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Quantum Bertrand systems

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Introduction

Since the appearance of Newton's equations of motion there seemed to be a kind of general hope in the air that every closed system could be solved exactly. An expression of this feeling was represented by the idea of Laplace that a hypotetical daemon could be able to embrace in a single formula anything set anywhere in the past or future. In the 19th century Liouville found a way to solve an Hamiltonian system which admits a maximal set of Poisson commuting invariants (functions on the phase space whose Poisson brackets with the Hamiltonian of the system, and with each other, vanish). However, the idea of predicting analytically the position of any point mass particle was overtaken when Henri Poincaré, approaching the so called "three body problem", realized that the conserved quantities are not preserved when an integrable system is perturbed in a generic way, hence proving that a system can be in general not integrable. If, on the one hand, this fact limited the possibility to solve physical systems through analytical methods, on the other hand it was remarkably amazing how most systems of interest known at the time could be considered as a perturbation of integrable systems. Let us think for example of the solar system or of the harmonic approximation or of other dynamical systems such as shallow water waves discovered in the same century by John Scott Russel whose mathematical wave equation, despite of its nonlinearity, showed immediately some particular solutions expressed in terms of analytical functions. These facts seem to be indicative of a new role played by integrable systems: though, after Poincare , they were in a sense downgraded to something like a Platonic Idea of geometrical picture which just an arbitrarily small perturbation could make unperfect (non-integrable), nevertheless, at the same time, they appear to be needed by nature, as a sort of alphabet to describe ordered phenomena existing in itself. So, from this point of view, integrable systems would turn out to be one of the best source of mathematical models in physics to represent those phenomena whose behaviour is far enough from Chaos. With the onset of quantum mechanics the situation became even hardest for integrable systems: Even a Hamiltonian 1-dimensional system, which in the classical realm is integrable by definition as its equation of motion is solvable at least by a quadrature, may not be solvable in the quantum context, inasmuch as there might not be any recipes to get analytically the physical interesting quantities, namely the spectrum and the eigenfunctions. Over the years the exactly solvable quantum models, as the classical ones, have represented an invaluable tool describing in an effective way a great variety of physical systems: let us quote for example the

exactly solvable quantum systems involved in the description of the energy levels of hydrogenoid atoms or in the study of vibrational effective potentials in molecules or in lattices [2], just to mention a few.
Unfortunately, most examples of such systems are restricted to spaces of constant scalar curvature or to lower dimensional cases (two-dimensional systems on nonconstant curvature [32], [33]); in addition to that the increasing importance of non-Euclidean geometries in modern physics makes worthwhile, both from a mathematical and a physical perspective, to investigate about the existence of an exactly solvable quantum family also in this framework, and indeed this issue can be considered as the main goal of the present thesis. We start the analysis giving in the first chapter a short review about the exactly solvable quantum systems descrbing them using the language of the shape invariance and underlining explicitly their link with the most general family of classical orthogonal polynomials, namely the Jacobi polynomials.
In the second chapter we introduce the concept of Maximal Superintegrability (M.S.) in classical mechanics, this property plays an essential role in the analysis of the exactly solvable systems since we can always determine its trajectory on the phase space up to the solution of algebraic equations. The classification of the M.S. systems is a hard work, anyway considering a radial symmetry (also in non-Euclidean frameworks) it is possible to classify all these systems in just two multiparametric families, namely the Bertrand-Perlick systems ([50] [51]), which in the flat limit coincide with the well known exactly solvable systems of the Kepler and Harmonic oscillator problem. The radial symmetry and the Bertrand classification of M.S. systems in a non-Euclidean framework defines therefore an ideal starting point to investigate about the existence of exactly solvable quantum systems on a non-Euclidean manifold with a number of dimensions greater than two. According to the aforementioned facts, the remaining part of the thesis is devoted to the "quantization" of the Bertrand-Perlick systems, to this aim we have dedicated a preliminary chapter (the third chapter) in order to introduce the problems that always arise when dealing with systems whose kinetic part turns out to be position dependent, and at the same time we give also a review about the most known quantization prescriptions present in literature.
Finally in the chapter four we tackle the quantization of the Bertrand Perlick systems making use of the mathematical tools (like the coupling constant metamorphosis or the gauge transformation) previously introduced in the first three chapters. As result of this analysis we have found a quantum version of the Bertrand-Perlick systems generalized to an arbitrary number of dimensions which preserves explicitly the exact solvability of the eigenvalue problem, moreover this quantum version has the remarkable property of exhibiting a spectrum characterized by the so called accidental degeneracy giving us a strong indication about the M.S. of this quantum family.
The remaining part of the thesis is dedicated to the analysis of some examples which turn out to be explicitly M.S. both in their classical and quantum version; in particular we describe in detail the quantum system defined on the non-Euclidean

Darboux-III space that can be regarded as well as a quantum system with a non-constant parabolic mass. We conclude the thesis giving in the appendix an example of how the Bertrand-Perlick systems can be a very flexible family of systems and approximate well some real physical problems like the motion on a Swarzschild metric.

Chapter 1

Orthogonal polynomials and shape invariance

1.1 Introduction to shape invariant systems

To solve exactly a (nonrelativistic) quantum system means to find explicitly the spectrum and the corresponding (generalized) eigenfunctions for the Schroedinger operator. Even though the 1-dimensional Schrodinger equation is just a second order linear ordinary differential equation, solving the associated spectral problem can be a very hard task. Actually, a natural approach to solve a second order linear differential equation would be trying to transform it into a first order linear differential equation; for our aims it is quite useful to tackle this problem following the "Dirac strategy", adopted by Dirac himself in order to get his most famous equation, namely to look for the square root of the second order linear differential operator [1]:

$$\begin{aligned} \hat{H} &= -\hbar^2 \frac{d^2}{dx^2} + V(x) \rightarrow (-i\hbar \hat{A} \frac{d}{dx} + \hat{B} W(x)')^2 \qquad (1.1) \\ &= -\hbar^2 \hat{A}^2 \frac{d^2}{dx^2} - i\hbar \hat{A}\hat{B} W(x)'' - i\hbar \hat{A}\hat{B} W(x)' \frac{d}{dx} - i\hbar \hat{B}\hat{A} W(x)' \frac{d}{dx} + \hat{B}^2 W(x)'^2. \end{aligned}$$

As in the case of the Dirac equation, the main drawback in order to factorize a second order differential operator is due to the presence of non commutative objects, for example matrices. This is a very delicate operation, because it entails a deep change in the space of solutions, from a scalar function to a spinor of functions, as it will be seen in the sequel; anyway it is possible performing such an operation provided that the square of this new first order matrix differential operator yields a second order diagonal operator, where at least one of the diagonal entries is the original second order differential operator.

[1] An analog approach, but for relativistic particles can be found also in [6]

These requirements turn into the following three conditions:

$$\hat{A}\hat{B} + \hat{B}\hat{A} = 0; \qquad \hat{A}^2 = \hat{B}^2 = I; \qquad \hat{A}\hat{B} = \hat{C}_{i,j} : \hat{C}_{i,i} \neq 0, \hat{C}_{i,j\neq i} = 0. \tag{1.2}$$

It is straightforward to recognize that the Pauli matrices fullfill the relations (1.2):

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \hat{B} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}; \qquad \hat{C} = -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.3}$$

the equation (1.1) turns into:

$$(-i\hbar\hat{A}\frac{d}{dx} + \hat{B}W(x)')^2 = \begin{pmatrix} 0 & -i\hbar\frac{d}{dx} - iW(x)' \\ -i\hbar\frac{d}{dx} + iW(x)' & 0 \end{pmatrix}^2. \tag{1.4}$$

It is easy to verify that $-i\hbar\hat{A}\frac{d}{dx} + i\hat{B}W(x)'$ is still an Hermitian operator in the spinor functional space defined by the proper boundary conditions:

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}; < \Psi_a|\Psi_b >= \int_{x_1}^{x_2} \Psi_a^\dagger(x)\Psi_b(x)dx; \quad \Psi(x_1) = \Psi(x_2) = 0. \tag{1.5}$$

Let's define the operators:

$$\hat{a} = -i\hbar\partial_x + iW(x)'; \qquad \hat{a}^\dagger = -i\hbar\partial_x - iW(x)'; \quad W(x)' \in \mathbb{R} \tag{1.6}$$

the square operator turns into:

$$\begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{pmatrix}^2 = \begin{pmatrix} \hat{a}^\dagger\hat{a} & 0 \\ 0 & \hat{a}\hat{a}^\dagger \end{pmatrix} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix} \tag{1.7}$$

$$\hat{H}_1 = \hat{a}^\dagger\hat{a} = -\hbar^2\frac{d^2}{dx^2} + \hbar W'' + W'^2$$

$$\hat{H}_2 = \hat{a}\hat{a}^\dagger = -\hbar^2\frac{d^2}{dx^2} - \hbar W'' + W'^2$$

A further step is needed in order to connect the original differential operator $\hat{H}$ to $\hat{H}_{1,2}$:

$$\pm\hbar W(x)'' + W(x)'^2 = V(x) \tag{1.8}$$

We recognize the above equation as the well known Riccati equation, and this fact is not unexpected because it is known as well that the Riccati equation is linked to the Schroedinger equation through algebraic manipulations. Summarizing, the effects of looking for the square root of a 2nd order linear differential operator have been twofold: on one hand one has got the transformation which reduces the order of the Schroedinger equation and on the other hand it is interesting to note that whenever the Riccati equation has an explicit solution (namely the Schroedinger equation is exactly solvable) it is possible as well to get an explicit factorization (1.7) for the Hamiltonian operator; furthermore, when the factorization is explicitly performed one gets for free a couple of different Hamiltonian operators which share the same spectrum and some additional properties:

$$\begin{pmatrix} 0 & \hat{a}^\dagger \\ \hat{a} & 0 \end{pmatrix} \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix} = \sqrt{E_n} \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix} \rightarrow \tag{1.9}$$

$$\rightarrow \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix} \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix} = E_n \begin{pmatrix} \psi_{1n} \\ \psi_{2n} \end{pmatrix}, \qquad E_n \geq 0. \tag{1.10}$$

Furthermore the eigenfunctions of the Hamiltonians $\hat{H}_1, \hat{H}_2$ turn out to be linked by the equation (1.9), explicitly :

$$\hat{a}^\dagger \psi_{2n} = \sqrt{E_n} \psi_{1n}; \qquad \hat{a}\psi_{1n} = \sqrt{E_n} \psi_{2n} \tag{1.11}$$

The equation (1.11) entails the so called intertwining relation [11] between the two Hamiltonians $\hat{H}_1, \hat{H}_2$

$$\begin{aligned} \hat{H}_1 \hat{a}^\dagger &= \hat{a}^\dagger \hat{H}_2 \\ \hat{a}\hat{H}_1 &= \hat{H}_2 \hat{a} \end{aligned} \tag{1.12}$$

Finally it is notable that if it is chosen $E_0 = 0$ as the groundstate eigenvalue, it becomes straightforward to get the ground state eigenfunctions

$$\begin{cases} \hat{a}\psi_{1,0} = -i\hbar \frac{d}{dx} \psi_{1,0} + iW'(x)\psi_{1,0} = 0 \rightarrow \psi_{1,0} = C_1 e^{\frac{W(x)}{\hbar}} \\ \hat{a}^\dagger \psi_{2,0} = -i\hbar \frac{d}{dx} \psi_{2,0} - iW'(x)\psi_{2,0} = 0 \rightarrow \psi_{2,0} = C_2 e^{-\frac{W(x)}{\hbar}} \end{cases}$$

The two spinor components turn out to be reciprocal to each other so they can't satisfy at the same time the boundary conditions (1.5). Therefore, if a normalizable solution exist, this has to be one of the following:

$$\begin{pmatrix} \psi_{1,0} \\ \psi_{2,0} \end{pmatrix} = \begin{pmatrix} C_1 e^{\frac{W(x)}{\hbar}} \\ 0 \end{pmatrix}; \quad \begin{pmatrix} 0 \\ C_2 e^{-\frac{W(x)}{\hbar}} \end{pmatrix}. \tag{1.13}$$

Let us say for instance that the first solution $\psi_{1,0} = C_1 e^{\frac{W(x)}{\hbar}}$ be normalizable, this means that the ground state of the Hamiltonian $\hat{H}_2$ is $\psi_{2,1}$ instead of $\psi_{2,0} = 0$, so let us define for $\hat{H}_2$ the eigenfunctions :

$$\tilde{\psi}_{2,n} = \psi_{2,n+1}, \rightarrow \tilde{E}_n = E_{n+1}, \qquad \tilde{\psi}_{2,n} = \frac{1}{\sqrt{E_{n+1}}} \hat{a}\psi_{1,n+1}. \tag{1.14}$$

Therefore the two Hamiltonians share the same spectrum except for the eigenvalue E_0 which belongs only to the Hamiltonian $\hat{H}_1$.

