposed are linearly independent.

Definition 1.12. A linear system $L = L_d(m_1, \dots, m_n)$ is said to be *special* when $\dim L > \operatorname{expdim} L$ and *nonspecial* when $\dim L = \operatorname{expdim} L$.

Proposition 1.13. *Let L be a linear system of type $L_d(m_1, \dots, m_n)$ and D be the 0-dimensional scheme defined by taking the points $p_1, \dots, p_n$ with multiplicities $m_1, \dots, m_n$. Let $\mathcal{L}$ be the ideal sheaf of D twisted by d, $\mathcal{I}_{D|\mathbb{P}^2}(d)$. Then* $\operatorname{vdim} L = \chi(\mathcal{L})$ *and the linear system L is special if and only if $h^1(\mathcal{L}) > 0$ and $L \neq \emptyset$.*

Proof. First note that $L = \mathbb{P}\left(H^0\left(\mathcal{I}_{D|\mathbb{P}^2}(d)\right)\right)$. Then consider the short exact sequence

$$0 \to \mathcal{L} \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_D \to 0$$

from which is immediate that

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_{\mathbb{P}^2}(d)) - \chi(\mathcal{O}_D) = \operatorname{vdim} \mathcal{L}.$$

Finally it is known that both $h^2(\mathcal{O}_{\mathbb{P}^2}(d)))$ and $h^1(\mathcal{O}_D)$ are zero, which means that $h^2(\mathcal{L})$ is zero as well and then

$$\operatorname{vdim} L = \chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) = \dim L - h^1(\mathcal{L}) \qquad \square$$

Example 1.14. Let's consider the linear system $L_2(2,2)$: the conics with two double points. The virtual dimension of this system is -1, so its expected dimension is -1 too and we expect the system to be empty. However there exists a line through the two points and this line counted with multiplicity two gives us a divisor in $L_2(2,2)$, so the actual dimension of the linear system is different from the expected one.

This example was known classically, among others. All of these examples involve some curve counted with multiplicity 2 or more, leading to the following conjecture (see [15]).

Conjecture 1.15 (Segre)**.** *If a linear system $L_d(m_1, \dots, m_n)$ is special, its general element is not reduced.*

One can consider $\pi : P \to \mathbb{P}^2$ the blowup of $\mathbb{P}^2$ at the points $p_1, \dots, p_n$. We will call E_i the divisor $\pi^{-1}(p_i)$ for each i. We have that

$$\pi^* L_d(m_1, \dots, m_n) = dH - \sum_{i=1}^{n} m_i E_i$$

which is a complete linear system. We will often abuse notation and confuse the original linear system with its pull-back and thus refer to things such as its associated line bundle or its Euler characteristic. Moreover when we compute its intersection numbers with some curve C it will be assumed that the intersection is computed on P rather than $\mathbb{P}^2$. Elsewhere it should be clear from the context whether we are referring to the original system on $\mathbb{P}^2$ or to its pull-back on P.

Definition 1.16. Let S be a surface. A (-1)-curve is a smooth rational curve C on S whose self-intersection number C^2 is -1.

Definition 1.17. Let $L = L_d(m_1, \ldots, m_n)$ be a linear system on a surface and $\pi : P \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $p_1, \ldots, p_n$. We say that L is (-1)-special if and only if there exists a (-1)-curve C on P such that $(\pi^* L).C \leq -2$

Conjecture 1.18 (Harbourne-Gimigliano-Hirshowitz)**.** *A linear system on* $\mathbb{P}^2$ *is special if and only if it is* (-1)*-special.*

This conjecture still stands, although one direction can be easily proved:

Proposition 1.19. *A* (-1)*-special linear system is special.*

Proof. Let L be the linear system and C the (-1)-curve such that $L.C = -m \leq -2$. Let L' be the linear system $|L - mC|$. Since mC is in the base locus of L the linear systems L and L' have the same dimension.

Moreover, the Euler characteristics of $\mathcal{O}(L)$ and $\mathcal{O}(L')$ are related. The short exact sequence

$$0 \to \mathcal{O}(L - C) \to \mathcal{O}(L) \to \mathcal{O}_C(L.C) \to 0$$

shows that $\chi(\mathcal{O}(L - C)) = \chi(\mathcal{O}(L) - (m - 1)$ and the same argument can be iterated m times, yielding that

$$\chi\left(\mathcal{O}(L')\right) = \chi\left(\mathcal{O}(L)\right) + \binom{m}{2}.$$

Putting it together, we have

$$\dim L = \dim L' \geq \chi\left(\mathcal{O}(L')\right) - 1 = \chi\left(\mathcal{O}(L)\right) + \binom{m}{2} - 1$$

which is greater than the virtual dimension of L, making it special. $\square$

Ciliberto and Miranda proved in [5] that the Segre conjecture and the Harbourne-Gimigliano-Hirshowitz conjecture are in fact equivalent. Conjecture 1.18 is then now the Segre-Harbourne-Gimigliano-Hirshowitz (or SHGH) conjecture.

Conjecture 1.18 has been proved to be true in a few particular cases.

- SHGH is true if the points are 9 or less [2]
- SHGH is true if the multiplicities are 11 or less [8]
- SHGH is true if the multiplicities are all equal and 42 or less [7]
- SHGH is true if the number of points is a square [10]

Moreover the homogeneous (-1)-special systems where the assigned points are 9 or less have been classified in [4]; since 1.18 is true for such linear systems that list is a comlete classification of homogeneous special systems with up to 9 assigned points. We will often refer to this list in chapter 3.

Polynomial interpolation and collisions of fat points

Degenerations are one of the most used tools in the study of polynomial interpolation. The idea is to arrange the points in a particular position, so that the dimension of limit linear system can be computed and proven to be equal to the expected dimension of the system. One can then say by semicontinuity that also the dimension of the linear system on the generic fiber cannot exceed the expected dimension, making the system nonspecial. This is the technique Evain used in [10] to prove conjecture 1.18 in the case where the number n of points is a square.

This technique requires some clever balance when choosing the degeneration because the degenerated linear system has to be both special enough that its dimension can be computed and general enough that its dimension still is the expected one. In this light the results contained in this work both use and hope to be useful tools for this study, since, while we will need to rely on the cases where 1.18 has been proved, knowing the general collision of fat points can help in devising degenerations that strike the balance between generality and computability.

Chapter 2

Matching Conditions

In this chapter we will provide the tools that we are going to use to study the collisions of fat points in a *general way*. We will make our constructions in the affine plane but since the study is local (in the analytic sense) around the limit point, what we say here can be applied to any smooth quasi-projective surface.

2.1 First degeneration of the plane

We want to take n points $p_1(t), \dots, p_n(t)$ moving in the affine plane $\mathbb{A}^2$ according to some parameter t. Let Δ be a complex disk, and let's consider the trivial family of affine planes over Δ, $\phi : A = \mathbb{A}^2 \times \Delta \to \Delta$, and let $p_1 \dots p_n$ be n sections of it. Later we will make some generality assumption on these sections but for now we will just assume that the sections are distinct and that $p_1(0) = \cdots = p_n(0) = p$.

Let $m_1, \dots, m_n$ be positive integers and let's consider the closed subscheme of A

$$\mathcal{D} := \bigcup_i m_i \mathrm{Im}(p_i).$$

For every $t \in \Delta$ the fiber $\mathcal{D}_t$ of the restriction of ϕ to $\mathcal{D}$ is a closed subscheme of the affine plane A_t. When $t \neq 0$ the fibers $\mathcal{D}_t$ consist of the points $p_i(t)$, each counted with its multiplicity m_i, while the fiber $\mathcal{D}_0$ is their flat limit.

A general surface $\mathcal{C}$ containing $\mathcal{D}$ can be thought of as a projective family of curves with parameter space Δ whose general element $\mathcal{C}_t$ passes through the points $p_1(t), \dots, p_n(t)$ with multiplicities at least $m_1, \dots, m_n$. This means that our limit scheme is the largest scheme that is contained in any curve $\mathcal{C}_0$ which is limit of curves $\mathcal{C}_t$ passing through $p_1(t), \dots, p_n(t)$ with multiplicities $m_1, \dots, m_n$; such limit scheme can be obtained as the intersection of all such

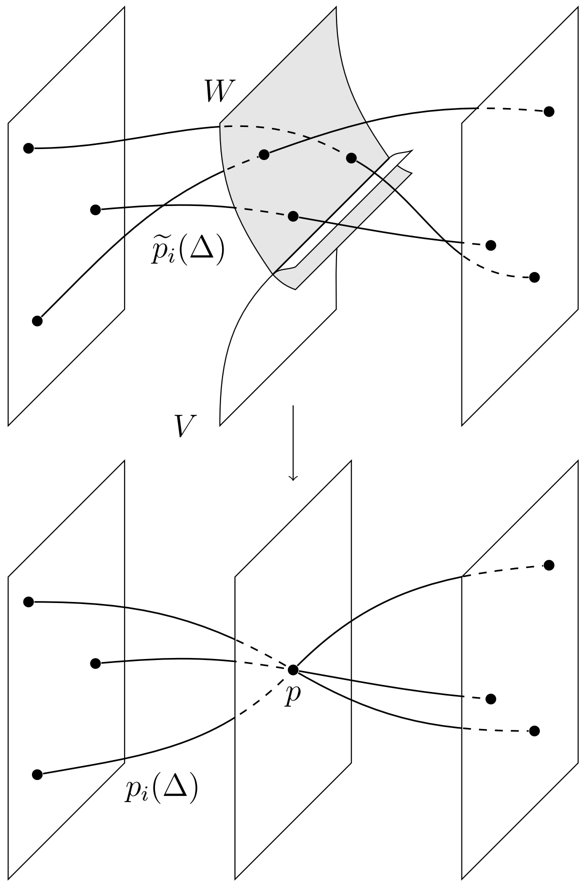


Figure 2.1: Blowing up p changes the central fiber

curves $\mathcal{C}_0$. For this reason our study of the limit scheme becomes the study of the linear system of the surfaces $\mathcal{C} \subset A$ containing $\mathcal{D}$.

Now we want to state what we mean by "n points coming together in a *general way*". Consider the blow up of A at the point p. The resulting 3-fold still has a flat morphism over Δ whose fiber over $t \neq 0$ is $\mathbb{A}^2$. The fiber over 0 has two irreducible components: a projective plane W, which is the exceptional divisor of the blowup, and a surface V, which is the original central fiber A_0 blown up at p. The intersection of these two components is a curve R which is a line on W and the exceptional divisor on V.

The sections $p_1, \ldots, p_n$ can be lifted to sections $\widetilde{p_1}, \ldots, \widetilde{p_n}$ of the blowup map. Since all the original sections pass through p, their liftings will meet the central fiber somewhere in W.

Definition 2.1. We will say that the sections $p_1, \ldots, p_n$ represent n points coming together in a general way if

1. $p_1(0) = \cdots = p_n(0) = p$
2. The points $\widetilde{p_1}(0), \ldots, \widetilde{p_n}(0)$ are in general position on W.

Remark 2.2: If the sections $p_1, \ldots, p_n$ represent n points coming together in a general way then for the generic $t \in \Delta$ the points $p_1(t), \ldots, p_n(t)$ are in general position on the affine plane A_t.

Remark 2.3: Part of the analysis we are going to make holds even if the sections are not general; in that case, however, several special cases arise that would need separate treatment.

Remark 2.4: Even if we drop the generality hypothesis on the sections, none of the points $p_1(0), \ldots, p_n(0)$ can be on the line R, which is the double locus of the fiber over 0.

The limit scheme is not, in general, just the point p counted with some multiplicity; however we can ask ourselves what the multiplicity of the limit scheme is, meaning by this the largest integer k such that kp is contained in it. To answer this question we will use the following result:

Proposition 2.5. *Let X be a smooth variety, L a complete linear system on X and Z a smooth subvariety of X. Let $\pi : \widetilde{X} \to X$ be the blowup of X along Z and E be the exceptional divisor. Then*

$$\mathrm{mult}_Z L \geq \min \left\{ j \in \mathbb{N} \ / \ h^0 \left(E, \mathcal{O}_X(\pi^{-1}L - jE)|_E \right) > 0 \right\}.$$

Proof. There is an obvious chain of inclusions $\pi_*(L-(j+1)E) \subseteq \pi_*(L-jE)$ since the first linear system contains the hypersurfaces of X whose multiplicity along Z is at least $(j+1)$ and the second one is made of those whose multiplicity along Z is at least j. These inclusions are equalities when all surfaces that have such multiplicity of at least j actually have it greater than j; this will be the case for all values of j up to the multiplicity of L along Z, which will then be the least value of j for which the above inclusion is proper; in short

$$\mathrm{mult}_Z L = \min \left\{ j \in \mathbb{N} \ / \pi_*(L-(j+1)E) \subsetneq \pi_*(L-jE) \right\}. \tag{2.1}$$

We have the short exact sequence

$$0 \to \mathcal{O}_X(L-(j+1)E) \to \mathcal{O}_X(L-jE) \to \mathcal{O}_X\left(L-jE\right) \otimes \mathcal{O}_E \to 0$$

If $h^0\left(\mathcal{O}_X\left(L-jE\right) \otimes \mathcal{O}_E\right) = 0$ we have that $H^0(\mathcal{O}_X(L-(j+1)E)) = H^0(\mathcal{O}_X(L-jE))$ and the isomorphism carries over to $\pi_*(L-(j+1)E)$ and

$\pi_*(L - jE)$, implying that they are equal. This means that the minimum in equation 2.1, and then the multiplicity, cannot be reached by values of j for which $h^0(\mathcal{O}_X(L - jE) \otimes \mathcal{O}_E) > 0$. $\square$

This proposition only provides a bound on the multiplicity rather than its exact value; however, in all the cases we have considered, this bound is later proven to be the actual value by a length computation.

The proposition only applies to complete linear systems, so we need to make our linear system of hypersurfaces containing $\mathcal{D}$ complete. To do so, we blow up the images of the $\widetilde{p_i}$'s. We call $\mathcal{X}$ the blowup and $\mathcal{E}_i$ the exceptional divisor over $\widetilde{p_i}(\Delta)$ for all $i = 1, \ldots, n$. We name the blowup maps and their compositions as in the following diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\psi''"] \arrow[drr, "\phi''"'] & Bl_{\mathcal{D}}A \arrow[r, "\psi'"] \arrow[dr, "\phi'"] & A \arrow[d, "\phi"] \\ & & \Delta \end{tikzcd}$$

The general fiber $\mathcal{X}_t$ of ϕ'' over t is now an affine plane blown up at n points and each exceptional divisor is a line of the corresponding ruled surface $\mathcal{E}_i$. The fiber $\mathcal{X}_0$ over 0 still has two irreducible components; we will denote by P the proper transform of W and keep the name V for the proper transform of V since it has not been changed by this blowup. We will denote by E_i the restrictions of $\mathcal{E}_i$ to P.

Theorem 2.6. *The multiplicity of the limit scheme is at least the minimum integer j such that the linear system $L_j(m_1, \ldots, m_n)$ on W, given by the projective plane curves of degree j having multiplicities at least $m_1, \ldots, m_n$ at $\widetilde{p_1}(0), \ldots, \widetilde{p_n}(0)$, is not empty.*

Proof. We use proposition 2.5 and just need to give an interpretation of the cohomology space that appears in its statement. The linear system we want to consider is that of the hypersurfaces of $\mathcal{X}$ containing the pull-back of $\mathcal{D}$), which is $\mathcal{O}_{\mathcal{X}}(-\sum_i m_i \mathcal{E}_i)$. Since $\mathcal{X}_t|_P \sim 0$ for all t,

$$P|_P \sim (\mathcal{X}_0 - V)|_P \sim -V|_P \sim -R.$$

Put together, we have that the sheaf appearing in proposition 2.5 is

$$\mathcal{O}_P\left(jR - \sum_i m_i E_i\right) = \psi''^* L_j(m_1, \ldots, m_n). \qquad \square$$

From now on we will call k this lower bound, L the linear system on $\mathcal{X}$ $|-\sum_i m_i\mathcal{E}_i - kP|$ and $\mathcal{L}$ the associated line bundle.

Example 2.7. We can compute in this way the limit of three single points coming together in a general way. Following the construction just described we have to consider the least k for which $h^0(\mathcal{O}_P(kR - E_1 - E_2 - E_3) \neq 0$, which means the least k such that there are curves of degree k passing through three assigned points in a general position. There are obviously no lines passing through them but $k = 2$ is already enough; then the limit has multiplicity at least 2, i.e. it contains the point p counted with multiplicity 2.

A point of multiplicity 2 in the plane is a scheme of length 3, which is the same as the length of the union of 3 single points; this means, by the proposition 1.7, that the limit scheme is exactly a double point.

Example 2.8. More generally, if we have $\frac{l(l+1)}{2}$ single points coming together in a general way, we have that $k = l$ since $\dim L_{l-1}(1^{\frac{l(l+1)}{2}}) = 0$ while $\dim L_l(1^{\frac{l(l+1)}{2}}) = l$. The length of a point of multiplicity l is exactly the length of the limit scheme so, by proposition 1.7, the limit scheme is a point of multiplicity l.

Example 2.9. We can compute the multiplicity of three points of multiplicity two coming together in a general way. Again, after following the construction just described, we have that there are no lines or conics with three double points in general position but there are such cubics since the virtual dimension of $\mathcal{L}_3(2^3)$ is greater than 0. Three double points make up a scheme of length 9 so the limit scheme must have length 9 while a point of multiplicity 4 has length 10; this means that the limit scheme contains a point of multiplicity 3 but not one of multiplicity 4 i.e. its multiplicity is 3.

The limit scheme has to be larger than just a triple point since the limit scheme has length 9 while a triple point has only length 6; we will deal with situations like this in the next section.

2.2 Matching Conditions

The surface V is isomorphic to the affine plane A (in which the limit scheme lies) blown up at the point p; the line R is the exceptional divisor of this

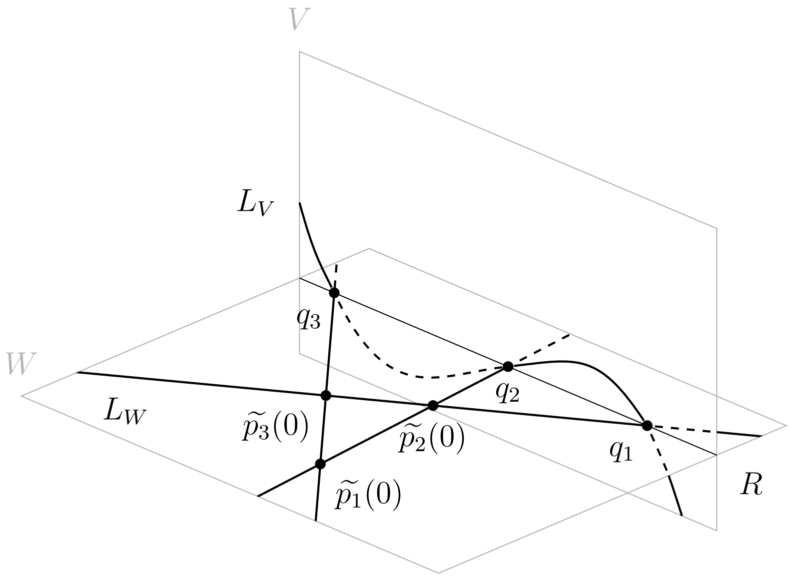


Figure 2.2: L_P consists of three lines (the picture represents L_W). All the elements of L_V must then pass through the points q_1, q_2, and q_3 where these three lines intersect R.

blowup and thus its closed points parametrize the tangent directions at p. This means that in order to look at the first infinitesimal neighborhood of the limit scheme, seen as the intersection of all curves passing through it, one has to look at the base locus of the restriction of L to V. Moreover the elements of $L|_V$ and $L|_P$ are related since they are both different restrictions of the elements of L.

In this section we assume to have a perfect knowledge of the linear system $L|_P$ and our aim is to find the conditions this implies on the elements of $L|_V$. We will call these conditions *matching conditions*.

Example 2.10. Consider again the collision of three points of multiplicity 2. We saw in Example 2.9 that the limit scheme has length 9 and its multiplicity at p is 3, meaning that the push-forward of an element of L_V not only has multiplicity 3 at p, but has to satisfy 3 more independent conditions.

The sheaf $\mathcal{L}$ is $\mathcal{O}_{\mathcal{X}}(-3P - 2\mathcal{E}_1 - 2\mathcal{E}_2 - 2\mathcal{E}_3)$; its restrictions are $\mathcal{L}_P \cong \mathcal{O}_P(3R - 2E_1 - 2E_2 - 2E_3)$ and $\mathcal{L}_V = \mathcal{O}_V(-3R)$ (in particular the linear system L_V is properly contained in $|\mathcal{L}_V|$). In the projecive plane W, let B_1 be the line through $\widetilde{p_2}(0)$ and $\widetilde{p_3}(0)$, B_2 be the line through $\widetilde{p_2}(0)$ and $\widetilde{p_3}(0)$ and B_3 be the line through $\widetilde{p_1}(0)$ and $\widetilde{p_1}(0)$ (see figure 2.2). The linear system

$|L_P|$ has only one element which is the strict transform of the three lines B_1, B_2 and B_3. Each of these three lines intersects the line R at one point; we call these intersection points q_1, q_2 and q_3.

