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## Contents

1 Introduction ..... 7
2 Preliminaries ..... 13
2.1 Basics ..... 13
Notation and conventions ..... 13
Some useful lemmas ..... 14
2.2 Some remarks about singularities of pairs ..... 17
LC centers ..... 18
Standard log-resolutions and applications ..... 20
Remarks on DLT pairs ..... 25
2.3 Asymptotic base loci ..... 30
2.4 Multiplier ideals on singular varieties ..... 32
Asymptotic multiplier ideals ..... 33
$2.5 \mathbb{Q}$-CKM Zariski decompositions ..... 35
2.6 Asymptotic and numerical orders of vanishing ..... 39
Numerical orders of vanishing: the non-nef locus ..... 39
Asymptotic orders of vanishing: the non nef-abundant locus ..... 40
3 On the semiampleness of the positive part of CKM Zariski de- compositions ..... 43
3.1 The case $N=0$ ..... 43
Weak $\log$ Fano pairs ..... 44
3.2 Direct consequences of a theorem of Fujino ..... 44
3.3 DLT case ..... 46
Generalizing Kawamata's proof ..... 47
Logbig DLT case ..... 58
3.4 LC pairs "approximated" by KLT pairs ..... 60
3.5 Reducing LC to DLT via DLT blow-up ..... 61
3.6 Relatively KLT case ..... 68
Ambro's theorem ..... 68
Using Ambro's theorem in the $\mathbb{Q}$-Gorenstein case ..... 74
3.7 Nklt and $\mathrm{Nklt}_{2}$ : Dimension 3 ..... 75
3.8 Dimension 4 ..... 85
3.9 Relatively DLT case ..... 88
Relatively DLT pairs ..... 88
Main theorems ..... 95
3.10 Alternative hypotheses ..... 101
3.11 Examples ..... 104
Basic construction ..... 104
Applications ..... 104
4 Asymptotic base loci on singular varieties ..... 107
4.1 Some special cases ..... 107
4.2 Main results ..... 109
4.3 Nef and abundant divisors ..... 113
Characterization of nef-abundant divisors ..... 113
Applications ..... 115

## Chapter 1

## Introduction

In algebraic geometry line bundles are particularly interesting as they can be used to define rational maps to projective spaces. In particular, given a line bundle (or equivalently a Cartier divisor) $D$ on a complex projective variety $X$, the first question arising is if $D$ defines a map (i.e. if it has global sections) and, if this is the case, what are the indeterminacies of the map (i.e. the points where all the global sections vanish). Note that in general we want to investigate the asymptotic behavior of $D$, so that we denote by $\phi_{D}$ the rational map induced by a sufficiently high and divisible multiple $m D$. The classical object that encloses the basic information about $\phi_{D}$ is the stable base locus

$$
\mathbb{B}(D)=\bigcap_{m \in \mathbb{N}} B s(|m D|)
$$

In fact $\mathbb{B}(D)$ is the (Zariski) closed subset of $X$ where $\phi_{D}$ is not defined, so that in particular $\mathbb{B}(D)=X$ if and only if no multiples of the given line bundle have sections, whilst it is empty if $m D$ defines an actual morphism for some $m \in \mathbb{N}$, i.e. $D$ is semiample.

Recently in [ELMNP06] some modifications of the stable case locus have been introduced to address more involved problems. For example the augmented base locus

$$
\mathbb{B}_{+}(D)=\bigcap_{\substack{E \mathbb{R} \text {-divisor, } \geq \geq 0 \\ D-E \text { ample }}} \operatorname{Supp}(E)
$$

is the closed subset where $\phi_{D}$ is not locally a closed immersion, or, in other words, where $D$ is not ample.
In particular $\mathbb{B}_{+}(D)=\emptyset$ if and only if $D$ is ample and $\mathbb{B}_{+}(D)=X$ if and only if $\phi_{D}$ is not an isomorphism on the generic point of $X$, i.e. $D$ is not big.
A natural generalization of the property of being ample leads to the notion of nefness. A divisor (or line bundle) $D$ is nef if and only if it non negatively intersects every irreducible curve on $X$, that is proved to be equivalent to say that its numerical class lies in the closure of the ample cone. Nef divisors are particularly interesting for many reasons: their top self-intersection, for example, asymptotically determine the number of their global sections (see [Laz04, corollary 1.4.41]), while the nefness of the canonical divisor $K_{X}$ is one of the main issues in the context of the Minimal Model Program, as it corresponds to the minimality of the variety $X$.

Hence many different tools have been introduced to study "where a divisor fails to be nef". In analogy with the augmented base locus we can consider the restricted base locus

$$
\mathbb{B}_{-}(D)=\bigcup_{A} \mathbb{B}(D+A)
$$

where the union is taken over all ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $X$ (see def. 2.3.5).

Note that $\mathbb{B}_{-}(D)=\emptyset$ if and only if $D$ is nef. Also $\mathbb{B}_{-}(D)$ contains every curve $C$ such that $(D \cdot C)<0$, but it can be strictly bigger than the union of such curves.
Moreover, by definition, $\mathbb{B}_{-}(D)$ is a countable union of closed subsets; we do not know whether it is closed in general or not.
A more classical approach is to look for a Zariski decomposition of $D$, i.e. to try to decompose it as the sum of a nef divisor $P$ that encodes the positivity of $D$ in some sense and an effective divisor $N$ that represents the non-nef part of D.

More precisely we say that a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ on a projective variety $X$ admits a $\mathbb{Q}$-Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a $\mathbb{Q}$-CKM Zariski decomposition) $D=P+N$ if

- $P$ and $N$ are $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors;
- $P$ is nef and $N$ is effective;
- There exists an integer $k>0$ such that $k D$ and $k P$ are integral divisors and for every $m \in \mathbb{N}$ the natural map

$$
H^{0}\left(X, \mathcal{O}_{X}(k m P)\right) \rightarrow H^{0}\left(X, \mathcal{O}_{X}(k m D)\right)
$$

is bijective.
Every pseudoeffective divisor on a smooth projective surface admits a Zariski decomposition (see [Fuj79]). On the other hand in higher dimension there exist big divisors such that no birational pullbacks admit a Zariski decomposition, as shown by Cutkosky in [Cut86], even if we allow $P$ and $N$ to be $\mathbb{R}$-divisors (see[Nak04]).
Note that the restricted base locus of $D$ is strictly related to the existence of such decompositions. In fact if $D$ is big and $D=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition, then $\operatorname{Supp}(N)=\mathbb{B}_{-}(D)$, which implies that $\mathbb{B}_{-}(D)$ is a divisor in this case.
If $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition, then, up to pass to a multiple, the graded ring $R(X, D):=\bigoplus_{m \in \mathbb{N}} H^{0}\left(X, \mathcal{O}_{X}(m D)\right)$ is isomorphic to $R(X, P)$, so that in order to check its finite generation it suffices to study the finite generation of the ring associated to the nef divisor $P$. This implies that the semiampleness of $P$, the so-called positive part of the Zariski decomposition of $D$, is a sufficient condition for the finite generation of $R(X, D)$. Moreover the semiampleness of $P$ gives that $\mathbb{B}(D)=\operatorname{Supp}(N)=\mathbb{B}_{-}(D)$. See [Mor87, (9.11)] for the importance of the Zariski decomposition in the context of the Abundance Conjecture.
In this work in chapter 3, whose main results appear in the paper [Cac10], we try to extend to the LC case the main theorem of Kawamata's paper [Kaw87], stating that if the canonical divisor $K_{X}$ of a smooth projective variety admits
a CKM Zariski decomposition then the positive part $P$ of the decomposition is semiample.
More precisely a simple generalization of this theorem says that:
Theorem 1.0.1 (Kawamata). Let $X$ be a normal projective variety and let $\Delta$ be a Weil effective $\mathbb{Q}$-divisor such that $(X, \Delta)$ is a KLT pair. If $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that

A: $D$ is big;
B: $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;
C: $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$;
then the positive part $P$ is semiample, so that $R(X, D)$ is finitely generated.
The same theorem is no longer true in general if $(X, \Delta)$ is an LC pair, as shown in section 3.11 , so that we have to assume more. For this reason we use the notion of logbig divisors, introduced by Miles Reid.
A big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor is logbig for an LC pair if its restriction to every LC center of the pair is still big (see definition 2.2.7).
We state the following conjecture:
Conjecture 1. Let $(X, \Delta)$ be an LC pair, with $\Delta$ effective.
If $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ which satisfies $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, and $P$ is logbig for the pair $(X, \Delta)$, then $P$ is semiample.

In the case $N=0$, a very similar result was stated by Miles Reid in [Rei93] and was proved by Florin Ambro in the more general setting of quasi-log varieties in [Amb01, theorem 7.2] (see also [Fuj09c, theorem 4.4]). Moreover, when $X$ is smooth and $\Delta$ and $N$ are SNCS divisors Conjecture 1 is true and follows from [Fuj07b, theorem 5.1].
Note that a sufficient condition for the positive part $P$ to be logbig for the pair $(X, \Delta)$ is that the augmented base locus $\mathbb{B}_{+}(D)$ does not contain any LC center of the pair. Hence conjecture 1 would imply the following:
Conjecture 2. Let $(X, \Delta)$ be an LC pair, with $\Delta$ effective.
If $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ which satisfies $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$, and $\mathbb{B}_{+}(D)$ does not contain any LC center of the pair $(X, \Delta)$, then $P$ is semiample.

On the other hand, note that, for a divisor $D$, the existence of a Zariski decomposition is a very strong property in general, while it is more likely that a birational pullback of $D$ admits one. In other words we want to replace hypothesis $\mathbf{C}$ with the following:
$\mathbf{C}_{f}$ : There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $f^{*}(D)=P+N$.

We can thus generalize Conjecture 1 and Conjecture 2 as follows (the "b" stands for "birational"):

Conjecture 1b. Let $(X, \Delta)$ be an LC pair, with $\Delta$ effective.
Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ satisfying $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}_{f}$, for some $f$ : $Z \rightarrow X$. If $P$ is logbig for the pair $\left(Z, \Delta_{Z}\right)$, where $\Delta_{Z}$ is $a \mathbb{Q}$-divisor on $Z$ such that $K_{Z}+\Delta_{Z}=f^{*}\left(K_{X}+\Delta\right)$, then $P$ is semiample.

Conjecture 2b. Let $(X, \Delta)$ be an LC pair, with $\Delta$ effective. Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ satisfying $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}_{f}$, for some $f: Z \rightarrow$ $X$. If $\mathbb{B}_{+}\left(f^{*} D\right)$ does not contain any $L C$ center of the pair $\left(Z, \Delta_{Z}\right)$, where $\Delta_{Z}$ is a $\mathbb{Q}$-divisor on $Z$ such that $K_{Z}+\Delta_{Z}=f^{*}\left(K_{X}+\Delta\right)$, then $P$ is semiample.
In this thesis we easily deduce from [Fuj07b, theorem 5.1] that Conjecture 1b holds if a suitable pair on $Z$ is DLT (see theorem 3.3.11), which in particular implies that Conjecture 1 holds if $(X, \Delta)$ is DLT. Moreover, as a corollary of more general results (see 3.7.5 and 3.7.7), we prove the following:

Theorem 1.0.2 (see corollary 3.7.8). Conjecture $1 b$ holds if $\operatorname{dim} X \leq 3$.
Also we prove that Conjecture 2 holds if we assume some standard very strong conjectures concerning the Minimal Model Program, namely the abundance conjecture (for semi divisorial log terminal pairs) and the existence of log-minimal models (in lower dimension) for DLT pairs of log-general type (see theorem 3.8.1). As a corollary we get that

Theorem 1.0.3 (see corollary 3.8.2). Conjecture 2 holds if $\operatorname{dim} X \leq 4$.
In section 3.9 we consider a relative version of DLT pair:
Definition 1.0.4 (see definition 3.9.1). Let $(X, \Delta)$ be a pair, with $\Delta=\sum a_{i} D_{i}$, where the $D_{i}$ 's are distinct prime divisors and $a_{i} \in \mathbb{Q}$ for every $i$. Let $D \in$ $\operatorname{Div}_{\mathbb{Q}}(X)$.
We define the non-simple normal crossing locus of $(X, \Delta)$ as the closed set

$$
N S N C(\Delta)=X \backslash U
$$

where $U$ is the biggest open subset of $X$ such that $\Delta_{\left.\right|_{U}}$ has simple normal crossing support.
We say that $(X, \Delta)$ is a $D$-DLT pair if

1. $V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$ for every LC center $V$ of the pair $(X, \Delta)$ such that $V \cap \mathbb{B}(D) \neq \emptyset$;
2. $a_{i} \leq 1$ for every $i$ such that $D_{i} \cap \mathbb{B}(D) \neq \emptyset$;

In particular a $D$-DLT pair is not necessarily LC. Inspired by [Amb05, Theorem 2.1] we prove the following:

Theorem 1.0.5 (see corollary 3.9.10). Let $(X, \Delta)$ be a pair such that $\Delta$ is effective. Let $D$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ satisfying $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}_{f}$, for some $f: Z \rightarrow X$, and let $\Delta_{Z}$ be $a \mathbb{Q}$-divisor on $Z$ such that $K_{Z}+\Delta_{Z}=f^{*}\left(K_{X}+\Delta\right)$. If $\left(Z, \Delta_{Z}\right)$ is $f^{*}(D)$-DLT and $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain any $L C$ center of the pair $\left(Z, \Delta_{Z}\right)$, then $P$ is semiample.

Note that in many statements of chapter 3 we do not need $a \geq 0$ in hypothesis B. Moreover, as shown in section 3.10, some of the theorems hold under the more usual hypotheses

B': $a D-\left(K_{X}+\Delta\right)$ big and nef for some $a \in \mathbb{Q}^{+} ;$
C: $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$.

In chapter 4 we consider a different approach to the asymptotic study of linear series, by means of geometric valuations. The results of this chapter come from a joint work with Lorenzo Di Biagio (see [CD11]).
In [ELMNP06] the authors, inspired by the work of Nakayama in [Nak04], define an asymptotic measure of the singularities of a linear series: if $v$ is a geometric discrete valuation on $X$ then the asymptotic order of vanishing of a big $\mathbb{Q}$-divisor $D$ along $v$ is

$$
v(\|D\|):=\lim _{p \rightarrow \infty} \frac{v(|p D|)}{p}
$$

(see def. 2.6.3).
They notice that $v(\|D\|)$ is a numerical invariant and, by passing to limits, it is possible to define a numerical order of vanishing on every pseudoeffective $\mathbb{R}$ Cartier divisor on $X$ that, following the notation of [BBP09], we will denote by $v_{\text {num }}(D)$ (see section 2.6). We define the non-nef locus $\operatorname{NNef}(D)$ as the subset of $X$ given by the union of all the centers of the valuations $v$ such that $v_{\text {num }}(D)>0$ (see definition 2.6.1 and [BBP09, definition 1.7]). Note that this corresponds to the numerical base locus $\operatorname{NBs}(D)$ defined by Nakayama in [Nak04, definition III.2.6] in the smooth case.

We know that $\operatorname{NNef}(D)=\emptyset$ if and only if $D$ is nef, as the name itself suggests, and $\operatorname{NNef}(D)=X$ if and only if $D$ is not psuedoeffective. Therefore, as the same holds for the restricted base locus $\mathbb{B}_{-}(D)$ and considering that $\operatorname{NNef}(D) \subseteq$ $\mathbb{B}_{-}(D)$ for every $\mathbb{R}$-Cartier divisor $D$ (cf.[BBP09, lemma 1.8]), it is natural to wonder if these two loci coincide in general.
Note that $\operatorname{NNef}(D)$ behaves well with respect to birational morphisms, meaning that $\operatorname{NNef}\left(\mu^{*}(D)\right)=\mu^{-1}(\operatorname{NNef}(D))$ for every $\mu$ birational, so that the same would hold for the restricted base locus if we knew the equality of the two sets. By [ELMNP06, theorem 2.8] we have the following:
Theorem 1.0.6 (ELMNP). Let $X$ be a smooth projective variety and let $D$ be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)$.

It is then trivial to show that, passing to limits, the same equality holds for pseudoeffective $\mathbb{R}$-divisors.
In this paper we generalize this result to normal varieties that are not too singular:

Theorem 1.0.7 (see theorem 4.2.7). Let $X$ be a normal projective variety and suppose that there exists an effective $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is a KLT pair.
Then for every $\mathbb{R}$-Cartier pseudoeffective divisor $D$ on $X$ we have that $\mathbb{B}_{-}(D)=$ $\operatorname{NNef}(D)$.

This is a partial answer to a conjecture of Boucksom, Broustet and Pacienza (see [BBP09, Conjecture 1.9]).
The hypothesis of the existence of a KLT boundary is necessary in our proof because this is the only context where asymptotic multiplier ideals are not strongly influenced by the singularities of $X$, so that they reflect the asymptotic behaviour of the base loci. We could avoid it only in the case of surfaces (corollary 4.1.5). See also cor. 4.2 .10 and cor. 4.2 .12 for slight generalizations.

On the other hand we define asymptotic orders of vanishing $v(\|D\|)$ for every effective divisor. These are in general different from numerical orders of vanishing (remark 2.6.5) and we have that $v(\|D\|)=0$ for every geometric discrete
valuation $v$ if and only if $D$ is nef and abundant (lemma 2.6.6). In analogy to the definition of the non-nef locus we use these asymptotic orders of vanishing to define a non-nef abundant locus NNA( $D$ ) (see def. 2.6.7). In particular we prove that for every effective Cartier divisor on a variety $X$ that admits a KLT boundary $\Delta$ it holds that

$$
\operatorname{NNA}(D)=\bigcup_{m \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|m D\|))
$$

(see theorems 4.2.5 and 4.3.1). Note that, when $D$ is big, this means that

$$
\mathbb{B}_{-}(D)=\mathcal{Z}(\mathcal{J}((X, \Delta) ;\|m D\|)),
$$

so that, in particular, we get a generalization of [ELMNP06, Cor. 2.10].
Moreover, as a corollary of this result, we give a characterization of nef and abundant divisors in terms of triviality of asymptotic multiplier ideals (see corollary 4.3.2), generalizing to the KLT case the main theorem of F. Russo's paper [Rus09].

## Chapter 2

## Preliminaries

### 2.1 Basics

## Notation and conventions

We will work over the field of complex numbers $\mathbb{C}$. As in [Laz04] a scheme is a separated algebraic scheme of finite type over $\mathbb{C}$ and a variety is a reduced, irreducible scheme. A curve is a variety of dimension 1. A surface is a variety of dimension 2.
Given a variety $X$ and a coherent sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_{X}$ we denote by $\mathcal{Z}(\mathcal{J})$ the closed subset of $X$ defined by $\mathcal{J}$, without any scheme structure.
We denote by $\mathrm{W} \operatorname{Div}(X)$ the group of Weil divisors and by $\operatorname{Div}(X)$ the group of Cartier divisors, while, for $\mathbb{K}=\mathbb{Q}, \mathbb{R}$, we denote by $\operatorname{Div}(X)_{\mathbb{K}}$ the group of $\mathbb{K}$-Cartier $\mathbb{K}$-divisors and by $N^{1}(X)_{\mathbb{K}}$ the vector space given by the quotient of $\operatorname{Div}(X)_{\mathbb{K}}$ by the subgroup of numerically trivial divisors. Note that $N^{1}(X)_{\mathbb{R}}$ is a finite dimensional $\mathbb{R}$-vector space. Moreover we consider a norm $\|\cdot\|$ on this space.
Given a variety $X$ and a Cartier divisor (or a line bundle) $D$ we denote by $\kappa(X, D)$ its Kodaira dimension. Given a smooth variety $X$ and $D \in \operatorname{Div}_{\mathbb{Q}}(X)$, we denote by $\operatorname{mult}_{x} D$ the multiplicity at $x \in X$ of $D$, in the sense of [Laz04, Def. 9.3.1].
Definition 2.1.1. Let $X$ be a normal projective variety over $\mathbb{C}$ and let $\Delta$ be a Weil $\mathbb{Q}$-divisor on $X$. We say that $(X, \Delta)$ is a pair if $K_{X}+\Delta \in \operatorname{Div}_{\mathbb{Q}}(X)$. A pair is effective if $\Delta \geq 0$.

Remark 2.1.2. Note that from now on we will always consider normal projective varieties. Then, in our setting, the cycle map

$$
\operatorname{Div}(X) \longrightarrow \mathrm{W} \operatorname{Div}(\mathrm{X})
$$

is injective.
Hence, with a slight abuse, we will use the same notation to indicate a Cartier divisor and the Weil divisor obtained as the image through the cycle map.

Definition 2.1.3. Let $X$ be a normal projective variety and let $D$ be a Weil divisor on $X$. We define the sheaf $\mathcal{O}_{X}(D)$ by putting, for every open subset $U \subseteq X$,

$$
\mathcal{O}_{X}(U)=\left\{f \in k(X): \operatorname{div}_{U} f+D_{\left.\right|_{U}} \geq 0\right\} .
$$

Definition 2.1.4. Let $X$ and $Y$ be normal varieties and let $\mu: Y \rightarrow X$ be a proper birational morphism.
If $U \subseteq Y$ is the biggest open subset such that $\mu_{\left.\right|_{U}}$ is an isomorphism, we define

$$
\operatorname{exc}(\mu)=Y \backslash U
$$

We say that a prime divisor $E$ on $Y$ is $\mu$-exceptional (or simply exceptional) if $\operatorname{Supp}(E) \subseteq \operatorname{exc}(\mu)$.
If $E_{1}, \ldots, E_{t}$ are all distinct the prime $\mu$-exceptional divisors on $Y$, we define

$$
\operatorname{divexc}(\mu)=\operatorname{Supp}\left(E_{1}+\ldots+E_{t}\right)
$$

Note that, with a slight abuse of notation, we will use the same the notation to indicate the reduced divisor $E_{1}+\cdots+E_{t}$ itself.
Note also that, by convention, we define $\operatorname{Supp}(0)=\emptyset$.
Remark 2.1.5. Note that if $E$ is an exceptional prime divisor for a proper birational morphism $\mu: Y \rightarrow X$ between normal varieties, then $\operatorname{codim} \mu(\operatorname{Supp}(E)) \geq$ 2 , that is $E$ is contracted by $\mu$, so that $\mu_{*}(E)=0$. Moreover note that $\mu^{-1}(\mu(\operatorname{exc}(\mu)))=\operatorname{exc}(\mu)$.
For the proof of this facts see [Deb01, 1.40].

## Some useful lemmas

In this section we recall some basic well-known lemmas that will be useful in the following chapters:

Lemma 2.1.6. Let $X, Y$ be normal varieties and let $f: Y \rightarrow X$ be a proper birational map. Let $\mathcal{J}$ be an ideal sheaf on $Y$. Then $\mathcal{Z}\left(f_{*} \mathcal{J}\right) \subseteq f(\mathcal{Z}(\mathcal{J}))$.

Proof. Let $W:=Y \backslash \mathcal{Z}(\mathcal{J})$ and $V:=X \backslash f(\mathcal{Z}(\mathcal{J}))$. For every $V^{\prime} \subseteq V$ open subset of $X$ we have that $f^{-1}\left(V^{\prime}\right) \subseteq W$, hence $\left(f_{*} \mathcal{J}\right)\left(V^{\prime}\right)=\mathcal{J}\left(f^{-1}\left(V^{\prime}\right)\right)=$ $\mathcal{O}_{Y}\left(f^{-1}\left(V^{\prime}\right)\right)=f_{*} \mathcal{O}_{Y}\left(V^{\prime}\right) \cong \mathcal{O}_{X}\left(V^{\prime}\right)$ by Zariski's main theorem (cf. [Har77, Cor. III.11.4]).
Let $x \in \mathcal{Z}\left(f_{*} \mathcal{J}\right)$. By contradiction suppose that $x \in V$. Since $\left(f_{*} \mathcal{J}\right)_{x} \subsetneq \mathcal{O}_{X, x}$ then there exists an open subset $P \subseteq V$ such that $x \in P$ and $\left(f_{*} \mathcal{J}\right)(P) \subsetneq$ $\mathcal{O}_{X}(P)$. But by the previous argument we have also that $\left(f_{*} \mathcal{J}\right)(P)=\mathcal{O}_{X}(P)$. Hence $x$ must be in $f(\mathcal{Z}(\mathcal{J}))$.

Lemma 2.1.7. Let $X$ and $Y$ be projective varieties such that $Y$ is smooth and $X$ is normal. Let $\mu: Y \rightarrow X$ be a proper birational morphism. If $D \in \operatorname{Div}(Y), E \in \operatorname{Div}(X)$ are such that

$$
\mu_{*} D \leq E,
$$

then there exists an effective $\mu$-exceptional Cartier divisor $\Gamma$ on $Y$ such that

$$
D \leq \mu^{*} E+\Gamma
$$

Proof. Let $E_{1}, \ldots, E_{t}$ be all the prime $\mu$-exceptional divisors on $Y$. Then, there exists a finite set of prime divisors on $X$, say $\left\{D_{1}, \ldots, D_{s}\right\}$, such that we can write

$$
D=\sum_{i=1}^{s} a_{i} D_{i}^{\prime}+\sum_{j=1}^{t} b_{j} E_{j}, \quad \mu^{*}(E)=\sum_{i=1}^{s} c_{i} D_{i}^{\prime}+\sum_{j=1}^{t} e_{j} E_{j},
$$

where the $D_{i}^{\prime}$ are the birational transforms of the $D_{i}$, and $a_{i}, c_{i}, b_{j}, e_{j} \in \mathbb{Z}$. By hypothesis we have that

$$
\sum_{i=1}^{s} a_{i} D_{i}=\mu_{*} D \leq E=\mu_{*}\left(\mu^{*}(E)\right)=\sum_{i=1}^{s} c_{i} D_{i} .
$$

which implies that $a_{i} \leq c_{i}$, for all $i=1, \ldots, s$.
Now, for each $j=1, \ldots, t$, we define $\gamma_{j}=\max \left\{0, b_{j}-e_{j}\right\}$, and let

$$
\Gamma:=\sum_{j=1}^{t} \gamma_{j} E_{j} .
$$

Then, by definition, $\Gamma$ is an effective, $\mu$-exceptional Cartier divisor on $Y$. Moreover

$$
\mu^{*}(E)+\Gamma \geq \sum_{i=1}^{s} c_{i} D_{i}^{\prime}+\sum_{j=1}^{t} b_{j} E_{j} \geq \sum_{i=1}^{s} a_{i} D_{i}^{\prime}+\sum_{j=1}^{t} b_{j} E_{j}=D
$$

Lemma 2.1.8. Let $X$ and $Y$ be normal projective varieties, such that $Y$ is smooth and let $\mu: Y \rightarrow X$ be a proper birational morphism.
Suppose $D_{Y} \in \operatorname{Div}(Y), D$ is a Weil divisor on $X$ and $B$ is a Cartier divisor on $X$. If $\mu_{*} D_{Y} \leq D$, then

$$
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(B)+D_{Y}\right)\right) \leq h^{0}\left(X, \mathcal{O}_{X}(B+D)\right)
$$

Proof. We use the standard fact that $\mu_{*} \mathcal{O}_{Y}\left(D_{Y}\right) \subseteq \mathcal{O}_{Y}\left(\mu_{*} D_{Y}\right)$.
Together with projection formula this implies that

$$
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(B)+D_{Y}\right)\right) \leq h^{0}\left(X, \mathcal{O}_{X}\left(B+\mu_{*}\left(D_{Y}\right)\right)\right)
$$

But $h^{0}\left(X, \mathcal{O}_{X}\left(B+\mu_{*}\left(D_{Y}\right)\right)\right) \leq h^{0}\left(X, \mathcal{O}_{X}(B+D)\right)$ by hypothesis, so that the lemma is proved.

Lemma 2.1.9. Let $Y$ and $X$ be normal projective varieties. Let $\mu: Y \rightarrow X$ be a composition of blowings-up of smooth centers of codimension greater than 1. If $A \in D i v_{\mathbb{Q}}(X)$ is ample and $E_{1}, \ldots, E_{k}$ are all the $\mu$-exceptional prime divisors on $Y$, then for every $i=1, \ldots, k$ there exist arbitrarily small positive rational numbers $\delta_{i}$, such that

$$
\mu^{*}(A)-\sum \delta_{i} E_{i}
$$

is an ample divisor on $Y$.

Proof. We work by induction on the number $k$ of the blowings-up composing $\mu$. To prove the base of the induction we consider $Z \subseteq X$ a smooth subvariety of codimension greater than one and $\mu: Y=B l_{Z} X \rightarrow X$ as in the hypothesis.
Let $\mathcal{J}=\mathcal{J}_{Z}$ be the ideal sheaf defining the subvariety $Z$. Then there exists an effective Cartier divisor $E$ on $Y$ such that $\mu^{-1} \mathcal{J}=\mathcal{O}_{Y}(-E)$, so that, in particular, $\operatorname{Supp}(E)=\mu^{-1}(Z)$.
Moreover, by [Har77, II ex. 7.14(b)], we have that $\mu^{*}(m A)-E$ is ample for every $m \gg 0$, so that

$$
\mu^{*} A-\frac{1}{m} E
$$

is ample for every $m \gg 0$.
Note that, by the properties of the blowings-up, $\mu$ is an isomorphism outside $E$, so that $\operatorname{exc}(\mu) \subseteq \operatorname{Supp}(E)$.
Moreover every prime divisor whose support is contained in $\operatorname{Supp}(E)$ is $\mu$ exceptional, because its image is contained in $Z$, so that it is contracted by $\mu$ on a subvariety of codimension greater than 1 . Hence $\operatorname{Supp}(E)=\operatorname{divexc}(\mu)=$ $\operatorname{exc}(\mu)$.
Then there exist some integers $a_{i}>0$ such that

$$
E=\sum a_{i}^{1} E_{i}^{1}
$$

where the $E_{i}^{1}$ are all the $\mu$-exceptional prime divisors, so that

$$
\mu^{*}(A)-\sum \frac{a_{i}^{1}}{m} E_{i}^{1}
$$

is ample for every integer $m \gg 0$. Thus we get the base of the induction.
Let us now prove the inductive step:
Let $\mu_{1}: X^{\prime} \rightarrow X$ be a composition of $k-1$ blowings-up as in the hypothesis, let $\mu_{2}: Y \rightarrow X^{\prime}$ be the blowing-up along a smooth subvariety $Z^{\prime} \subseteq X^{\prime}$ of codimension greater than 1 and let $\mu=\mu_{1} \circ \mu_{2}$.
If $E_{1}, \ldots, E_{s}$ are all the prime exceptional divisors of $\mu_{1}$, by induction we can suppose that there exists arbitrarily small rational numbers $\delta_{1}, \ldots, \delta_{s}>0$ such that

$$
\mu_{1}^{*}(A)-\delta_{1} E_{1}-\cdots-\delta_{s} E_{s}
$$

is ample.
If we denote $F_{i}$ all the exceptional prime divisors of $\mu_{2}$, we have to prove that

$$
\mu^{*}(A)-\delta_{1} \mu_{2 *}^{-1} E_{1}-\cdots+\delta_{s} \mu_{2_{*}^{*}}^{-1} E_{s}-\sum \epsilon_{i} F_{i}
$$

is ample for positive arbitrarily small rational numbers $\epsilon_{i}$.
The base of the induction shows that

$$
\mu_{2}^{*}\left(\mu_{1}^{*}(A)-\delta_{1} E_{1}-\cdots-\delta_{s} E_{s}\right)-\sum e_{i} F_{i}
$$

is ample for $0<\delta_{1}, \ldots, \delta_{s} \ll 1, \delta_{i} \in \mathbb{Q}$ and $0<e_{i} \ll 1, e_{i} \in \mathbb{Q}$.
But

$$
\begin{gathered}
\mu_{2}^{*}\left(\mu_{1}^{*}(A)-\delta_{1} E_{1}-\cdots-\delta_{s} E_{s}\right)-\sum e_{i} F_{i}= \\
\mu^{*}(A)-\sum_{j=1}^{s} \delta_{j} \mu_{2 *}^{-1} E_{j}-\sum_{i}\left(\sum_{j=1}^{s} a_{j i} \delta_{j}+e_{i}\right) F_{i}
\end{gathered}
$$

where the $a_{j i}$ are non-negative integer numbers not depending on the $\delta_{j}$. Thus if for each $i$ we put

$$
\epsilon_{i}=\sum_{j=1}^{s} a_{j i} \delta_{j}+e_{i}
$$

we are done because each $\epsilon_{i}$ is a positive rational number if the $\delta_{j}$ 's and $e_{i}$ are such and we can make the $\epsilon_{i}$ 's arbitrarily small by suitably choosing the $\delta_{j}$ 's and the $e_{i}$ 's.

Lemma 2.1.10. (cf. [Cac08, lemma 5.3]). Let $X$ be a normal projective variety and let $D \in \operatorname{Div} \mathbb{R}^{( }(X)$. Then there exists a sequence $\left\{A_{m}\right\}_{m \geq 1}$ of ample $\mathbb{R}$ Cartier divisors such that $\left\|A_{m}\right\| \rightarrow 0$ in $N^{1}(X)_{\mathbb{R}}$ and $D+A_{m}$ is a $\mathbb{Q}$-Cartier divisor for every $m \geq 1$.

Proof. Let $A$ be an ample $\mathbb{R}$-divisor and let $r \in \mathbb{R}^{+}$be such that, if we denote by $B_{r}(A)$ the ball in $N^{1}(X)$ centered at the class of the divisor $A$ with ray $r$, then $B_{r}(A) \subseteq \operatorname{Amp}(X) \subseteq N^{1}(X)$.
Since $B_{r}(D+A) \cap N^{1}(X)_{\mathbb{Q}} \neq \emptyset$ then there exists a $\mathbb{Q}$-Cartier divisor $D^{\prime}$ such that $\left\|D^{\prime}-(D+A)\right\|<r$. Therefore $D^{\prime}-D$ is ample and can be written as $\sum_{i=1}^{s} c_{i} P_{i}$ with $c_{i} \in \mathbb{R}^{+}$and $P_{i}$ is an ample Cartier divisors for every $i$.
Let $q_{i, m} \in \mathbb{Q}^{+}$be such that for every $i, q_{i, m}<c_{i}$ and $\lim _{m \rightarrow \infty} q_{i, m}=c_{i}$. Let $A_{m}:=\sum_{i=1}^{s}\left(c_{i}-q_{i, m}\right) P_{i}$. It is then easily seen that $\left\|A_{m}\right\| \rightarrow 0$ and that $D+A_{m}$ is $\mathbb{Q}$-Cartier.

### 2.2 Some remarks about singularities of pairs

Definition 2.2.1. Let $X$ be a normal projective variety of dimension $n$. A reduced divisor $D=\sum D_{i}$ has simple normal crossings (and it is an SNC divisor) if each $D_{i}$ is smooth and if $D$ is defined in a neighborhood of any point by an equation in local analytic coordinates of type

$$
z_{1} \cdots \cdot z_{k}=0
$$

for some $k \leq n$. A Weil $\mathbb{Q}$-divisor $\sum a_{i} D_{i}$ has simple normal crossing support (and it is an SNCS divisor) if $\sum D_{i}$ is an SNC divisor.
Definition 2.2.2. Let $(X, D)$ be a pair, let $|V|$ be a non-empty linear series on $X$ and let $\mathcal{J} \subseteq \mathcal{O}_{X}$ be a nonzero ideal sheaf on $X$. Then a log-resolution of the pair $(X, D)$ is a projective birational morphism

$$
\mu: X^{\prime} \rightarrow X
$$

such that the exceptional locus $\operatorname{exc}(\mu)$ is a divisor, $X^{\prime}$ is smooth and the $\mathbb{Q}$ divisor $\mu_{*}^{-1}(D)+\operatorname{exc}(\mu)$ is SNCS.
A log-resolution of $|V|$ is a birational morphism $\nu: X^{\prime \prime} \rightarrow X$ such that $\operatorname{exc}(\nu)$ is a divisor, $X^{\prime \prime}$ is smooth and $\nu^{*}|V|=|W|+F$, where $|W|$ is a linear series without base points and $F+\operatorname{exc}(\nu)$ is SNCS.
A log-resolution of $\mathcal{J} \subseteq \mathcal{O}_{X}$ is a projective birational morphism $\phi: X^{\prime \prime \prime} \rightarrow X$ such that $\operatorname{exc}(\phi)$ is a divisor, $X^{\prime \prime \prime}$ is smooth and

$$
\phi^{-1}(\mathcal{J}):=\mathcal{J} \cdot \mathcal{O}_{X^{\prime \prime \prime}}=\mathcal{O}_{X^{\prime \prime \prime}}(-F)
$$

where $F$ is an effective Cartier divisor on $X$ such that $F+\operatorname{exc}(\mu)$ is SNCS.

## LC centers

Definition 2.2.3. Let $X$ and $X^{\prime}$ be normal projective varieties and let $E \subseteq X^{\prime}$ be a prime divisor. We say that $E$ is a prime divisor over $X$ if there exists a proper birational morphism $\mu: X^{\prime} \rightarrow X$.

Definition 2.2.4. Let $(X, \Delta)$ be a pair. Then, for every prime divisor $E$ over $X$ we consider the discrepancy $a(E, X, \Delta) \in \mathbb{Q}$ as in [KM00, 2.3] (note that we will use the same notation of [KM00, Notation 2.26]).
We say that the pair $(X, \Delta)$ is Kawamata log terminal or KLT (resp. log canonical or $L C$ ) if $a(E, X, \Delta)>-1$ (resp. $a(E, X, \Delta) \geq-1$ ) for every prime divisor $E$ over $X$. We say that the pair $(X, \Delta)$ is purely log terminal or PLT if $(X, \Delta)$ is LC and $a(E, X, \Delta)>-1$ for every exceptional prime divisor $E$ over $X$.
We say that a subvariety $V \subseteq X$ is a $\log$ canonical center or a $L C$ center of the pair $(X, \Delta)$ if it is the image, through a proper birational morphism, of a divisor $E$ over $X$ such that $a(E, X, \Delta) \leq-1$.
We define $C L C(X, \Delta)=\{\mathrm{LC}$ centers of the pair $(X, \Delta)\}$ and we consider the non-klt locus of the pair $(X, \Delta)$ defined as the closed subset of $X$

$$
\operatorname{Nklt}(X, \Delta)=\bigcup_{V \in C L C(X, \Delta)} V
$$

Definition 2.2.5. Let $X$ be a normal variety and let $D$ be a Weil $\mathbb{Q}$-divisor on $X$. If we write $D=\sum d_{i} D_{i}$, where the $D_{i}$ are distinct prime divisors, we define

$$
D^{\geq 1}=\sum_{d_{i} \geq 1} d_{i} D_{i}, \quad D^{=1}=\sum_{d_{i}=1} D_{i} .
$$

Remark 2.2.6. Let $(X, \Delta)$ be a pair such that $X$ is smooth and $\Delta$ is SNCS. Then, set-theoretically,

$$
\operatorname{Nklt}(X, \Delta):=\bigcup_{V \in C L C(X, \Delta)} V=\operatorname{Supp}\left(\Delta^{\geq 1}\right)
$$

In fact on the one hand if $E$ is a prime divisor in $\operatorname{Supp}\left(\Delta^{\geq 1}\right)$ then $a(E, X, \Delta)=$ $-\operatorname{ord}_{E}(\Delta) \leq-1$.
On the other hand if $V \subseteq X$ is a subvariety that is not contained in $\operatorname{Supp}\left(\Delta^{\geq 1}\right)$, $\mu: X^{\prime} \rightarrow X$ is a proper birational morphism and $E \subseteq X^{\prime}$ is a prime divisor such that $\mu(E)=V$, then

$$
a(E, X, \Delta)=a\left(E, X, \Delta-\Delta^{\geq 1}\right)-\operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\geq 1}\right)\right)=a\left(E, Z, \Delta-\Delta^{\geq 1}\right)
$$

because $V \nsubseteq \operatorname{Supp}\left(\Delta^{\geq 1}\right)$ implies that $\operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\geq 1}\right)\right)=0$. But $\left(X, \Delta-\Delta^{\geq 1}\right)$ is a KLT pair because $X$ is smooth, $\Delta-\Delta^{\geq 1}$ is SNCS and all its coefficients are smaller than 1.
Thus $a\left(E, X, \Delta-\Delta^{\geq 1}\right)>-1$, so that $V \notin C L C(X, \Delta)$.
Now let $(X, \Delta)$ be any LC pair, let $\mu: Y \rightarrow X$ be a log-resolution of the pair $(X, \Delta)$ and let $\Delta_{Y} \in \operatorname{Div}_{\mathbb{Q}}(Y)$ be such that

$$
K_{Y}+\Delta_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right) \quad \text { and } \quad \mu_{*} \Delta_{Y}=\Delta
$$

Then $\Delta_{Y}{ }^{\geq 1}=\Delta_{Y}=1$ and the LC centers of the pair $(X, \Delta)$ are exactly the images on $X$ of the irreducible components of the finite intersections of prime divisors in the support of $\Delta_{\bar{Y}}{ }^{1}$.

We give here the definition of logbig divisor with respect to a pair.
Definition 2.2.7. Let $(X, \Delta)$ be a pair and let $L \in \operatorname{Div}_{\mathbb{Q}}(X)$.
We say that $L$ is logbig for the pair $(X, \Delta)$ if $L$ is big and $L_{\left.\right|_{V}}$ is big for every $V \in C L C(X, \Delta)$.
Moreover given an integer $k \in\{1, \ldots, n\}$ we say that $L$ is logbig in codimension $k$ if $L$ is big and $L_{\left.\right|_{V}}$ is big for every $V \in C L C(X, \Delta)$ such that $\operatorname{codim}_{X} V=k$.

In the following three lemmas we remark how the LC centers of a log canonical pair $(X, \Delta)$ change when we modify $\Delta$ by subtracting an effective divisor.

Lemma 2.2.8. Let $(X, \Delta)$ be a pair.
If $N \in \operatorname{Div}_{\mathbb{Q}}(X)$ is effective, then

1. $(X, \Delta-N)$ is LC if $(X, \Delta)$ is such;
2. $C L C(X, \Delta-N) \subseteq C L C(X, \Delta)$.

Proof. Let $F$ be a prime divisor over $X$, say $F$ is contained in a normal projective variety $Y$ and $\mu: Y \rightarrow X$ is a proper birational morphism.
We have that

$$
\begin{gathered}
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E \\
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta-N\right)+\sum a(E, X, \Delta-N) E
\end{gathered}
$$

Hence

$$
a(F, X, \Delta-N)=a(F, X, \Delta)+\operatorname{ord}_{F} \mu^{*}(N),
$$

which implies that

$$
a(F, X, \Delta-N) \geq a(F, X, \Delta)
$$

thanks to the effectivity of $N$.
If $a(F, X, \Delta-N) \leq-1$, then $a(F, X, \Delta) \leq-1$ so that every LC center of $(X, \Delta-N)$ is also an LC center of $(X, \Delta)$.
Moreover if $(X, \Delta)$ is LC then $a(G, X, \Delta) \geq-1$, for every prime divisor $G$ over $X$, so that, by the previous inequality, we get the same inequalities for the pair $(X, \Delta-N)$, that is $(X, \Delta-N)$ is an LC pair.

Lemma 2.2.9. Let $(X, \Delta)$ be an LC pair, let $B \in \operatorname{Div}_{\mathbb{Q}}(X)$ be effective, $B \neq 0$, and take $V \in C L C(X, \Delta)$.
Then the following statements are equivalent:

1. $\operatorname{Supp}(B) \supseteq V$;
2. $V \notin C L C(X, \Delta-c B)$ for every rational number $c>0$;
3. $V \notin C L C(X, \Delta-c B)$ for some rational number $c>0$.

Proof. $(\mathbf{1} \Rightarrow \mathbf{2})$ Let $E$ be a prime divisor over $X$ such that $V$ is the center of $E$ on $X$.
In other words there exists a projective normal variety $Y$ and a proper birational morphism $\mu: Y \rightarrow X$ such that $E \subseteq Y$ and $\mu(E)=V$.
We have to show that $a(E, X, \Delta-c B)>-1$ for every $c \in \mathbb{Q}^{+}$:
We can write

$$
\begin{gathered}
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E \\
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta-c B\right)+\sum a(E, X, \Delta-c B) E
\end{gathered}
$$

Hence

$$
a(E, X, \Delta-c B)=a(E, X, \Delta)+c \operatorname{ord}_{E}\left(\mu^{*}(B)\right)
$$

But $V=\mu(E) \subseteq \operatorname{Supp}(B)$, so that $\operatorname{ord}_{E}\left(\mu^{*}(B)\right)>0$. Thus

$$
a(E, X, \Delta-c B)>a(E, X, \Delta) \geq-1
$$

$(2 \Rightarrow 3)$ Trivial.
$(\mathbf{3} \Rightarrow \mathbf{1})$ As $V \in C L C(X, \Delta)$ there exists a prime divisor $E$ over $X$ such that $a(E, X, \Delta)=-1$. Using the same notation as before we find again that

$$
a(E, X, \Delta-c B)=a(E, X, \Delta)+c \operatorname{ord}_{E}\left(\mu^{*}(B)\right)=-1+c \operatorname{ord}_{E}\left(\mu^{*}(B)\right)
$$

Hence $a(E, X, \Delta-c B)>-1$ for some $c>0$ implies that $\operatorname{ord}_{E}\left(\mu^{*}(B)\right)>0$, which in turn implies that $V=\mu(E) \subseteq \operatorname{Supp}(B)$.

Lemma 2.2.10. Let $(X, \Delta)$ be an LC pair and let $B \in \operatorname{Div}_{\mathbb{Q}}(X)$ be effective, $B \neq 0$.
Then the following statements are equivalent:

1. Supp $(B)$ contains all the $L C$ centers of the pair $(X, \Delta)$;
2. $(X, \Delta-c B)$ is a KLT pair for every rational number $c>0$;
3. $(X, \Delta-c B)$ is a KLT pair for some rational number $c>0$.

Proof. The lemma follows from lemma 2.2 .9 and the fact that $C L C(X, \Delta-$ $c B) \subseteq C L C(X, \Delta)$ for every $c>0$ because $B$ is effective.

## Standard log-resolutions and applications

It can be sometimes useful to consider log-resolutions of pairs (or ideals) that are isomorphisms on the locus where the pair (or the ideal) is already "log-resolved". For this reason in this subsection we introduce the notion of standard logresolutions. With this purpose we begin by giving the definition of non-simple normal crossing locus of a divisor:

Definition 2.2.11. Let $X$ be a normal variety and let $D$ be a Weil $\mathbb{R}$-divisor on $X$. If $U$ is the biggest open subset of $X$ such that $D_{\left.\right|_{U}}$ is SNCS we define the non-simple normal crossing locus of $D$ as the closed set

$$
N S N C(D)=X \backslash U
$$

Note that $N S N C(D)$ only depends on the support of $D$ and we always have that $N S N C(D) \subseteq \operatorname{Supp}(D)$.

Lemma 2.2.12. Let $(X, \Delta)$ be an $L C$ pair, let $N \in \operatorname{Div}_{\mathbb{Q}}(X)$ be effective, let $W \subseteq X$ be an (irreducible) subvariety. If $W \in C L C(X, \Delta-N)$ and $W \subseteq$ $N S N C(\Delta-N)$, then $W \subseteq N S N C(\Delta)$.

Proof. We begin by noticing that $(X, \Delta-N)$ is LC by lemma 2.2.8. Moreover note that $N S N C(X, \Delta-N) \subseteq N S N C(\Delta) \cup \operatorname{Supp}(N)$.
Then, as $W$ is irreducible by hypothesis, it suffices to show that $W \nsubseteq \operatorname{Supp}(N)$. Let $\mu: Y \rightarrow X$ be a proper birational morphism and let $F \subseteq Y$ be a prime divisor such that $\mu(F)=W$ and $a(F, X, \Delta-N)=-1$. We can write

$$
\begin{gathered}
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E \\
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta-N\right)+\sum a(E, X, \Delta-N) E
\end{gathered}
$$

As $N \in \operatorname{Div}_{\mathbb{Q}}(X)$ we can deduce that

$$
\sum a(E, X, \Delta) E=\sum a(E, X, \Delta-N) E-\mu^{*}(N)
$$

Hence

$$
-1 \leq a(F, X, \Delta)=a(F, X, D-N)-\operatorname{ord}_{F} \mu^{*}(N)=-1-\operatorname{ord}_{F} \mu^{*}(N)
$$

Thanks to the effectivity of $N$ this implies that $\operatorname{ord}_{F} \mu^{*}(N)=0$, so that $W \nsubseteq$ $\operatorname{Supp}(N)$.

Definition 2.2.13. Let $X$ be a normal projective variety, let $L$ be a Cartier divisor on $X$ and let $V \subseteq H^{0}\left(X, \mathcal{O}_{X}(L)\right)$ be a subspace. Consider the evaluation morphism

$$
e_{V}: V \otimes \mathcal{O}_{X}(-L) \rightarrow \mathcal{O}_{X}
$$

Then the base ideal of the linear series $|V|$, denoted by $b(|V|)$, is the ideal sheaf defined as the image of the map $e_{V}$.
The base scheme of $|V|, B s(|V|)$, is the scheme defined by the base ideal $b(|V|)$, so that, in particular, it is supported on the set $\mathcal{Z}(b(|V|))$

Before giving the definition of standard log-resolution we recall the following easy lemma:

Lemma 2.2.14. Let $X$ be a normal projective variety, let $|L|$ be a non empty linear series on $X$ and let $b(|L|)$ be its base ideal. Suppose $\mu: X^{\prime} \rightarrow X$ is a log-resolution of the ideal $b(|L|)$. Then

1. $b\left(\left|\mu^{*}(L)\right|\right)=\mu^{-1}(b(|L|))$;
2. $\mu$ is a log-resolution of the linear series $|L|$.

Proof. 1. By [Gro60, 4.4.5] we have that the ideal $\mu^{-1}(b(|L|))$ defines the scheme $\mu^{-1}(B s(|L|))$. Now let $|L|=<s_{1}, \ldots, s_{k}>$. Then

$$
\mu^{*}(L)=<\mu^{*}\left(s_{1}\right), \ldots, \mu^{*}\left(s_{k}\right)>=<s_{1} \circ \mu, \ldots, s_{k} \circ \mu>
$$

Then $B s\left(\left|\mu^{*}(L)\right|\right)$ is the scheme defined by $s_{1} \circ \mu, \ldots, s_{k} \circ \mu$, so that it is the inverse image of the scheme defined by $<s_{1}, \ldots, s_{k}>$.
In other words $B s\left(\left|\mu^{*}(L)\right|\right)=\mu^{-1}(B s(|L|))$.