1.1.1 SUperSYmmetric Quantum Mechanics

An elegant way to explain the content of last section with a more compact formalism is to introduce the concept of supersymmetric quantum mechanics [3],[4], [5]. In this new formalism the Hamiltonians $\hat{H}_1, \hat{H}_2$ are known also as supersymmetric partners, the origin of this name being linked to the degeneracy of their spectra, that can be understood in terms of the SUSY algebra. Let us consider the Hamiltonian (1.10) $\hat{H}_{susy}$ and the other operators $\hat{Q}, \hat{Q}^\dagger$:

$$\hat{H}_{susy} = \begin{pmatrix} \hat{H}_1 & 0 \\ 0 & \hat{H}_2 \end{pmatrix}, \qquad \hat{Q} = \begin{pmatrix} 0 & 0 \\ \hat{a} & 0 \end{pmatrix}, \qquad \hat{Q}^\dagger = \begin{pmatrix} 0 & \hat{a}^\dagger \\ 0 & 0 \end{pmatrix}. \tag{1.15}$$

They are part of a closed superalgebra which contains both bosonic and fermionic operators with commutation and anticommutation relations:

$$[\hat{H}_{susy}, \hat{Q}] = [\hat{H}_{susy}, \hat{Q}^\dagger] = 0, \tag{1.16}$$

$$\{\hat{Q}, \hat{Q}^\dagger\} = \hat{H}_{susy}, \qquad \{\hat{Q}, \hat{Q}\} = \{\hat{Q}^\dagger, \hat{Q}^\dagger\} = 0. \tag{1.17}$$

In case there exists a solution of the type (1.13) with eigenvalue $E = 0$, then this is unique, and the supersymmetry is said to be unbroken; otherwise, if doesn't exist any groundstate with $E = 0$ the ground state energy of the system turns out to be $E_1 \neq 0$, thus the two spinor solutions are linked by the relation (1.11) and there exist two different ground states. In this case the supersymmetry is said to be broken because of the presence of two different ground states:

$$\psi_{2,1} = \frac{\hat{a}\psi_{1,1}}{\sqrt{E_1}} \rightarrow \Psi_{1,0} = \begin{pmatrix} \psi_{1,1} \\ 0 \end{pmatrix} \quad ; \quad \Psi_{2,0} = \begin{pmatrix} 0 \\ \frac{\hat{a}\psi_{1,1}}{\sqrt{E_1}} \end{pmatrix} \tag{1.18}$$

$$\hat{H}_{susy}\Psi_{1,0} = \hat{H}_{susy}\Psi_{2,0} = E_1. \tag{1.19}$$

Moreover using this formalism it is easy to explain the degeneracy observed in the spectrum of $\hat{H}_{susy}$, thanks to the symmetry established by the commutation of $\hat{Q}$ and $\hat{Q}^\dagger$ with $\hat{H}_{susy}$.

1.1.2 Shape invariance and exactly solvable Schroedinger equation

In the previous section we have shown how to factorize an Hamiltonian operator. The factorization is the key idea in the Dirac method to solve the Harmonic oscillator problem. The purpose of this section is to generalize that idea to the most known exactly solvable quantum Hamiltonians. Let us rewrite explicitly the supersymmetric partner Hamiltonians $\hat{H}_1$, $\hat{H}_2$ with the potential V_1, V_2:

$$\hat{a} = -i\hbar\frac{d}{dx} + iW'(x); \qquad \hat{a}^\dagger = -i\hbar\frac{d}{dx} - iW'(x) \tag{1.20}$$

$$\hat{H}_1 = \hat{a}^\dagger \hat{a} = -\hbar^2 \frac{d^2}{dx^2} + V_1(x); \qquad V_1(x) = W'(x)^2 + \hbar W''(x) \tag{1.21}$$

$$\hat{H}_2 = \hat{a}\hat{a}^\dagger = -\hbar^2 \frac{d^2}{dx^2} + V_2(x); \qquad V_2(x) = W'(x)^2 - \hbar W''(x) \tag{1.22}$$

Let us now introduce the concept of shape invariance: if the pair of supersymmetric partner potentials $V_{1,2}$ defined in (1.21),(1.22), have the same functional form and differ only in the parameters that appear in them, then they are said to be shape invariant[2] (in [14] can be found an exhaustive review on the argument) . More precisely, if the supersymmetric partner potentials $V_{1,2}$ satisfy the condition:

$$V_2(x, \alpha) = V_1(x, \alpha_1) + \epsilon(\alpha), \tag{1.23}$$

where α is a set of parameters, α_1 is another set of parameters of the same dimension of α which is a function of α (say $\alpha_1 = f(\alpha)$) and the remainder $\epsilon(\alpha)$ is a function of α but indipendent of x then $V_1(x, \alpha)$ and $V_2(x, \alpha_1)$ are said to be shape invariant. The shape invariant condition (1.23) is an integrability condition. Using this condition and the hierarchy of Hamiltonians discussed in the past section, one can easily obtain the energy eigenvalues and the eigenfunctions of any shape invariant potential when SUSY is unbroken. Let us see now explicitly how this integrability condition works: by definition of shape invariance the following equation hold:

$$\hat{H}_2 = \hat{a}_\alpha \hat{a}_\alpha^\dagger = -\hbar^2 \frac{d^2}{dx^2} + V_2(x, \alpha) = -\hbar^2 \frac{d^2}{dx^2} + V_1(x, f(\alpha)) + \epsilon(\alpha) = \tag{1.24}$$

$$= \hat{a}_{f(\alpha)}^\dagger \hat{a}_{f(\alpha)} + \epsilon(\alpha)$$

as we have seen in the previous section the ground state is determined by the operators $\hat{a}, \hat{a}^\dagger$
so we define:

$$\phi_{0,\alpha} = e^{-\frac{W_\alpha}{\hbar}}; \quad \alpha \in \mathcal{D}_\alpha : \int \phi_\alpha^* \phi_\alpha dx < \infty; \quad \hat{a}_\alpha \phi_{0,\alpha} = 0 \tag{1.25}$$

The relation (1.24) entails the following equation:

$$\hat{a}_{f(\alpha)}^\dagger \hat{a}_{f(\alpha)} \phi_{0,f(\alpha)} = 0 \rightarrow$$

$$(\hat{a}_\alpha \hat{a}_\alpha^\dagger - \epsilon(\alpha))\phi_{0,f(\alpha)} = 0 \tag{1.26}$$

multiplying $\hat{a}_\alpha^\dagger$ by left, the equation (1.26) turns into:

[2] The main Idea behind the shape invariant technique, date back to the pioneering papers of Crum, Infield et Hull [12] [13]

$$\hat{a}^\dagger_\alpha \hat{a}_\alpha (\hat{a}^\dagger_\alpha \phi_{0,f(\alpha)}) = \epsilon(\alpha)(\hat{a}^\dagger_\alpha \phi_{0,f(\alpha)}) \tag{1.27}$$

where $(\hat{a}^\dagger_\alpha \phi_{0,f(\alpha)})$ defines another eigenfunction.
Because $\hat{a}^\dagger \hat{a}$ has got a positive spectrum $E_n \geq 0, \forall n \in \mathbb{N}$ if $f(\alpha) \in \mathcal{D}$ then $\epsilon(\alpha) > 0$, so we can regard to $\epsilon(\alpha)$ as the first excited level. Iterating this procedure we can get all the eigenvectors and the spectra of $\hat{a}^\dagger_\alpha \hat{a}_\alpha$

$$\phi_{n,\alpha} \propto \prod_{i=0}^{n-1} \hat{a}^\dagger_{f^{(i)}(\alpha)} \phi_{0,f^{(n)}(\alpha)} \qquad ; \qquad E_n = \sum_{i=0}^{n-1} \epsilon(f^{(i)}(\alpha)) \tag{1.28}$$

where:

$$f^{(n)}(\alpha) = \underbrace{f(f(...f(\alpha)))}_{\times n}. \tag{1.29}$$

Moreover because of $\epsilon(f^i(\alpha)) > 0, \forall i \in \mathbb{N}$, it is very simple to verify that E_n is a monotonically increasing function of n. Yet the eigenvalues obtained so far span the whole bounded spectrum of the Hamiltonian. Indeed, let us consider the Hamiltonian:

$$\hat{H}_{f^{-2}(\alpha)} = \hat{a}^\dagger_{f^{-2}(\alpha)} \hat{a}_{f^{-2}(\alpha)} \tag{1.30}$$

thanks to the relation (1.28) it is known (at least) part of the spectrum :

$$E_0 = 0, \quad E_1 = \epsilon(f^{-2}(\alpha)), \quad E_2 = \epsilon(f^{-2}(\alpha)) + \epsilon(f^{-1}(\alpha)), \quad E_3 = ... \tag{1.31}$$

Now let us suppose the existence of another eigenvalue $\tilde{E}$ such that

$$E_1 < \tilde{E} < E_2 \tag{1.32}$$

and:

$$\hat{a}^\dagger_{f^{-2}(\alpha)} \hat{a}_{f^{-2}(\alpha)} \tilde{\psi} = \tilde{E} \tilde{\psi} \tag{1.33}$$

the relation (1.24) can be generalized as follows:

$$\hat{a}_{f^{-n-1}(\alpha)} \hat{a}^\dagger_{f^{-n-1}(\alpha)} = \hat{a}^\dagger_{f^{-n}(\alpha)} \hat{a}_{f^{-n}(\alpha)} + \epsilon(f^{-n-1}(\alpha)) \tag{1.34}$$

now let us multiply $\hat{a}_{f^{-2}(\alpha)}$ by left in both sides of (1.33)

$$\hat{a}_{f^{-2}(\alpha)} \hat{a}^\dagger_{f^{-2}(\alpha)} (\hat{a}_{f^{-2}(\alpha)} \tilde{\psi}) = \tilde{E} (\hat{a}_{f^{-2}(\alpha)} \tilde{\psi}) \tag{1.35}$$

applying the (1.34)

$$\hat{a}^\dagger_{f^{-1}(\alpha)} \hat{a}_{f^{-1}(\alpha)} (\hat{a}_{f^{-2}(\alpha)} \tilde{\psi}) = (\tilde{E} - \epsilon(f^{-2}(\alpha))(\hat{a}_{f^{-2}(\alpha)} \tilde{\psi}) \tag{1.36}$$

the eigenvalue $(\tilde{E} - \epsilon(f^{-2}(\alpha))$ is greater than zero thanks the relation (1.32) and thus is still admissible, but if we iterate another time:

$$\hat{a}_{f^{-1}(\alpha)}\hat{a}^{\dagger}_{f^{-1}(\alpha)}(\hat{a}_{f^{-1}(\alpha)}\hat{a}_{f^{-2}(\alpha)}\tilde{\psi}) = (\tilde{E} - \epsilon(f^{-2}(\alpha))(\hat{a}_{f^{-1}(\alpha)}\hat{a}_{f^{-2}(\alpha)}\tilde{\psi}) \tag{1.37}$$

namely:

$$\hat{a}^{\dagger}_{f(\alpha)}\hat{a}_{f(\alpha)}(\hat{a}_{f^{-1}(\alpha)}\hat{a}_{f^{-2}(\alpha)}\tilde{\psi}) = (\tilde{E} - \epsilon(f^{-1}(\alpha)) - \epsilon(f^{-2}(\alpha)))(\hat{a}_{f^{-1}(\alpha)}\hat{a}_{f^{-2}(\alpha)}\tilde{\psi}), \tag{1.38}$$

but this is an absurd because the new eigenvalue $(\tilde{E} - \epsilon(f^{-1}(\alpha)) - \epsilon(f^{-2}(\alpha)))$ is negative by ipothesis (1.32); this is a strong evidence that it is not possible to get other eigenvalues outside of the scheme (1.28). Before concluding the section we show an explicit remarkable example.