The general element of $|\mathcal{L}_V|$ intersects R in three points; we claim that for the general element of L_V these three points have to be q_1, q_2 and q_3. This means that any curve in A_0 containing the limit scheme not only needs to have a multiplicity of at least 3 at p but, if the multiplicity is exactly 3, its three tangent lines at p need to be those corresponding to the points q_1, q_2 and q_3. This is a total of 9 linearly independent conditions, which is the length of the limit scheme. This means that the set of the equations of the curves that satisfy these conditions is the ideal defining the limit scheme.

To prove the claim, note that any element C of L_V has to be the restriction to V of some element $\mathcal{C}$ of L; this means that the restriction of C to R is the restriction of $\mathcal{C}$ to R, which can also be found by restricting $\mathcal{C}$ to P and then to R; the restriction of $\mathcal{C}$ to P has to be the unique element of $|\mathcal{L}_P|$ and then its subsequent restriction to R consists of the three points q_1, q_2 and q_3.

We saw that the fiber $\mathcal{X}_0$ has two irreducible components, P and V, meeting along a line R; we will denote by L_P, L_V and L_R the restrictions of L to them and will use a similar notation $\mathcal{L}_P$, $\mathcal{L}_V$ and $\mathcal{L}_R$ for the restrictions of the sheaf $\mathcal{L}$. While the linear system L is complete, i.e. $L = |\mathcal{L}| = \mathbb{P}\left(H^0(\mathcal{L})\right)$, this is not true in general for its restrictions.

The sheaf $\mathcal{L}_P$ is $\mathcal{L}\left(kR - \sum_i m_i E_i\right)$. We can observe that the associated linear system is the pull-back from W of the linear system of curves of degree k having multiplicity at least m_i at the point $\widetilde{p_i}(0)$ for each i. In other words, it is of the type $\mathcal{L}_k(m_1, \dots, m_n)$. Moreover, the linear system L_P is complete, as proved below.

Lemma 2.11. *The linear system L_P is complete.*

Proof. Consider the exact sequence:

$$0 \to \mathcal{L}(-P) \to \mathcal{L} \to \mathcal{L}_P \to 0$$

The system L_P is the projectivization of the image of the map $H^0(\mathcal{L}) \to H^0(\mathcal{L}_P)$. The system is then complete if the map is surjective. We can prove it by showing that $h^1(\mathcal{L}(-P)) = 0$.

The sheaf $\mathcal{L}(-P)$ is the pull-back of the ideal sheaf $\mathcal{I}$ on A of the scheme-theoretic union of $\mathcal{D}$ and $(k+1)p$; this means that $H^1(\mathcal{L}(-p)) \cong H^1(\mathcal{I}) = 0$ since A is affine. $\square$

The sheaf $\mathcal{L}_V$ is $\mathcal{O}_V(-kR)$. The corresponding linear system consists of all curves whose push-down (contracting R back to the point p) have multiplicity at least k at p. This is usually larger than L_V. See for instance the last example 2.10.

The sheaf $\mathcal{L}_R$ is $\mathcal{O}_R(k)$, as can be seen by further restricting $\mathcal{L}_P$. As is the case for L_V, the linear system L_R is not complete in general.

We have the following commutative diagram of restriction maps:

$$\begin{tikzcd} & \lvert\mathcal{L}\rvert \arrow[dl, "\rho_P"'] \arrow[dr, "\rho_V"] & \\ \lvert\mathcal{L}_P\rvert \arrow[dr, "\rho_R^P"'] & & \lvert\mathcal{L}_V\rvert \arrow[dl, "\rho_R^V"] \\ & \lvert\mathcal{L}_R\rvert & \end{tikzcd}$$

We are interested in the base locus of L_V and the diagram shows that all the elements of L_V have the property that their restriction is in L_R. L_R can be written as either $\left(\rho_R^V \circ \rho_V\right)(L)$ or $\left(\rho_R^P \circ \rho_P\right)(L)$ since the diagram is commutative; the latter expression is equivalent to $\rho_R^P(L_P)$ since ρ_P is surjective and this allows L_R to be computed from L_P.

One might hope to infer all the conditions defining the elements of L_V in $|\mathcal{L}_V|$ by looking at $|L_R|$. Unfortunately the restriction map ρ_R^V loses some relevant information and when we deal with nonreduced elements of L_R this loss is also visible on the number of conditions imposed to the elements of L_V.

Example 2.12. Consider the situations depicted in figure 2.3 where we have that L_P consists of a line B counted twice and the only divisor of L_R is the intersection point q counted twice (this situation arises when one considers the limit of two points of multiplicity 2 coming together and we will see that in that particular case the left part of the picture is the correct one). It could be that all the elements of L are singular along B (case (a)), implying that all the elements of L_V have a node at q, or it could be that the general element of L is just tangent to P along B, which would imply that the elements of L_V need to only have multiplicity 1 at q and be tangent to B there. The first possibility would impose 3 conditions on the elements of L_V while the second one only 2.

As we saw in the example above, the cases where the multiplicity of the elements of L_R at a point q is 2 or more require a deeper analysis. By Bertini's theorem, the general element of L_R has multiplicity 0 or 1 at any point q that is not a base point for L_R. The base points of L_R must also be base points for

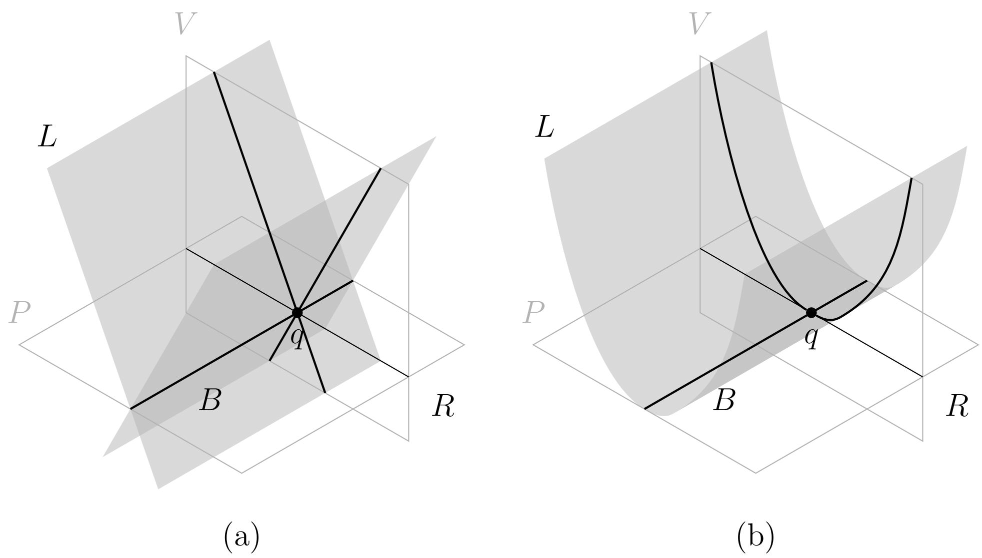


Figure 2.3: In these pictures L_P consists of the line B counted twice. The general element of L could be singular along B (a) or tangent to P along B (b)

L_P and the assumption of generality we made on the points $\widetilde{p_1}(0), \dots, \widetilde{p_n}(0)$ assures that the linear system L_P has no isolated base points on the line R. This means that when the multiplicity of the elements of L_R at a point q is 2 or more, q is in the intersection with R of some fixed component of L_P. We will address these cases in the next sections.

Let M be the movable part of L_P and M_R the movable part of L_R; what we said above means that $M_R = \rho^P_R(M)$. Since L_P was of the type $L_k(m_1, \dots, m_n)$ M will be of type $L_{\overline{k}}(\overline{m_1}, \dots, \overline{m_n})$, where $\overline{k}$ is the *residual degree* and the integers $\overline{m_1}, \dots, \overline{m_n}$ are the *residual multiplicities*, obtained by subtracting the degree and the multiplicities at $\widetilde{p_i}(0)$ of the fixed part of L_P from the corresponding integers (some of the residual multiplicities might be 0). We have the following proposition:

Proposition 2.13. *The movable part of $\rho^V_R(L_V)$ is $\rho^P_R(L_{\overline{k}}(\overline{m_1}, \dots, \overline{m_n}))$. This amounts to a total of $\overline{k}+1-h^0(\mathcal{L}_{\overline{k}}(\overline{m_1}, \dots, \overline{m_n}))$ independent conditions on the elements of L_V.*

Proof. The first part of the statement comes from what we said in this section since $\rho^V_R(L_V) = L_R$, its movable part M_R is $\rho^P_R(M)$ and M can be written as $L_{\overline{k}}(\overline{m_1}, \dots, \overline{m_n})$.

The number of independent conditions that are imposed to elements of $|\mathcal{L}_V|$ by the fact of having their image via ρ_R^V lie in M_R is equal to the dimension of the quotient vector space $\frac{|\mathcal{O}_R(\bar{k})|}{M_R}$ which is $\bar{k}+1-(\dim M_R+1)$. Finally, we have the exact sequence

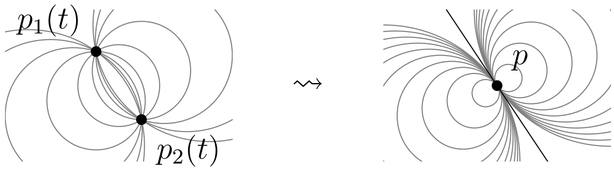
$$H^0(\mathcal{O}_P(M-R)) \longrightarrow H^0(\mathcal{O}_P(M)) \longrightarrow H^0(\mathcal{O}_R(M)).$$

The sheaf $\mathcal{O}_P(M-R)$ has no global sections because of the minimality of k, which means that the second map is injective. This means that the map ρ_R^P is also injective and then $\rho_R^P(M)$, which is M_R, has the same dimension as M, completing the proof. □

When there are no fixed components in the linear system L_P proposition 2.13 is enough to identify the limit scheme. Unfortunately this is not going to be true in general: the linear system $\pi_* L_P$ is the linear system of curves of minimal degree that pass through n ponts with assigned multiplicites; this system will often be special and conjecture 1.18 would require it to have at least one (-1)-curve as a multiple fixed component. Since conjecture 1.18 has been already proved in several cases, we know that fixed (-1)-curves are going to appear quite often.

Example 2.14. Let's compute in this way the limit of two points coming together. The minimum k for which we can find a curve passing through two points is 1. L_P has only one element: the line through $\widetilde{p_1}(0)$ and $\widetilde{p_2}(0)$, which intersects R at a point q. The exact location of q is determined by where the points $\widetilde{p_1}(0)$ and $\widetilde{p_2}(0)$ lie on W, which in turn is determined by the first order behavior around 0 of the two originally given sections p_1 and p_2.

The matching condition is that any curve in L_V has to pass through q. The points of R represent the tangent directions at p in A_0 so this means that any curve C containing the limit scheme contains this direction in its tangent space at p.



Our candidate limit scheme is then the scheme of length 2 supported at p having tangent space generated by that direction. This scheme is contained in the limit scheme and has length 2, which is the same as the limit scheme, so we can use proposition 1.7 to conclude that this is the limit scheme.

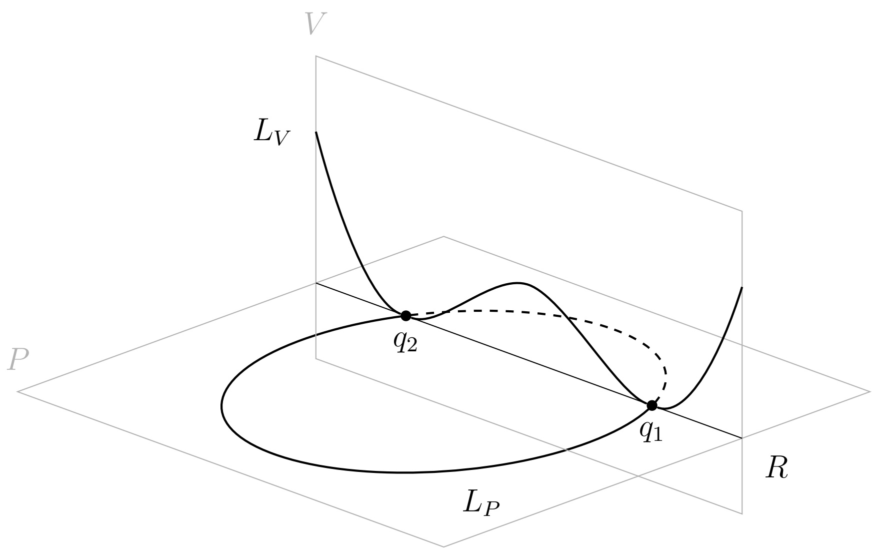


Figure 2.4: The presence in L_P of the fixed double conic B implies that the curves in L_V pass through q_1 and q_2 and are tangent to R there. In this case a general surface L is tangent to P along B.

Example 2.15. Consider the limit of 4 points coming together. The lowest degree for a curve on the projective plane W passing through the 4 assigned points is 2, so the limit contains p with multiplicity two.

The linear system L_P is a pencil of conics passing through the 4 points; its restriction to R is then a linear system of degree 2 and dimension 1, which is not complete. The general element of $|\mathcal{L}_V|$ meets R in two points but the elements of L_V need to cut on R an element of L_R; since $|\mathcal{O}_R(2)|$ has dimension 2 while L_R has dimension 1 we cannot pick the two points in any way we want: we can pick one of them arbitrarily but then the other point is fixed. Looking at the situation on W, this chosen point together with the $\widetilde{p_i}(0)$s will determine a conic in W and then the other point has to be the other intersection of such conic with R. There is exactly one matching condition because the codimension of L_R in $|\mathcal{L}_R|$ is one. The exact expression of this condition depends on the positions of the four points on W, which are in turn determined by the first order behavior around 0 of the sections $p_1, \ldots, p_4$ in A.

The candidate limit scheme is then a double point (3 conditions) together with this one further condition on the two tangent directions at this double point; this scheme has length four which, again, is the degree of the limit scheme which then has to coincide with the candidate limit.

2.3 Elimination of the fixed (-1)-curves

Let B be a (smooth, irreducible) curve on P that appears in L_P with multiplicity σ. We assume that B meets R transversally at τ points $q_1^1, \ldots, q_\tau^1$. As we saw in the previous section, the presence of such curves with $\sigma > 1$ poses a problem since then the restriction to R does not keep enough information. In this and the next section we will deal with such curves when they are (-1)-curves (i.e. rational curves such that $(B^2)_P = -1$) or curves of genus 1 and such that $(B^2)_P = 0$. This limitation is motivated by the fact that, if conjecture 1.18 is true, these are the only cases that will arise for general collisions.

Notation 2.16. In this and later sections we will make several blowups along curves. We will not introduce a different notation for the proper transform of something that is not changed by the blowup, such as the surfaces that contain the blown-up curves and the curves and surfaces that do not intersect them.

2.3.1 Fixed infinitely near points

We will look at the higher-order ifinitesimal nieghborhoods of the points q_i^1. Roughly speaking, we will proceed as follows:

1. Blow up B.

2. Called T the exceptional divisor of the blowup, find the minimum α such that the total transform of L minus αT is effective. Then the system L has multiplicity at least α at the points q_i^1.

3. If the intersection of P and T still has negative intersection with L call that B_1, define the points q_i^2 to be its intersection with the proper transform of V and repeat.

The results of the analysis differ depending on the genus of B; in this work we treat the genus 0 and genus 1 cases.

We will have go through the cycle sketched above a number of times. Let h and e be integers such that $\sigma = \tau h - e$ and $0 \leq e < \tau$.

First step: Let $\pi^{(1)} : \mathcal{X}^{(1)} \to \mathcal{X}$ be the blow up of $\mathcal{X}$ along B. The exceptional divisor of $\pi^{(1)}$ is a rational ruled surface T_1 that intersects P along B transversally; we will call B_1 the curve B on T_1 and F_1 a fiber. We call $V^{(1)}$ the proper transform of V; $V^{(1)}$ is isomorphic to V blown up at

$q_1^1, \ldots, q_\tau^1$. The triple point formula (proposition 1.10) applied to B_1 tells us that $(B_1^2)_{T_1} = 1 - \tau$, meaning that $T_1 \cong \mathbb{F}_{\tau-1}$.

We use proposition 2.5 to have a lower bound on the multiplicity of L along B. We have to find the minimum integer j such that $h^0(T_1, \mathcal{O}(\pi^{(1)*}(L) - jT_1)) > 0$.

The restriction of T_1 to itself is $T_1|_{T_1} \sim -B_1 - \tau F_1$ and the restriction of $\pi^{(1)*}(L)$ to T_1 is

$$\pi^{(1)*}(L)|_{T_1} \sim \pi^{(1)*}(L)|_{\pi^{(1)*}(B)} \sim \pi^{(1)*}(L|_B) \sim -\sigma F_1$$

We can then write

$$\pi^{(1)*}(L) - jT_1 \sim -\sigma F_1 + j(B_1 + \tau F_1) = jB_1 + (\tau(j-h) + e)F_1.$$

This is effective if and only if $j \geq h$, which means that L has multiplicity at least h along B and, in particular, its restriction $L|_V$ has multiplicity at least h at the points $q_1^1, \ldots, q_\tau^1$.

From now on we will call $L^{(1)}$ the linear system $\pi^{(1)*}(L) - hT_1$.

Finally, to tell wether we need to take further steps we compute the the intersection number

$$(L^{(1)}|_{T_1}.B_1)_{T_1} = hB_1^2 + eB_1.F_1 = -h(\tau - 1) + e.$$

If $\tau = 1$ this number is nonnegative for any possible value of h and e, meaning that $L^{(1)}$ does not have B in its base locus, and we are done. If $\tau \geq 2$ we repeat the process.

Remark 2.17: If $\tau \geq 2$ but $h = 1$ and $e = \tau - 1$ the intersection number is also 0. It is however convenient to continue with the process until the intersection number is nonnegative for any value of h, in order to keep the case $h = 1$ into the general case rather than to give it separate treatment.

***i*-th step:** In the previous steps we constructed a sequence of blowups

$$\mathcal{X}^{(i-1)} \xrightarrow{\pi^{(i-1)}} \cdots \xrightarrow{\pi^{(1)}} \mathcal{X}$$

$$\begin{tikzcd} \mathcal{X}^{(n-1)} \arrow[r, "\pi^{(i-1)}"] & \cdots \arrow[r, "\pi^{(1)}"] & \mathcal{X} \arrow[d, "\psi''"] \\ & & \Delta \end{tikzcd}$$

For each $j < i$, T_j is the exceptional divisor of $\pi^{(j)}$ and it is isomorphic to the Hirzebruch surface $\mathbb{F}_{\tau-i}$; B_j is its base curve, F_j is a fiber and B_{i-1} coincides with B. $V^{(j)}$ is the proper transform of V via $\pi^{(j)} \circ \cdots \circ \pi^{(1)}$, $L^{(j)}$ is the linear system obtained by subtracting from the transform of L a suitable number of copies of the exceptional divisors and $\mathcal{L}^{(j)}$ is the associated invertible sheaf. The fiber over $t = 0$ of the composition of all these blowups with ϕ'' is

$$\mathcal{X}_0^{(i-1)} = P + \sum_{l=1}^{i-1} lT_l + V^{(i-1)}.$$

Moreover in step $(i-1)$ we computed the intersection number

$$L^{(i-1)}.B_{i-1} = \begin{cases} -h(\tau - (i-1)) + e & \text{if } i-1 < \tau - e \\ -(h-1)(\tau - (i-1)) & \text{if } i-1 \geq \tau - e \end{cases}$$

Note that if $\tau = e$ the two expressions coincide and at the end of step 1 we came out with an expression that matches the top one.

Let $q_1^i, \ldots, q_\tau^i$ be the points that B cuts on $V^{(i-1)}$. Let $\pi^{(i)} : \mathcal{X}^{(i)} \to \mathcal{X}^{(i-1)}$ be the blow up of $\mathcal{X}^{(i-1)}$ along B. The exceptional divisor of $\pi^{(i)}$ is a rational ruled surface T_i that intersects transversally P in B and T_{i-1} in B_{i-1}; we will call B_i its base curve, which coincides with B, and F_i a fiber. We call $V^{(i)}$ the proper transform of $V^{(i-1)}$; $V^{(i)}$ is isomorphic to $V^{(i-1)}$ blown up at $q_1^i, \ldots, q_\tau^i$. The triple point formula applied to B_i tells us that $(B_i^2)_{T_i} = i - \tau$, meaning that $T_i \cong \mathbb{F}_{\tau-i}$.