Therefore we obtain that $\mu^{-1}(b(\mid L))$ defines the base scheme $B s\left(\left|\mu^{*}(L)\right|\right)$, that is $\mu^{-1}(b(\mid L))=b\left(\left|\mu^{*}(L)\right|\right)$.
2. By hypothesis $X^{\prime}$ is smooth and there exists a SNCS divisor $F$ on $X^{\prime}$ such that $\mu^{-1}(b(|L|))=\mathcal{O}_{X^{\prime}}(-F)$. Thus, by the first part we have that $b\left(\left|\mu^{*}(L)\right|\right)=$ $\mathcal{O}_{X^{\prime}}(-F)$.
In other words the base scheme $B s\left(\left|\mu^{*}(L)\right|\right)=F$.
Thus we can write $\left|\mu^{*}(L)\right|=\left|\mu^{*}(L)-F\right|+F$, where the linear series $\left|\mu^{*}(L)-F\right|$ is free.

Definition 2.2.15. Let $X$ be a normal projective variety and let $D$ be a reduced Weil divisor on $X$.
A standard log-resolution of the pair $(X, D)$ is a log-resolution $f$ of the pair $(X, D)$ such that

- $f$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 up to isomorphisms;
- $f_{\left.\right|_{f^{-1}(U)}}$ is an isomorphism, where $U=X \backslash(N S N C(D) \cup \operatorname{Sing}(X))$.

Definition 2.2.16. Let $X$ be a smooth projective variety and let $\mathcal{J} \subseteq \mathcal{O}_{X}$ be a non zero ideal sheaf. A standard log-resolution of the ideal sheaf $\overline{\mathcal{J}}$ is a log-resolution $g$ of $\mathcal{J}$ such that $g$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 contained in $\mathcal{Z}(\mathcal{J})$ up to isomorphisms. In particular $g_{\left.\right|_{g^{-1}(X \backslash \mathcal{Z}(\mathcal{J}))}}$ is an isomorphism.

Remark 2.2.17. Note that, by definition, a standard log-resolution of a pair or an ideal sheaf has a divisorial exceptional set.
In fact this is true for every composition of blowings-up of smooth subvarieties of codimension greater than 1:
It suffices to prove that for a single blowing-up $\pi$ of a smooth subvariety $Z$ of codimension greater than 1 .
But, in this case, by the properties of the blowing-up we have that $\pi^{-1}(Z)$ is a divisor and $\operatorname{exc}(\pi) \subseteq \pi^{-1}(Z)$. On the other hand every prime divisor in $\pi^{-1}(Z)$ is contracted by $\pi$ because its image, being contained in $Z$, has codimension greater than 1.
Thus $\pi^{-1}(Z) \subseteq \operatorname{divexc}(\pi) \subseteq \operatorname{exc}(\pi)$, so that $\pi^{-1}(Z)=\operatorname{divexc}(\pi)=\operatorname{exc}(\pi)$.
Theorem 2.2.18. Let $X$ be a normal projective variety and let $D$ be a reduced Weil divisor on $X$. Then there exists a standard log-resolution of the pair $(X, D)$.

Proof. Let $f: X^{\prime} \rightarrow X$ be a resolution of singularities of $X$ obtained by blowing up smooth subvarieties of $X$ contained in $\operatorname{Sing}(X)$ (see for example [Laz04, theorem 4.1.3]). As $X$ is normal $\operatorname{codimSing}(X) \geq 2$, so that $f$ is a composition of blowings-up of subvarieties of codimension greater than 1.
Now, as $X^{\prime}$ is smooth, thanks to [Fuj07a, theorem 3.5.1] there exists a logresolution of the pair $\left(X^{\prime}, f_{*}^{-1}(D)+\operatorname{divexc}(f)\right)$, say $g: Y \rightarrow X^{\prime}$, such that $g$ is a composition of blowings-up of smooth subvarieties and $g$ is an isomorphism outside $N S N C\left(f_{*}^{-1}(D)+\operatorname{divexc}(f)\right)$.

Let $\mu=f \circ g$. Then $\mu$ is a log-resolution of $(X, D)$ because

$$
\mu_{*}^{-1}(D)+\operatorname{divexc}(\mu)=g_{*}^{-1}\left(f_{*}^{-1} D+\operatorname{divexc}(f)\right)+\operatorname{divexc}(g)
$$

is SNC by definition of $g$.
We have that $\mu$ is a composition of blowings-up of smooth subvarieties because the same holds both for $f$ and $g$ and we can suppose their codimension is greater than 1:
In fact we have already proved this for $f$. As for $g$, note that every smooth subvariety $Z$ of $X^{\prime}$ of codimension 1 is a Cartier divisor because $X^{\prime}$ is smooth. Hence the blowing-up along $Z$ is an isomorphism.
Let us prove now that $\mu$ is an isomorphism outside $\mu^{-1}(N S N C(D) \cup \operatorname{Sing}(X))$ : First of all we note that

$$
N S N C\left(f_{*}^{-1}(D)+\operatorname{divexc}(f)\right) \subseteq \operatorname{divexc}(f) \cup f^{-1}(N S N C(D)):
$$

In fact if $f_{*}^{-1}(D)+\operatorname{divexc}(f)$ is not simple normal crossing in a point $x$ outside divexc $(f)$, then $f_{*}^{-1}(D)$ is not simple normal crossing in $x$. But $f$ is an isomorphism in a neighbourhood of $x$ because $x \notin \operatorname{divexc}(f)=\operatorname{exc}(f)$ by remark 2.2.17, hence $D$ is not simple normal crossing in $f(x)$.

Now, by definition, $\mu$ is an isomorphism outside

$$
\operatorname{exc}(\mu)=\operatorname{divexc}(\mu)=\operatorname{divexc}(g) \cup g^{-1}(\operatorname{divexc}(f))
$$

On the other hand we have that $\operatorname{divexc}(f) \subseteq f^{-1}(\operatorname{Sing}(X))$. Hence we get that $\mu$ is an isomorphism outside

$$
\begin{gathered}
\operatorname{divexc}(g) \cup g^{-1}(\operatorname{divexc}(f)) \subseteq \operatorname{divexc}(g) \cup g^{-1}\left(f^{-1}(\operatorname{Sing}(X))\right) \subseteq \\
\subseteq g^{-1}\left(N S N C\left(f_{*}^{-1}(D)+\operatorname{divexc}(f)\right)\right) \cup \mu^{-1}(\operatorname{Sing}(X))
\end{gathered}
$$

because of the definition of $g$. Thus, using the previous observation we have that

$$
\begin{gathered}
\operatorname{exc}(\mu) \subseteq g^{-1}(\operatorname{divexc}(f)) \cup g^{-1}\left(f^{-1}(N S N C(D))\right) \cup \mu^{-1}(\operatorname{Sing}(X))= \\
=\mu^{-1}(N S N C(D)) \cup \mu^{-1}(\operatorname{Sing}(X))
\end{gathered}
$$

Theorem 2.2.19. Let $X$ be a smooth projective variety and let $\mathcal{J} \subseteq \mathcal{O}_{X}$ be a non zero ideal sheaf. Then there exists a standard log-resolution of $\mathcal{J}$.

Proof. Apart from [Hir64], this is proved, for example, in [Kol05, theorem 35 ] or [Kol05, theorem 21] (together with theorem 2.2.18). Note that we can suppose the log-resolution being composed, up to isomorphisms, by blowingsup of varieties of codimension greater than 1 because every prime divisor on $X$ is a Cartier divisor, by the smoothness of $X$ itself. Thus blowing up a smooth codimension one subvariety on $X$ gives rise to an isomorphism.

The following lemma is based on the existence of standard log-resolutions:

Lemma 2.2.20. Let $(X, D)$ be a pair and let $|L|$ be a linear series on $X$. Then there exists a common log-resolution of the pair $(X, D)$ and of the base ideal $b(|L|)$, say $\mu: Z \rightarrow X$, such that

1. $\mu$ is an isomorphism on $Z \backslash \mu^{-1}(B s(|L|) \cup \operatorname{Sing}(X) \cup N S N C(D))$;
2. $\mu$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1, up to isomorphisms.

In particular, by lemma 2.2.14, $\mu$ is a log-resolution of the linear series $|L|$.
Proof. Let $b=b(|L|)$. Let $f: X^{\prime} \rightarrow X$ be a resolution of singularities of $X$ obtained by blowing up smooth subvarieties of $X$ contained in $\operatorname{Sing}(X)$ (see for example [Laz04, theorem 4.1.3]). Note that all the varieties we blow up have codimension greater than 1 , because $\operatorname{codim} \operatorname{Sing}(X) \geq 2$ as $X$ is normal.
As $X^{\prime}$ is smooth, by theorem 2.2.19 there exists $g: X^{\prime \prime} \rightarrow X^{\prime}$ a standard logresolution of the ideal sheaf $f^{-1}(b)$. Then $g$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 contained in $\mathcal{Z}\left(f^{-1}(b)\right)$. Moreover there exists an effective divisor $F$ on $X^{\prime \prime}$ such that $g^{-1}\left(f^{-1}(b)\right)=$ $\mathcal{O}_{X^{\prime \prime}}(-F)$.
Now, by theorem 2.2.18 we can consider $h: Z \rightarrow X^{\prime \prime}$, a standard log-resolution of the pair $\left(X^{\prime \prime}, \operatorname{Supp}(F)+\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)+\operatorname{divexc}(f \circ g)\right)$.
Again $h$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1.
Let $\mu=f \circ g \circ h: Z \rightarrow X$. Then $\mu$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 because the same holds for $f, g$ and $h$.

Let us show that $\mu$ is a common log-resolution of $(X, D)$ and $|L|$ : We begin by noting that $Z$ is smooth by definition.
Now we have that $\mu^{-1}(b)=h^{-1}\left(\mathcal{O}_{X^{\prime \prime}}(-F)\right)=h^{*}\left(\mathcal{O}_{X^{\prime \prime}}(-F)\right)$ by [Har77, II 7.12.2] because $\mathcal{O}_{X^{\prime \prime}}(-F)$ is an invertible sheaf. Then we have to show that

$$
\operatorname{Supp}\left(\mu_{*}^{-1}(D)\right) \cup \operatorname{Supp}\left(h^{*}(F)\right) \cup \operatorname{divexc}(\mu)
$$

is SNC. But

$$
\begin{gathered}
\operatorname{Supp}\left(\mu_{*}^{-1}(D)\right) \cup \operatorname{Supp}\left(h^{*}(F)\right) \cup \operatorname{divexc}(\mu)= \\
=h_{*}^{-1}\left(\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1}(D)\right)\right)\right) \cup h_{*}^{-1}(\operatorname{Supp}(F)) \cup \operatorname{divexc}(h) \cup h_{*}^{-1} \operatorname{divexc}(f \circ g),
\end{gathered}
$$

so that it is SNC thanks to the choice of $h$.
It remains to show that $\mu$ is an isomorphism on $Z \backslash \mu^{-1}(B s(|L|) \cup \operatorname{Sing}(X) \cup$ $N S N C(D)$ ).
We will equivalently show that

$$
\operatorname{divexc}(\mu)=\operatorname{exc}(\mu) \subseteq \mu^{-1}(B s(|L|) \cup \operatorname{Sing}(X) \cup N S N C(D))
$$

We use that, by definition,

$$
\operatorname{divexc}(\mu)=\operatorname{divexc}(h) \cup h^{-1}(\operatorname{divexc}(g)) \cup h^{-1}\left(g^{-1}(\operatorname{divexc}(f))\right)
$$

$\operatorname{But} \operatorname{divexc}(f) \subseteq f^{-1}(\operatorname{Sing}(X))$ by definition of $f$, so that $h^{-1}\left(g^{-1}(\operatorname{divexc}(f))\right) \subseteq$ $\mu^{-1}(\operatorname{Sing}(X))$.

On the other hand, by the definition of $g$ and using [Gro60, 4.4.5], we have that $h^{-1}(\operatorname{divexc}(g)) \subseteq h^{-1}\left(g^{-1}\left(\mathcal{Z}\left(f^{-1}(b)\right)\right)\right) \subseteq h^{-1}\left(g^{-1}\left(f^{-1}(\mathcal{Z}(b))\right)\right)=\mu^{-1}(B s(|L|))$.

It remains to study the exceptional locus of $h$ :
Let $U \subseteq X^{\prime \prime}$ be the biggest open subset such that

$$
\left(\operatorname{Supp}(F)+\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)+\operatorname{divexc}(f \circ g)\right)_{\left.\right|_{U}}
$$

is SNC.
Then, as $h$ is a standard log-resolution, we have that divexc $(h) \subseteq Z \backslash h^{-1}(U)$.
Now let $V$ be the biggest open subset of $X$ such that $D_{\left.\right|_{V}}$ is SNCS, so that $N S N C(D)=X \backslash V$.
We have to show that

$$
Z \backslash h^{-1}(U) \subseteq\left(Z \backslash \mu^{-1}(V)\right) \cup \mu^{-1}(\operatorname{Sing}(X)) \cup \mu^{-1}(B s(|L|))
$$

This is equivalent to say that

$$
\begin{array}{r}
h^{-1}(U) \supseteq \mu^{-1}(V) \cap\left(Z \backslash \mu^{-1}(\operatorname{Sing}(X))\right) \cap\left(Z \backslash \mu^{-1} B s(|L|)\right)= \\
=h^{-1}\left(g^{-1}\left(f^{-1}(V)\right)\right) \cap h^{-1}\left(X^{\prime \prime} \backslash g^{-1}\left(f^{-1}(\operatorname{Sing}(X))\right)\right) \cap \\
\cap h^{-1}\left(X^{\prime \prime} \backslash g^{-1}\left(f^{-1} B s(|L|)\right)\right) .
\end{array}
$$

Hence we have to show that

$$
\begin{array}{r}
U \supseteq g^{-1}\left(f^{-1}(V)\right) \cap\left(X^{\prime \prime} \backslash g^{-1}\left(f^{-1}(\operatorname{Sing}(X))\right)\right) \cap\left(X^{\prime \prime} \backslash g^{-1}\left(f^{-1} B s(|L|)\right)\right)= \\
=: S
\end{array}
$$

In other words this is equivalent to say that

$$
\left(\operatorname{Supp}(F)+\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)+\operatorname{divexc}(f \circ g)\right)_{\mid S}
$$

is SNC. But

$$
\begin{aligned}
\operatorname{divexc}(f \circ g)=\operatorname{divexc}(g) \cup g^{-1}(\operatorname{divexc}(f)) & \subseteq g^{-1}\left(f^{-1}(B s(|L|))\right) \cup \\
& \cup g^{-1}\left(f^{-1}(\operatorname{Sing}(X))\right)
\end{aligned}
$$

and $\operatorname{Supp}(F) \subseteq g^{-1}\left(f^{-1}(B s(|L|))\right)$ by definition. This implies that

$$
\left(\operatorname{Supp}(F)+\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)+\operatorname{divexc}(f \circ g)\right)_{\left.\right|_{S}}=\left(\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)\right)_{\left.\right|_{s}} .
$$

Now note that $\operatorname{Supp}(D)_{\left.\right|_{(f \circ g)(S)}}$ is SNC, as $(f \circ g)(S) \subseteq V$.
On the other hand we have just noticed that divexc $(f \circ g) \subseteq X^{\prime \prime} \backslash S$, so that $f \circ g$ is an isomorphism on $S$. Thus $\left(\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} D\right)\right)\right)_{\mid S}$ remains SNC.

## Remarks on DLT pairs

We recall, first of all, the definition of a DLT pair:
Definition 2.2.21. (cf. [KM00, definition 2.37]). Let $(X, \Delta)$ be a pair such that if we write $\Delta=\sum a_{i} D_{i}$, with the $D_{i}$ 's distinct prime divisors, then $0 \leq a_{i} \leq 1$ for every $i$. We say that ( $X, \Delta$ ) is divisorial log terminal (or DLT) if there exists a closed subset $Z \subseteq X$ such that:

1. $X \backslash Z$ is smooth and $\Delta_{\left.\right|_{X \backslash Z}}$ is SNCS
2. If $f: Y \rightarrow X$ is birational and $E \subseteq Y$ is an irreducible divisor such that $f(E) \subseteq Z$ then the discrepancy $a(E, X, \Delta)>-1$.

Definition 2.2.22. Let $(X, \Delta)$ be a pair, let $Y$ be a normal variety and let $f: Y \rightarrow X$ be a projective birational morphism. We say that $f$ is a $D L T$ morphism for the pair $(X, \Delta)$ if

- $f$ is a composition of blowings-up of smooth centers of codimension greater than 1;
- We have that

$$
a(E, X, \Delta)>-1 \quad \text { for every prime divisor } E \subseteq Y \text { exceptional on } X .
$$

Following [Sza95] we give in the next theorem a useful multiple characterization of the property of DLTness:

Theorem 2.2.23 ([Sza95]). Let $(X, \Delta)$ be an effective LC pair. Then the following statements are equivalent:

1. $(X, \Delta)$ is DLT;
2. There exists a log-resolution of $(X, \Delta)$ that is a DLT morphism for the same pair;
3. There exists a standard log-resolution of $(X, \Delta)$ that is a DLT morphism for the same pair;
4. For every $V \in C L C(X, \Delta)$ we have that $V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$.

Proof. $(\mathbf{1} \Rightarrow 4)$ By definition there exists a closed subset of $X$, say $Z$, such that $X \backslash Z$ is smooth, $\Delta_{\left.\right|_{X \backslash Z}}$ is SNCS and $Z$ does not contain any LC center of $(X, \Delta)$. Then $\operatorname{NSNC}(\Delta) \subseteq Z$ because $\Delta_{\left.\right|_{X \backslash Z}}$ is $\operatorname{SNCS}$ and $\operatorname{Sing}(X) \subseteq Z$ because $X \backslash Z$ is smooth. Thus if $V \in C L C(X, \Delta)$, since $V \nsubseteq Z$ we have that $V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$.
$(4 \Rightarrow 3)$ Let $\mu$ be a standard log-resolution of the pair $(X, \operatorname{Supp}(\Delta))$, whose existence is assured by theorem 2.2.18.
Then, by definition, $\mu$ is an isomorphism on $\mu^{-1}(X \backslash(N S N C(\Delta) \cup \operatorname{Sing}(X)))$. Hence divexc $(\mu) \subseteq \mu^{-1}(N S N C(\Delta) \cup \operatorname{Sing}(X))$, so that if $E$ is a $\mu$-exceptional prime divisor we have that $\mu(E) \subseteq N S N C(\Delta) \cup \operatorname{Sing}(X)$. Thanks to the hypothesis, this implies that $\mu(E) \notin C L C(X, \Delta)$, so that $a(E, X, \Delta)>-1$. In other words $\mu$ is a DLT morphism for the pair $(X, \Delta)$.
(3 $\Rightarrow 2$ ) Trivial.
$(\mathbf{2} \boldsymbol{7} \mathbf{1 )}$ Let $f: Y \rightarrow X$ be the a log-resolution as in the hypothesis and let $U$ be the biggest open subset of $X$ such that $f_{\left.\right|_{f^{-1}(U)}}$ is an isomorphism. Then

$$
Y \backslash f^{-1}(U)=\operatorname{exc}(f)=\operatorname{divexc}(f)
$$

thanks to the properties of DLT morphisms.
Define $Z=X \backslash U$. Then $\Delta_{\left.\right|_{X \backslash Z}}=\Delta_{\left.\right|_{U}}$ is SNCS because $f$ is an isomorphism over $U$ and $f_{*}^{-1}(\Delta)$ is SNCS. Moreover $X \backslash Z=U \simeq f^{-1}(U)$ is smooth because $Y$ is such.

In order to prove that $(X, \Delta)$ is DLT it remains to show that there are no LC centers of $(X, \Delta)$ contained in $Z$.
Suppose, by contradiction, there exists $V \in C L C(X, \Delta)$ such that $V \subseteq Z$. Define

$$
\Delta_{Y}=-\sum a(E, X, \Delta) E
$$

where the sum is taken on all prime divisors $E$ on $Y$. Then, by remark 2.2.6, $V=f(W)$, where $W$ is an irreducible component of a finite intersection of prime divisors in the support of $\Delta_{\bar{Y}}^{1}$. But, by definition of DLT morphism, $a(E, X, \Delta)>-1$ if $E$ is $f$-exceptional.
Then $W$ is an irreducible component of a finite intersection of strict transforms of divisors in the support of $\Delta$, so that $W \nsubseteq \operatorname{divexc}(f)$ because $\mu_{*}^{-1}(\Delta)+\operatorname{divexc}(f)$ is a SNCS divisor.
Thus $W \nsubseteq \operatorname{exc}(f)$, so that $V=f(W) \nsubseteq f(\operatorname{exc}(f))=f(Y) \backslash U=X \backslash U=Z$, and we get the contradiction.

In the rest of this subsection we list some good properties of DLT pairs.
Lemma 2.2.24. Let $X$ be a normal projective $\mathbb{Q}$-Gorenstein variety, and let $\Delta \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that the pair $(X, \Delta)$ is DLT. Then $(X,(1-\epsilon) \Delta)$ is a KLT pair for every rational number $\epsilon>0$.
Proof. By theorem 2.2.23 there exists a birational morphism $\mu: Y \rightarrow X$ such that $\mu$ is a log-resolution of the pair $(X, \Delta)$ and a DLT morphism for the same pair.
As $\Delta$ is $\mathbb{Q}$-Cartier we have that, for every rational number $\epsilon>0$, the $\mathbb{Q}$-divisor $K_{X}+(1-\epsilon) \Delta$ is $\mathbb{Q}$-Cartier.
Then we can write

$$
\begin{gathered}
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E, \\
K_{Y} \equiv \mu^{*}\left(K_{X}+(1-\epsilon) \Delta\right)+\sum a(E, X,(1-\epsilon) \Delta)= \\
=\mu^{*}\left(K_{X}+\Delta\right)-\epsilon \mu^{*}(\Delta)+\sum a(E, X,(1-\epsilon) \Delta) .
\end{gathered}
$$

Hence, if $\Delta_{\epsilon}:=-\sum a(E, X,(1-\epsilon) \Delta) E$, then, thanks to [KM00, lemma 2.30], we have that

$$
a(D, X,(1-\epsilon) \Delta)=a\left(D, Y, \Delta_{\epsilon}\right) \quad \forall D \text { prime divisor over } X
$$

Thus, for every $\epsilon \in \mathbb{Q}^{+},(X,(1-\epsilon) \Delta)$ is KLT if and only if the same holds for $\left(Y, \Delta_{\epsilon}\right)$.
But $Y$ is smooth and $\Delta_{\epsilon}$ is SNCS, whence $\left(Y, \Delta_{\epsilon}\right)$ is KLT if and only if all the coefficients of $\Delta_{\epsilon}$ are less than 1 , that is $a(E, X,(1-\epsilon) \Delta)>-1$ for each prime divisor $E$ on $Y$.
But, for every prime divisor $E$ on $Y$ we have that

$$
a(E, X,(1-\epsilon) \Delta)=a(E, X, \Delta)+\epsilon \operatorname{ord}_{E} \mu^{*}(\Delta)
$$

Now, as $\mu$ is a DLT morphism, if $a(E, X, \Delta)=-1$, then $E$ is not $\mu$-exceptional. Then, by definition of discrepancies, if $a(E, X, \Delta)=-1$ we have that

$$
E \subseteq \operatorname{Supp}\left(\mu_{*}^{-1}(\Delta)\right) \subseteq \operatorname{Supp}\left(\mu^{*}(\Delta)\right)
$$

so that $\operatorname{ord}_{E} \mu^{*}(\Delta)>0$.
Thus we obtain that

- if $a(E, X, D)>-1$, then $a(E, X,(1-\epsilon) \Delta) \geq a(E, X, D)>-1$;
- if $a(E, X, D)=-1$, then $a(E, X,(1-\epsilon) \Delta)>a(E, X, D) \geq-1 ;$
so that we are done.
Remark 2.2.25. It is very easy to see that the converse of the last lemma does not hold, even if we add the effectiveness of the boundary.
In other words there exist pairs $(X, \Delta)$ such that $\Delta$ is effective and $(X,(1-\epsilon) \Delta)$ is KLT for every $\epsilon>0$, but $(X, \Delta)$ is not DLT.
It suffices to consider a smooth variety $X$ and $\Delta \in \operatorname{Div}_{\mathbb{Q}}(X)$ effective, such that $(X, \Delta)$ is LC but not DLT. For example take $X$ to be the plane and $\Delta=\frac{5}{6} A$, where $A$ is the cuspidal cubic curve (see [Laz04, Example 9.2.15]).
Lemma 2.2.26. Let $(X, \Delta)$ be an LC pair.
If $\Delta^{\prime} \in \operatorname{Div}_{\mathbb{Q}}(X)$ is effective and Supp $\left(\Delta^{\prime}\right)$ does not contain LC centers of the pair $(X, \Delta)$, then there exists $\lambda_{0}>0$ such that for all $\lambda \in \mathbb{Q} \cap\left[0, \lambda_{0}\right)$, we have

1. $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ is an $L C$ pair and $C L C\left(X, \Delta+\lambda \Delta^{\prime}\right)=C L C(X, \Delta)$;
2. If $(X, \Delta)$ is DLT then $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ is also DLT.

Proof. 1. Let $\mu: Y \rightarrow X$ be a common log-resolution of $(X, \Delta)$ and $\left(X, \Delta^{\prime}\right)$, so that $Y$ is smooth and for every $\lambda \geq 0$ we have that

$$
\operatorname{Supp}\left(\mu^{*}(\Delta)\right) \cup \operatorname{Supp}\left(\mu^{*}\left(\lambda \Delta^{\prime}\right)\right) \cup \operatorname{exc}(\mu)
$$

is SNC.
We can write

$$
\begin{aligned}
& K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E \\
K_{Y} \equiv & \mu^{*}\left(K_{X}+\Delta+\lambda \Delta^{\prime}\right)+\sum a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E
\end{aligned}
$$

Hence, if $\Delta_{\lambda}:=-\sum a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E$, thanks to [KM00, lemma 2.30] we have that

$$
a\left(D, X, \Delta+\lambda \Delta^{\prime}\right)=a\left(D, Y, \Delta_{\lambda}\right) \quad \forall D \text { prime divisor over } X
$$

Thus, if we show that $\left(Y, \Delta_{\lambda}\right)$ is LC for $0 \leq \lambda \ll 1$, then the same holds for $\left(X, \Delta+\lambda \Delta^{\prime}\right)$.
But $Y$ is smooth and $\Delta_{\lambda}$ is $\operatorname{SNCS}$, because $\operatorname{Supp}\left(\Delta_{\lambda}\right) \subseteq \operatorname{Supp}\left(\mu^{*}(\Delta)\right) \cup$ $\operatorname{Supp}\left(\mu^{*}\left(\lambda \Delta^{\prime}\right)\right) \cup \operatorname{exc}(\mu)$, whence it suffices to show that all the coefficients of $\Delta_{\lambda}$ are less than or equal to 1 , that is $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) \geq-1$ for each prime divisor $E$ on $Y$.
But we have that

$$
a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=a(E, X, \Delta)-\lambda \operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\prime}\right)\right)
$$

and, moreover, if $a(E, X, \Delta)=-1$ then $\mu(E) \in C L C(X, \Delta)$, so that $\mu(E) \nsubseteq$ $\operatorname{Supp}\left(\Delta^{\prime}\right)$, that is $\operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\prime}\right)\right)=0$.
Hence, if we define

$$
\lambda_{0}=\min _{\operatorname{ord}_{E}\left(\mu^{*} \Delta^{\prime}\right)>0}\left\{\frac{1+a(E, X, \Delta)}{\operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\prime}\right)\right)}\right\}
$$

then $\lambda_{0}>0$, and for each $\lambda \in\left[0, \lambda_{0}\right)$ we have that

- If $a(E, X, \Delta)=-1$ then $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=-1$;
- If $a(E, X, \Delta)>-1$ then $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)>-1$.

Thus $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ is LC and $C L C\left(X, \Delta+\lambda \Delta^{\prime}\right)=C L C(X, \Delta)$ thanks to remark 2.2.6, because $\Delta_{\lambda}^{=1}=\Delta_{0}^{=1}$.
2. If ( $X, \Delta$ ) is DLT we have that $\Delta+\lambda \Delta^{\prime}$ is effective and we know, by the first part, that the pair $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ is LC. Hence, by theorem 2.2 .23 , it suffices to show that there exists a log-resolution of the pair $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ that is a DLT morphism for the same pair.
As the pair $(X, \Delta)$ is DLT, by theorem 2.2.23 there exists a log-resolution $f: X^{\prime} \rightarrow X$ of the pair $(X, \Delta)$ that is a DLT morphism for $(X, \Delta)$.
Let $g: Y \rightarrow X^{\prime}$ be a standard $\log$-resolution of the pair $\left(X^{\prime}, \operatorname{Supp}\left(f^{*}(\Delta)+\right.\right.$ $\left.\left.f^{*}\left(\Delta^{\prime}\right)\right) \cup \operatorname{divexc}(f)\right)$.
Then $\operatorname{Supp}\left(g^{*}\left(f^{*}\left(\Delta+\Delta^{\prime}\right)\right)\right) \cup \operatorname{divexc}(f \circ g)$ is SNC, so that $f \circ g$ is a log-resolution of the pair $\left(X, \Delta+\lambda \Delta^{\prime}\right)$.
Note also that $f \circ g$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1 , because the same holds for $f$ and $g$.
Moreover, if $E$ is a prime $(f \circ g)$-exceptional divisor, then, $E$ is $g$-exceptional or $f$-exceptional (identifying $E$ with $g_{*}(E)$ in the latter case).
If $E$ is $f$-exceptional, then $a(E, X, \Delta)>-1$ because $f$ is a DLT morphism for $(X, \Delta)$, that is $f(E)$ is not a LC center of the pair $(X, \Delta)$. Hence (by part 1) $f(E)$ is not a LC center of $\left(X, \Delta+\lambda \Delta^{\prime}\right)$, so that $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)>-1$.
Let now $E$ be $g$-exceptional. Then, by definition of standard log-resolution, as

$$
\begin{aligned}
& \left(\operatorname{Supp}\left(f^{*}(\Delta)+f^{*}\left(\Delta^{\prime}\right)\right) \cup \operatorname{divexc}(f)\right)_{\left.\right|_{X^{\prime} \backslash \operatorname{Supp}\left(f^{*}\left(\Delta^{\prime}\right)\right)}}= \\
& \quad=\left(\operatorname{Supp}\left(f^{*}(\Delta)\right) \cup \operatorname{divexc}(f)\right)_{\left.\right|_{X^{\prime} \backslash \operatorname{Supp}\left(f^{*}\left(\Delta^{\prime}\right)\right)}}
\end{aligned}
$$

is SNC, we have that $g$ is an isomorphism on $g^{-1}\left(X^{\prime} \backslash \operatorname{Supp}\left(f^{*}\left(\Delta^{\prime}\right)\right)\right)$.
Hence $g(E) \subseteq \operatorname{Supp}\left(f^{*}\left(\Delta^{\prime}\right)\right)$, that is $f(g(E)) \subseteq \operatorname{Supp}\left(\Delta^{\prime}\right)$. Then $f(g(E))$ is not a LC center of $(X, \Delta)$, which implies, by step 1 , that $f(g(E))$ is not a LC center of $\left(X, \Delta+\lambda \Delta^{\prime}\right)$, that is $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)>-1$.
We have thus shown that $f \circ g$ is a DLT morphism for the pair $\left(X, \Delta+\lambda \Delta^{\prime}\right)$.
Lemma 2.2.27. Let $(X, \Delta)$ be a DLT pair and let $N \in \operatorname{Div}(X)$ be effective. If $|L|$ is a linear series on $X$ such that $B s(|L|)$ does not contain $L C$ centers of the pair $(X, \Delta)$, then there exists $\mu: Z \rightarrow X$, a common log-resolution of the pair $(X, \Delta-N)$ and of the linear series $|L|$, such that $\mu$ is a DLT morphism for the pair $(X, \Delta-N)$.
In particular $(X, \Delta-N)$ is DLT if $\Delta \geq N$.
Proof. Let $\mu: Y \rightarrow X$ be a common resolution of $(X, \Delta-N)$ and $|L|$ as in lemma 2.2.20. In order to prove that $\mu$ is a DLT morphism for $(X, \Delta-N)$ we have only to show that every $\mu$-exceptional prime divisor $E$ on $Y$ has discrepancy $a(E, X, \Delta-N)>-1$, the other property following by lemma 2.2.20.
Let $E$ be a $\mu$-exceptional prime divisor on $Y$. Lemma 2.2.20 implies that

$$
\mu(E) \subseteq B s(|L|) \cup \operatorname{Sing}(X) \cup N S N C(X, \Delta-N)
$$

Hence, as $E$ is irreducible, we have that $\mu(E)$ is contained in $B s(|L|)$ or in $\operatorname{Sing}(X)$ or in $\operatorname{NSNC}(X, \Delta-N)$.

If $\mu(E) \subseteq B s(|L|)$, then $\mu(E) \notin C L C(X, \Delta)$ by hypothesis, so that $\mu(E) \notin$ $C L C(X, \Delta-N)$ by lemma 2.2.8. This implies that $a(E, X, \Delta-N)>-1$.
If $\mu(E) \subseteq \operatorname{Sing}(X)$, then $\mu(E) \notin C L C(X, \Delta)$ thanks to theorem 2.2.23, because $(X, \Delta)$ is DLT. Then, as before, $\mu(E) \notin C L C(X, \Delta-N)$ by lemma 2.2.8, so that $a(E, X, \Delta-N)>-1$.
Suppose now $\mu(E) \subseteq \operatorname{NSNC}(\Delta-N)$. If, by contradiction, $\mu(E) \in$ $C L C(X, \Delta-N)$, then, by lemma 2.2.12, $\mu(E) \subseteq N S N C(\Delta)$. This implies, by theorem 2.2 .23 that $\mu(E) \notin C L C(X, \Delta)$ so that $\mu(E) \notin C L C(X, \Delta-N)$ using again lemma 2.2.8, and we reach a contradiction.
Therefore $a(E, X, \Delta-N)>-1$ because $\mu(E) \notin C L C(X, \Delta-N)$.

### 2.3 Asymptotic base loci

We recall the following well-known definitions:
Definition 2.3.1. Let $X$ be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{R}}(X)$.

2. $|D|_{\mathbb{R}}:=\left\{E \in \operatorname{Div}_{\mathbb{R}}(X), E\right.$ effective, $\left.E \sim_{\mathbb{R}} D\right\}$, where $E \sim_{\mathbb{R}} D$ means that $E-D$ is an $\mathbb{R}$-linear combination of principal divisors $(f), f \in \mathbb{C}(X)$;
3. $|D|_{\mathbb{Q}}:=\left\{E \in \operatorname{Div}_{\mathbb{R}}(X), E\right.$ effective, $\left.E \sim_{\mathbb{Q}} D\right\}$.

Definition 2.3.2. (cf. [BCHM10, Def. 3.5.1]). Let $X$ be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{R}}(X)$. The (real) stable base locus of $D$ is

$$
\mathbb{B}(D):=\bigcap_{E \in|D|_{\mathbb{R}}} \operatorname{Supp}(E)
$$

where, by convention, we put $\mathbb{B}(D)=X$ if $|D|_{\mathbb{R}}=\emptyset$.
Remark 2.3.3. Notice that if $D$ is $\mathbb{Q}$-Cartier then by [BBP09, Prop. 1.1] the real stable base locus coincide with the usual stable base locus defined as in [Laz04, Rmk. 2.1.24].

Following [ELMNP06] we introduce two perturbations of the stable base locus: the augmented and the restricted base locus.

Definition 2.3.4. (cf. [ELMNP06, Def. 1.2]). Let $X$ be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{R}}(X)$. The augmented base locus of $D$ is

$$
\mathbb{B}_{+}(D):=\bigcap_{\substack{E \in \operatorname{Dive}_{\mathbb{R}}(X), E \geq 0 \\ D-E \text { ample }}} \operatorname{Supp}(E)
$$

if $D$ is big; otherwise $\mathbb{B}_{+}(D):=X$ by convention.
Definition 2.3.5. (cf. [ELMNP06, Def. 1.12]). Let $X$ be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{R}}(X)$. The restricted base locus of $D$ is

$$
\mathbb{B}_{-}(D):=\bigcup_{\substack{A \in \operatorname{Div}_{\mathbb{R}}(X) \\ A \text { ample }}} \mathbb{B}(D+A)
$$

In the rest of the section we gather some lemmas involving augmented and restricted base locus that we will use in the next chapters.

Lemma 2.3.6. Let $X$ be a normal projective variety. Let $D \in \operatorname{Div}_{\mathbb{R}}(X)$. Let $\left\{A_{m}\right\}_{m \geq 1}$ be any sequence of ample $\mathbb{R}$-Cartier divisors such that $\left\|A_{m}\right\| \rightarrow 0$ in $N^{1}(X)_{\mathbb{R}}$. Then

$$
\mathbb{B}_{-}(D)=\cup_{m \geq 1} \mathbb{B}_{-}\left(D+A_{m}\right)
$$

Proof. By definition $\mathbb{B}_{-}(D)=\bigcup_{A}$ ample $\mathbb{B}(D+A)$ and hence it can be easily seen that we also have that $\mathbb{B}_{-}(D)=\bigcup_{A \text { ample }} \mathbb{B}_{-}(D+A)$. For any $A$ ample divisor let $m_{A}$ be sufficiently large so that $A-A_{m_{A}}$ is still ample. Hence $\mathbb{B}_{-}(D+A)=\mathbb{B}_{-}\left(D+A_{m_{A}}+A-A_{m_{A}}\right) \subseteq \mathbb{B}_{-}\left(D+A_{m_{A}}\right)$.

The following lemma allows us to turn big and nef divisors on LC pairs, whose augmented base locus has good properties with respect to the pair itself, into ample ones, by perturbing the pair without worsening its singularities:

Lemma 2.3.7. Let $(X, \Delta)$ be a $L C$ pair. Let $L \in \operatorname{Div}_{\mathbb{Q}}(X)$ be big and nef and such that $\mathbb{B}_{+}(L)$ does not contain any $L C$ center of the pair $(X, \Delta)$.
Then there exists an effective Cartier divisor $\Gamma$ on $X$, not containing any $L C$ center of $(X, \Delta)$ in its support, and a rational number $\lambda_{0}>0$ such that $B s(|\Gamma|)=$ $\mathbb{B}_{+}(L)$ and for each $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{0}\right)$, we have that

1. $L-\lambda \Gamma \in \operatorname{Div}_{\mathbb{Q}}(X)$ and it is ample;
2. $(X, \Delta+\lambda \Gamma)$ is an LC pair;
3. $C L C(X, \Delta+\lambda \Gamma)=C L C(X, \Delta)$.
4. $(X, \Delta+\lambda \Gamma)$ is DLT if $(X, \Delta)$ is such.

Moreover $\Gamma$ can be chosen generically in its linear series.
Proof. By [ELMNP06, Prop. 1.5] there exists $H$, an ample $\mathbb{Q}$-divisor on $X$, such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)
$$

Moreover there exists $m_{0} \in \mathbb{N}$ such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)=B s\left(\left|m_{0}(L-H)\right|\right) .
$$

Then $B s\left(\left|m_{0}(L-H)\right|\right)$ does not contain LC centers of the pair $(X, \Delta)$. Hence, as $C L C(X, \Delta)$ is a finite set (see for example [Kol07, par. 8.1]), we can choose a general divisor $\Gamma$ in $\left|m_{0}(L-H)\right|$ such that $\operatorname{Supp}(\Gamma)$ does not contain LC centers of $(X, \Delta)$.
Thus, by lemma 2.2.26, there exists $\lambda_{1}>0$ such that if $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{1}\right)$, then $(X, \Delta+\lambda \Gamma)$ is an LC pair and $C L C(X, \Delta+\lambda \Gamma)=C L C(X, \Delta)$.
Moreover, by part 2 of the same lemma, it follows that $(X, \Delta+\lambda \Gamma)$ is DLT if $(X, \Delta)$ is such.
Now, for all $\lambda \in \mathbb{Q} \cap(0,1]$, we have that

$$
L-\lambda \Gamma \sim_{\mathbb{Q}}\left(1-\lambda m_{0}\right) L+\lambda m_{0} H
$$

is ample if $\lambda \leq \frac{1}{m_{0}}$ because $L$ is nef and $H$ is ample.
Thus, if we define $\lambda_{0}=\min \left\{\lambda_{1}, \frac{1}{m_{0}}\right\}$, we are done.

Lemma 2.3.8. Let $\mu: Y \rightarrow X$ be a proper birational morphism of normal projective varieties and let $D \in \operatorname{Div}_{\mathbb{R}}(X)$ be a big divisor. Then

$$
\mathbb{B}_{+}\left(\mu^{*}(D)\right)=\operatorname{exc}(\mu) \cup \mu^{-1}\left(\mathbb{B}_{+}(D)\right)
$$

Proof. See [BBP09, proposition 1.5].

Lemma 2.3.9. Let $X$ be a normal variety and let $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ be big and nef.

1. If $V \subseteq X$ is a subvariety such that $\mathbb{B}_{+}(P) \nsupseteq V$, then $P_{\left.\right|_{V}}$ is big.
2. If $E$ is a prime divisor on $X$ such that $P_{\left.\right|_{E}}$ is big, then $\mathbb{B}_{+}(P) \nsupseteq E$.

Proof. 1.) As $V \nsubseteq \mathbb{B}_{+}(P)$ we can find an ample divisor $A \in \operatorname{Div}_{\mathbb{R}}(X)$ and an effective divisor $F \in \operatorname{Div}_{\mathbb{R}}(X)$ such that $P=A+F$ and $V \nsubseteq \operatorname{Supp}(F)$. Then $P_{\left.\right|_{V}}=A_{\left.\right|_{V}}+F_{\left.\right|_{V}}$, where $A_{\left.\right|_{V}}$ is ample and $F_{\left.\right|_{V}}$ is effective. Thus $P_{\left.\right|_{V}}$ is big.
2.) Let $\mu: X^{\prime} \rightarrow X$ be a resolution of singularities of $X$ and let $\widetilde{E}$ be the strict transform of $E$ through $\mu$. Then

$$
\left(\mu^{*}(P)^{n-1} \cdot \widetilde{E}\right)=\left(P^{n-1} \cdot E\right)=\left(P_{\left.\right|_{E}}\right)^{n-1}>0,
$$

because $P_{\left.\right|_{E}}$ is big and nef.
Now, by [Laz04, 10.3.6], as $\mu^{*}(P)$ is big and nef, $N u l l\left(\mu^{*}(P)\right)$ is a closed proper subset of $X^{\prime}$. Again by [Laz04, 10.3.6], $\widetilde{E}$ is not an irreducible component of $\operatorname{Null}\left(\mu^{*}(\underset{\sim}{P})\right)$.
But $\operatorname{dim} \widetilde{E}=\operatorname{dim} X^{\prime}-1 \geq \operatorname{dim} N u l l\left(\mu^{*}(P)\right)$, so that $\widetilde{E}$ cannot be strictly contained in an irreducible component of $N u l l\left(\mu^{*}(P)\right)$. Thus $\widetilde{E} \nsubseteq N u l l\left(\mu^{*}(P)\right)$. Thanks to Nakamaye's theorem (see [Laz04, theorem 10.3.5]) this implies that $\widetilde{E} \nsubseteq \mathbb{B}_{+}\left(\mu^{*}(P)\right)$.
Then, by lemma 2.3.8, $\widetilde{E} \nsubseteq \mu^{-1}\left(\mathbb{B}_{+}(P)\right)$, so that $E=\mu(\widetilde{E}) \nsubseteq \mathbb{B}_{+}(P)$.
Remark 2.3.10. The first part of the last lemma implies that if $(X, \Delta)$ is a pair and $P$ is a big and nef divisor on $X$ such that $\mathbb{B}_{+}(P)$ does not contain any LC center of the pair $(X, \Delta)$ then $P$ is logbig with respect to the pair $(X, \Delta)$.

### 2.4 Multiplier ideals on singular varieties

We recall here the standard definition of multiplier ideal associated to a pair and a divisor or a line bundle (see [Laz04, Section 9.3 G])

Definition 2.4.1. (see [Laz04, Definition 9.3.56]) Let $(X, \Delta)$ be a pair, let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ and let us denote by $\mu: Y \rightarrow X$ a log-resolution of the pair $(X, \Delta)$ such that $\mu^{*}(D)+\operatorname{divexc}(\mu)$ is SNCS. Then for every prime divisor $E$ on $Y$ we can consider the discrepancy $a(E)=a(E, X, \Delta) \in \mathbb{Q}$ and uniquely defined rational numbers $b(E)$ such that $\mu^{*}(-D)=\sum b(E) E$, so that

$$
K_{Y}-\mu^{*}\left(K_{X}+\Delta+D\right) \equiv \sum(a(E)+b(E)) E
$$

We define the multiplier ideal of $D$ on the pair $(X, \Delta)$ as the sheaf

$$
\mathcal{J}((X, \Delta) ; D)=\mu_{*} \mathcal{O}_{Y}\left(\sum\ulcorner a(E)+b(E)\urcorner E\right) .
$$

With a slight abuse of notation we can also write

$$
\mathcal{J}((X, \Delta) ; D)=\mu_{*} \mathcal{O}_{Y}\left(K_{Y}-\left[\mu^{*}\left(K_{X}+\Delta+D\right)\right]\right)
$$

Note that we will use the standard notation $\mathcal{J}(X, \Delta):=\mathcal{J}((X, \Delta) ; 0)$.
Definition 2.4.2. Let $(X, \Delta)$ be a pair, let $c$ be a non negative rational number and let $|L|$ be a non empty linear series on $X$. Consider $\mu: Y \rightarrow X$ a logresolution of the pair $(X, \Delta)$ and of the linear series $|L|$, so that we can write

$$
\mu^{*}|L|=|W|+F,
$$

where $|W|$ is a base-point free linear series and $F+\operatorname{divexc}(\mu)$ is SNCS.
For every prime divisor $E$ on $Y$ we can consider the discrepancy $a(E)=$ $a(E, X, \Delta) \in \mathbb{Q}$ and uniquely defined rational numbers $b(E)$ such that $-F=$ $\sum b(E) E$.
We define the multiplier ideal associated to $c$ and $|L|$ on the pair $(X, \Delta)$ as

$$
\mathcal{J}((X, \Delta) ; c|L|)=\mu_{*} \mathcal{O}_{Y}\left(\sum\ulcorner a(E)+c b(E)\urcorner\right) .
$$

With a slight abuse of notation we can also write

$$
\mathcal{J}((X, \Delta) ; c|L|)=\mu_{*} \mathcal{O}_{Y}\left(K_{Y}-\left[\mu^{*}\left(K_{X}+\Delta\right)+c F\right]\right)
$$

Note that we will use the standard notation $\mathcal{J}(X, c|L|):=\mathcal{J}((X, 0) ; c|L|)$ and we use the convention that $\mathcal{J}((X, \Delta) ; c|L|)=0$ if $|L|=\emptyset$.

## Asymptotic multiplier ideals

In this subsection we define asymptotic multiplier ideals on (possibly singular) pairs. See [Laz04, chapter 11] for details about the same construction in the case of smooth varieties.

Lemma 2.4.3. (cf. [Laz04, Lemma 11.1.1]) Let $(X, \Delta)$ be an effective pair, $c>0$ a rational number, $p \in \mathbb{N}$ and $L$ a Cartier divisor on $X$ such that $\kappa(X, L) \geq 0$. For every integer $k \geq 1$ one has the inclusion

$$
\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right) \subseteq \mathcal{J}\left((X, \Delta) ; \frac{c}{p k}|p k L|\right)
$$

Proof. We may assume that $|p L| \neq \emptyset$ (and hence also $|p k L| \neq \emptyset$ ). Choose $\mu: Y \rightarrow X$ a log-resolution of the pair $(X, \Delta)$ and of the linear series $|p L|$ and $|p k L|$. Hence we can write

$$
\begin{aligned}
\mu^{*}(|p L|) & =\left|W_{p}\right|+F_{p} \\
\mu^{*}(|p k L|) & =\left|W_{p k}\right|+F_{p k}
\end{aligned}
$$

where $W_{p}$ and $W_{p k}$ are base-point free linear series. Let us write $F_{p}=$ $-\sum b_{p}(E) E$ and $F_{p k}=-\sum b_{p k}(E) E$, where the sums are taken on all prime
divisors on $Y$. Note that $k \cdot F_{p} \geq F_{p k}$, so that $b_{p}(E) \leq \frac{b_{p k}(E)}{k}$ for every prime divisor $E$.
Thus, if we denote by $a(E)=a(E, X, \Delta)$ the discrepancies of the pair $(X, \Delta)$, we find that

$$
\begin{gathered}
\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right)=\mu_{*} \mathcal{O}_{Y}\left(\sum\left\ulcorner a(E)+\frac{c}{p} b_{p}(E)\right\urcorner E\right) \subseteq \\
\subseteq \mu_{*} \mathcal{O}_{Y}\left(\sum\left\ulcorner a(E)+\frac{c}{p k} b_{p k}(E)\right\urcorner E\right)=\mathcal{J}\left((X, \Delta) ; \frac{c}{p k}|p k L|\right) .
\end{gathered}
$$

Lemma 2.4.4. Let $(X, \Delta)$ be a pair, $c>0$ a rational number and $L$ a Cartier divisor on $X$ such that $\kappa(X, L) \geq 0$. Then the family of ideals

$$
\left\{\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right)\right\}_{p \in \mathbb{N}}
$$

has a unique maximal element.
Proof. We first consider the case when $(X, \Delta)$ is an effective pair. In this case each of the sheaves $\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right)$ is an actual ideal sheaf, i.e. it is contained in $\mathcal{O}_{X}$. Hence the existence of at least one maximal member follows from the ascending chain condition on ideals. On the other hand if $\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right)=$ $\mathcal{J}\left((X, \Delta) ; \frac{c}{q}|q L|\right)$ for some $p, q \in \mathbb{N}$, then, by lemma 2.4.3 they must both coincide with $\mathcal{J}\left((X, \Delta) ; \frac{c}{p q}|p q L|\right)$.
Let us consider now the general case. Let $A$ be an ample Cartier divisor on $X$ and take $m \in \mathbb{N}$ sufficiently high such that $\mathcal{O}_{X}(m A) \otimes \mathcal{O}_{X}([\Delta]) \simeq \mathcal{O}_{X}(m A+[\Delta])$ is a globally generated coherent sheaf. This implies that there exists a Cartier divisor $E \sim m A$ such that $E+[\Delta] \geq 0$, so that $E+\Delta \geq 0$ as well.
Hence if we consider consider the family of ideals

$$
\left\{\mathcal{J}\left((X, \Delta+E) ; \frac{c}{p}|p L|\right)\right\}_{p \in \mathbb{N}}
$$

by the previous part of the proof we know that it has a unique maximal element, whence there exists $p_{0} \in \mathbb{N}$ such that for every integer $p>0$

$$
\mathcal{J}\left((X, \Delta+E) ; \frac{c}{p}|p L|\right) \subseteq \mathcal{J}\left((X, \Delta+E) ; \frac{c}{p_{0}}\left|p_{0} L\right|\right)
$$

Now note that for every integer $p>0$ we have that $\mathcal{J}\left((X, \Delta+E) ; \frac{c}{p}|p L|\right)=$ $\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right) \otimes \mathcal{O}_{X}(-E)$ by [Laz04, Prop. 9.2.31] (see also pag.185, volume $2)$, so that we have

$$
\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right) \otimes \mathcal{O}_{X}(-E) \subseteq \mathcal{J}\left((X, \Delta) ; \frac{c}{p_{0}}\left|p_{0} L\right|\right) \otimes \mathcal{O}_{X}(-E)
$$

Thus, tensoring by $\mathcal{O}_{X}(E)$, we get that

$$
\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right) \subseteq \mathcal{J}\left((X, \Delta) ; \frac{c}{p_{0}}\left|p_{0} L\right|\right)
$$

whence $\mathcal{J}\left((X, \Delta) ; \frac{c}{p_{0}}\left|p_{0} L\right|\right)$ is the unique maximal element of our family.