1.1.3 The Coulomb potential

Let us see as a concrete example how the hydrogen atom can be solved by applying the operatorial mechanism:
let us define the "creator" and "annihilator" operators:

$$\hat{a}_l = -i\hbar\frac{d}{dx} + i(-\frac{\mu}{\hbar l} + \frac{\hbar l}{x}) \quad ; \quad \hat{a}^{\dagger}_l = -i\hbar\frac{d}{dx} - i(-\frac{\mu}{\hbar l} + \frac{\hbar l}{x}), \tag{1.39}$$

and the associated Hamiltonian:

$$\hat{H}_{hyd} = \hat{a}^{\dagger}_l\hat{a}_l = -\hbar^2\frac{d^2}{dx^2} + \frac{\hbar^2 l(l-1)}{x^2} - \frac{2\mu}{x} + \frac{\mu^2}{\hbar^2 l^2}; \qquad V_{1,l} = \frac{\hbar^2 l(l-1)}{x^2} - \frac{2\mu}{x} + \frac{\mu^2}{\hbar^2 l^2} \tag{1.40}$$

this problem fulfills the shape invariance condition:

$$\hat{a}_l\hat{a}^{\dagger}_l = -\hbar^2\frac{d^2}{dx^2} + \frac{\hbar^2 l(l+1)}{x^2} - \frac{2\mu}{x} + \frac{\mu^2}{\hbar^2 l^2}; \qquad V_2 = \frac{\hbar^2 l(l+1)}{x^2} - \frac{2\mu}{x} + \frac{\mu^2}{\hbar^2 l^2} \tag{1.41}$$

$$V_2 = V_{1,l+1} + \frac{\mu^2}{\hbar^2 l^2} - \frac{\mu^2}{\hbar^2(l+1)^2} \tag{1.42}$$

following our formalism we find out:

$$f(l) = l + 1; \qquad f^{(n)}(l) = l + n \qquad \epsilon(l) = \frac{\mu^2}{\hbar^2 l^2} - \frac{\mu^2}{\hbar^2(l+1)^2} \tag{1.43}$$

therefore it is readily understood that the n-th eigenstate and its relative eigenvalue is:

$$\psi_n \propto \prod_{i=0}^{n-1} \hat{a}^{\dagger}_{l+i} e^{-\frac{\mu x}{\hbar^2(l+n)}} x^{\hbar(l+n)}; \qquad E_n = \sum_{i=0}^{n-1} \epsilon(l+i) = \frac{\mu^2}{\hbar^2 l^2} - \frac{\mu^2}{\hbar^2(l+n)^2} \tag{1.44}$$

Before ending the section it has to be pointed out that a complete classification of the shape invariant potential is not available yet [14], and remarkably enough, all well known analytically solvable potentials found in most text books on nonrelativistic quantum mechanics belong to the shape invariance class characterized by $f(\alpha) = \alpha + \delta$ (traslational shape invariant potentials). In the early nineties a further class of shape invariant potential was discovered [15], [16] characterized by $f(\alpha) = q\alpha$. Yet, more recently, a new impetus has been given to the research for shape invariant potentials, in relation to the introduction of a new family of orthogonal polynomials, the so called "exceptional orthogonal polynomials" [7], [8], [9], [10]. Another important remark is that the shape invariance condition goes well beyond the simple structure defined by the SUSY quantum mechanics which establishes just a link between two different partner Hamiltonians $\hat{H}_1, \hat{H}_2$, while for the shape invariants the two Hamiltonians are intimately connected and this fact can be seen as the presence of an additional symmetry [17].

Concluding the section we propose a list of the traslational shape invariant potentials:

Name of potential	prepotential	domain	potential
3-dim. Harmonic oscillator	$-\frac{\omega x^2}{2} + (l+1)\ln(x)$	$0 < x < \infty, l \geq 0$	$\omega^2 x^2 + \frac{l(l+1)}{x^2}$ $-2\omega(l+1)$
Coulomb	$-\frac{\mu x}{l+1} + (l+1)\ln(x)$	$0 < x < \infty$ $\mu > 0, l \geq 0$	$-\frac{2\mu}{x} + \frac{l(l+1)}{x^2} + \frac{\mu^2}{(l+1)^2}$
Poschl-Teller	$g \ln\sin(x) + h\ln\cos(x)$	$0 < x < \frac{\pi}{2}$ $g > 0, h > 0$	$\frac{g(g-1)}{\sin(x)^2} + \frac{h(h-1)}{\cos(x)^2}$ $-(g+h)^2$
Spherical Coulomb	$-\frac{\mu x}{g} + g\ln\sin(x)$	$0 < x < \pi, g > 0$	$-2\mu\cot x + \frac{g(g-1)}{\sin x^2}$ $-g^2 - \frac{\mu^2}{g^2}$
Symmetric Top	$g\ln\sin x + h\ln\cot\frac{x}{2}$	$0 < x < \pi,$ $g > h > 0$	$\frac{g(g-1)+h^2-h(2g-1)\cos x}{\sin x^2}$ $-g^2$
Soliton	$-h\ln\cosh x$	$-\infty < x < \infty,$ $h > 0$	$-\frac{h(h+1)}{\cosh x^2} + h^2$
Hyperbolic Poschl-Teller	$g\ln\sinh x - h\ln\cosh x$	$0 < x < \infty$ $, h > g > 0$	$\frac{g(g-1)}{\sinh^2 x} - \frac{h(h+1)}{\cosh^2 x} +$ $+(g-h)^2$
Hyperbolic Symmetric Top I	$-g\ln\sinh x + h\ln\tanh\frac{x}{2}$	$0 < x < \infty$ $h > g > 0$	$\frac{g(g+1)+h^2-h(2g+1)\cosh x}{\sinh^2 x}$ $+g^2$
Hyperbolic Symmetric Top II	$-g\ln\cosh x +$ $-h\ln\arctan\sinh x$	$-\infty < x < \infty$ $g > 0$	$\frac{-g(g+1)+h^2+h(2g+1)\sinh x}{\cosh^2 x}$ $+g^2$
Kepler in spherical space	$g\ln\sinh x - \frac{\mu x}{g}$	$0 < x < \infty$ $\mu > g^2, g > 0$	$-2\mu\coth x + \frac{g(g-1)}{\sinh^2 x}$ $+g^2 + \frac{\mu^2}{g^2}$
Rosen Morse	$-g\ln\cosh x - \frac{\mu x}{g}$	$-\infty < x < \infty$ $g > 0, \mu > 0$	$2\mu\tanh x - \frac{g(g+1)}{\cosh^2 x}$ $+g^2 + \frac{\mu^2}{g^2}$
Morse	$-ge^q + \mu q$	$-\infty < x < \infty$ $g > 0, \mu > 0$	$g^2 e^{2x} - g(2\mu+1)e^x$ $+\mu^2$

where $\hbar = 1$

1.2 Jacobi orthogonal polynomials and shape invariant systems

It is very well known that any shape invariant system can be associated to a family of orthogonal polynomials. The orthogonal polynomials constitute a very active branch of research in mathematical physics especially in the search of integrable systems where they play a fundamental role [18] [19]. Very well known theorems ensure that the most general family of classical orthogonal polynomials is the class of

Jacobi orthogonal polynomials [20]. The crucial point is that the Jacobi orthogonal polynomials can be defined as solutions of the following second order differential equation:

$$(1-x^2)\frac{d^2}{dx^2}P_n^{\alpha,\beta}(x)+(\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx}P_n^{\alpha,\beta}(x)+n(n+\alpha+\beta+1)P_n^{\alpha,\beta}(x)=0. \tag{1.45}$$

The orthogonality condition turns out to be satisfied by:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)dx=\delta_{n,m} \tag{1.46}$$

where the $\delta_{n,m}$ is the usual Kronecker delta and the polynomials turn out to be normalized:

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{\alpha,\beta}(x)^2dx=1,\forall n. \tag{1.47}$$

The advantage in the definition (1.45) is twofold: on the one hand it is given a compact and smart definition of the orthogonal polynomials as eigenfunctions of a Sturm Liouville operator which entails the orthogonality condition, on the other hand it provides a quite general example of a second order differential operator whose discrete spectrum and eigenfunctions are exactly known.

1.2.1 Jacobi orthogonal polynomials as eigenfunctions of a 1-dimensional Schrodinger equation

It is possible to manipulate algebraically the differential equation (1.45) in order to get a Schrodinger-like differential equation:

$$-\frac{d^2}{dx^2}\psi+V\psi=E\psi. \tag{1.48}$$

The first step consists in transforming the "kinetic" energy term $(1-x^2)\frac{d^2}{dx^2}$ (1.45) in the "standard" term $\frac{d^2}{dx^2}$ of the equation (1.48). To this aim let us apply the following change of variables:

$$x=f(x'):\to(1-x^2)\frac{d^2}{dx^2}=\frac{1-f(x')^2}{f'(x')}\frac{d}{dx'}\left(\frac{1}{f'(x')}\frac{d}{dx'}\right) \tag{1.49}$$

$$\frac{1-f(x')^2}{f'(x')^2}=1 \tag{1.50}$$

Let us consider the particular solution $f(x')=\sin(x')$

$$\frac{d}{dx}=\frac{1}{\cos(x')}\frac{d}{dx'} \tag{1.51}$$

the differential equation (1.45) in the new variables turns out to be the following:

$$\hat{J} = \left(\frac{d^2}{dx'^2} + \left(\frac{\beta - \alpha}{\cos(x')} - (\alpha + \beta + 1) \tan(x') \right) \frac{d}{dx'} + n(n + \alpha + \beta + 1) \right) \quad (1.52)$$

$$\hat{J} P_n^{\alpha,\beta}(\sin(x')) = 0 \quad (1.53)$$

The eigenfunctions $P_n^{\alpha,\beta}(\sin(x))$ turn out to be orthogonal in the interval $[-\frac{\pi}{2}; \frac{\pi}{2}]$ with the proper weight function:

$$w(x') = (1 - \sin(x'))^\alpha (1 + \sin(x'))^\beta \cos(x') \quad (1.54)$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} w(x') P_n^{\alpha,\beta}(\sin(x')) P_m^{\alpha,\beta}(\sin(x')) dx' = \delta_{n,m} \quad (1.55)$$

Finally it is possible to transform the operator (1.52) in a (1.48) -like operator by performing a similarity transformation which simply absorbs the weight function in the eigenfunctions

$$\psi_n^{\alpha,\beta}(x) = \sqrt{w(x)} P_n^{\alpha,\beta}(\sin(x')). \quad (1.56)$$

It is straightforward to prove that this new set of functions turn out to be the eigenfunctions of the differential operator:

$$\hat{H}_j - E_n = -\sqrt{w(x)}\ \hat{J}\ \frac{1}{\sqrt{w(x)}} \quad (1.57)$$

explicitly it is got:

$$\hat{H}_j = -\partial_x^2 + \frac{1}{8} \left(\frac{4\alpha^2 - 1}{1 - \sin(x)} + \frac{4\beta^2 - 1}{1 + \sin(x)} \right) \quad (1.58)$$

$$E_n = (n + \frac{\alpha + \beta + 1}{2})^2$$

formally we have got an exactly solvable 1-dimensional quantum system :

$$\hat{H}_j \psi_n^{\alpha,\beta} = (n + \frac{\alpha + \beta + 1}{2})^2 \psi_n^{\alpha,\beta} \quad (1.59)$$

1.2.2 Jacobi shape invariant system

In section 1 we have pointed out that any 1-dimensional quantum Hamiltonian can be factorized, and this factorization can be obtained directly from the knowledge of the ground state:

$$(\hat{H} - E_0)\psi_0 = \hat{a}^\dagger \hat{a} \psi_0 = 0 \rightarrow \hat{a} = i\frac{d}{dx} - i\frac{\psi_0'}{\psi_0}. \quad (1.60)$$

Let us compute the operator $\hat{a}$ related to the hamiltonian $\hat{H}_j$(1.58)

$$\hat{a}_{\alpha,\beta} = i\frac{d}{dx} - i\frac{(\sqrt{w(x)})'}{\sqrt{w(x)}} = i\frac{d}{dx} - \frac{i}{2}\left(\frac{\beta-\alpha}{\cos(x)} - (\alpha+\beta+1)\tan(x)\right) \tag{1.61}$$