Again, we want to use proposition 2.5 to have a bound on the multiplicity of $L^{(i-1)}$ along B. We have to find the minimum j such that $h^0(T_i, \mathcal{O}(\pi^{(i)*}L^{(i-1)} - jT_i)) > 0$.

The restriction of T_i to itself is $\frac{1}{i}(-B_i - \tau F_i - (i-1)B_{i-1})$. The equivalence class of B_{i-1} can be computed: $B_{i-1} \sim aB_i + bF_i$ for some integers a and b; the fact that $B_{i-1}.F_i = 1$ implies that $a = 1$ and then $B_{i-1}.B_i = 0$ implies $b = \tau - i$. Then $B_{i-1} \sim B_i + (\tau - i)F_i$ on T_i and

$$T_i|_{T_i} \sim -(B_i + (\tau - i + 1)F_i).$$

The restriction of $\pi^{(i)*}(L^{(i-1)})$ to T_i is

$$\begin{aligned} \pi^{(i)*}\left(L^{(i-1)}\right)|_{T_i} &= \pi^{(i)*}\left(L^{(i-1)}\right)|_{\pi^{(i-1)*}(B_{i-1})} \\ &= \pi^{(i-1)*}\left(L^{(i-1)}|_{B_{i-1}}\right) \\ &\sim \left(L^{(i-1)}.B_{i-1}\right) F_i \end{aligned}$$

Now we need to separate two cases depending on the expression for $L^{(i-1)}.B_{i-1}$. If $i < \tau - e + 1$

$$\begin{aligned} L^{(i)}{}_l|_{T_i} &\sim \left(L^{(i-1)}.B_{i-1}\right) F_i + lB_i + l(\tau - i + 1)F_i \\ &= -h(\tau - i + 1)F_i + eF_i + lB_i + l(\tau - i + 1)F_i \\ &= lB_i + (\tau - i + 1)(l - h)F_i + eF_i \end{aligned}$$

since in this case $e < \tau - i + 1$ this is effective when $l \geq h$, meaning that L^{i-1} has multiplicity at least h along B and then that $L^{i-1}|_{V^{(i-1)}}$ has multiplicity at least h at $q_1^1, \ldots, q_\tau^i$. From now on we will call $L^{(i)}$ the linear system $L^{(i)}{}_h$.

Let's compute the intersection number

$$L^{(i)}|_{T_i}.B_i = hB_i^2 + eB_i.F_i = -h(\tau - i) + e$$

If $i = \tau$ this number is nonnegative for all possible values of h and e; this means that B is not in the base locus of L^i and we are done. If $i < \tau$ this is not the case and we need to repeat the process again. In this case note that the expression we found for the intersection number matches the one we stated earlier: if $i < \tau - e$ this is immediate and if $i = \tau - e$ the intersection number can be also written as $-(h-1)(\tau - i)$.

if $i \geq \tau - e + 1$

$$\begin{aligned} L^{(i)}{}_l|_{T_i} &\sim \left(L^{(i-1)}.B_{i-1}\right) F_i + lB_i + l(\tau - i + 1)F_i \\ &= -(h-1)(\tau - i + 1)F_i + lB_i + l(\tau - i + 1)F_i \\ &= (\tau - i + 1)(l - h + 1)F_i + lB_i \end{aligned}$$

which is effective when $l \geq h - 1$, meaning that L^{i-1} has multiplicity at least $h - 1$ along B and then that $L^{i-1}|_{V^{(i-1)}}$ has multiplicity at least $h - 1$ at $q_1^i, \ldots, q_\tau^i$. From now on we will call $L^{(i)}$ the linear system $L^{(i)}{}_{h-1}$.

Let's compute the intersection number

$$L^{(i)}|_{T_i}.B_i = (h-1)B_i^2 = -(h-1)(\tau - i)$$

If $i = \tau$ this number is nonnegative for all possible values of h and e; this means that B is not in the base locus of L^i and we are done. If $i < \tau$ this is not the case and we need to repeat the process again; in this case note that the expression we found for the intersection number matches the one we stated earlier. It cannot happen that $i > \tau$ since we saw that we stop iterating process when $i = \tau$.

Remark 2.18: If $h = 1$ the curve B ceases to be contained in the base locus of $L^{(i-1)}$ for $i > \tau - e$. Again, it is more convenient to make some unnecessary steps until the intersection number is nonnegative for any h rather than to deal with the $h = 1$ case separatedly. Consistently, we find that in such cases the bound on the multiplicity at the points $q_1^i, \ldots, q_\tau^i$ is 0.

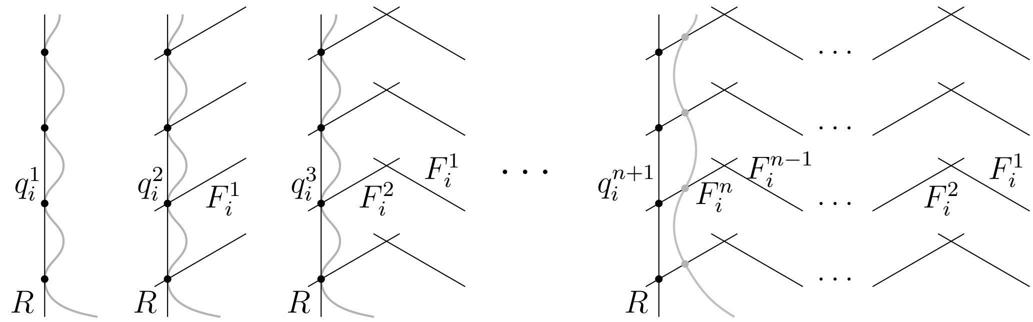


Figure 2.5: The curve R and its total transforms after 0, 1, 2 and n blowups. The gray curves are (the proper transforms of) a curve that contains the τ-ple of points $(q_1^1, \ldots, q_\tau^1)$ of type (1^n): it passes once through all the points q_i^j for $j \leq n$.

Sets of points of type $(\alpha_1, \ldots, \alpha_n)$

During the process described above we found many conditions on the elements of L. These conditions translate into more conditions for the surfaces containing $\mathcal{D}$ and therefore into a larger candidate limit scheme. Here we find a 0-dimensional subscheme of A such that the condition of containing this subscheme is equivalent to the conditions we have found so far. This form will also have the benefit of being intrinsic to the affine plane A rather than requiring to reference to the whole degeneration.

Let S be a surface, R a curve on S and $q_1^1, \ldots, q_\tau^1$ be τ distinct points on R (cf. Figure 2.5). Let $\pi_1 : S^{(1)} \to S$ be the blow up of S at $q_1^1, \ldots, q_\tau^1$. This creates an exceptional divisors consisting of τ components which we will indicate as F_i^1, for $1 \leq i \leq \tau$, labeled so that $\pi_1(F_i^1) = q_i^1$. Let $q_1^2, \ldots, q_\tau^2$ be the intersections of $F_1^1, \ldots, F_\tau^1$ with (the proper transform of) R.

Repeat this n times: the j-th repetition consists of making the blow up $\pi_j : S^{(j)} \to S^{(j-1)}$ of $S^{(j-1)}$ at $q_1^j, \ldots, q_\tau^j$, defining $F_1^j, \ldots, F_\tau^j$, as the new exceptional divisors and $q_1^{j+1}, \ldots, q_\tau^{j+1}$ as their intersection with R. Let Π_j be the composition map $\pi_j \circ \cdots \circ \pi_1$.

Finally, once we completed all the steps up to $S^{(n)}$, we will denote by $\overline{F}_i^j$ the total transform of F_i^j on $S^{(n)}$; $\overline{F}_i^j = \sum_{l=j}^n F_i^l$.

Definition 2.19. Under the construction just exposed, we define the set of points $q_1^1, \ldots, q_\tau^1$ of type $(\alpha_1, \ldots \alpha_n)$ along R to be the subscheme of S defined by the ideal sheaf

$$\Pi_{n*}\mathcal{O}_{S^{(n)}}\left(\sum_{i=1}^{\tau}\sum_{j=1}^{n}-\alpha_j\overline{F}_i^j\right)$$

If in the type there appears a string of equal consecutive values $\alpha_j = \cdots = \alpha_{j+k} = m$, we might instead write m^k.

Definition 2.20. Let S be a surface, $p \in S$ a smooth point, and X a closed subscheme of S supported at p. Let $\pi : S' \to S$ be the blow up of S at p. By adding τ infinitely near points of type $(\alpha_1, ...\alpha_n)$ in $q_1^1, \ldots, q_\tau^1$ to the scheme X we mean to take the scheme

$$\pi\left(\pi^{-1}(X) \cup N\right)$$

where N is the subscheme of S' given by the points $q_1^1, \ldots, q_\tau^1$ of type $(\alpha_1, ...\alpha_n)$ along the exceptional divisor of π.

Example 2.21. The image of an immersion of $\mathrm{Spec}\frac{\mathbb{C}[\epsilon]}{(\epsilon^n)}$ in S is the set of a single point of type (1^n) or a point of multiplicity one together with one infinitely near point of type (1^{n-1}).

We can then express the results of this section as the following proposition:

Proposition 2.22. *Let $p_1(t), \ldots, p_n(t)$ be n fat points of multiplicities $m_1, \ldots, m_n$ coming together at a point p and make the construction exposed in proposition 2.1. Let k be the minimum integer such that $h^0\left(\mathcal{L}_k(m_1, \ldots, m_n)\right) > 0$ and L be the linear system on P associated to the sheaf $\mathcal{L}_k(m_1, \ldots, m_n)$.*

If there are s rational curves $B_1, \ldots, B_s \subset P$ such that $B_i^2 = -1$, $B_i.L = -\sigma_i < 0$ and B_i meets V transversally at τ_i points $q_{i,1}, \ldots, q_{i,\tau_i}$ let h_i and e_i be integers such that $\sigma_i = \tau_i h_i - e_i$ and $0 \leq e_i < \tau_i$. Then the limit scheme contains the point p with multiplicity k together with s sets of τ_i infinitely near points of type $({h_i}^{\tau_i - e_i}, (h_i-1)^{e_i})$ at $q_{i,1}, \ldots, q_{i,\tau_i}$.

Example 2.23. Let's consider the case of five points of multiplicity m coming together. Those colliding points translate into five points of the same multiplicity on P. The minimal degree k of a curve having multiplicity m at all of them is $2m$ and the linear system on W has only one element: the conic B passing through those five points counted with multiplicity m (see figure 2.4). B is a rational curve and $B^2 = -1$, so we can apply the process we just described.

In this situation $\sigma = -(mB).B = m$, $\tau = 2$ and we can write $m = 2h - e$ and define h and e. We need to blow up B twice; the first time we create an exceptional divisor $T_1 \cong \mathbb{F}_1$ and the second time we get $T_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$.

After the first blow up there are two fixed points for the system L_V: the two points where B intersects R. Blowing up B results in blowing up V in both points and we know that this time we had to remove the exceptional divisor h times, implying that there were two fixed points of multiplicity h in the first infinitesimal neighborhood of p. Since the curve B on P is still a fixed component for $L_P^{(1)}$, with the same argument each of those two points has another fixed point of multiplicity $\alpha_2 = h - e$ in its infinitesimal neighborhood.

The candidate limit so far is a point of multiplicity $2m$ with two infinitely near points of type $(h, h-e)$. However, even for $m = 2$, the length of this candidate is too low to be the limit scheme, meaning that there are more conditions to be found.

2.3.2 Further matching conditions

The linear system on the tower

In the previous section we created by blowup several ruled surfaces; we will call the exceptional divisor of the composition of these blowups the *tower over B*. In this section we want to find expressions for the restrictions of the linear system $\mathcal{L}^{(\tau)}$ to the various components of the tower and to the surface P.

Lemma 2.24. *The following are the restrictions of T_j to iT_i for all suitable values of i and j:*

1. *if $|i-j| \geq 2$, $T_j|_{iT_i} \sim 0$*
2. *$T_{i+1}|_{iT_i} \sim B_i$*
3. *$T_{i-1}|_{iT_i} \sim B_i + (\tau - i)F_i$*
4. *$T_\tau|_{\tau T_\tau} \sim -B_\tau - F_\tau$*
5. *if $j = i$ and $1 \leq i < \tau$, $T_i|_{iT_i} \sim -2B_i - (\tau - i + 1)F_i$*

Proof. If $|i-j| \geq 2$, T_i and T_j are disjoint; hence $T_j|_{iT_i} \sim 0$. The second and third cases are also immediate from the construction, remembering that we found that, on T_i, $B_{i-1} \sim B_i + (\tau - i)F_i$.

The other two cases can be computed using the fact that the iT_is are irreducible components of a fiber of a degeneration over Δ. This means that we can write their normal class as the linear equivalence class of their intersections with the other components of the fiber.

The surface T_τ intersects $T_{\tau-1}$ and $V^{(\tau)}$, so

$$\begin{aligned} T_\tau|_{\tau T_\tau} &\sim -\frac{1}{\tau}((\tau-1)T_{\tau-1}|_{\tau T_\tau} + V^{(\tau)}|_{\tau T_\tau} + P|_{\tau T_\tau}) \\ &\sim -\frac{1}{\tau}((\tau-1)B_\tau + \tau F_\tau + B_\tau) = -B_\tau - F_\tau. \end{aligned}$$

If $\tau = 1$ the surface $T_{\tau-1}$ does not exist. Coherently, in this expression it appears with multiplicity $\tau - 1$, so the same formula holds even if $\tau = 1$.

The surface T_1 intersects T_2 and $V^{(\tau)}$, so $T_1|_{T_1} \sim -(2T_2)|_{T_1} - V^{(\tau)}|_{T_1} \sim -2B_1 - \tau F_1$.

If $j = i$ and $1 < i < \tau$, T_i intersects T_{i-1}, T_{i+1} and $V^{(\tau)}$.

$$\begin{aligned} T_i|_{iT_i} &\sim -\frac{1}{i}((i-1)T_{i-1}|_{iT_i} + (i+1)T_{i+1}|_{iT_i} + V^{(\tau)}|_{iT_i}) \\ &\sim -\frac{1}{i}((i-1)B_i + (i-1)(\tau-i)F_i + (i+1)B_i + \tau F_i) \\ &= -2B_i - (\tau - i + 1)F_i \end{aligned}$$

□

Proposition 2.25. *The linear system $L^{(\tau)}$ has the following restrictions (cf. Figure 2.6):*

1. $L^{(\tau)}|_P \sim L_P - \sigma B$.
2. *if* $e \neq 0$
$$L^{(\tau)}|_{iT_i} \sim \begin{cases} eF_i & \text{for } i < \tau - e \\ B_i + eF_i & \text{for } i = \tau - e \\ 0 & \text{for } \tau - e < i < \tau \\ (h-1)B_i & \text{for } i = \tau \end{cases}.$$
3. *if* $e = 0$
$$L^{(\tau)}|_{iT_i} \sim \begin{cases} 0 & \text{for } i < \tau \\ hB_i & \text{for } i = \tau. \end{cases}.$$

Proof. Let α_l be the number of times we subtracted T_l from the linear system during the l-th step of the construction of the tower over B. The linear system we have at the end of the construction is

$$L^{(\tau)} \sim \pi^*(L) - \sum_{j=1}^{\tau} \left(\sum_{l=1}^{j} \alpha_l \right) T_j.$$

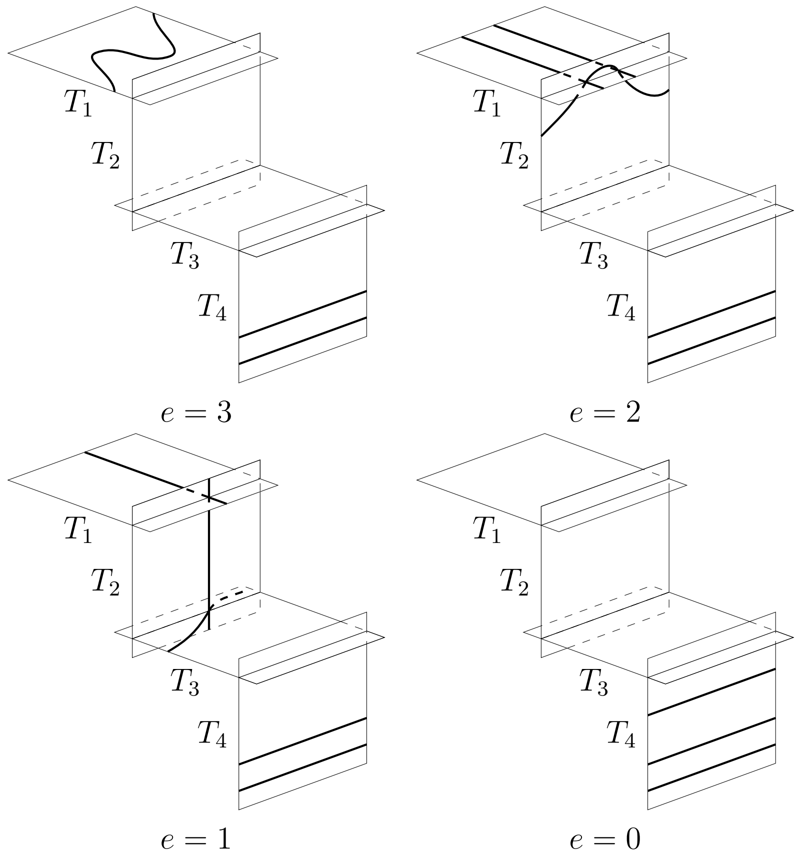


Figure 2.6: The tower of blowups and the linear system $L^{(\tau)}|_T$ for $\tau = 4$, $h = 3$ and the four possible values for e

We found that $\alpha_l = h$ for $1 \leq l \leq \tau - e$ and $\alpha_l = h - 1$ for $\tau - e < l \leq \tau$. Putting in these values we obtain:

$$\begin{aligned} L^{(\tau)} &\sim \pi^{-1}(L) - \sum_{j=1}^{\tau-e} jhT_j - \sum_{j=\tau-e+1}^{\tau} (jh - (j - (\tau - e)))T_j \\ &= \pi^{-1}(L) - \sum_{j=1}^{\tau} jhT_i + \sum_{j=\tau-e+1}^{\tau} (j - (\tau - e))T_j. \end{aligned} \tag{2.2}$$

First, consider the restriction to P. The only surface of the T_is that intersects P is T_τ, then we can write

$$L^{(\tau)}|_P \sim L_P - \tau hT_\tau|_P + eT_\tau|_P = L_P - (\tau h - e)B = L_P - \sigma B.$$

If $\tau = 1$ the rest of the proof is immediate since then the (2.2) reduces to

$$L^{(1)}|_{T_1} \sim \pi^{-1}(L)|_{T_1} - \sigma T_1|_{T_1} \sim -\sigma F_1 + \sigma B_1 + \sigma F_1 = \sigma B_1 = hB_1.$$

In what follows we will then suppose $\tau \geq 2$.

To compute the restrictions to the T_is we will first consider $\pi^{-1}(L)$ and then the two sums appearing in (2.2) one at a time.

It is fairly easy to compute $L^{(\tau)}|_{iT_i}$. If $T = \sum_i iT_i$ and F is the fiber of π over a point of B, $F = \sum_i iF_i$, we have that

$$\pi^{-1}(L)|_T = \pi^{-1}(L)|_{\pi^{-1}(B)} = \pi^{-1}(L|_B) = ((L.B)_X)\, F = -\sigma F$$

and the further restriction to iT_i is $-\sigma F_i$.

Now let's subtract from it the first of the two sums appearing in (2.2). We need to consider the cases $i = 1$ and $i = \tau$ separatedly.
If $i = 1$

$$\begin{aligned} & \pi^{-1}(L)|_{T_1} - hT_1|_{T_1} - 2hT_2|_{T_1} \\ \sim\ & -\tau hF_1 + eF_1 + 2hB_1 + \tau hF_1 - 2hB_1 \\ =\ & eF_1 \end{aligned}$$

If $1 < i < \tau$

$$\begin{aligned} & \pi^{-1}(L)|_{iT_i} - (i-1)hT_{i-1}|_{iT_i} - hiT_i|_{iT_i} - (i+1)hT_{i+1}|_{iT_i} \\ \sim\ & -\tau hF_1 + eF_i - h(i-1)B_i - h(i-1)(\tau - i)F_i + \\ & +2hiB_i + hi(\tau - i + 1)F_i - h(i+1)B_i) \\ =\ & eF_i \end{aligned}$$

If $i = \tau$

$$\begin{aligned} & \pi^{-1}(L)|_{\tau T_\tau} - (\tau - 1)hT_{\tau-1}|_{\tau T_\tau} - \tau hT_\tau|_{\tau T_\tau} \\ \sim\ & -\tau hF_\tau + eF_\tau - (\tau - 1)hB_\tau + \tau hB_\tau + \tau hF_\tau \\ =\ & hB_\tau + eF_\tau \end{aligned}$$

If $e = 0$ the second sum appearing in (2.2) is empty and the proof is complete. If $e \geq 1$ we go on and add the second sum.