Definition 2.4.5. Let $(X, \Delta)$ be a pair, consider a complete linear series $|L|$ on $X$ such that $\kappa(X, L) \geq 0$ and a rational number $c>0$. The asymptotic multiplier ideal sheaf associated to $c$ and $|L|$ on the pair $(X, \Delta)$, denoted by

$$
\mathcal{J}((X, \Delta) ; c\|L\|),
$$

is defined as the unique maximal member in the family of ideals $\left\{\mathcal{J}\left((X, \Delta) ; \frac{c}{p}|p L|\right)\right\}$.

Note that $\mathcal{J}((X, \Delta) ; c\|L\|) \subseteq \mathcal{O}_{X}$ if the pair $(X, \Delta)$ is effective.
In the following definition we associate to every effective pair $(X, \Delta)$ two bdivisors $\mathbf{A}(\Delta)$ and $\mathbf{L}(\Delta)$. This will be useful every time we consider birational transformations of pairs.

Definition 2.4.6. Let $(X, \Delta)$ be an effective pair. We define the b-divisors $\mathbf{A}(\Delta)$ and $\mathbf{L}(\Delta)$ :
For every proper birational morphism $f: Z \rightarrow X$, if $E$ and $F$ are effective Weil $\mathbb{Q}$-divisors on $Z$ without common components such that

$$
K_{Z}+E \equiv f^{*}\left(K_{X}+\Delta\right)+F \quad \text { and } \quad f_{*}(E-F)=\Delta
$$

we put the trace $\mathbf{A}(\Delta)_{Z}=E-F$ and the trace $\mathbf{L}(\Delta)_{Z}=E$.
Lemma 2.4.7 (Birational transformation rule for asymptotic multiplier ideals). (cf. [Laz04, Prop. 9.3.62]). Let $(X, \Delta)$ be a pair and let $f: Y \rightarrow X$ be a proper birational morphism. If $D$ is any Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$, then

$$
\mathcal{J}((X, \Delta) ;\|D\|)=f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)
$$

Proof. By definition of asymptotic multiplier ideal we have that $\mathcal{J}((X, \Delta) ;\|D\|)$ $=\sup _{k}\left\{\mathcal{J}\left((X, \Delta) ; \frac{1}{k}|k D|\right)\right\}$. When $k \geq 2$ and $|k D| \neq \emptyset$, by [Laz04, Prop. 9.2.26] (see also [Laz04, volume 2, p. 185]) we have that $\mathcal{J}\left((X, \Delta) ; \frac{1}{k}|k D|\right)=$ $\mathcal{J}\left((X, \Delta) ; \frac{1}{k} D_{k}\right)$, where $D_{k}$ is a general element in $|k D|$. By [Laz04, Prop. 9.3.62] we have that, for every $k \in \mathbb{N}$ such that $|k D| \neq \emptyset, \mathcal{J}\left((X, \Delta) ; \frac{1}{k} D_{k}\right)=$ $f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ; \frac{1}{k} f^{*} D_{k}\right)\right)$. Since $D_{k}$ is general in $|k D|$ then $f^{*} D_{k}$ is general in $\left|k f^{*} D\right|$, so that $\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ; \frac{1}{k} f^{*} D_{k}\right)=\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ; \frac{1}{k}\left|k f^{*} D\right|\right)$.
Since $f_{*}$ preserves inclusions we also have that
$\sup _{k}\left\{f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ; \frac{1}{k}\left|k f^{*} D\right|\right)\right)\right\}=f_{*}\left(\sup _{k}\left\{\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ; \frac{1}{k}\left|k f^{*} D\right|\right)\right\}\right)$.
Hence the lemma follows.

## 2.5 $\mathbb{Q}$-CKM Zariski decompositions

For our purposes we need to extend the classical definition of Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki to some non $\mathbb{Q}$-Cartier cases. Hence we will adopt the following definition:

Definition 2.5.1. Let $X$ be a normal projective variety and let $D$ be a Weil $\mathbb{Q}$-divisor on $X$. We say that $D$ admits a $\mathbb{Q}$-Zariski decomposition in the sense of Cutkosky-Kawamata-Moriwaki (or a $\mathbb{Q}$-CKM Zariski decomposition) $D=P+N$ if

- $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ and $N$ is a Weil $\mathbb{Q}$-divisor;
- $P$ is nef and $N$ is effective;
- There exists an integer $k>0$ such that $k P$ is Cartier, $k D$ is an integral Weil divisor and for every $m \in \mathbb{N}$ we have an isomorphism

$$
H^{0}\left(X, \mathcal{O}_{X}(k m P)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(k m D)\right)
$$

Note that $k m D$ might not be a Cartier divisor but it still makes sense to consider the reflexive sheaf $\mathcal{O}_{X}(k m D)$ and its $H^{0}$ (see definition 2.1.3). In particular if $D$ is $\mathbb{Q}$-Cartier and it admits a CKM Zariski decomposition in the usual sense (see for example [Cut86]) with $P$ and $N$ in $\operatorname{Div}_{\mathbb{Q}}(X)$ then it admits a $\mathbb{Q}$-CKM Zariski decomposition in our sense.

Lemma 2.5.2. Let $X$ be a normal projective variety and let $D \in \operatorname{Div}(X)$. If $D=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and $D^{\prime} \sim_{\mathbb{Q}} D$ then $D^{\prime}=P^{\prime}+N$ is a $\mathbb{Q}$-CKM Zariski decomposition, where $P^{\prime}=\left(D^{\prime}-D\right)+P$.

Proof. As $D^{\prime} \sim_{\mathbb{Q}} D$ there exists an integer $k_{0} \in \mathbb{N}$ such that $k_{0} D^{\prime} \sim k_{0} D$, so that $k_{0} P^{\prime} \sim k_{0} P$, in particular note that $P^{\prime}$ is nef. Also by our definition of $\mathbb{Q}$-CKM Zariski decomposition there exists $m_{0} \in \mathbb{N}$ such that $m_{0} D$ and $m_{0} P$ are Cartier divisors and $H^{0}\left(X, m m_{0} D\right) \simeq H^{0}\left(X, m m_{0} P\right)$ for every $m \in \mathbb{N}$.
But for every $m \in \mathbb{N}$ we have that $m k_{0} m_{0} D^{\prime} \sim m k_{0} m_{0} D$ and $m k_{0} m_{0} P^{\prime} \sim$ $m k_{0} m_{0} P$, so that $H^{0}\left(X, m k_{0} m_{0} D^{\prime}\right) \simeq H^{0}\left(X, m k_{0} m_{0} D\right) \simeq H^{0}\left(X, m k_{0} m_{0} P\right)$ $\simeq H^{0}\left(X, m k_{0} m_{0} P^{\prime}\right)$.

The following technical lemma will be very useful to treat the case when a birational pullback of a given divisor admits a Zariski decomposition.

Lemma 2.5.3. Let $(X, \Delta)$ be an effective pair, take $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ and $a \in \mathbb{Q}$. If there exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=$ $P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition, then there exist Weil $\mathbb{Q}$-divisors $D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ such that

- $\Delta_{Z}=\mathbf{L}(\Delta)_{Z}$ if $a \geq 0$ and $\Delta_{Z}=\mathbf{L}(\Delta)_{Z}-a N$ if $a<0$, so that, in any case, $\Delta_{Z}$ is effective;
- $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition;
- $\Delta_{Z}-N^{\prime}=\mathbf{A}(\Delta)_{Z}-a N$, so that in particular $\left(Z, \Delta_{Z}-N^{\prime}\right)$ is a pair;
- $P^{\prime}=b P$ for some $b>0$;
- $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)=P+f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$ for some $t_{0} \in \mathbb{Q}$.

In particular if $D$ is big and $a D-\left(K_{X}+\Delta\right)$ is nef or if a $D-\left(K_{X}+\Delta\right)$ is big and nef, then

$$
t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)
$$

is big and nef.

Proof. Define $a^{\prime}=-\min \{0, a\}$ and $a^{\prime \prime}=\max \{0, a\}$, so that $a^{\prime} \geq 0, a^{\prime \prime} \geq 0$ and $a=a^{\prime \prime}-a^{\prime}$. Moreover we can write $\mathbf{A}(\Delta)_{Z}=A^{+}-A^{-}$, where $A^{+}$and $A^{-}$are effective and without common components, so that $A^{-}$is $f$-exceptional. We define $\Delta_{Z}:=A^{+}+a^{\prime} N, N^{\prime}:=a^{\prime \prime} N+A^{-}, P^{\prime}:=\left(a^{\prime \prime}+1\right) P$ and $D^{\prime}:=P^{\prime}+N^{\prime}$. Then it is immediate to compute $\Delta_{Z}$ and to see that $\Delta_{Z}-N^{\prime}=\mathbf{A}(\Delta)_{Z}-a N$ and $P^{\prime}$ is a positive rational multiple of $P$.
Moreover $P^{\prime}$ is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, $N^{\prime}$ is effective and by using the hypothesis and Fujita's lemma (see for example [KMM85, lemma 1.3.2]) we can see that there exists $k^{\prime} \in \mathbb{N}$ such that

$$
H^{0}\left(Z, \mathcal{O}_{Z}\left(k^{\prime} m P^{\prime}\right)\right) \simeq H^{0}\left(Z, \mathcal{O}_{Z}\left(k^{\prime} m D^{\prime}\right)\right)
$$

## for every $m \in \mathbb{N}$.

In fact we know that there exists $k \in \mathbb{N}$ such that $k D$ and $k P$ are Cartier and $H^{0}\left(Z, \mathcal{O}_{Z}\left(f^{*}(k m D)\right)\right) \simeq H^{0}\left(Z, \mathcal{O}_{Z}(k m P)\right)$ for every $m \in \mathbb{N}$. Let $c \in \mathbb{N}$ be such that $l:=c\left(a^{\prime \prime}+1\right) \in \mathbb{N}$ and $c N^{\prime}$ is integral and define $k^{\prime}=k c$, so that $k P^{\prime}$ is Cartier and $k N^{\prime}$ is an integral Weil divisor.
Then for every $m \in \mathbb{N}$

$$
\begin{aligned}
& H^{0}\left(Z, \mathcal{O}_{Z}\left(k^{\prime} m P^{\prime}\right)\right) \simeq H^{0}\left(Z, \mathcal{O}_{Z}(k m l P)\right) \simeq H^{0}\left(Z, \mathcal{O}_{Z}\left(f^{*}(k m l D)\right)\right) \simeq \\
&\left.\simeq H^{0}\left(Z, \mathcal{O}_{Z}\left(f^{*}(k m l D)+k m l A^{-}\right)\right)\right)
\end{aligned}
$$

by Fujita's lemma, because $A^{-}$is effective and $f$-exceptional. On the other hand

$$
\begin{array}{r}
H^{0}\left(Z, \mathcal{O}_{Z}\left(k^{\prime} m P^{\prime}\right)\right) \subseteq H^{0}\left(Z, \mathcal{O}_{Z}\left(k^{\prime} m D^{\prime}\right)\right)=H^{0}\left(Z, \mathcal{O}_{Z}\left(k m l P+k c m a^{\prime \prime} N+\right.\right. \\
\left.\left.\left.+k c m A^{-}\right)\right) \subseteq H^{0}\left(Z, \mathcal{O}_{Z}\left(f^{*}(k m l D)+k m l A^{-}\right)\right)\right)
\end{array}
$$

Hence $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition.
Now note that

$$
\begin{gathered}
P+f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)=(a+1) P+a N-\left(K_{Z}+\mathbf{A}(\Delta)_{Z}\right)= \\
=(a+1) P+a^{\prime \prime} N-\left(K_{Z}+\mathbf{A}(\Delta)_{Z}+a^{\prime} N\right)=(a+1) P-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)= \\
=t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)
\end{gathered}
$$

where $t_{0}=\frac{a+1}{a^{\prime \prime}+1} \in \mathbb{Q}$.

The following lemma shows that the augmented base locus has good properties with respect to $\mathbb{Q}$-CKM Zariski decompositions.

Lemma 2.5.4. Let $X$ be a normal projective variety, let $D \in \operatorname{Div} v_{\mathbb{Q}}(X)$ and suppose there exists $a \mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

Then $\mathbb{B}_{+}(D)=\mathbb{B}_{+}(P)$ and $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$.
Proof. If $D$ is not big then $P$ is also not big, so that $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)=X$. We can thus assume that $D$ is big.
Then the decomposition $D=P+N$ is also a Fujita-Zariski decomposition (see for example [Pro02, proposition 7.4]). Hence, given an ample $\mathbb{R}$-divisor $A$ such
that $A \leq D$, by definition of Fujita-Zariski decomposition we have that $A \leq P$, so that $D-A \geq N$. Now

$$
\mathbb{B}_{+}(D)=\bigcap \operatorname{Supp}(D-A)
$$

where the intersection is taken over all the ample $\mathbb{R}$-divisors $A$ such that $A \leq D$. Then

$$
\begin{aligned}
\mathbb{B}_{+}(D) & =\bigcap_{A \leq D} \operatorname{Supp}(D-A)=\bigcap_{A \leq D}(\operatorname{Supp}(D-A-N) \cup \operatorname{Supp}(N))= \\
& =\operatorname{Supp}(N) \cup \bigcap_{A \leq P} \operatorname{Supp}(P-A)=\operatorname{Supp}(N) \cup \mathbb{B}_{+}(P) .
\end{aligned}
$$

In order to complete the proof it suffices to prove that $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(P)$ :
We first prove this in the case when $X$ is smooth:
In particular, denoted by $n$ the dimension of $X$, we will show that for every divisorial component $F$ of the support of $N$ we have that $\left(P^{n-1} \cdot F\right)=0$. This will imply, as is well known, that $F \subseteq \mathbb{B}_{+}(P)$, so that $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(P)$.
Let $a \in \mathbb{N}$ be such that $a D, a P$ and $a N$ are Cartier divisors and $H^{0}\left(X, \mathcal{O}_{X}(m a D)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m a P)\right)$ for all $m \in \mathbb{N}$.
Let us consider now, for all $m \in \mathbb{N}$, the long exact sequence

$$
\begin{aligned}
0 \rightarrow H^{0}\left(X, \mathcal{O}_{X}(m a P)\right) \rightarrow & H^{0}\left(X, \mathcal{O}_{X}(m a P+F)\right) \rightarrow H^{0}\left(F, \mathcal{O}_{F}\left((m a P+F)_{\left.\right|_{F}}\right)\right) \rightarrow \\
& \rightarrow H^{1}\left(X, \mathcal{O}_{X}(m a P)\right) \rightarrow
\end{aligned}
$$

By the choice of $a$, as $0 \leq F \leq m a N$, we have that $H^{0}\left(X, \mathcal{O}_{X}(m a P)\right) \simeq$ $H^{0}\left(X, \mathcal{O}_{X}(m a P+F)\right)$.
This implies that the map $H^{0}\left(F, \mathcal{O}_{F}\left((m a P+F)_{\left.\right|_{F}}\right)\right) \rightarrow H^{1}\left(X, \mathcal{O}_{X}(m a P)\right)$ is an injection, so that

$$
h^{0}\left(F, \mathcal{O}_{F}\left((m a P+F)_{\left.\right|_{F}}\right)\right) \leq h^{1}\left(X, \mathcal{O}_{X}(m a P)\right)
$$

for all $m \in \mathbb{N}$.
But the bigness of $D$ implies that $P$ is also big (because $\kappa(X, P)=\kappa(X, D)=n)$. Then, by a theorem of Fujita, there exists a constant $C>0$ such that $h^{1}\left(X, \mathcal{O}_{X}(m a P)\right) \leq C \cdot m^{n-2}$ for all $m \in \mathbb{N}($ see [Fta83, corollary 6.7]).
Hence, for every $m \in \mathbb{N}$,

$$
h^{0}\left(F, \mathcal{O}_{F}\left((m a P+F)_{\left.\right|_{F}}\right)\right) \leq C \cdot m^{n-2}
$$

On the other hand by asymptotic Riemann-Roch (see [Laz04, 1.4.41]) we get that

$$
h^{0}\left(F, \mathcal{O}_{F}\left((m a P+F)_{\left.\right|_{F}}\right)\right)=\frac{\left(a P_{\left.\right|_{F}}\right)^{n-1} \cdot m^{n-1}}{(n-1)!}+O\left(m^{n-2}\right) .
$$

Therefore $0=\left(a P_{\left.\right|_{F}}\right)^{n-1}=a\left(P^{n-1} \cdot F\right)$ and we are done.
In the general case take $\mu: X^{\prime} \rightarrow X$ to be a resolution of singularities of $X$.
Note that $\mu^{*}(D)=\mu^{*}(P)+\mu^{*}(N)$ is a $\mathbb{Q}$-CKM Zariski decomposition of $\mu^{*}(D)$, so that, by the previous case, we get that $\operatorname{Supp}\left(\mu^{*}(N)\right) \subseteq \mathbb{B}_{+}\left(\mu^{*}(P)\right)$.

Now take $F$ to be a prime divisor contained in the support of $N$. Then

$$
\mu_{*}^{-1}(F) \subseteq \operatorname{Supp}\left(\mu^{*}(N)\right) \subseteq \mathbb{B}_{+}\left(\mu^{*}(P)\right)=\operatorname{exc}(\mu) \cup \mu^{-1}\left(\mathbb{B}_{+}(P)\right)
$$

thanks to lemma 2.3.8.
But $\mu_{*}^{-1}(F) \nsubseteq \operatorname{exc}(\mu)$ because it is a strict transform, so that, thanks to its irreducibility, $\mu_{*}^{-1}(F) \subseteq \mu^{-1}\left(\mathbb{B}_{+}(P)\right)$, that is $F=\mu\left(\mu_{*}^{-1}(F)\right) \subseteq \mathbb{B}_{+}(P)$ and we are done.

### 2.6 Asymptotic and numerical orders of vanishing

Let $X$ be a normal projective variety. We can consider a geometric discrete valuation $v$ on $X$ as in [dFH09, section 2].
In other words a geometric discrete valuation on $X$ is a discrete valuation of the function field of $X$ of the form $v=q \operatorname{val}_{F}$, where $q>0$ is an integer and $\operatorname{val}_{F}$ is the valuation associated to a prime divisor $F$ over $X$.
Suppose $\mu: X^{\prime} \rightarrow X$ is a projective birational morphism such that $F \subseteq X^{\prime}$. We denote by $c_{X}(v)$ the center of the valuation $v$ on $X$, which corresponds to the image on $X$ of the divisor $F$.
Given $D \in \operatorname{Div}_{\mathbb{R}}(X)$ we define the valuation $v(D)$ of $D$ as $q$ times the coefficient of $F$ in the divisor $\mu^{*}(D)$.
If $|L|$ is a non-empty linear series on $X$ we define the valuation $v(|L|)$ as the valuation of a general divisor in $|L|$.
Moreover if $\mathcal{J} \subseteq \mathcal{O}_{X}$ is a non-trivial ideal sheaf, we define the valuation $v(\mathcal{J})$ of $\mathcal{J}$ as

$$
v(\mathcal{J})=\min \left\{v(\phi) \mid \phi \in \mathcal{J}(U), U \cap c_{X}(v) \neq \emptyset\right\}
$$

Note that for every linear series $|L| \neq \emptyset$ we have that $v(|L|)=v(b(|L|))$, where $b(|L|)$ is the base ideal of the linear series $|L|$.

## Numerical orders of vanishing: the non-nef locus

If $D \in \operatorname{Div}_{\mathbb{R}}(X)$ is big we define, as in [BBP09], the numerical order of vanishing of $D$ along $v$ as

$$
v_{\text {num }}(D):=\inf \left\{v(E), E \in|D|_{\equiv}\right\} .
$$

When $D \in \operatorname{Div}_{\mathbb{R}}(X)$ is pseudoeffective we set

$$
v_{\mathrm{num}}(D):=\lim _{\varepsilon \rightarrow 0} v_{\mathrm{num}}(D+\varepsilon A),
$$

with $A$ ample. It is easy to see that the limit exists and the definition does not depend on the choice of the ample divisor $A$. See $[\mathrm{BBP} 09, \S 1.3]$.

Definition 2.6.1. (cf. [BBP09, Def. 1.7]). Let $X$ be a normal projective variety, let $D \in \operatorname{Div}_{\mathbb{R}}(X)$ and let us denote by $c_{X}(v)$ the center on $X$ of a given geometric discrete valuation $v$ on $X$. The non-nef locus of $D$ is

$$
\operatorname{NNef}(D):=\bigcup\left\{c_{X}(v), v_{\text {num }}(D)>0\right\}
$$

if $D$ is pseudoeffective. When $D$ is not pseudoeffective we put $\operatorname{NNef}(D)=X$.

Note that, as the name itself suggests, $D \in \operatorname{Div}_{\mathbb{R}}(X)$ is nef if and only if $\operatorname{NNef}(D)=\emptyset$, i.e., if and only if $v_{\text {num }}(D)=0$ for every geometric discrete valuation $v$ on $X$ (see [BBP09, §1.3]).

Note also that the notation is not univocal and the restricted base locus itself, $\mathbb{B}_{-}(D)$, is called "non-nef locus" on some papers.

Lemma 2.6.2. Let $X$ be a normal projective variety and let $D \in \operatorname{Div}_{\mathbb{R}}(X)$. Let $\left\{A_{m}\right\}_{m \geq 1}$ be any sequence of ample $\mathbb{R}$-Cartier divisors such that $\left\|A_{m}\right\| \rightarrow 0$ in $N^{1}(X)_{\mathbb{R}}$. Then

$$
\operatorname{NNef}(D)=\cup_{m \geq 1} \operatorname{NNef}\left(D+A_{m}\right)
$$

Proof. Since clearly $\operatorname{NNef}(D)=\cup_{A}$ ample $\operatorname{NNef}(D+A)$, then we can just apply the proof of lemma 2.3.6, substituting $\mathbb{B}_{-}$with NNef.

## Asymptotic orders of vanishing: the non nef-abundant locus

Definition 2.6.3. (cf. [ELMNP06, Def. 2.2] and [Cac08, Def. 5.20]). Let $X$ be a normal projective variety and let $D$ be a Cartier divisor such that $\kappa(X, D) \geq 0$. Let $v$ be a geometric discrete valuation on $X$. If $e=e(D)$ is the exponent of $D$ (cf. [Laz04, Def. 2.1.1]), we define the asymptotic order of vanishing of $D$ along $v$ as

$$
v(\|D\|):=\lim _{p \rightarrow \infty} \frac{v(|p e D|)}{p e}
$$

Remark 2.6.4. The above definition can be generalized to $\mathbb{Q}$-Cartier divisors and it can be easily seen that the limit is also the inf, so that for every divisor $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ we have that

$$
v(\|D\|)=\inf \left\{v(E), E \in|D|_{\mathbb{Q}}\right\}
$$

See [ELMNP06] for details.
Remark 2.6.5. If $D$ is a big $\mathbb{Q}$-Cartier divisor then by [ELMNP06, Lemma 3.3] we have that $v(\|D\|)=v_{\text {num }}(D)=\inf \left\{v(E), E \in|D|_{\mathbb{R}}\right\}$. More generally when $D$ is abundant we have that $v(\|D\|)=v_{\text {num }}(D)$ by [Leh11a, Prop. 6.4]. See [Nak04, Def. V 2.23] for the definition of abundant divisor, see [Leh11a] and [Leh11b] for some equivalent definitions and remarks.
In general, if $D$ is an effective $\mathbb{Q}$-Cartier divisor then by definition $v(\|D\|) \geq$ $v_{\text {num }}(D)$ but equality does not hold in general. Take for example a nef irreducible curve $D$ on a smooth surface as in [Rus09, Ex. 1] and set $v:=\operatorname{ord}_{D}$. We have that $v_{\text {num }}(D)=0$ by the nefness of $D$, while $v(\|D\|)=1$.

Recall that given a nef Cartier divisor $D$ on a normal projective variety $X$ we can define its numerical dimension as $\nu(D):=\max \left\{k \in \mathbb{N}: D^{k} \not \equiv 0\right\}$. If $D$ is nef we have that it is abundant if and only if the Kodaira dimension of $D$ equals its numerical dimension (see [Nak04, Prop 2.22 (5)]).
In this paper we will only deal with abundance of nef divisors. The following lemma is a translation of [Rus09, Thm. 1] in terms of discrete valuations:
Lemma 2.6.6. Let $D$ be a Cartier divisor on a projective normal variety $X$. Then $D$ is nef and abundant if and only if $v(\|D\|)=0$ for every geometric discrete valuation $v$ on $\mathbb{C}(X)$.

Proof. If $D$ is nef and abundant, then by [MR97, Lemma 1] (see also [Kaw85, Prop. 2.1]) there exist a birational morphism $f: Z \rightarrow X$, where $Z$ is a smooth projective variety, an integer $k_{0}>0$ and a Cartier divisor $N$ on $Z$ such that $B_{m}:=m k_{0} f^{*}(D)-N$ is semiample for every $m \in \mathbb{N}$. Moreover we can suppose that $k_{0}$ is a multiple of the exponent $e(D)$.
Now, given any geometric discrete valuation $v$ on $\mathbb{C}(X)$, we have that $v(\|D\|)=$ $v\left(\left\|f^{*} D\right\|\right)$ by [ELMNP06]. But

$$
\begin{aligned}
v\left(\left\|f^{*} D\right\|\right) & =\frac{1}{k_{0}} v\left(\left\|f^{*}\left(k_{0} D\right)\right\|\right)=\frac{1}{k_{0}} \cdot \lim _{m \rightarrow+\infty} \frac{v\left(\left|f^{*}\left(m k_{0} D\right)\right|\right)}{m} \leq \\
& \leq \frac{1}{k_{0}} \cdot \lim _{m \rightarrow+\infty}\left(\frac{v\left(\left|B_{m}\right|\right)}{m}+\frac{v(|N|)}{m}\right)=0
\end{aligned}
$$

because $v\left(\left|B_{m}\right|\right)=0$ by the semiampleness of $B_{m}$ and $v(|N|)$ does not depend on $m$.
Now suppose $v(\|D\|)=0$ for every geometric discrete valuation $v$ on $\mathbb{C}(X)$. If $\mu: X^{\prime} \rightarrow X$ is a desingularization of $X$, then $v\left(\left\|\mu^{*} D\right\|\right)=v(\|D\|)=0$ for every $v$. On the other hand, as $X^{\prime}$ is smooth, we have that for every (not necessarily closed) point $x \in X^{\prime}$ we can consider the geometric discrete valuation or $d_{x}$ given by the order of vanishing at $x$, so that $\operatorname{ord}_{x}\left(\left\|\mu^{*} D\right\|\right)=0$ for every $x \in X^{\prime}$, which is equivalent to saying that $D$ is almost base-point free (cf. [Rus09, Def. 1]). By [Rus09, Thm. 1] this implies that $\mu^{*} D$ is nef and abundant, which in turn implies that $D$ is nef and abundant.

The previous lemma suggests the following definition:
Definition 2.6.7. Let $X$ be a normal projective variety and let $D$ be a Cartier divisor such that $\kappa(X, D) \geq 0$. The non nef-abundant locus of $D$ is

$$
\operatorname{NNA}(D):=\bigcup_{v \in V}\left\{c_{X}(v): v(\|D\|)>0\right\}
$$

where $V$ is the set of all geometric discrete valuations on $\mathbb{C}(X)$ and, for any $v \in V, c_{X}(v)$ is the center of $v$ on $X$.

Remark 2.6.8. Trivially, by lemma 2.6.6, NNA $(D)=\emptyset$ if and only if $D$ is nef and abundant. When $D$ is a big divisor $\operatorname{NNef}(D)=\operatorname{NNA}(D)$ by remark 2.6.5. By the same remark we see that if $D$ is effective then in general NNA $(D) \supseteq$ $\operatorname{NNef}(D)$ and that equality does not hold in general if $D$ is not big.

Lemma 2.6.9. Let $X$ be a normal projective variety and let $D$ be a Cartier divisor such that $\kappa(X, D) \geq 0$. Let $f: X^{\prime} \rightarrow X$ be any birational map from a normal projective variety $X^{\prime}$. Then $f\left(\operatorname{NNA}\left(f^{*} D\right)\right)=\operatorname{NNA}(D)$.

Proof. The lemma follows from the easy fact that for any discrete geometric valuation $v$ on $\mathbb{C}(X)$ we have that $v\left(\left\|f^{*} D\right\|\right)=v(\|D\|)$ (cf. [ELMNP06]).

## Chapter 3

## On the semiampleness of the positive part of CKM Zariski decompositions

In this chapter we work to some extensions of theorem 1.0.1 to the case when the given pair $(X, \Delta)$ is not KLT. In particular the main results appear in section 3.3 , where we prove Conjecture 1b for DLT pairs, in section 3.7 and 3.8 , where we consider lower dimensional cases and in section 3.9, where we give some results for relatively DLT pairs. All the main theorems appear in [Cac10].

### 3.1 The case $N=0$

In this section we show that in the case when $D$ is nef, that is $D$ admits the trivial Zariski decomposition given by $P=D$ and $N=0$, Conjecture 2 follows easily by a theorem of Fujino.
We begin the section by stating Fujino's theorem, which generalizes the basepoint free theorem to the LC case.

Theorem 3.1.1. [Fuj09a, theorem 1.2]
Let $(X, B)$ be an LC pair, with $B$ effective. Let $L$ be a nef Cartier divisor on $X$. Assume that $a L-\left(K_{X}+B\right)$ is ample for some $a>0$. Then the linear system $|m L|$ is base-point free for every $m \gg 0$.

As a simple consequence of this theorem we find the following, which in particular implies Conjecture 2 in the case $N=0$.

Theorem 3.1.2. Let $(X, \Delta)$ be an effective LC pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X), b \in \mathbb{N}$ is such that bD is a Cartier divisor and

1. $D$ is big and nef;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some $a \in \mathbb{Q}^{+}$;
3. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;
then $|m b D|$ is base-point free for every $m \gg 0$.

Proof. We can assume without loss of generality that $D$ is a Cartier divisor. By lemma 2.3.7 there exist an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $D-\lambda \Gamma$ is ample and $(X, \Delta+\lambda \Gamma)$ is an LC pair.
Then

$$
(1+a) D-\left(K_{X}+\Delta+\lambda \Gamma\right)=(D-\lambda \Gamma)+\left(a D-\left(K_{X}+\Delta\right)\right)
$$

is ample because it is the sum of an ample divisor and a nef one.
Thus we can apply theorem 3.1.1 to the pair $(X, \Delta+\lambda \Gamma)$ and we have the assert.

Theorem 3.1.3. Let $(X, \Delta)$ be an effective LC pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $D$ is nef;
2. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some $a \in \mathbb{Q}^{+}$;
3. $\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)$ does not contain any $L C$ center of the pair $(X, \Delta)$;
then $D$ is semiample.
The proof is similar to the previous theorem.
Note that we cannot remove hypothesis 3. In fact [KMM85, remark 3-1-2] gives a counterexample.
On the other hand note that theorem 3.1.2 and theorem 3.1.3 also follow by [Amb01, theorem 7.2](cfr. [Fuj09c, theorem 4.4]). In particular we can lighten hypothesis 3 in both theorems by assuming in theorem 3.1.2 that $D$ is logbig with respect to $(X, \Delta)$ and in theorem 3.1.3 that $a D-\left(K_{X}+\Delta\right)$ is logbig with respect to $(X, \Delta)$.

## Weak log Fano pairs

Definition 3.1.4. Let $(X, \Delta)$ be an effective pair. We say that $(X, \Delta)$ is a weak log Fano pair if $-\left(K_{X}+\Delta\right)$ is big and nef.

By theorem 3.1.3 we obtain a sufficient condition for the semiampleness of the anticanonical divisors on a LC weak log Fano pair:

Corollary 3.1.5. Let $(X, \Delta)$ be a weak log Fano LC pair.
If $\mathbb{B}_{+}\left(-\left(K_{X}+\Delta\right)\right)$ does not contain any $L C$ center of the pair $(X, \Delta)$, then $-\left(K_{X}+\Delta\right)$ is semiample.

Remark 3.1.6. Again in the above corollary the hypothesis on the $\mathbb{B}_{+}$is necessary: it is not sufficient that some LC centers of $(X, \Delta)$ are not contained in $\mathbb{B}_{+}\left(-\left(K_{X}+\Delta\right)\right)$. See example 3.11.1 and example 3.11.2.

### 3.2 Direct consequences of a theorem of Fujino

In this section we easily get from theorem 6.1 of Fujino's paper [Fuj07b] a generalization of Kawamata's theorem 1.0.1 in the LC case that goes in a different direction with respect to Conjecture 1 and Conjecture 2. We require, in fact, as additional hypothesis, that the positive part of the Zariski decomposition is
semiample when restricted to the locus where the given pair is not KLT (see corollary 3.2 .3 ).
The following is a simplified version of theorem 6.1 of [Fuj07b]:
Theorem 3.2.1. Let $(X, B)$ be an LC pair such that $X$ is smooth and $B$ is SNCS.
Write $B=B_{+}-B_{-}$, where $B_{+}$and $B_{-}$are effective $\mathbb{Q}$-divisors, and they have not common components. Let $P \in \operatorname{Div} v_{\mathbb{Q}}(X)$ be such that

1. $P$ is nef;
2. $a P-\left(K_{X}+B\right)$ is big and nef for some rational number $a \geq 0$;
3. There exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner B_{-}\right\urcorner\right)\right)
$$

4. If $(X, B)$ is not $K L T$ then $P_{\left.\right|_{B=1}}$ is semiample.

Then $P$ is semiample.
Corollary 3.2.2. Let $X$ be a normal projective variety and let $\Delta$ be an effective Weil $\mathbb{Q}$-divisor. Suppose $D$ is a Weil $\mathbb{Q}$-divisor that admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$ such that

1. $(X, \Delta-a N)$ is an LC pair some rational numbers $a \geq 0$;
2. If $(X, \Delta-a N)$ is not $K L T$ then $P_{\left.\right|_{\mathrm{Nklt}(X, \Delta-a N)}}$ is semiample;
3. $m_{0} P-\left(K_{X}+\Delta-a N\right)$ is big and nef for some $m_{0} \in \mathbb{Q}$.

Then $P$ is semiample.
Note that in particular we can replace hypotheses 1 and 2 by asking that $(X, \Delta)$ is an LC pair such that $\operatorname{Nklt}(X, \Delta)=\emptyset$ or or $P_{\left.\right|_{\mathrm{Nklt}(X, \Delta)}}$ is semiample.

Proof. Let $\mu: Y \rightarrow X$ be a log-resolution of the pair $(X, \Delta-a N)$ and let $B:=\mathbf{A}(\Delta-a N)_{Y} \in \operatorname{Div}_{\mathbb{Q}}(Y)$.
We will show that we can apply theorem 3.2 .1 to the pair $(Y, B)$ and the divisor $\mu^{*}(P)$.
Note that $Y$ is smooth, $B$ is SNCS and $(Y, B)$ is KLT if $(X, \Delta-a N)$ is such. Moreover $\mu^{*}(P)$ is nef and

$$
m_{0} \mu^{*}(P)-\left(K_{Y}+B\right) \equiv \mu^{*}\left(m_{0} P-\left(K_{X}+\Delta-a N\right)\right)
$$

so that it is big and nef.
Now, by definition of $\mathbb{Q}$-CKM Zariski decomposition, we can choose $k_{0} \in \mathbb{N}$ such that $k_{0} \geq a, k_{0} P$ is Cartier, $k_{0} D$ is integral and

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} D\right)\right)
$$

for all $m \in \mathbb{N}$.

Furthermore note that if we write $B=B_{+}-B_{-}$, where $B_{+}$and $B_{-}$are effective divisors and they have not common components, then $\mu_{*}\left(\left\ulcorner B_{-}\right\urcorner\right) \leq\ulcorner a N\urcorner$, because $\Delta$ is effective.
Thus we can apply lemma 2.1.8 and we get that for all $m \in \mathbb{N}$

$$
\begin{gathered}
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(m k_{0} P\right)+\left\ulcorner B_{-}\right\urcorner\right) \leq h^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)+\ulcorner a N\urcorner\right) \leq\right. \\
\leq h^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} D\right)\right)=h^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right)=h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(m k_{0} P\right)\right)\right) .
\end{gathered}
$$

Hence property 3 is verified.
Now suppose $(Y, B)$ is not KLT, define $T=B^{=1}$ and consider the following commutative diagram:


Then $\mu^{*}(P)_{\left.\right|_{T}}=\mu_{\left.\right|_{T}}^{*}\left(P_{\left.\left.\right|_{\text {Nklt }(X, \Delta-a N)}\right)}\right)$ is semiample, being the pullback of a semiample divisor.
Thus we can apply theorem 3.2.1 and we get that $\mu^{*}(P)$ is semiample, which in turn implies that $P$ is semiample.

Corollary 3.2.3. Let $(X, \Delta)$ be an effective LC pair and let $D \in \operatorname{Div}(X)$. consider the following assumptions:

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;

2'. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^{*}(D)=P+N$ and $P_{\left.\right|_{\mathrm{Nklt}(Z, \mathbf{A}(\Delta) Z)}}$ is semiample if $(X, \Delta)$ is not $K L T$.
If $D$ satisfies 1, 2 and 3, or $D$ satisfies 2' and 3, then $P$ is semiample.
Proof. Let us apply lemma 2.5.3 and consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma, so that $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition, $\Delta_{Z}$ is effective and $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)$ is big and nef.
Note that the pair $\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ is LC because $(X, \Delta)$ is such, so that the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)=\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is also LC by lemma 2.2.8. By the same lemma we get that $P_{l_{\text {Nklt }\left(Z, \Delta_{Z}-N^{\prime}\right)}}$ is semiample whenever $\left(Z, \Delta_{Z}-N^{\prime}\right)$ is not KLT. Therefore the assert follows by applying corollary 3.2.2.

### 3.3 DLT case

In this section we consider some generalizations of Kawamata's thereom 1.0.1 in the case of DLT pairs. In particular by adapting the proof in [Kaw87] we get Conjecture 2 for DLT pairs (see corollary 3.3.4). On the other hand in the second part of the section, as a corollary of [Fuj07b, theorem 5.1], we prove a stronger result (see theorem 3.3.11) which, in particular, implies Conjecture 1 for DLT pairs.

## Generalizing Kawamata's proof

Lemma 3.3.1. Let $X$ be a normal projective variety and let $A$ be a Weil $\mathbb{Q}$ divisor on $X$ such that $(X,-A)$ is a KLT pair.
Let $P$ and $N$ be Weil $\mathbb{Q}$-divisors such that $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ and it is nef, $N$ is effective and $A \leq a N$ for some $a \in \mathbb{Q}$.
If $D$ is a Weil $\mathbb{Q}$-divisor such that

1. $D=P+N$;
2. There exists $t_{0} \in \mathbb{Q}$ such that

$$
t_{0} P+A-K_{X}
$$

is big and nef;
then $h^{0}(X, m D) \geq 0$ for some $m \in \mathbb{N}$.
Proof. Thanks to the nefness of $P$ we can assume $t_{0}>0$.
Now let $k_{0}$ be a positive integer such that $k_{0} P$ is a Cartier divisor. We have that $k_{0} P$ is nef and

$$
\frac{t_{0}}{k_{0}}\left(k_{0} P\right)+A-K_{X}=t_{0} P+A-K_{X}
$$

is big and nef.
Then, by Shokurov nonvanishing theorem ([KM00, theorem 3.4]) we get that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\ulcorner A\urcorner\right) \neq 0\right.
$$

for every integer $m \gg 0$.
Consider $k_{1} \in \mathbb{N}$ such that $k_{1} N$ is an integral divisor and $k_{1} \geq a$. Hence $\ulcorner A\urcorner \leq\ulcorner a N\urcorner \leq\left\ulcorner k_{1} N\right\urcorner=k_{1} N$, so that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+k_{1} N\right) \neq 0\right.
$$

for every $m \gg 0$.
Now take a sufficiently large integer $m_{0}$ such that $m_{0} k_{0} \geq k_{1}$. Then

$$
m_{0} k_{0} P+k_{1} N \leq m_{0} k_{0} P+m_{0} k_{0} N=m_{0} k_{0} D
$$

Thus $H^{0}\left(X, \mathcal{O}_{X}\left(m_{0} k_{0} D\right)\right) \neq 0$, so that the lemma is proved.

Theorem 3.3.2. Let $(X, \Delta)$ be a DLT pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$ such that

1. there exist two rational numbers $t_{0}$ and $a$ with $a \geq 0$ and

$$
t_{0} P+a N-\left(K_{X}+\Delta\right)
$$

is ample;
2. $\mathbb{B}(P)$ does not contain $L C$ centers of the pair $(X, \Delta)$;
then $P$ is semiample.

Proof.
We define $A:=a N-\Delta$ so that $A \leq a N$.
By lemma 2.2.8, the pair $(X,-A)$ is LC and $C L C(X,-A) \subseteq C L C(X, \Delta)$. Then $\mathbb{B}(P)$ does not contain LC centers of the pair $(X,-A)$. Moreover

$$
t_{0} P+A-K_{X}=t_{0} P+a N-\left(K_{X}+\Delta\right)
$$

is ample.
Now if $(X,-A)$ is KLT then $\kappa(X, D) \geq 0$ by lemma 3.3.1. Hence, as $\kappa(X, P)=$ $\kappa(X, D)$ by definition of CKM Zariski decomposition, we have that $\kappa(X, P) \geq 0$, too.
If $(X,-A)$ is not KLT then $C L C(X,-A) \neq \emptyset$, so that $\mathbb{B}(P) \neq X$, that is $\kappa(X, P) \geq 0$.
Then, if we denote

$$
\mathbb{N}(P)=\left\{m \in \mathbb{N} \text { such that } m P \in \operatorname{Div}(X) \text { e } H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \neq 0\right\}
$$

we have that $\mathbb{N}(P) \neq \emptyset$.
We suppose, by contradiction, that $\mathbb{B}(P) \neq \emptyset$ and we consider $m_{1} \in \mathbb{N}(P)$ such that $B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P) \neq \emptyset$.
We will find $m \in \mathbb{N}(P)$ and a subvariety $V$ such that $V \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$ but $V \nsubseteq B s(|m P|)$, leading, in such a way, to a contradiction.
Let $\left\{D_{j}\right\}_{j=1}^{k}$ be the finite set of the prime divisors appearing in the support of $A$ or as base components of $\left|m_{1} P\right|$. We write $A=\sum_{j=1}^{k} a_{j} D_{j}$, where the $a_{j}$ are possibly zero rational numbers.
As $B s\left(\left|m_{1} P\right|\right)$ does not contain any element in $C L C(X, \Delta)$, by lemma 2.2.27, there exists $\mu: Y \rightarrow X$ such that $\mu$ is a log-resolution of the pair $(X,-A)$ and of the linear series $\left|m_{1} P\right|$ and $\mu$ is a DLT morphism for the pair $(X,-A)$.
Let $\left\{F_{j}=\tilde{D}_{j}\right\}_{j=1}^{k}$ be the finite set of the strict transforms of the divisors $D_{j}$ and let $\left\{F_{j}=E_{j}\right\}_{j=k+1}^{l}$ be the finite set of the $\mu$-exceptional prime divisors on $Y$, so that $\sum_{j=1}^{l} F_{j}$ is a SNC divisor.
Moreover we can write

$$
K_{Y} \equiv \mu^{*}\left(K_{X}-A\right)+\sum b_{j} F_{j}
$$

where

$$
b_{j} \geq-1 \quad \text { if } \quad 1 \leq j \leq k
$$

because the pair $(X,-A)$ is LC and

$$
b_{j}>-1 \quad \text { if } \quad k+1 \leq j \leq l
$$

because $\mu$ is a DLT morphism for $(X,-A)$.
On the other hand there exists an integral base-point free divisor $L$ and coefficients $r_{j} \in \mathbb{N} \cup\{0\}$ such that $\mu^{*}\left(m_{1} P\right)=L+\sum r_{j} F_{j}$ and $\mu^{*}\left|m_{1} P\right|=$ $|L|+\sum r_{j} F_{j}$.
Hence we have that $B s\left(\left|m_{1} P\right|\right)=\mu\left(\bigcup_{r_{j} \neq 0} F_{j}\right)$, so that we can suppose $r_{j}>0$ for some $j$ because $B s\left(\left|m_{1} P\right|\right) \neq \emptyset$.
Moreover if $r_{j} \neq 0$, then $\mu\left(F_{j}\right) \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$, which implies that $b_{j}>-1$, as $\mathbb{B}(P)$ does not contain any LC center of the pair $(X,-A)$.

Now, as $t_{0} P+A-K_{X}$ is ample and $\mu$ is a DLT morphism, by lemma 2.1.9 there exist, for each $j=k+1, \ldots, l$, rational, arbitrarily small numbers $\delta_{j}>0$, such that

$$
\mu^{*}\left(t_{0} P+A-K_{X}\right)-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is still ample.
Thanks to the openness of the ample cone there exist also, for each $j=1, \ldots, k$, positive rational numbers $\delta_{j} \in \mathbb{Q}$ such that if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ then

$$
\mu^{*}\left(t_{0} P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample.
Now we define

$$
c=\min _{\left\{j: r_{j} \neq 0\right\}} \frac{b_{j}+1-\delta_{j}}{r_{j}} .
$$

By choosing the $\delta_{j}$ 's small enough we can suppose that

$$
b_{j}+1-\delta_{j}>0 \text { for all } j \text { such that } b_{j}>-1
$$

Hence $c>0$ because $b_{j}>-1$ if $r_{j} \neq 0$.
Moreover, perturbing slightly the $\delta_{j}$ 's if necessary, we can suppose that the minimum is attained on a unique $j$, say $j=j_{0}$. Define $B:=F_{j_{0}}$.
Now take $s:=t_{0}+c m_{1}$, and

$$
B_{m}:=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{j=k+1}^{l} \delta_{j} F_{j} .
$$

Then, if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ for each $j=1, \ldots, k$, we have that for every integer $m \geq s$

$$
B_{m}-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample, because $m-c m_{1} \geq t_{0}$ and $\mu^{*}(P)$ is nef.
Let us consider now, for each $j$ such that $b_{j}=-1$, rational, arbitrarily small numbers $\epsilon_{j}>0$, and define

$$
\begin{gathered}
A^{\prime}:=\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}, \\
A^{\prime \prime}:=\sum_{\substack{b_{j}>-1 \\
j \neq j_{0}}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}+\sum_{b_{j}=-1}\left(-1+\epsilon_{j}\right) F_{j} .
\end{gathered}
$$

As $r_{j}=0$ if $b_{j}=-1$ we have that

$$
A^{\prime \prime}-A^{\prime}=\sum_{b_{j}=-1}\left(\epsilon_{j}+\delta_{j}\right) F_{j}
$$

Now we define, for every $m \in \mathbb{N}(P)$, the divisor

$$
Q_{m}:=\mu^{*}(m P)+A^{\prime \prime}-B-K_{Y} .
$$

Then

$$
\begin{gathered}
Q_{m}:=\mu^{*}(m P)+A^{\prime}-B-K_{Y}+\left(A^{\prime \prime}-A^{\prime}\right) \equiv \\
\equiv \mu^{*}(m P)+\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}-F_{j_{0}}-\sum b_{j} F_{j}-\mu^{*}\left(K_{X}-A\right)+\left(A^{\prime \prime}-A^{\prime}\right)= \\
=\mu^{*}\left(m P+A-K_{X}\right)-\sum_{j \neq j_{0}} c r_{j} F_{j}-\sum_{j \neq j_{0}} \delta_{j} F_{j}+F_{j_{0}}\left(-1-b_{j_{0}}\right)+\left(A^{\prime \prime}-A^{\prime}\right)= \\
=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)+c L+F_{j_{0}}\left(c r_{j_{0}}-1-b_{j_{0}}\right)-\sum_{j \neq j_{0}} \delta_{j} F_{j}+\left(A^{\prime \prime}-A^{\prime}\right)= \\
=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)+c L-\sum \delta_{j} F_{j}+\sum_{b_{j}=-1}\left(\epsilon_{j}+\delta_{j}\right) F_{j}= \\
=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{b_{j}>-1} \delta_{j} F_{j}+c L+\sum_{b_{j}=-1} \epsilon_{j} F_{j}= \\
=B_{m}-\sum_{j \in J} \delta_{j} F_{j}+c L+\sum_{b_{j}=-1} \epsilon_{j} F_{j},
\end{gathered}
$$

where $J=\left\{j \in \mathbb{N}\right.$ such that $1 \leq j \leq k$ and $\left.b_{j}>-1\right\}$.
Let $m_{2}=\min \{m \in \mathbb{N}(P)$ such that $m \geq s\}$. Then $B_{m_{2}}-\sum_{j \in J} \delta_{j} F_{j}$ is ample, so that $Q_{m_{2}}$ is also ample if the $\epsilon_{j}$ are small enough because $L$ is nef.
Hence $Q_{m}$ is ample for every $m \in \mathbb{N}(P)$ such that $m \geq s$.
Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that $H^{1}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner-B\right)\right)=0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.
This implies that the restriction homomorphism

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right) \rightarrow H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right)
$$

is surjective in this case.
Now we notice that $\mu^{*}\left(m_{2} P\right)_{\left.\right|_{B}}+A_{\left.\right|_{B}}^{\prime \prime}-K_{B}=Q_{\left.m_{2}\right|_{B}}$ is an ample $\mathbb{Q}$-divisor.
Moreover $A_{\left.\right|_{B}}^{\prime \prime}$ is SNCS because $A^{\prime \prime}$ is such and $B$ intersects transversally all the $F_{j}$ 's with $j \neq j_{0}$.
Hence it suffices to verify that all the coefficients of $A^{\prime \prime}$ are greater than -1 to show that the pair $\left(B,-A_{\left.\right|_{B}}^{\prime \prime}\right)$ is KLT.
But we have that

- if $b_{j}=-1$ then $-1+\epsilon_{j}>-1$;
- if $b_{j}>-1$ and $r_{j}=0$ then $-c r_{j}+b_{j}-\delta_{j}=b_{j}-\delta_{j}>-1$ (by the choice of the $\delta_{j}$ 's);
- if $b_{j}>-1, r_{j} \neq 0$ and $j \neq j_{0}$ then $-c r_{j}+b_{j}-\delta_{j}>-\frac{b_{j}+1-\delta_{j}}{r_{j}} r_{j}+b_{j}-\delta_{j}=$ -1 .