The crucial point is that this exactly solvable system fulfills the shape invariance request:

$$\hat{a}_{\alpha,\beta}\hat{a}^\dagger_{\alpha,\beta} = \hat{a}^\dagger_{\alpha+1,\beta+1}\hat{a}_{\alpha+1,\beta+1} + cost \tag{1.62}$$

explicitly, the following equations hold:

$$\hat{a}^\dagger_{\alpha,\beta}\hat{a}_{\alpha,\beta} = -\frac{d^2}{dx^2} + \frac{1}{8}\left(\frac{4\alpha^2-1}{1-\sin(x)} + \frac{4\beta^2-1}{1+\sin(x)}\right) - \frac{(a+b+a)^2}{4} \tag{1.63}$$

$$\hat{a}_{\alpha,\beta}\hat{a}^\dagger_{\alpha,\beta} = -\frac{d^2}{dx^2} + \frac{1}{8}\left(\frac{4(\alpha+1)^2-1}{1-\sin(x)} + \frac{4(\beta+1)^2-1}{1+\sin(x)}\right) - \frac{(a+b+a)^2}{4} \tag{1.64}$$

The system (1.58) can be further generalized, keeping at the same time the Schroedinger-like structure (1.48), through a scale-shift transformation:

$$x = sx' + \delta \tag{1.65}$$

In the new variables the eigenvalue equation becomes:

$$\hat{H}'_j = -\frac{d^2}{dx^2} + \frac{s^2}{8}\left(\frac{4\alpha^2-1}{1-\sin(sx'+\delta)} + \frac{4\beta^2-1}{1+\sin(sx'+\delta)}\right) \tag{1.66}$$

$$\hat{H}'_j\psi_n^{\alpha,\beta} = s^2(n+\frac{\alpha+\beta+1}{2})^2\psi_n^{\alpha,\beta} \tag{1.67}$$

Since our systems originates from a very general class of orthogonal polynomials, one may expect that tuning the free parameters one could get some shape invariant systems already known in literature, and this is indeed the case.
Let us present a list of well known shape-invariant systems which can be regarded as subcases of the general Jacobi shape invariant system:

general Jacobi	α	β	δ	s
Poschl-Teller	$g-\frac{1}{2}$	$h-\frac{1}{2}$	$\frac{\pi}{2}$	2
Symmetric Top	$g-h-\frac{1}{2}$	$g+h-\frac{1}{2}$	$\frac{\pi}{2}$	1
Soliton System	$-l-\frac{1}{2}$	$-l-\frac{1}{2}$	0	$\sqrt{-1}$

general Jacobi	$-\frac{d^2}{dx^2} + \frac{s^2}{8}\left(\frac{4\alpha^2-1}{1-\sin(sx'+\delta)} + \frac{4\beta^2-1}{1+\sin(sx'+\delta)}\right)$	$s^2(n+\frac{\alpha+\beta+1}{2})^2$
Poschl-Teller	$-\frac{d^2}{dx^2} + \frac{g(g-1)}{\sin(x)^2} + \frac{h(h-1)}{\cos(x)^2}, \quad 0 < x < \frac{\pi}{2}$	$(2n+g+h)^2$
Symmetric Top	$-\frac{d^2}{dx^2} + \frac{g(g-1)+h^2-h(2g-1)\cos(x)}{\sin(x)^2}, 0 < x < \pi, g > h > 0$	$(n+g)^2$
Soliton System	$-\frac{d^2}{dx^2} - \frac{l(l+1)}{\cosh(x)^2}, \quad -\infty < x < \infty$	$-(l-n)^2$

1.3 Coupling Constant Metamorphosis equivalent systems

In the previous section we have emphasized the generality of the Jacobi orthogonal polynomials as a source of shape invariant systems. In fact, we have identified of a set of well known shape invariant systems as subcases of the Jacobi system. For a full understanding of the versatility of the Jacobi system it would be interesting to look for possible existing relationships with other already known shape invariant systems. To this aim it is needed to enlarge the equivalent class of Jacobi shape invariant systems obtained in previous section ; as it has been seen previously the Poschl-Teller, Symmetric top and Soliton systems can be regarded as particular cases of the Jacobi system up to the application of a suitable rescaling-shift operation; as this operation doesn't change the integrability properties of the original systems, the goal of this section is to present a transformation which maps the Jacobi system to the other shape invariant systems that are not in the previous list. To this end, we introduce the so called Coupling Constant Metamorphosis (CCM) [21], [22] [23]. Let the Hamiltonian $\hat{H}$ be an exactly solvable quantum problem in the sense that the eigenfunctions and the spectrum are known:

$$\hat{H}\psi_{n,\alpha} = (\hat{T} + \alpha V)\psi_{n,\alpha} = E_{n,\alpha}\psi_{n,\alpha}; \quad \hat{T} = -\frac{d^2}{dx^2}. \tag{1.68}$$

Of course the spectrum depends on the quantum number n and on the parameter α which plays the role of the coupling constant. Equation (1.68) can be obviously rewritten in the following form:

$$\left(\frac{1}{V}\hat{T} - \frac{E_{n,\alpha}}{V}\right)\psi_{n,\alpha} = -\alpha\psi_{n,\alpha} \tag{1.69}$$

the final step is a redefinition of the Coupling Constant (that's the origin of the name CCM):

$$\mu = E(n,\alpha) \to \alpha = \alpha(n,\mu) \tag{1.70}$$

so that it is obtained a new quantum Hamiltonian system which turns out to be an exactly solvable one:

$$\hat{H}_{CCM} = \frac{1}{V}(\hat{T} - \mu)\psi_{n,\mu} = -\alpha_{n,\mu}\psi_{n,\mu}. \tag{1.71}$$

Let us come back to the general Jacobi Hamiltonian:

$$\left(-\frac{d^2}{dx^2} + s^2\frac{2(\alpha^2+\beta^2)-1}{4\cos(sx+\delta)^2} + s^2\frac{(\alpha^2-\beta^2)\sin(sx+\delta)}{2\cos(sx+\delta)^2}\right)\psi_n^{\alpha,\beta} = s^2(n+\frac{\alpha+\beta+1}{2})^2\psi_n^{\alpha,\beta}. \tag{1.72}$$

This Hamiltonian system has two coupling constants: $\alpha^2+\beta^2$ and $\alpha^2-\beta^2$ associated respectively to the two potentials:

$$\frac{1}{\cos(sx+\delta)^2}, \qquad \frac{\sin(sx+\delta)}{\cos(sx+\delta)^2} \tag{1.73}$$

Let us apply this machinery to the eigenvalue equation (1.72) considering the potential $\frac{1}{\cos(sx+\delta)^2}$ as the function V

$$\cos(sx+\delta)^2\left(-\frac{d^2}{dx^2} - s^2(n+\frac{\alpha+\beta+1}{2})^2 + s^2(\alpha^2-\beta^2)\frac{\sin(sx+\delta)}{2\cos(sx+\delta)^2}\right)\psi_n^{\alpha,\beta} = \tag{1.74}$$

$$= -s^2\frac{2(\alpha^2+\beta^2)-1}{4}\psi_n^{\alpha,\beta}$$

This new differential equation has a non constant coefficient in front of the second derivative term that makes it different from a standard 1-dimensional Schroedinger equation. Let us now generalize the algebraic transformations already seen in section 3. Whenever we have a term:

$$g(x)\frac{d^2}{dx^2} \tag{1.75}$$

in a second order differential equation, it is possible to look for a new coordinate system in which (1.75) become $\frac{d^2}{dx^2}$, so let us apply the following general change of variable:

$$x = f(x'); \qquad \frac{d}{dx} = \frac{1}{f'(x')}\frac{d}{dx'}. \tag{1.76}$$

applying this change to the term (1.75) it is straightforwardly found:

$$g(x)\frac{d^2}{dx'^2} = \frac{g(f(x'))}{(f'(x'))^2}\frac{d^2}{dx'^2} - \frac{g(f(x'))}{f'(x')^3}f''(x')\frac{d}{dx'}. \tag{1.77}$$

Eq. (1.77) determines the point canonical transformation which turns the differential operator (1.74) in a second order ODE with a constant term in front of the second order term:

$$\frac{\sqrt{g(f(x'))}}{f'(x')} = \pm 1 \rightarrow x' = \pm\int^f \frac{df}{\sqrt{g}} \tag{1.78}$$

The inversion of this relation yields the point canonical transformation we are looking for. In this case this transformation turns out to be:

$$x = \frac{2}{s}\tan^{-1}(\tanh(\frac{x'}{2})) - \frac{\delta}{s}. \tag{1.79}$$

The eigenvalue equation (1.74) turns into the following equation:

$$\hat{H} = \left(-\frac{d^2}{dx^2} - \tanh(x)\frac{d}{dx} - \frac{(n+\frac{\alpha+\beta+1}{2})^2}{\cosh(x)^2} + \frac{(\alpha^2-\beta^2)}{2}\tanh(x)\right), \tag{1.80}$$

$$\hat{H}\psi_n^{\alpha,\beta} = -\frac{2(\alpha^2+\beta^2)-1}{4}\psi_n^{\alpha,\beta}. \tag{1.81}$$

The eigenfunctions in the new variables are:

$$\psi_n^{\alpha,\beta} = (1-\tanh(x))^{\frac{\alpha}{2}}(1+\tanh(x))^{\frac{\beta}{2}}\frac{P_n^{\alpha,\beta}(\tanh(x))}{\sqrt{\cosh(x)}}. \tag{1.82}$$

Now let us apply just another little make up to the (1.80) in order to get a Schroedinger-like equation. Any differential operator of the form:

$$\hat{A} = \frac{d^2}{dx^2} + g(x)\frac{d}{dx} \tag{1.83}$$

can be reduced to:

$$\hat{A}' = \frac{d^2}{dx^2} + V(x) \tag{1.84}$$

by means of a suitable similarity transformation:

$$e^{-f(x)}\hat{A}e^{f(x)} = \frac{d^2}{dx^2} + 2f'(x)\frac{d}{dx} + f''(x) + f'(x)^2 + g(x)\frac{d}{dx} + g(x)f'(x) \tag{1.85}$$

so that we need:

$$f(x) = -\frac{1}{2}\int^x g(x')dx'. \tag{1.86}$$

then:

$$e^{-f(x)}\hat{A}e^{f(x)} = \frac{d^2}{dx^2} - \frac{g'(x)}{2} - \frac{g(x)^2}{4}. \tag{1.87}$$

following the strategy outlined above we get finally the similarity transformation which allows us to obtain a 1-dim Schrodinger equation

$$\tilde{\psi}_n^{\alpha,\beta}(x) = \sqrt{\cosh(x)}\psi_n^{\alpha,\beta}(x) \tag{1.88}$$

modifying at the same time the Hamiltonian operator:

$$\hat{H}' = \sqrt{\cosh(x)}\hat{H}\frac{1}{\sqrt{\cosh(x)}} - \frac{1}{4}, \tag{1.89}$$

$$\hat{H}' = \left(-\frac{d^2}{dx^2} - \frac{(n+\frac{\alpha+\beta}{2})(n+\frac{\alpha+\beta}{2}+1)}{\cosh(x)^2} + \frac{(\alpha^2-\beta^2)}{2}\tanh(x) \right), \tag{1.90}$$

$$\hat{H}'\tilde{\psi}_n^{\alpha,\beta}(x) = -\frac{\alpha^2+\beta^2}{2}\tilde{\psi}_n^{\alpha,\beta}(x). \tag{1.91}$$

It should be remarked that the above algebraic transformations (similarity and point canonical) play a central role as well in the construction of the so called QES (Quasi Exactly Solvable systems) [24].
Once got a Schroedinger-like equation it is possible to apply the core of the CCM, namely the redefinition of the parameters:

$$\begin{cases} n+\frac{\alpha+\beta}{2} = l \\ \alpha^2-\beta^2 = 4\mu \end{cases} \rightarrow \begin{cases} \alpha = (l-n)+\frac{\mu}{l-n} \\ \beta = (l-n)-\frac{\mu}{l-n} \end{cases} . \tag{1.92}$$

This leads to the following eigenvalue equation:

$$\left(-\frac{d^2}{dx^2} - \frac{l(l+1)}{\cosh(x)^2} + 2\mu\tanh(x) \right)\psi_n^{\alpha,\beta}(x) = -\left((l-n)^2 + \frac{\mu^2}{(l-n)^2} \right)\psi_n^{\alpha,\beta}(x). \tag{1.93}$$