If $i < \tau - e$ the second sum is empty ad the proof is complete.

If $i = \tau - e$ only the first term of the second sum is relevant and it is $T_{i+1}|_{T_i}$, which is linearly equivalent to B_i. Adding it to what we already computed we have

$$L^{(\tau)}|_{T_{\tau-e}} \sim B_{\tau-e} + eF_{\tau-e}.$$

If $\tau - e + 1 < i < \tau$ the sum contains three relevant terms: those for $j = i - 1$, $j = i$ and $j = i + 1$.

$$\begin{aligned} & (i - 1 - \tau + e)T_{i-1}|_{T_i} + (i - \tau + e)T_i|_{T_i} + (i + 1 - \tau + e)T_{i+1}|_{T_i} \\ =\ & (i - 1 - \tau + e)B_i + (i - 1 - \tau + e)(\tau - i)F_i - 2(i - \tau + e)B_i \\ & -(i - \tau + e)(\tau - i + 1)F_i + (i + 1 - \tau + e)B_i \\ =\ & -eF_i \end{aligned}$$

which gives

$$L^{(\tau)}|_{T_i} \sim eF_i - eF_i = 0.$$

If $i = \tau - e + 1$ the sum contains two relevant terms: those for $j = i$ and $j = i + 1$. This case folds back into the previous one because then the coefficient of the $j = i-1$ term that appears there is zero when we substitute i with $\tau - e + 1$.

If $i = \tau$ and $e \geq 2$ the second sum has two relevant terms:

$$\begin{aligned} & (e-1)T_{\tau-1}|_{T_\tau} + eT_\tau|_{T_\tau} \\ = & \ (e-1)B_\tau - eB_\tau - eF_\tau \\ = & \ -B_\tau - eF_\tau \end{aligned}$$

which gives

$$L^{(\tau)}|_{T_\tau} \sim hB_\tau + eF_\tau - B_\tau - eF_\tau = (h-1)B_\tau.$$

If $e = 1$ the sum only has one relevant term. This case folds back into the previous one because then the coefficient of the $j = \tau - 1$ term that appears there is zero when we substitute i with τ. $\square$

Matching conditions of a higher order

For each of the curves $B_1, \cdots, B_r$, let $T^i = \sum_{j=1}^{\tau_i} jT_j$ be the tower of ruled surfaces obtained in the previous section. Let $\widetilde{V}$ be the transform of V after all these blowups; let $Y = P + \sum_i T^i$. Any element of $\mathcal{L}|_{\widetilde{V}}$ has to match with an element of $\mathcal{L}|_Y$ along the intersection scheme $Z = Y \cap \widetilde{V}$; we know the system $\mathcal{L}|_Y$ and in this section we will compute how many conditions this imposes on the elements of $\mathcal{L}|_{\widetilde{V}}$.

We want to find out how many conditions we have to impose on a global section of $\mathcal{L}|_Z$ to be able to lift it to a global section of $\mathcal{L}|_Y$, i.e. the dimension of the cokernel of the linear map $\phi : H^0(\mathcal{L}|_Y) \to H^0(\mathcal{L}|_Z)$. We have an exact sequence

$$0 \to H^0(\mathcal{L}|_Y(-Z)) \to H^0(\mathcal{L}|_Y) \xrightarrow{\phi} H^0(\mathcal{L}|_Z) \to \mathrm{coker}(\phi) \to 0$$

and we claim that $H^0(\mathcal{L}|_Y(-Z))$=0. To prove it, remember that we obtained L by making several blowups, considering its total transform and subtracting from it the exceptional divisor the minimum number of times. Now note that $\mathcal{L}|_Y(-Z) \cong \mathcal{L}(Y)|_Y$ and Y is the sum of the exceptional divisors, taken with some positive multiplicity. This means that $\mathcal{L}|_Y(-Z)$ is what we would have obtained instead of $\mathcal{L}|_Y$ if we had subtracted each of the exceptional divisors with a lower multiplicity. We know that this sheaf cannot have sections by construction because of the minimality.

Then

$$\begin{aligned}&\#\{\text{new matching conditions}\} = \dim\operatorname{coker}(\phi) = \\ &= h^0\left(\mathcal{L}|_Z\right) - h^0\left(\mathcal{L}|_Y\right)\end{aligned}$$

both terms can be decomposed, since the irreducible components of Z and Y intersect transversally

$$\begin{aligned}&= h^0(\mathcal{L}|_R) + \sum_{i=1}^{r} h^0(\mathcal{L}|_{\widetilde{V}\cap T^i}) - \sum_{i=1}^{r} h^0(\mathcal{L}|_{R\cap\widetilde{V}\cap T^i}) - \\ &\quad - h^0\left(\mathcal{L}|_P\right) - \sum_{i=1}^{r} h^0(\mathcal{L}|_{T^i}) + \sum_{i=1}^{r} h^0(\mathcal{L}|_{T^i\cap P})\end{aligned}$$

P and $\widetilde{V}$ also intersect transversally. We can then collect three of the summations above and have

$$= \left(h^0(\mathcal{L}|_R) - h^0\left(\mathcal{L}|_P\right)\right) + \sum_{i=1}^{r}\left(h^0(\mathcal{L}|_{T^i\cap(\widetilde{V}\cup P)} - h^0(\mathcal{L}|_{T^i}))\right).$$

The first difference is the same we already considered in proposition 2.13. For the second term we will deal with each tower separatedly.

Let B be one of the curves $B_1, \ldots, B_r$; in the following computations we will use the same notation we used while making the blowups in the previous section. Let $T = \sum_{j=1}^{\tau} jT_j$ be the tower over B; we have

$$\begin{aligned}&h^0\left(\mathcal{L}|_{T\cap(\widetilde{V}\cup P)}\right) - h^0(\mathcal{L}|_T)) = \\ &\quad = \sum_{i=1}^{\tau} h^0(\mathcal{L}|_{iT_i\cap\widetilde{V}}) - \sum_{i=1}^{\tau-1} \tau i(i+1) + h^0(\mathcal{L}|_{\tau T_\tau\cap P}) - \tau^2 \\ &\qquad - \sum_{i=1}^{\tau} h^0(\mathcal{L}|_{iT_i}) + \sum_{i=1}^{\tau-1} h^0(\mathcal{L}|_{iT_i\cap(i+1)T_{i+1}})\end{aligned}$$

Now we need to compute the restrictions of $\mathcal{L}$ and the dimensions of the cohomology spaces that appear here.

Computation of the dimensions

Lemma 2.26. *Let $T \cong \mathbb{F}_e \overset{\pi}{B}$ be a rational ruled surface of base curve B and let F be a fiber. For any integer values a and b with $a \geq 0$ we have that*

$$h^i(\mathcal{O}_T(aB + bF)) = \sum_{j=0}^{a} h^i(\mathcal{O}_{\mathbb{P}^1}(b - je)).$$

Proof. T is a ruled surface and $(aB + bF).F = a \leq 0$, so we know that

$$h^i(T, \mathcal{O}_T(aB + bF)) = h^i\left(B, \pi_* \mathcal{O}_T(aB + bF)\right)$$

we can write the push-forward using the symmetric product

$$\begin{aligned} &= h^i\left(S^a\left(\mathcal{O}_B \oplus \mathcal{O}_B(-e)\right) \otimes \mathcal{O}_B(b)\right) \\ &= h^i\left(\bigoplus_{j=0}^{a} \left(\mathcal{O}_B(-je)\right) \otimes \mathcal{O}_B(b)\right) \\ &= \sum_{j=0}^{a} h^i(\mathcal{O}_{\mathbb{P}^1}(b - je)). \end{aligned}$$

□

Lemma 2.27. *Let $0 \leq i \leq \tau$ and D a Cartier divisor on iT_i. Suppose that for any $j < i$ we have that $h^1\left(\mathcal{O}_{T_i}(D) \otimes \left(\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}}\right)^j\right) = 0$. Then*

$$h^0(\mathcal{O}_{iT_i}(D)) = \sum_{j=0}^{i-1} h^0\left(\mathcal{O}_{T_i}(D) \otimes \left(\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}}\right)^j\right)$$

Proof. By induction: if $i = 0$ the statement is an identity; if $i > 1$ the result comes inductively via the exact sequence

$$0 \to \mathcal{O}_{T_i}(D) \otimes \left(\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}}\right)^{i-1} \to \mathcal{O}_{iT_i}(D) \to \mathcal{O}_{(i-1)T_i}(D) \to 0$$

thanks to the assumption that the first cohomology space of the sheaf on the left is zero. □

Unfortunately, $h^1(\mathcal{O}_{T_i}(\mathcal{L}) \otimes (\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}})^j) = 0$ only for $\tau \leq 4$. If $\tau \geq 5$ there are examples where this is not true so we will limit ourselves to $\tau \leq 4$ for the time being.

Example 2.28. Let's suppose to have curve B such that $\tau = 5$ and $e = 0$ (i.e. $\sigma = 5h$). The surface $T_2 \cong \mathbb{F}_3$ appears with multiplicity 2, $\mathcal{L}^{(\tau)}|_{T_2} \cong \mathcal{O}_{T_2}$ and $\mathcal{N}^*_{T_2|\mathcal{X}^{(\tau)}} \cong \mathcal{O}_{T_2}(2B_2 + 4F_2)$. Then for $j = 1$

$$\begin{aligned} & h^1\left(\mathcal{O}_{T_2} \otimes \left(\mathcal{N}^*_{T_2|\mathcal{X}^{(\tau)}}\right)\right) = h^1(\mathcal{O}_{T_2}(2B + 4F)) = \\ & = h^1(\mathcal{O}_{\mathbb{P}^1}(4)) + h^1(\mathcal{O}_{\mathbb{P}^1}(1)) + h^1(\mathcal{O}_{\mathbb{P}^1}(-2)) = 1. \end{aligned}$$

We need to make two different computations for $i = \tau$ and $0 \leq i < \tau$ because in those two cases the expression we found in 2.24 for the normal sheaf of T_i is different.

We know that $\mathcal{L}|_{\tau T_\tau} = aB_\tau$ where a is either h or $h - 1$. Both cases are covered by the following lemma:

Lemma 2.29. *One has*

$$h^0\left(\mathcal{O}_{\tau T_\tau}(aB_\tau)\right) = \frac{\tau(\tau+1)(3a+2\tau+1)}{6}.$$

Proof. By 2.24 we know that $\mathcal{N}^*_{T_\tau}$ is effective and since $T_\tau \cong \mathbb{P}^1 \times \mathbb{P}^1$, $h^1(\mathcal{L}|_{T_i} \otimes (\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}})^j)$ is going to be 0 for any positive j. We can then conclude, by 2.27, that

$$\begin{aligned} h^0\left(\mathcal{O}_{\tau T_\tau}(aB_\tau)\right) &= \sum_{j=0}^{\tau-1} h^0((a+j)B_\tau + jF_\tau) = \\ &= \sum_{j=0}^{\tau-1} (j+1)(a+j+1) = \sum_{j=1}^{\tau} j(a+j) = \\ &= a\frac{\tau(\tau+1)}{2} + \frac{\tau(\tau+1)(2\tau+1)}{6} = \\ &= \frac{\tau(\tau+1)(3a+2\tau+1)}{6} \end{aligned}$$

□

For $1 \leq i < \tau$, $i \neq \tau - e$ we know that $\mathcal{L}|_{iT_i}$ is either eF_i or 0. Both cases are covered by the following lemma.

Lemma 2.30. *Let $1 \leq i < \tau$ and $b < \tau$. Suppose also that $\tau \leq 4$. Then*

$$h^0\left(\mathcal{O}_{iT_i}(bF_i)\right) = \frac{4i^3 + 3(2b+1)i^2 - i}{6}$$

Proof. Since $i < \tau$, $\mathcal{N}^*_{T_i} = \mathcal{O}_{T_i}(2B_i + (\tau - i + 1)F_i)$.

$$\begin{aligned} h^l\left(\mathcal{O}_{T_i}(eF_i) \otimes \left(\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}}\right)^j\right) &= h^l(\mathcal{O}_{T_i}(2jB_i + (j(\tau - i) + j + b)F_i)) \\ &= \sum_{u=0}^{2j} h^l(\mathcal{O}_{\mathbb{P}^1}(j(\tau - i) + j + b - u(\tau - i))) \end{aligned}$$

We start by checking the condition on the h^1 for all $j < i$. We have $h^1 > 0$ if and only if $j(\tau - i) + j + b - u(\tau - i) \leq -2$; u varies between 0 and $2j$, so this happens only if it happens for $u = 2j$

$$\begin{aligned} j(\tau - i) + j + b - 2j(\tau - i) &\geq -1 \\ j &\leq \frac{j + b + 1}{\tau - i} \end{aligned}$$

If $\tau \leq 4$, $\tau - i$ can only be 1, 2 or 3. If $\tau - i = 1$ the inequality is always satisfied, if $\tau - i = 2$ then i is at most (if $\tau = 4$) 2, j is either 0 or 1 and in both cases the inequality is satisfied, if $\tau - i = 3$, i can be at most 1 and j has to be zero which satisfies the inequality too.

Now we can apply 2.27 and write

$$\begin{aligned} h^0\left(\mathcal{O}_{iT_i}(bF_i)\right) &= \sum_{j=0}^{i-1} \sum_{u=0}^{2j} j(\tau - i + 1) + b - u(\tau - i) + 1 \\ &= \sum_{j=0}^{i-1} (2j+1)(j(\tau - i + 1) + b + 1) - (\tau - i)j(2j+1) \\ &= \sum_{j=0}^{i-1} 2j^2 + (2b+3)j + b + 1 \\ &= i(b+1) + \frac{i(i+1)}{2}(2b+3) + 2\frac{i(i-1)(2i-1)}{6} \\ &= \frac{4i^3 + 3(2b+1)i^2 - i}{6}. \end{aligned}$$

□

If $e \neq 0$ the restriction of $\mathcal{L}$ to $T_{\tau - e}$ cannot be written as some number of fibers and needs to be treated separatedly:

Lemma 2.31. *Let's suppose that we are in a case where $e \neq 0$. Let $i = \tau - e$. Then*

$$h^0\left(\mathcal{L}|_{iT_{\tau-e}}\right) = \frac{i(i+1)(4i + 3e + 2)}{6}$$

Proof. Since $i = \tau - e$, $\mathcal{N}^*_{T_i} = \mathcal{O}_{T_i}(2B_i + (e+1)F_i)$.

$$\begin{aligned} h^l\left(\mathcal{O}_{T_i}(\mathcal{L}) \otimes \left(\mathcal{N}^*_{T_i|\mathcal{X}^{(\tau)}}\right)^j\right) &= h^l(\mathcal{O}_{T_i}((2j+1)B_i + (je + j + e)F_i)) \\ &= \sum_{u=0}^{2j+1} h^l(\mathcal{O}_{\mathbb{P}^1}(je + j + e - ue)) \end{aligned}$$

First we check that the h^1 is zero for all $j < i$. We have $h^1 > 0$ if and only if $je + j + e - ue \leq -2$; this can only happen if it happens for the greatest possible value of u, which is $2j+1$

$$\begin{aligned} je + j + e - 2je - e &\geq -1 \\ j(e-1) &\leq 1 \end{aligned}$$

If $\tau \leq 4$, e can only be 1, 2 or 3. If $e = 1$ the inequality is always satisfied, if $e = 2$ then i is at most 2 (and $i = 2$ can be reached only when $\tau = 4$), j is either 0 or 1 and in both cases the inequality is satisfied, if $e = 3$, i can be at most 1 and j has to be zero which satisfies the inequality too.

Now we can apply 2.27 and write

$$\begin{aligned} h^0\left(\mathcal{O}_{iT_{\tau-e}}(B_i + eF_i)\right) &= \sum_{j=0}^{i-1}\sum_{u=0}^{2j+1}(j+1)(e+1) - ue \\ &= \sum_{j=0}^{i-1}(2j+2)(j+1)(e+1) - e(j+1)(2j+1) \\ &= \sum_{j=0}^{i-1} 2(j+1)^2 + e(j+1) \\ &= 2\frac{i(i+1)(2i+1)}{6} + e\frac{i(i+1)}{2} \\ &= \frac{i(i+1)(4i+3e+2)}{6}. \end{aligned}$$

□

The following lemma covers the intersections between the T_i's; the number b stands either for 0 or for e according to the restriction of $\mathcal{L}$ to $T_i \cap T_{i+1}$. We can infer from 2.25 that

$$\mathcal{L}_{T_i \cap T_{i+1}} \cong \begin{cases} \mathcal{O}_{T_i \cap T_{i+1}}(e) & \text{if } i < \tau - e \\ \mathcal{O}_{T_i \cap T_{i+1}} & \text{otherwise} \end{cases}$$

Lemma 2.32. *Let $i < \tau$ and $0 \leq b < \tau$. Then*

$$h^0(\mathcal{O}_{iT_i \cap (i+1)T_{i+1}}(b)) = \frac{i(i+1)}{2}(\tau + 2b + 1).$$

Proof. In the exact sequence

$$0 \to \mathcal{O}_{iT_i}(bF_i - (i+1)B_i) \to \mathcal{O}_{iT_i}(bF_i) \to \mathcal{O}_{iT_i \cap (i+1)T_{i+1}}(b) \to 0$$

we know the cohomology of the middle sheaf and its H^1 and H^2 spaces are zero, so if $h^2(\mathcal{O}_{iT_i}(bF_i - (h+1)B_i)) = 0$ we have that

$$h^0(\mathcal{O}_{iT_i \cap (i+1)T_{i+1}}(b)) = h^0(\mathcal{O}_{iT_i}(bF_i)) + \chi(\mathcal{O}_{iT_i}(bF_i - (i+1)B_i)).$$

One can see that $h^2(\mathcal{O}_{iT_i}(bF_i - (i+1)B_i)) = 0$ by induction on the multiplicity of T_i, using 1.3 and the fact that, for any $j \geq 0$, $h^2(\mathcal{O}_{T_i}(-jB_i + bF_i) \otimes \left(\mathcal{N}|_{T_i|\mathcal{X}^{(\tau)}}\right)^*) = 0$. This last vanishing can be verified by noting that the conormal bundle is effective and that this could not happen if the divisor associated to that bundle were smaller than the anticanonical divisor of T_i.

$$\begin{aligned}
&\chi\left(\mathcal{O}_{iT_i}(-(i+1)B_i + bF_i)\right) = \text{ (by 1.3)} \\
&= \sum_{j=0}^{i-1} \chi\left(\mathcal{O}_{iT_i}(-(i+1)B_i + bF_i \otimes \left(\mathcal{N}|^*_{T_i|\mathcal{X}^{(\tau)}}\right)^j\right) \\
&= \sum_{j=0}^{i-1} \chi\left(\mathcal{O}_{T_i}((2j-i-1)B_i + (j(\tau-i+1) + (\tau-i) + b + 2)F_i)\right) \\
&= \sum_{j=0}^{i-1} \frac{1}{2}(-(\tau-i)(2j-i+1)(2j-i-1) + (2j-i+1)(\tau-i+b+2)) + \\
&\quad +(2j-i-1)(j(\tau-i+1) + (\tau-i) + b + 2) + 1 \\
&= \sum_{j=0}^{i-1} 2j^2 + j((\tau-i-1)(i+1) + 2b + 3) - (\tau-i)\frac{i(i+1)}{2} - i(b+1)
\end{aligned}$$

We found earlier that, if $\tau \leq 4$,

$$h^0\left(\mathcal{O}_{iT_i}(bF_i)\right) = \sum_{j=0}^{i-1} 2j^2 + j(2b+3) + b + 1;$$

then:

$$\begin{aligned}
h^0(\mathcal{O}_{iT_i \cap (i+1)T_{i+1}}(b)) &= h^0(\mathcal{O}_{iT_i}(bF_i)) + \chi\left(\mathcal{O}_{iT_i}(-(i+1)B_i + bF_i)\right) = \\
&= \sum_{j=0}^{i-1} -j((\tau-i-1)(i+1)) + (\tau-i)\frac{i(i+1)}{2} + (b+1)(i+1) \\
&= \frac{i(i+1)}{2}(\tau + 2b + 1).
\end{aligned}$$

□

The following is the computation for the intersection of T with P.