Hence the pair $\left(B,-A_{\left.\right|_{B}}^{\prime \prime}\right)$ is KLT. Thus, by Shokurov's nonvanishing theorem ([KM00, theorem 3.4]) for every integer $k>0$ there exists $\mu_{k} \in \mathbb{N}$, such that $\mu_{k} \geq a, \mu_{k} \geq s, \mu_{k}$ is a multiple of $k$ and

$$
H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}\left(\mu_{k} P\right)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right) \neq 0 .
$$

In fact this cohomology group is non zero for every sufficiently large multiple of $m_{2}$.
Now, by the definition of $\mathbb{Q}$-CKM Zariski decomposition, there exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ and $k_{0} D$ are integral and $H^{0}\left(X, t k_{0} P\right) \simeq H^{0}\left(X, t k_{0} D\right)$ for every $t \in \mathbb{N}$. Let us define $m:=\mu_{k_{0}}$. Then

$$
B \nsubseteq B s\left(\left|\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right|\right) \text { and } H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m D)\right)
$$

Now we verify the inequalities $\left\ulcorner A^{\prime \prime}\right\urcorner \geq 0$ and $\mu_{*}\left\ulcorner A^{\prime \prime}\right\urcorner \leq\ulcorner a N\urcorner \leq m N$. The first one holds because all the coefficients of $A^{\prime \prime}$ are greater than -1 .
The third one is verified because $m \geq a$ and $m N$ is integral (because $m P$ and $m D$ are such).
As for the second one we have that

$$
\begin{gathered}
\mu_{*}\left(\left\ulcorner A^{\prime \prime}\right\urcorner\right)=\sum_{\substack{b_{j}>-1 \\
j \neq j_{0}}}\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner \mu_{*}\left(F_{j}\right)+\sum_{b_{j}=-1}\left\ulcorner-1+\epsilon_{j}\right\urcorner \mu_{*}\left(F_{j}\right)= \\
=\sum_{\substack{b_{j}>-1 \\
j \neq j_{0}}}\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner \mu_{*}\left(F_{j}\right)=\sum_{j \in J^{\prime}}\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner D_{j},
\end{gathered}
$$

where $J^{\prime}=\left\{j \in \mathbb{N}\right.$ such that $1 \leq j \leq k, b_{j}>-1$ e $\left.j \neq j_{0}\right\}$.
But, if $j \in J^{\prime}$ then $\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner=\left\ulcorner-c r_{j}+a_{j}-\delta_{j}\right\urcorner \leq\left\ulcorner a_{j}\right\urcorner$.
Hence

$$
\mu_{*}\left(\left\ulcorner A^{\prime \prime}\right\urcorner\right) \leq \sum_{j \in J^{\prime}}\left\ulcorner a_{j}\right\urcorner D_{j} \leq \sum_{j=1}^{k} \max \left\{0,\left\ulcorner a_{j}\right\urcorner\right\} D_{j} \leq\ulcorner a N\urcorner
$$

as $\ulcorner a N\urcorner \geq\ulcorner A\urcorner=\sum\left\ulcorner a_{j}\right\urcorner D_{j}$ and $\ulcorner a N\urcorner \geq 0$.
From these inequalities, thanks to lemma 2.1.7, it follows that

$$
\left\ulcorner A^{\prime \prime}\right\urcorner \leq \mu^{*}(m N)+T,
$$

for a suitable $\mu$-exceptional effective Cartier divisor $T$.
Moreover, as $\left\ulcorner A^{\prime \prime}\right\urcorner \geq 0$, we have that

$$
\begin{gathered}
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right) \hookrightarrow H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right) \hookrightarrow \\
\hookrightarrow H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P+m N)+T\right)\right) \simeq \\
\simeq H^{0}\left(X, \mathcal{O}_{X}(m P+m N) \otimes \mu_{*} \mathcal{O}_{Y}(T)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m D)\right)
\end{gathered}
$$

by Fujita's lemma. But

$$
H^{0}\left(X, \mathcal{O}_{X}(m D)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \simeq H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right)
$$

Then the injection

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right) \hookrightarrow H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right)
$$

is an isomorphism. This implies that $B \nsubseteq B s\left(\left|\mu^{*}(m P)\right|\right)$.
Hence $\mu(B) \nsubseteq B s(|m P|)$.
But $\mu(B)=\mu\left(F_{j_{0}}\right) \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$ and we have the contradiction.

Theorem 3.3.3. Let $(X, \Delta)$ be a DLT pair.
If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $D$ is big;
2. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;
3. $D$ admits $a \mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

then there exists $\beta>0$ such that if

$$
a D-\left(K_{X}+\Delta\right) \text { is nef for some rational number } a>-\beta
$$

then $P$ is semiample.
Proof. We begin by observing that $0 \leq \operatorname{dim} X=\kappa(X, D)$. Moreover, by definition of CKM Zariski decomposition, $\kappa(X, D)=\kappa(X, P)$, that is $P$ is big.
Note also that, as $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)$ by lemma 2.5.4, we have that $\mathbb{B}_{+}(P)$ does not contain any LC center of the pair $(X, \Delta)$.
Hence, by lemma 2.3.7, there exists an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $P-\lambda \Gamma$ is ample, $(X, \Delta+\lambda \Gamma)$ is DLT and $C L C(X, \Delta+\lambda \Gamma)=C L C(X, \Delta)$.
Now, as $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in \mathbb{Q}^{+}$such that if $0<\beta^{\prime}<\beta$, then $\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)$ is still DLT and $C L C\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)=C L C(X, \Delta)$.
Let $a$ be a rational number such that $a>-\beta$ and $a D-\left(K_{X}+\Delta\right)$ is nef.
We have that

$$
\begin{gathered}
(1+a) P+a N-\Delta-\lambda \Gamma-K_{X}=(1+a) P+a N-\lambda \Gamma-a D+\left(a D-\left(K_{X}+\Delta\right)\right)= \\
=(a P+a N)+(P-\lambda \Gamma)-a D+\left(a D-\left(K_{X}+\Delta\right)\right)= \\
=(P-\lambda \Gamma)+\left(a D-\left(K_{X}+\Delta\right)\right)
\end{gathered}
$$

is ample because it is the sum of an ample and a nef divisor.
If $a \geq 0$ we conclude by applying theorem 3.3 .2 to the pair $(X, \Delta+\lambda \Gamma)$. In fact, as $\mathbb{B}(P) \subseteq \mathbb{B}_{+}(P)$, it does not contain any element of $C L C(X, \Delta)=$ $C L C(X, \Delta+\lambda \Gamma)$.
If $-\beta<a<0$ we can apply theorem 3.3.2 to the pair $(X, \Delta+\lambda \Gamma-a N)$. In fact, again, as $\mathbb{B}(P) \subseteq \mathbb{B}_{+}(P)$, it does not contain any element of $C L C(X, \Delta)=$ $C L C(X, \Delta+\lambda \Gamma-a N)$.

The following corollary corresponds to Conjecture 2 in the case when the pair $(X, \Delta)$ is DLT. Note that in the next subsection we prove a stronger result in a different way (see theorem 3.3.11).

Corollary 3.3.4. Let $(X, \Delta)$ be a DLT pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;
3. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;
4. $D$ admits $a \mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

then $P$ is semiample.
Remark 3.3.5. Note that in the above theorem the hypothesis that $\mathbb{B}_{+}(D)$ does not contain any LC center of the pair $(X, \Delta)$ is necessary. It is not enough to assume that some LC centers of $(X, \Delta)$ are not contained in $\mathbb{B}_{+}(D)$. See example 3.11.1 and example 3.11.2.

The following is a variant of corollary 3.3.4
Theorem 3.3.6. Let $(X, \Delta)$ be an effective LC pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
2. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition;
3. $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is a DLT pair;
4. $\mathbb{B}_{+}\left(f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)\right)$ does not contain any LC center of the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$;
5. $\mathbb{B}(P)$ does not contain any $L C$ center of the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$;
then $P$ is semiample.
Proof. Let us apply lemma 2.5 .3 and consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma. Note that $\left(Z, \Delta_{Z}\right)$ is a pair because $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is such. Hence $N^{\prime}$ and $D^{\prime}$ are in $\operatorname{Div}_{\mathbb{Q}}(X)$. Take $L:=f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$, so that by lemma 2.3.7 we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $L-\lambda \Gamma$ is ample, $\left(Z, \mathbf{L}(\Delta)_{Z}+\lambda \Gamma\right)$ is DLT and $C L C\left(Z, \mathbf{L}(\Delta)_{Z}\right)=C L C\left(Z, \mathbf{L}(\Delta)_{Z}+\lambda \Gamma\right)$. Now we have that

$$
t_{0} P^{\prime}+N^{\prime}-\left(K_{Z}+\Delta_{Z}+\lambda \Gamma\right)=P+L-\lambda \Gamma
$$

is ample. Thus we conclude by applying theorem 3.3 .2 to the $\mathbb{Q}$-Cartier $\mathbb{Q}$ divisor $D^{\prime}=P^{\prime}+N^{\prime}$ and the DLT pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$.

## Nonvanishing theorems

By using the same ingredients of the proof of the main theorem of [Kaw87] (see also [Mor86, step 2 of the proof of thm. 0]), we prove the following nonvanishing, again in the context of DLT pairs.

Theorem 3.3.7. Let $(X, \Delta)$ be a DLT pair.
Let $P, N \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that $P$ is nef and $N$ is effective. If $D \in \operatorname{Div} \mathbb{Q}_{\mathbb{Q}}(X)$ is such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
2. $\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)$ does not contain any LC center of the pair $(X, \Delta)$;
3. $D=P+N$;
then $\kappa(X, D) \geq 0$.
Proof. Let $L:=a D-\left(K_{X}+\Delta\right)$.
Thanks to lemma 2.3.7 we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $L-\lambda \Gamma$ is ample, the pair $(X, \Delta+\lambda \Gamma)$ is DLT and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$.
Now we define $A:=a N-\Delta-\lambda \Gamma$. Then, thanks to lemma 2.2.8, we have that $(X,-A)$ is an LC pair and $C L C(X,-A) \subseteq C L C(X, \Delta)$.
Moreover if $A \neq 0$, we write $A=\sum_{j=1}^{k} a_{j} D_{j}$, where the $D_{j}$ 's are prime distinct divisors and the $a_{j}$ 's are non zero rational numbers.
Now for every integer $t \geq a$ we have that

$$
\begin{gathered}
t P+A-K_{X}=t P+a N-\Delta-\lambda \Gamma-K_{X}= \\
(t-a) P+(a P+a N)-a D+a D-\Delta-\lambda \Gamma-K_{X}=(t-a) P+L-\lambda \Gamma
\end{gathered}
$$

is ample because it is the sum of a nef and an ample divisor.
By lemma 2.2.27 (applied for example to the trivial linear series $\left|\mathcal{O}_{X}\right|$ ) there exists $\mu: Y \rightarrow X$, a log-resolution of the pair $(X,-A)$ such that $\mu$ is a DLT morphism for $(X,-A)$.
Let $\left\{F_{j}=\tilde{D}_{j}\right\}_{j=1}^{k}$ be the finite set of the strict transforms of the divisors $D_{j}$ and let $\left\{F_{j}=E_{j}\right\}_{j=k+1}^{l}$ be the finite set of the $\mu$-exceptional prime divisors on $Y$, so that $\sum_{j=1}^{l} F_{j}$ is a SNC divisor.
Moreover we can write

$$
K_{Y} \equiv \mu^{*}\left(K_{X}-A\right)+\sum b_{j} F_{j}
$$

where

$$
b_{j} \geq-1 \quad \text { if } \quad 1 \leq j \leq k
$$

because the pair $(X,-A)$ is LC and

$$
b_{j}>-1 \quad \text { if } \quad k+1 \leq j \leq l
$$

because $\mu$ is a DLT morphism for $(X,-A)$.
Now, as $a P+A-K_{X}$ is ample and $\mu$ is a DLT morphism, by lemma 2.1.9 there exist, for each $j=k+1, \ldots, l$, rational, arbitrarily small numbers $\delta_{j}>0$, such that

$$
\mu^{*}\left(a P+A-K_{X}\right)-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample.
Thanks to the openness of the ample cone there exist also, for each $j=1, \ldots, k$, positive rational numbers $\delta_{j} \in \mathbb{Q}$ such that if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ then

$$
\mu^{*}\left(a P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample.
Moreover, by choosing the $\delta_{j}$ 's small enough we can suppose that

$$
b_{j}+1-\delta_{j}>0 \quad \forall j: b_{j}>-1
$$

Now for every $m \in \mathbb{N}$ we define

$$
B_{m}:=\mu^{*}\left(m P+A-K_{X}\right)-\sum_{j=k+1}^{l} \delta_{j} F_{j} .
$$

Hence, if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ for every $j=1, \ldots, k$, then for every integer $m \geq a$

$$
B_{m}-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}=\mu^{*}\left(m P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample, as $\mu^{*}(P)$ is nef.
Now, for each $j$ such that $b_{j}=-1$, we consider arbitrarily small rational numbers $\epsilon_{j}>0$, and we define

$$
A^{\prime}:=\sum_{b_{j}>-1}\left(b_{j}-\delta_{j}\right) F_{j}+\sum_{b_{j}=-1}\left(-1+\epsilon_{j}\right) F_{j} .
$$

We also define, for every $m \in \mathbb{N}$, the $\mathbb{Q}$-divisor

$$
Q_{m}:=\mu^{*}(m P)+A^{\prime}-K_{Y} .
$$

Then
$Q_{m} \equiv \mu^{*}(m P)+\sum_{b_{j}>-1}\left(b_{j}-\delta_{j}\right) F_{j}+\sum_{b_{j}=-1}\left(-1+\epsilon_{j}\right) F_{j}-\mu^{*}\left(K_{X}-A\right)-\sum_{j=1}^{l} b_{j} F_{j}=$
$=\mu^{*}\left(m P+A-K_{X}\right)-\sum_{b_{j}>-1} \delta_{j} F_{j}+\sum_{b_{j}=-1} \epsilon_{j} F_{j}=B_{m}-\sum_{j \in J} \delta_{j} F_{j}+\sum_{b_{j}=-1} \epsilon_{j} F_{j}$,
where $J=\left\{j \in \mathbb{N}\right.$ such that $1 \leq j \leq k$ e $\left.b_{j}>-1\right\}$.
Let $m_{1}=\min \{m \in \mathbb{N} \backslash\{0\}: m \geq a$ and $m P \in \operatorname{Div}(X)\}$. Then $B_{m_{1}}-$ $\sum_{j \in J} \delta_{j} F_{j}$ is ample, so that $Q_{m_{1}}=\mu^{*}\left(m_{1} P\right)+A^{\prime}-K_{Y}$ is also ample if we take the $\epsilon_{j}$ small enough.
Now we verify that the pair $\left(Y,-A^{\prime}\right)$ is KLT: as $-A^{\prime}$ is a SNCS divisor it suffices to verify that all the coefficients of $-A^{\prime}$ are less than 1, or, equivalently, that all the coefficients of $A^{\prime}$ are greater than -1 .
But we have that

- if $b_{j}=-1$ then $-1+\epsilon_{j}>-1$;
- if $b_{j}>-1$ then $b_{j}-\delta_{j}>-1$ thanks to our choice of the $\delta_{j}$ 's.

Hence, by Shokurov nonvanishing theorem ([KM00, theorem 3.4]), we find that $H^{0}\left(Y, \mu^{*}\left(s m_{1} P\right)+\left\ulcorner A^{\prime}\right\urcorner\right) \neq 0$ for every integer $s \gg 0$.
In particular, there exists a positive integer $d \geq a$ such that $d P$ and $d N$ are Cartier divisors (hence also $d D$ ) and

$$
H^{0}\left(Y, \mu^{*}(d P)+\left\ulcorner A^{\prime}\right\urcorner\right) \neq 0
$$

Now we verify the inequalities $0 \leq \mu_{*}\left\ulcorner A^{\prime}\right\urcorner \leq\ulcorner a N\urcorner \leq d N$.
The first one holds because all the coefficients of $A^{\prime}$ are greater than -1 . The third one holds because $d \geq a$ and $d N$ is integral.
As for the second we have that, if $A=0$, then $\mu_{*}\left(\left\ulcorner A^{\prime}\right\urcorner\right)=0$ because $A^{\prime}$ is $\mu$-exceptional, due to the fact that all the $F_{j}$ 's are $\mu$-exceptional prime divisors. Hence the inequality holds because $\ulcorner a N\urcorner \geq 0$.
If $A \neq 0$ then

$$
\begin{gathered}
\mu_{*}\left(\left\ulcorner A^{\prime}\right\urcorner\right)=\sum_{b_{j}>-1}\left\ulcorner b_{j}-\delta_{j}\right\urcorner \mu_{*}\left(F_{j}\right)+\sum_{b_{j}=-1}\left\ulcorner-1+\epsilon_{j}\right\urcorner \mu_{*}\left(F_{j}\right)= \\
=\sum_{b_{j}>-1}\left\ulcorner b_{j}-\delta_{j}\right\urcorner \mu_{*}\left(F_{j}\right)=\sum_{\substack{j \leq k \\
b_{j}>-1}}\left\ulcorner b_{j}-\delta_{j}\right\urcorner D_{j} \leq \sum_{\substack{j \leq k \\
a_{j}>-1}}\left\ulcorner a_{j}\right\urcorner D_{j} \leq \\
\leq \sum_{j=1}^{k} \max \left\{0,\left\ulcorner a_{j}\right\urcorner\right\} D_{j} \leq\ulcorner a N\urcorner
\end{gathered}
$$

because $\ulcorner a N\urcorner \geq\ulcorner A\urcorner=\sum\left\ulcorner a_{j}\right\urcorner D_{j}$ and $\ulcorner a N\urcorner \geq 0$.
Now, as $\mu_{*}\left\ulcorner A^{\prime}\right\urcorner \leq d N$, by lemma 2.1.7 we get that there exists a $\mu$-exceptional divisor $T \geq 0$ on $Y$ such that

$$
\left\ulcorner A^{\prime}\right\urcorner \leq \mu^{*}(d N)+T \text {. }
$$

Hence

$$
\begin{gathered}
0 \neq h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(d P)+\left\ulcorner A^{\prime}\right\urcorner\right)\right) \leq h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(d P+d N)+T\right)=\right. \\
=h^{0}\left(X, \mathcal{O}_{X}(d D) \otimes \mu_{*} \mathcal{O}_{Y}(T)\right)=h^{0}\left(X, \mathcal{O}_{X}(d D)\right)
\end{gathered}
$$

thanks to Fujita's lemma. Thus $\kappa(X, D) \geq 0$.
Definition 3.3.8. [Bir09, Def 1.3]
Let $X$ be a normal projective variety and let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$.
We say that $D$ admits a weak $\mathbb{Q}$-Zariski decomposition if there exists a projective birational morphism $g: Y \rightarrow X$, where $Y$ is a normal projective variety, such that

1. $g^{*}(D)=P+N$;
2. $P, N \in \operatorname{Div}_{\mathbb{Q}}(X)$;
3. $P$ is nef and $N$ is effective.

Corollary 3.3.9. Let $(X, \Delta)$ be an effective PLT pair. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a>0$;
2. $\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)$ does not contain any LC center of the pair $(X, \Delta)$;
3. D a admits a weak $\mathbb{Q}$-Zariski decomposition;
then $\kappa(X, D) \geq 0$.
Proof. Let $g: Y \rightarrow X$ be a projective birational morphism of normal projective varieties such that $g^{*}(D)=P+N$ is a weak $\mathbb{Q}$-Zariski decomposition.
Let $h: Z \rightarrow Y$ be a log-resolution of the pair $\left(Y, g_{*}^{-1}(\Delta)+\operatorname{exc}(g)\right)$, so that $Z$ is smooth and $g \circ h: Z \rightarrow X$ is a $\log$-resolution of the pair $(X, \Delta)$.
Let us denote $f:=g \circ h: Z \rightarrow X$.
Then

$$
f^{*}(D)=h^{*}\left(g^{*}(D)\right)=h^{*}(P)+h^{*}(N)
$$

where $h^{*}(P)$ is nef and $h^{*}(N)$ is effective.
Now let $F:=\mathbf{L}(\Delta)_{Z}-\mathbf{A}(\Delta)_{Z}$. Then we have that

- $\mathbf{A}(\Delta)_{Z}=\mathbf{L}(\Delta)_{Z}-F ;$
- $\mathbf{L}(\Delta)_{Z}$ and $F$ are both effective;
- $\operatorname{Supp}(F) \subseteq \operatorname{divexc}(f)$;
- $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is a PLT pair and $C L C\left(Z, \mathbf{L}(\Delta)_{Z}\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}\right)$.

The first three assertions are obvious. As for the last one we have that $C L C\left(Z, \mathbf{L}(\Delta)_{Z}\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ by remark 2.2 .6 because $\mathbf{L}(\Delta)_{Z}$ and $\mathbf{A}(\Delta)_{Z}$ are SNCS and $\left(\mathbf{L}(\Delta)_{Z}\right)^{=1}=\left(\mathbf{A}(\Delta)_{Z}\right)^{=1}$. Hence $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is PLT because $\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ is such by [KM00, 2.30].
In particular $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is a DLT pair, because $\mathbf{L}(\Delta)_{Z}$ is effective.
Now $\left(f^{*}(a D)+F\right)-\left(K_{Z}+\mathbf{L}(\Delta)_{Z}\right)=f^{*}(a D)-\left(K_{Z}+\mathbf{A}(\Delta)_{Z}\right) \equiv f^{*}(a D-$ $\left.\left(K_{X}+\Delta\right)\right)$ is big and nef.
Moreover we can write

$$
f^{*}(a D)+F=h^{*}(a P)+\left(h^{*}(a N)+F\right)=P^{\prime}+N^{\prime}
$$

where $P^{\prime}:=h^{*}(a P)$ is nef and $N^{\prime}:=h^{*}(a N)+F$ is effective.
We show now that $\mathbb{B}_{+}\left(\left(f^{*}(a D)+F\right)-\left(K_{Z}+\mathbf{L}(\Delta)_{Z}\right)\right)$ does not contain any LC center of the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$.
We use that, as the $\mathbb{B}_{+}$only depends by the numerical equivalence class,

$$
\begin{gathered}
\mathbb{B}_{+}\left(\left(f^{*}(a D)+F\right)-\left(K_{Z}+\mathbf{L}(\Delta)_{Z}\right)\right)=\mathbb{B}_{+}\left(f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)\right) \subseteq \\
\subseteq \operatorname{exc}(f) \cup f^{-1}\left(\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)\right),
\end{gathered}
$$

thanks to lemma 2.3.8.
Let $V \in C L C\left(Z, \mathbf{L}(\Delta)_{Z}\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}\right)$. Then $V$ is a prime divisor because $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is PLT and it is not contracted by $f$ because $f(V) \in$ $C L C(X, \Delta)$ and $(X, \Delta)$ is also PLT. Then, by remark 2.1.5, $V \nsubseteq \operatorname{exc}(f)$.

Moreover, as $f(V) \nsubseteq \mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)$ by hypothesis, we have that $V \nsubseteq$ $f^{-1}\left(\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)\right)$.
Thus, as $V$ is irreducible, it follows that $V \nsubseteq \mathbb{B}_{+}\left(\left(f^{*}(a D)+F\right)-\left(K_{Z}+\mathbf{L}(\Delta)_{Z}\right)\right)$. We can then apply theorem 3.3.7 and we find that $\kappa\left(Z, f^{*}(a D)+F\right) \geq 0$, so that there exists $m \in \mathbb{N}$ such that $m\left(f^{*}(a D)+F\right)$ is integral and

$$
H^{0}\left(Z, \mathcal{O}_{Z}\left(m\left(f^{*}(a D)+F\right)\right)\right) \neq 0
$$

Hence

$$
\begin{aligned}
0 \neq h^{0}\left(Z, \mathcal{O}_{Z}\left(f^{*}(m a D)\right.\right. & +m F))=h^{0}\left(X, \mathcal{O}_{X}(m a D) \otimes f_{*} \mathcal{O}_{Z}(m F)\right)= \\
& =h^{0}\left(X, \mathcal{O}_{X}(m a D)\right)
\end{aligned}
$$

thanks to Fujita's lemma because $F$ is $f$-exceptional. Thus $\kappa(X, D) \geq 0$.

## Logbig DLT case

The following theorem is a simplified version of theorem 5.1 of [Fuj07b]:
Theorem 3.3.10. Let $(X, B)$ be an $L C$ pair such that $X$ is smooth and $B$ is SNCS.
Write $B=B_{+}-B_{-}$, where $B_{+}$and $B_{-}$are effective $\mathbb{Q}$-divisors, and they have not common components. Let $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $P$ is nef;
2. aP $-\left(K_{X}+B\right)$ is logbig for the pair $(X, B)$ and nef for some rational number $a \geq 0$;
3. There exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner B_{-}\right\urcorner\right)\right)
$$

Then $P$ is semiample.
From the above theorem we deduce in the DLT case the following result, that generalizes corollary 3.3.4:
Theorem 3.3.11. Let $(X, \Delta)$ be an effective LC pair and let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$. Consider the following assumptions:

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;
(2') $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that

$$
f^{*}(D)=P+N
$$

is a $\mathbb{Q}$-CKM Zariski decomposition and the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is a DLT pair;
4. $P$ is logbig for the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$;
(4') $f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$ is logbig for the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$.
If $D$ satisfies $1,2,3,4$ or $2^{\prime}, 3,4^{\prime}$, then $P$ is semiample.
Proof. Let us apply lemma 2.5.3 and take $t_{0} \in \mathbb{Q}$ and $D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ Weil $\mathbb{Q}$ divisors on $Z$ as in the lemma, so that in particular $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition.
Define $B:=\Delta_{Z}-N^{\prime}=\mathbf{A}(\Delta)_{Z}-a N \leq \mathbf{L}(\Delta)_{Z}$, so that $t_{0} P^{\prime}-\left(K_{Z}+B\right)=$ $P+f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$ is big and nef.
Moreover $t_{0} P^{\prime}-\left(K_{Z}+B\right)$ is logbig for the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ because, by hypothesis, its restriction to every LC center of the pair $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is the sum of a big and a nef divisor. Then we have that $t_{0} P^{\prime}-\left(K_{Z}+B\right)$ is logbig for the pair $(Z, B)$, because $C L C(Z, B) \subseteq C L C\left(Z, \mathbf{L}(\Delta)_{z}\right)$ by lemma 2.2.8.
Now thanks to the theorem 2.2.23, the DLTness of $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ implies that every LC center of $\left(Z, \mathbf{L}(\Delta)_{Z}\right)$ is not contained in $\operatorname{Sing}(Z) \cup \operatorname{NSNC}\left(\mathbf{L}(\Delta)_{Z}\right)$. Thus by lemma 2.2 .12 we have that every LC center of the pair ( $Z, B$ ) is not contained in $\operatorname{Sing}(Z) \cup N S N C(B)$.
Let $\mu: Z^{\prime} \rightarrow Z$ be a standard $\log$-resolution of the pair $(Z, B)$, so that $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$ is an LC pair, $Z^{\prime}$ is smooth and $\mathbf{A}(B)_{Z^{\prime}}$ is SNCS.
Now we choose $k_{0} \in \mathbb{N}$ such that $k_{0} P^{\prime}$ is a Cartier divisor, $k_{0} D^{\prime}$ is integral and

$$
H^{0}\left(Z, \mathcal{O}_{Z}\left(m k_{0} P^{\prime}\right)\right) \simeq H^{0}\left(Z, \mathcal{O}_{Z}\left(m k_{0} D^{\prime}\right)\right)
$$

for all $m \in \mathbb{N}$.
Moreover if we write $\mathbf{A}(B)_{Z^{\prime}}=\left(B^{\prime}\right)_{+}-\left(B^{\prime}\right)_{-}$, where $\left(B^{\prime}\right)_{+}$and $\left(B^{\prime}\right)_{-}$are effective divisors and they have not common components, then $\mu_{*}\left(\left\ulcorner\left(B^{\prime}\right)-\right\urcorner\right) \leq$ $\left\ulcorner N^{\prime}\right.$, because $\Delta_{Z}$ is effective. Thus by lemma 2.1.8 we get that for all $m \in \mathbb{N}$

$$
\begin{aligned}
& h^{0}\left(Z^{\prime}, \mathcal{O}_{Z^{\prime}}\left(\mu^{*}\left(m k_{0} P^{\prime}\right)+\left\ulcorner\left(B^{\prime}\right)-\right\urcorner\right)\right) \leq h^{0}\left(Z, \mathcal{O}_{Z}\left(m k_{0} P^{\prime}+\left\ulcorner N^{\prime}\right\urcorner\right)\right) \leq \\
\leq & h^{0}\left(Z, \mathcal{O}_{Z}\left(m k_{0} D^{\prime}\right)\right)=h^{0}\left(Z, \mathcal{O}_{Z}\left(m k_{0} P^{\prime}\right)\right)=h^{0}\left(Z^{\prime}, \mathcal{O}_{Z^{\prime}}\left(\mu^{*}\left(m k_{0} P^{\prime}\right)\right)\right),
\end{aligned}
$$

so that $H^{0}\left(Z^{\prime}, \mathcal{O}_{Z^{\prime}}\left(\mu^{*}\left(m k_{0} P^{\prime}\right)\right)\right) \simeq H^{0}\left(Z^{\prime}, \mathcal{O}_{Z^{\prime}}\left(\mu^{*}\left(m k_{0} P^{\prime}\right)+\left\ulcorner\left(B^{\prime}\right)-\right\urcorner\right)\right)$.
Note also that

$$
t_{0} \mu^{*}\left(P^{\prime}\right)-\left(K_{Z^{\prime}}+\mathbf{A}(B)_{Z^{\prime}}\right) \equiv \mu^{*}\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)
$$

is big and nef, being the birational pullback of a big and nef divisor.
We will prove that $\mu^{*}\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)$ is logbig for the pair $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$ :
Let $V \in C L C\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$. Then $\mu(V) \nsubseteq \operatorname{Sing}(Z) \cup N S N C(B)$. Thanks to the choice of $\mu$ this implies that $V \nsubseteq \operatorname{exc}(\mu)$. Then $\mu_{\left.\right|_{V}}$ is birational.
Consider the following commutative diagram:


We know that $t_{0} P^{\prime}-\left(K_{Z}+B\right)$ is logbig for the pair $(Z, B)$, which implies that $\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)_{\left.\right|_{\mu(V)}}$ is big.
Then, by birationality of $\mu_{\mid V}$, we have that $\mu_{\left.\right|_{V}}^{*}\left(\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)_{\left.\right|_{\mu(V)}}\right)$ is a big $\mathbb{Q}$-divisor on $V$.

But, by commutativity of the diagram, we have that

$$
\mu_{\left.\right|_{V}}^{*}\left(\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)_{\left.\right|_{\mu(V)}}\right)=\left(\mu^{*}\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)\right)_{\left.\right|_{V}}
$$

Thus we have proved that $\mu^{*}\left(t_{0} P^{\prime}-\left(K_{Z}+B\right)\right)$ is big when restricted to each LC center of the pair $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$, whence it is logbig for the pair $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$. Hence $t_{0} \mu^{*}\left(P^{\prime}\right)-\left(K_{Z^{\prime}}+\mathbf{A}(B)_{Z^{\prime}}\right)$ is logbig for the pair $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$.
Therefore we can apply theorem 3.3.10 to the divisor $\mu^{*}\left(P^{\prime}\right)$ and the pair $\left(Z^{\prime}, \mathbf{A}(B)_{Z^{\prime}}\right)$, so that $\mu^{*}\left(P^{\prime}\right)$ is semiample, which implies that $P$ is semiample.

### 3.4 LC pairs "approximated" by KLT pairs

The following theorem implies Conjecture 2 when $X$ is a $\mathbb{Q}$-Gorenstein variety and we can "approximate" the pair $(X, \Delta)$ by KLT pairs.

Theorem 3.4.1. Let $X$ be a normal projective $\mathbb{Q}$-Gorenstein variety and let $\Delta \in \operatorname{Div}(X)$ be effective and such that $(X, \Delta)$ is an LC pair and $(X,(1-b) \Delta)$ is a KLT pair for some rational number $b>0$. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $D$ is big;
2. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;
3. $D$ admits $a \mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

then there exists $\beta>0$ such that if

$$
a D-\left(K_{X}+\Delta\right) \text { is nef for some rational number } a>-\beta
$$

then $P$ is semiample.
Proof. First of all we can assume that $\Delta \neq 0$ :
In fact, if $\Delta=0$ the hypotheses imply that the pair $(X, 0)$ is KLT. In particular $(X, 0)$ is a DLT pair, so that we are under the hypotheses of theorem 3.3.3 and we are done.

Being $\Delta \neq 0$, we can apply lemma 2.2 .10 and we have that $\operatorname{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta)$.
Moreover note that $P$ is big because $D$ is such and $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)$ by lemma 2.5.4. Then, thanks to lemma 2.3.7, we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $P-\lambda \Gamma$ is ample, the pair $(X, \Delta+\lambda \Gamma)$ is LC and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$.
Now, as $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$ by lemma 2.5.4, thanks to lemma 2.2 .26 there exists $\beta \in \mathbb{Q}^{+}$such that if $0 \leq \beta^{\prime}<\beta$, then the pair $\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)$ is LC and $C L C\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)=C L C(X, \Delta)$.
This implies that $\operatorname{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta+\lambda \Gamma+$ $\beta^{\prime} N$ ).

Hence, by applying again lemma 2.2.10, we have that for every rational number $\epsilon \in(0,1)$, for every $\beta^{\prime} \in[0, \beta)$, the pair $\left(X,(1-\epsilon) \Delta+\lambda \Gamma+\beta^{\prime} N\right)$ is KLT and $(1-\epsilon) \Delta+\lambda \Gamma+\beta^{\prime} N$ is effective.
Now we have that
$(1+a) P+a N-\left(K_{X}+(1-\epsilon) \Delta+\lambda \Gamma\right)=(P-\lambda \Gamma)+\left(a D-\left(K_{X}+\Delta\right)\right)+\epsilon \Delta$ is ample, thanks to the openness of the ample cone, for $\epsilon>0$ small enough. In other words there exists a rational number $\epsilon_{0} \in(0,1)$ such that

$$
(1+a) P+a N-\left(K_{X}+\left(1-\epsilon_{0}\right) \Delta+\lambda \Gamma\right)
$$

is ample.
If $a \geq 0$ we conclude by applying theorem 3.3.2 to the pair $\left(X,\left(1-\epsilon_{0}\right) \Delta+\lambda \Gamma\right)$. If $-\beta<a<0$ we can apply theorem 3.3.2 to the pair $\left(X,\left(1-\epsilon_{0}\right) \Delta+\lambda \Gamma-\right.$ $a N)$.

Remark 3.4.2. Note that lemma 2.2 .24 implies that, in the $\mathbb{Q}$-factorial case, theorem 3.4.1 is a generalization of corollary 3.3.4.

### 3.5 Reducing LC to DLT via DLT blow-up

In this section we try to reduce Conjecture 2 to the case when the pair $(X, \Delta)$ is DLT, already proved in section 3.3. In order to do this we use a DLT blow-up (see theorem 3.5.1) to turn our pair into a DLT one. The existence of DLT blowups has been recently proved as a consequence of the main results of the paper [BCHM10]. Unfortunately in the end we need to add supplementary hypotheses on the LC centers of the pair (see theorem 3.5.3).

Theorem 3.5.1 (DLT blow-up). Let $(X, \Delta)$ be an effective $L C$ pair. Then there exists a projective birational morphism $\mu: Y \rightarrow X$, from a normal projective variety, with the following properties:

1. $Y$ is $\mathbb{Q}$-factorial;
2. $a(E, X, \Delta)=-1$ for every $\mu$-exceptional divisor $E$ on $Y$;
3. If we put

$$
\Gamma=\mu_{*}^{-1} \Delta+\sum_{E \subseteq \operatorname{exc}(\mu)} E
$$

then $(Y, \Gamma)$ is a DLT pair and $K_{Y}+\Gamma \sim_{\mathbb{Q}} \mu^{*}\left(K_{X}+\Delta\right)$.
$\mu$ is called a DLT blow-up of the pair $(X, \Delta)$.
For the proof see [Fuj09b, theorem 10.5].
Lemma 3.5.2. Let $\mu: Y \rightarrow X$ be a projective birational morphism of normal projective varieties and suppose $X$ is $\mathbb{Q}$-factorial. Let $L$ be an effective Cartier divisor on $Y$ such that $L$ is $\mu$-exceptional and $-L$ is $\mu$-ample. Then there exists $m_{0} \in \mathbb{N}$ such that for all $m \geq m_{0}$ we have that

1. $\mu(\operatorname{Supp}(L))=\operatorname{Supp}\left(\mu_{*}\left(\mathcal{O}_{L}\right)\right)$;
2. $R^{j} \mu_{*} \mathcal{O}_{Y}(-m L)=0$ for every $j>0$;
3. $\mu(\operatorname{Supp}(L))=\mathcal{Z}\left(\mu_{*}\left(\mathcal{O}_{Y}(-m L)\right)\right)$.

Proof. 1) We prove separately the two inclusions:
$\supseteq)$ Take $x \in \operatorname{Supp}\left(\mu_{*}\left(\mathcal{O}_{L}\right)\right)$. Then $\left(\mu_{*} \mathcal{O}_{L}\right)_{x} \neq 0$.
This implies that for every open set $V$ containing $x$ there exists an open subset $U \subseteq V$ such that $x \in U$ and $\mathcal{O}_{L}\left(\mu^{-1}(U)\right)=\mu_{*} \mathcal{O}_{L}(U) \neq 0$.
If, by contradiction, $x \notin \mu(\operatorname{Supp}(L))$, then take $V=X \backslash \mu(\operatorname{Supp}(L))$, and take $U \subseteq V$.
Hence $\mu^{-1}(U) \subseteq \mu^{-1}(V)=\mu^{-1}(X \backslash \mu(\operatorname{Supp}(L))) \subseteq Y \backslash \operatorname{Supp}(L)$.
Thus $\mathcal{O}_{L}\left(\mu^{-1}(U)\right)=0$ and we find the contradiction.
$\subseteq)$ Consider the morphism of schemes

$$
\mu_{\left.\right|_{L}}: L \rightarrow \mu(L) .
$$

and consider the induced morphism

$$
\mu_{\left.\right|_{L}}^{\sharp}: \mathcal{O}_{\mu(L)} \rightarrow\left(\mu_{\left.\right|_{L}}\right)_{*} \mathcal{O}_{L}
$$

Then, by [Gro60, 9.5.2] we have that $\operatorname{ker} \mu_{\left.\right|_{L}}^{\sharp}=0$, that is $\mu_{\left.\right|_{L}}^{\sharp}$ is injective.
Now note that $\operatorname{Supp}(L) \supseteq \operatorname{divexc}(\mu)$, because $L$ is effective and $-L$ is $\mu$-ample, and $\operatorname{Supp}(L) \subseteq \operatorname{divexc}(\mu)$ by hypothesis.
Then $\operatorname{Supp}(L)=\operatorname{divexc}(\mu)=\operatorname{exc}(\mu)$ because $X$ is $\mathbb{Q}$-factorial (see [KM00, corollary 2.63 ]).
Hence $\operatorname{Supp}(L)=\mu^{-1}(\mu(\operatorname{Supp}(L)))$ by remark 2.1.5, which implies that $\mu_{*} \mathcal{O}_{L}=$ $\left(\mu_{\left.\right|_{L}}\right)_{*} \mathcal{O}_{L}$.
Thus

$$
\operatorname{Supp}\left(\mu_{*} \mathcal{O}_{L}\right)=\operatorname{Supp}\left(\left(\mu_{\left.\right|_{L}}\right)_{*} \mathcal{O}_{L}\right) \supseteq \operatorname{Supp}\left(\mathcal{O}_{\mu(L)}\right)=\mu(\operatorname{Supp}(L))
$$

2) Thanks to [KM00, Prop. 1.45], given an ample divisor $A$ on $X$, we have that $\mu^{*}\left(k_{0} A\right)-L$ is ample for some $k_{0} \in \mathbb{N}$. Moreover $\mu^{*}\left(k_{0} A\right)$ is nef, as $k_{0} A$ is such. Then, thanks to Fujita's vanishing theorem ( see[Laz04, theorem 1.4.35]), there exists $m_{0} \in \mathbb{N}$ such that for all $m \geq m_{0}$

$$
H^{j}\left(Y, \mathcal{O}_{Y}\left(m\left(k_{0} \mu^{*}(A)-L\right)+s \mu^{*}\left(k_{0} A\right)\right)\right)=0 \quad \forall j>0, \forall s \in \mathbb{N}
$$

Note that the integer $m_{0}$ does not depend on $s$.
Equivalently we have that, for all $m \geq m_{0}$,

$$
H^{j}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left((m+s) k_{0} A\right)-m L\right)\right)=0 \quad \forall j>0, \forall s \in \mathbb{N}
$$

that is nothing but saying that for all $m \geq m_{0}$ we have

$$
H^{j}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(t k_{0} A\right)\right) \otimes \mathcal{O}_{Y}(-m L)\right)=0 \quad \forall j>0, \quad \forall t \geq m
$$

Hence, as $k_{0} A$ is ample, by [Laz04, lemma 4.3.10] we get that $R^{j} \mu_{*} \mathcal{O}_{Y}(-m L)=$ 0 for all $j>0$ and for all $m \geq m_{0}$.
3) Fix an integer $m \geq m_{0}$ and consider the short exact sequence

$$
0 \rightarrow \mathcal{O}_{Y}(-m L) \rightarrow \mathcal{O}_{Y} \rightarrow \mathcal{O}_{m L} \rightarrow 0
$$

By applying $\mu_{*}$ we get a long exact sequence

$$
0 \rightarrow \mu_{*} \mathcal{O}_{Y}(-m L) \rightarrow \mu_{*} \mathcal{O}_{Y} \rightarrow \mu_{*} \mathcal{O}_{m L} \rightarrow R^{1} \mu_{*} \mathcal{O}_{Y}(-m L) \rightarrow \ldots
$$

But $\mu_{*} \mathcal{O}_{Y}=\mathcal{O}_{X}$ because $\mu$ is birational and $R^{1} \mu_{*} \mathcal{O}_{Y}(-m L)=0$ by part 2 of the lemma. Hence we get a short exact sequence

$$
0 \rightarrow \mu_{*} \mathcal{O}_{Y}(-m L) \rightarrow \mathcal{O}_{X} \rightarrow \mu_{*} \mathcal{O}_{m L} \rightarrow 0
$$

Thus, by the first part of the lemma it follows that $\mathcal{Z}\left(\mu_{*}\left(\mathcal{O}_{Y}(-m L)\right)\right)=$ $\operatorname{Supp}\left(\mu_{*} \mathcal{O}_{m L}\right)=\mu(\operatorname{Supp}(m L))=\mu(\operatorname{Supp}(L))$.

Theorem 3.5.3. Let $(X, \Delta)$ be an effective LC pair such that $X$ is $\mathbb{Q}$-factorial. Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$.
If

1. $D$ is big;
2. D has a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$ such that

- $\mathbb{B}_{+}(D) \nsupseteq V$, for every $V \in C L C(X, \Delta)$;
- $\mathbb{B}_{+}(D) \cap V=\emptyset$, for every $V \in C L C(X, \Delta)$ such that $V \nsubseteq \operatorname{Supp}(\Delta)$;
then there exists $\beta>0$ such that if

$$
a D-\left(K_{X}+\Delta\right) \text { is nef for some rational number } a>-\beta
$$

then $P$ is semiample.
Proof. Note that, by definition of Zariski decomposition, $\kappa(X, P)=\kappa(X, D)=$ $\operatorname{dim} X$, so that $P$ is big.
Moreover by lemma 2.5 .4 we have that $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)$, so that $\mathbb{B}_{+}(P)$ does not contain any LC center of the pair $(X, \Delta)$.
Then, thanks to lemma 2.3.7 there exist a Cartier divisor $M>0$ and $\lambda \in \mathbb{Q}^{+}$ such that $P-\lambda M$ is ample, $(X, \Delta+\lambda M)$ is an LC pair and $C L C(X, \Delta)=$ $C L C(X, \Delta+\lambda M)$.
Now, as $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$ by lemma 2.5.4, thanks to lemma 2.2.26 there exists $\beta \in(0,1)$ such that if $0 \leq \beta^{\prime}<\beta$, then the pair $\left(X, \Delta+\lambda M+\beta^{\prime} N\right)$ is LC and $C L C(X, \Delta)=C L C\left(X, \Delta+\lambda M+\beta^{\prime} N\right)$.
Suppose $a>-\beta$ is a rational number such that $a D-\left(K_{X}+\Delta\right)$ is nef and define $a^{\prime}=-\min \{0, a\}$, so that $0 \leq a^{\prime}<\beta$. Note that $a+1>0$, because $a>-\beta>-1$.
If $\Delta \neq 0$, then, thanks to lemma 2.2.9, for every $\delta \in \mathbb{Q}^{+}$, if $V \in C L C(X, \Delta+$ $\left.\lambda M+a^{\prime} N\right)$ is such that $V \subseteq \operatorname{Supp}(\Delta)$, then $V \notin C L C\left(X, \Delta+\lambda M-\delta \Delta+a^{\prime} N\right)=$ $C L C\left(X,(1-\delta) \Delta+\lambda M+a^{\prime} N\right)$.
The same thing trivially holds if $\Delta=0$, because $\operatorname{Supp}(0)=\emptyset$.
Let us define

$$
\Delta_{\delta}^{\prime}=(1-\delta) \Delta+\lambda M+a^{\prime} N
$$

and let

$$
\left\{V_{1}, \ldots, V_{s}\right\}=\{V \in C L C(X, \Delta): V \nsubseteq \operatorname{Supp}(\Delta)\}
$$

Then, by lemma 2.2 .8 , for every $\delta \in \mathbb{Q}^{+}$we get that $\left(X, \Delta_{\delta}^{\prime}\right)$ is an LC pair such that

$$
C L C\left(X, \Delta_{\delta}^{\prime}\right) \subseteq\left\{V_{1}, \ldots, V_{s}\right\}
$$

Moreover if we define $a^{\prime \prime}=\max \{0, a\}$ we have that $a^{\prime \prime} \geq 0$ and $a^{\prime \prime}-a^{\prime}=a$, so that

$$
\begin{gathered}
(1+a) P+a^{\prime \prime} N-\left(K_{X}+\Delta_{\delta}^{\prime}\right)=(1+a) P+a^{\prime \prime} N-\left(K_{X}+(1-\delta) \Delta+\lambda M+a^{\prime} N\right)= \\
=(P-\lambda M)+\left(a D-\left(K_{X}+\Delta\right)\right)+\delta \Delta
\end{gathered}
$$

is ample for $\delta$ sufficiently small thanks to the openness of the ample cone. Let us denote $\Delta^{\prime}=\Delta_{\delta}^{\prime}$, for a sufficiently small fixed $\delta$ such that the above ampleness holds and $\Delta^{\prime}$ is effective.
Let $\mu: Y \rightarrow X$ be a DLT blow-up of the pair $\left(X, \Delta^{\prime}\right)$, whose existence is assured by theorem 3.5.1.
Take

$$
\Delta_{Y}^{\prime}=\mu_{*}^{-1} \Delta^{\prime}+\sum_{E \mu-e x c} E
$$

so that $K_{Y}+\Delta_{Y}^{\prime} \sim_{\mathbb{Q}} \mu^{*}\left(K_{X}+\Delta^{\prime}\right)$ and $\left(Y, \Delta_{Y}^{\prime}\right)$ is a DLT pair.
If $\mu$ is an isomorphism, then the pair $\left(X, \Delta^{\prime}\right)$ is DLT. Hence we can apply theorem 3.3.2 to the pair $\left(X, \Delta^{\prime}\right)$ because we have verified that all the hypotheses are satisfied and we are done.
Thus we can assume from now on that $\mu$ is not an isomorphism.
By [KM00, lemma 2.62] there exists an effective $\mu$-exceptional Cartier divisor $G$ on $Y$ such that $-G$ is $\mu$-ample.
Hence $\operatorname{Supp}(G) \subseteq \operatorname{divexc}(\mu)$ and $G \neq 0$ because $-G$ is $\mu$-ample and $\mu$ contracts some curve.
Thus there exists a set $J \subseteq\{1, \ldots, s\}$ such that $J \neq \emptyset$ and

$$
\mu(\operatorname{Supp}(G))=\bigcup_{j \in J} V_{j}
$$

because, thanks to the properties of the DLT blow-up, every $\mu$-exceptional divisor on $Y$ maps onto a LC center of the pair $\left(X, \Delta^{\prime}\right)$.
Take

$$
L=\mu^{*}(D)-\epsilon G,
$$

for a sufficiently small rational number $\epsilon>0$.
We will show that the pair $\left(Y, \Delta_{Y}^{\prime}\right)$ and the $\mathbb{Q}$-divisor $L$ on $Y$ satisfy the hypotheses of theorem 3.3.2.

We begin by proving that the decomposition

$$
L=\left(\mu^{*}(P)-\epsilon G\right)+\left(\mu^{*}(N)\right)
$$

is a $\mathbb{Q}$-CKM Zariski decomposition.
The effectivity of $\mu^{*}(N)$ is trivial. Let us prove that $\mu^{*}(P)-\epsilon G$ is nef:
Let $A^{\prime}$ be an ample divisor on $X$ such that

$$
\mathbb{B}_{+}(P)=\mathbb{B}\left(P-A^{\prime}\right)=B s\left(\left|m_{0}\left(P-A^{\prime}\right)\right|\right)
$$

for some $m_{0} \in \mathbb{N}$, so that, by lemma 2.5.4, $\mathbb{B}_{+}(D)=B s\left(\left|m_{0}\left(P-A^{\prime}\right)\right|\right)$.