This exactly solvable quantum system is known as the Rosen-Morse system, and it is important to point out that this system is still a shape invariant one. It is interesting to remark that the spectrum of Rosen-Morse system has been got by solving just an algebraic system in the two variables α, β , whose roots, once replaced in the old coupling constant $\alpha^2+\beta^2$, determine the whole spectra of the new system. Following this line we could apply the CCM systematically to any coupling constant existing in the Hamiltonian operator, so getting new Hamiltonian quantum systems whose spectra is exactly determined by an algebraic equation. Since the Jacobi system has got two coupling constant this can be transformed in two different CCM systems. In the following we show the generalization of the Jacobi system that is obtained by adding a proper constant and those that can be generated after a CCM.

$$\hat{H}\psi_n = (\hat{T} + (\alpha^2 + \beta^2)V_1 + (\alpha^2 - \beta^2)V_2)\psi_n = E_n\psi_n$$

$$(\hat{T}+(\alpha^2+\beta^2)V_1+(\alpha^2-\beta^2)(V_2+c))\psi_n=(E_n+c(\alpha^2-\beta^2))\psi_n$$

$$\downarrow$$

$$\left(\frac{\hat{T}-(E_n+c(\alpha^2-\beta^2))+(\alpha^2+\beta^2)V_1}{c+V_2}\right)\psi_n = (\beta^2 - \alpha^2)\psi_n$$

$$\downarrow$$

$$\begin{cases} E_n(\alpha,\beta)+c(\alpha^2 - \beta^2)=l \\ \alpha^2 + \beta^2 = \mu \end{cases} \rightarrow \begin{cases} \alpha = \alpha(n,l,\mu) \\ \beta = \beta(n,l,\mu) \end{cases}$$

$$\downarrow$$

$$\tilde{E}_n = \beta(n,l,\mu)^2 - \alpha(n,l,\mu)^2$$

$$(\hat{T}+(\alpha^2+\beta^2)(V_1+c)+(\alpha^2-\beta^2)V_2)\psi_n=(E_n+c(\alpha^2+\beta^2))\psi_n$$

$$\downarrow$$

$$\left(\frac{\hat{T}-(E_n+c(\alpha^2+\beta^2))+(\alpha^2-\beta^2)V_2}{c+V_1}\right)\psi_n = -(\alpha^2 + \beta^2)\psi_n$$

$$\downarrow$$

$$\begin{cases} E_n(\alpha,\beta)+c(\alpha^2 + \beta^2)=l \\ \alpha^2 - \beta^2 = \mu \end{cases} \rightarrow \begin{cases} \alpha = \alpha(n,l,\mu) \\ \beta = \beta(n,l,\mu) \end{cases}$$

$$\downarrow$$

$$\tilde{E}_n = -\alpha(n,l,\mu)^2 - \beta(n,l,\mu)^2$$

Of course it is also possible to get other shape invariant systems carrying on with this approach, and in the following we report some examples :

1.3.1 Shape invariant CCM systems

The Rosen-Morse system:

$$\left(-\frac{d^2}{dx^2} - \frac{l(l+1)}{\cosh(x)^2} + 2\mu\tanh(x)\right)\psi_n^{\alpha,\beta}(x) = -\left((l-n)^2 + \frac{\mu^2}{(l-n)^2}\right)\psi_n^{\alpha,\beta}(x) \tag{1.94}$$

can be used as a starting point to generate a lot of shape invariant systems.
Let us start doing a shift and a rescaling:

$$x = sx' + \delta \tag{1.95}$$

$$\left(-\frac{d^2}{dx^2} - s^2\frac{l(l+1)}{\cosh(sx+\delta)^2} + 2\mu s^2\tanh(sx+\delta)\right)\psi_n^{\alpha,\beta}(x) = -s^2\left((l-n)^2 + \frac{\mu^2}{(l-n)^2}\right)\psi_n^{\alpha,\beta}(x) \tag{1.96}$$

now let us set the parameters as follows:

$$s = ik, \quad \delta = i\frac{\pi}{2}, \quad \mu = \frac{-i\mu'}{k}, \quad l = -g, \quad \sqrt{-1} = i$$

the eigenvalue equation becomes:

$$\left(-\frac{d^2}{dx^2} + k^2\frac{g(g-1)}{\sin(kx)^2} - 2\mu' k\cot(kx)\right)\psi_n^{\alpha,\beta}(x) = \left(+k^2(g+n)^2 - \frac{{\mu'}^2}{(g+n)^2}\right)\psi_n^{\alpha,\beta}(x) \tag{1.97}$$

The system above is the Kepler problem on the sphere. By taking the limit

$$\lim_{k\to 0}\left(-\frac{d^2}{dx^2} + k^2\frac{g(g-1)}{\sin(kx)^2} - 2\mu' k\cot(kx)\right)\psi_n^{\alpha,\beta}(x) = \lim_{k\to 0}\left(+k^2(g+n)^2 - \frac{{\mu'}^2}{(g+n)^2}\right)\psi_n^{\alpha,\beta}(x)$$

which leads to the subcase :

$$\left(-\frac{d^2}{dx^2} + \frac{l(l-1)}{x^2} - \frac{2\mu'}{x}\right)\psi_n^{\alpha,\beta}(x) = -\frac{{\mu'}^2}{(l+n)^2}\psi_n^{\alpha,\beta}(x) \tag{1.98}$$

namely the radial part of the Kepler Schroedinger equation. This one dimensional system has two different potential parts (centrifugal and the Kepler potential part) therefore it is possible to apply two differents CCM transformations to this system.

Rosen system In order to apply the CCM to the centrifugal part let us divide by $\frac{1}{x^2}$ both sides of the equation.

$$\left(-x^2\frac{d^2}{dx^2} - 2\mu' x + \frac{{\mu'}^2}{(l+n)^2}x^2\right)\psi_n^{\alpha,\beta}(x) = -l(l-1)\psi_n^{\alpha,\beta}(x) \tag{1.99}$$

now let us apply a change of variables:

$$x \to e^x \tag{1.100}$$

the eigenvalue equation turns into:

$$\hat{H}\psi_n^{\alpha,\beta}(x) = \left(-\frac{d^2}{dx^2} + \frac{d}{dx} - 2\mu' e^x + \frac{{\mu'}^2}{(l+n)^2}e^{2x}\right)\psi_n^{\alpha,\beta}(x) = -l(l-1)\psi_n^{\alpha,\beta}(x) \tag{1.101}$$

which after the following similarity transformation become:

$$e^{-\frac{x}{2}}\hat{H}e^{\frac{x}{2}} = \left(-\frac{d^2}{dx^2} - 2\mu' e^x + \frac{\mu'^2}{(l+n)^2}e^{2x}\right) + \frac{1}{4} \tag{1.102}$$

and redefining the parameters $\mu' \to g(p+\frac{1}{2}), \quad l \to p + \frac{1}{2} - n$, the Rosen system is obtained:

$$\hat{H} = \left(-\frac{d^2}{dx^2} - g(2p+1)e^x + g^2 e^{2x}\right) \tag{1.103}$$

Harmonic oscillator system Let us apply the S.T. to the Kepler potential of the Hamiltonian (1.98).

$$x\left(-\frac{d^2}{dx^2} + \frac{l(l-1)}{x^2} - \frac{2\mu'}{x}\right)\psi_n^{\alpha,\beta}(x) = -x\frac{\mu'^2}{(l+n)^2}\psi_n^{\alpha,\beta}(x) \tag{1.104}$$

now let us apply the algebraic manipulations already used for the other systems :

$$x \to \frac{x'^2}{2}$$

$$\hat{H}\psi_n^{\alpha,\beta}(x) = \left(-\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{4l(l-1)}{x^2} + \frac{\mu^2}{(l+n)^2}x^2\right)\psi_n^{\alpha,\beta}(x) = 4\mu\psi_n^{\alpha,\beta}(x) \tag{1.105}$$

$$\frac{1}{\sqrt{x}}\hat{H}\sqrt{x} = \left(-\frac{d^2}{dx^2} + \frac{4l(l-1)+\frac{3}{4}}{x^2} + \frac{\mu^2}{(l+n)^2}x^2\right) \tag{1.106}$$

$$l = \frac{p}{2} + \frac{1}{4}, \qquad \mu = \omega(l+n)$$

$$\left(-\frac{d^2}{dx^2} + \frac{p(p-1)}{x^2} + \omega^2 x^2\right)\psi = \omega(4n+2p+1)\psi. \tag{1.107}$$

Chapter 2

Classical Maximally Superintegrable systems on non-Euclidean manifolds

2.1 Introduction

In the first chapter it has been remarked that in quantum physics there is an exceptional class of 1-dimensional exactly solvable quantum Hamiltonians. With the exact solvability we mean the possibility of describing the spectrum and the eigenfunctions in terms of standard trascendental functions. Also, in the first chapter, it has been pointed out that a vast majority of exactly solvable quantum systems can be solved in an algebraic way by applying the so called operatorial method for shape invariant systems. In classical mechanics to solve exactly a system means knowing its trajectory on the phase space and describing it as a time parametrical function. Analogously to what happens in quantum mechanics for the shape invariant systems, where there exists a mechanism that allows us to solve the spectral problem, in classical mechanics there exists a class of systems named "maximally superintegrable" whose trajectory on the phase space can be determined by solving just algebraic equations: a Maximally Superintegrable system (M.S.) is a Hamiltonian system with N degrees of freedom equipped with a maximum number of independent observables Poisson commuting with the Hamiltonian, namely $2N-1$ indipendent constant of motion and at least a set of N integral of motion in involution (any M.S. system is also Liouville Integrable). Fixing the value of the $2N-1$ constants of motion the 2N-dimensional phase space is reduced to a 1-dimensional space which, by definition, coincides with the trajectory of the system itself. The M.S. systems constitute an exceptional class of systems exactly on the same footing of the shape invariant systems introduced in the first chapter. The exceptional nature of the M.S. systems was remarked for the first time by Bertrand in the 19th century in the studies he made on the central potential fields: looking for the class of potentials that admit stable closed orbit around the centre of the field, he proved that the only potentials whose bounded orbits are closed and stable correspond to the Harmonic Oscillator potential or the Kepler Coulomb potential. The main consequence of this theorem is that cannot exist M.S. systems other than the Kepler -Coulomb or Harmonic oscillator potential since any other new M.S. system would define a new system with the closed orbit property. Therefore the Bertrand theorem entails that the only two candidates to be maximally superintegrable systems with a radial symmetry are the harmonic oscillator problem and the Kepler problem, and this is indeed the case. In fact, other than the angular momentum conserved, they have also the Fradkin tensor (harmonic oscillator), and the Laplace Runge Lenz (Kepler potential) [53]. It is amazing that these two problems are exactly solvable both in in classical and quantum mechanics where they are classified as shape invariant potentials. It is worthy to say that the exceptional class of the radial M.S. systems are uniquely determined by the Bertrand theorem in contrast with the shape invariant systems (in which we find also the harmonic oscillator and the Kepler problem) which at the moment, are still an open list. It is also quite remarkable, as we will see in the next chapter, that the vast majority of shape invariant systems can be linked to a maximally superintegrable system. Aim of this chapter is to provide a survey of the maximally superintegrable systems (MS) giving the most exhaustive classification of systems which exibit such a property.

2.2 Hamiltonian systems on non-Euclidean spaces and Maximal superintegrability

The birth of non-Euclidean geometry, marked by the first works by Gauss, Boylay and Lobachevsky, was one of the greatest breakthrough in the history of mathematic. It had a deep influence on the way of thinking and allowed scientists to look at natural phenomena by new perspectives, and can be considered as the first step to one of the most relevant achievements of modern physics like the Einsteinian theory of general relativity. Before generalizing the studies of M.S. systems to physical systems defined on non-Euclidean manifolds let us introduce briefly the main concepts necessary to defining an Hamiltonian system in such a new context.