Lemma 2.33. *One has:*

$$h^0\left(\mathcal{O}_{\tau T_\tau \cap P}(\mathcal{L})\right) = \frac{\tau(\tau+1)}{2}.$$

Proof. We know that $\mathcal{O}_{sT_\tau}(\mathcal{L}) = \mathcal{O}_{\tau T_\tau}(hB_\tau)$. In the exact sequence

$$0 \to \mathcal{O}_{\tau T_\tau}((h-1)B_\tau) \to \mathcal{O}_{\tau T_\tau}(hB_\tau) \to \mathcal{O}_{\tau T_\tau \cap P}(\mathcal{L}) \to 0$$

we know the cohomology of the first two sheaves which is zero in dimension 1 and 2. Therefore

$$\begin{aligned} h^0(\mathcal{L}|_{\tau T_\tau \cap P}) &= h^0(\mathcal{O}_{\tau T_\tau}(hB_\tau)) - h^0(\mathcal{O}_{\tau T_\tau}((h-1)B_\tau)) \\ &= h\frac{\tau(\tau+1)}{2} + \frac{\tau(\tau+1)(\tau+1)}{6} - \\ &\quad -(h-1)\frac{\tau(\tau+1)}{2} - \frac{\tau(\tau+1)(2\tau+1)}{6} \\ &= \frac{\tau(\tau+1)}{2}. \end{aligned}$$

□

Now we move to make the computations at the intersections between the components of T and $\widetilde{V}$. We need to make diferent cases according to the expressions we found for $h^0(\mathcal{L}|_{iT_i})$.

Lemma 2.34. *Let $i \leq \tau$ and suppose $\tau \leq 4$.*

$$\begin{aligned} h^0\left(\mathcal{L}|_{iT_i \cap \widetilde{V}}\right) &= h^0\left(\mathcal{L}|_{iT_i}\right) + \chi(\mathcal{L}|_{iT_i} \otimes \mathcal{O}_{iT_i}(-\tau F_i)) \\ h^l\left(\mathcal{L}|_{iT_i \cap \widetilde{V}}\right) &= 0 \textit{ for } l > 0 \end{aligned}$$

Proof. In the exact sequence

$$0 \to \mathcal{L}(-\widetilde{V})|_{iT_i} \to \mathcal{L}|_{iT_i} \to \mathcal{L}|_{iT_i \cap \widetilde{V}} \to 0$$

we know the cohomology of the middle sheaf and its H^1 and H^2 spaces are zero, so we only need to prove that $h^2(\mathcal{L}(-\widetilde{V})|_{iT_i}) = 0$. Let D be the Cartier divisor associated to the restriction of $\mathcal{L}$; D is of the form $aB_i + bF_i$ for suitable a and b and the computation made in 2.25 shows that $a \geq 0$. $h^2(\mathcal{L}(-\widetilde{V})|_{iT_i}) = h^2(\mathcal{O}_{iT_i}(aB_i + (b-\tau)F_i))$ and it is easy to see that this is 0 by induction on s, using 1.3 and the fact that $h^2(\mathcal{O}_{T_i}(aB_i + (b-\tau)F_i) \otimes \left(\mathcal{N}|_{T_i|\mathcal{X}^{(\tau)}}\right)^*)$ is 0 for any $j \geq 0$. □

Lemma 2.35. *Let $1 \leq i < \tau \leq 4$, $i \neq \tau - e$, $\mathcal{L}|_{iT_i} = bF_i$. Then*

$$h^0\left(\mathcal{L}|_{iT_i \cap \widetilde{V}}\right) = \tau i^2.$$

Note that the result does not depend on b.

Proof.

$$
\begin{aligned}
&\chi\left(\mathcal{L}\otimes\mathcal{O}_{iT_i}(-\tau F_i)\right) = \\
&= \sum_{j=0}^{i-1}\chi\left(\mathcal{L}\otimes\mathcal{O}_{T_i}(-\tau F_i)\otimes\left(\mathcal{N}|^*_{T_i|\mathcal{X}(\tau)}\right)^j\right)\\
&= \sum_{j=0}^{i-1}\chi\left(\mathcal{O}_{T_i}(2jB_i+(b-\tau+j(\tau-i)+j)F_i)\right)\\
&= \sum_{j=0}^{i-1}\frac{1}{2}(-(\tau-i)2j(2j+2)+2j(b-\tau+j(\tau-i+1)+(\tau-i)+2)+\\
&\quad +2(j+1)(e-\tau+j(\tau-i+1))+1\\
&= \sum_{j=0}^{i-1}2j^2+j(2b+3-2\tau)+b-\tau+1
\end{aligned}
$$

We found earlier that, if $\tau\leq 4$,

$$h^0\left(\mathcal{O}_{iT_i}(bF_i)\right)=\sum_{j=0}^{i-1}2j^2+j(2b+3)+b+1;$$

the difference is

$$\sum_{j=0}^{i-1}\tau(2j+1)=\tau i^2. \qquad \square$$

Lemma 2.36. *Same hypothesis as above, except that $i=\tau-e$. Then*

$$h^0\left(\mathcal{L}|_{(\tau-e)T_{\tau-e}\cap\widetilde{V}}\right)=\tau(\tau-e)^2+\tau(\tau-e).$$

Proof.

$$\begin{aligned}
&\chi\left(\mathcal{L}\otimes\mathcal{O}_{(\tau-e)T_{\tau-e}}(-\tau F_{\tau-e})\right) = \\
&= \sum_{j=0}^{\tau-e-1}\chi\left(\mathcal{L}\otimes\mathcal{O}_{T_{\tau-e}}(-\tau F_{\tau-e})\otimes\left(\mathcal{N}|^*_{T_{\tau-e}|\mathcal{X}^{(\tau)}}\right)^j\right)\\
&= \sum_{j=0}^{\tau-e-1}\chi\left(\mathcal{O}_{T_\tau-e}((2j+1)B_{\tau-e}+(b-\tau+j(e+1))F_{\tau-e})\right)\\
&= \sum_{j=0}^{\tau-e-1}\frac{1}{2}(-e(2j+1)(2j+3)+(2j+1)((e+1)(j+2)-\tau)+\\
&\quad +(2j+3)(e-\tau+j(e+1))+1\\
&= \sum_{j=1}^{\tau-e}\frac{1}{2}(-e(2j-1)(2j+1)+(2j-1)((e+1)(j+1)-\tau)+\\
&\quad +(2j+1)(e-\tau+(j-1)(e+1))+1\\
&= \sum_{j=1}^{\tau-e}2j^2+(e-2\tau)j
\end{aligned}$$

We found earlier that, if $\tau\le 4$,

$$h^0\left(\mathcal{O}_{(\tau-e)T_{\tau-e}}(B_{\tau-e}+eF_{\tau-e})\right)=\sum_{j=1}^{\tau-e}2j^2+ej;$$

therefore

$$h^0\left(\mathcal{L}|_{(\tau-e)T_{\tau-e}\cap\widetilde{V}}\right)=\sum_{j=1}^{\tau-e}2\tau j=\tau(\tau-e)(\tau-e+1). \qquad \square$$

Lemma 2.37. *Same hypothesis as above, except that $i=\tau$. Let a stand for either h or $h-1$ according to the expression of $\mathcal{L}|_{T_\tau}$. Then*

$$h^0\left(\mathcal{L}|_{iT_\tau\cap\widetilde{V}}\right)=\tau\frac{i(i+1)}{2}+\tau ai.$$

Proof.

$$\begin{aligned}
&\chi\left(\mathcal{L}\otimes\mathcal{O}_{iT_\tau}(-\tau F_\tau)\right) = \\
&= \sum_{j=0}^{i-1}\chi\left(\mathcal{L}\otimes\mathcal{O}_{T_\tau}(-\tau F_\tau)\otimes\left(\mathcal{N}|^*_{T_\tau|\mathcal{X}^{(\tau)}}\right)^j\right) \\
&= \sum_{j=0}^{i-1}\chi\left(\mathcal{O}_{T_i}((a+j+2)B_\tau+(j-\tau+2)F_\tau)\right) \\
&= \sum_{j=0}^{i-1}(a+j+1)(j+1-\tau) \\
&= \sum_{j=1}^{i}(a+j)(j-\tau)
\end{aligned}$$

We found earlier that, if $\tau \leq 4$,

$$h^0\left(\mathcal{O}_{iT_\tau}(aB_\tau)\right) = \sum_{j=1}^{i} j(a+j);$$

and therefore

$$h^0\left(\mathcal{L}|_{iT_\tau\cap\widetilde{V}}\right) = \sum_{j=1}^{i}\tau(a+j) = \tau\frac{i(i+1)}{2}+\tau a i.$$

□

We can now complete the computation from the beginning of the section.

If $e = 0$ then

$$
\begin{aligned}
&h^0\left(\mathcal{L}|_{T\cap(P\cup\widetilde{V})}\right) - h^0\left(\mathcal{L}|_T\right) = \\
&= \sum_{i=1}^{\tau-1} h^0\left(\mathcal{O}_{iT_i\cap\widetilde{V}}\right) + h^0\left(\mathcal{O}_{\tau T_\tau\cap\widetilde{V}}(h)\right) - \sum_{i=1}^{\tau-1}\tau i(i+1) + h^0\left(\mathcal{O}_{\tau T\tau\cap P}\right) - \tau^2 \\
&\quad - \sum_{i=1}^{\tau-1} h^0\left(\mathcal{O}_{iT_i}\right) - h^0\left(\mathcal{O}_{\tau T_\tau}(hB_\tau)\right) + \sum_{i=1}^{\tau-1} h^0\left(\mathcal{O}_{iT_i\cap(i+1)T_{i+1}}\right) \\
&= \sum_{i=1}^{\tau-1}\left(h^0\left(\mathcal{O}_{iT_i\cap\widetilde{V}}\right) - \tau i(i+1) - h^0(\mathcal{O}|_{iT_i}) + h^0(\mathcal{O}|_{iT_i\cap(i+1)T_{i+1}})\right) + \\
&\quad + h^0\left(\mathcal{O}_{\tau T_\tau\cap\widetilde{V}}(h)\right) + h^0\left(\mathcal{O}_{\tau T_\tau\cap P}\right) - \tau^2 - h^0\left(\mathcal{O}_{\tau T_\tau}(hB_\tau)\right) \\
&= \sum_{i=1}^{\tau-1}\left(\frac{i(i+1)}{2} + \tau^2 h - \tau i(i+1) - \frac{4i^3+3i^2-i}{6} + \tau i^2\right) + \\
&\quad + \frac{\tau^2(\tau+1)}{2} + \frac{\tau(\tau+1)}{2} - \tau^2 - \frac{\tau(\tau+1)(3h+2\tau+1)}{6} \\
&= \sum_{i=1}^{\tau-1}\left(-\frac{2}{3}i^3 + \frac{1}{2}\tau i^2 + \left(\frac{2}{3} - \frac{\tau}{2}\right)i\right) + \frac{\tau^3 + (3h-3)\tau^2 + (2-3h)\tau}{6} \\
&= -\frac{\tau(\tau^2-3\tau+2)}{6} + \frac{\tau^3+(3h-3)\tau^2+(2-3h)\tau}{6} \\
&= h\frac{\tau(\tau-1)}{2}
\end{aligned}
$$

If $e > 0$

$$\begin{aligned}
& h^0\left(\mathcal{L}|_{T\cap(P\cup\widetilde{V})}\right) - h^0\left(\mathcal{L}|_T\right) = \\
= \;& \underline{\sum_{i=1}^{\tau-1} h^0(\mathcal{O}_{iT_i\cap\widetilde{V}})} + \left(h^0(\mathcal{L}|_{(\tau-e)T_{\tau-e}\cap\widetilde{V}}) - h^0(\mathcal{O}_{(\tau-e)T_{\tau-e}\cap\widetilde{V}})\right) + \\
& +\underline{h^0(\mathcal{O}_{\tau T_\tau\cap\widetilde{V}}((h-1)B_\tau))} - \underline{\sum_{i=1}^{\tau-1} \tau i(i+1)} + \underline{h^0(\mathcal{O}_{\tau T_\tau\cap P})} - \underline{\tau^2} - \\
& -\underline{\sum_{i=1}^{\tau-1} h^0(\mathcal{O}_{iT_i})} + \sum_{i=1}^{\tau-e-1}\left(h^0(\mathcal{O}_{iT_i}) - h^0(\mathcal{O}_{iT_i}(eF_i))\right) + \\
& +(h^0(\mathcal{O}_{(\tau-e)T_{\tau-e}}) - h^0(\mathcal{O}_{(\tau-e)T_{\tau-e}}(B_{\tau-e} + eF_{\tau-e}))) - \\
& -\underline{h^0(\mathcal{O}_{\tau T_\tau}((h-1)B_\tau))} + \underline{\sum_{i=1}^{\tau-1} h^0(\mathcal{O}_{iT_i\cap(i+1)T_{i+1}})} - \\
& - \sum_{i=1}^{\tau-e-1}\left(h^0(\mathcal{O}_{iT_i\cap(i+1)T_{i+1}}) - h^0(\mathcal{O}_{iT_i\cap(i+1)T_{i+1}}(e))\right)
\end{aligned}$$

The shaded terms are exactly what appeared for $e = 0$, except that $h - 1$ appears where h used to. We replace them with $(h-1)\frac{\tau(\tau-1)}{2}$ and we get

$$\begin{aligned}
& h^0\left(\mathcal{L}|_{T\cap(P\cup\widetilde{V})}\right) - h^0\left(\mathcal{L}|_T\right) = \\
= \;& (h-1)\frac{\tau(\tau-1)}{2} + \left(\tau(\tau-e)^2 + \tau(\tau-e) - \tau(\tau-e)^2\right) + \\
& + \sum_{i=1}^{\tau-e-1}\left(\frac{4i^3+3i^2-i}{6} - \frac{4i^3+(3+6e)i^2-i}{6}\right) \\
& + \left(\frac{4(\tau-e^3)+3(\tau-e)^2-(\tau-e)}{6} - \right. \\
& \left. -\frac{(\tau-e)(\tau-e+1)(4(\tau-e)+3e+2)}{6}\right) - \\
& - \sum_{i=1}^{\tau-e-1}\left(\frac{i(i+1)}{2}(\tau+1) - \frac{i(i+1)}{2}(\tau+2e+1)\right) \\
= \;& (h-1)\frac{\tau(\tau-1)}{2} + \sum_{i=1}^{\tau-e-1}(ei) - (e+1)\frac{(\tau-e)(\tau-e+1)}{2} + \tau(\tau-e) \\
= \;& (h-1)\frac{\tau(\tau-1)}{2} + \frac{(\tau-e)(\tau-e-1)}{2}
\end{aligned}$$

Note that if we put $e = 0$ in this formula we obtain exactly the one we found for $e = 0$.

Results

Putting the above lemmas together we obtain the following proposition:

Proposition 2.38. *Let $p_1(t), \ldots, p_n(t)$ be n points coming together for $t = 0$ in a general way and let's make the construction described earlier. If there is a (smooth, irreducible) rational curve B on P meeting R transversally at the points $q_1^1, \ldots, q_\tau^1$, with $\tau \leq 4$, and such that $L.B = -\sigma < 0$ and $(B^2)_P = -1$, then the limit scheme contains in its first infinitesimal neighborhood a set of τ points of type $(h^{\tau-e}; (h-1)^e)$ at $q_1^1, \ldots, q_\tau^1$ together with $(h-1)\frac{(\tau-1)\tau}{2} + \frac{(\tau-e-1)(\tau-e)}{2}$ extra conditions that are visible on the restriction of the linear system $L^{(\tau)}$ to the succesive blowups of B.*

This explicits a number of conditions on the curves of $L|_{X_0}$. Adding those up we have that:

Corollary 2.39. *Every such curve B contributes to the length of the candidate limit scheme by $\frac{\sigma(\sigma+2\tau-1)}{2}$.*

Proof. The proof is a straightforward computation, remembering that $\sigma = \tau h - e$ by the definition of τ and e.

$$\begin{aligned}
&\tau\left(e\frac{(h+1)h}{2} + (\tau-e)\frac{h(h+1)}{2}\right) + (h-1)\frac{(\tau-1)\tau}{2} + \frac{(\tau-e-1)(\tau-e)}{2} \\
&= \tau^2\frac{h(h+1)}{2} - \tau he + \frac{\tau^2 h - \tau h - \tau^2 + \tau + \tau^2 - 2\tau e + e^2 - \tau + e}{2} \\
&= \frac{\tau^2 h + 2 + \tau^2 h - 2\tau he + \tau^2 h - \tau h - 2\tau e + e^2 + e}{2} \\
&= \frac{\tau^2 h^2 - 2\tau he + e^2 + 2\tau^2 h - 2\tau e - \tau h + e}{2} \\
&= \frac{(\tau h - e)(\tau h - e + 2\tau - 1)}{2} \\
&= \frac{\sigma(\sigma + 2\tau - 1)}{2}
\end{aligned}$$

□

We have come to the following strengthening of 2.22

Theorem 2.40. *Let $p_1(t), \ldots, p_n(t)$ be n fat points of multiplicities $m_1, \ldots, m_n$ coming together at a point p in a general way and suppose that after the first blowup there are s rational curves $B_1, \ldots, B_s$ on P such that $B_i^2 = -1$, $B_i.L = -\sigma_i < 0$, and B_i meets V transversally at τ_i points $q_{i,1}, \ldots, q_{i,\tau_i}$. Let h_i and e_i be integers such that $0 \le e_i < \tau_i$ and $\sigma_i = \tau_i h_i - e_i$. Let k be the minimum integer such that there is a curve of degree k in the projective plane passing through n points in general position of multiplicities $m_1, \ldots, m_n$ and let L be the linear system we obtain after the process of removing the fixed components.*

Then the limit scheme contains the scheme defined by the point p with multiplicity k together with s sets of τ_i infinitely near points of type $({h_i}^{\tau_i - e_i, (h_i-1)^{e_i}})$ in $q_{i,1}, \ldots, q_{i,\tau_i}$ and the further matching conditions. If all $\tau_i \le 4$, this scheme has length

$$\frac{k(k+1)}{2} + \deg|L_R| - \dim|L_R| + \sum_{i=1}^{s}\left(\tau_i(\tau_i - e_i)\frac{h_i(h_i+1)}{2} + \right.$$
$$\left. + \tau_i e_i \frac{h_i(h_i-1)}{2} + (h_i - 1)\frac{\tau_i(\tau_i-1)}{2} + \frac{(\tau_i - e_i)(\tau_i - e_i - 1)}{2}\right)$$

Note that in order to compute this number effectively one has to know the value of k and have perfect knowledge of the system L_P, including its dimension, its degree and all its fixed components B_i. All this is related to the so called Harbourne-Hirschowitz conjecture.

Writing explicit expressions for those conditions is still a work in progress. However some of these conditions are easy to understand since they are visible by looking at the restriction of L to T_{red}. They appear and are described in the following examples.

Even if our comprehension of the candidate limit scheme is not complete yet, being able to count how many conditions come from matching is enough to narrow considerably the possible limit schemes for a general collision of fat points. Since we know the length of this candidate limit scheme we can compare it with the length of the limit scheme and prove that it is the limit, as we will do in the following applications.

Example 2.41. Continuing on example 2.23, the computations made in this section call for a total of $(h-1)+(1-e) = h-e$ conditions coming from the matching of L. We can see that $L^{(2)}|_{T_1} \sim eB_1 + eF$ and $L^{(2)}|_{T_2} \sim (h-e)B_2$; the first system intersects F_1 at e points and the second one intersects F_2 at $h-e$ points. If $e = 0$ the system ${L^{(2)}}_P|_{T_1}$ is trivial; if $e = 1$ there is one point on each of the two fibers over q_1 and q_2 and however we choose those points there is a curve in $|B_1 + F_1|$ that contains both, so restricting to this linear system does not pose any extra conditions on the positions of those points.

On the other hand, on $T_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ we have two fibers with $h-e$ points on each and the system is $L^{(2)}{}_P|_{T_2} \sim (h-e)B_2$, that is $h-e$ lines from the fibration other than the one of the fibers. This means that any given position of the $h-e$ points that lie on one of the two fibers already determines a unique element of the linear system and thus forces where the points on the other fiber have to lie. Restricting to an element of that linear system then poses for a curve $h-e$ more conditions to the second order.

Let's now compute the length of the limit scheme. Since e is either 0 or 1, we have that $e^2 - e = 0$

$$5\frac{m(m+1)}{2} = 5\frac{4h^2 - 4eh + e^2 + 2h - e}{2} = 10h^2 - 10eh + 5h$$

while the candidate limit scheme has length

$$\begin{aligned} & \frac{2m(2m+1)}{2} + 2(2-e)\frac{h(h+1)}{2} + 2e\frac{h(h-1)}{2} + h - e = \\ = \; & 8h^2 - 8he + 2e^2 + 2h - e + 2h^2 + 2h - 2eh + h - e \\ = \; & 10h^2 - 10eh + 5h \end{aligned}$$

then the candidate limit is the limit scheme.

For $\tau \geq 3$ there are conditions that are only visible looking at the nonreduced structure of iT_i.