Let $\Gamma$ be a general divisor in the linear series $\left|m_{0}\left(P-A^{\prime}\right)\right|$, so that $B s(|\Gamma|)=$ $\mathbb{B}_{+}(P)$.
Thanks to the hypotheses on $\mathbb{B}_{+}(D)$, we can suppose that $\operatorname{Supp}(\Gamma)$ does not contain any LC center of the pair $(X, \Delta)$.
Take $A=P-\frac{1}{m_{0}} \Gamma$, so that

$$
\mu^{*}(P)-\epsilon G=\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}(\Gamma)-\epsilon G
$$

Then $A \equiv A^{\prime}$, so it is ample.
Hence, thanks to [KM00, 1.45], we get that $\mu^{*}(s A)-G$ is ample for every $s \gg 0$, which implies that $\mu^{*}(A)-\epsilon G$ is ample for sufficiently small $\epsilon \in \mathbb{Q} \cap(0,1)$.
Moreover, $\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}(\Gamma)=\mu^{*}(P)$ is nef.
Let $C$ be an irreducible curve on $Y$. We want to verify the nefness of $\mu^{*}(P)-\epsilon G$ by checking that its intersection with $C$ is non-negative.
If $C \nsubseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right)$, then, by effectivity of $\Gamma$, we get that

$$
\left(\left(\mu^{*}(P)-\epsilon G\right) \cdot C\right)=\left(\left(\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}(\Gamma)-\epsilon G\right) \cdot C\right) \geq\left(\left(\mu^{*}(A)-\epsilon G\right) \cdot C\right)>0
$$

because $\mu^{*}(A)-\epsilon G$ is ample.
If $C \subseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right)$ and $C$ is contracted by $\mu$, then

$$
\begin{gathered}
\quad\left(\left(\mu^{*}(P)-\epsilon G\right) \cdot C\right)=\left(\left(\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}(\Gamma)-\epsilon G\right) \cdot C\right)= \\
=\left(\left(\mu^{*}(A)-\epsilon G\right) \cdot C\right)+\frac{1}{m_{0}}\left(\Gamma \cdot \mu_{*}(C)\right)=\left(\left(\mu^{*}(A)-\epsilon G\right) \cdot C\right)>0 .
\end{gathered}
$$

Suppose $C \subseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right)$ and $C$ is not contracted by $\mu$. If $C \cap \operatorname{Supp}(G)=\emptyset$, then

$$
\left(\left(\mu^{*}(P)-\epsilon G\right) \cdot C\right)=\left(\left(\mu^{*}(P) \cdot C\right) \geq 0\right.
$$

In order to conclude the check we just lack the case when $C \subseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right), C$ is not contracted by $\mu$ and $C \cap \operatorname{Supp}(G) \neq \emptyset$ :
In this case

$$
\mu(C \cap \operatorname{Supp}(G)) \subseteq \mu(\operatorname{Supp}(G))=\bigcup_{j \in J} V_{j} \subseteq \bigcup_{j=1}^{s} V_{j}
$$

Hence
$B s(|\Gamma|) \cap \mu(C \cap \operatorname{Supp}(G))=\mathbb{B}_{+}(P) \cap \mu(C \cap \operatorname{Supp}(G)) \subseteq \mathbb{B}_{+}(P) \cap\left(\bigcup_{j=1}^{s} V_{j}\right)=\emptyset$
by hypothesis.
Then we can choose an effective divisor $\Gamma^{\prime} \sim \Gamma$ such that $\operatorname{Supp}\left(\Gamma^{\prime}\right) \nsupseteq$ $\mu(C \cap \operatorname{Supp}(G))$, so that $\operatorname{Supp}\left(\Gamma^{\prime}\right) \nsupseteq \mu(C)$, which in turn implies that $\left(\Gamma^{\prime} \cdot \mu_{*}(C)\right) \geq 0$.
Hence

$$
\left(\left(\mu^{*}(P)-\epsilon G\right) \cdot C\right)=\left(\left(\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}\left(\Gamma^{\prime}\right)-\epsilon G\right) \cdot C\right)=
$$

$$
=\left(\left(\mu^{*}(A)-\epsilon G\right) \cdot C\right)+\frac{1}{m_{0}}\left(\Gamma^{\prime} \cdot \mu_{*}(C)\right) \geq\left(\left(\mu^{*}(A)-\epsilon G\right) \cdot C\right)>0
$$

Thus we have that $\mu^{*}(P)-\epsilon G$ is nef.
In order to prove that we have a $\mathbb{Q}$-CKM Zariski decomposition it remains to show that there exists an integer $k>0$ such that $k L$ and $k\left(\mu^{*}(P)-\epsilon G\right)$ are Cartier divisors on $Y$ and for every $p \in \mathbb{N}$ the natural injective maps

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(p k\left(\mu^{*}(P)-\epsilon G\right)\right)\right) \longrightarrow H^{0}\left(Y, \mathcal{O}_{Y}(p k L)\right)
$$

are bijective.
But we know, by hypothesis, that there exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ and $k_{0} D$ are Cartier divisors on $X$ and, for every integer $p>0$ we have that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(p k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(p k_{0} D\right)\right)
$$

Moreover, by lemma 3.5.2 there exists an integer $k_{1}$ such that for all $t \geq k_{1}$

$$
\mathcal{Z}\left(\mu_{*}\left(\mathcal{O}_{Y}(-t G)\right)\right)=\mu(\operatorname{Supp}(G))
$$

We choose an integer $b$ such that $\frac{b}{\epsilon}$ is an integer and we define $k:=\frac{b}{\epsilon} \cdot k_{0} \cdot k_{1}$. Then $k \mu^{*}(P), k \epsilon G$ and $k \mu^{*}(D)$ are Cartier divisors. Hence, the same holds for $k L$ and $k\left(\mu^{*}(P)-\epsilon G\right)$.
Moreover, for every $m \in k \mathbb{N}$, we get that

$$
\begin{gathered}
H^{0}\left(Y, \mathcal{O}_{Y}\left(m\left(\mu^{*}(P)-\epsilon G\right)\right)\right) \simeq H^{0}\left(X, \mu_{*}\left(\mathcal{O}_{Y}\left(m\left(\mu^{*}(P)-\epsilon G\right)\right)\right)\right) \simeq \\
\simeq H^{0}\left(X, \mathcal{O}_{X}(m P) \otimes \mu_{*} \mathcal{O}_{Y}(-m \epsilon G)\right)
\end{gathered}
$$

In the same way

$$
H^{0}\left(Y, \mathcal{O}_{Y}(m L)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m D) \otimes \mu_{*} \mathcal{O}_{Y}(-m \epsilon G)\right)
$$

Now, define $\mathcal{J}=\mu_{*} \mathcal{O}_{Y}(-m \epsilon G)$, so that, thanks to our choice of $k$, for all $m \in k \mathbb{N}$ we have that

$$
\mathcal{Z}(\mathcal{J})=\mu(\operatorname{Supp}(G))=\bigcup_{j \in J} V_{j}
$$

In other words, if we denote by $\mathcal{Z}$ the scheme defined by the sheaf of ideals $\mathcal{J}$, then, set-theoretically,

$$
\mathcal{Z}=\bigcup_{j \in J} V_{j}
$$

Now note that $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$ by lemma 2.5.4. Then, thanks to the hypotheses, we find that $\operatorname{Supp}(N) \nsupseteq V_{j}$ for all $j \in J$. Hence $m N_{\mid \mathcal{Z}}$ is effective for $m \in k \mathbb{N}$.
Thus, for every $m \in k \mathbb{N}$, we obtain the following commutative diagram

where the rows are exact, $g$ is an isomorphism thanks to the choice of $k$ and $h$ is injective.
Commutativity of the above diagram leads, on the one hand, to the existence of a map

$$
f: H^{0}\left(X, \mathcal{O}_{X}(m P) \otimes \mathcal{J}\right) \longrightarrow H^{0}\left(X, \mathcal{O}_{X}(m D) \otimes \mathcal{J}\right)
$$

obtained by restricting $g$.
On the other hand if we denote by $\Phi \subseteq H^{0}\left(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(m D)\right)$ the image of $\phi$ and by $\Psi \subseteq H^{0}\left(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}}(m P)\right)$ the image of $\psi$, it provides the existence of an injective map $h^{\prime}: \Psi \rightarrow \Phi$, obtained by restricting $h$.
We obtain, in a such a way, the existence of the following commutative diagram:

where again the rows are exact, $g$ is an isomorphism and $h^{\prime}$ is injective because $h$ is such.
Therefore, thanks to the snake lemma, we get that the map $f$ is surjective. This implies that for every $m \in k \mathbb{N}$

$$
\begin{aligned}
h^{0}\left(Y, \mathcal{O}_{Y}\left(m\left(\mu^{*}(P)-\epsilon G\right)\right)\right)= & h^{0}\left(X, \mathcal{O}_{X}(m P) \otimes \mathcal{J}\right) \geq h^{0}\left(X, \mathcal{O}_{X}(m D) \otimes \mathcal{J}\right)= \\
& =h^{0}\left(Y, \mathcal{O}_{Y}(m L)\right)
\end{aligned}
$$

Thus, for all $m \in k \mathbb{N}$, the injective maps

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(m\left(\mu^{*}(P)-\epsilon G\right)\right)\right) \longrightarrow H^{0}\left(Y, \mathcal{O}_{Y}(m L)\right)
$$

are bijective.
Now we prove the additional hypothesis of theorem 3.3.2:
We have that

$$
\begin{aligned}
& (1+a)\left(\mu^{*} P-\epsilon G\right)+a^{\prime \prime} \mu^{*}(N)-\left(K_{Y}+\Delta_{Y}^{\prime}\right) \equiv \\
\equiv & (1+a)\left(\mu^{*} P-\epsilon G\right)+a^{\prime \prime} \mu^{*}(N)-\mu^{*}\left(K_{X}+\Delta^{\prime}\right)= \\
= & \mu^{*}\left[(1+a) P+a^{\prime \prime} N-\left(K_{X}+\Delta^{\prime}\right)\right]-(1+a) \epsilon G .
\end{aligned}
$$

Then it is ample if $\epsilon$ is sufficiently small thanks to [KM00, 1.45] because we checked that $(1+a) P+a^{\prime \prime} N-\left(K_{X}+\Delta^{\prime}\right)$ is ample, $-G$ is $\mu$-ample and $1+a>0$.

In order to apply theorem 3.3.2 it remains to show that $\mathbb{B}\left(\mu^{*}(P)-\epsilon G\right)$ does not contain any LC center of the pair $\left(Y, \Delta_{Y}^{\prime}\right)$ :
Let $V \in C L C\left(Y, \Delta_{Y}^{\prime}\right)$. Then $\mu(V) \in C L C\left(X, \Delta^{\prime}\right) \subseteq C L C(X, \Delta)$.
Now

$$
\mathbb{B}\left(\mu^{*}(P)-\epsilon G\right) \subseteq \mathbb{B}_{+}\left(\mu^{*}(P)-\epsilon G\right)=\bigcap \operatorname{Supp}\left(\mu^{*}(P)-\epsilon G-H\right)
$$

where the intersection is taken over all the ample $\mathbb{R}$-divisors $H$ such that $\mu^{*}(P)-$ $\epsilon G-H \geq 0$.
We proved that $\mu^{*}(P)-\epsilon G=\mu^{*}(A)+\frac{1}{m_{0}} \mu^{*}(\Gamma)-\epsilon G$, where $\mu^{*}(A)-\epsilon G$ is ample and $\Gamma$ is effective.
Hence

$$
\mathbb{B}\left(\mu^{*}(P)-\epsilon G\right) \subseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right)
$$

But we have that $\operatorname{Supp}(\Gamma)$ does not contain any LC center of the pair $(X, \Delta)$.
Hence $\mu(V) \nsubseteq \operatorname{Supp}(\Gamma)$, so that $V \nsubseteq \operatorname{Supp}\left(\mu^{*}(\Gamma)\right)$. Thus $\mathbb{B}\left(\mu^{*}(P)-\epsilon G\right) \nsupseteq V$.
All the hypotheses being satisfied, we can now apply theorem 3.3.2, so that we get the semiampleness of $\mu^{*}(P)-\epsilon G$.
In other words, there exists $l \in \mathbb{N}$ such that $l\left(\mu^{*}(P)-\epsilon G\right)$ is a Cartier divisor and $B s\left(\left|l\left(\mu^{*}(P)-\epsilon G\right)\right|\right)=\emptyset$. This implies that

$$
B s\left(\left|l \mu^{*}(P)\right|\right) \subseteq \operatorname{Supp}(G) \Rightarrow \mu^{-1}(B s(|l P|)) \subseteq \operatorname{Supp}(G)
$$

But $\mu$ is surjective because it is a projective birational morphism. Thus

$$
\mathbb{B}(P) \subseteq B s(|l P|) \subseteq \mu(\operatorname{Supp}(G))=\bigcup_{j \in J} V_{j}
$$

But $\mathbb{B}(P) \subseteq \mathbb{B}_{+}(P)$. Hence, by hypothesis, $\mathbb{B}(P) \cap V_{j}=\emptyset$ for all $j \in J$. Therefore $\mathbb{B}(P)=\emptyset$, that is $P$ is semiample.

### 3.6 Relatively KLT case

In this section we present a simplified version of the main theorem of [Amb05] (see theorem 3.6.4) that corresponds to a generalization of Kawamata's theorem 1.0.1 in the particular setting of relatively KLT pairs (see definition 3.6.2).

In the second part of the section we use Ambro's theorem to generalize our results of the previous sections in the case of $\mathbb{Q}$-Gorenstein varieties.
Note that we will apply Ambro's theorem 3.6.4 also in the proof of theorem 3.7.4.

Definition 3.6.1. Let $(X, \Delta)$ be a pair and let $S \subseteq X$ be a closed subset. We define

$$
C L C(X, \Delta, S)=\{V \in C L C(X, \Delta): V \cap S \neq \emptyset\}
$$

Definition 3.6.2. Let $(X, \Delta)$ be a pair, let $S \subseteq X$ be a closed subset, let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$.
We say that the pair $(X, \Delta)$ is $S$-KLT if $C L C(X, \Delta, S)=\emptyset$.
We say that $(X, \Delta)$ is $D$-KLT if $C L C(X, \Delta, \mathbb{B}(D))=\emptyset$
We remark that, given a closed subset $S \subsetneq X$, an $S$-KLT pair might not be LC in general.

## Ambro's theorem

Lemma 3.6.3. Let $X$ be a normal variety and let $V \subseteq X$ be a subvariety. Suppose $P, L$ and $M$ are Cartier divisors on $X$ such that $P \leq M$ and $P+L \leq$ $M$.

Let $v \in H^{0}(M-P)$ be a section defining the effective divisor $M-P$ and let $\tau \in H^{0}(M-P-L)$ be a section defining the effective divisor $M-P-L$. Denote by

$$
\Phi_{v}: H^{0}(P) \hookrightarrow H^{0}(M)
$$

and

$$
\Phi_{\tau}: H^{0}(P+L) \hookrightarrow H^{0}(M)
$$

the injective maps defined by multiplication by $v$ and by $\tau$ respectively. If

1. $\Phi_{v}\left(H^{0}(P)\right) \supseteq \Phi_{\tau}\left(H^{0}(P+L)\right)$;
2. $V \nsubseteq B s(|P+L|)$;
3. $V \nsubseteq \operatorname{Supp}(L)$;
then $V \nsubseteq B s(|P|)$.
Proof. By hypothesis there exists an effective divisor $\Gamma \in|P+L|$ such that $\operatorname{ord}_{V}(\Gamma)=0$. Let $\sigma \in H^{0}(P+L)$ be a section defining $\Gamma$.
Then $\sigma \tau \in \Phi_{\tau}\left(H^{0}(P+L)\right) \subseteq \Phi_{v}\left(H^{0}(P)\right)$, that is there exists $u \in H^{0}(P)$ such that $\sigma \tau=u v$. This implies that

$$
\Gamma+M-P-L=\{\sigma \tau=0\}=\{u v=0\}=\{u=0\}+M-P
$$

Hence $P^{\prime}:=\{u=0\}$ is a divisor in $|P|$ such that $P^{\prime}=\Gamma-L$, so that $\operatorname{ord}_{V} P^{\prime}=$ $\operatorname{ord}_{V} \Gamma-\operatorname{ord}_{V} L=0$. Thus $V \nsubseteq B s(|P|)$.

The following theorem is a simplified version of [Amb05, theorem 2.1]:
Theorem 3.6.4 (Ambro). Let $(X, \Delta)$ be a pair and let $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $P$ is nef
2. There exists a rational number $t_{0}$ such that

$$
t_{0} P-\left(K_{X}+\Delta\right)
$$

is big and nef;
3. There exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right),
$$

for some $\mathbb{Q}$-divisor $\Delta_{-}$such that $\Delta_{-} \geq 0$ and $\Delta+\Delta_{-} \geq 0$;
4. $(X, \Delta)$ is $P-K L T$;
then $P$ is semiample.

Proof. Let $\mathbb{N}(P)=\left\{m \in \mathbb{N}\right.$ such that $m P \in \operatorname{Div}(X)$ and $H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \neq$ $0\}$. Then $\mathbb{N}(P) \neq \emptyset$ :
In fact if $C L C(X, \Delta) \neq \emptyset$, then $\mathbb{B}(P) \neq X$, while if $C L C(X, \Delta)=\emptyset$, that is $(X, \Delta)$ is KLT, then $H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right) \neq 0$ for all $m \gg 0$ by Shokurov's nonvanishing (see [KM00, theorem 3.4]), so that, by hypothesis, $H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \neq 0$.

Now suppose, by contradiction, $\mathbb{B}(P) \neq \emptyset$ and let $m_{1} \in \mathbb{N}(P)$ be such that $B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P) \neq \emptyset$.
We claim that there exists $\mu: Y \rightarrow X$, a log-resolution of the pair $(X, \Delta)$ and of the linear series $\left|m_{1} P\right|$, and a SNC divisor $F=\sum_{j=1}^{N} F_{j}$ on $Y$ such that

1. $\operatorname{Supp}\left(\mu_{*}^{-1}(\Delta)\right) \cup \operatorname{divexc}(\mu) \subseteq \operatorname{Supp}(F)$;
2. For every $j=1, \ldots, N$ there exist $r_{j} \geq 0$ such that $\mu^{*}\left(m_{1} P\right)=L+\sum r_{j} F_{j}$ and $\mu^{*}\left|m_{1} P\right|=|L|+\sum r_{j} F_{j}$, where $L$ is an integral base-point free divisor on $Y$.
3. For every $j=1, \ldots, N$ there exist arbitrarily small rational numbers $\delta_{j}>$ 0 such that $\mu^{*}\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\sum \delta_{j} F_{j}$ is ample.

In fact, thanks to [KMM85, corollary 0.3.6], there exists a proper birational morphism $f: X^{\prime} \rightarrow X$, with $X^{\prime}$ smooth, and a SNC divisor $\sum A_{k}$ on $X^{\prime}$ such that $f^{*}\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\sum \epsilon_{k} A_{k}$ is ample for $\epsilon_{k} \in \mathbb{Q}^{+}$arbitrarily small.
Let $g: X^{\prime \prime} \rightarrow X^{\prime}$ be a log-resolution of the linear series $\left|f^{*}\left(m_{1} P\right)\right|$, so that $\left|g^{*}\left(f^{*}\left(m_{1} P\right)\right)\right|=|M|+\sum s_{l} B_{l}$, where $M$ is an integral free divisor on $X^{\prime \prime}$ and $\sum B_{l}$ is a SNC divisor on $X^{\prime \prime}$.
Now take
$F^{\prime}=\operatorname{Supp}\left(g_{*}^{-1}\left(f_{*}^{-1} \Delta\right)\right)+\operatorname{Supp}\left(\sum s_{l} B_{l}\right)+\operatorname{Supp}\left(g_{*}^{-1} \sum \epsilon_{k} A_{k}\right)+\operatorname{divexc}(f \circ g)$
and let $h: Y \rightarrow X^{\prime \prime}$ be a log-resolution of the pair $\left(X^{\prime \prime}, F^{\prime}\right)$.
We define $F=h_{*}^{-1} F^{\prime}+\operatorname{divexc}(h)$ and let $\mu=f \circ g \circ h: Y \rightarrow X$. Then $F$ is SNC thanks to its definition, $\mu$ is a log-resolution of $(X, \Delta)$ because $\operatorname{Supp}\left(\mu_{*}^{-1}(\Delta)\right) \cup$ divexc $\mu \subseteq \operatorname{Supp}(F)$. In particular note that the assertion 1 holds. Moreover

$$
\left|\mu^{*}\left(m_{1} P\right)\right|=\left|h^{*}(M)\right|+h^{*}\left(\sum s_{l} B_{l}\right)
$$

where $\left|h^{*}(M)\right|$ is free and $\operatorname{Supp}\left(h^{*}\left(\sum s_{l} B_{l}\right)\right) \subseteq \operatorname{Supp}(F)$. Hence $\mu$ is a logresolution of $\left|m_{1} P\right|$ and 2 holds.
As for the assertion 3, let divexc $(g \circ h)=\sum C_{s}$. Note that

$$
\begin{aligned}
& h^{*} g^{*}\left(f^{*}\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\sum \epsilon_{k} A_{k}\right)-\sum \gamma_{s} C_{s}= \\
= & \mu^{*}\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\left(h^{*}\left(g^{*}\left(\sum \epsilon_{k} A_{k}\right)\right)+\sum \gamma_{s} C_{s}\right)
\end{aligned}
$$

is ample for rational numbers $0 \leq \epsilon_{k} \ll 1,0 \leq \gamma_{s} \ll 1$, because $g \circ h$ is a morphism between smooth projective varieties (so that we can apply [KM00, lemma $2.62+$ Prop. 1.45]).
But $\operatorname{Supp}\left(h^{*}\left(g^{*}\left(\sum \epsilon_{k} A_{k}\right)\right)+\sum \gamma_{s} C_{s}\right) \subseteq \operatorname{Supp}(F)$, and the coefficients can be made arbitrarily small by suitably choosing the $\epsilon_{k}$ 's and the $\gamma_{s}$ 's.

In particular we can suppose all the $\delta_{j}$ 's are strictly positive thanks to the openness of the ample cone. Thus the claim is proved.

Now, thanks to the claim, we can write

$$
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum_{j=1}^{N} b_{j} F_{j}
$$

where $\mu\left(F_{j}\right) \in C L C(X, \Delta)$ if $b_{j} \leq-1$.
On the other hand we have that $B s\left(\left|m_{1} P\right|\right)=\mu\left(\bigcup_{r_{j} \neq 0} F_{j}\right)$, so that we can suppose $r_{j}>0$ for some $j$ because $B s\left(\left|m_{1} P\right|\right) \neq \emptyset$.
Moreover if $r_{j} \neq 0$, then $\mu\left(F_{j}\right) \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$, which implies that $b_{j}>-1$, as $\mathbb{B}(P)$ does not contain any LC center of the pair $(X, \Delta)$, as $(X, \Delta)$ is $P$-KLT.
Now we define

$$
c=\min _{\left\{j: r_{j} \neq 0\right\}} \frac{b_{j}+1-\delta_{j}}{r_{j}} .
$$

By choosing the $\delta_{j}$ 's small enough we can suppose that

$$
b_{j}+1-\delta_{j}>0 \text { for all } j \text { such that } b_{j}>-1
$$

Hence $c>0$ because $b_{j}>-1$ if $r_{j} \neq 0$.
Moreover, perturbing slightly the $\delta_{j}$ 's if necessary, we can suppose that the minimum is attained on a unique $j$, say $j=j_{0}$. Define $B:=F_{j_{0}}$.
Now let $s:=t_{0}+c m_{1}$, and let

$$
B_{m}:=\mu^{*}\left(\left(m-c m_{1}\right) P-\left(K_{X}+\Delta\right)\right)-\sum \delta_{j} F_{j} .
$$

Then $B_{m}$ is ample for every integer $m \geq s$, because $m-c m_{1} \geq t_{0}$ and $\mu^{*}(P)$ is nef.
Define now

$$
A^{\prime}:=\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}
$$

and, for every $m \in \mathbb{N}(P)$, let

$$
Q_{m}:=\mu^{*}(m P)+A^{\prime}-B-K_{Y}
$$

Then

$$
\begin{gathered}
Q_{m}:=\mu^{*}(m P)+A^{\prime}-B-K_{Y} \equiv \\
\equiv \mu^{*}(m P)+\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}-F_{j_{0}}-\sum b_{j} F_{j}-\mu^{*}\left(K_{X}+\Delta\right)= \\
=\mu^{*}\left(m P-\left(K_{X}+\Delta\right)\right)-\sum_{j \neq j_{0}} c r_{j} F_{j}-\sum_{j \neq j_{0}} \delta_{j} F_{j}+F_{j_{0}}\left(-1-b_{j_{0}}\right)= \\
=\mu^{*}\left(\left(m-c m_{1}\right) P-\left(K_{X}+\Delta\right)\right)+c L+F_{j_{0}}\left(c r_{j_{0}}-1-b_{j_{0}}\right)-\sum_{j \neq j_{0}} \delta_{j} F_{j}= \\
=\mu^{*}\left(\left(m-c m_{1}\right) P-\left(K_{X}+\Delta\right)\right)+c L-\sum \delta_{j} F_{j}=B_{m}+c L .
\end{gathered}
$$

Hence $Q_{m}$ is ample for every integer $m \geq s$ because $B_{m}$ is such and $L$ is free.

Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that $H^{1}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime}\right\urcorner-B\right)\right)=0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.
This implies that the restriction homomorphism

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime}\right\urcorner\right)\right) \rightarrow H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime}\right\urcorner\right)\right)
$$

is surjective in this case.
Let $m_{2}=\min \{m \in \mathbb{N}(P)$ such that $m \geq s\}$. Then $\mu^{*}\left(m_{2} P\right)_{\left.\right|_{B}}+A_{\left.\right|_{B}}^{\prime}-K_{B}=$ $Q_{\left.m_{2}\right|_{B}}$ is an ample $\mathbb{Q}$-divisor.
Moreover $A_{\left.\right|_{B}}^{\prime}$ is SNCS because $A^{\prime}$ is such and $B$ intersects transversally all the $F_{j}$ 's with $j \neq j_{0}$. Hence, in order to show that the pair $\left(B,-A_{\left.\right|_{B}}^{\prime}\right)$ is KLT, it suffices to verify that all the coefficients of $A_{\left.\right|_{B}}^{\prime}$ are greater than -1 .
But we have that

- if $b_{j}>-1$ and $r_{j}=0$ then $-c r_{j}+b_{j}-\delta_{j}=b_{j}-\delta_{j}>-1$ (by the choice of the $\delta_{j}$ 's);
- if $b_{j}>-1, r_{j} \neq 0$ and $j \neq j_{0}$ then $-c r_{j}+b_{j}-\delta_{j}>-\frac{b_{j}+1-\delta_{j}}{r_{j}} r_{j}+b_{j}-\delta_{j}=$ -1 .
- if $b_{j} \leq-1$ then $\mu\left(F_{j}\right) \in C L C(X, \Delta)$. On the other hand $\mu(B) \subseteq \mathbb{B}(P)$, so that $\mu(B) \cap \mu\left(F_{j}\right)=\emptyset$ because $C L C(X, \Delta, \mathbb{B}(P))=\emptyset$. Then $B \cap F_{j}=\emptyset$, that is $F_{\left.j\right|_{B}}=0$.
Hence the pair $\left(B,-A_{\left.\right|_{B}}^{\prime}\right)$ is KLT.
Thus, by Shokurov's nonvanishing theorem ([KM00, theorem 3.4]), for every integer $k>0$ there exists $\mu_{k} \in \mathbb{N}$, such that $\mu_{k} \geq a, \mu_{k} \geq s, \mu_{k}$ is a multiple of $k$ and

$$
H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}\left(\mu_{k} P\right)+\left\ulcorner A^{\prime}\right\urcorner\right)\right) \neq 0
$$

In fact this cohomology group is non zero for every sufficiently large multiple of $m_{2}$.
Now, by hypothesis, there exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is integral and

$$
H^{0}\left(X, \mathcal{O}_{X}\left(t k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(t k_{0} P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)
$$

for every $t \in \mathbb{N}$.
Let us define $m:=\mu_{k_{0}}$. Then
$B \nsubseteq B s\left(\left|\mu^{*}(m P)+\left\ulcorner A^{\prime}\right\urcorner\right|\right)$ and $H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)$.
Now define

$$
T=\sum_{j \neq j_{0}} \max \left\{0,\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner\right\} F_{j} .
$$

Then $T$ is an effective Cartier divisor on $X^{\prime},\left\ulcorner A^{\prime}\right\urcorner \leq T$ by definition and we have that $\mu_{*} T \leq\left\ulcorner\Delta_{-}\right\urcorner$:
In fact note that, by its choice in the hypothesis, $\left\ulcorner\Delta_{-}\right\urcorner \geq\ulcorner-\Delta\urcorner$ and $\Delta_{-} \geq 0$. Moreover by definition of the $b_{j}$ 's it follows that

$$
-\Delta=\sum_{j=1}^{N} b_{j} \mu_{*}\left(F_{j}\right)
$$

Hence

$$
\left\ulcorner\Delta_{-}\right\urcorner \geq \sum_{j=1}^{N} \max \left\{0,\left\ulcorner b_{j}\right\urcorner\right\} \mu_{*}\left(F_{j}\right)
$$

Thus
$\mu_{*}(T)=\sum_{j \neq j_{0}} \max \left\{0,\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner\right\} \mu_{*}\left(F_{j}\right) \leq \sum_{j \neq j_{0}} \max \left\{0,\left\ulcorner b_{j}\right\urcorner\right\} \mu_{*}\left(F_{j}\right) \leq\left\ulcorner\Delta_{-}\right\urcorner$
From these inequalities, thanks to lemma 2.1.8 it follows that

$$
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+T\right) \leq h^{0}\left(X, \mathcal{O}_{X}\left(m P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)\right.
$$

But, by hypothesis,

$$
h^{0}\left(X, \mathcal{O}_{X}\left(m P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)=h^{0}\left(X, \mathcal{O}_{X}(m P)\right)=h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right)
$$

Then the injection

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right) \hookrightarrow H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+T\right)\right)
$$

is an isomorphism. Note also that $B \nsubseteq \operatorname{Supp}\left(\left\ulcorner A^{\prime}\right\urcorner\right)$ by definition.
Hence we can apply lemma 3.6.3 and we get that $B \nsubseteq B s\left(\left|\mu^{*}(m P)\right|\right)$, which implies that $\mu(B) \nsubseteq B s(|m P|)$. Therefore, as $\mu(B) \subseteq B s\left(\left|m_{1} P\right|\right)$, we get that $B s(|m P|)$ does not contain $B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$, and we find a contradiction.

We can easily apply the previous theorem to the positive part of a $\mathbb{Q}$-CKM Zariski decomposition:
Theorem 3.6.5 (Ambro). Let $X$ be a normal projective variety and let $\Delta$ be an effective Weil $\mathbb{Q}$-divisor. Let $D$ be a Weil $\mathbb{Q}$-divisor such that

1. There exists a $\mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

2. There exist a and $t_{0}$, rational numbers, with $a \geq 0$, such that $(X, \Delta-a N)$ is a $P$-KLT pair and

$$
t_{0} P-\left(K_{X}+\Delta-a N\right)
$$

is big and nef,
then $P$ is semiample.
Proof. Let $B=\Delta-a N$ and let $B_{-}=a N$. Then $B+B_{-}=\Delta$ is effective and $t_{0} P-\left(K_{X}+B\right)$ is big and nef.
Moreover, by definition of $\mathbb{Q}$-CKM Zariski decomposition there exists $k_{0} \in \mathbb{N}$ such that $k_{0}>a, k_{0} P$ is a Cartier divisor, $k_{0} D$ is integral and

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} D\right)\right)
$$

for all $m \in \mathbb{N}$.
But $\left\ulcorner B_{-}\right\urcorner=\ulcorner a N\urcorner \leq k_{0} N$. Hence, for all $m \in \mathbb{N}$, we get that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner B_{-}\right\urcorner\right)\right)
$$

Thus we can apply theorem 3.6.4 and we get the semiampleness of $P$.

Corollary 3.6.6. Let $(X, \Delta)$ be an effective pair and let $D \in \operatorname{Div}(X)$. Consider the following assumptions:

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \in \mathbb{Q}$;
(2') $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \in \mathbb{Q}$;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)$ admits $a \mathbb{Q}$-CKM Zariski decomposition

$$
f^{*}(D)=P+N
$$

and $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is $P-K L T$.
If $D$ satisfies 1,2, and 3, or $D$ satisfies 2' and 3, then $P$ is semiample.
Proof. Let us apply lemma 2.5.3, and consider $t_{0} \in \mathbb{Q}$ and $D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ Weil $\mathbb{Q}$-divisors on $Z$ as in the lemma. Then $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)$ is big and nef and $\left(Z, \Delta_{Z}-N^{\prime}\right)$ is $P^{\prime}$-KLT.
Thus we can apply theorem 3.6.5 and we are done.

## Using Ambro's theorem in the $\mathbb{Q}$-Gorenstein case

Let us consider now the $\mathbb{Q}$-Gorenstein case. Making use of Ambro's theorem we can give the following more general version of theorem 3.5.3:

Theorem 3.6.7. Let $(X, \Delta)$ be an effective LC pair such that $X$ is $\mathbb{Q}$-Gorenstein. Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $D$ is big;
2. $\mathbb{B}_{+}(D) \nsupseteq V$, for every $V \in C L C(X, \Delta)$;
3. D has a $\mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

such that $\mathbb{B}(P) \cap V=\emptyset$ for every $V \in C L C(X, \Delta)$ such that $V \nsubseteq \operatorname{Supp}(\Delta)$;
then there exists $\beta>0$ such that if

$$
a D-\left(K_{X}+\Delta\right) \text { is nef for some rational number } a>-\beta
$$

then $P$ is semiample.
Proof. Note that $P$ is big because $D$ is such and $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)$ by lemma 2.5.4. Then, thanks to lemma 2.3.7, we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $P-\lambda \Gamma$ is ample, the pair $(X, \Delta+\lambda \Gamma)$ is LC and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$.
Now, as $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}(D)$ by lemma 2.5.4, thanks to lemma 2.2 .26 there exists $\beta \in \mathbb{Q}^{+}$such that if $0 \leq \beta^{\prime}<\beta$, then the pair $\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)$ is LC and $C L C\left(X, \Delta+\lambda \Gamma+\beta^{\prime} N\right)=C L C(X, \Delta)$.
Suppose $a>-\beta$ is a rational number such that $a D-\left(K_{X}+\Delta\right)$ is nef.

Define $a^{\prime}=-\min \{0, a\}, a^{\prime \prime}=\max \{0, a\}$, so that $a=a^{\prime \prime}-a^{\prime}, a^{\prime \prime} \geq 0,0 \leq a^{\prime}<\beta$. Moreover we define $\Delta^{\prime}=\Delta+\lambda \Gamma+a^{\prime} N$, so that $\Delta^{\prime}$ is effective, $\left(X, \Delta^{\prime}\right)$ is LC and $C L C\left(X, \Delta^{\prime}\right)=C L C(X, \Delta)$.
Hence, if $\Delta \neq 0$, thanks to lemma 2.2.9 and lemma 2.2.8, we get that for every $\epsilon \in \mathbb{Q}^{+}$

$$
C L C\left(X, \Delta^{\prime}-\epsilon \Delta-a^{\prime \prime} N\right) \subseteq\{V \in C L C(X, \Delta) \text { such that } V \nsubseteq \operatorname{Supp}(\Delta)\}
$$

so that, by hypothesis, $\mathbb{B}(P)$ does not intersect any LC center of the pair $\left(X, \Delta^{\prime}-\right.$ $\left.\epsilon D-a^{\prime \prime} N\right)$.
If $\Delta=0$, then $\operatorname{Supp}(0)=\emptyset$, so that the same result holds by hypothesis because, by lemma 2.2.8, $C L C\left(X, \Delta^{\prime}-\epsilon \Delta-a^{\prime \prime} N\right) \subseteq C L C\left(X, \Delta^{\prime}\right)=C L C(X, 0)$.
Moreover

$$
\begin{gathered}
(1+a) P+a^{\prime \prime} N-\left(K_{X}+\Delta^{\prime}-\epsilon \Delta\right)=(1+a) P+a^{\prime \prime} N-\left(K_{X}+\Delta+\lambda \Gamma+a^{\prime} N-\epsilon \Delta\right)= \\
=(P-\lambda \Gamma)+\left(a D-\left(K_{X}+\Delta\right)\right)+\epsilon \Delta
\end{gathered}
$$

is ample if $\epsilon$ is sufficiently small thanks to the openness of the ample cone.
Thus we obtain the semiampleness of $P$ by applying theorem 3.6.5 to the pair $\left(X, \Delta^{\prime}-\epsilon \Delta\right)$.

### 3.7 Nklt and $\mathrm{Nklt}_{2}$ : Dimension 3

In this section we generalize the results obtained so far towards Conjecture 1 and Conjecture 2 by making a more careful study of the non-klt locus of the given pair $(X, \Delta)$. As a corollary we obtain Conjecture 1 b for varieties of dimension less than or equal to 3 (cf. corollary 3.7.8).

Definition 3.7.1. Let $(X, \Delta)$ be a pair, then we define

$$
\widetilde{\operatorname{Nklt}}(X, \Delta)=\bigcup_{\substack{V \in C L C(X, \Delta) \\ V \subseteq N S N C(\Delta) \cup \operatorname{Sing}(X)}} V .
$$

Definition 3.7.2. Let $(X, \Delta)$ be a pair such that $\operatorname{dim} X=n$. For every integer $k \in\{1, \ldots, n\}$ we define

$$
\operatorname{Nklt}_{k}(X, \Delta)=\bigcup_{\substack{V \in C L C(X, \Delta) \\ \operatorname{dim} V \leq n-k}} V
$$

Remark 3.7.3. Given a pair $(X, \Delta)$ note that, by definition, $\operatorname{Nklt}(X, \Delta)=$ $\operatorname{Nklt}_{1}(X, \Delta)$.
Note also that $\widetilde{\operatorname{Nklt}}(X, \Delta) \subseteq \operatorname{Nklt}_{2}(X, \Delta)$ because of the normality of $X$. Moreover if $(X, \Delta)$ is DLT then $\widetilde{\operatorname{Nklt}}(X, \Delta)=\emptyset$ by theorem 2.2.23.

Theorem 3.7.4. Let $(X, \Delta)$ be a pair and suppose that $\Delta=\sum_{i \in I} d_{i} D_{i}$, where all the $D_{i}$ 's are distinct prime divisors and $d_{i} \leq 1$ for every $i \in I$.
Moreover suppose that $P \in \operatorname{Div}_{\mathbb{Q}}(X)$ and we can write $\Delta=\Delta_{+}-\Delta_{-}$, where $\Delta_{+}$and $\Delta_{-}$are effective $\mathbb{Q}$-divisors and the following properties are satisfied:

1. $P$ is nef;
2. $t_{0} P-\left(K_{X}+\Delta\right)$ is ample for some $t_{0} \in \mathbb{Q}^{+}$;
3. There exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is a Cartier divisor and for all $m \in \mathbb{N}$ it holds that

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)
$$

4. $\widetilde{\operatorname{Nklt}}(X, \Delta)=\emptyset$, or $P_{\left.\right|_{\widetilde{\operatorname{NkIt}}(X, \Delta)}}$ is semiample;
5. There exists $\mu: X^{\prime} \rightarrow X$, a standard log-resolution of the pair $(X, \Delta)$ such that $a(E, X, \Delta)>-2$ for every prime divisor $E \subseteq X^{\prime}$.

Then $P$ is semiample.
Proof. Let $\mu$ be as in the hypothesis. Note that $\operatorname{Nklt}\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right)=$ $\operatorname{Supp}\left(\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{\geq 1}\right)$, because $X^{\prime}$ is smooth and $\mathbf{A}(\Delta)_{X^{\prime}}$ is SNCS.
Now by the ampleness of $t_{0} P-\left(K_{X}+\Delta\right)$, thanks to lemma 2.1.9, for all $\mu$ exceptional divisors $E_{1}, \ldots, E_{s}$ on $X^{\prime}$ there exist arbitrarily small coefficients $\delta_{1}, \ldots, \delta_{s} \in \mathbb{Q}^{+}$, such that

$$
\mu^{*}\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\sum_{j=1}^{s} \delta_{j} E_{j}
$$

is ample.
Then, if $0 \leq \epsilon \ll 1$ we have that

$$
\mu^{*}\left(t_{0} P\right)-\left(K_{X^{\prime}}+(1-\epsilon) \mathbf{A}(\Delta)_{X^{\prime}}+\sum_{j=1}^{s} \delta_{j} E_{j}\right)
$$

is still ample.
Now for every $\epsilon$ sufficiently small such that the above condition holds we define

$$
\widehat{\Delta}_{\epsilon}=(1-\epsilon) \mathbf{A}(\Delta)_{X^{\prime}}+\sum \delta_{j} E_{j}
$$

so that $\mu^{*}(m P)-\left(K_{X^{\prime}}+\widehat{\Delta}_{\epsilon}\right)$ is ample for every integer $m \geq t_{0}$ thanks to the nefness of $P$.
Now we can write

$$
\mathbf{A}(\Delta)_{X^{\prime}}=\sum_{k \in K} c_{k} X_{k}+\sum_{l \in L} a_{l} Y_{l}-\sum_{m \in M} b_{m} Z_{m}
$$

where, for every $k \in K, l \in L$ and $m \in M$, we have that $X_{k}, Y_{l}, Z_{m}$ are pairwise distinct prime divisors, and

- $b_{m}>0 \quad \forall m \in M$;
- $0 \leq a_{l}<1 \quad \forall l \in L$;
- $1 \leq c_{k}<2 \quad \forall k \in K$ :

In fact all the coefficients of $\mathbf{A}(\Delta)_{X^{\prime}}$ are smaller than 2 because of the choice of $\mu$.
Moreover we can suppose that $\operatorname{divexc}(\mu) \subseteq \operatorname{Supp}\left(\sum X_{k}+\sum Y_{l}+\sum Z_{m}\right)$, by considering among the $Y_{l}$ 's also the prime $\mu$-exceptional divisors not appearing in $\operatorname{Supp}\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)$, with coefficient 0 .
Now let us define

$$
\Delta_{+}^{\prime}:=\sum_{k \in K} c_{k} X_{k}+\sum_{l \in L} a_{l} Y_{l} ; \quad \Delta_{-}^{\prime}:=\sum_{m \in M} b_{m} Z_{m},
$$

so that $\Delta_{+}^{\prime}$ and $\Delta_{-}^{\prime}$ are effective, they have no common components and $\mathbf{A}(\Delta)_{X^{\prime}}=\Delta_{+}^{\prime}-\Delta_{+}^{\prime}$. Moreover for every $k \in K, l \in L$ and $m \in M$ we define
$\gamma_{k}=\left\{\begin{array}{ll}\delta_{j} & \text { if } X_{k}=E_{j} \\ 0 & \text { otherwise }\end{array} ; \quad \gamma_{l}=\left\{\begin{array}{ll}\delta_{j} & \text { if } Y_{l}=E_{j} \\ 0 & \text { otherwise }\end{array} ; \quad \gamma_{m}= \begin{cases}\delta_{j} & \text { if } Z_{m}=E_{j} \\ 0 & \text { otherwise }\end{cases}\right.\right.$
so that we can write
$\widehat{\Delta}_{\epsilon}=\sum_{k \in K}\left((1-\epsilon) c_{k}+\gamma_{k}\right) X_{k}+\sum_{l \in L}\left((1-\epsilon) a_{l}+\gamma_{l}\right) Y_{l}-\sum_{m \in M}\left((1-\epsilon) b_{m}-\gamma_{m}\right) Z_{m}$.
Now we choose $\epsilon$ and the $\delta_{j}$ 's small enough such that the following inequalities hold:

- $c_{k}^{\prime}:=(1-\epsilon) c_{k}+\gamma_{k}<2 \quad \forall k \in K ;$
- $a_{l}^{\prime}:=(1-\epsilon) a_{l}+\gamma_{l}<1 \quad \forall l \in L ;$
- $b_{m}^{\prime}:=(1-\epsilon) b_{m}-\gamma_{m}>0 \quad \forall m \in M$,
and we define $\widehat{\Delta}:=\widehat{\Delta}_{\epsilon}$. Hence $\widehat{\Delta}=\sum c_{k}^{\prime} X_{k}+\sum a_{l}^{\prime} Y_{l}-\sum b_{m}^{\prime} Z_{m}$, and
- $0<c_{k}^{\prime}<2 \quad \forall k \in K ;$
- $0 \leq a_{l}^{\prime}<1 \quad \forall l \in L ;$
- $0<b_{m}^{\prime} \leq b_{m} \quad \forall m \in M$.

Note that

$$
\mu^{*}(m P)-\left(K_{X^{\prime}}+\widehat{\Delta}\right)
$$

is ample for every integer $m \geq t_{0}$ and $\widehat{\Delta}$ is a SNCS divisor because $\operatorname{Supp}(\widehat{\Delta}) \subseteq$ $\operatorname{Supp}\left(\mathbf{A}(\Delta)_{X^{\prime}}\right) \cup \operatorname{divexc}(\mu)$, so that $\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)=\operatorname{Supp}((\widehat{\Delta}) \geq 1)$.
Now let us define

$$
\widehat{\Delta}_{+}:=\sum c_{k}^{\prime} X_{k}+\sum a_{l}^{\prime} Y_{l} ; \quad \widehat{\Delta}_{-}:=\sum b_{m}^{\prime} Z_{m}
$$

so that $\widehat{\Delta}_{+}$and $\widehat{\Delta}_{-}$are effective and $\widehat{\Delta}=\widehat{\Delta}_{+}-\widehat{\Delta}_{-}$.
We claim that $\mu_{*}\left\ulcorner\widehat{\Delta}_{-}\right\urcorner \leq\left\ulcorner\Delta_{-}\right\urcorner$:
In fact, note that $\widehat{\Delta}_{-} \leq \Delta_{-}^{\prime}$, so that it suffices to show that $\mu_{*}\left\ulcorner\Delta_{-}^{\prime}\right\urcorner \leq\left\ulcorner\Delta_{-}\right\urcorner$. In particular we will show that $\mu_{*} \Delta_{-}^{\prime} \leq \Delta_{-}$, which implies that $\mu_{*}\left\ulcorner\Delta_{-}^{\prime}\right\urcorner=$ $\left\ulcorner\mu_{*} \Delta_{-}^{\prime}\right\urcorner \leq\left\ulcorner\Delta_{-}\right\urcorner$.

Now the required inequality holds because, by definition,

$$
\Delta_{-}^{\prime}=\sum_{a(E, X, \Delta)>0} a(E, X, \Delta) E .
$$

Hence

$$
\mu_{*}\left(\Delta_{-}^{\prime}\right)=\sum_{a\left(\mu_{*}^{-1} D_{i}, X, \Delta\right)>0} a\left(\mu_{*}^{-1} D_{i}, X, \Delta\right) D_{i}=\sum_{d_{i}<0}-d_{i} D_{i} \leq \Delta_{-}
$$

because $\Delta_{-}$is effective and $\Delta_{-}=\Delta_{+}-\Delta$, so that, for every $i$, we have that

$$
\operatorname{ord}_{D_{i}} \Delta_{-}=\operatorname{ord}_{D_{i}} \Delta_{+}-d_{i} \geq-d_{i}
$$

Thus the claim is proved.
Thanks to the claim we can use lemma 2.1.8 and we obtain that if $k_{0}$ is as in the hypothesis, then

$$
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{0} m P\right)+\left\ulcorner\widehat{\Delta}_{-}\right\urcorner\right)\right) \leq h^{0}\left(X, \mathcal{O}_{X}\left(k_{0} m P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)
$$

for all $m \in \mathbb{N}$. But, by hypothesis,

$$
h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{0} m P\right)\right)\right)=h^{0}\left(X, \mathcal{O}_{X}\left(k_{0} m P\right)\right)=h^{0}\left(X, \mathcal{O}_{X}\left(k_{0} m P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)
$$

for all $m \in \mathbb{N}$. Therefore

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{0} m P\right)\right)\right) \simeq H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{0} m P\right)+\left\ulcorner\widehat{\Delta}_{-}\right\urcorner\right)\right)
$$

for all $m \in \mathbb{N}$.
We will show the semiampleness of $P$ by applying theorem 3.6.4 to the pair $\left(X^{\prime}, \widehat{\Delta}\right)$ and the divisor $\mu^{*}(P)$. This will imply the semiampleness of $\mu^{*}(P)$, leading, in such a way, to the semiampleness of $P$ itself.
In particular, in order to apply the theorem it remains to show that $\left(X^{\prime}, \widehat{\Delta}\right)$ is a $\mu^{*}(P)$-KLT pair. In other words we have to prove that $\mathbb{B}\left(\mu^{*}(P)\right) \cap$ $\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)=\emptyset:$
Note that

$$
\begin{array}{r}
\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)=\operatorname{Supp}\left((\widehat{\Delta})^{\geq 1}\right) \subseteq \bigcup_{k \in K} X_{k}=\operatorname{Supp}\left(\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{\geq 1}\right)= \\
=\operatorname{Nklt}\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right)
\end{array}
$$

Moreover $\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right) \subseteq \operatorname{exc}(\mu)$ :
In fact if $k \in K$ is such that $X_{k}$ is not exceptional, then $X_{k}=\mu_{*}^{-1} G$, for some prime divisor $G$ on $X$. Then $c_{k}=a\left(\mu_{*}^{-1} G, X, \Delta\right)=-\operatorname{ord}_{G} \Delta \geq-1$, thanks to the hypotheses on $\Delta$.
On the other hand $\gamma_{k}=0$ because $X_{k}$ is not exceptional, so that

$$
c_{k}^{\prime}=(1-\epsilon) c_{k}<c_{k} \leq 1
$$

Thus we get that $\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right) \subseteq \operatorname{Nklt}\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right) \cap \operatorname{exc}(\mu)$.