2.2.1 Differentiable manifolds

The non euclidean geometry can be described by introducing the elegant and abstract concept of differential manifold: following Schutz [25] we can define a differential manifold as a set of "points" $\mathcal{M}$ where any point of this set has an open neighborhood which has a continuous 1-1 map onto an open set of $\mathbb{R}^N$ for some N. In other words this means that $\mathcal{M}$ is locally like $\mathbb{R}^N$ and the dimension of $\mathcal{M}$ is obviously N. By definition, the map associates to a point P of $\mathcal{M}$ an n-tuple $(x_1(P), ...x_N(P))$. These numbers $x_1(P), ..., x_N(P)$ are called the coordinates of P under this map. The pair consisting of a neighborhood of P say U and its map f: $(f : U \subset \mathbb{R}^N)$ is said to be a chart (U, f). Let us consider now another neighborhood

V of $\mathcal{M}$ such that : $U \cap V \neq 0$, the chart (V,g) defines another set of coordinate $Y(P)$ for the point P. In the intersection $U \cap V$ we have two coordinate systems $X(P)$ and $Y(P)$ therefore there exist a functional relationship between the two coordinate systems:

$$y_1 = y_1(x_1, ..., x_N) \tag{2.1}$$

$$\vdots \tag{2.2}$$

$$y_N = y_N(x_1, ..., x_N) \tag{2.3}$$

If the partial derivatives of order k (or less then k) of all these functions $\{y_i\}$ with respect to all the $\{x_i\}$ exist and are continuous, then the maps f and g (strictly the charts (U, f) and (V, g)) are said to be $\mathcal{C}^k$ -related . If it is possible to construct a whole system of charts (called ,appropriately enough, atlas) so that every point of $\mathcal{M}$ is in at least one neighborhood and every chart is $\mathcal{C}^k$related to every other one it overlaps with, then the manifold $\mathcal{M}$ is said to be a $\mathcal{C}^k$ manifold, a manifold $\mathcal{C}^1$ is called a differentiable manifold. Now let us introduce the concept of a curve as a differentiable mapping from an open set of $\mathbb{R}^1$ into $\mathcal{M}$. Thus, one associates with each point of $\mathbb{R}^1$ (which is a real number, say λ) a point in $\mathcal{M}$, which is called the image point of λ . The set of all image point is the ordinary notion of a curve. For differentiable mapping we mean again that our parametrized curve described by $\{x_i(\lambda), i = 1, ...n\}$ are differentiable functions of λ.

2.2.2 Vectors on a differentiable manifold

In physics the vectors are fundamental objects so it is crucial to restore this concept also in the general context of a differentiable manifold. Let us consider a curve passing through the point P of $\mathcal{M}$, described by the equations $x_i = x_i(\lambda), i = 1, ..., n$. Consider also a differentiable function $f(x_1, ..., x_n)$ abbreviated hereafter with $f(\mathbf{x})$ on $\mathcal{M}$. At each point of the curve, $f(\mathbf{x})$ has a value. Therefore, along the curve there is a differentiable function $g(\lambda)$ which gives the value of $f(\mathbf{x})$ at the point whose parameter value is λ:

$$g(\lambda) = f(x_1(\lambda)...x_n(\lambda)) = f(\mathbf{x}(\lambda))$$

differentiating and using the chain rule gives :

$$\frac{dg}{d\lambda} = \sum_i \frac{dx_i}{d\lambda} \frac{\partial f}{\partial x_i} \tag{2.4}$$

This is true for any function g, so we can write :

$$\frac{d}{d\lambda} = \sum_i \frac{dx_i}{d\lambda} \frac{\partial}{\partial x_i} \tag{2.5}$$

Now, in the ordinary picture of vectors in Euclidean space, one would say that the set of numbers $\{\frac{dx_i}{d\lambda}\}$ are components of a vector tangent to the curve $x_i(\lambda)$; one can see this by realizing that $\{dx_i\}$ are infinitesimal displacements along the curve, and that dividing that by $d\lambda$ only change the scale, not the direction, of this displacement. In fact, since a curve has a unique parameter, to every curve there is a unique $\{\frac{dx_i}{d\lambda}\}$, which are then said to be components of the tangent to the curve. Thus, with our definition of a curve, every curve has a unique tangent vector. This use of the term "vector" relies on familiar concepts from Euclidean space, where vectors are defined by analogy with displacements Δx_i. However, since we have not defined a notion of distance between points, we shall need a definition of vector which relies only on infinitesimal neighborhoods of points of $\mathcal{M}$. Suppose a and b two numbers, and $x_i = x_i(\mu)$ is another curve through P. Then at P we have :

$$\frac{d}{d\mu} = \sum_i \frac{dx_i}{d\mu}\frac{\partial}{\partial x_i}, \tag{2.6}$$

and :

$$a\frac{d}{d\lambda} + b\frac{d}{d\mu} = \sum_i \left(a\frac{dx_i}{d\lambda} + b\frac{dx_i}{d\mu}\right)\frac{\partial}{\partial x_i}. \tag{2.7}$$

Now the numbers $\{a\frac{dx_i}{d\lambda} + b\frac{dx_i}{d\mu}\}$ are components of a new vector, which is certainly the tangent to some curve through P. So there must exist a curve with parameter, say, ϕ such that at P:

$$\frac{d}{d\phi} = \sum_i \left(a\frac{dx_i}{d\lambda} + b\frac{dx_i}{d\mu}\right)\frac{\partial}{\partial x_i}. \tag{2.8}$$

Collecting these results, we get, at P,

$$a\frac{d}{d\lambda} + b\frac{d}{d\mu} = \frac{d}{d\phi}. \tag{2.9}$$

Therefore, the directional derivatives along curves, like $\frac{d}{d\lambda}$, form a vector space at P. There are in any coordinate system special curves, the coordinate lines themselves, the derivation along them are clearly $\frac{\partial}{\partial x_i}$, and equation (2.5) shows that any $\frac{d}{d\lambda}$ can be written as a linear combination of the particular derivatives $\frac{\partial}{\partial x_i}$. It follows that $\{\frac{\partial}{\partial x_i}\}$ are a basis for this vector space. Then (2.5) shows that $\frac{d}{d\lambda}$ has components $\frac{dx_i}{d\lambda}$ on this basis. Therefore we have the remarkable result that the space of all tangent vectors at P and the space of all derivatives along curves at P are in 1-1 correspondence. Now, however, one must realize that only vectors at the same point P can be added together. Vectors at two different points have no relation with one another; in fact the vectors lie, not in $\mathcal{M}$, but in the tangent space to $\mathcal{M}$ at P, which is called T_p.

2.3 Hamiltonian systems on Riemannian manifolds

In the previous section we have stressed that in a differentiable manifold the notion of vector is no more a globally defined notion, but it is restricted to a so called "tangent vector space" at each point of the manifold. Now let us define a metric on each tangent space. This is possible by introducing the idea of Riemannian manifold: a Riemannian manifold is a real differentiable manifold $\mathcal{M}$ in which each tangent space is equipped with an inner product g, called a Riemannian metric, which varies smoothly from point to point. Once defined a metric on this space it also possible to define dinamical variables like the velocity and therefore also to define a Lagrangian for a particle moving on a Riemannian manifold. Let us consider a Riemannian manifold with the general metric defined by the following fundamental quadratic form:

$$ds^2 = g(\mathbf{x})_{i,j} dx^i dx^j. \tag{2.10}$$

The Lagrangian of a particle bounded to move on such a manifold in absence of any potential is:

$$L = \frac{1}{2}\left(\frac{ds}{dt}\right)^2 = \frac{1}{2} g(\mathbf{x})_{i,j} \dot{x}^i \dot{x}^j \tag{2.11}$$

this classical problem can be recasted using the Hamiltonian formalism, namely performing a Legendre transformation:

$$\mathcal{H} = \frac{1}{2} g(\mathbf{x})^{i,j} p_i p_j; \quad p_i = \frac{\partial L}{\partial \dot{x}^i}, \qquad g(\mathbf{x})_{i,l} g(\mathbf{x})^{l,j} = \delta_i^j \tag{2.12}$$

The main difference between a free particle moving on a Riemannian manifold and a free particle moving in a Euclidean manifold ($g_{ij} = \delta_{ij}$) is that the first one breaks in general the simmetry properties holding in an Euclidean space such as the translation and rotation invariance, namely the classical conserved quantities as linear momentum and angular momentum. So the question is: Are there some Riemannian spaces that preserve the Maximal superintegrability property, similarly to what happens trivially in the Euclidean space with the classical conserved quantities (linear momentum and angular momentum) ? A partial answer to this question, with regard to the two dimensional spaces, was provided by a note written by Koenigs on the Vol IV of the treatise of Darboux [26], who listed all those Riemannian spaces characterized by the existence of three constants of the motion at most quadratic in linear momenta and with a non-constant scalar curvature. They are known as Darboux spaces, and are a finite set of systems like the Bertrand systems. We report such complete list of spaces with the corresponding constants of the motion:

	$ds^2 =$ $\mathcal{H}$	I_1	I_2	I_3
I	$2u(du^2+dv^2)$; $\frac{1}{2u}(p_u^2+p_v^2)$	p_v	$p_up_v - \frac{v}{2u}(p_u^2+p_v^2)$	$p_v(vp_u - up_v)+$ $-\frac{v^2}{4u}(p_u^2+p_v^2)$
II	$\frac{u^2+1}{u^2}(du^2+dv^2)$; $\frac{u^2}{u^2+1}(p_u^2+p_v^2)$	p_v	$\frac{2v(p_v^2-u^2p_u^2)}{u^2+1}+2up_up_v$	$\frac{(v^2-u^4)p_v^2+u^2(1-v^2)p_u^2}{u^2+1}$ $+2uvp_up_v$
III	$\frac{4(e^u+1)}{e^{2u}}(du^2+dv^2)$ $\frac{e^{2u}}{4(e^u+1)}(p_u^2+p_v^2)$	p_v	$-\frac{(e^{2u}+2e^u)\cos(v)}{4(e^u+1)}p_v^2+$ $\frac{e^{2u}\cos(v)}{1+e^u}p_u^2+\frac{e^u\sin(v)}{2}p_up_v$	$\frac{e^{2u}\sin(v)}{4(e^u+1)}p_u^2$ $-\frac{e^{2u}+2e^u}{4(e^u+1)}\sin(v)p_v^2$ $-\frac{e^u\cos(v)}{2}p_up_v$
IV	$\frac{2\cos(2u)+a}{\sin^2(2u)}(du^2+dv^2)$ $\frac{\sin^2(2u)}{2\cos(2u)+a}(p_u^2+p_v^2)$	p_v	$\frac{e^{2v}\sin^2(2u)}{2\cos(2u)+a}(p_u^2+p_v^2)$ $e^{2v}\cos(2u)p_u^2+$ $+\sin(2u)p_up_v$	$\frac{e^{-2v}\sin^2(2u)}{2\cos(2u)+a}(p_u^2+p_v^2)$ $+e^{-2v}(\cos(2u)p_up_v$ $-\sin(2u)p_up_v)$

The Darboux spaces are a genuine new class of Riemannian spaces and they cannot be reduced to neither Euclidean, nor spherical spaces through change of variables; the main consequence of this fact is that they have to define a class of maximally superintegrable spaces with a nonconstant intrinsic Gaussian curvature.

2.4 Maximally Superintegrable potentials and Bertrand systems

In the past section it has been provided a classification of the two dimensional spaces which admit the maximal number of symmetries, namely the Darboux (non-constant curvature) together with the Euclidean and spherical spaces (zero and constant curvature). These spaces admit an Hamiltonian free motion which has got the maximal number of conserved quantities at most quadratic in the momenta. Now we would draw our attention to the classification of the potentials which preserve maximal superintegrability for a particle moving on these spaces. Historically speaking the first classification in Euclidean space was given by Bertrand [27], he didn't request directly the maximal superintegrability, but we can find it as a consequence of his original request, namely to have a periodic motion.
As any central potential which admits bounded motion admits as well a circular orbit, then asking for its stability under small perturbations he found a series of constraints that at the end entailed that the only central potentials with a stable closed orbit are the Kepler and the harmonic oscillator potentials. Many other classification of potentials on two dimensional spaces have been given over the years, but in most cases they assume as starting point the existence of three conserved quantities with some special characteristics such as a quadratic momentum dependence, or by requiring the superseparability of the Hamiltonian systems, feature intimately related with the M.S. systems; the literature about these systems is quite large; I try to give a survey following an historical order. The first of such a Hamiltonian systems

can be found in the papers by Drach [28] (1935) and Fris et al. [29] (1965) they managed to find a class of M.S. potentials on the Euclidean space with extra integral of motion quadratic and cubic in the momenta; such a list was enlarged by the more recent works in the '90s [30] [31], and by systematic studies of superintegrable systems conducted for spaces of constant curvature in two and three dimensions [35] [36]. With regard to M.S. Hamiltonian systems on non-Euclidean manifold with non-constant scalar curvature a classification has been carried out by Kalnins Kress and Winternitz [32], [33] who using the coupling constant metamorphosis (or also the Stackel transformation) found a class of potentials which preserve the maximal superintegrability property of the Darboux spaces. The examples of maximally superintegrable systems aforementioned are really general from a geometrical perspective, they have been obtained without requiring any geometrical symmetry, but on the other hand, the main drawback is that we are limiting our studies to additional constants of the motion with a definite order in the momenta, a way to overcome this restriction is to consider the Bertrand approach, in other words we require the closed orbit property without caring about the characteristics of the additional constant of motion, which in turn have to be found later: unfortunately in this case we loose the generality on the space, and in fact the Bertrand approach has been used, up to now, to studying perturbation of circular orbits, meaning that a spherical setting is supposed.