Example 2.42. Let's consider the case of one point of multiplicity 6 and six points of multiplicity 3 coming together. Those colliding points translate into points of the same multiplicity on P. The minimal degree k needed to have a curve with the assigned multiplicities is 9 and the linear system on W is the unique cubic having one node at the multiplicity 6 point and passing through the other six, counted three times. Let B be the corresponding reduced curve on P; this means the linear system on P is $3B$.

In this situation $\sigma = -(3B).B = 3$, $\tau = 3$ and e and h assume value 0 and 1 respectively. We need to blow up B three times; the first time we create an exceptional divisor $T_1 \cong \mathbb{F}_2$, the second time the exceptional divisor is $T_2 \cong \mathbb{F}_1$ counted twice and the third and final time we get $T_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ surface counted three times. The linear system on T_3 is B_3 and the restriction to $2T_2$ and T_1 is empty.

Now let's look at the conditions that this imposes to the corresponding system on $V^{(2)}$. With the same arguments of the previous example the construction tells us that the limit scheme contains a point of multiplicity 9 with three infinitely near points of type $(1, 1, 1)$. This adds up to

$$\frac{9 \cdot 10}{2} + 3(1 + 1 + 1) = 54$$

conditions. The statement of 2.40 calls for a total of 3 more conditions, for a total of 57. And the length of the limit scheme is the same as the general fiber of $\mathcal{D}$, which is one point of multiplicity 6 (21 conditions) and 6 points of multiplicity 3 (6 conditions each, 36 total) for a total of 57 conditions; the candidate limit described by 2.40 is then the flat limit.

Among the 3 "extra" conditions , 2 are visible looking at the reduced level: Any element of $L|_{\widetilde{V}}$ intersects each of the fibers over the points q_1, q_2 and q_3 in a point of $T_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the linear system there is made by a single line of the fibration of B_3. This means that one of these points is free but then the choice of the other two is forced, giving two more conditions. The missing condition is only visible on the nonreduced structure on $3T_3$.

2.4 Elimination of fixed genus 1 curves

Let now B be a (smooth, irreducible) genus 1 curve on P appearing as a fixed component of L_P with multiplicity σ. We assume that B meets R transversally at τ points $q_1^1, \ldots, q_\tau^1$, that B does not meet the other components of L_P and that $(B^2)_P = 0$ but its self-intersection divisor is not trivial. The curve B is the proper transform from the projective plane W of a (possibly singular) curve of degree τ; let's write the self-intersection divisor of B as $\tau H - D$, where H is the pull-back of the hyperplane section.

As we did in the rational case, we proceed iteratively

First step: Let $\mathcal{X}^{(1)} \xrightarrow{\pi^{(1)}} \mathcal{X}$ be the blowup of $\mathcal{X}$ along B and let T_1 be the exceptional divisor; the surface T_1 is isomorphic to a scroll with base curve B_1. The surfaces T_1 and P intersect transversally along a curve which is the base curve of T_1 and the curve B on P; we will refer to it either way depending on which surface we are considering. Let $V^{(1)}$ be the proper transform of V; $V^{(1)}$ is isomorphic to the blowup of V at the points $q_1^1, \ldots, q_\tau^1$.

We apply proposition 2.5 again; The bound on the multiplicity of L along B is the least j such that $h^0\left(T_1, {\pi^{(1)}}^*(\mathcal{L}) \otimes \mathcal{O}(-jT_1)|_{T_1}\right) > 0$.

The restriction of the first factor can be done as follows:

$$ {\pi^{(1)}}^*(L)|_{T_1} \sim \pi^{(1)*}(L)|_{\pi^{(1)*}(B)} \sim \pi^{(1)*}(L|_B) \sim \sigma(\tau H - D) f_1 $$

while the second factor can be computed using the fact that

$$ T_1|_{T_1} \sim (\mathcal{X}_0^{(1)} - P - V^{(1)})|_{T_1} \sim -B_1 - H f_1 $$

This tells us that

$$ \left({\pi^{(1)}}^*(\mathcal{L}) \otimes \mathcal{O}(-jT_1)\right)|_{T_1} \sim \sigma(\tau H - D) f_1 + j(B_1 + H f_1) $$

For $j = 0$ this is not an effective divisor on T_1 but $j = 1$ is already enough since then the f_1 part is a divisor on B of degree $\tau > 0$ and, by Riemann-Roch theorem, every divisor of positive degree on a curve of genus 1 has an effective divisor in its linear equivalence class. This means that the multiplicity of L along B is at least 1.

Let $L^{(1)}$ be the linear system $|\pi^{(1)^*}L - T_1|$ on $\mathcal{X}^{(1)}$ and let us now check whether B is a fixed component of $L^{(1)}$; we have that

$$L^{(1)}|_{B_1} \sim L^{(1)}|_{T_1}|_{B_1} \sim (B_1|_{B_1})|_{T_1} + (\sigma\tau + 1)H - \sigma D$$

here the divisor $(B_1|_{B_1})|_{T_1}$ can be computed via the triple point formula (proposition 1.10) as $-(\tau + 1)H + D$ to obtain

$$L^{(1)}|_B \sim (\sigma - 1)(\tau H - D).$$

Then if $\sigma = 1$ we are done; otherwise we have to apply the procedure again.

i-th step: In the previous steps we have constructed a sequence of blowups

$$\mathcal{X}^{(i-1)} \xrightarrow{\pi^{(i-1)}} \cdots \xrightarrow{\pi^{(1)}} \mathcal{X}.$$

For each $j < i$, T_j is the exceptional divisor of $\pi^{(j)}$ and it is isomorphic to a ruled surface; we called B_j its base curve. Where they are both defined T_j intersects T_{j-1} transversally along B_{j-1} and T_j and T_k are disjoint if j and k are not consecutive; the surfaces P and T_{i-1} intersect transversally along B.

We also called $V^{(j)}$ the proper transform of V via $\pi^{(j)} \circ \cdots \circ \pi^{(1)}$, $L^{(j)}$ the linear system obtained by subtracting from the transform of L a suitable number of copies of the exceptional divisors and $\mathcal{L}^{(j)}$ the associated invertible sheaf.

The fiber over $t = 0$ of the composition $\pi^{(j)} \circ \cdots \circ \pi^{(1)} \circ \psi''$ is

$$\mathcal{X}_0^{(i-1)} = P + \sum_{l=1}^{i-1} lT_l + V^{(i-1)}.$$

Moreover in step $(i-1)$ we computed the intersection divisor

$$L^{(i-1)}.B = (\sigma - (i-1))(\tau H - D).$$

Let $\mathcal{X}^{(i)} \xrightarrow{\pi^{(i-1)}} \mathcal{X}^{(i-1)}$ be the blowup of $\mathcal{X}^{(i-1)}$ along B; the proper transform of P under this blowup is isomorphic to P itself so we still call it P and we can still talk about B meaning by that the corresponding curve on P. Let T_i be the exceptional divisor; T_i is isomorphic to a scroll with base curve B. The surfaces T_i and P intersect transversally along a curve which is the base curve B_i of T_i and the curve B on P; we will refer to it either way depending on which surface we are considering. Let $V^{(i)}$ be the proper transform of $V^{(i-1)}$; $V^{(1)}$ is isomorphic to the blowup of $V^{(i-1)}$ at the points $q_1^i, \dots, q_\tau^i$.

We apply proposition 2.5 again; The bound on the multplicity of $L^{(i-1)}$ along B is the least j such that $h^0\left(T_i, \pi^{(i)^*}(\mathcal{L}^{(i-1)}) \otimes \mathcal{O}(-jT_i)|_{T_i}\right) > 0$.

The restriction to T_i of the first factor can be done as follows:

$$\begin{aligned} \pi^{(i)^*}(L^{(i-1)})|_{T_i} &\sim \pi^{(i)*}(L^{(i-1)})|_{\pi^{(i)*}(B)} \\ &\sim \pi^{(1)*}(L^{(i-1)}|_B) \\ &\sim (\sigma-(i-1))(\tau H - D)f_i \end{aligned}$$

the computation of the second factor is slightly more complicated since this time the T_i component of $\mathcal{X}_0^{(i)}$ is not reduced.

$$\begin{aligned} -iT_i|_{T_i} &\sim -\left(\mathcal{X}_0^{(i)} - P - V^{(1)} - \sum_{l=1}^{i-1} lT_l\right)\Bigg|_{T_i} \\ &\sim Hf_i + B_i + B_{i-1} \end{aligned} \tag{2.3}$$

here we need to substitute to B_{i-1} a divisor of the form $aB_i + Mf_i$, where M is a divisor on B_i. Since B_i meets any fiber in one point we have that $a = 1$ and the fact that B_{i-1} and B_i are disjoint implies that $M = -(B_i|_{B_i})_{T_i}$. This divisor M can be computed via the triple point formula (proposition 1.10) as $-(\tau i+1)H + iD$. Plugging this result into the 2.3 and dividing by i we obtain that

$$-T_i|_{T_i} \sim B_i + ((\tau i - \tau + 1)H - (i-1)D)f_i$$

This tells us that

$$\left(\pi^{(i)^*}(\mathcal{L}^{(i-1)}) \otimes \mathcal{O}(-jT_i)\right)\Big|_{T_i} \sim (\sigma-i+1)(\tau H-D)f_i + jB_i + j((\tau i-\tau+1)H-(i-1)D)f_i$$

For $j = 0$ this is not an effective divisor on T_1 but $j = 1$ is already enough since then the f_1 part is a divisor on B of degree $\tau > 0$ and, by Riemann-Roch theorem, every divisor of positive degree on a curve of genus 1 has an effective divisor in its linear equivalence class. This means that the multiplicity of $L^{(i-1)}$ along B is at least 1.

Let $L^{(i)}$ be the linear system $|\pi^{(i)^*}L^{(i-1)} - T_i|$ on $\mathcal{X}^{(())}i$ and let's now check whether B is a fixed component of $L^{(i)}$; we have that

$$\begin{aligned} L^{(i)}|_B &\sim L^{(i)}|_{T_i}|_{B_i} \\ &\sim (B_i|_{B_i})_{T_i} + (\tau\sigma - i\tau + \tau + i\tau - \tau + 1)H - \sigma D \\ &\sim -(\tau i+1)H + iD + \tau\sigma H + H - \sigma D \\ &\sim (\sigma - i)(\tau H - D). \end{aligned}$$

Then if $i = \sigma$ we are done; otherwise this divisor is not effective and we have to apply the procedure again. This also means that the procedure is repeated exactly σ times, regardless of τ.

We can summarize the results of this section in the following proposition:

Proposition 2.43. *Let $p_1(t), \dots, p_n(t)$ be n points coming together for $t = 0$ in a general way and let's make the construction described in the previous section. If there is a (smooth, irreducible) curve B of genus 1 on P appearing with multiplicity σ as a fixed component of L_P such that B does not meet any other component of L_P, it meets R transversally at the points $q_1^1, \dots, q_\tau^1$ and $(B^2)_P = 0$ but the self-intersection divisor is not trivial, then the limit scheme contains in its first infinitesimal neighborhood a set of τ points of type (1^σ) at $q_1^1, \dots, q_\tau^1$.*

This accounts for $\sigma\tau$ conditions on the curves of $L|_{X_0}$ and thus every such curve B contributes to the length of the candidate limit scheme by $\sigma\tau$.

Example 2.44. Let's consider the case of nine points of multiplicity m coming together. Those colliding points translate into nine points of the same multiplicity on P. The minimal degree k of a curve having multiplicity m at all of them is $3m$ and the linear system on W has only one element: the cubic B passing through those points counted with multiplicity m. B is curve of genus one and the self-intersection divisor of B is $3H - D$, where D is the divisor $\sum_{i=1}^{9} \widetilde{p}_i(0)$; $3H - D$ has degree 0 so we can apply the process we just described.

In this situation $\sigma = -(mB).B = m$ and $\tau = 3$. We need to blow up B m times. The candidate limit is then a point of multiplicity $3m$ with three infinitely near points of type (1^m). This scheme has length

$$\frac{3m(3m+1)}{2} + 3 \cdot m \cdot 1 = 9\frac{m(m+1)}{2}$$

which is the same as the limi scheme, meaning that it is indeed the limit.

Example 2.45. Let a, m be positive integers, with $a \geq 3$. Let's consider the collision of a point of multiplicity $m(a-2)$, $a-3$ points of multiplicity $2m$ and 8 points of multiplicity m.

One can see that the minimum degree of a plane curve that passes through all those points with the assigned multiplicities is am by induction on m.

If $m = 1$ we claim that the the minimum degree is a. To prove this we note that the (projective) virtual dimension of the linear system $L_a(a-2; 2^{a-3}; 1^8)$ is 0, meaning that there exist a curve B of degree a passing through the assigned points with those multiplicities. If degree $a - 1$ were enough we would have that the linear system $L_{a-1}(a-2;2^{a-3};1^8)$ would contain all the $a-3$ lines from the multiplicity $a-2$ point to each of the nodes as fixed components; the remaining part would only have degree 2 but would have to pass through $a+6$ points in general position, which is impossible.

If $m \geq 2$ then degree am is enough (just take the curve B counted with multiplicity m) while degree $am-1$ is not; in fact the intersection number $|L_{am-1}((a-2)m;2m^{a-3};m^8)|.B=-a$, meaning that B is a fixed component of the system which can then be rewritten as $|L_{a(m-1)}((a-2)(m-1);2(m-1)^{a-3};(m-1)^8)|+B$ which is empty by the inductive hypothesis. This proves that the minimal degree is always am.

Now we observe that for $a=3$ the problem is the collision of nine points of multiplicity m which we treated in example 2.44; speifically, we know that the linear system $L_{3m}(m^9)$ has a unique element which is the cubic curve passing through the 9 points counted m times.

If $a>3$ we can use Cremona transformations on W to reduce ourselves back to this case. Cremona transformations preserve the dimension of linear systems, therefore that will prove that the curve is unique, and they induce a birational morphism between corresponding curves, therefore the curve will be irreducible. This would then prove that the linear system $L_{am}(m(a-2),2m^{d-3},m^8)$ has a unique irreducible element which is the (unique) curve B in $L_a(a-2,2^{d-3},1^8)$ counted m times.

If $a=4$ the points have multiplicities $(2m,2m,m^8)$. We can apply a Cremona transformation centered at the two points of multiplicity $2m$ and one of the points of multiplicity m which gives us the reduction

$$\begin{array}{r|llll} \text{degree} & \text{multiplicities} \\ 4m & 2m & 2m & m & m^7 \\ 3m & m & m & 0 & m^7 \end{array}$$

If $a \geq 5$ we can apply a Cremona transformation centered at the point of high multiplicity and at two of the points of multiplicity $2m$; this yields the reduction

$$\begin{array}{r|lllll} \text{degree} & \text{multiplicities} \\ am & (a-2)m & 2m & 2m & 2m^{d-5} & m^8 \\ (a-2)m & (a-4)m & 0 & 0 & 2m^{d-5} & m^8 \end{array}$$

where the second line is the same problem where a has been reduced by 2. We can then proceed iteratively picking two other of the points of multiplicity $2m$ until we lower a to either 4 or 3.

Following the construction outlined the linear system L on P is just mB.

It is easy to see that the curve B has genus 1, meets the line R at am points and that these intersections are transversal for the genericity of the collision. Our candidate limit scheme is then a point of multiplicity am with a points of type (1^m) in its first order neighborhood. The length of such a scheme is

$$\frac{am(am+1)}{2} + am = \frac{am(am+3)}{2}$$

while the length of the limit scheme is

$$\tfrac{m(a-2)(m(a-2)+1)}{2} + (a-3)\tfrac{2m(2m+1)}{2} + 8\tfrac{m(m+1)}{2} = \\ = \tfrac{am(am+3)}{2}$$

Proving that the candidate is the limit scheme.

The last example covers cases of a curve B of genus 1 for any values of τ and σ; this means that, differently from the rational case, fixed curves of genus one do not impose in general any condition to the limit scheme other than having τ points of type (1^σ) in the first infinitesimal neighborhood.

Chapter 3

Limits of a small number of fat points

In this chapter we will use the techniques developed so far to compute the limit of n points $p_1, \ldots, p_n$ of multiplicities $m_1, \ldots, m_n$. We will treat each n separately.

Our general procedure will be the following:

1. Determine the minimum k such that $\dim L_k(m_1, \ldots, m_n) > 0$.

2. Find the fixed components of $L = L_k(m_1, \ldots, m_n) > 0$.

3. Look at the movable part of L; compare its dimension with its degree.

4. Use the results in chapter 2 to produce a candidate limit scheme.

5. Verify that the length of the candidate limit scheme matches the length of the actual limit scheme and close the argument with proposition 1.7.

We will use these method to get the limit of up to four points of any multiplicity, the limit of to nine points having the same multiplicity and a few other easy cases. For a higher number of points it is considerably more difficult to execute the first part of our strategy because there are fewer results for the polynomial interpolation problem.

3.1 Non-homogeneous Collisions

Proposition 3.1. *The flat limit of two points p_1, p_2 of multiplicities m_1, m_2 respectively, with $m_1 \geq m_2$, coming together in a general way is a point of multiplicity m_1 with an infinitely near point of multiplicity m_2*

Proof. After the first construction we find two points of multiplicity m_1 and m_2 on W. The lowest k such that $L_k(m_1, m_2) \neq \emptyset$ is m_1 and the linear system consists of the line B through p_1 and p_2 counted m_2 times plus $m_1 - m_2$ more lines through the point of multiplicity m_1. This linear system has thus dimension $m_1 - m_2$, which is also the degree of its movable part, and this means that the moveable part of L_P imposes no conditions. The only fixed component is the line B. The only condition on the first infinitesimal neighborhood is then to have a fixed point of multiplicity m_2.

The candidate limit scheme is then a point of multiplicity m_1 with a point of multiplicity m_2 in its first infinitesimal neighborhood. The length of this scheme is the same as the length of a point of multiplicity m_1 plus the length of a point of multiplicity m_2, so, by proposition 1.7, the candidate limit scheme is the limit scheme. □

3.1.1 Collisions of three points

Lemma 3.2. *Let m_1, m_2, m_3 be positive integers, $m_1 \geq m_2 \geq m_3$. The lowest degree d for which the linear system $L_d(m_1, m_2, m_3)$ is not empty is*

$$\max\left\{m_1; \left\lceil \frac{m_1 + m_2 + m_3}{2} \right\rceil\right\} = \begin{cases} m_1 & \text{if } \mathrm{m_1 \geq m_2 + m_3} \\ \left\lceil \dfrac{m_1 + m_2 + m_3}{2} \right\rceil & \text{if } m_1 < m_2 + m_3 \end{cases}$$

Moreover, if B_{ij} is the line through p_i and p_j, the linear system is of the following form:

If $m_1 \geq m_2 + m_3$, the fixed part is $m_2 B_{12} + m_3 B_{13}$ and the movable part consists of $m_1 - m_2 - m_3$ lines through p_1.

If $m_1 < m_2 + m_3$, the fixed part is $(m_1 + m_2 - d)B_{12} + (m_1 + m_3 - d)B_{13} + (m_2 + m_3 - d)B_{23}$. If $m_1 + m_2 + m_3$ is even there is no movable part; if $m_1 + m_2 + m_3$ is odd the movable part is the linear system of the conics pasisng through p_1, p_2 and p_3.

Proof. The minimum degree is at least m_1 since no curve of a lower degree can have a point of multiplicity m_1. If $m_1 \geq m_2 + m_3$ there exist a suitable linear system of degree exactly m_1, so the minimum degree in this case is m_1. Comparing the intersection numbers with the dimension of the system one finds that the elements of the linear system realizing this minimum consist of m_1 lines through the point p_1, so that m_2 of them also pass through p_2 and m_3 others pass through p_3. This leaves $m_1 - m_2 - m_3$ lines passing through p_1 as the movable part of the system.

If $m_1 < m_2 + m_3$ it is easy to see that the linear system described in the statement has the stated degree and has multiplicity m_i at each p_i. To prove that that is the one having minimum degree, let k be an integer such that $L_k(m_1, m_2, m_3) \neq \emptyset$. We can apply a quadratic Cremona transformation with base points p_1, p_2 and p_3; the transformed system has degree $2k - m_1 - m_2 - m_3$ and is not empty, meaning that $k \geq \left\lceil \frac{m_1+m_2+m_3}{2} \right\rceil$.