Now we define

$$
T=\sum_{c_{k}^{\prime} \geq 1} X_{k}
$$

so that $T$ is reduced and $T=\operatorname{Supp}\left((\widehat{\Delta})^{\geq 1}\right)=\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)$.
In particular $T \subseteq \operatorname{Nklt}\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right) \cap \operatorname{exc}(\mu)$.
Let $T_{0}$ be a prime divisor in the support of $T$. Then, on the one hand, $T_{0} \subseteq \operatorname{Supp}\left(\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{\geq 1}\right)$, that is $a\left(T_{0}, X, \Delta\right) \leq-1$, which implies that $\mu\left(T_{0}\right) \in$ $C L C(X, \Delta)$.
On the other hand $T_{0} \subseteq \operatorname{exc}(\mu)$ implies that $\mu\left(T_{0}\right) \subseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$, because $\mu$ is a standard log-resolution of the pair $(X, \Delta)$.
Hence we get that $\mu\left(T_{0}\right) \subseteq \widetilde{\operatorname{Nklt}}(X, \Delta)$. But the same holds for every component of $T$, so that we have

$$
\mu(T) \subseteq \widetilde{\operatorname{Nklt}}(X, \Delta)
$$

If $\widetilde{\operatorname{Nklt}}(X, \Delta)=\emptyset$, then $\mu(T)=\emptyset$, so that $T=\operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)=\emptyset$ and there is nothing to prove. We can thus assume that $\widetilde{\operatorname{Nklt}}(X, \Delta) \neq \emptyset$.
Then, as by hypothesis $P_{l_{\sqrt[N k l t]{ }(X, \Delta)}}$ is semiample, we get that $P_{\left.\right|_{\mu(T)}}$ is semiample. Now we consider the commutative diagram:


As $P_{\left.\right|_{\mu(T)}}$ is semiample, we have that $\mu_{\left.\right|_{T}}^{*}\left(P_{\left.\right|_{\mu(T)}}\right)$ is semiample, which, by commutativity of the diagram, implies that $\mu^{*}(P)_{\left.\right|_{T}}$ is semiample.

Now we claim that $\ulcorner-\widehat{\Delta}\urcorner=\left\ulcorner\widehat{\Delta}_{-}\right\urcorner-T$ :
In fact

$$
\ulcorner-\widehat{\Delta}\urcorner=\sum_{m \in M}\left\ulcorner b_{m}^{\prime}\right\urcorner Z_{m}+\sum_{k \in K}\left\ulcorner-c_{k}^{\prime}\right\urcorner X_{k}+\sum_{l \in L}\left\ulcorner-a_{l}^{\prime}\right\urcorner Y_{l} .
$$

But, for all $l \in L$, we have that $0 \geq-a_{l}^{\prime}>-1$, so that $\left\ulcorner-a_{l}^{\prime}\right\urcorner=0$.
Moreover for all $k \in K, 0>-c_{k}^{\prime}>-2$, so that

$$
\left\ulcorner-c_{k}^{\prime}\right\urcorner= \begin{cases}-1 & \text { if } c_{k}^{\prime} \geq 1 \\ 0 & \text { if } c_{k}^{\prime}<1\end{cases}
$$

Thus

$$
\ulcorner-\widehat{\Delta}\urcorner=\sum_{m \in M}\left\ulcorner b_{m}^{\prime}\right\urcorner Z_{m}-\sum_{c_{k}^{\prime} \geq 1} X_{k}=\left\ulcorner\widehat{\Delta}_{-}\right\urcorner-T,
$$

and the claim is proved.
Take $k_{1} \in \mathbb{N}$ such that $k_{1}>t_{0}$ and $k_{1}$ is a multiple of $k_{0}$, so that $k_{1} P$ is a Cartier divisor and

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{1} m P\right)\right)\right) \simeq H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}\left(k_{1} m P\right)+\left\ulcorner\widehat{\Delta}_{-}\right\urcorner\right)\right)
$$

for every $m \in \mathbb{N}$. Let us consider, for every $k \in k_{1} \mathbb{N}$, the following commutative diagram:

where the vertical arrow on the left is an isomorphism thanks to the choice of $k_{1}$.
Note that $i_{k}$ is injective for every $k \in k_{1} \mathbb{N}$ because $\left\ulcorner\widehat{\Delta}_{-}\right\urcorner_{\left.\right|_{T}}$ is effective:
In fact $\left\ulcorner\widehat{\Delta}_{-}\right\urcorner$is effective and $\operatorname{Supp}\left(\left\ulcorner\widehat{\Delta}_{-}\right\urcorner\right)=\operatorname{Supp}\left(\widehat{\Delta}_{-}\right)=\cup Z_{m}$ does not contain any component of $T$.
Let us prove that $\beta_{k}$ is surjective for every $k \in k_{1} \mathbb{N}$. In particular we prove that $H^{1}\left(X^{\prime}, \mathcal{O}_{X^{\prime}}\left(\mu^{*}(k P)+\left\ulcorner\widehat{\Delta}_{-}\right\urcorner-T\right)\right)=0:$
Note that $\mu^{*}(k P)-\left(K_{X^{\prime}}+\widehat{\Delta}\right)$ is ample, thanks to the choice of $k_{1}$, and $\left\{\mu^{*}(k P)-\right.$ $\left.\left(K_{X^{\prime}}+\widehat{\Delta}\right)\right\}=\{-\widehat{\Delta}\}$ is SNCS.
Then, by Kawamata-Viehweg vanishing theorem (see [Laz04, 9.1.20]), we get that $H^{1}\left(X^{\prime}, \mathcal{O}_{X^{\prime}}\left(\mu^{*}(k P)+\ulcorner-\widehat{\Delta}\urcorner\right)\right)=0$.
But $\ulcorner-\widehat{\Delta}\urcorner=\left\ulcorner\widehat{\Delta}_{-}\right\urcorner-T$. Then $H^{1}\left(X^{\prime}, \mathcal{O}_{X^{\prime}}\left(\mu^{*}(k P)+\left\ulcorner\widehat{\Delta}_{-}\right\urcorner-T\right)\right)=0$, as required.

By the commutativity of the diagram, the surjectivity of $\beta_{k}$ implies that $i_{k}$ is surjective, that is $i_{k}$ is an isomorphism. Thus $\alpha_{k}$ is also surjective for every $k \in k_{1} \mathbb{N}$.
But $\mu^{*}(P)_{\left.\right|_{T}}$ is semiample, whence there exists $k_{2} \in k_{1} \mathbb{N}$ such that $\mu^{*}\left(k_{2} P\right)_{\left.\right|_{T}}$ is base-point free.
Then the surjectivity of $\alpha_{k_{2}}$ implies that $B s\left(\mu^{*}\left(k_{2} P\right)\right) \cap T=\emptyset$.
Therefore $\mathbb{B}\left(\mu^{*}(P)\right) \cap \operatorname{Nklt}\left(X^{\prime}, \widehat{\Delta}\right)=\emptyset$, so that we can apply theorem 3.6.4 to the pair $(X, \widehat{\Delta})$ and the divisor $\mu^{*}(P)$ and we are done.

Corollary 3.7.5. Let $(X, \Delta)$ be an effective pair.
Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some $a \in \mathbb{Q}$;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is an LC pair;
- $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain any LC center of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-\right.$ $a N)$;
- $\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)=\emptyset$, or $P_{\left.\right|_{\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)}}$ is semiample.

Then $P$ is semiample.
We remark that if $a \geq 0$ the LCness of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ holds if we suppose that $(X, \Delta)$ is an LC pair.

Proof. Let us apply lemma 2.5 .3 and consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma, so that $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)$ is big and nef.
Note that $\mathbb{B}_{+}\left(P^{\prime}\right)=\mathbb{B}_{+}(P)=\mathbb{B}_{+}\left(f^{*}(D)\right)$ by lemma 2.5.4. Hence we can apply lemma 2.3.7 to the big and nef $\mathbb{Q}$-divisor $P^{\prime}$ and to the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)=$ $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ and we find a Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $P^{\prime}-\lambda \Gamma$ is ample, $\left(Z, \Delta_{Z}-N^{\prime}+\lambda \Gamma\right)$ is LC and $C L C\left(Z, \Delta_{Z}-N^{\prime}+\lambda \Gamma\right)=$ $C L C\left(Z, \Delta_{Z}-N^{\prime}\right)$.
Furthermore, we can choose $\Gamma$ generically in its linear series and we have that $B s(|\Gamma|)=\mathbb{B}_{+}\left(P^{\prime}\right)$. Then, by Bertini's theorem, we can suppose that, outside $\mathbb{B}_{+}\left(P^{\prime}\right), \Gamma$ is smooth and it intersects $\Delta_{Z}-N^{\prime}$ in a simple normal crossing way. Let us put $B=\Delta_{Z}-N^{\prime}+\lambda \Gamma$. We will show that the pair $(Z, B)$ and the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $P^{\prime}$ satisfy the hypotheses of theorem 3.7.4.
First of all we have that $\left(t_{0}+1\right) P^{\prime}-\left(K_{Z}+B\right)=\left(P^{\prime}-\lambda \Gamma\right)+\left(t_{0} P^{\prime}-\left(K_{Z}+\right.\right.$ $\left.\Delta_{Z}-N^{\prime}\right)$ ) is ample, so that property 2 holds.
By the LCness of the pair $(Z, B)$ we get that all the coefficients of $B$ are less than or equal to 1 and property 5 holds. Moreover property 1 is trivially verified and property 3 follows by the definition of $\mathbb{Q}$-CKM Zariski decomposition because $\Delta_{Z}$ is effective.

In order to prove that property 4 holds we will show that $\widetilde{\operatorname{Nklt}}(Z, B) \subseteq$ $\widetilde{\operatorname{Nklt}}\left(Z, \Delta_{Z}-N^{\prime}\right)=\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$, so that we can use the hypothesis of the corollary:
By the choice of $\Gamma$ we have that $C L C\left(Z, \Delta_{Z}-N^{\prime}\right)=C L C(Z, B)$ and $N S N C(B) \subseteq N S N C\left(\Delta_{Z}-N^{\prime}\right) \cup \mathbb{B}_{+}\left(P^{\prime}\right)$.
Then, if $V \in C L C(Z, B)$ and $V \subseteq \operatorname{Sing}(Z) \cup N S N C(B)$, we get that $V \in$ $C L C\left(Z, \Delta_{Z}-N^{\prime}\right)$ and $V \subseteq \operatorname{Sing}(Z) \cup N S N C\left(\Delta_{Z}-N^{\prime}\right) \cup \mathbb{B}_{+}\left(P^{\prime}\right)$. This implies that $V \subseteq \operatorname{Sing}(Z) \cup N S N C\left(\Delta_{Z}-N^{\prime}\right)$. Hence $V \subseteq \widetilde{\operatorname{Nklt}}\left(Z, \Delta_{Z}-N^{\prime}\right)$, and we get the required inclusion. Therefore we can apply theorem 3.7.4.

Theorem 3.7.6. Let $(X, \Delta)$ be an $L C$ pair, with $\operatorname{dim} X \geq 2$. Suppose that $P \in \operatorname{Div} v_{\mathbb{Q}}(X)$ and we can write $\Delta=\Delta_{+} \Delta_{-}$, where $\Delta_{+}$and $\Delta_{-}$are effective $\mathbb{Q}$-divisors, and the following are satisfied:

1. $P$ is nef;
2. $t_{0} P-\left(K_{X}+\Delta\right)$ is nef for some $t_{0} \in \mathbb{Q}^{+}$;
3. There exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is a Cartier divisor and for all $m \in \mathbb{N}$ we have

$$
H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P\right)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}\left(m k_{0} P+\left\ulcorner\Delta_{-}\right\urcorner\right)\right)
$$

4. $\operatorname{Nklt}_{2}(X, \Delta)=\emptyset$, or $P_{\mathrm{Nklt}_{2}(X, \Delta)}$ is semiample.
5. $P$ is logbig in codimension 1 for the pair $(X, \Delta)$, or $t_{0} P-\left(K_{X}+\Delta\right)$ is logbig in codimension 1 for the pair $(X, \Delta)$

Then $P$ is semiample.

## Proof. Let

$$
L= \begin{cases}P & \text { if } P \text { is logbig in codimension } 1 \text { for the pair }(X, \Delta) \\ t_{0} P-\left(K_{X}+\Delta\right) & \text { otherwise }\end{cases}
$$

Then $L$ is nef and logbig in codimension 1 for the pair $(X, \Delta)$, so that, thanks to lemma 2.3.9, we have that $\mathbb{B}_{+}(L)$ does not contain any divisorial LC center of the pair $(X, \Delta)$.
By [ELMNP06, Prop. 1.5] there exists $H \in \operatorname{Div}_{\mathbb{Q}}(X)$, ample, such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)
$$

Moreover there exists $m_{0} \in \mathbb{N}$ such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)=B s\left(\left|m_{0}(L-H)\right|\right)
$$

Then $B s\left(\left|m_{0}(L-H)\right|\right)$ does not contain any divisorial LC center of the pair $(X, \Delta)$. Hence, we can choose a general divisor $\Gamma$ in $\left|m_{0}(L-H)\right|$ such that $\operatorname{Supp}(\Gamma)$ does not contain any divisorial LC center of $(X, \Delta)$.
Note that we have

$$
L-\lambda \Gamma \sim_{\mathbb{Q}}\left(1-\lambda m_{0}\right) L+\lambda m_{0} H
$$

is ample if $\lambda \in\left(0, \frac{1}{m_{0}}\right]$ because $L$ is nef and $H$ is ample.
Now, for every $\lambda \in\left(0, \frac{1}{m_{0}}\right]$, let us define $\Delta_{\lambda}=\Delta+\lambda \Gamma$. We will prove that there exists $\lambda_{0} \in \mathbb{Q}^{+}$such that if $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{0}\right)$, then $P$ and the pair $\left(X, \Delta_{\lambda}\right)$ satisfy the hypotheses of the theorem 3.7.4.
First of all note that
$\left(t_{0}+1\right) P-\left(K_{X}+\Delta_{\lambda}\right)=\left(t_{0}+1\right) P-\left(K_{X}+\Delta+\lambda \Gamma\right)=P+\left(t_{0} P-\left(K_{X}+\Delta\right)\right)-\lambda \Gamma=$

$$
= \begin{cases}L-\lambda \Gamma+\left(t_{0} P-\left(K_{X}+\Delta\right)\right) & \text { if } P \text { is logbig in codimension } 1 \\ P+(L-\lambda \Gamma) & \text { otherwise }\end{cases}
$$

is ample in both cases for every $\lambda \in\left(0, \frac{1}{m_{0}}\right]$, being the sum of an ample and a nef divisor.
Now let us define

$$
\left(\Delta_{\lambda}\right)_{+}:=\Delta_{+}+\lambda \Gamma ; \quad\left(\Delta_{\lambda}\right)_{-}:=\Delta_{-} .
$$

Then $\Delta_{\lambda}=\left(\Delta_{\lambda}\right)_{+}-\left(\Delta_{\lambda}\right)_{-}$, and $\left(\Delta_{\lambda}\right)_{+}$and $\left(\Delta_{\lambda}\right)_{-}$are effective $\mathbb{Q}$-divisors for every $\lambda>0$, because $\Gamma$ is effective.
Moreover note that, with these definitions, hypotheses 1 and 3 of theorem 3.7.4 are trivially verified.

Now take a rational number $\lambda^{\prime}>0$ such that $\operatorname{Supp}(\Delta)+\operatorname{Supp}(\Gamma)=\operatorname{Supp}(\Delta+$ $\lambda \Gamma)$ for every $\lambda \in\left(0, \lambda^{\prime}\right)$. and let $\mu: X^{\prime} \rightarrow X$ be a standard log-resolution of the pair $(X, \Delta+\lambda \Gamma)$.
For every prime divisor $E \subseteq X^{\prime}$ we have that

$$
a\left(E, X, \Delta_{\lambda}\right)=a(E, X, \Delta+\lambda \Gamma)=a(E, X, \Delta)-\lambda_{\operatorname{ord}}^{E}\left(\mu^{*}(\Gamma)\right),
$$

where $a(E, X, \Delta) \geq-1$ because $(X, \Delta)$ is an LC pair.

Suppose $E$ is a divisor on $X^{\prime}$ such that $E$ is not $\mu$-exceptional and $a(E, X, \Delta)=$ -1 .
Then $\mu(E)$ is a divisorial LC center of $(X, \Delta)$, so that $\operatorname{ord}_{\mu(E)} \Gamma=0$, that is $\operatorname{ord}_{E}\left(\mu^{*}(\Gamma)\right)=0$, which implies $a\left(E, X, \Delta_{\lambda}\right)=-1$.
Now define

$$
\lambda_{1}:=\min _{\substack{\operatorname{ord}_{E}\left(\mu^{*}(\Gamma)\right)>0 \\ a(E, X, \Delta)>-1}}\left\{\frac{1+a(E, X, \Delta)}{\operatorname{ord}_{E}\left(\mu^{*}(\Gamma)\right)}, 1\right\} .
$$

Then $\lambda_{1}>0$ and, if $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{1}\right)$, we have that $a\left(E, X, \Delta_{\lambda}\right)>-1$ for every prime divisor $E \subseteq X^{\prime}$ such that $a(E, X, \Delta)>-1$.
Define

$$
\lambda_{2}:=\min _{\substack{\operatorname{ord}_{E}\left(\mu^{*}(\Gamma)\right)>0 \\ a(E, X, \Delta)=-1}}\left\{\frac{2+a(E, X, \Delta)}{\operatorname{ord}_{E}\left(\mu^{*}(\Gamma)\right)}, 1\right\} .
$$

Then $\lambda_{2}>0$ and, if $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{2}\right)$, we have that $a\left(E, X, \Delta_{\lambda}\right)>-2$ for every prime divisor $E \subseteq X^{\prime}$ such that $a(E, X, \Delta)=-1$.
We put $\lambda_{0}=\min \left\{\lambda^{\prime}, \lambda_{1}, \lambda_{2}, \frac{1}{m_{0}}\right\}$, so that if $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{0}\right)$ then $\left(X, \Delta_{\lambda}\right)$ satisfies hypothesis 5 of theorem 3.7.4.
Furthermore we can write

$$
\Delta_{\lambda}=\sum-a\left(\mu_{*}^{-1} B_{i}, X, \Delta_{\lambda}\right) B_{i}
$$

where the $B_{i}$ 's are distinct prime divisors on $X$. By definition, for every $i, \mu_{*}^{-1} B_{i}$ is not an exceptional divisors, so that, it follows by the previous calculation that $-a\left(\mu_{*}^{-1} B_{i}, X, \Delta_{\lambda}\right) \leq 1$.

Thus, in order to apply theorem 3.7.4, it just remains to prove that hypothesis 4 is satisfied for every $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{0}\right)$ :
Let us consider

$$
\mathbf{A}(\Delta)_{X^{\prime}}=\sum_{E \subseteq X^{\prime}}-a(E, X, \Delta) E ; \quad \mathbf{A}\left(\Delta_{\lambda}\right)_{X^{\prime}}=\sum_{E \subseteq X^{\prime}}-a\left(E, X, \Delta_{\lambda}\right) E .
$$

Thanks to the choice of $\mu$ and $\lambda$ we have that $\Delta^{\prime}$ and $\Delta_{\lambda}^{\prime}$ are SNCS.
Let us put

$$
\begin{gathered}
F:=\sum_{a\left(E, X, \Delta_{\lambda}\right)<-1}\left(-a\left(E, X, \Delta_{\lambda}\right)-1\right) E \\
\widetilde{\Delta}:=\mathbf{A}\left(\Delta_{\lambda}\right)_{X^{\prime}}-F=\sum_{a\left(E, X, \Delta_{\lambda}\right) \geq-1}-a\left(E, X, \Delta_{\lambda}\right) E+\sum_{a\left(E, X, \Delta_{\lambda}\right)<-1} E .
\end{gathered}
$$

Then we have that $F$ is effective, $\operatorname{Supp}(\widetilde{\Delta}) \subseteq \operatorname{Supp}\left(\mathbf{A}\left(\Delta_{\lambda}\right)_{X^{\prime}}\right)$ and all the coefficients of $\widetilde{\Delta}$ are less than or equal to 1 . In particular the pair $(X, \widetilde{\Delta})$ is LC.
Moreover, by the previous calculations, we have that $F$ is exceptional, $\operatorname{Supp}(F) \subseteq$ $\operatorname{Supp}\left(\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{=1}\right)$ and $\widetilde{\Delta}=1=\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{=1}$.

Let us show that $\widetilde{\operatorname{Nklt}}\left(X, \Delta_{\lambda}\right) \subseteq \operatorname{Nklt}_{2}(X, \Delta)$ :
We will actually prove that $\operatorname{Nklt}_{2}\left(X, \Delta_{\lambda}\right) \subseteq \operatorname{Nklt}_{2}(X, \Delta)$.
Let $V$ be an LC center of the pair $\left(X, \Delta_{\lambda}\right)$ of codimension greater than one.
Then $V=\mu(W)$ for some $W \in C L C\left(X, \mathbf{A}\left(\Delta_{\lambda}\right)_{X^{\prime}}\right)$.

If $W \nsubseteq \operatorname{Supp}(F)$, then $W \in C L C\left(X^{\prime}, \widetilde{\Delta}\right)$. But $X^{\prime}$ is smooth, $\widetilde{\Delta}$ is SNCS and the pair $(X, \widetilde{\Delta})$ is LC, whence $W$ is an irreducible component of a finite intersection of prime divisors in the support of $\widetilde{\Delta}^{=1}=\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{=1}$.
Hence $W \in C L C\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right)$, which implies that $V=\mu(W) \in C L C(X, \Delta)$, so that $V \subseteq \operatorname{Nklt}_{2}(X, \Delta)$, because the codimension of $V$ is greater than 1 .
If $W \subseteq \operatorname{Supp}(F)$ then there exists a prime divisor $F_{0} \subseteq \operatorname{Supp}(F)$ such that $W \subseteq F_{0}$.
Then $F_{0} \subseteq \operatorname{Supp}(F) \subseteq \operatorname{Supp}\left(\left(\mathbf{A}(\Delta)_{X^{\prime}}\right)^{=1}\right)$. Hence $F_{0} \in C L C\left(X^{\prime}, \mathbf{A}(\Delta)_{X^{\prime}}\right)$, so that $\mu\left(F_{0}\right) \in C L C(X, \Delta)$. Moreover $\operatorname{codim} \mu\left(F_{0}\right) \geq 2$, because $F_{0}$ is exceptional, as $F$ is exceptional.
Thus

$$
V=\mu(W) \subseteq \mu\left(F_{0}\right) \subseteq \operatorname{Nklt}_{2}(X, \Delta)
$$

This shows that $\widetilde{\operatorname{Nklt}}\left(X, \Delta_{\lambda}\right) \subseteq \operatorname{Nklt}_{2}\left(X, \Delta_{\lambda}\right) \subseteq \operatorname{Nklt}_{2}(X, \Delta)$, which implies, by the hypotheses, that $\widetilde{\operatorname{Nklt}}\left(X, \Delta_{\lambda}\right)=\emptyset$ or $P_{\sqrt{\operatorname{Nklt}\left(X, \Delta_{\lambda}\right)}}$ is semiample.
Therefore all the hypotheses of theorem 3.7.4 are satisfied and we get the semiampleness of $P$.

Corollary 3.7.7. Let $(X, \Delta)$ be a pair and $\operatorname{dim} X \geq 2$ and let $a \in \mathbb{Q}$.
Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that:

1. $D$ is big
2. $a D-\left(K_{X}+\Delta\right)$ is nef;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that

$$
f^{*}(D)=P+N
$$

is a $\mathbb{Q}$-CKM Zariski decomposition and

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is an LC pair;
- $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain divisorial LC centers of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$;
- $\operatorname{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)=\emptyset$, or $P_{\mathrm{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)}$ is semiample;

Then $P$ is semiample.
Note that in the case $a \geq 0$ we can just assume that the pair $(X, \Delta)$ is LC in order to have the LCness of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$.
Proof. Let us apply lemma 2.5.3 and consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma, so that $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition and $\left(Z, \Delta_{Z}-\right.$ $\left.N^{\prime}\right)=\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$. Then the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $P^{\prime}$ and the pair $\left(Z, \Delta_{Z}-\right.$ $N^{\prime}$ ) satisfy all the hypotheses of theorem 3.7.6. In fact we can take $\Delta_{Z}$ as the positive part $\left(\Delta_{Z}-N^{\prime}\right)_{+}$and $N^{\prime}$ as the negative part. Conditions 1 and 2 are direct consequences of lemma 2.5 .3 , condition 3 holds because $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition and condition 4 holds by hypothesis. As for condition 5 note that $\mathbb{B}_{+}\left(P^{\prime}\right)=\mathbb{B}_{+}(P)=\mathbb{B}_{+}\left(f^{*}(D)\right)$ by lemma 2.5.4, so that $P^{\prime}$ is logbig for the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)$ thanks to lemma 2.3.9. Therefore theorem 3.7.6 applies and we are done.

Corollary 3.7.8. Let $(X, \Delta)$ be an effective pair such that $\operatorname{dim} X \leq 3$, let $a \in \mathbb{Q}$. Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and

- $(X, \Delta)$ is an $L C$ pair and $a \geq 0$ (resp. $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is an $L C$ pair);
- $P$ is logbig for the pair $\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ (resp. $P$ is logbig for the pair $\left.\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)\right)$.
Then $P$ is semiample.
Proof. Begin by noting that if $\operatorname{dim} X \leq 1$ then the theorem is trivial because every big divisor on a curve is ample. We can thus assume that $2 \leq \operatorname{dim} X \leq 3$. Note also that if $a \geq 0$ and $(X, \Delta)$ is LC then, by lemma 2.2.8, $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is LC and $C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right) \subseteq C L C\left(Z, \mathbf{A}(\Delta)_{Z}\right)$. Thus we can assume that $P$ is logbig for the LC pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$.
Hence by lemma 2.3.9 we get that $\mathbb{B}_{+}(P)$ does not contain divisorial LC centers of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$, so that the same holds for $\mathbb{B}_{+}\left(f^{*}(D)\right)$ by lemma 2.5.4.

Then, in order to apply corollary 3.7.7, it just remains to show that $P_{l_{\mathrm{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)}}$ is semiample if $\operatorname{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right) \neq \emptyset$.
Equivalently we will show that $P$ is semiample when restricted to each connected component of this closed subset:
Let $C$ be a connected component of $\operatorname{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$. Then, by hypothesis, we have that $0 \leq \operatorname{dim} C \leq 1$.
If $\operatorname{dim} C=0$ then $P_{\left.\right|_{C}}$ is trivially semiample.
If $\operatorname{dim} C=1$ then we can write $C=\cup_{j=1}^{k} C_{j}$, where the $C_{j}$ 's are irreducible curves.
Then we have that $C_{j} \in C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ for every $j \in\{1, \ldots, k\}$, so that $P_{C_{C_{j}}}$ is big, because $P$ is logbig for the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$.
But, as $C_{j}$ is an irreducible curve, this implies that $P_{C_{C_{j}}}$ is ample for every $j=\{1, \ldots, k\}$. Hence $P_{\left.\right|_{C}}$ is ample, because ampleness can be checked on the irreducible components (see [Laz04, 1.2.16]), so that in particular it is semiample.
Therefore we can apply corollary 3.7.7 and we are done.

### 3.8 Dimension 4

In this section we show in theorem 3.8.1 that Conjecture 2 holds if we assume some strong standard conjectures in the field of the Minimal Model Program. By using that these conjectures hold true in low dimension we obtain Conjecture 2 in dimension less than or equal to 4 (cf. corollary 3.8.2).
Before stating the theorems let us fix some notation and definitions:

- We say that a pair $(X, \Delta)$ is of log-general type if $K_{X}+\Delta \in \operatorname{Div}_{\mathbb{Q}}(X)$ is big;
- We refer to [KM00, definition 3.50] for the definition of log minimal model of a DLT pair. More precisely given a DLT pair $(X, \Delta)$ we say that the pair $\left(X^{\prime}, \Delta^{\prime}\right)$ is a $\log$ minimal model of the pair $(X, \Delta)$ if there exists a birational map $\phi: X \rightarrow X^{\prime}$ such that

1. $\phi^{-1}$ has no exceptional divisors;
2. $\Delta^{\prime}=\phi_{*}(\Delta)$;
3. $K_{X^{\prime}}+\Delta^{\prime}$ is nef;
4. $a(E, X, \Delta)<a\left(E, X^{\prime} \Delta^{\prime}\right)$ for every $\phi$-exceptional divisor $E$.

- We refer to [Fuj00, Definition 1.1] for the definition of semi divisorial log terminal (or $s D L T$ ) $n$-fold: More precisely let $X$ be a reduced $S_{2}$ scheme and assume it is pure $n$-dimensional and normal crossing in codimension 1. Let $\Delta$ be a Weil $\mathbb{Q}$-divisor on $X$ such that $K_{X}+\Delta$ is $\mathbb{Q}$-Cartier. Let $X=\bigcup X_{i}$ be a decomposition into irreducible components and let $\mu: \coprod X_{i}^{\prime} \rightarrow X=\bigcup X_{i}$ be the normalization. A $\mathbb{Q}$-divisor $\Theta$ on $X^{\prime}$ is defined by $K_{X}^{\prime}+\Theta:=\mu^{*}\left(K_{X}+\Delta\right)$ and a $\mathbb{Q}$-divisor $\Theta_{i}$ on $X_{i}^{\prime}$ by $\Theta_{i}=\Theta_{\left.\right|_{X_{i}^{\prime}}}$.
We say that $(X, \Delta)$ is semi divisorial $\log$ terminal $n$-fold if $X_{i}$ is normal for every $i$ (so that $X_{i}^{\prime}$ is isomorphic to $X_{i}$ ) and ( $X^{\prime}, \Theta$ ) is DLT.
- We say that $s D L T$-abundance holds in dimension $n$ if for every sDLT $n$-fold $(X, \Delta)$ such that $K_{X}+\Delta$ is nef we have that $K_{X}+\Delta$ is semiample.
Theorem 3.8.1. Let $(X, \Delta)$ be an effective LC pair of dimension n. Let $D \in$ $\operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;
3. $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$;
4. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;

Also suppose that log minimal models exist for every $\mathbb{Q}$-factorial DLT pair of dimension $n$ of log-general type and that sDLT-abundance holds in dimension $n-1$.
Then $P$ is semiample.
Proof. Note that $\mathbb{B}_{+}(P)=\mathbb{B}_{+}(D)$ by lemma 2.5.4. Then we can apply lemma 2.3.7 to $P$ and we find a Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $P-\lambda \Gamma$ is ample, $(X, \Delta+\lambda \Gamma)$ is LC and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$. Then $a D+P-\left(K_{X}+\Delta+\lambda \Gamma\right)=\left(a D-\left(K_{X}+\Delta\right)\right)+(P-\lambda \Gamma)$ is ample.
Thus if $D^{\prime}=a D+P$, then $D^{\prime}$ is big and $D^{\prime}$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D^{\prime}=P^{\prime}+N^{\prime}$, where $P^{\prime}:=(a+1) P$ and $N^{\prime}:=a N$. Furthermore $D^{\prime}-\left(K_{X}+\Delta+\lambda \Gamma\right)$ is ample, $P$ is semiample if and only if $P^{\prime}$ is such and $\mathbb{B}_{+}(D)=\mathbb{B}_{+}\left(D^{\prime}\right)$.
This implies that, if we replace $D$ by $D^{\prime}$ and $\Delta$ by $\Delta+\lambda \Gamma$, we can suppose that $D-\left(K_{X}+\Delta\right)$ is ample.
Hence by [KM00, lemma 5.17] there exists an effective ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $H$ such that

$$
D-\left(K_{X}+\Delta\right) \sim_{\mathbb{Q}} H
$$

and $(X, \Delta+H)$ is an LC pair.
In other words, if we put $\Delta_{0}:=\Delta+H$, then $D \sim_{\mathbb{Q}} K_{X}+\Delta_{0}$ and $\left(X, \Delta_{0}\right)$ is an LC pair.
Therefore, thanks to lemma 2.5.2, we are reduced to show that if $K_{X}+\Delta=$ $P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition, $K_{X}+\Delta$ is big and $(X, \Delta)$ is LC, then $P$ is semiample. Moreover up to performing a DLT blow-up (see theorem 3.5.1) we can suppose that $X$ is $\mathbb{Q}$-factorial and the pair $(X, \Delta)$ is DLT.

Then by hypothesis we have a $\log$ minimal model $\left(X^{\prime}, \Delta^{\prime}\right)$ of the pair $(X, \Delta)$, so that there exists $\phi: X \rightarrow X^{\prime}$ a birational map, $\Delta^{\prime}=\phi_{*}(\Delta), K_{X^{\prime}}+\Delta^{\prime}$ is nef and ( $X^{\prime}, \Delta^{\prime}$ ) is LC. If we resolve the indeterminacies of $\phi$ we find two birational morphisms $f: Z \rightarrow X$ and $g: Z \rightarrow Y$ such that

$$
f^{*}\left(K_{X}+\Delta\right)=g^{*}\left(K_{X^{\prime}}+\Delta^{\prime}\right)+E
$$

where $E$ is $g$-exceptional and effective.
Hence, by Fujita's lemma (see for example [KMM85, lemma 1.3.2]), this is a $\mathbb{Q}$-CKM Zariski decomposition of $f^{*}\left(K_{X}+\Delta\right)$. By the uniqueness of the $\mathbb{Q}$-CKM Zariski decomposition for big divisors (see [Pro02, Proposition 7.4]) this implies that $f^{*}(P)=g^{*}\left(K_{X^{\prime}}+\Delta^{\prime}\right)$. Thus we are reduced to prove that $K_{X^{\prime}}+\Delta^{\prime}$ is semiample. Note that we can assume that the pair $\left(X^{\prime}, \Delta^{\prime}\right)$ is DLT by performing again a DLT blow-up if necessary.
Let $V=\operatorname{Nklt}\left(X^{\prime}, \Delta^{\prime}\right)$. Then there exists a $\mathbb{Q}$-divisor $\Delta_{V}^{\prime}$ on $V$ such that

$$
\left(K_{X^{\prime}}+\Delta^{\prime}\right)_{\left.\right|_{V}}=K_{V}+\Delta_{V}^{\prime}
$$

and $\left(V, \Delta_{V}^{\prime}\right)$ is an $\operatorname{sDLT}(n-1)$-fold (see for example [Fuj00, Remark 1.2(3)]). Hence by sDLT-abundance we have that $\left(K_{X^{\prime}}+\Delta^{\prime}\right)_{\left.\right|_{V}}=K_{V}+\Delta_{V}^{\prime}$ is semiample. Moreover for every sufficiently divisible $m \geq 2$ we have that

$$
H^{1}\left(X^{\prime}, \mathcal{I}_{V}\left(m\left(K_{X^{\prime}}+\Delta^{\prime}\right)\right)\right)=0
$$

by Nadel vanishing (see [Laz04, theorem 9.4.17]), because $K_{X^{\prime}}+\Delta^{\prime}$ is big and nef.
Therefore we can lift sections and we find that $\mathbb{B}\left(K_{X^{\prime}}+\Delta^{\prime}\right) \cap \operatorname{Nklt}\left(X^{\prime}, \Delta^{\prime}\right)=\emptyset$. Thus $K_{X^{\prime}}+\Delta^{\prime}$ is semiample (see for example theorem 3.6.5) and we are done.

Corollary 3.8.2. Let $(X, \Delta)$ be an LC pair of dimension less than or equal to 4. If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some rational number $a \geq 0$;
3. $D$ admits $a \mathbb{Q}$-CKM Zariski decomposition $D=P+N$;
4. $\mathbb{B}_{+}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$;

Then $P$ is semiample.
Proof. sDLT-abundance in dimension 3 holds by [Fuj00, theorem 0.1], while every DLT $\mathbb{Q}$-factorial pair of dimension 4 of log-general type has a log minimal model by [AHK07, corollary 3.6]. Hence we can apply theorem 3.8.1 and we are done.

### 3.9 Relatively DLT case

In this section we consider a relative version of DLT pairs (see definition 3.9.1). After investigating some properties of this notion we prove some generalizations of Kawamata's theorem 1.0.1 in the context of relatively DLT pairs (cf. theorem 3.9.9 and 3.9.10).

## Relatively DLT pairs

Definition 3.9.1. Let $(X, \Delta)$ be a pair, with $\Delta=\sum a_{i} D_{i}$, where the $D_{i}$ 's are distinct prime divisors and $a_{i} \in \mathbb{Q}$ for every $i$. Suppose $S \subseteq X$ is a closed subset.
We say that $(X, \Delta)$ is a $S$-DLT pair if

1. $V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$ for every $V \in C L C(X, \Delta, S)$;
2. $a_{i} \leq 1$ for every $i$ such that $D_{i} \cap S \neq \emptyset$;

Now let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$. We say that $(X, \Delta)$ is a $D$-DLT pair if $(X, \Delta)$ is a $\mathbb{B}(D)$-DLT pair.

Remark 3.9.2. Let $S \subseteq X$ be a closed subset. Then it is immediate to see that a $S$-KLT pair is $S$-DLT. Moreover by theorem 2.2.23 a DLT pair is $S$-DLT.

Lemma 3.9.3. Let $(X, \Delta)$ be a pair and let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$. Suppose $m \in \mathbb{N}$ is such that $m D$ is a Cartier divisor and, set-theoretically, $B s(|m D|)=\mathbb{B}(D)$. If $(X, \Delta)$ is a $D$-DLT pair and $\mathbb{B}(D)$ does not contain any $L C$ center of the pair $(X, \Delta)$, then there exists a common log-resolution of $(X, \Delta)$ and of the linear series $|m D|$, say $\mu: Y \rightarrow X$, such that

1. a $(E, X, \Delta)>-1$ for every $\mu$-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset ;$
2. $a(E, X, \Delta) \geq-1$ for every non- $\mu$-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$;
3. $\mu$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1

Proof. Fix $m \in \mathbb{N}$ as in the hypothesis and consider $\mu: Y \rightarrow X$ a log-resolution of $(X, \Delta)$ and $|m D|$ as in lemma 2.2.20.
Let us prove that $\mu$ satisfies all the given conditions:
Write $\Delta=\sum a_{i} D_{i}$, where the $D_{i}$ 's are distinct prime divisors and $a_{i} \in \mathbb{Q}$.
We have that

$$
K_{Y} \equiv \mu^{*}\left(K_{X}+\Delta\right)+\sum a(E, X, \Delta) E .
$$

1) Suppose $E$ is a $\mu$-exceptional prime divisor on $Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$. As $E$ is $\mu$-exceptional, by the choice of $\mu$, we get that

$$
\mu(E) \subseteq \mathbb{B}(D) \cup \operatorname{Sing}(X) \cup N S N C(\Delta)
$$

If, by contradiction, $a(E, X, \Delta) \leq-1$, then $\mu(E) \in C L C(X, \Delta, \mathbb{B}(D))$. Then, by hypothesis, $\mu(E) \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$. Thus, by the irreducibility of
$\mu(E)$, we get that $\mu(E) \subseteq \mathbb{B}(D)$, so that we get a contradiction, because $\mathbb{B}(D)$ does not contain LC centers of the pair $(X, \Delta)$.
2) Let $E$ be a non $\mu$-exceptional prime divisor on $Y$ such that $\mu(E) \cap \mathbb{B}(D) \neq \emptyset$, then $E=\mu_{*}^{-1}\left(D_{i}\right)$ for some $i$, whence $\mu(E)=D_{i}$, so that $D_{i} \cap \mathbb{B}(D) \neq \emptyset$.
Then $-a(E, X, \Delta)=a_{i} \leq 1$, that is $a(E, X, \Delta) \geq-1$.
3) Follows by the choice of $\mu$.

In the following lemmas we prove some good properties of $S$-DLT pairs:
Lemma 3.9.4. Let $(X, \Delta)$ be a pair and let $S \subseteq X$ be a closed subset. If $(X, \Delta)$ is a $S$-DLT pair, then $C L C(X, \Delta, S)$ is a finite set.

Proof. Write $\Delta=\sum a_{i} D_{i}$, where the $a_{i}$ 's are rational numbers and the $D_{i}$ 's are distinct prime divisors on $X$.
Let $\mu: Y \rightarrow X$ be a standard log-resolution of the pair $(X, \Delta)$.
If $E \subseteq Y$ is a $\mu$-exceptional prime divisor, then $\mu(E) \subseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$, so that, by hypothesis $\mu(E) \notin C L C(X, \Delta, S)$.
Thus, if $\mu(E) \cap S \neq \emptyset$, then $a(E, X, \Delta)>-1$.
Now define
$\Delta_{Y}:=-\sum a(E, X, \Delta) E=-\sum a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right) \mu_{*}^{-1}\left(D_{i}\right)-\sum_{E \text { exc. }} a(E, X, \Delta) E$,
so that for every $V \in C L C(X, \Delta, S)$ we have that $V=\mu(Z)$, where $Z \in$ $C L C\left(Y, \Delta_{Y}\right)$ is such that $\mu(Z) \cap S \neq \emptyset$.
But, if $Z$ is such, then $\mu(Z) \nsubseteq \mu(E)$ for all prime divisor $E$ on $Y$ such that $\mu(E) \cap S=\emptyset$, so that $Z \nsubseteq E$ if $\mu(E) \cap S=\emptyset$.
Now, if $F$ is a prime divisor on a normal variety $Y^{\prime}$ such that $h: Y^{\prime} \rightarrow Y$ is birational and $h(F)=Z$, then

$$
\begin{gathered}
a\left(F, Y, \Delta_{Y}\right)=a\left(F, Y,-\sum a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right) \mu_{*}^{-1}\left(D_{i}\right)-\sum_{E \text { exc. }} a(E, X, \Delta) E\right)= \\
=a\left(F, Y,-\sum_{D_{i} \cap S \neq \emptyset} a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right) \mu_{*}^{-1}\left(D_{i}\right)-\sum_{\substack{E \text { exc. } \\
\mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E\right)+ \\
\quad+\sum_{\mu(E) \cap S=\emptyset} a(E, X, \Delta) \operatorname{ord}_{F}\left(h^{*}(E)\right) .
\end{gathered}
$$

But, if $\mu(E) \cap S=\emptyset$, then we have seen that $Z=h(F) \nsubseteq E$, that is $\operatorname{ord}_{Z}(E)=0$, so that $\operatorname{ord}_{F}\left(h^{*}(E)\right)=0$. Thus if we define

$$
\Delta_{Y}^{\prime}:=-\sum_{D_{i} \cap S \neq \emptyset} a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right) \mu_{*}^{-1} D_{i}-\sum_{\substack{E \text { exc. } \\ \mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E,
$$

then

$$
\begin{aligned}
a\left(F, Y, \Delta_{Y}\right)=a(F, Y, & -\sum_{D_{i} \cap S \neq \emptyset} a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right) \mu_{*}^{-1} D_{i}- \\
& \left.-\sum_{\substack{E \text { exc. } \\
\mu(E) \cap S \neq \emptyset}} a(E, X, \Delta) E\right)=a\left(F, Y, \Delta_{Y}^{\prime}\right) .
\end{aligned}
$$

## CHAPTER 3. ON THE SEMIAMPLENESS OF THE POSITIVE PART OF

 CKM ZARISKI DECOMPOSITIONSHence, $Z \in C L C\left(Y, \Delta_{Y}\right)$ implies that $Z \in C L C\left(Y, \Delta_{Y}^{\prime}\right)$, so that $V$ is the image of an element in $C L C\left(Y, \Delta_{Y}^{\prime}\right)$.
But $\left(Y, \Delta_{Y}^{\prime}\right)$ is an LC pair because $Y$ is smooth, $\Delta_{Y}^{\prime}$ is SNCS and its coefficients are all less than or equal to 1 :
In fact, by hypothesis of S-DLTness of $(X, \Delta)$, we have that $-a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta\right)=$ $a_{i} \leq 1$ if $D_{i} \cap S \neq \emptyset$.
On the other hand we have proved that if $E$ is $\mu$-exceptional and $\mu(E) \cap S \neq \emptyset$, then $a(E, X, \Delta)>-1$.
Therefore $C L C\left(Y, \Delta_{Y}^{\prime}\right)$ is finite, so that $C L C(X, \Delta, S)$ is finite as well, because every element in $C L C(X, \Delta, S)$ is the image of an element in $C L C\left(Y, \Delta_{Y}^{\prime}\right)$.

Lemma 3.9.5. Let $(X, \Delta)$ be a pair, let $S \subseteq X$ be a closed subset and let $\Delta^{\prime} \in \operatorname{Div}_{\mathbb{Q}}(X)$ be an effective divisor.
If $(X, \Delta)$ is $S$-DLT and, $\operatorname{Supp}\left(\Delta^{\prime}\right) \nsupseteq V$, for every $V \in C L C(X, \Delta, S)$, then there exists a rational number $\lambda_{0}>0$ such that for every $\lambda \in \mathbb{Q} \cap\left[0, \lambda_{0}\right]$, we have that $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ is $S$-DLT and $C L C\left(X, \Delta+\lambda \Delta^{\prime}, S\right)=C L C(X, \Delta, S)$.

Proof. Let us write

$$
\Delta=\sum a_{i} D_{i}, \quad \Delta^{\prime}=\sum b_{i} D_{i}
$$

where the $D_{i}$ 's are distinct prime divisors on $X$, the $a_{i}$ 's and the $b_{i}$ 's are (possibly zero) rational numbers. In particular, as $\Delta^{\prime}$ is effective, $b_{i} \geq 0$ for all $i$. Hence, for every $\lambda>0$,

$$
\Delta+\lambda \Delta^{\prime}=\sum\left(a_{i}+\lambda b_{i}\right) D_{i}
$$

Now let $\mu: Y \rightarrow X$ be a standard $\log$-resolution of the pair $(X, \operatorname{Supp}(\Delta)+$ $\operatorname{Supp}\left(\Delta^{\prime}\right)$ ).
Define, for every $\lambda \geq 0$,

$$
\widetilde{\Delta_{\lambda}}:=-\sum_{E \subseteq Y} a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E
$$

so that, the LC centers of $\left(X, \Delta+\lambda \Delta^{\prime}\right)$ are the images of the LC centers of the pair $\left(Y, \widetilde{\Delta_{\lambda}}\right)$.
We have that

$$
\widetilde{\Delta_{\lambda}}=-\sum_{E \subseteq Y} a(E, X, \Delta) E+\mu^{*}\left(\lambda \Delta^{\prime}\right)
$$

so that, for every prime divisor $E \subseteq Y$, we get

$$
a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=a(E, X, \Delta)-\lambda \operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\prime}\right)\right)
$$

Claim 1 There exists $\lambda^{\prime} \in \mathbb{Q} \cap(0,1]$ such that for every rational number $\lambda \in\left[0, \lambda^{\prime}\right)$ we have the following:
If $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S \neq \emptyset$, then

1. $a(E, X, \Delta) \geq-1$;
2.     - $a(E, X, \Delta)>-1 \Rightarrow a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)>-1$;

- $a(E, X, \Delta)=-1 \Rightarrow a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=-1$.


## Proof of claim 1.

1) If $E \subseteq Y$ is a $\mu$-exceptional prime divisor and $\mu(E) \cap S \neq \emptyset$, then, by choice of $\mu$, we have that $\mu(E) \subseteq \operatorname{Sing}(X) \cup N S N C\left(\operatorname{Supp}(\Delta)+\operatorname{Supp}\left(\Delta^{\prime}\right)\right) \subseteq$ $\operatorname{Sing}(X) \cup N S N C(\Delta) \cup \operatorname{Supp}\left(\Delta^{\prime}\right)$.
If, by contradiction $a(E, X, \Delta)<-1$, then $\mu(E) \in C L C(X, \Delta, S)$.
By $S$-DLTness of $(X, \Delta)$ this implies that $\mu(E) \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$. But $\mu(E) \nsubseteq \operatorname{Supp}\left(\Delta^{\prime}\right)$ by hypothesis, so that we find a contradiction.
On the other hand if $E \subseteq Y$ is a prime divisor, it is not $\mu$-exceptional and $\mu(E) \cap S \neq \emptyset$, then $E=\mu_{*}^{-1}\left(D_{i}\right)$ for some prime divisor $D_{i}$ on $X$ such that $D_{i} \cap S \neq \emptyset$. Then, as $(X, \Delta)$ is $S$-DLT, we have that $a(E, X, \Delta)=-a_{i} \geq-1$.
2) Let $A=\{E \subseteq Y: E$ is a prime divisor, $a(E, X, \Delta)>-1, \mu(E) \cap S \neq$ $\left.\emptyset, \operatorname{ord}_{E} \mu^{*}\left(\Delta^{\prime}\right) \neq 0\right\}$.
Then we put

$$
\lambda^{\prime}= \begin{cases}\min _{E \in A}\left\{\frac{1+a(E, X, \Delta)}{\operatorname{ord}_{E} \mu^{*}\left(\Delta^{\prime}\right)}\right\} & \text { if } A \neq \emptyset \\ 1 & \text { if } A=\emptyset\end{cases}
$$

Let $E$ be a prime divisor on $Y$ such that $\mu(E) \cap S \neq \emptyset$ and $a(E, X, \Delta)>-1$. If $0 \leq \lambda<\lambda^{\prime}$, then $a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=a(E, X, \Delta)-\lambda \operatorname{ord}_{E} \mu^{*}\left(\Delta^{\prime}\right)>-1$.

Now suppose $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S \neq \emptyset$ and $a(E, X, \Delta)=$ -1 . Then $\mu(E) \in C L C(X, \Delta, S)$. Hence $\operatorname{Supp}\left(\Delta^{\prime}\right) \nsupseteq \mu(E)$, that is $\operatorname{ord}_{E}\left(\mu^{*}\left(\Delta^{\prime}\right)\right)=$ 0 . Thus, for every $\lambda \geq 0$,

$$
a\left(E, X, \Delta+\lambda \Delta^{\prime}\right)=a(E, X, \Delta)=-1
$$

This proves claim 1.
Claim 2 For every $\lambda \in \mathbb{Q} \cap\left[0, \lambda^{\prime}\right)$, we have that

$$
C L C\left(X, \Delta+\lambda \Delta^{\prime}, S\right) \subseteq C L C(X, \Delta, S)
$$

If the claim holds, then, for every $\lambda \in \mathbb{Q} \cap\left[0, \lambda^{\prime}\right)$, we have that

$$
C L C\left(X, \Delta+\lambda \Delta^{\prime}, S\right)=C L C(X, \Delta, S)
$$

because $C L C\left(X, \Delta+\lambda \Delta^{\prime}\right) \supseteq C L C(X, \Delta)$ by lemma 2.2.8, so that $C L C(X, \Delta+$ $\left.\lambda \Delta^{\prime}, S\right) \supseteq C L C(X, \Delta, S)$.