2.4.1 Bertrand spacetimes

In the nineties some authors rediscovered this approach [37], [38], [50] and the result was a complete classification of Hamiltonian systems on non-Euclidean 3-dimensional spaces whose bounded trajectories are closed. Since the most general class of such systems was obtained by Perlick [50] let us retrace briefly his steps :
Let us consider a spherically symmetric and static spacetime $(\mathcal{M} \times R, g)$ where $\mathcal{M}$ is a 3-manifold. This ensure that the Lorentzian metric g can be written as:

$$g = e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - e^{2\nu(r)}dt^2 \tag{2.13}$$

although the space is (3+1)-dimensional, without losing of generality we can consider a (2+1)-dimensional space, if we restrict ourselves to the equatorial plane $\theta = \frac{\pi}{2}$. Then the Lagrangian for the geodesic equation reads:

$$\mathcal{L} = \frac{1}{2}(e^{2\lambda(r)}\dot{r}^2 + r^2\dot{\phi}^2 - e^{2\nu}\dot{t}^2) \tag{2.14}$$

where the dot denotes derivative with respect to the curve parameter: from a physical point of view this parameter can be regarded as a proper time. This Lagrangian admits of course three constant of motion:

$$E = \frac{1}{2}(e^{2\lambda(r)}\dot{r}^2 + r^2\dot{\phi}^2 - e^{2\nu}\dot{t}^2) \tag{2.15}$$

$$L = r^2\dot{\phi} \tag{2.16}$$

$$C = e^{2\nu(r)}\dot{t}. \tag{2.17}$$

It is not restrictive to impose a reparametrization of the curve parameter such that $C = 1$. A timelike geodesic in such a space defines a trajectory that actually corresponds to a trajectory (in configuration space) [39] of the following Hamiltonian:

$$\mathcal{H}(r, P_r, \phi, P_\phi) := \frac{e^{-2\lambda(r)}}{2}P_r^2 + \frac{P_\phi^2}{2r^2} - \frac{e^{-2\nu(r)}}{2} \tag{2.18}$$

The orbit equation for such a system is :

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{P_\phi^2}e^{-2\lambda(r)}\left(2E - \frac{P_\phi^2}{r^2} + e^{-2\nu(r)}\right) \tag{2.19}$$

Once the closed orbit property is imposed, this Hamiltonian system is expected to have three constant of motion functionally indipendent, namely to be a M.S. system. Given this setting and following the Bertrand strategy, Perlick considers a circular trajectory, namely:

$$P_r = 0 \cup \dot{P}_r = 0 \rightarrow E = \frac{P_\phi^2}{2r^2} - \frac{e^{-2\nu(r)}}{2}; \quad \dot{P}_r = \frac{P_\phi^2}{r^3} - \nu(r)'e^{-2\nu(r)} = 0 \tag{2.20}$$

thus for a circular orbit holds:

$$P_\phi^2 = r^3\nu'(r)e^{-2\nu(r)} > 0 \tag{2.21}$$

$$E_0 = \frac{e^{-2\nu(r)}}{2}(r\nu'(r) - 1) < 0 \tag{2.22}$$

now the idea is to perturb this circular trajectory setting for example $E = E_0 + \epsilon$ so that the radial distance is no more constant and becomes to oscillate : $R_-(\epsilon) < r < R_+(\epsilon)$. The angular distance between $R_-(\epsilon)$ and $R_+(\epsilon)$ is given by (2.19):

$$\Phi(\epsilon) = \int_{R_-(\epsilon)}^{R_+(\epsilon)} \frac{P_\phi e^{\lambda(r)}}{r^2\sqrt{2E - \frac{L^2}{r^2} + e^{-2\nu(r)}}}dr. \tag{2.23}$$

The crucial point is that a closed orbit entails:

$$\Phi(\epsilon) = \frac{\pi}{\beta}, \quad \beta \in \mathbb{Q} \tag{2.24}$$

Because β can assume only rational values the function $\Phi(\epsilon)$ must be a constant because is a continuous function of ϵ. This request is very strong and entails many constraints to the form of $\lambda(r)$ and $\nu(r)$, so after a quite long series of calculations (for the details see [50]) it is possible to work out the so called "Bertrand Spacetimes"

as named by Perlick :
$\bullet TypeI$:

$$g = \frac{dr^2}{\beta^2(1+Kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dt^2}{\sqrt{\frac{1}{r^2}+K+G}}. \tag{2.25}$$

$\bullet TypeII$:

$$g = \frac{2(1-Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})}{\beta^2((1-Dr^2)^2 - Kr^4)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + \tag{2.26}$$

$$-\frac{dt^2}{G \mp r^2(1-Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})^{-1}}.$$

To these two metric spaces it is possible to link two Hamiltonian systems whose equations of motion define trajectories that are coincident with the timelike geodesic determined by the metrics (2.25) (2.26). Follow (2.18), hereafter we will refer to these Hamiltonians as Perlick type I and II:

$$\mathcal{H}_I = \frac{1}{2}\beta^2(1+Kr^2)P_r^2 + \frac{L^2}{2r^2} - \sqrt{\frac{1}{r^2}+K} + G \tag{2.27}$$

$$\mathcal{H}_{II} = \frac{1}{2}\frac{\beta^2((1-Dr^2)^2 - Kr^4)}{2(1-Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})}P_r^2 + \frac{L^2}{2r^2} + \tag{2.28}$$

$$-G \pm r^2(1-Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})^{-1}$$

where:

$$L^2 = P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta} \tag{2.29}$$

Exact solvability of Bertrand spacetimes

In analogy with the shape invariant systems, the Perlick's Hamiltonians (2.27) (2.28) show the possibility to be described in terms of known trascendental functions, in other words they are exactly solvable [51].
Let us compute the orbit equation for the general cases:

$$\mathcal{H} = \frac{f(r)}{2}P_r^2 + \frac{P_\phi^2}{2r^2} + V(r) \tag{2.30}$$

$$P_r = \sqrt{\frac{2}{f(r)}\left(E - \frac{P_\phi^2}{2r^2} - V(r)\right)} \tag{2.31}$$

$$\begin{matrix} \dot{r} = f(r)P_r \\ \dot{\phi} = \frac{P_\phi}{r^2} \end{matrix} \rightarrow d\phi = \frac{P_\phi dr}{r^2\sqrt{2f(r)\left(E - \frac{P_\phi^2}{2r^2} - V(r)\right)}} \tag{2.32}$$

now let us compute explicitly the orbit equation for the two Hamiltonians (2.27) (2.28)

Perlick Hamiltonian I

$$\beta d\phi = \frac{P_\phi dr}{r^2\sqrt{2(1+Kr^2)\left(E - \frac{P_\phi^2}{2r^2} + \sqrt{\frac{1}{r^2}+K} - G\right)}} \tag{2.33}$$

(2.33) simplifies dramatically with the change of variables:

$$u = \sqrt{\frac{1}{r^2}+K} \tag{2.34}$$

the orbit equation is then given by:

$$\beta d\phi = \frac{-P_\phi du}{\sqrt{2\left(E - \frac{P_\phi^2 u^2}{2} + \frac{KP_\phi^2}{2} + u - G\right)}} \tag{2.35}$$

which can be readily integrated to yield:

$$\sin(\beta\phi - \phi_0) = \frac{1 - P_\phi^2\sqrt{\frac{1}{r^2}+K}}{\sqrt{1+2P_\phi^2(E-G)+KP_\phi^4}} \tag{2.36}$$

Perlick Hamiltonian II

In this second case the treatment is analogous. Now the orbit equation reads:

$$\beta d\phi = \frac{P_\phi dr}{r^2\sqrt{\frac{(1-Dr^2)^2-Kr^4}{1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}}\left(E+G-\frac{P_\phi^2}{2r^2}\mp\frac{r^2}{1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}}\right)}} \tag{2.37}$$

and it is convenient to introduce the variable:

$$v = \frac{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}{r^2} \tag{2.38}$$

$$\beta d\phi = \frac{-P_\phi dv}{\sqrt{4v(E+G) - P_\phi^2(v^2+2Dv+K) \mp 4}} \tag{2.39}$$

the integration leads to:

$$\cos(\beta\phi - \phi_0) = \frac{P_\phi^2(v+D) - 2G - 2E}{\sqrt{(2E+2G-DP_\phi^2)^2 \mp 4P_\phi^2 - KP_\phi^4}} \tag{2.40}$$

2.5 Bertrand spaces as the most general spherical superintegrable spaces

In the previous section the Perlick Hamiltonians have been introduced; these Hamiltonians can be regarded also as describing particles moving on a Non-Euclidean tridimensional space subjected to a potential:

$$g_I = \frac{dr^2}{\beta^2(1+Kr^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2); \quad V_I(r) = -\sqrt{\frac{1}{r^2} + K} \tag{2.41}$$

$$g_{II} = \frac{2(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})}{\beta^2((1-Dr^2)^2 - Kr^4)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2); \tag{2.42}$$

$$V_{II}(r) = \pm \frac{r^2}{(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4})}$$

The scalar curvature of these systems turn out to be :

$$R(r)_I = -\frac{2}{r^2}(3\beta^2 Kr^2 + \beta^2 - 1) \tag{2.43}$$

$$R(r)_{II} = \frac{3}{r^2}\left(\frac{2}{3}(1-\beta^2) + \beta^2 \frac{(K - D^2)r^4 + 1}{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}\right) \tag{2.44}$$

Before going on let us analyze some special cases:

2.5.1 Flat spaces

$$\beta = 1; \quad K = 0 \rightarrow R(r)_I = 0 \tag{2.45}$$

we get the Euclidean case and the Hamiltonian (2.27) turns into the Kepler problem:

$$\mathcal{H} = P_r^2 + \frac{L^2}{r^2} - \frac{1}{r} \tag{2.46}$$

while for the second family :

$$\beta = 2; \quad K = 0; \quad D = 0 \rightarrow R(r)_{II} = 0 \tag{2.47}$$

the (2.28) turns into the harmonic oscillator Hamiltonian:

$$\mathcal{H} = P_r^2 + \frac{L^2}{r^2} + \frac{r^2}{2} \tag{2.48}$$

in perfect agreement with the Bertrand theorem. Moreover this means that the Perlick I / II spaces can be regarded as the only multiparametric deformations of the Kepler / Coulomb systems able to preserve the closed orbit property.

2.5.2 Constant curvature spaces

Let us set the parameters so that the scalar curvatur be constant ($R > 0$ spherical space, $R < 0$ pseudospherical case)
Type I:

$$\beta = 1 \rightarrow R_I(r) = -6K \tag{2.49}$$

the Hamiltonian (2.27) turns into:

$$\mathcal{H} = (1 + Kr^2)P_r^2 + \frac{L^2}{r^2} - \sqrt{\frac{1}{r^2} + K} \tag{2.50}$$

this is the well known Kepler problem on the sphere, and this is Maximally super-integrable as well [54] [55].
It is possible to get another M.S. system on the sphere from the space II:

$$\beta = 2; \quad K = 0 \rightarrow R_{II} = 6D \tag{2.51}$$

this choice of parameters transform the (2.28) in:

$$\mathcal{H} = (1 - Dr^2)P_r^2 + \frac{L^2}{r^2} + \frac{r^2}{2(1 - Dr^2)} \tag{2.52}$$

the kinetical part is obviously equal to the kinetical part of (2.50), but the potential part is clearly a deformation of on oscillator potential, in effect this problem has been classified as well in the M.S. systems on spaces of constant curvature [54] [55] [52].