To prove that the system has the stated fixed part, consider that, on the blowup of $\mathbb{P}^2$ at p_1, p_2 and p_3, $L_k(m_1, m_2, m_3).B_{ij} = k - m_i - m_j$; since the lines B_{ij} are (-1)-curves, they need to be contained in the system with multiplicity $m_j + m_i - k$. □

Proposition 3.3. *Let p_1, p_2 and p_3 be three points of positive multiplicities m_1, m_2 and m_3 respectively, with $m_1 \geq m_2 \geq m_3$, coming together in a general way.*

1. *If $m_1 \geq m_2 + m_3$, the flat limit is a point of multiplicity m_1 with two infinitely near points of multiplicities m_2 and m_3.*

2. *If $m_1 > m_2 + m_3$, let $m_1 + m_2 + m_3 = 2k - e$, with e either 0 or 1. The flat limit is a point of multiplicity k with three infinitely near points of multiplicities $k - e - m_1$, $k - e - m_2$ and $k - e - m_3$.*

Proof. The multiplicity of the limit point is the minimum degree found earlier and each line which is a fixed component of the system on W gives a fixed infinitely near point of the same multiplicity.

Each line present in the fixed part of L_W with multiplicity σ requires L_V to have multiplicity σ at its intersection point with R. The multiplicities stated are the ones found in the previous lemma, remembering that $m_1 + m_2 + m_3 = 2k - e$. If m is even there is nothing more to say. If m is odd the presence of the conics still does not impose any conditions since they cut a complete linear series on R. □

3.1.2 Collisions of four points

Throughout this section, we will consider the four multiplicities ordered as $m_1 \geq m_2 \geq m_3 \geq m_4$.

Lemma 3.4. *Let $d = d(m_1, m_2, m_3, m_4)$ be the minimum degree d for which the linear system $L_d(m_1, m_2, m_3, m_4)$ is not empty. We have that*

$$d \geq \left\lceil \frac{m_1 + m_2 + m_3 + m_4}{2} \right\rceil .$$

Proof. Let k be any degree such that $L_k(m_1, m_2, m_3, m_4) \neq \emptyset$. We can apply a Cremona transformation based in p_1, p_2 and p_3 to the system and the transformed system is of the type $L_{2k-m_1-m_2-m_3}(k-m_1-m_2, k-m_1-m_3, k-m_2-m_3, m_4)$. The original system is not empty if and only if the transformed one is and this system is empty if its degree is less than m_4 since it is required to have a fixed point of multiplicity m_4. This means that $2k-m_1-m_3-m_3 \geq m_4$ or $k \geq \frac{m_1+m_2+m_3+m_4}{2}$ and, since k is an integer, $k \geq \lceil \frac{m_1+m_2+m_3+m_4}{2} \rceil$.

In both cases the number of conditions imposed match the length of the limit scheme, meaning that the canddate limit scheme is the limit scheme. □

The bound we found in this lemma is the minimum degree except in the case where one point has multiplicity greater than the sum of the other three. We prove this in three cases and then will reduce all the others to one of these three.

Lemma 3.5. *The minimum d such that the system $L_d(m^4)$ is not empty is $2m$. The system that realizes this minimum has dimension m and its members are the unions of m conics, each passing through the points $p_1, \dots, p_4$.*

Proof. The union of m conics each passing through $p_1, \dots, p_4$ realizes the lower bound on the degree found in 3.4, so $2m$ is the minimum degree. These unions of conics form a linear system of dimension m so we only need to prove that this is the dimension of $L_{2m}(m^4)$; we can do so by applying a Cremona transformation of base points three of the four points and find that $\dim L_{2m}(m^4) = \dim L_m(m) = m$. □

Lemma 3.6. *The minimum d such that the system $L_k(m+1, m^3)$ is not empty is $2m+1$. The system that realizes this minimum has dimension equal to its degree and has no fixed components.*

Proof. $2m+1$ is the lower bound found in 3.4; we need to prove that it is realized. Consider the linear system $L_{2m+1}(m+1, m^3)$ and apply a Cremona transformation based at the points p_1, p_2 and p_3 to it; the transformed system is of type $L_{m+1}(1, m)$ and we know that it is nonspecial since it is obtained by adding a single point to $L_{m+1}(m)$, which is not. The dimension of both systems is then $\frac{(m+1)(m+4)}{2} - \frac{(m-1)m}{2} - 1 = 2m+1$.

Moreover, the original linear system does not have any of the fundamental lines of the Cremona transformation as fixed components since its intersection number with them is not negative. Any other fixed component should correspond to a fixed component of the transformed system but the general element of $L_{m+1}(m)$ is irreducible, preventing that. □

Lemma 3.7. *The minimum d such that the system $L_k(m^3, m-1)$ is not empty is $2m$. The system that realizes this minimum has dimension equal to its degree and has no fixed components.*

Proof. $2m$ is the lower bound found in 3.4; we need to prove that it is realized. Consider the linear system $L_{2m}(m^3, m-1)$ and apply to it a Cremona transformation centered at the three multiplicity m points; the transformed system is of type $L_m(m-1)$ and we know that it has dimension m, so the original system has dimension m as well.

Moreover, the original linear system does not have any of the fundamental lines of the Cremona transformation as fixed components since its intersection number with them is not negative. Any other fixed component should correspond to a fixed component of the transformed system but the general element of $L_m(m-1)$ is irreducible, preventing that. □

Proposition 3.8. *Let m_1, m_2, m_3 and m_4 be positive integers, $m_1 \geq m_2 \geq m_3 \geq m_4$. The linear system $L_d(m_1, m_2, m_3, m_4)$, where d is the minimum degree for which the system is not empty, is of the following form:*

1. *if $m_1 \geq m_2+m_3+m_4$ them $d = m_1$ and the system consists of $m_2B_{12} + m_3B_{13} + m_4B_{14}$ as the fixed part and $m_1 - m_2 - m_3 - m_4$ lines through p_1 as the movable part.*
2. *if $m_1 < m_2 + m_3 + m_4$ then $d = \left\lceil \frac{m_1+m_2+m_3+m_4}{2} \right\rceil$. The system realizing this minimum has fixed part given by the sum of:*
   - *(a)* $(m_1 + m_2 - d)B_{12}$
   - *(b)* $(m_1 + m_3 - d)B_{13}$
   - *(c) If $m_1+m_4 \geq d$, $(m_1+m_4-d)B_{14}$. If $m_1+m_4 < d$, $(m_2+m_3-d)B_{23}$.

   and

   - *(i) If $m_1 + m_2 + m_3 + m_4$ is odd, the movable part of the system has degree equal to its dimension equal to n; n is $2(d - m_1) + 1$ if $m_1 + m_4 \geq d$ or $2m_4 + 2$ if $m_1 + m_4 < d$.*
   - *(ii) If $m_1 + m_2 + m_3 + m_4$ is even, the movable part of the system is made of n conics, each passing through the four points. n is $d - m_1$ if $m_1 + m_4 \geq d$ and m_4 if $m_1 + m_4 < d$.*

Proof. In the case where $m_1 \geq m_2 + m_3 + m_4$ the proof is immediate.

If $m_1 < m_2+m_3+m_4$, let k and e be integers such that $m_1+m_2+m_3+m_4 = 2k - e$ and $0 \leq e \leq 1$. Let us consider the system $L = L_k(m_1, m_2, m_3, m_4)$. The intersection numbers of L with B_{12} and B_{13} are less than or equal to 0 because of the ordering of the m_i's, proving that those lines are contained in the fixed part of L with multiplicities $k - m_1 - m_2$ and $k - m_1 - m_3$ respectively. For the same reason, if $m_1 + m_4 \geq k$ the line B_{14} appears in the fixed part of L with multiplicity $k - m_1 - m_4$; if $m_1 + m_4 < k$ then $m_2 + m_3 > k - e$, which implies $m_2 + m_3 \geq k - e + 1 \geq k$, allowing us to say the same thing on B_{23}.

Subtracting those fixed lines, the residual system has degree k' and has to pass through the points $p_1, \ldots, p_4$ with multiplicities $m'_1, \ldots, m'_4$.

If $m_1 + m_4 \geq k$

$$\begin{aligned}
k' &= k - (m_1 + m_2 - k) - (m_1 + m_3 - k) - (m_1 + m_4 - k) \\
&= 4k - 2m_1 - m_1 - m_2 - m_3 - m_4 = 2k - 2m_1 + e \\
m'_1 &= m_1 - (m_1 + m_2 - k) - (m_1 + m_3 - k) - (m_1 + m_4 - k) \\
&= 3k - 2m_1 - m_2 - m_3 - m_4 = k - m_1 + e \\
m'_2 &= m_2 - (m_1 + m_2 - k) = k - m_1 \\
m'_3 &= m_3 - (m_1 + m_3 - k) = k - m_1 \\
m'_4 &= m_4 - (m_1 + m_4 - k) = k - m_1
\end{aligned}$$

otherwise, if $m_1 + m_4 < k$

$$\begin{aligned}
d' &= k - (m_1 + m_2 - k) - (m_1 + m_3 - k) - (m_2 + m_3 - k) \\
&= 4k - 2m_1 - 2m_2 - 2m_3 = 2m_4 + 2e \\
m'_1 &= m_1 - (m_1 + m_2 - k) - (m_1 + m_3 - k) = \\
&= 2k - m_1 - m_2 - m_3 = m_4 + e \\
m'_2 &= m_2 - (m_1 + m_2 - k) - (m_2 + m_3 - k) = \\
&= 2k - m_1 - m_2 - m_3 = m_4 + e \\
m'_3 &= m_3 - (m_1 + m_3 - k) - (m_2 + m_3 - k) = \\
&= 2k - m_1 - m_2 - m_3 = m_4 + e \\
m'_4 &= m_4
\end{aligned}$$

If $e = 0$ we get to a case where all multiplicities are equal and their degree is double that number, which is not empty and is described 3.5. If $e = 1$, we have that all multiplicities are equal except for one which is either greater or smaller than the others by one; those are the cases covered by 3.6 and 3.7 and in both cases $L_{k'}(m'_1, m'_2, m'_3, m'_4)$ is not empty.

The system $L_k(m_1, m_2, m_3, m_4)$ realizes the lower bound found in 3.4, then k is the minimum degree d for which the system is not empty. □

Theorem 3.9. *Consider the case where p_1, p_2, p_3 and p_4 are four points of positive multiplicities m_1, m_2, m_3 and m_4 respectively, with $m_1 \geq m_2 \geq m_3 \geq m_4$, coming together in a general way.*

1. *If $m_1 \geq m_2 + m_3 + m_4$, the flat limit is a point of multiplicity m_1 with three infinitely near points of multiplicities m_2, m_3 and m_4.*

2. *If $m_1 < m_2 + m_3 + m_4$, let $m_1 + m_2 + m_3 + m_4 = 2k - e$, with e either 0 or 1. The flat limit is a point of multiplicity k and in its first infinitesimal neighborhood there are*

   (a) *If $m_1 + m_2 > k$, one point of multiplicity $(m_1 + m_2 - d)$*

   (b) *If $m_1 + m_3 > k$, one point of multiplicity $(m_1 + m_3 - d)$*

   (c) *If $m_1 + m_4 > k$, one point of multiplicity $(m_1 + m_4 - d)$. If $m_2 + m_3 > k$, $(m_2 + m_3 - d)$. Note that only one of them may exist.*

   (d) *If $e = 0$, there is a correspondence between the points on the first neighborhood and the points of the limit scheme on the first infinitesimal neighborhood that are not fixed by the conditions above form a set that is invariant under this correspondence. Those points are $2m_4$ if $m_1+m_4 < d$ and $2k-2m_1$ otherwise and the correspondence depends on the way in which the four general points were coming together.*

Now we want to write a characterization for the polynomials belonging to the ideal of the limit scheme; for simpicity we do this only in the homogeneous case (i.e. $m_1 = m_2 = m_3 = m_4 = m$). We denote by Z the limit scheme; we suppose that its support, p is the origin.

What theorem 3.9 tells us is that the limit scheme is a point p of multiplicity $2m$ and that there is an involution ι, defined on R, such that the $2m$ points on R that correspond to the tangent directions at p are divided into m pairs of points and each pair is sent into itself by ι.

Let's put projective coordinates on R such that the line $\lambda_1 X + \lambda_2 Y = 0$ has coordinates $[\lambda_1 : \lambda_2]$. Any involution on R is represented by an element $A_{a,b,c} \in PGL_n(\mathbb{C})$ of type

$$A_{a,b,c} = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

where $a, b, c \in \mathbb{C}$ and $a^2 + bc \neq 0$, so the datum of ι is equivalent to the datum of a suitable triple (a, b, c), and let $f = \sum_{i,j\geq 0} \alpha_{i,j} X^i Y^j$ be a polynomial in X and Y. We want to characterize the polynomials f contained in the ideal I_Z defining Z i.e. the polynomials f such that the curve defined by f contains Z.

If $Z \subset V(f)$ then the fact that $V(f)$ has a point of multiplicity at least $2m$ at the origin means that $\alpha_{i,j} = 0$ whenever $i + j < 2m$. To describe the condition on the tangents at this point we write the part of degree $2m$ (supposing for now that it is not zero) as

$$f_{2m}(X,Y) = \prod_{h=1}^{m} (\gamma_{2,0}^h X^2 + \gamma_{1,1}^h XY + \gamma_{0,2}^h Y^2)$$

or

$$f_{2m}(X,Y) = \prod_{h=1}^{m} (\lambda_1^h X + \lambda_2^h Y)(\mu_1^h X + \mu_2^h Y),$$

where $\gamma_{2,0}^h X^2 + \gamma_{1,1}^h XY + \gamma_{0,2}^h Y^2 = (\lambda_1^h X + \lambda_2^h Y)(\mu_1^h X + \mu_2^h Y)$, so that the points $[\underline{\lambda^h}]$ and $[\underline{\mu^h}]$ are exchanged by ι.

We have to impose that, for every h, $[\underline{\mu^h}] = [A_{a,b,c} \cdot \underline{\lambda^h}]$ i.e. $\mu_1^h = a\lambda_1^h + b\lambda_2^h$ and $\mu_2^h = c\lambda_1^h - a\lambda_2^h$. It follows that, up to a common factor, the coefficients $\gamma_{i,j}^h$ are

$$\begin{aligned} \gamma_{2,0}^h &= a(\lambda_1^h)^2 + b\lambda_1^h\lambda_2^h \\ \gamma_{1,1}^h &= c(\lambda_1^h)^2 + b(\lambda_2^h)^2 \\ \gamma_{0,2}^h &= -a(\lambda_2^h)^2 + c\lambda_1^h\lambda_2^h. \end{aligned}$$

and we can find the following relation among them:

$$c\gamma_{2,0}^h - a\gamma_{1,1}^h - b\gamma_{0,2}^h = 0. \tag{3.1}$$

What we have proved is that if a polynomial f is contained in I_Z, then there exists a factorization of its homogeneous part of degree $2m$ in m polynomials of degree two whose coefficients satisfy 3.1 (this is also true for the polynomials of order greater than $2m$). These conditions, together with the fact that the order of f needs to be at least $2m$, are all linearly independent and their total number is exactly the length of Z, so they define the ideal of Z.

Corollary 3.10. *The limit of four multiple points of multiplicities $m_1 \geq \cdots \geq m_4$ coming together in a general way is a point of some multiplicity k with no extra conditions in its first infinitesimal neighborhood if and only if $m_2 = m_3 = m_4 = m_1 - 1$ or $m_2 = m_3 = m_1$ and $m_4 = m_1 - 1$. In the first case the multiplicity of the limit point is $k = 2m_1 + 1$ and in the second case it is $k = 2m_1$.*

Proof. The limit of four points of multiplicities $m, m-1, m-1, m-1$ and $m, m, m, m-1$ can be computed via 3.9 or directly via lemmas 3.6 and 3.7 and it has to contain a point of multiplicity respectively $2m-1$ and $2m$. The length of a point with multiplicity k is already the length of the limit scheme therefore the limit scheme is just a point of multiplicity k.

Conversely, we know that $m_1+m_2+m_3+m_4$ has to be odd since otherwise there would be conditions on the first infinitesimal neighborhood of the limit point given by the last part of 3.9; this means that $k = \frac{m_1+m_2+m_3+m_4+1}{2}$.

In order to have no conditions on the first infinitesimal neighborhood, none of the conditions in the statement of 3.9 has to be satisfied. The ordering of the points and the expression found for the mutliplicity k implies that $m_1 + m_2 \geq k$ and having to not satisfy the first condition in 3.9 tell us that $m_1 + m_2 \leq k$ as well. This means that

$$m_1 + m_2 = m_3 + m_4 + 1$$

and the same argument holds for m_3, yielding (after reordering)

$$m_1 - m_2 = m_4 + 1 - m_3.$$

Adding up the two we find that $m_4 = m_1 - 1$ and subtracting the second one form the first we get that $m_2 = m_3$; given the ordering we gave to the multiplicities only $m_2 = m_3 = m_1$ or $m_2 = m_3 = m_1 - 1$ are possible. □

3.2 Homogeneous collisions

In this section we treat the cases where the colliding points have the seme multiplicity m. This assumption greatly simplifies the things because it is easier to find the minimum degree k that makes the linear system $L_k(M^n)$ not empty and because there exist a complete classification of the homogeneous (-1)-special linear systems for up to 9 points in [4]. Homogeneous collisions of 5 points or less have already been described in [3]; we won't repeat them here.

3.2.1 Simple points

Proposition 3.11. *Let's consider any number n of points of multiplicity 1 coming together in a general way. Then the limt scheme is a point of multiplicity $r = \left\lceil \frac{-3+\sqrt{9+8n}}{2} \right\rceil$ together with $\frac{r(r+3)}{2} - n$ conditions given by the fact that the k points forming the first ininitesimal neighborhood of the limit scheme have to lie on the restriction to R of the linear system $L_k(1^n)$ on W.*

Proof. The least d such that $h^0(L_d(1^n)) > 0$ can be computed by looking at the virtual dimension of the system since linear systems of that kind are never special; this leads to find k as stated. The conditions mentioned in the statement then come from the analysis made in section 2.2. □

Corollary 3.12. *Any scheme which is a point with multiplicity m can be obtained as a collision of $\frac{m(m+1)}{2}$ simple points coming together in a general way.*

3.2.2 Nodes

Proposition 3.13. *Let's consider any number $n \neq 2, 5$ of double points coming together in a general way. The limit scheme is a point of multiplicity $k = \left\lceil \frac{-3+\sqrt{9+24n}}{2} \right\rceil$ together with $k - \left(\frac{k(k+3)}{2} - 3n \right)$ conditions given by the fact that the k points in the first infinitesimal neighborhood have to lie in the restriction to R of $L_k(2^n)$.*

If $n = 2$ the limit is a double point with an infinitely near double point and if $n = 5$ the limit is a point of multiplicity 4 with a pair of points of type $(1, 1)$ with matching second order tangents in its first infinitesimal neighborhood.

Proof. We use the fact that the linear system $L_d(2^n)$ is not special for any $n \neq 2, 5$. This means that the minimum degree k such that $h^0(L_d(2^n)) > 0$ can be found by looking at the virtual dimension as the least integer d such that $\frac{d(d+3)}{2} - 3n \geq 0$, which is $k = \left\lceil \frac{-3+\sqrt{9+24n}}{2} \right\rceil$.

The k points in the first infinitesimal neighborhood need to lie on the restriction to R of the linear system on P $L_k(2^n)$. The number of independent conditions that this imposes can be obtained as $h^0\left(R, (L_k(2^n))|_R\right) - h^0\left(P, L_k(2^n)\right)$; the first number is $k+1$ and the second one is the virtual dimension of the system plus 1.

The length of the candidate limit scheme found in this way is

$$\frac{k(k+1)}{2} + k - \left(\frac{k(k+3)}{2} - 3n \right) = 3n$$

which is the same as the limit scheme.

If $n = 2$ the minimum degree for which $h^0(L_d(2, 2)) > 0$ is 2, so the candidate limit contains a double point; moreover the system $L_2(2, 2)$ consists of the line through the two points counted twice; we can then apply the results from section 2.3 and conclude that the limit scheme needs to also contain a double point in its first infinitesimal neighborhood. This gives us a candidate limit scheme of length 6, which is the same as the limit scheme.

If $n = 4$ the minimum degree for which $h^0(L_d(2^5)) > 0$ is 4, so the candidate limit contains a point of multiplicity 4; moreover the system $L_4(2^5)$ consists of the conic through the five points counted twice; we can then apply the results from section 2.3 and conclude that the candidate limit scheme is as described. This candidate limit scheme has length 15, which is again the same as the limit scheme. □

Corollary 3.14. *A point of multiplicity m can be obtained as the limit of n general points of multiplicity 2 coming together in a general way if and only if either $m \equiv 0$ or $m \equiv 2$ (mod 3). In this case it can be realized as* *the limit of $\frac{m(m+3)}{6}$ double points. Conversely, the limit of n double points coming together in a general way is a point of multiplicity m with no other conditions if and only if n can be written as $\frac{m(m+1)}{6}$ for some m, which is then the multiplicity of the limit point.*

Proof. If $m \equiv 1 \pmod 3$ we have that the length of a point of multiplicity m is not a multiple of 3 and thus cannot match the length of a union of double points. Conversely, if $\frac{m(m+3)}{6}$ is a natural number, plugging it as n into the result above tells us that the limit of that many double points is a point of multiplicity m with no other conditions.