Moreover we can deduce the $S$-DLTness of $\left(X, \Delta+\lambda \Delta^{\prime}\right)$, for $\lambda \in \mathbb{Q} \cap\left[0, \lambda^{\prime}\right)$ :
Property 1 holds because if $V \in C L C\left(X, \Delta+\lambda \Delta^{\prime}, S\right)$, then $V \in C L C(X, \Delta, S)$. Then, thanks to the hypotheses,

$$
V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta) \cup \operatorname{Supp}\left(\Delta^{\prime}\right)
$$

But $N S N C\left(\Delta+\lambda \Delta^{\prime}\right) \subseteq N S N C\left(\operatorname{Supp}(\Delta)+\operatorname{Supp}\left(\Delta^{\prime}\right)\right) \subseteq N S N C(\Delta) \cup \operatorname{Supp}\left(\Delta^{\prime}\right)$.
Therefore $V \nsubseteq \operatorname{Sing}(X) \cup N S N C\left(\Delta+\lambda \Delta^{\prime}\right)$.
Now suppose $D_{i} \cap S \neq \emptyset$. Then

$$
a_{i}+\lambda b_{i}=-a\left(\mu_{*}^{-1}\left(D_{i}\right), X, \Delta+\lambda \Delta^{\prime}\right) \leq 1
$$

thanks to claim 1 . Thus property 2 is satisfied for $\lambda \in \mathbb{Q} \cap\left[0, \lambda^{\prime}\right)$.
The lemma follows by choosing $\lambda_{0} \in \mathbb{Q}$ such that $0 \leq \lambda_{0}<\min \left\{\frac{1}{m_{0}}, \lambda^{\prime}\right\}$.
Proof of claim 2. Suppose $V \in C L C\left(X, \Delta+\lambda \Delta^{\prime}, S\right)$.
Then $V=\mu(W)$, for some $W \in C L C\left(Y, \widetilde{\Delta_{\lambda}}\right)$, and $\mu(W) \cap S \neq \emptyset$.
Hence, if $E \subseteq Y$ is a prime divisor such that $\mu(E) \cap S=\emptyset$, then $W \nsubseteq E$, that is $\operatorname{ord}_{W}(E)=0$.
This implies that, for every prime divisor $F$ over $W$, for every $x \in \mathbb{Q}$,

$$
a\left(F, Y, \widetilde{\Delta_{\lambda}}+x E\right)=a\left(F, Y, \widetilde{\Delta_{\lambda}}\right)
$$

Therefore, if we define

$$
\widetilde{\Delta_{\lambda}^{\prime}}:=-\sum_{\mu(E) \cap S \neq \emptyset} a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E
$$

we get that

$$
\begin{gathered}
W \in C L C\left(Y, \widetilde{\Delta_{\lambda}}+\sum_{\mu(E) \cap S=\emptyset} a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E\right)= \\
=C L C\left(Y,-\sum_{\mu(E) \cap S \neq \emptyset} a\left(E, X, \Delta+\lambda \Delta^{\prime}\right) E\right)=C L C\left(Y, \widetilde{\Delta_{\lambda}^{\prime}}\right) .
\end{gathered}
$$

Note that the pair $\left(Y, \widetilde{\Delta_{\lambda}^{\prime}}\right)$ is LC, because $Y$ is smooth, $\widetilde{\Delta_{\lambda}^{\prime}}$ is SNCS and all its coefficients are less than or equal to 1 by claim 1 .
Thus, by remark 2.2.6, all the LC centers of this pair are irreducible components of intersections of prime divisors in the support of $\left(\widetilde{\Delta_{\lambda}^{\prime}}\right)=1$.
But, again by claim 1, we get that

$$
\operatorname{Supp}\left(\left(\widetilde{\Delta_{\lambda}^{\prime}}\right)^{=1}\right) \subseteq \operatorname{Supp}\left(\left(-\sum a(E, X, \Delta) E\right)^{=1}\right)=\operatorname{Supp}\left(\left(\widetilde{\Delta_{0}}\right)^{=1}\right)
$$

Thus $W$ is an irreducible component of a finite intersection of prime divisors in $\operatorname{Supp}\left(\left(\widetilde{\Delta_{0}}\right)^{=1}\right)$, that is $W \in C L C\left(Y,\left(\widetilde{\Delta_{0}}\right)^{=1}\right)$.
Moreover, as $\widetilde{\Delta_{0}}$ is SNCS, $W$ is not contained in any other prime divisor in the support of $\widetilde{\Delta_{0}}$, so that $W \in C L C\left(Y, \widetilde{\Delta_{0}}\right)$.
This implies that $V=\mu(W) \in C L C(X, \Delta, S)$, because we had chosen $V$ such that $V \cap S \neq \emptyset$.

The following lemma is an improvement of lemma 2.3.7.
Lemma 3.9.6. Let $(X, \Delta)$ be a pair and let $S \subseteq X$ be a closed subset such that $(X, \Delta)$ is an $S$-DLT pair. Suppose $L \in \operatorname{Div}_{\mathbb{Q}}(X)$ is big and nef and $\mathbb{B}_{+}(L)$ does not contain any element in $C L C(X, \Delta, S)$.
Then there exists an effective Cartier divisor $\Gamma$ on $X$, and a rational number $\lambda_{0}>0$ such that $B s(|\Gamma|)=\mathbb{B}_{+}(L)$ and for each $\lambda \in \mathbb{Q} \cap\left(0, \lambda_{0}\right]$, we have that

1. $L-\lambda \Gamma \in \operatorname{Div}_{\mathbb{Q}}(X)$ and is ample;
2. $C L C(X, \Delta+\lambda \Gamma, S)=C L C(X, \Delta, S)$;
3. $(X, \Delta+\lambda \Gamma)$ is an $S$-DLT pair.

Proof. By [ELMNP06, Prop. 1.5] there exists $H$, an ample $\mathbb{Q}$-divisor on $X$, such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)
$$

Moreover there exists $m_{0} \in \mathbb{N}$ such that

$$
\mathbb{B}_{+}(L)=\mathbb{B}(L-H)=B s\left(\left|m_{0}(L-H)\right|\right)
$$

Let $\Gamma \in\left|m_{0}(L-H)\right|$ be a general divisor, so that $B s(|\Gamma|)=\mathbb{B}_{+}(L)$.
Now, for all $\lambda \in \mathbb{Q} \cap(0,1]$, we have that

$$
L-\lambda \Gamma \sim_{\mathbb{Q}}\left(1-\lambda m_{0}\right) L+\lambda m_{0} H
$$

is ample if $\lambda \leq \frac{1}{m_{0}}$ because $L$ is nef and $H$ is ample.
As $C L C(X, \Delta, S)$ is a finite set by lemma 3.9.4, and as, by hypothesis, $B s(|\Gamma|)=$ $\mathbb{B}_{+}(L)$ does not contain any element in $C L C(X, \Delta, S)$, we can choose $\Gamma$ such that $\operatorname{Supp}(\Gamma)$ does not contain any element of $C L C(X, \Delta, S)$, as well.
Then we can apply lemma 3.9.5 and we find a rational number $\lambda_{0}>0$ such that for every rational number $\lambda \in\left(0, \lambda_{0}\right]$ the pair $(X, \Delta+\lambda \Gamma)$ is $S$-DLT and $C L C(X, \Delta+\lambda \Gamma, S)=C L C(X, \Delta, S)$.

Lemma 3.9.7. Let $(X, \Delta)$ be a pair, let $S \subseteq X$ be a closed subset and let $N \in \operatorname{Div}_{\mathbb{Q}}(X)$ be effective.
If $(X, \Delta)$ is $S$-DLT, then $(X, \Delta-N)$ is also $S-D L T$.
Proof. Trivially, $(X, \Delta-N)$ satisfies the property 2, because, by effectivity of $N$, the coefficients of $\Delta-N$ are not bigger than the coefficients of $\Delta$.

Let us prove that $(X, \Delta-N)$ satisfies the property 1 :
Suppose there exists $V \in C L C(X, \Delta-N, S)$, (otherwise there is nothing to prove). Then, as $V \in C L C(X, \Delta, S), V \nsubseteq \operatorname{Sing}(X) \cup N S N C(\Delta)$.
In order to prove the property 1 we have to show that $V \nsubseteq N S N C(\Delta-N)$ :
Claim There exists a proper birational morphism $f: Y \rightarrow X$, and an irreducible divisor $F$ on $Y$ such that $f(F)=V$ and

$$
a(F, X, \Delta)=a(F, X, \Delta-N)=-1
$$

If the claim holds, then, as usual, we have that

$$
a(F, X, \Delta)=a(F, X, \Delta-N)-\operatorname{ord}_{F}\left(f^{*}(N)\right)
$$

Then, by the claim, $\operatorname{ord}_{F}\left(f^{*}(N)\right)=0$, so that $V=f(F) \nsubseteq \operatorname{Supp}(N)$.
Hence $V \nsubseteq \operatorname{Supp}(N) \cup N S N C(\Delta)$. As $N S N C(\Delta-N) \subseteq \operatorname{Supp}(N) \cup N S N C(\Delta)$, we get that $V \nsubseteq N S N C(\Delta-N)$, so that property 1 holds. Thus the lemma will be proved once we prove the claim.

Proof of the claim. Let $\mu: X^{\prime} \rightarrow X$ be a standard log-resolution of the pair $(X, \operatorname{Supp}(\Delta))$.

Let $E_{1}, \ldots, E_{k}$ be prime divisors on $X^{\prime}$ such that, for all $j \in\{1, \ldots, k\}$, we have

$$
a\left(E_{j}, X, \Delta\right) \neq 0 \quad \text { and } \quad \mu\left(E_{j}\right) \supseteq V
$$

Note that the set of the prime divisors on $X^{\prime}$ with this properties is nonempty because $V \in C L C(X, \Delta)$.
Suppose, furthermore, that $E_{1}, \ldots, E_{k}$ are the only prime divisor on $X^{\prime}$ with both these properties.
Now suppose there exists $j \in\{1, \ldots, k\}$ such that $E_{j}$ is $\mu$-exceptional.
Then, by definition of standard log-resolution,

$$
\mu\left(E_{j}\right) \subseteq \operatorname{Sing}(X) \cup N S N C(\Delta) \Longrightarrow V \subseteq \operatorname{Sing}(X) \cup N S N C(\Delta)
$$

But we have proved before the claim that this is not possible.
Thus, all the $E_{j}$ are non $\mu$-exceptional. Moreover, as $V \cap S \neq \emptyset$, we have that $\mu\left(E_{j}\right) \cap S \neq \emptyset$ for all $j=1, \ldots, k$.
Then

$$
a\left(E_{j}, X, \Delta\right) \geq-1 \quad \forall j=1, \ldots k
$$

thanks to the $S$-DLTness of $(X, \Delta)$ (property 2 ).
Let $\nu: X^{\prime \prime} \rightarrow X^{\prime}$ be a proper birational morphism such that there exists a prime divisor $F \subseteq X^{\prime \prime}$ such that $\mu(\nu(F))=V$ and $a(F, X, \Delta-N) \leq-1$.
Note that we can find such a morphism because $V \in C L C(X, \Delta-N)$.
Moreover, composing, if necessary, $\nu$ with a suitable log-resolution, we can suppose that $X^{\prime \prime}$ is smooth and that the divisor

$$
\nu_{*}^{-1} \mu_{*}^{-1} \Delta+\nu_{*}^{-1} \operatorname{exc}(\mu)+\operatorname{exc}(\nu)
$$

is SNCS.
Now, as usual, we find that

$$
\begin{gathered}
a(F, X, \Delta)=a\left(F, X^{\prime},-\sum a(E, X, \Delta) E\right)= \\
a\left(F, X^{\prime},-\sum_{j=1}^{k} a\left(E_{j}, X, \Delta\right) E_{j}\right)+\sum_{E \neq E_{j}} a(E, X, \Delta) \operatorname{ord}_{F} \nu^{*}(E)
\end{gathered}
$$

Let us show that if $E \subseteq X^{\prime}$ is a prime divisor such that $E \neq E_{j}$ for all $j=$ $\{1, \ldots, k\}$, then, either $a(E, X, \Delta)=0$, or $\operatorname{ord}_{F} \nu^{*}(E)=0$ : In fact, if $\operatorname{ord}_{F} \nu^{*}(E) \neq 0$ then

$$
F \subseteq \operatorname{Supp}\left(\nu^{*}(E)\right) \subseteq \nu^{-1}(E) \Longrightarrow \nu(F) \subseteq E \Longrightarrow V=\mu(\nu(F)) \subseteq \mu(E)
$$

Then $a(E, X, \Delta)=0$ because of the choice of the $E_{j}$ 's. This shows that

$$
a(F, X, \Delta)=a\left(F, X^{\prime},-\sum_{j=1}^{k} a\left(E_{j}, X, \Delta\right) E_{j}\right)
$$

But $\left(X^{\prime},-\sum_{j=1}^{k} a\left(E_{j}, X, \Delta\right) E_{j}\right)$ is an LC pair, because the divisor $-\sum_{j=1}^{k} a\left(E_{j}, X, \Delta\right) E_{j}$ is SNCS (as $\mu$ is a log-resolution of $(X, \Delta)$ ), and we have seen that $a\left(E_{j}, X, \Delta\right) \geq-1$ for all $j \in\{1, \ldots, k\}$.

Therefore

$$
-1 \geq a(F, X, \Delta-N) \geq a(F, X, \Delta)=a\left(F, X^{\prime},-\sum_{j=1}^{k} a\left(E_{j}, X, \Delta\right) E_{j}\right) \geq-1
$$

so that $a(F, X, \Delta-N)=a(F, X, \Delta)=-1$, and the claim is proved by putting $f:=\mu \circ \nu$ and $Y:=X^{\prime \prime}$.

## Main theorems

Theorem 3.9.8. Let $X$ be a normal projective variety and let $\Delta$ be an effective Weil $\mathbb{Q}$-divisor. If $D$ is a Weil $\mathbb{Q}$-divisor such that

1. $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$;
2. $(X, \Delta-a N)$ is a $P$-DLT pair;
3. There exist two rational numbers $t_{0}$ and $a$, with $a \geq 0$ such that

$$
t_{0} P-\left(K_{X}+\Delta-a N\right)
$$

is ample;
4. For every $V \in C L C(X, \Delta-a N)$ we have that $V \nsubseteq \mathbb{B}(P)$;
then $P$ is semiample.
Proof. We define $A:=a N-\Delta$. Then $(X,-A)$ is a pair and

$$
t_{0} P+A-K_{X}=t_{0} P-\left(K_{X}+\Delta-a N\right)
$$

is ample.
Now if $(X,-A)$ is KLT then $\kappa(X, D) \geq 0$ by lemma 3.3.1. Hence, as $\kappa(X, P)=$ $\kappa(X, D)$ by definition of CKM Zariski decomposition, we have that $\kappa(X, P) \geq 0$, as well.
If $(X,-A)$ is not KLT then $C L C(X,-A) \neq \emptyset$, so that, $\mathbb{B}(P) \neq X$, because of 4. Hence, again, $\kappa(X, P) \geq 0$.

Then, if we denote

$$
\mathbb{N}(P)=\left\{m \in \mathbb{N} \text { such that } m P \in \operatorname{Div}(X) \text { e } H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \neq 0\right\}
$$

we have that $\mathbb{N}(P) \neq \emptyset$. Take $m_{1} \in \mathbb{N}$ such that $m_{1} P$ is a Cartier divisor and $B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$, as closed sets.
We suppose, by contradiction, that $\mathbb{B}(P) \neq \emptyset$.
We will find $m \in \mathbb{N}(P)$ and a subvariety $V \subseteq X$ such that, set-theoretically, $V \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$ but $V \nsubseteq B s(|m P|)$, leading, in such a way, to a contradiction.
Let $\left\{D_{j}\right\}_{j=1}^{k}$ be the finite set of the prime divisors appearing in the support of $A$ or as base components of $\left|m_{1} P\right|$. We write $A=\sum_{j=1}^{k} a_{j} D_{j}$, where the $a_{j}$ are possibly zero rational numbers.
Now, as $(X,-A)$ is $P$-DLT and $\mathbb{B}(P)$ does not contain LC centers of the pair $(X, \Delta-a N)$, we can apply lemma 3.9 .3 , so that we find $\mu: Y \rightarrow X$, a logresolution of the pair $(X,-A)$ and of the linear series $\left|m_{1} P\right|$ such that:

- $\mu$ is a composition of blowings-up of smooth subvarieties of codimension greater than 1.
- $a(E, X,-A)>-1$ for every $\mu$-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(P) \neq \emptyset ;$
- $a(E, X,-A) \geq-1$ for every non- $\mu$-exceptional prime divisor $E \subseteq Y$ such that $\mu(E) \cap \mathbb{B}(P) \neq \emptyset$;
Let $\left\{F_{j}=\tilde{D}_{j}\right\}_{j=1}^{k}$ be the finite set of the strict transforms of the divisors $D_{j}$ and let $\left\{F_{j}=E_{j}\right\}_{j=k+1}^{l}$ be the finite set of the $\mu$-exceptional prime divisors on $Y$, so that $\sum_{j=1}^{l} F_{j}$ is a SNC divisor.
We can write

$$
K_{Y} \equiv \mu^{*}\left(K_{X}-A\right)+\sum_{j=1}^{l} b_{j} F_{j}
$$

where $b_{j}=a\left(F_{j}, X,-A\right)$ for every $j=1, \ldots, l$.
Moreover we can consider an integral base-point free divisor $L$ and coefficients $r_{j} \in \mathbb{N} \cup\{0\}$ such that $\mu^{*}\left(m_{1} P\right)=L+\sum r_{j} F_{j}$ and $\mu^{*}\left|m_{1} P\right|=|L|+\sum r_{j} F_{j}$. Hence we have that $B s\left(\left|m_{1} P\right|\right)=\mu\left(\bigcup_{r_{j} \neq 0} F_{j}\right)$, so that we can suppose $r_{j}>0$ for some $j$ because $B s\left(\left|m_{1} P\right|\right) \neq \emptyset$.
Moreover if $r_{j} \neq 0$, then $\mu\left(F_{j}\right) \subseteq B s\left(\left|m_{1} P\right|\right)=\mathbb{B}(P)$, which implies that $b_{j}>-1$, as, by hypothesis, $\mathbb{B}(P)$ does not contain any LC center of the pair $(X,-A)$.
Now, as $t_{0} P+A-K_{X}$ is ample and $\mu$ satisfies the hypotheses of lemma 2.1.9, there exist, for each $j=k+1, \ldots, l$, arbitrarily small, rational numbers $\delta_{j}>0$, such that

$$
\mu^{*}\left(t_{0} P+A-K_{X}\right)-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is still ample.
Thanks to the openness of the ample cone there exist also, for each $j=1, \ldots, k$, positive rational numbers $\delta_{j}$ such that if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ then

$$
\mu^{*}\left(t_{0} P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample.
Now we define

$$
c=\min _{\left\{j: r_{j} \neq 0\right\}} \frac{b_{j}+1-\delta_{j}}{r_{j}} .
$$

By choosing the $\delta_{j}$ 's small enough we can suppose that

$$
b_{j}+1-\delta_{j}>0 \text { for all } j \text { such that } b_{j}>-1
$$

Hence $c>0$ because $b_{j}>-1$ for every $j$ such that $r_{j} \neq 0$.
Moreover, perturbing slightly the $\delta_{j}$ 's if necessary, we can suppose that the minimum is attained on a unique $j$, say $j=j_{0}$. Let $B:=F_{j_{0}}$.
Now we define

- $J_{1}=\left\{j \in\{1, \ldots, k\}\right.$ such that $\left.b_{j}>-1\right\}$,
- $J_{2}=\left\{j \in\{1, \ldots, k\}\right.$ such that $b_{j}=-1$ and $\left.D_{j} \cap \mathbb{B}(P) \neq \emptyset\right\}$,
- $J_{3}=\{k+1, \ldots, l\}$,
- $J_{4}=\left\{j \in\{1, \ldots, k\}\right.$ such that $b_{j} \leq-1$ and $\left.D_{j} \cap \mathbb{B}(P)=\emptyset\right\}$.

Note that $J_{1} \amalg J_{2} \amalg J_{3} \amalg J_{4}=\{1, \ldots, l\}$, because, by choice of $\mu, b_{j} \geq-1$ if $1 \leq j \leq k$ and $D_{j} \cap \mathbb{B}(P) \neq \emptyset$.
Moreover $j_{0} \in J_{1} \amalg J_{3}$ as $b_{0}>-1$, being $r_{0} \neq 0$.
Now let $s:=t_{0}+c m_{1}$, and let

$$
B_{m}:=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{j \in J_{3}} \delta_{j} F_{j} .
$$

Then, if $0 \leq \delta_{j}^{\prime} \leq \delta_{j}$ for each $j \in\{1, \ldots, k\}=J_{1} \amalg J_{2} \amalg J_{4}$, we have that for every integer $m \geq s$

$$
B_{m}-\sum_{J_{1} \amalg J_{2} \amalg J_{4}} \delta_{j}^{\prime} F_{j}=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{j=1}^{k} \delta_{j}^{\prime} F_{j}-\sum_{j=k+1}^{l} \delta_{j} F_{j}
$$

is ample, because $m-c m_{1} \geq t_{0}$ and $\mu^{*}(P)$ is nef.
Let us consider now, for each $j \in J_{2}$, rational, arbitrarily small numbers $\epsilon_{j}>0$, and define

$$
\begin{aligned}
& A^{\prime}:=\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}, \\
& A^{\prime \prime}:=A^{\prime}+\sum_{j \in J_{2}}\left(\epsilon_{j}+\delta_{j}\right) F_{j} .
\end{aligned}
$$

As $r_{j}=0$ if $b_{j} \leq-1$ we get that

$$
A^{\prime \prime}=\sum_{\left(J_{1} \amalg J_{3} \amalg J_{4}\right) \backslash\left\{j_{0}\right\}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}+\sum_{J_{2}}\left(-1+\epsilon_{j}\right) F_{j},
$$

Now we define, for every $m \in \mathbb{N}(P)$, the divisor

$$
Q_{m}:=\mu^{*}(m P)+A^{\prime \prime}-B-K_{Y} .
$$

$$
\begin{aligned}
& \qquad Q_{m}:=\mu^{*}(m P)+A^{\prime}-B-K_{Y}+\left(A^{\prime \prime}-A^{\prime}\right) \equiv \\
& \equiv \mu^{*}(m P)+\sum_{j \neq j_{0}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j}-F_{j_{0}}-\sum b_{j} F_{j}-\mu^{*}\left(K_{X}-A\right)+\left(A^{\prime \prime}-A^{\prime}\right)= \\
& =\mu^{*}\left(m P+A-K_{X}\right)-\sum_{j \neq j_{0}} c r_{j} F_{j}-\sum_{j \neq j_{0}} \delta_{j} F_{j}+F_{j_{0}}\left(-1-b_{j_{0}}\right)+\left(A^{\prime \prime}-A^{\prime}\right)= \\
& =\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)+c L+F_{j_{0}}\left(c r_{j_{0}}-1-b_{j_{0}}\right)-\sum_{j \neq j_{0}} \delta_{j} F_{j}+\left(A^{\prime \prime}-A^{\prime}\right)= \\
& =\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)+c L-\sum \delta_{j} F_{j}+\sum_{J_{2}}\left(\epsilon_{j}+\delta_{j}\right) F_{j}=
\end{aligned}
$$

$$
\begin{gathered}
=\mu^{*}\left(\left(m-c m_{1}\right) P+A-K_{X}\right)-\sum_{J_{1} \amalg J_{3} \amalg J_{4}} \delta_{j} F_{j}+c L+\sum_{J_{2}} \epsilon_{j} F_{j}= \\
=B_{m}-\sum_{J_{1} \amalg J_{4}} \delta_{j} F_{j}+c L+\sum_{J_{2}} \epsilon_{j} F_{j} .
\end{gathered}
$$

Let $m_{2}=\min \{m \in \mathbb{N}(P)$ such that $m \geq s\}$. Then $B_{m_{2}}-\sum_{J_{1} \amalg J_{4}} \delta_{j} F_{j}$ is ample, so that $Q_{m_{2}}$ is also ample if the $\epsilon_{j}$ are small enough because $L$ is nef.
Hence $Q_{m}$ is ample for every $m \in \mathbb{N}(P)$ such that $m \geq s$ thanks to the nefness of $P$.
Thus, by Kawamata-Viehweg vanishing theorem (see [Laz04, Cor. 9.1.20]), we find that $H^{1}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner-B\right)\right)=0$ if $m \geq s$ and $m \in \mathbb{N}(P)$.
This implies that the restriction homomorphism

$$
H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right) \rightarrow H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right)
$$

is surjective in this case.
Now we notice that $\mu^{*}\left(m_{2} P\right)_{\left.\right|_{B}}+A_{\left.\right|_{B}}^{\prime \prime}-K_{B}=Q_{\left.m_{2}\right|_{B}}$ is an ample $\mathbb{Q}$-divisor.
Moreover $A_{\left.\right|_{B}}^{\prime \prime}$ is SNCS because $A^{\prime \prime}$ is such and $B$ intersects transversally all the $F_{j}$ 's with $j \neq j_{0}$.
Hence it suffices to verify that all the coefficients of $A_{\left.\right|_{B}}^{\prime \prime}$ are greater than -1 to show that the pair $\left(B,-A_{\left.\right|_{B}}^{\prime \prime}\right)$ is KLT:
Note that if $j \in J_{4}$, then $\mu\left(F_{j}\right) \cap \mathbb{B}(P)=D_{j} \cap \mathbb{B}(P)=\emptyset$, so that $\mu\left(F_{j}\right) \cap \mu(B)=\emptyset$, as $\mu(B) \subseteq \mathbb{B}(P)$.
This implies that $B \cap F_{j}=\emptyset$, that is $F_{j_{\left.\right|_{B}}}=0$.
Moreover if $j \in J_{3}$ and $b_{j} \leq-1$, then $\mu\left(F_{j}\right) \cap \mathbb{B}(P)=\emptyset$, because of the properties of $\mu$, so that, as before, we obtain again that $F_{j_{\left.\right|_{B}}}=0$.
Thus

$$
\begin{gathered}
A_{\left.\right|_{B}}^{\prime \prime}=\sum_{\left(J_{1} \amalg J_{3}\right) \backslash\left\{j_{0}\right\}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{\left.j\right|_{B}}+\sum_{J_{2}}\left(-1+\epsilon_{j}\right) F_{\left.j\right|_{B}}= \\
=\sum_{J_{1} \backslash\left\{j_{0}\right\}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{j_{\left.\right|_{B}}}+\sum_{\substack{J_{3} \backslash\left\{j_{0}\right\} \\
b_{j}>-1}}\left(-c r_{j}+b_{j}-\delta_{j}\right) F_{\left.\right|_{\left.\right|_{B}}}+\sum_{J_{2}}\left(-1+\epsilon_{j}\right) F_{j_{\left.\right|_{B}}} .
\end{gathered}
$$

In particular we can suppose that $j \in J_{2}$ or $b_{j}>-1$. But we have that

- if $b_{j}>-1$ and $r_{j}=0$ then $-c r_{j}+b_{j}-\delta_{j}=b_{j}-\delta_{j}>-1$ (by the choice of the $\delta_{j}$ 's);
- if $b_{j}>-1, r_{j} \neq 0$ and $j \neq j_{0}$ then $-c r_{j}+b_{j}-\delta_{j}>-\frac{b_{j}+1-\delta_{j}}{r_{j}} r_{j}+b_{j}-\delta_{j}=$ -1 .
- if $j \in J_{2}$ then $-1+\epsilon_{j}>-1$;

Therefore the pair $\left(B,-A_{\left.\right|_{B}}^{\prime \prime}\right)$ is KLT.
This enables us to use Shokurov's nonvanishing theorem ([KM00, theorem 3.4]), so that for every integer $k>0$ we can find $\mu_{k} \in \mathbb{N}(P)$, such that $\mu_{k} \geq m_{2}$, $\mu_{k} \geq a, \mu_{k}$ is a multiple of $k$ and

$$
H^{0}\left(B, \mathcal{O}_{B}\left(\mu^{*}\left(\mu_{k} P\right)+\left\ulcorner A^{\prime \prime}\right\urcorner\right)\right) \neq 0
$$

In fact this cohomology group is non zero for every sufficiently large multiple of $m_{2}$.
Now, by the definition of Zariski decomposition, there exists $k_{0} \in \mathbb{N}$ such that $k_{0} P$ is Cartier, $k_{0} D$ is integral and $H^{0}\left(X, t k_{0} P\right) \simeq H^{0}\left(X, t k_{0} D\right)$ for every $t \in \mathbb{N}$. Let us define $m:=\mu_{k_{0}}$. Then

$$
B \nsubseteq B s\left(\left|\mu^{*}(m P)+\left\ulcorner A^{\prime \prime}\right\urcorner\right|\right) \text { and } H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \simeq H^{0}\left(X, \mathcal{O}_{X}(m D)\right)
$$

Now let us write $\left\ulcorner A^{\prime \prime}\right\urcorner=A^{+}-A^{-}$, where $A^{+}$and $A^{-}$are effective divisors without common components. Note that $\left\ulcorner A^{\prime \prime\urcorner}=\sum_{J_{1} \amalg J_{3} \amalg J_{4} \backslash\left\{j_{0}\right\}}\left\ulcorner-c r_{j}+b_{j}-\right.\right.$ $\left.\delta_{j}\right\urcorner F_{j}$, so that if we put $x_{j}:=\left\ulcorner-c r_{j}+b_{j}-\delta_{j}\right\urcorner$ for every $j=1, \ldots, l$, we have that

$$
A^{+}=\sum_{\substack{J_{1} \amalg J_{3} \amalg J_{4} \backslash\left\{j_{0}\right\} \\ x_{j}>0}} x_{j} F_{j}, \quad A^{-}=-\sum_{\substack{J_{1} \amalg J_{3} \amalg J_{4} \backslash\left\{j_{0}\right\} \\ x_{j}<0}} x_{j} F_{j} .
$$

Note that $B \nsubseteq \operatorname{Supp}\left(A^{+}\right)$and $B \nsubseteq \operatorname{Supp}\left(A^{-}\right)$, so that in particular $B \nsubseteq$ $B s\left(\left|\mu^{*}(m P)+A^{+}\right|\right)$.
Moreover we have that $\mu_{*}\left(A^{+}\right) \leq\ulcorner a N\urcorner \leq m N$ :
In fact

$$
\mu_{*}\left(A^{+}\right)=\sum_{\substack{J_{1} \amalg J_{3} \amalg J_{4} \backslash\left\{j_{0}\right\} \\ x_{j}>0}} x_{j} \mu_{*}\left(F_{j}\right)=\sum_{\substack{J_{1} \amalg J_{4} \backslash\left\{j_{0}\right\} \\ x_{j}>0}} x_{j} D_{j} .
$$

But, if $j \in J_{1} \amalg J_{4}$ then $x_{j}=\left\ulcorner-c r_{j}+a_{j}-\delta_{j}\right\urcorner \leq\left\ulcorner a_{j}\right\urcorner$. Hence

$$
\mu_{*}\left(A^{+}\right) \leq \sum_{\substack{J_{1} \amalg J_{4} \backslash\left\{j_{0}\right\} \\ x_{j}>0}}\left\ulcorner a_{j}\right\urcorner D_{j} \leq \sum_{j=1}^{k} \max \left\{0,\left\ulcorner a_{j}\right\urcorner\right\} D_{j} \leq\ulcorner a N\urcorner
$$

as $\ulcorner a N\urcorner \geq\ulcorner A\urcorner=\sum\left\ulcorner a_{j}\right\urcorner D_{j}$ and $\ulcorner a N\urcorner \geq 0$.
From these inequalities it follows that $h^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+A^{+}\right)\right) \leq$ $h^{0}\left(X, \mathcal{O}_{X}(m D)\right)$, by using lemma 2.1.8.
But

$$
\begin{aligned}
H^{0}\left(X, \mathcal{O}_{X}(m D)\right) & \simeq H^{0}\left(X, \mathcal{O}_{X}(m P)\right) \simeq H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right) \hookrightarrow \\
& \hookrightarrow H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+A^{+}\right)\right)
\end{aligned}
$$

so that $H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)\right)\right) \simeq H^{0}\left(Y, \mathcal{O}_{Y}\left(\mu^{*}(m P)+A^{+}\right)\right)$.
Therefore we see that $B \nsubseteq B s\left(\left|\mu^{*}(m P)\right|\right)$, which implies that $\mu(B) \nsubseteq B s(|m P|)$, giving a contradiction.

Theorem 3.9.9. Let $(X, \Delta)$ be an effective pair. If $D \in \operatorname{Div} v_{\mathbb{Q}}(X)$ is such that

1. $D$ is big;
2. $a D-\left(K_{X}+\Delta\right)$ is nef for some $a \in \mathbb{Q}$;
3. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)$ admits a $\mathbb{Q}$-CKM Zariski decomposition $f^{*}(D)=P+N$ and

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is a $P$-DLT pair;
- For every $V \in C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N, \mathbb{B}(P)\right)$ we have that $V \nsubseteq$ $\mathbb{B}_{+}\left(f^{*}(D)\right)$;
then $P$ is semiample.
Proof. We apply lemma 2.5 .3 and we consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma, so that in particular $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition and $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)$ is big and nef. Moreover $P^{\prime}$ is big and nef, $\mathbb{B}_{+}\left(P^{\prime}\right)=$ $\mathbb{B}_{+}(P)=\mathbb{B}_{+}\left(f^{*}(D)\right)$ by lemma 2.5.4 and the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)=\left(Z, \mathbf{A}(\Delta)_{Z}-\right.$ $a N)$ is $P^{\prime}$-DLT.
Then, by lemma 3.9.6, there exists an effective Cartier divisor $\Gamma$ on $Z$ and a rational number $\lambda>0$ such that $P^{\prime}-\lambda \Gamma$ is ample, $\left(Z, \Delta_{Z}+\lambda \Gamma-N^{\prime}\right)$ is $P^{\prime}$-DLT and $C L C\left(Z, \Delta_{Z}+\lambda \Gamma-N^{\prime}\right)=C L C\left(Z, \Delta_{Z}-N^{\prime}\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$. Thus

$$
\left(1+t_{0}\right) P^{\prime}-\left(K_{Z}+\Delta_{Z}+\lambda \Gamma-N^{\prime}\right)=P^{\prime}-\lambda \Gamma+t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)
$$

is ample, being the sum of an ample and a nef divisor.
Moreover, as $\mathbb{B}\left(P^{\prime}\right) \subseteq \mathbb{B}_{+}\left(P^{\prime}\right)=\mathbb{B}_{+}\left(f^{*}(D)\right)$, we have that $\mathbb{B}\left(P^{\prime}\right)$ does not contain any element in $C L C\left(Z, \Delta_{Z}+\lambda \Gamma-N^{\prime}, \mathbb{B}\left(P^{\prime}\right)\right)$, so that $\mathbb{B}\left(P^{\prime}\right)$ does not contain any LC center of the pair $\left(Z, \Delta_{Z}+\lambda \Gamma-N^{\prime}\right)$.
Therefore we can apply theorem 3.9.8 and we get that $P$ is semiample.
The following corollary is a generalization of corollary 3.3.4.
Corollary 3.9.10. Let $(X, \Delta)$ be an effective pair. If $D \in \operatorname{Div} v_{\mathbb{Q}}(X)$ is such that

## 1. $D$ is big;

2. There exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and

- $\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ is a $f^{*}(D)$-DLT pair;
- For every $V \in C L C\left(Z, \mathbf{A}(\Delta)_{Z}, \mathbb{B}\left(f^{*}(D)\right)\right)$ we have that $V \nsubseteq$ $\mathbb{B}_{+}\left(f^{*}(D)\right)$;
then there exists $\beta>0$ such that if

$$
a D-\left(K_{X}+\Delta\right) \text { is nef for some rational number } a>-\beta
$$

then $P$ is semiample.
Proof. Note that $\operatorname{Supp}(N) \subseteq \mathbb{B}_{+}\left(f^{*}(D)\right)$ by lemma 2.5.4. Then by lemma 3.9.5 we can find $\beta>0$ such that if $0 \geq a>-\beta$, then the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is $f^{*}(D)$-DLT and $C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N, \mathbb{B}\left(f^{*}(D)\right)\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}, \mathbb{B}\left(f^{*}(D)\right)\right)$. On the other hand if $a>0$ then $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is a $f^{*}(D)$-DLT pair by lemma 3.9.7 and $C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N, \mathbb{B}\left(f^{*}(D)\right)\right) \subseteq C L C\left(Z, \mathbf{A}(\Delta)_{Z}, \mathbb{B}\left(f^{*}(D)\right)\right)$ by lemma 2.2.8 because of the effectivity of $N$.
Thus, as $\mathbb{B}(P) \subseteq \mathbb{B}\left(f^{*}(D)\right)$, for every $a>-\beta$ we have that $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is a $P$-DLT pair and $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain any LC center in $C L C\left(Z, \mathbf{A}(\Delta)_{Z_{-}}\right.$ $a N, \mathbb{B}(P))$. Therefore we can apply theorem 3.9.9.

Remark 3.9.11. Note that in theorem 3.9.9 we may change our hypothesis 3 by assuming that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^{*}(D)=P+N$ such that

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is a $f^{*}(D)$-DLT pair;
- $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain any element in $C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right.$, $\left.\mathbb{B}\left(f^{*}(D)\right)\right)$.

In fact we have that $\mathbb{B}(P) \subseteq \mathbb{B}\left(f^{*}(D)\right)$, so that these assumptions imply hypothesis 3 of the theorem.
On the other hand note that in corollary 3.9.10 we may replace the hypothesis 2 with the assumptions that there exists a $\mathbb{Q}$-CKM Zariski decomposition $f^{*}(D)=P+N$ such that $\left(Z, \mathbf{A}(\Delta)_{Z}\right)$ is a $P$-DLT pair and $\mathbb{B}_{+}\left(f^{*}(D)\right)$ does not contain any LC center in $C L C\left(Z, \mathbf{A}(\Delta)_{Z}, \mathbb{B}(P)\right)$. This follows by the proof of the corollary itself.

Corollary 3.9.12. Let $(X, \Delta)$ be a weak log Fano pair. Suppose that

- $(X, \Delta)$ is a $-\left(K_{X}+\Delta\right)$-DLT pair;
- for every $V \in C L C\left(X, \Delta, \mathbb{B}\left(-\left(K_{X}+\Delta\right)\right)\right.$ we have $V \nsubseteq \mathbb{B}_{+}\left(-\left(K_{X}+\Delta\right)\right)$;
then $-\left(K_{X}+\Delta\right)$ is semiample.


### 3.10 Alternative hypotheses

In this section we state some of the theorems in the previous chapters with more common "base-point free type" hypotheses. Proofs are very similar.
The following is a different version of theorem 3.4.1:
Theorem 3.10.1. Let $X$ be a normal projective $\mathbb{Q}$-Gorenstein variety and let $\Delta \in \operatorname{Div}(X)$ be effective and such that $(X, \Delta)$ is an LC pair and $(X,(1-b) \Delta)$ is a KLT pair for some rational number $b>0$.
If $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ is such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
2. $\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right)$ does not contain any LC center of the pair $(X, \Delta)$
3. $D$ admits $a \mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

such that $\mathbb{B}(P)$ does not contain any $L C$ center of the pair $(X, \Delta)$.
Then $P$ is semiample.
Proof. First of all we can assume that $\Delta \neq 0$ :
In fact, if $\Delta=0$ the hypotheses implies that the pair $(X, 0)$ is KLT. In particular $(X, 0)$ is a DLT pair, so that we are under the hypotheses of theorem 3.3.6.

Being $\Delta \neq 0$, we can apply lemma 2.2 .10 and we have that $\operatorname{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta)$.
Now, thanks to lemma 2.3.7, we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $a D-\left(K_{X}+\Delta\right)-\lambda \Gamma$ is ample, the pair $(X, \Delta+\lambda \Gamma)$ is LC and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$.

This implies that $\operatorname{Supp}(\Delta)$ contains all the LC centers of the pair $(X, \Delta+\lambda \Gamma)$. Hence, by applying again lemma 2.2.10, we have that for every $\epsilon \in \mathbb{Q}^{+}$the pair $(X,(1-\epsilon) \Delta+\lambda \Gamma)$ is KLT.
Now we have that

$$
a P+a N-\left(K_{X}+(1-\epsilon) \Delta+\lambda \Gamma\right)=\left(a D-\left(K_{X}+\Delta\right)-\lambda \Gamma\right)+\epsilon \Delta
$$

is ample, thanks to the openness of the ample cone, for $\epsilon>0$ small enough. In other words there exists $\epsilon_{0} \in \mathbb{Q}^{+}$such that

$$
a P+a N-\left(K_{X}+\left(1-\epsilon_{0}\right) \Delta+\lambda \Gamma\right)
$$

is ample.
Thus we can apply theorem 3.3.2 to the pair $\left(X,\left(1-\epsilon_{0}\right) \Delta+\lambda \Gamma\right)$.

The following is a different version of theorem 3.6.7:
Theorem 3.10.2. Let $(X, \Delta)$ be an effective LC pair such that $X$ is $\mathbb{Q}$-Gorenstein. Let $D \in D i v_{\mathbb{Q}}(X)$ be such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some $a \in \mathbb{Q}^{+} \cup\{0\}$;
2. $\mathbb{B}_{+}\left(a D-\left(K_{X}+\Delta\right)\right) \nsupseteq V$, for every $V \in C L C(X, \Delta)$;
3. $D$ has a $\mathbb{Q}$-CKM Zariski decomposition

$$
D=P+N
$$

such that $\mathbb{B}(P) \cap V=\emptyset$ for every $V \in C L C(X, \Delta)$ such that $V \nsubseteq \operatorname{Supp}(\Delta)$;
then $P$ is semiample.
Proof. Thanks to lemma 2.3.7, we can find an effective Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $a D-\left(K_{X}+\Delta\right)-\lambda \Gamma$ is ample, the pair $(X, \Delta+\lambda \Gamma)$ is LC and $C L C(X, \Delta)=C L C(X, \Delta+\lambda \Gamma)$.
Hence, thanks to lemma 2.2.9 and lemma 2.2.8, we get that for every $\epsilon \in \mathbb{Q}^{+}$

$$
C L C(X,(1-\epsilon) \Delta+\lambda \Gamma-a N) \subseteq\{V \in C L C(X, \Delta) \text { such that } V \nsubseteq \operatorname{Supp}(\Delta)\}
$$

so that, by hypothesis, $\mathbb{B}(P)$ does not intersect any LC center of the pair $(X,(1-\epsilon) \Delta+\lambda \Gamma)$. Moreover

$$
a P+a N-\left(K_{X}+(1-\epsilon) \Delta+\lambda \Gamma\right)=\left(a D-\left(K_{X}+\Delta\right)-\lambda \Gamma\right)+\epsilon \Delta
$$

is ample if $\epsilon$ is sufficiently small thanks to the openness of the ample cone. Thus we obtain the semiampleness of $P$ by applying theorem 3.6.5 to the pair $(X,(1-\epsilon) \Delta+\lambda \Gamma)$.

The following is a different version of corollary 3.7.5:
Corollary 3.10.3. Let $(X, \Delta)$ be an effective pair. Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some $a \in \mathbb{Q}$;
2. there exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=$ $P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is an LC pair;
- $\mathbb{B}_{+}\left(f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)\right)$ does not contain any LC center of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$;
- $\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)=\emptyset$, or $P_{\left.\right|_{\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)}}$ is semiample.

Then $P$ is semiample.
Proof. Define $L:=f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$. Then we can apply lemma 2.3.7 to the big and nef $\mathbb{Q}$-divisor $L$ and to the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ and we find a Cartier divisor $\Gamma$ and a rational number $\lambda>0$ such that $L-\lambda \Gamma$ is ample, $\left(Z, \mathbf{A}(\Delta)_{Z}-\right.$ $a N+\lambda \Gamma)$ is LC and $C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N+\lambda \Gamma\right)=C L C\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$.
Furthermore, we can choose $\Gamma$ generically in its linear series and we have that $B s(|\Gamma|)=\mathbb{B}_{+}(L)$. Then, by Bertini's lemma, we can suppose that, outside $\mathbb{B}_{+}(L), \Gamma$ is smooth and it intersects $\mathbf{A}(\Delta)_{Z}-a N$ in a simple normal crossing way.
Now we apply lemma 2.5 .3 , we consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma and define $B:=\Delta_{Z}-N^{\prime}+\lambda \Gamma$. We will show that the pair $(Z, B)$ and the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $P^{\prime}$ satisfy the hypotheses of theorem 3.7.4:
First of all we have that $t_{0} P^{\prime}-\left(K_{Z}+B\right)=t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}+\lambda \Gamma-N^{\prime}\right)=$ $P+L-\lambda \Gamma$ is ample, being the sum of a nef and an ample divisor.
By the LCness of the pair $(Z, B)$ we get that all the coefficients of $B$ are less than or equal to 1 and property 5 holds. Moreover property 1 is trivially verified and property 3 follows by the definition of $\mathbb{Q}$-CKM Zariski decomposition because $\Delta_{Z}$ is effective.

In order to prove that property 4 holds we will show that $\widetilde{\operatorname{Nklt}}(Z, B) \subseteq$ $\widetilde{\operatorname{Nklt}}\left(Z, \Delta_{Z}-N^{\prime}\right)=\widetilde{\operatorname{Nklt}}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$, so that we can use the hypothesis of the corollary:
By the choice of $\Gamma$ we have that $C L C\left(Z, \Delta_{Z}-N^{\prime}\right)=C L C(Z, B)$ and $N S N C(B) \subseteq N S N C\left(\Delta_{Z}-N^{\prime}\right) \cup \mathbb{B}_{+}(L)$.
Then, if $V \in C L C(Z, B)$ and $V \subseteq \operatorname{Sing}(Z) \cup N S N C(B)$, we get that $V \in$ $C L C\left(Z, \Delta_{Z}-N^{\prime}\right)$ and $V \subseteq \operatorname{Sing}(Z) \cup N S N C\left(\Delta_{Z}-N^{\prime}\right) \cup \mathbb{B}_{+}(L)$. This implies that $V \subseteq \operatorname{Sing}(Z) \cup N S N C\left(\Delta_{Z}-N^{\prime}\right)$, because by hypothesis $\mathbb{B}_{+}(L)$ does not contain LC centers of the pair ( $Z, \Delta_{Z}-N^{\prime}$ ). Hence $V \subseteq \widetilde{\operatorname{Nklt}}\left(Z, \Delta_{Z}-N^{\prime}\right)$, and we get the required inclusion. Therefore we can apply theorem 3.7.4 and we are done.

The following is an alternative version of corollary 3.7.7:
Corollary 3.10.4. Let $(X, \Delta)$ be an effective pair, with $\operatorname{dim} X \geq 2$.
Let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that:

1. $a D-\left(K_{X}+\Delta\right)$ is big and nef for some rational number $a \geq 0$;
2. there exists a projective birational morphism $f: Z \rightarrow X$ such that $f^{*}(D)=$ $P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and

- $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is an LC pair;
- $\mathbb{B}_{+}\left(f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)\right)$ does not contain divisorial LC centers of the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$;
- $\operatorname{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)=\emptyset$, or $P_{\mathrm{Nklt}_{2}\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)}$ is semiample.

Then $P$ is semiample.
Note that if $a \geq 0$ it suffices that $(X, \Delta)$ is an LC pair for the pair $\left(Z, \mathbf{A}(\Delta)_{Z}-\right.$ $a N)$ to be LC.

Proof. Let us apply lemma 2.5 .3 and consider $t_{0}, D^{\prime}, P^{\prime}, N^{\prime}, \Delta_{Z}$ as in the lemma, so that $D^{\prime}=P^{\prime}+N^{\prime}$ is a $\mathbb{Q}$-CKM Zariski decomposition, $\Delta_{Z}$ is effective and the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)=\left(Z, \mathbf{A}(\Delta)_{Z}-a N\right)$ is LC.
Note that by lemma 2.3.9, we have that $f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$ is logbig in codimension 1 for the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)$, so that $t_{0} P^{\prime}-\left(K_{Z}+\Delta_{Z}-N^{\prime}\right)=P+$ $f^{*}\left(a D-\left(K_{X}+\Delta\right)\right)$ is nef and logbig in codimension 1 for the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)$. Thus we can apply theorem 3.7.6 to the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $P^{\prime}$ and to the pair $\left(Z, \Delta_{Z}-N^{\prime}\right)$ because all the hypotheses are satisfied and we get the semiampleness of $P$.

### 3.11 Examples

## Basic construction

The following general construction is due to Hacon and McKernan (see [Laz09, theorem A.6]). The choice of the surface $S$ is due to Gongyo (see [Gon09, Example 5.2]).

Let $S$ be the surface obtained by blowing up $\mathbb{P}^{2}$ in 9 very general points, so that $-K_{S}$ is nef but not semiample. Let $S \subseteq \mathbb{P}^{N}$ be a projectively normal embedding.
Let $X_{0}$ be the cone over $S$ and let $\phi: X \rightarrow X_{0}$ be the blowing-up at the vertex. We have that $X \simeq \mathbb{P}_{S}\left(\mathcal{O}_{S} \oplus \mathcal{O}_{S}(-H)\right)$, where $H$ is a sufficiently ample divisor on $S$. Now we denote by $\pi: X \rightarrow S$ the natural projection, and by $E$ the $\phi$-exceptional divisor, so that $E \simeq S$.
Note that $-\left(K_{X}+E\right)$ is big and nef. Hence $(X, E)$ is a weak log Fano DLT pair of dimension 3 and $E$ is the only LC center of $(X, E)$; in particular it is a PLT pair.
Now, by adjunction, we have that

$$
-\left(K_{X}+E\right)_{\left.\right|_{E}}=-K_{E},
$$

whence $-\left(K_{X}+E\right)$ is not semiample because $-K_{S}$ is not semiample.

## Applications

In example 3.11.1 we will show that, with the notation of the previous subsection, $E \subseteq \mathbb{B}_{+}\left(-\left(K_{X}+E\right)\right)$, but $E \nsubseteq \mathbb{B}\left(-\left(K_{X}+E\right)\right)$.
Then we have that $(X, E)$ is a PLT (hence DLT) pair such that

1. $-\left(K_{X}+E\right)$ is big and nef;
2. $\mathbb{B}\left(-\left(K_{X}+E\right)\right)$ does not contain the only LC center of the pair $(X, E)$;
3. $-\left(K_{X}+E\right)$ is not semiample.

Thus it shows that in many of our theorems, e.g. theorem 3.3.3, theorem 3.3.6, theorem 3.1.2, theorem 3.5.3 and corollary 3.7.5, the hypothesis about the $\mathbb{B}_{+}$ cannot be removed or replaced with the same hypothesis on the stable base locus.

In example 3.11.2 we will construct, for arbitrary large $k \in \mathbb{N}$, a $\mathbb{Q}$-divisor $P$ and a $\mathbb{Q}$-divisor $\Delta$ on $X$ such that $(X, \Delta)$ is DLT and the following conditions are satisfied:

1. $P$ is big and nef;
2. $P-\left(K_{X}+\Delta\right)$ is big and nef;
3. The pair $(X, \Delta)$ has $m \geq k$ LC centers and $m-1$ LC centers among these are not contained in $\mathbb{B}_{+}(P)$;
4. $P$ is not semiample.

Note that, by lemma 2.3.9, property 3 implies that there is one LC center of $(X, \Delta)$, say $V$, such that $P$ remains big when restricted to every LC center in $C L C(X, \Delta) \backslash\{V\}$.