The second part is just a particular case of the previous result. □

3.2.3 6 points

Proposition 3.15. *Let's consider 6 points of multiplicity m coming together in a general way. Let k and r be the integers such that $12m = 5k - r$ with $0 \leq r < 5$ and n be $5m - 2k$. Let us also suppose that m is not 1 or 3. Then the limt scheme is a point of multiplicity k with six infinitely near matching couples of points of type $\left(\left\lfloor \frac{n+1}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor\right)$ and the remaining $5r$ points in the first order neighborhood lie on the restriction of $L_{5r}((2r)^6)$.*

Proof. Let C_i be the conic through the five points other than $\widetilde{p_i}$. Let's look at the linear system $L_k(m^6)$. C_i is a (-1)-curve, its intersection number (in P) with the system is $2k - 5m = -n$; since $n \geq 0$ for $m \neq 1, 3$, we have that C_i is contained in the system as a fixed component at least n times. If we remove these conics from the system, the residual part has degree $d' = k - 12n = 5(5k - 12m) = 5r$ and will have to pass through each of the six points with multiplicity $m - 5n = 2(5k - 12m) = 2r$. For each suitable value of r the linear system $L_{5r}((2r)^6)$ is nonspecial and is not empty, so $L_k(m^6)$ is nonempty as well.

Now we need to prove that if d is an integer such that the system $L_d(m^6)$ is nonempty then $d \geq k$. For any such d we have that $L_d(m^6).C_i = 2d - 5m$ for all the six curves C_i, which means that either $d \geq \frac{5}{2}m$ or each of the C_is is a fixed component of the system; if we are in the latter case we can remove the six conics and repeat the process for the system $L_{d-12}((m-5)^6)$ until, after some number j of such steps, we fall in the first case and have that $d - 12j \geq \frac{5}{2}(m - 5j)$. Now

$$d \geq \frac{5}{2}(m - 5j) + 12j \geq \frac{12}{5}(m - 5j) + 12j = \frac{12}{5}m$$

and since k is the least integer greater or equal than $\frac{12}{5}m$, we have that $d \geq k$.

To finish the proof we need to count the conditions imposed by the passage through the six points and compare it with the length of the scheme that is described by theorem 2.40. By remark 2.39 each pair of points coming from the fixed conics imposes $\frac{n(n+3)}{2}$ conditions; if we subtract these conditions and those imposed by the passage through p with multiplicity k from the ones imposed by the six original multiple points we have

$$6\frac{m(m+1)}{2} - \frac{k(k+1)}{2} - 6\frac{n(n+3)}{2} = \frac{r(7-r)}{2}$$

which, for the values of r between 0 and 4, is the difference between the degree ad the dimension of the residual system $L_{5r}((2r)^6)$. This means that the candidate limit scheme obtained through theorem 2.40 is the limit scheme. □

The only cases we have left to deal with are $m = 1$ and $m = 3$. For $m = 1$ we saw in proposition 3.11 that the limit scheme is a point of multiplicity 3. For $m = 3$ it is easy to see that the minimum degree is 8 and a point of multiplicity 8 has length 36, which is also the length of the limit scheme, so the limit scheme is exactly a point of multiplicity 8.

3.2.4 7 points

Proposition 3.16. *Let's consider 7 points of multiplicity m coming together in a general way. Let k and r be the integers such that $21m = 8k - r$ with $0 \leq r < 8$ and n be $8m-3k$. Let us also suppose that m is not 1,2,4,5,7,10 or 13. Then the limt scheme is a point of multiplicity k with seven infinitely near matching sets of three points of type $\left(\left\lfloor\frac{n+2}{3}\right\rfloor, \left\lfloor\frac{n+1}{3}\right\rfloor, \left\lfloor\frac{n}{3}\right\rfloor\right)$ and the remaining $8r$ points in the first order neighborhood lie on the restriction of $L_{8r}((3r)^7)$.*

Proof. Let C_i be the cubic having a double point at $\widetilde{p}_i$ and passing through the other six points. Let's look at the linear system $L_k(m^7)$. C_i is a (-1)-curve, its intersection number (in P) with the system is $3k - 8m = -n$; since $n \geq 0$ except for the values of m excluded in the statement, we have that C_i is contained in the system as a fixed component at least n times. If we remove these curves from the system, the residual part has degree $d' = k - 21n = 8(8k - 21m) = 8r$ and will have to pass through each of the fixed points with multiplicity $m - 8n = 3(8k - 21m) = 3r$. For each suitable value of r the linear system $L_{8r}((3r)^7)$ is nonspecial and is not empty, so $L_k(m^7)$ is nonempty as well.

Now we need to prove that if d is an integer such that the system $L_d(m^7)$ is nonempty then $d \geq k$. For any such d we have that $L_d(m^7).C_i = 3d - 8m$

for all seven curves C_i, which means that either $d \geq \frac{8}{3}m$ or each of the C_is is a fixed component of the system; if we are in the latter case we can remove the seven curves and repeat the process for the system $L_{d-21}((m-8)^7)$ until, after some number j of such steps, we fall in the first case and have that $d - 21j \geq \frac{8}{3}(m-8j)$. Now

$$d \geq \frac{8}{3}(m-8j) + 21j \geq \frac{21}{8}(m-8j) + 21j = \frac{21}{8}m$$

and since k is the least integer greater or equal than $\frac{21}{8}m$, we have that $d \geq k$.

To finish the proof we need to count the conditions imposed by the passage through the seven points and compare it with the length of the scheme that is described by theorem 2.40. By remark 2.39 each triple of points coming from the fixed cubics imposes $\frac{n(n+5)}{2}$ conditions; if we subtract these conditions and those imposed by the passage through p with multiplicity k from the ones imposed by the six original multiple points we have

$$7\frac{m(m+1)}{2} - \frac{k(k+1)}{2} - 7\frac{n(n+5)}{2} = \frac{r(13-r)}{2}$$

which, for the values of r between 0 and 7, is the difference between the degree ad the dimension of the rfesidual system $L_{8r}((3r)^7)$. This means that the candidate limit scheme obtained through theorem 2.40 is the limit scheme. □

We are left with the special cases that have been excluded in the statement; it is not hard to deal with them. First let's note that in all those cases the minimum degree is still $d = k$ as defined in the statement and that the linear system $L_k(m^7)$ is not special. We can then fill out a table

m	k	$\dim L_k(m^7)$	extra conditions	conditions given by kp	length of the limit scheme
1	3	2	1	6	7
2	6	6	0	21	21
4	11	7	4	66	70
5	14	14	0	105	105
7	19	13	6	190	196
10	27	20	7	378	385
13	35	28	7	630	637

The first column reports the multiplicity m of the colliding points. The second column is the least degree k such that $L_k(m^7)$ is not empty and also the multiplicity of the limit scheme at p; it is calculated as $\frac{21}{8}m$ rounded up. The third column is the dimension of the linear system on P; it is calculated as the virtual dimension since we know that it is not special. The fourth column lists how many conditions come from the matching of the restrictions to R of L_P and L_V; it is calculated as $k - \dim L_k(m^7)$. The fifth column lists the length of the scheme kp. The last column lists the length of the limit scheme.

It can be checked that the last column corresponds to the sum of the previous two, meaning that the conditions found are represented by a candidate limit scheme which has the same length as the actual limit, proving that also in these cases the candidate limit is the limit.

3.2.5 8 points

In this case we will have to deal with some rational (-1)-curves with $\tau = 6$. Due to the restriction we had to assume in section 2.3.2 our result is not complete in this case and we can only show a candidate limit scheme that has to be contained in the limit.

Proposition 3.17. *Let's consider 8 points of multiplicity m coming together in a general way. Let k and r be the integers such that $48m = 17k - r$ with $0 \leq r < 17$ and n be $17m - 6k$. Let us also suppose that m is not such that $n > 0$ (see remark below).*

Then the limt scheme contains a point of multiplicity k with 8 infinitely near sets of 6 points of type $\left(\left\lfloor \frac{n+5}{6} \right\rfloor, \left\lfloor \frac{n+4}{6} \right\rfloor, \left\lfloor \frac{n+3}{6} \right\rfloor, \left\lfloor \frac{n+2}{6} \right\rfloor, \left\lfloor \frac{n+1}{6} \right\rfloor, \left\lfloor \frac{n}{6} \right\rfloor\right)$ and where the remaining $17r$ points in the first order neighborhood lie on the restriction of $L_{17r}((6r)^8)$.

Proof. Let S_i be the sextic having a triple point at $\widetilde{p_i}$ and double points at the other seven points. Let's look at the linear system $L_k(m^8)$. S_i is a (-1)-curve, its intersection number (in P) with the system is $6k - 17m = -n$; since $n \geq 0$ we have that S_i is contained in the system as a fixed component at least n times. If we remove these curves from the system, the residual part has degree $d' = k - 48n = 17r$ and will have to pass through each of the fixed points with multiplicity $m - 17n = 6r$. For each suitable value of r the linear system $L_{17r}((6r)^8)$ is not empty (it has a positive expected dimension), so $L_k(m^8)$ is nonempty as well.

Now we need to prove that if d is an integer such that the system $L_d(m^8)$ is nonempty then $d \geq k$. For any such d we have that $L_d(m^8).S_i = 6d - 17m$ for all the eight curves S_i, which means that either $d \geq \frac{17}{6}m$ or each of the S_is is a fixed component of the system; if we are in the latter case we can separate the eight curves and repeat the process for the system $L_{d-48}((m-17)^8)$

until, after some number j of such steps, we fall in the first case and have that $d - 48j \geq \frac{17}{6}(m - 17j)$. Now

$$d \geq \frac{17}{6}(m - 17j) + 48j \geq \frac{48}{17}(m - 17j) + 48j = \frac{48}{17}m$$

and since k is the least integer greater or equal than $\frac{48}{17}m$, we have that $d \geq k$. □

Again, we are left with several (but finitely many) special cases that have been excluded in the statement.

The same approach we used in the last section still works and we list the results in another table:

m	k	$\dim L_k(m^8)$	extra conditions	conditions given by kp	length of the limit scheme
1	3	1	2	6	8
2	6	3	3	21	24
3	9	6	3	45	48
4	12	10	2	78	80
5	15	15	0	120	120
7	20	6	14	210	224
8	23	11	12	276	288
9	26	17	9	351	360
10	29	24	5	435	440
11	32	32	0	528	528
13	37	12	25	703	728
14	40	20	20	820	840
15	43	29	14	946	960
16	46	39	7	1081	1088
19	54	19	35	1485	1520
20	57	30	27	1653	1680
21	60	42	18	1830	1848
22	63	55	8	2016	2024
25	71	27	44	2556	2600
26	74	41	33	2775	2808
27	77	56	21	3003	3024
28	80	72	8	3240	3248
31	88	36	52	3916	3968
32	91	53	38	4186	4224
33	94	71	23	4465	4488

Continued on Next Page…

m	k	$\dim L_k(m^8)$	extra conditions	conditions given by kp	length of the limit scheme
37	105	46	59	5565	5624
38	108	66	42	5886	5928
39	111	87	24	6216	6240
43	122	57	65	7503	7568
44	125	80	45	7875	7920
45	128	104	24	8256	8280
49	139	69	70	9730	9800
50	142	95	47	10153	10200
55	156	82	74	12246	12320
56	159	111	48	12720	12768
61	173	96	77	15051	15128
62	176	128	48	15576	15624
67	190	111	79	18145	18224
73	207	127	80	21528	21608
79	224	144	80	25200	25280

The first column reports the multiplicity m of the eight colliding points. The second column is the least degree k such that $L_k(m^8)$ is not empty and also the multiplicity of the limit scheme at p; it is calculated as $\frac{48}{17}m$ rounded up. The third column is the dimension of the linear system on P; it is calculated as the virtual dimension since we know that it is not special. The fourth column lists how many conditions come from the matching of the restrictions to R of L_P and L_V; it is calculated as $k - \dim L_k(m^8)$. The fifth column lists the length of the scheme kp. The last column lists the length of the limit scheme.

It can again be checked that the last column corresponds to the sum of the previous two, meaning that the conditions found are represented by a candidate limit scheme which has the same length as the actual limit, proving that also in these cases the candidate limit is the limit.

Remark 3.18: It is worth noting that if, the obvious generalization of remark 2.39 were true, the length of the (new, larger) candidate limit scheme given by the number of conditions imposed by the sextics plus the ones imposed by the residual linear system $L_{17r}((6r)^8)$ would be the same as the length of the limit scheme. Each set of six points coming from the fixed sextics would impose $\frac{n(n+5)}{2}$ conditions; if we detract these conditions and those imposed by the passage through p with multiplicity k from the ones imposed by the six original multiple points we have

$$7\frac{m(m+1)}{2} - \frac{k(k+1)}{2} - 7\frac{n(n+5)}{2} = \frac{r(13-r)}{2}$$

which, for the values of r between 0 and 7, is the difference between the degree ad the dimension of the residual system $L_{17r}((6r)^8)$, meaning that the candidate limit would indeed be the limit scheme.

3.2.6 9 points

We already dealt with this case in chapter 2 as example 2.44. We copy the result here for completeness of this section

Proposition 3.19. *The flat limit of nine fat points of multiplicity m coming together in a general way is a point of multiplicity $3m$ together with a set of nine points of type (1^m) in its first infinitesimal neighborhood.*

3.3 Applications

Most of this chapter consists of results that can be used in various ways to construct degenerations. Some limits can be quite complicated and of difficult use until their internal structure is better understood while others can be worked with with ease. Among these the easiest to use will be those cases where the limit scheme is a fat point. Here by a fat point we intend a point counted with some positive multiplicity k rather than any 0-dimensional subscheme of $\mathbb{A}^2$ supported at a single point.

Proposition 3.20. *The limit of n fat points coming together in a general way for $2 \leq n \leq 4$ is a fat point if and only if either $n = 3$ and all the multiplicities are 1 or $n = 4$ and the multiplicities are either $m, m, m, m-1$ or $m+1, m, m, m$ for some positive integer m.*

Proof. The case where $n = 4$ has been proved as Corollary 3.10. There are no such cases for $n = 2$ or $n = 3$ because, as seen in the relative subsections, their limits never take that form. The only exception is the limit of three simple points which can be seen as being a collision of type $m, m, m, m-1$ with $m = 1$. □

Proposition 3.21. *The only cases where the limit of $n \leq 9$ fat points of the same multiplicity m coming together in a general way is a fat point are the following:*

number of points	*multiplicity*	*multiplicity of the limit*
3	1	2
6	1	3
6	3	8
7	2	6
7	5	14
8	5	15
8	11	32

Proof. Looking through the results of the previous section these are the only cases appearing. □

These cases are relatively few. This can be ascribed to the fact that the minimum degree is often realized by a special linear system. The fixed components that the linear system has will then translate into some conditions in the first infinitesimal neighborhood. This argument only applies to general degenerations, meaning that one could conceivably construct a degeneration of fat points that does not appear in this list whose limit is a fat point.

All this cases are sumrized in Table 3.5. In particular the first lines in the table lend themselves to be used as a base for more constructions. Note that these constructions might not be general collisions in the sense we intend it in this work. We present here a few application of this idea to prove conjecture 1.18 in some cases.

Proposition 3.22. *The following linear systems satisfy SHGH conjectureare for any positive value of m and n.*

1. $L_d(m^{10n^2}, (m+1)^{6n^2})$
2. $L_d(m^{6n^2}, (m+1)^{10n^2})$
3. $L_d(m^{36n^2}, (m+1)^{28n^2})$
4. $L_d(m^{28n^2}, (m+1)^{36n^2})$
5. $L_d(m^(3n^2), (m+1)^{12n^2}, (m+2)^{n^2})$

Proof. We observe in Table 3.5 that the limit of four points of multiplicities m^3,$(m+1)$ is a single point of multiplicity $2m+1$. We can then construct a degeneration of 12 points of multiplicities m^9, $(m+1)^3$ to three points of multiplicity $2m+1$ or, by also using the other limit for 4 points appearing in the table, a degeneration of 16 points of multiplicities m^{10}, $(m+1)^6$ to three points of multiplicity $2m+1$ and one point of multiplicity $2m+2$. We can then construct a degeneration of the 16 points to a single point of multiplicity $4m+3$ by having the four limit points collide.

The linear systems appearing in case 1 can then be degenerated to $\mathcal{L}_d\left((4m+3)^{n^2}\right)$ by taking n^2 groups of 16 points of multiplicities $m^10, (m+1)^6$ and have each group of fat points collide as described above. The limit linear system has the same expected dimension than the original one and 1.18 holds for it because it has a square number of assigned points. We can then say by semicontinuity that conjecture 1.18 holds for the original linear system as well.

All the other cases are done in the same way, taking other combination of points that can degenerate to a single point. In the second case the first degeneration yields four points of multiplicities $(2m+2)^3, 2m+1$; cases 3 and 4 are done starting with 64 points and then making three degenerations first to 16, then to 4 and finally to a single point. Case 5 is similar to the first two. $\square$

Many other cases can be constructed, that can be treated in a similar way. Moreover it is known that conjecture 1.18 holds and there are no (-1)-special systems if the points are either 4 or 9 and the points have the same multiplicity. One can reason along the same lines to produce examples of nonspecial linar systems with many assigned points.

Example 3.23. The linear system $L_d((2m)^2, m^6, (m-1)^2)$ is nonspecial for any npositive value of m. Indeed one can degenerate the linear system to $L_d((2m)^4)$, which has the same expected dimension and is nonspecial. By semicontinuity the system in the statement has the expected dimension as well.

number of points	multiplicity	multiplicity of the limit	notes
$\frac{m(m+1)}{2}$	1	m	for any positive integer m
$\frac{m(m+3)}{6}$	2	m	for $m \not\equiv 1 \bmod 3$
4	$m+1$, m^3	$2m+1$	for any positive integer m
4	m^3, $m-1$	$2m$	for any $m \geq 2$
6	1	3	
6	3	8	
7	2	6	
7	5	14	
8	5	15	
8	11	32	

Table 3.5: This is the list of all the collections of fat points whose general flat limit is a fat point. No other examples exist with 4 points or less or with 9 points or less if the colliding points have the same multiplicity.

Bibliography

[1] Alberto Calabri, Ciro Ciliberto, Flaminio Flammini, and Rick Miranda. On the K^2 of degenerations of surfaces and the multiple point formula. *Ann. of Math. (2)*, 165(2):335–395, 2007.

[2] Guido Castelnuovo. Ricerche generali sopra i sistemi lienari di curve piane. *Mem. Acc. Sci. Torino*, II(42), 1891.

[3] Ciro Ciliberto and Rick Miranda. Degenerations of planar linear systems. *J. Reine Angew. Math.*, 501:191–220, 1998.

[4] Ciro Ciliberto and Rick Miranda. Linear systems of plane curves with base points of equal multiplicity. *Trans. Amer. Math. Soc.*, 352(9):4037–4050, 2000.

[5] Ciro Ciliberto and Rick Miranda. The Segre and Harbourne-Hirschowitz conjectures. In *Applications of algebraic geometry to coding theory, physics and computation (Eilat, 2001)*, volume 36 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 37–51. Kluwer Acad. Publ., Dordrecht, 2001.

[6] Ciro Ciliberto and Rick Miranda. Matching conditions for degenerating plane curves and applications. In *Projective varieties with unexpected properties*, pages 177–197. Walter de Gruyter GmbH & Co. KG, Berlin, 2005.

[7] Marcin Dumnicki. Reduction method for linear systems of plane curves with base fat points, 2006.

[8] Marcin Dumnicki and Witold Jarnicki. New effective bounds on the dimension of a linear system in $\mathbb{P}^2$. *J. Symbolic Comput.*, 42(6):621–635, 2007.

[9] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[10] Laurent Evain. Computing limit linear series with infinitesimal methods. *Ann. Inst. Fourier (Grenoble)*, 57(6):1947–1974, 2007.

[11] Alessandro Gimigliano. *On linear systems of plane curves*. PhD thesis, Queen's University, Kingston, 1987.

[12] Brian Harbourne. The geometry of rational surfaces and Hilbert functions of points in the plane. Algebraic geometry, Proc. Conf., Vancouver/B.C. 1984, CMS Conf. Proc. 6, 95-111 (1986)., 1986.

[13] A. Hirschowitz. Une conjecture pour la cohomologie des diviseurs sur les surfaces rationelles g'en'eriques. *J. reine angew. Math*, (397):208–213, 1989.

[14] Joaquim Roé. Limit linear systems and applications. arXiv:math/0602213v2 [math.AG].

[15] Beniamino Segre. Alcune questioni su insiemi finiti di punti in geometria algebrica. In *Atti Convegno Intern. di Geom. Alg. di Torino*, pages 15–33, 1961.