This shows that in many of our theorems, e.g. theorem 3.3.3, theorem 3.5.3 and corollary 3.7 .5 , we cannot lighten the hypothesis on $\mathbb{B}_{+}$, in the sense that we must assume that it does not contain any LC center.
Similarly we cannot sharpen the hypothesis of logbigness of $P$ in corollary 3.3.11, in theorem 3.7.6 and in corollary 3.7.8.

Example 3.11.1. Let us show that $E \subseteq \mathbb{B}_{+}\left(-\left(K_{X}+E\right)\right)$ :
We identify $E \simeq S=B l_{p_{1}, \ldots, p_{9}} \mathbb{P}^{2}$, we denote by $\epsilon: E \rightarrow \mathbb{P}^{2}$ the blowing-up in 9 very general points, we denote by $H$ an hyperplane section on $\mathbb{P}^{2}$ and by $F_{1}, \ldots, F_{9}$ the 9 distinct prime $\epsilon$-exceptional divisors. As $-\left(K_{X}+E\right)$ is big and nef, thanks to Nakamaye's theorem we have that

$$
\mathbb{B}_{+}\left(-\left(K_{X}+E\right)\right)=\operatorname{Null}\left(-\left(K_{X}+E\right)\right) .
$$

Hence, it suffices to show that $\left(-\left(K_{X}+E\right)^{2} \cdot E\right)=0$.
But $\left(-\left(K_{X}+E\right)^{2} \cdot E\right)=\left(-\left(K_{X}+E\right)_{\left.\right|_{E}}\right)^{2}=\left(-K_{E}\right)^{2}=\left(3 \epsilon^{*}(H)-F_{1}-\ldots-F_{9}\right)^{2}=$ 0 .

On the other hand we have that $E \nsubseteq \mathbb{B}\left(-\left(K_{X}+E\right)\right)$ :
Note that

$$
\begin{aligned}
h^{0}\left(E,-\left(K_{X}+E\right)_{\left.\right|_{E}}\right) & =h^{0}\left(E,-K_{E}\right)=h^{0}\left(E, 3 \epsilon^{*}(H)-F_{1}-\ldots-F_{9}\right)= \\
& =h^{0}\left(\mathbb{P}^{2}, \mathcal{J}_{\left\{p_{1}, \ldots p_{9}\right\}}(3)\right) \neq 0 .
\end{aligned}
$$

Now consider the exact sequence

$$
\begin{gathered}
0 \rightarrow H^{0}\left(X,-\left(K_{X}+E\right)-E\right) \rightarrow H^{0}\left(X,-\left(K_{X}+E\right)\right) \rightarrow H^{0}\left(E,-\left(K_{X}+E\right)_{\left.\right|_{E}}\right) \rightarrow \\
\rightarrow H^{1}\left(X,-\left(K_{X}+E\right)-E\right) \rightarrow \ldots
\end{gathered}
$$

We have that $-\left(K_{X}+E\right)-E=K_{X}-2\left(K_{X}+E\right)$, where $-2\left(K_{X}+E\right)$ is big and nef. Hence, by Kawamata-Viehweg vanishing (see [Laz04, theorem 4.3.1]), $H^{1}\left(X,-\left(K_{X}+E\right)-E\right)=0$.
Thus we obtain the surjectivity of the restriction map

$$
H^{0}\left(X,-\left(K_{X}+E\right)\right) \rightarrow H^{0}\left(E,-\left(K_{X}+E\right)_{\left.\right|_{E}}\right) \neq 0
$$

which implies that $E \nsubseteq B s\left(\left|-\left(K_{X}+E\right)\right|\right)$, so that in particular $E \nsubseteq$ $\mathbb{B}\left(-\left(K_{X}+E\right)\right)$.

Example 3.11.2. Let $A_{1}, \ldots, A_{k}$ be smooth hyperplane sections on $X_{0}$ such that $v \notin A_{i}$ for every $i=1, \ldots, k$ and the ample divisor $A:=\sum A_{i}$ is SNC. Bertini's theorem assures the existence of such divisors. Let

$$
P:=-\left(K_{X}+E\right)+\phi^{*}(A) .
$$

Moreover define $\Delta:=E+\phi^{*}(A)=E+\phi_{*}^{-1}(A)$.
Note that the pair $(X, \Delta)$ is DLT, because $X$ is smooth and $E+\phi_{*}^{-1}(A)$ is a SNC divisor, and the LC centers of $(X, \Delta)$ are exactly the irreducible components of finite intersections of prime divisors in the support of $\Delta$, namely $E$ and $\phi^{*}\left(A_{i}\right)$ for every $i \in\{1, \ldots k\}$.
Note also that $P$ is big and nef because it is the sum of two big and nef divisors and

$$
P-\left(K_{X}+\Delta\right)=-\left(K_{X}+E\right)+\phi^{*}(A)-\left(K_{X}+E+\phi^{*}(A)\right)=2\left(-\left(K_{X}+E\right)\right)
$$

is also big and nef.
Now, by lemma 2.1.9 there exists $\epsilon>0$ such that $\phi^{*}(A)-\epsilon E$ is ample.
Then we can write

$$
P=\left(-\left(K_{X}+E\right)+\phi^{*}(A)-\epsilon E\right)+\epsilon E
$$

where $-\left(K_{X}+E\right)+\phi^{*}(A)-\epsilon E$ is ample.
This implies that $\mathbb{B}_{+}(P) \subseteq E$. On the other hand $\phi^{*}(A) \cap E=\emptyset$, so that the only LC center of the pair $(X, \Delta)$ contained in $E$ is $E$ itself.
Thus the only LC center of the pair $(X, \Delta)$ contained in $\mathbb{B}_{+}(P)$ is $E$.
Moreover $\phi^{*}(A)_{\left.\right|_{E}}=0$ because $\phi^{*}(A) \cap E=\emptyset$.
Then $P_{\left.\right|_{E}}=-\left(K_{X}+E\right)_{\left.\right|_{E}}=-K_{E}$ is not semiample, because $E \simeq S$. Therefore $P$ is not semiample.

## Chapter 4

## Asymptotic base loci on singular varieties

The aim of this chapter is to generalize theorem 1.0.6 to the case when the variety $X$ is not smooth. In particular in section 4.1 we study some particular cases, we prove the main theorems in section 4.2 , while in section 4.3 we give, as an application, a characterizarion of nef and abundant divisors. All the main results of this chapter appear in [CD11].
Throughout the chapter, unless otherwise stated, by divisor we mean a Cartier divisor and, for $\mathbb{K}=\mathbb{Q}, \mathbb{R}$, by $\mathbb{K}$-divisor we mean an element of $\operatorname{Div}_{\mathbb{K}}(X)$.

### 4.1 Some special cases

In this section we investigate the relationship between $\mathbb{B}_{-}(D)$ and $\operatorname{NNef}(D)$ just exploiting the fact that, by [ELMNP06, Prop. 2.8], we already know that they are equal on smooth varieties. After few lemmas about the behaviour of the restricted base locus under birational maps, we prove that $\mathbb{B}_{-}(D)$ and $\operatorname{NNef}(D)$ agree on the smooth locus of $X$. Some considerations will then allow us to conclude that $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)$ on any normal surface.

Lemma 4.1.1. Let $X$ and $Y$ be normal projective varieties and let $f: Y \rightarrow X$ be a birational morphism. If $A$ is an ample $\mathbb{R}$-divisor on $X$ and $x \in X$ is not contained in the image of the exceptional locus exc $(f)$, then $f^{-1}(x) \notin \mathbb{B}_{+}\left(f^{*} A\right)$.

Proof. The lemma immediately follows from lemma 2.3.8.
Lemma 4.1.2. Let $X$ and $Y$ be normal projective varieties and let $f: Y \rightarrow X$ be a birational morphism. If $D$ is an $\mathbb{R}$-divisor on $X$ and $x \in X$ is not contained in $f(\operatorname{exc}(f))$, then $x \in \mathbb{B}_{-}(D) \Rightarrow f^{-1}(x) \in \mathbb{B}_{-}\left(f^{*} D\right)$.

Proof. Take $x \in \mathbb{B}_{-}(D)$. By definition, there exists $A_{X}$, an ample $\mathbb{R}$-divisor on $X$, such that $x \in \mathbb{B}\left(D+A_{X}\right)$. This implies that $f^{-1}(x) \in \mathbb{B}\left(f^{*} D+f^{*} A_{X}\right)$ : In fact, if by contradiction there exists $E^{\prime}$, an effective $\mathbb{R}$-divisor on Y , such that $f^{-1}(x) \notin \operatorname{Supp}\left(E^{\prime}\right)$ and $E^{\prime} \sim_{\mathbb{R}} f^{*}\left(D+A_{X}\right)$ then $f_{*} E^{\prime} \sim_{\mathbb{R}} D+A_{X}$ as $\mathbb{R}$-Weil divisor; but $D+A_{X}$ is an $\mathbb{R}$-Cartier divisor, hence $f_{*} E^{\prime}$ is an effective $\mathbb{R}$-Cartier divisor such that $x \notin \operatorname{Supp}\left(f_{*} E^{\prime}\right)$, i.e., $x \notin \mathbb{B}\left(D+A_{X}\right)$.

By lemma 4.1.1 there exists $A_{Y}$, an ample $\mathbb{R}$-divisor on $Y$, and $E_{Y}$, an effective $\mathbb{R}$-divisor on $Y$, such that $f^{*} A_{X}=A_{Y}+E_{Y}$ and $f^{-1}(x) \notin \operatorname{Supp}\left(E_{Y}\right)$. Hence $f^{-1}(x) \in \mathbb{B}\left(f^{*} D+A_{Y}\right) \subseteq \mathbb{B}_{-}\left(f^{*} D\right)$.

We can now compare $\mathbb{B}_{-}(D)$ and $\operatorname{NNef}(D)$ on the smooth locus of $X$. To this purpose define $X_{\mathrm{sm}}$ to be the smooth locus of $X$, i.e., $X_{\mathrm{sm}}:=X \backslash \operatorname{Sing}(X)$. The following holds:

Proposition 4.1.3. Let $X$ be a normal projective variety and let $D$ be an $\mathbb{R}$ divisor on $X$. We have that $\mathbb{B}_{-}(D) \cap X_{\mathrm{sm}}=\operatorname{NNef}(D) \cap X_{\mathrm{sm}}$.

Proof. In general $\operatorname{NNef}(D) \subseteq \mathbb{B}_{-}(D)$ (see [BBP09, lemma 1.8]), thus it is enough to show that $\mathbb{B}_{-}(D) \cap X_{\text {sm }} \subseteq \operatorname{NNef}(D)$.
Let $f: Y \rightarrow X$ be a resolution of the singularities of $X$ constructed as a series of blowings-up along smooth centers contained in $\operatorname{Sing}(X)$ (this is possible by Hironaka's theorem - cf. [Laz04, Th. 4.1.3]). By lemma 4.1.2, $f\left(\mathbb{B}_{-}\left(f^{*}(D)\right) \supseteq\right.$ $\mathbb{B}_{-}(D) \cap X_{\mathrm{sm}}$. Since $Y$ is smooth then $\mathbb{B}_{-}\left(f^{*} D\right)=\operatorname{NNef}\left(f^{*} D\right)$ by [ELMNP06, Prop. 2.8], therefore $\operatorname{NNef}(D)=f\left(\operatorname{NNef}\left(f^{*}(D)\right)\right)=f\left(\mathbb{B}_{-}\left(f^{*} D\right)\right) \supseteq \mathbb{B}_{-}(D) \cap$ $X_{\mathrm{sm}}$, where the first equality is a straightforward consequence of [BBP09, Lemma 1.6].

Note that in the following section we will give a generalization of this result (see corollary 4.2.10 and remark 4.2.11).
Recall that for every normal variety $X$ and $\mathbb{R}$-divisor $D$ on $X$ we have that both $\mathbb{B}_{-}(D)$ and $\operatorname{NNef}(D)$ are at most a countable union of Zariski closed subsets of $X$ (see, for example, [BBP09]).
In particular if the $W_{i}$ 's are irreducible subvarieties of $X$ such that $\mathbb{B}_{-}(D)=$ $\bigcup_{i \in \mathbb{N}} W_{i}$, we say that the components of $\mathbb{B}_{-}(D)$ are the subvarieties $W_{i}$ that are maximal with respect to inclusion. It is easy to see that this definition does not depend on initial the choice of the $W_{i}$ 's. In the same way we define the components of $\operatorname{NNef}(D)$.
We have that the following holds:
Corollary 4.1.4. Let $X$ be a normal projective variety and let $D$ be an $\mathbb{R}$-divisor on $X$. If $\mathbb{B}_{-}(D)$ has only divisorial components, then $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)$.

Proof. As before, we need only to show that $\mathbb{B}_{-}(D) \subseteq \operatorname{NNef}(D)$. Take $x \in$ $\mathbb{B}_{-}(D)$. By hypothesis, there exists a prime divisor $E$ such that $x \in E$ and $E \subseteq \mathbb{B}_{-}(D)$, upon identifying $E$ with its support. Write $\operatorname{NNef}(D)=\cup_{i \in \mathbb{N}} V_{i}$, where the $V_{i}$ 's are the components of $\operatorname{NNef}(D)$. By prop. 4.1.3, $E \cap X_{\mathrm{sm}} \subseteq$ $\cup_{i \in \mathrm{~N}}\left(V_{i} \cap E\right)$, but, since $\operatorname{codim}(\operatorname{Sing}(X)) \geq 2$, we also have that $E \cap X_{\mathrm{sm}}$ is countably dense in $E$ (cf. [DiB11, Def. 2.1, Lemma 2.2(2)]). Therefore there must exist $j \in \mathbb{N}$ such that $V_{j}=E$, i.e., $E \subseteq \operatorname{NNef}(D)$. Thus $x \in \operatorname{NNef}(D)$.

Corollary 4.1.5. Let $X$ be a normal projective surface and let $D$ be an $\mathbb{R}$ divisor on $X$. Then $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)$.

Proof. We can assume that $D$ is pseudoeffective, so that $\mathbb{B}_{-}(D)$ is strictly contained in $X$. We can write $\mathbb{B}_{-}(D)=\bigcup \mathbb{B}(D+A)$, where the union is taken over all ample $\mathbb{R}$-divisors $A$ such that $D+A$ is a $\mathbb{Q}$-divisor. By [ELMNP09, Prop. 1.1] it follows that for every ample $A$ as above $\mathbb{B}(D+A)$ has no isolated points, that is $\mathbb{B}(D+A)$ has no irreducible components of dimension 0 , which implies that the same holds for $\mathbb{B}_{-}(D)$. Since $X$ is a surface this is equivalent to saying
that $\mathbb{B}_{-}(D)$ has only divisorial components, so that we can conclude by cor. 4.1.4.

We can also prove the following:
Proposition 4.1.6. Let $X$ be a normal projective variety and let $D \in \operatorname{Div}_{\mathbb{Q}}(X)$ be such that $D$ is big and $D$ admits a $\mathbb{Q}$-CKM Zariski decomposition $D=P+N$. Then $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)=\operatorname{Supp}(N)$.

Proof. We have that $\operatorname{NNef}(D) \subseteq \mathbb{B}_{-}(D)$ by [BBP09, lemma 1.8$]$, whence it is enough to show that $\mathbb{B}_{-}(D) \subseteq \operatorname{Supp}(N)$ and $\operatorname{Supp}(N) \subseteq \operatorname{NNef}(D)$.
We begin by showing that $\mathbb{B}_{-}(D) \subseteq \operatorname{Supp}(N)$ :
In fact if $A \in D i v_{\mathbb{Q}}(X)$ is ample, then $P+A$ is also ample, so that the stable base locus $\mathbb{B}(P+A)=\emptyset$. Hence $\mathbb{B}(D+A) \subseteq \mathbb{B}(P+A) \cup \operatorname{Supp}(N) \subseteq \operatorname{Supp}(N)$. But $\mathbb{B}_{-}(D)=\bigcup_{A} \mathbb{B}(D+A)$, where the union is taken over all ample $A \in D i v_{\mathbb{Q}}(X)$, so that $\mathbb{B}_{-}(D) \subseteq \operatorname{Supp}(N)$.
Let us show now that $\operatorname{Supp}(N) \subseteq \operatorname{NNef}(D)$ :
We can suppose $N \neq 0$, otherwise there is nothing to prove. Write $N=\sum a_{i} N_{i}$, where the $a_{i}$ 's are rational positive numbers and the $N_{i}$ 's are prime divisors. We will show that, for every $i,\left(\operatorname{ord}_{N_{i}}\right)_{\text {num }}(D)>0$, which implies that $\operatorname{Supp}(N)=$ $\bigcup N_{i} \subseteq \operatorname{NNef}(D)$.
As $D$ is big by [Pro02, proposition 7.4] we know that $D=P+N$ is a FujitaZariski decomposition. Moreover if $E \in|D|_{\equiv}$, Then $D-E \equiv 0$ is nef and $D-E \leq D$. Hence $D-E \leq P$, so that $E \geq N$. Thus for every $i$ we have that $\operatorname{ord}_{N_{i}}(E) \geq \operatorname{ord}_{N_{i}}(N)=a_{i}>0$, which implies that $\left(\operatorname{ord}_{N_{i}}\right)_{\text {num }}(D)>0$.

Corollary 4.1.7. Let $X$ be a normal projective variety and suppose $D \in$ $\operatorname{Div}_{\mathbb{Q}}(X)$ is big. If $f: Z \rightarrow X$ is a projective birational morphism from a normal variety $Z$ such that $f^{*}(D)=P+N$ is a $\mathbb{Q}$-CKM Zariski decomposition and $P$ is semiample, then $\operatorname{NNef}(D)=\mathbb{B}_{-}(D)=\mathbb{B}(D)=f(\operatorname{Supp}(N))$.

Proof. $\operatorname{NNef}(D)=f(\operatorname{Supp}(N))$ because $\operatorname{NNef}(D)=f\left(\operatorname{NNef}\left(f^{*}(D)\right)\right)$ by $[\operatorname{BBP} 09$, lemma 1.6] and $\operatorname{NNef}\left(f^{*}(D)\right)=\operatorname{Supp}(N)$ by proposition 4.1.6. Moreover $\mathbb{B}(D) \subseteq f(\operatorname{Supp}(N))$ because $\mathbb{B}(D)=f\left(\mathbb{B}\left(f^{*}(D)\right)\right)$ and $\mathbb{B}\left(f^{*}(D)\right) \subseteq \operatorname{Supp}(N)$ by the semiampleness of $P$. Therefore the statement follows by noting that by $\left[\operatorname{BBP} 09\right.$, lemma 1.8] we always have that $\operatorname{NNef}(D) \subseteq \mathbb{B}_{-}(D) \subseteq \mathbb{B}(D)$.

Remark 4.1.8. In most of the main theorems of chapter 3 we prove the semiampleness of the positive part of a $\mathbb{Q}$-CKM Zariski decomposition of a birational pullback of a big $\mathbb{Q}$-divisor $D$ enjoying some good given properties.
Corollary 4.1.7 shows that in all these cases the stable base locus of $D$ coincides with the restricted base locus and the non-nef locus.

### 4.2 Main results

In this section we will prove theorem 1.0.7, this is done in theorem 4.2.7.
The idea is to prove that given an effective KLT pair $(X, \Delta)$ and an effective integral divisor $D$ then we have that

$$
\mathbb{B}_{-}(D) \subseteq \bigcup_{p} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)) \subseteq \operatorname{NNA}(D)
$$

The former inclusion is a consequence of Nadel's vanishing theorem and the proof is just an easy generalization to singular varieties of some arguments in [ELMNP06, Prop. 2.8]. This is the content of lemma 4.2.1.
To prove the latter inclusion we notice that by considering a suitable logresolution we can reduce to the smooth case and get rid of the boundary $\Delta$ at the same time (see lemma 4.2.2 and proposition 4.2.3), so that the result follows by some considerations in [ELMNP06] (see prop. 4.4 and theorem 4.2.5). The rest of the section is devoted to slight generalizations and resumptive corollaries.

Lemma 4.2.1. Let $(X, \Delta)$ be an effective pair. Let $D$ be an integral divisor. Then $\mathbb{B}_{-}(D) \subseteq \bigcup_{p} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|))$.

Proof. We will follow [ELMNP06, proof of Prop. 2.8], taking into account the fact that $X$ may be singular. Take $x \in X$ and suppose that, for every $p \geq 1$, $\mathcal{J}((X, \Delta) ;\|p D\|)_{x}=\mathcal{O}_{X, x}$. Let $A$ be a fixed very ample divisor such that $A-\left(K_{X}+\Delta\right)$ is ample.
Recall that for every $p \geq 1, \mathcal{J}((X, \Delta) ;\|p D\|)=\mathcal{J}\left((X, \Delta) ; \frac{1}{k} D_{p k}\right)$ where $k$ is sufficiently large and divisible and $D_{p k}$ is a general element of $|p k D|$. In particular $\frac{1}{k} D_{p k} \sim_{\mathbb{Q}} p D$. Hence, if $\operatorname{dim} X=n$, by Nadel's vanishing theorem in the singular setting (see [Laz04, Theorem 9.4.17]), we have that for every $i \geq 1$

$$
H^{i}\left(X, \mathcal{O}_{X}((n+1) A+p D) \otimes \mathcal{O}_{X}(-i A) \otimes \mathcal{J}((X, \Delta) ;\|p D\|)\right)=0
$$

Thus, by Mumford's theorem (see [Laz04, Thm. 1.8.5] or [Laz04, volume 2, p. 194]), we have that $\mathcal{O}_{X}((n+1) A+p D) \otimes \mathcal{J}((X, \Delta) ;\|p D\|)$ is globally generated. In particular, since $\mathcal{J}((X, \Delta) ;\|p D\|)_{x}=\mathcal{O}_{X, x}$ and $\mathcal{J}((X, \Delta) ;\|p D\|)$ is an ideal sheaf, this implies that $x \notin \operatorname{Bs}|(n+1) A+p D|$ for every $p \geq 1$, i.e., $x \notin \mathbb{B}_{-}(D)$.

Lemma 4.2.2. Let $X$ be a normal variety and let $\Delta$ be $a \mathbb{Q}$-Weil divisor such that $(X, \Delta)$ is a KLT pair. Then there exists a log-resolution of $(X, \Delta)$, say $f: Y \rightarrow X$, such that for every $y \in Y$

$$
\operatorname{mult}_{y}\left(\mathbf{L}(\Delta)_{Y}\right)<1
$$

Proof. By [KM00, Proposition 2.36] there exists a $\log$-resolution $f: Y \rightarrow X$ such that $\operatorname{Supp}\left(\mathbf{L}(\Delta)_{Y}\right)$ is smooth. By definition

$$
\mathbf{L}(\Delta)_{Y}=\sum_{\substack{E \subseteq Y \\ a(E, X, \Delta)<0}}-a(E, X, \Delta) E
$$

and, by hypothesis, $-a(E, X, \Delta)<1$ for any prime divisor $E \subseteq Y$. Since the support of $\mathbf{L}(\Delta)_{Y}$ is smooth, for any $y \in Y$ either $y$ does not belong to $\operatorname{Supp}\left(\mathbf{L}(\Delta)_{Y}\right)$ or $y$ belongs to only one irreducible component of $\operatorname{Supp}\left(\mathbf{L}(\Delta)_{Y}\right)$, say $E_{y}$. In the latter case this means that $\operatorname{mult}_{y}\left(\mathbf{L}(\Delta)_{Y}\right)=-a\left(E_{y}, X, \Delta\right)<1$.

Proposition 4.2.3. Let $(X, \Delta)$ be an effective KLT pair such that $X$ is smooth of dimension $n$ and for every $x \in X \operatorname{mult}_{x}(\Delta)<1$. Let $D$ be a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$. For every $x \in \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|D\|))$ there exists $p_{x} \in \mathbb{N}$ such that $x \in \mathcal{Z}\left(\mathcal{J}\left(X,\left\|p_{x} D\right\|\right)\right)$. In particular $\mathcal{Z}(\mathcal{J}((X, \Delta) ;\|D\|)) \subseteq$ $\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}(X,\|p D\|))$.

Proof. Take $x \in \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|D\|))$. By definition, $\mathcal{J}((X, \Delta) ;\|D\|)$ is the unique maximal element of the family of ideals $\left\{\mathcal{J}\left((X, \Delta) ; \frac{1}{k}|k D|\right)\right\}_{k \in \mathbb{N}^{+}}$. Moreover for every $k \geq 2$, by [Laz04, Prop. 9.2.26] (see also p. 185, volume 2) and [Laz04, Ex. 9.3.57(i)], we have that $\mathcal{J}\left((X, \Delta) ; \frac{1}{k}|k D|\right)=\mathcal{J}\left(X, \Delta+\frac{1}{k} D_{k}\right)$ where $D_{k}$ is general in $|k D|$. Therefore $x \in \mathcal{Z}\left(\mathcal{J}\left(X, \Delta+\frac{1}{k} D_{k}\right)\right)$ for every $k \geq 2$.
Hence, by [Laz04, Prop. 9.5.13], we must have that $\operatorname{mult}_{x}\left(\Delta+\frac{1}{k} D_{k}\right) \geq 1$. Since mult $_{x}$ is additive and, by the hypothesis on $\Delta, \operatorname{mult}_{x}(\Delta)=1-c_{x}$ for a certain $c_{x}>0$, then for any $k \geq 2, \operatorname{mult}_{x}\left(\frac{1}{k} D_{k}\right) \geq c_{x}$.
Set $p_{x}$ be any positive natural number such that $p_{x} \geq \frac{n}{c_{r}}$. By [Laz04, Th. 11.1.8(i)] and arguing as before $\mathcal{J}\left(X,\left\|p_{x} D\right\|\right)=\mathcal{J}\left(X, p_{x}\|D\|\right)=\mathcal{J}\left(X, \frac{p_{x}}{h}|h D|\right)=$ $\mathcal{J}\left(X, \frac{p_{x}}{h} D_{h}\right)$ for $h$ sufficiently large. Since $\operatorname{mult}_{x}\left(\frac{p_{x}}{h} D_{h}\right)=p_{x} \operatorname{mult}_{x}\left(\frac{1}{h} D_{h}\right) \geq$ $p_{x} c_{x}=n$ then by [Laz04, Prop. 9.3.2], $\mathcal{J}\left(X, \frac{p_{x}}{h} D_{h}\right)$ is non-trivial at $x$, i.e., $x \in \mathcal{Z}\left(\mathcal{J}\left(X,\left\|p_{x} D\right\|\right)\right)$.
Proposition 4.2.4. Let $X$ be a smooth variety and let $D$ be a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$. We have that, for every $p \in \mathbb{N}, \mathcal{Z}(\mathcal{J}(X,\|p D\|)) \subseteq$ NNA $(D)$.

Proof. Note that by [ELMNP06, section 2] we have that for every geometric discrete valuation $v$ on $X$

$$
\sup _{p} \frac{v(\mathcal{J}(X,\|p D\|))}{p}=\lim _{p \rightarrow \infty} \frac{v(\mathcal{J}(X,\|p D\|))}{p} \leq v(\|D\|)
$$

where the first equality strongly relies on the subadditivity theorem of Demailly, Ein, Lazarsfeld (cf. [DEL00]). Hence if $x \in \mathcal{Z}(\mathcal{J}(X,\|p D\|))$ for some $p \in \mathbb{N}$, then $\operatorname{ord}_{x}(\mathcal{J}(X,\|p D\|))>0$, so that $\operatorname{ord}_{x}(\|D\|)>0$, i.e., $x \in \operatorname{NNA}(D)$.

Theorem 4.2.5. Let $(X, \Delta)$ be an effective KLT pair. If $D$ is a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$, then

$$
\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)) \subseteq \operatorname{NNA}(D)
$$

Proof. We will show that for every integer $p$ we have that

$$
\begin{equation*}
\mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)) \subseteq \operatorname{NNA}(D) \tag{4.1}
\end{equation*}
$$

Since by definition $\operatorname{NNA}(p D)=\operatorname{NNA}(D)$, without loss of generality we can furthermore assume that $p=1$.
Let $f: Y \rightarrow X$ be a log-resolution of $(X, \Delta)$ chosen as in lemma 4.2.2. By lemma 2.4.7, $\mathcal{J}((X, \Delta) ;\|D\|)=f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)$. Moreover since $\mathbf{L}(\Delta)_{Y} \geq$ $\mathbf{A}\left(\Delta_{Y}\right)$ then $\mathcal{J}\left(\left(Y, \mathbf{A}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right) \supseteq \mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)$. Recall that $f_{*}$ preserves inclusions, therefore $\mathcal{J}((X, \Delta) ;\|D\|) \supseteq f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)$ and hence, as by effectivity of $\mathbf{L}(\Delta)_{Y}$ the latter is an actual ideal sheaf, we have that $\mathcal{Z}(\mathcal{J}((X, \Delta) ;\|D\|)) \subseteq \mathcal{Z}\left(f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)\right)$. Since, by lemma 2.1.6, $\mathcal{Z}\left(f_{*}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)\right) \subseteq f\left(\mathcal{Z}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right)\right)$ and, by lemma 2.6.9, $f\left(\mathrm{NNA}\left(f^{*} D\right)\right)=\mathrm{NNA}(D)$, then in order to prove (4.1) it is sufficient to prove that

$$
\mathcal{Z}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right) \subseteq \operatorname{NNA}\left(f^{*} D\right)
$$

This last inclusion comes from the combination of proposition 4.2.3 and proposition 4.2.4.

If the effective pair $(X, \Delta)$ is not KLT, the same statement does not hold in general, because the zeros of the asymptotic multiplier ideals depend also on the singularities of the pair. In fact we have that $\operatorname{Nklt}(X, \Delta)=\mathcal{Z}(\mathcal{J}(X, \Delta))$. Anyway, we can still recover the previous result outside the non-klt locus.
The following holds:
Corollary 4.2.6. Let $(X, \Delta)$ be an effective pair. If $D$ is a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$, then

$$
\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)) \backslash \operatorname{Nklt}(X, \Delta) \subseteq \operatorname{NNA}(D)
$$

Proof. As in the proof of theorem 4.2.5 we can suppose $p=1$, i.e., it is enough to show that $\mathcal{Z}(\mathcal{J}((X, \Delta) ;\|D\|) \backslash \operatorname{Nklt}(X, \Delta) \subseteq \operatorname{NNA}(D)$. Let $f: Y \rightarrow X$ be a log-resolution of $(X, \Delta)$. Set $\mathbf{A}(\Delta)_{Y}=\sum a(E) E$, where the sum is taken on all prime divisors on $Y$, and define

$$
\Delta_{\bar{Y}}^{\geq 1}:=\sum_{a(E) \geq 1} a(E) E, \Delta_{Y}^{+}:=\sum_{0 \leq a(E)<1} a(E) E, \Delta_{Y}^{-}=\sum_{a(E)<0}-a(E) E,
$$

so that $\mathbf{L}(\Delta)_{Y}:=\Delta_{Y}^{\geq 1}+\Delta_{Y}^{+}$, and $\mathbf{A}(\Delta)_{Y}=\mathbf{L}(\Delta)_{Y}-\Delta_{Y}^{-}$. Note also that $f\left(\operatorname{Nklt}\left(Y, \mathbf{L}(\Delta)_{Y}\right)\right)=f\left(\operatorname{Nklt}\left(Y, \mathbf{A}(\Delta)_{Y}\right)\right)=\operatorname{Nklt}(X, \Delta)$.
As in the proof of theorem 4.2.5, by the birational transformation rule, lemma 2.1.6 and lemma 2.6.9, it is then enough to prove that

$$
\mathcal{Z}\left(\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)\right) \backslash \operatorname{Nklt}\left(Y, \mathbf{L}(\Delta)_{Y}\right) \subseteq \operatorname{NNA}\left(f^{*} D\right)
$$

At this point, notice that for any $y \notin \operatorname{Nklt}\left(Y, \mathbf{L}(\Delta)_{Y}\right)=\operatorname{Supp}\left(\Delta_{Y}^{\geqslant 1}\right)$ we have that $\mathcal{J}\left(\left(Y, \mathbf{L}(\Delta)_{Y}\right) ;\left\|f^{*} D\right\|\right)_{y} \cong \mathcal{J}\left(\left(Y, \Delta_{Y}^{+}\right) ;\left\|f^{*} D\right\|\right)_{y}$ (see [DiB10, proposition 1.35]). Therefore we are just left to prove that

$$
\mathcal{Z}\left(\mathcal{J}\left(\left(Y, \Delta_{Y}^{+}\right) ;\left\|f^{*} D\right\|\right)\right) \subseteq \operatorname{NNA}\left(f^{*} D\right)
$$

but, since $\left(Y, \Delta_{Y}^{+}\right)$is an effective KLT pair, this is just an instance of thm. 4.2.5, and we are done.

Theorem 4.2.7. Let $(X, \Delta)$ be an effective pair and let $D$ be an $\mathbb{R}$-divisor on $X$. Then $\operatorname{NNef}(D) \backslash \operatorname{Nklt}(X, \Delta)=\mathbb{B}_{-}(D) \backslash \operatorname{Nklt}(X, \Delta)$. In particular if $(X, \Delta)$ is an effective KLT pair then $\operatorname{NNef}(D)=\mathbb{B}_{-}(D)$.

Proof. If $D$ is not pseudoeffective there is nothing to prove. Thus let us assume that $D$ is pseudoeffective. Let $\left\{A_{m}\right\}$ be a sequence of ample $\mathbb{R}$-divisors as in lemma 2.1.10. By lemmas 2.3.6, 2.6.2 it is then clear that we can furthermore assume that $D$ is a big $\mathbb{Q}$-divisor. Since, clearly, for every $c>0, \mathbb{B}_{-}(c D)=$ $\mathbb{B}_{-}(D)$, and analogously for NNef, we can also assume that $D$ is integral. By [BBP09, Lemma 1.8] and lemma 4.2.1 we have that $\operatorname{NNef}(D) \subseteq \mathbb{B}_{-}(D) \subseteq$ $\bigcup_{p} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)$. Since $D$ is big, by remark 2.6.8, $\operatorname{NNA}(D)=\operatorname{NNef}(D)$, whence by corollary 4.2 .6 we have that $\bigcup_{p} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|)) \backslash \operatorname{Nklt}(X, \Delta) \subseteq$ $\operatorname{NNef}(D)$ and we are done.

Remark 4.2.8. When $X$ is smooth theorem 4.2 .7 follows by [ELMNP06, theorem 2.8].
If $(X, \Delta)$ is a KLT pair the theorem 4.2 .7 has been proved to hold for the divisor $K_{X}+\Delta$ by Boucksom, Broustet, Pacienza in [BBP09, Proposition 1.10] using [BCHM10].
In general it has been conjectured that theorem 4.2.7 holds for every normal projective variety: see, for example, [BBP09, Conj. 1.9].
Remark 4.2.9. Note also that theorem 4.2.7 implies that $\mathbb{B}_{-}(D)=\operatorname{NNef}(D)$ for every $\mathbb{R}$-divisor $D$ on a variety $X$ such that $(X, 0)$ is a log-terminal pair in the sense of [dFH09, definition 7.1]. In fact by [dFH09, proposition 7.2] this property is equivalent to the existence of an effective $\mathbb{Q}$-Weil divisor $\Delta$ such that $(X, \Delta)$ is a KLT pair.

Since $\operatorname{NNef}(D)$ and $\mathbb{B}_{-}(D)$ do not depend on the chosen boundary divisor $\Delta$, if we set $X_{\text {Nklt }}:=\bigcap_{\Delta \in \mathcal{F}} \operatorname{Nklt}(X, \Delta)$, where

$$
\mathcal{F}:=\{E \mathbb{Q} \text {-Weil divisor s.t. }(X, E) \text { is an effective pair }\},
$$

then the following holds:
Corollary 4.2.10. Let $X$ be a normal projective variety and let $D$ be an $\mathbb{R}$ divisor on $X$. Then $\operatorname{NNef}(D) \backslash X_{\text {Nklt }}=\mathbb{B}_{-}(D) \backslash X_{\text {Nklt }}$.
Remark 4.2.11. As Y. Gongyo kindly pointed out to us, for every normal variety $X$ and any smooth closed point $x \in X$ we can find an effective $\mathbb{Q}$-Weil divisor $\Delta_{x}$ such that $\left(X, \Delta_{x}\right)$ is a pair and $x \notin \operatorname{Supp}\left(\Delta_{x}\right)$.
In fact if $H$ is an ample Cartier divisor then there exists $m \in \mathbb{N}$ such that the coherent sheaf $\mathcal{O}_{X}\left(-K_{X}+m H\right)=\mathcal{O}_{X}\left(-K_{X}\right) \otimes \mathcal{O}_{X}(m H)$ is globally generated. On the other hand, as $x \in X$ is a smooth point, we have that $\mathcal{O}_{X}\left(-K_{X}+\right.$ $m H)_{x} \simeq \mathcal{O}_{X, x}$, so that there exists a section $s_{x} \in H^{0}\left(X, \mathcal{O}_{X}\left(-K_{X}+m H\right)\right)$ such that $s_{x}$ does not vanish on $x$. Take $\Delta_{x}=\left\{s_{x}=0\right\}$. Then $K_{X}+\Delta_{x} \sim m H$ is a Cartier divisor, so that $\left(X, \Delta_{x}\right)$ is a pair and $x \notin \operatorname{Supp}\left(\Delta_{x}\right)$. Therefore prop. 4.1.3 follows also directly by cor. 4.2.10.
Note that as $\mathbb{B}_{-}(D)$ does not contain isolated points (see [ELMNP09, Prop.1.1]) by corollary 4.2.10 we deduce the following:
Corollary 4.2.12. Let $X$ be a normal projective variety such that $X_{\mathrm{Nklt}}$ has dimension 0. Then for every $\mathbb{R}$-divisor $D$ on $X$ we have that $\operatorname{NNef}(D)=\mathbb{B}_{-}(D)$.

### 4.3 Nef and abundant divisors

## Characterization of nef-abundant divisors

In [Rus09, Theorem 2] F. Russo states a characterization of nef and abundant divisors on a smooth projective variety $X$ by means of asymptotic multiplier ideals. Given theorem 4.2.5 and its counterpart below (theorem 4.3.1) we can extend the same characterization to KLT pairs.

Theorem 4.3.1. Let $(X, \Delta)$ be an effective pair. If $D$ is a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$, then

$$
\operatorname{NNA}(D) \subseteq \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|))
$$

Proof. Note that if $e=e(D)$ is the exponent of $D$, then, by definition, $v(\|e D\|)=$ $e \cdot v(\|D\|)$, so that $v(\|D\|)>0$ if and only if $v(\|e D\|)>0$. Moreover

$$
\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p(e D)\|)) \subseteq \bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|))
$$

Hence, potentially multiplying $D$ by its exponent $e(D)$, we can suppose that $D$ has exponent 1.
Let $x \in X$ be such that there exists a discrete geometric valuation $v$ with $v(\|D\|)=S>0$ and $x \in c_{X}(v)$.
Then

$$
0<S=v(\|D\|))=\inf _{k \in \mathbb{N}}\left\{\frac{v(|k D|)}{k}\right\}
$$

We can suppose that $v$ corresponds to a prime divisor $E_{v}$ over $X$, so that there exists a birational morphism $\mu: X^{\prime} \rightarrow X$ such that $E_{v} \subseteq X^{\prime}$ and for every $k \in \mathbb{N}$, if $D_{k} \in|k D|$ is a general divisor, we have that $v(|k D|)=v\left(D_{k}\right)=\operatorname{ord}_{E_{v}} \mu^{*}\left(D_{k}\right)$. Thus $\inf _{k \in \mathbb{N}}\left\{\frac{\operatorname{ord}_{E_{v}} \mu^{*}\left(D_{k}\right)}{k}\right\}=S>0$, so that for every $k \in \mathbb{N}, \operatorname{ord}_{E_{v}} \mu^{*}\left(D_{k}\right) \geq k S$. Let $a\left(E_{v}, X, \Delta\right)$ be the discrepancy of the pair $(X, \Delta)$ along the divisor $E_{v}$ and let

$$
p_{0}= \begin{cases}\left\lceil\frac{1+a}{S}\right\rceil & \text { if } a:=a\left(E_{v}, X, \Delta\right)>-1 \\ 1 & \text { if } a\left(E_{v}, X, \Delta\right) \leq-1\end{cases}
$$

Then, in any case, $a\left(E_{v}, X, \Delta\right)-p_{0} S \leq-1$. Let $k \in \mathbb{N}$ be such that $\mathcal{J}\left((X, \Delta) ;\left\|p_{0} D\right\|\right)=\mathcal{J}\left((X, \Delta) ; \frac{1}{k} D_{k p_{0}}\right)$, where $D_{k p_{0}}$ is a general divisor in $\left|k p_{0} D\right|$. We have that

$$
\begin{array}{r}
a\left(E_{v}, X, \Delta+\frac{1}{k} D_{k p_{0}}\right)=a\left(E_{v}, X, \Delta\right)-\frac{1}{k} \operatorname{ord}_{E_{v}} \mu^{*}\left(D_{k p_{0}}\right) \leq \\
\leq a\left(E_{v}, X, \Delta\right)-S p_{0} \leq-1
\end{array}
$$

This implies that the subvariety $\mu\left(E_{v}\right)=c_{X}(v)$ is an LC center of the pair $\left(X, \Delta+\frac{1}{k} D_{k p_{0}}\right)$, so that $\mathcal{Z}\left(\mathcal{J}\left(\left(X, \Delta+\frac{1}{k} D_{k p_{0}}\right) ; 0\right)\right) \supseteq c_{X}(v)$.
Therefore $x \in \mathcal{Z}\left(\mathcal{J}\left(\left(X, \Delta+\frac{1}{k} D_{k p_{0}}\right) ; 0\right)\right)=\mathcal{Z}\left(\mathcal{J}\left((X, \Delta) ; \frac{1}{k} D_{k p_{0}}\right)\right)$.
Corollary 4.3.2. Let $(X, \Delta)$ be an effective KLT pair. Suppose $D$ is a Cartier divisor on $X$ such that $\kappa(X, D) \geq 0$. Then $D$ is nef and abundant if and only if $\mathcal{J}((X, \Delta) ;\|p D\|)=\mathcal{O}_{X}$ for every $p \in \mathbb{N}$.

Proof. We know that $D$ is nef and abundant if and only if $\operatorname{NNA}(D)=\emptyset$. On the other hand by theorem 4.3.1 and corollary 4.2.6 it follows that $\operatorname{NNA}(D)=$ $\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|))$.

It is also interesting to notice that, by the same token, also the following holds:
Corollary 4.3.3. Let $D$ be a Cartier divisor on a normal projective variety $X$ such that $\kappa(X, D) \geq 0$. Let $\Delta, \Delta^{\prime}$ be any two effective divisors such that $(X, \Delta)$, $\left(X, \Delta^{\prime}\right)$ are KLT pairs. Then

$$
\bigcup_{p \in \mathbb{N}} \mathcal{Z}(\mathcal{J}((X, \Delta) ;\|p D\|))=\bigcup_{p \in \mathbb{N}} \mathcal{Z}\left(\mathcal{J}\left(\left(X, \Delta^{\prime}\right) ;\|p D\|\right)\right)
$$

## Applications

It is a general simple trick, when dealing with big and nef divisors up to linear equivalence on a klt pair $(X, \Delta)$, to reduce oneself to studying a slightly different klt pair, in which the boundary $\Delta$ "has absorbed" the divisor. In view of corollary 4.3 .2 this is pretty much the same if we just have nef and abundant divisors instead of big and nef ones. In fact when $L$ is nef and abundant we get, in particular, that $\mathcal{J}((X, \Delta),\|L\|)=\mathcal{O}_{X}$, so that the hypotheses of the following lemma are verified:

Lemma 4.3.4. Let $(X, \Delta)$ be an effective KLT pair and let $L$ be a line bundle on $X$. Then $\mathcal{J}((X, \Delta),\|L\|)=\mathcal{O}_{X}$ if and only if there exists an effective $\mathbb{Q}$-Cartier divisor $D$ such that $D \sim_{\mathbb{Q}} L$ and $(X, \Delta+D)$ is a KLT pair.

Proof. By definition $\mathcal{J}((X, \Delta),\|L\|)=\mathcal{J}\left((X, \Delta), \frac{1}{p}|p L|\right)$ for some $p \in \mathbb{N}$. Moreover by [Laz04, Prop. 9.2.26] (see also p. 185, volume 2) and [Laz04, Ex. 9.3.57(ii)], we have that $\mathcal{J}\left((X, \Delta) ; \frac{1}{p}|p L|\right)=\mathcal{J}\left(X,\left(\Delta+\frac{1}{p} L_{p}\right) ; 0\right)$, where $L_{p}$ is a general divisor in $|p L|$. Hence $\frac{L_{p}}{p} \sim_{\mathbb{Q}} L$ and, if $\mathcal{J}((X, \Delta),\|L\|)=\mathcal{O}_{X}$, then $\mathcal{J}\left(X,\left(\Delta+\frac{1}{p} L_{p}\right) ; 0\right)=\mathcal{O}_{X}$, so that $\left(X, \Delta+\frac{1}{p} L_{p}\right)$ is a KLT pair.
On the other hand if there exists $D$ such that $D \sim_{\mathbb{Q}} L$ and $(X, \Delta+L)$ is a KLT pair, then $k D \sim k L$ for some $k \in \mathbb{N}$. Thus $\mathcal{O}_{X}=\mathcal{J}(X,(\Delta+D) ; 0)=$ $\mathcal{J}\left((X, \Delta), \frac{1}{k} k D\right) \subseteq \mathcal{J}\left((X, \Delta), \frac{1}{k}|k L|\right)$. In fact if $\mu: X^{\prime} \rightarrow X$ is a suitable logresolution, so that in particular $\left|\mu^{*}(k L)\right|=|W|+F$, where $|W|$ is base point free and $F$ is effective, we have that $\mu^{*}(k D) \geq F$, so that

$$
\begin{aligned}
& \mathcal{J}\left((X, \Delta), \frac{1}{k} k D\right)=\mu_{*} \mathcal{O}_{X^{\prime}}\left(\left\ulcorner\sum_{E} a(E, X, \Delta) E-\frac{1}{k} \mu^{*}(k D)\right\urcorner\right) \subseteq \\
& \subseteq \mu_{*} \mathcal{O}_{X^{\prime}}\left(\left\ulcorner\sum_{E} a(E, X, \Delta) E-\frac{1}{k} F\right\urcorner\right)=\mathcal{J}\left((X, \Delta), \frac{1}{k}|k L|\right) .
\end{aligned}
$$

Hence $\mathcal{J}\left((X, \Delta), \frac{1}{k}|k L|\right)=\mathcal{O}_{X}$, which implies that $\mathcal{J}((X, \Delta),\|L\|)=\mathcal{O}_{X}$ as well.

To illustrate the general principle touched on before we give a slight generalization of a theorem by F. Campana, V. Koziarz, M. Păun, but only in the case of KLT pairs (see [CKP10, corollary 1]).

Theorem 4.3.5 (Campana, Koziarz, Păun). Let $(X, \Delta)$ be an effective KLT pair, let $\rho$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $\rho \equiv 0$ and let $L$ be a nef and abundant line bundle on $X$. Then $\kappa\left(K_{X}+\Delta+L\right) \geq \kappa\left(K_{X}+\Delta+L+\rho\right)$.

Proof. Since $L$ is nef and abundant then by lemma 4.3 .4 there exists $\Delta^{\prime}$ such that $\left(X, \Delta^{\prime}\right)$ is an effective KLT pair and $K_{X}+\Delta^{\prime} \sim_{\mathbb{Q}} K_{X}+\Delta+L$. Hence $\kappa\left(K_{X}+\Delta+L\right)=\kappa\left(K_{X}+\Delta^{\prime}\right) \geq \kappa\left(K_{X}+\Delta^{\prime}+\rho\right)$, by [CKP10, corollary 1].

Therefore we just have the following corollary:
Corollary 4.3.6. Let $(X, \Delta)$ be an effective KLT pair of dimension n. Let L be a nef line bundle such that $\kappa(L) \geq n-1$ and let $\rho$ be a numerically trivial $\mathbb{Q}$-Cartier divisor. Then $\kappa\left(K_{X}+\Delta+L\right) \geq \kappa\left(K_{X}+\Delta+L+\rho\right)$.

Proof. The hypothesis on the Kodaira dimension actually implies that $L$ is nef and abundant. Hence the corollary follows by theorem 4.3.5.

Notice that the hypothesis $\kappa(L) \geq n-1$ is necessary. In fact for every $n$ we can find examples of smooth varieties of dimension $n$ and line bundles of Kodaira dimension $n-2$ for which corollary 4.3.6, with $\Delta=0$, does not hold:

Example 4.3.7. We will first of all construct an example for $n=2$.
Let $C$ be a smooth elliptic curve and let $\eta \in \operatorname{Pic}^{0}(C)$ be a non-torsion divisor on $C$. Let $\mathcal{E}:=\mathcal{O}_{C} \oplus \mathcal{O}_{C}(-\eta)$. Take $X:=\mathbb{P}(\mathcal{E})$ and let $\pi: X \rightarrow C$ be the related projection. As in [Har77, V.2.8.1] let $C_{0}$ be a section $\sigma_{0}: C \rightarrow X$. Set $\rho:=-\pi^{*}(\eta)$ and $L:=-K_{X}+\pi^{*}(\eta)$.
By [Har77, Lemma V.2.10], $K_{X} \sim-2 C_{0}+\rho$ and $L \sim 2 C_{0}+2 \pi^{*}(\eta)$, so that $L$ is nef because $C_{0}^{2}=0$ and $\eta \equiv 0$.
Hence for any $m \geq 1, H^{0}(X, m L)=H^{0}\left(X, 2 m C_{0}+2 m \pi^{*}(\eta)\right)$. By projection formula $H^{0}(X, m L)=H^{0}\left(C, S^{2 m}(\mathcal{E}) \otimes 2 m \eta\right)=H^{0}\left(C,\left(\mathcal{O}_{C} \oplus \mathcal{O}_{C}(-\eta) \oplus \cdots \oplus\right.\right.$ $\left.\left.\mathcal{O}_{C}(-2 m \eta)\right) \otimes 2 m \eta\right)=\mathbb{C}$. Therefore $\kappa(L)=0$.
Moreover $K_{X}+L+\rho=0$, hence $\kappa\left(K_{X}+L+\rho\right)=0$. On the contrary $\kappa\left(K_{X}+L\right)=$ $-\infty$, because for any $m \geq 1, H^{0}\left(X, m \pi^{*}(\eta)\right)=H^{0}(C, m \eta)=0$.
To produce examples in every dimension just build them up inductively taking products $X \times C$. Call $\pi_{1}$ the first projection and $\pi_{2}$ the second one. Fix any point $q$ on $C$ and define $\rho_{X \times C}:=\pi_{1}^{*}(\rho), L_{X \times C}:=\pi_{1}^{*}(L) \otimes \pi_{2}^{*}(q)$.

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