2.5.3 Iway - Katayama spaces

As said in the introduction by the Bertrand approach Iway and Katayama found a class of metric spaces on which it is possible to obtain Hamiltonian systems characterized as well by the closed orbit property. These systems are a generalization of the MIC-Kepler and Taub-NUT systems [40, 48]. This class of spaces is known as "multifold Kepler systems", but we have shown that such family of systems is indeed a subcase of the Perlick spaces [59], the so called Iway-Katayama spaces, given by:

$$ds^2 = \tilde{r}^{\frac{1}{\nu}-2}(a + b\tilde{r}^{\frac{1}{\nu}})(d\tilde{r}^2 + \tilde{r}^2 d\Omega) \tag{2.53}$$

applying the substitution:

$$\tilde{r} = \left(\frac{-a \pm \sqrt{a^2 + 4br^2}}{2b} \right)^{\nu} \tag{2.54}$$

the metric space turns into:

$$ds^2 = 2\nu^2 \frac{a^2 + 2br^2 \pm a\sqrt{a^2 + 4br^2}}{a^2 + 4br^2} dr^2 + r^2 d\Omega^2 \tag{2.55}$$

This correspond to the Perlick space of type II, by setting:
$\beta^2 = \frac{1}{\nu^2}$; $K = D^2$; $D = -\frac{2b}{a^2}$

2.5.4 Darboux spaces

As remarked in the introduction of this section, the Darboux spaces represent the complete classification of all those spaces with a variable intrinsic scalar curvature which admit the maximum number of symmetries at most quadratic in the momenta. Kalnins, Kress, Miller and Winternitz (KKMW) have managed to find all those potentials for which a particle moving on a Darboux space is described by a M.S. Hamiltonian whose additional constants of motion are at most quadratic in the momenta [32] [33], this means that whenever in this classification is found an Hamiltonian system with a spherical symmetry than this system must be in the Perlick classification as well, since all the KKMW have the closed orbit property.
This occurence is verified in the cases of Darboux III and Darboux IV.

Darboux IV space

The Darboux IV space as reported by Koenigs turns out to be :

$$ds^2 = \frac{a(e^{\frac{x-y}{2}} + e^{\frac{y-x}{2}}) + b}{(e^{\frac{x-y}{2}} - e^{\frac{y-x}{2}})^2} dxdy \tag{2.56}$$

let us rewrite such a metric in a diagonal form:

$$\begin{aligned} x = u + iv \rightarrow dx = du + idv \\ y = -u + iv \rightarrow dy = -du + idv \end{aligned} \tag{2.57}$$

now the metric has the form:

$$ds^2 = -\frac{2a\cosh(u) + b}{4\sinh^2(u)}(du^2 + dv^2) \tag{2.58}$$

this metric shows a radial symmetry once that u is read as the radial variable and v as the angular variable. In order to recast this metric in a radial explicit form let us define:

$$r^2 = -\frac{2a\cosh(u) + b}{4\sinh^2(u)} \rightarrow u = \cosh^{-1}\left(\frac{-a \pm \sqrt{a^2 - 4br^2 + 16r^2}}{4r^2}\right) \tag{2.59}$$

$$ds^2 = \frac{2a^2 - 4br^2 \pm 2a\sqrt{a^2 - 4br^2 + 16r^4}}{a^2 - 4br^2 + 16r^4} dr^2 + r^2 dv^2 \tag{2.60}$$

let us set:

$$\begin{aligned} b = \tfrac{8D}{D^2 - K} \\ a = \tfrac{4}{\sqrt{D^2 - K}} \end{aligned} \tag{2.61}$$

in these new variables the Darboux IV metric turns out to be:

$$ds^2 = \frac{2(1 - Dr^2 \pm \sqrt{(1 - Dr^2)^2 - Kr^4})}{((1 - Dr^2)^2 - Kr^4)} dr^2 + r^2 dv^2 \tag{2.62}$$

this is the Perlick metric of type II with the parameter $\beta = 1$, anyway, it should be remarked that the substitution (2.61) is not defined whenever $D^2 = K$.

2.5.5 Darboux III space

Darboux III space, as well as Darboux IV space, is expected to be a subcase of the Perlick spaces, so let us follow the same strategy outlined for the Darboux IV:
Darboux III space turns out to be:

$$ds^2 = (ae^{-\frac{x+y}{2}} + be^{-x-y})dxdy \tag{2.63}$$

following the same algebraic transformation introduced in paragraph (2.5.4):

$$\begin{aligned} x = u + iv \rightarrow dx = du + idv \\ y = u - iv \rightarrow dy = du - idv \end{aligned} \tag{2.64}$$

$$ds^2 = (ae^{-u} + be^{-2u})(du^2 + dv^2) \tag{2.65}$$

$$r^2 = (ae^{-u} + be^{-2u}) \rightarrow u = -\log\left(\frac{-a \pm \sqrt{a^2 + 4br^2}}{2b}\right) \tag{2.66}$$

$$\rightarrow ds^2 = \frac{2(a^2 + 2br^2 \pm a\sqrt{a^2 + 4br^2})}{a^2 + 4br^2} dr^2 + r^2 dv^2 \tag{2.67}$$

if it is set $D = \frac{2b}{a^2}$:

$$ds^2 = \frac{2(1 + Dr^2 \pm \sqrt{1 + 2Dr^2})}{1 + 2Dr^2} dr^2 + r^2 dv^2. \tag{2.68}$$

The metric (2.68) turns out to be exactly the Perlick space of type II when $\beta = 1, K = D^2$, namely the particular case not admissible in the Darboux IV parametrization. This means that DIII and DIV together represent the whole Perlick family of Type II with $\beta = 1$.

2.5.6 Beyond Darboux systems

These results entail as a main consequence that, the Perlick spaces with $\beta \neq 1$ can be regarded also as the β deformation of all those radial symmetric spaces whose additional constants of motion are at most quadratic in the momenta. In the light of the above considerations let us focus on the role played by the parameter β.
Let us define:

$$ds^2 = \frac{dr^2}{\beta^2 g(r)} + r^2 d\phi^2, \tag{2.69}$$

be one of the Perlick spaces, now let us compute its associated Perlick Hamiltonian system for $\beta = 1$:

$$\mathcal{H} = g(r)P_r^2 + \frac{P_\phi}{r^2} + V_{I,II}(r), \tag{2.70}$$

by definition of Perlick Hamiltonian system it is known that any bounded state describes a periodic motion:

$$\begin{aligned} r(t+T) &= r(t) \\ \phi(t+T) &= \phi(t) + 2\pi, \end{aligned} \tag{2.71}$$

where T must be regarded as the period.
Now let us going on applying a rescaling on the angular variable:

$$\begin{aligned} P_\phi &= \beta P'_\phi \\ \phi &= \tfrac{\phi'}{\beta} \end{aligned} \tag{2.72}$$

after this canonical transformation the Hamiltonian turns into:

$$\mathcal{H} = g(r)P_r^2 + \beta^2 \frac{{P'_\phi}^2}{r^2} + V_{I,II}(r) \tag{2.73}$$

Up to a constant factor β^2, this Hamiltonian formally describes a motion on the metric (2.69). The crucial point is that in these new variables the relation (2.71) turns into:

$$\begin{aligned} r(t+T) &= r(t) \\ \phi'(t+T) = \beta\phi(t+T) = \beta(\phi(t) + 2\pi) &= \beta\phi(t) + 2\beta\pi = \phi'(t) + 2\beta\pi, \end{aligned} \tag{2.74}$$

this means that when β is a ratio of two coprime numbers $\beta = \frac{m}{n}$ after nT the orbit close after m laps around the origin, that's why it is needed a $\beta \in \mathbb{Q}$, we stress also, that such a request entails not only the closed orbit property but even the M.S. of the Perlick systems:

A generalized Runge-Lenz vector for the Perlick's Hamiltonians

Studying the reduced two-dimensional Perlick spaces, namely the orbital "plane", we have noticed that the Perlick spaces collect as particular cases the great part of M.S. systems known. It is interesting to note that it is possible to extend such a property to the entire Perlick family, as proved recently in [51], by defining the so called generalized Runge-Lenz vector, namely the analougue of the Runge Lenz vector responsible of the hidden symmetries in the Kepler problem. The idea of looking for generalizations of the Runge Lenz vector in order to find an additional integral of motion is not new: a rather complete review of the related literature can be found in [60].The rigorous demonstration is quite long, anyway in order to be selfconsistent, let us recall the main ideas that allow us to find this extra integral of

motion:
let us start by recalling the Fradkin's construction [61] of a local vector first integral for a central Hamiltonian system:
the Fradkin Runge-Lenz vector is defined as:

$$\mathbf{a} = \frac{\cos\phi}{r}\mathbf{q} + \frac{\sin\phi}{rP_\phi}\mathbf{q}\times(\mathbf{q}\times\mathbf{p}) \tag{2.75}$$

where $r = |\mathbf{q}|$, if we consider $\phi(0) = 0$, then it is trivial to show that the vector is a constant $\mathbf{a} = (1, 0, 0)$.
Fradkin's observation is that if $\cos\phi$, $\sin\phi$ can be expressed in terms of $\mathbf{q}$, $\mathbf{p}$ in a domain $\Omega \subset \mathbb{R}^3 \setminus \{0\}$, then the resulting vector field is a first integral of $\mathcal{H}$ in Ω. When $\mathcal{H}$ is the Kepler Hamiltonian, the generalized Runge-Lenz vector field essentially coincides with the classical Runge-Lenz vector divided by its norm. Let us start from the solution obtained for the Perlick systems (2.36) (2.40) with an appropriate ϕ_0:

$$\cos(\frac{m}{n}\phi) = \chi(r^2, P_\phi^2, E) \tag{2.76}$$

where $\chi(r^2, P_\phi^2, E)$ is defined following (2.36)(2.40):

$$\chi(r^2, P_\phi^2, E) \equiv \begin{cases} \dfrac{1-P_\phi^2\sqrt{\frac{1}{r^2}+K}}{\sqrt{1+2P_\phi^2(E-G)+KP_\phi^4}} & (Perlick\ \ I) \\ \dfrac{P_\phi^2(\frac{1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}}{r^2}+D)-2G-2E}{\sqrt{(2E+2G-DP_\phi^2)^2\mp4P_\phi^2-KP_\phi^4}} & (Perlick\ \ II) \end{cases} \tag{2.77}$$

Moreover the chain rule immediately yields:

$$\sin\frac{m\phi}{n} = -\frac{n}{m}\frac{d}{d\phi}\left(\cos\frac{m\phi}{n}\right) = -\frac{n}{m}\frac{\dot{r}}{\dot{\phi}}\partial_r\chi(r^2, P_\phi^2, E) = \Theta(r\dot{r}, r^2, P_\phi, E) \tag{2.78}$$

Using the properties of the Chebyshev polynomials it is trivial to express $\cos m\phi$ and $\sin m\phi$ in terms of $r, \dot{r}, P_\phi, E$ as:

$$\cos m\phi = T_n\left(\cos\frac{m\phi}{n}\right) = T_n\left(\chi(r^2, P_\phi^2, E)\right) \tag{2.79}$$

$$\sin m\phi = \sin\frac{m\phi}{n}U_{n-1}\left(\cos\frac{m\phi}{n}\right) = \Theta(r\dot{r}, r^2, P_\phi, E)U_{n-1}\left(\chi(r^2, P_\phi^2, E)\right) \tag{2.80}$$

Here T_m and U_m respectively stand for the Chebyschev polynomials of the first and second kind and degree n. Now we can define the function:

$$\varepsilon_m(r\dot{r}, r^2, P_\phi, E) = T_n\left(\chi(r^2, P_\phi^2, E)\right) + i\Theta(r\dot{r}, r^2, P_\phi, E)U_{n-1}\left(\chi(r^2, P_\phi^2, E)\right) \tag{2.81}$$

in terms of which the orbit is characterized as

$$e^{im\phi} = \varepsilon_m(r\dot{r}, r^2, P_\phi, E) \tag{2.82}$$

from which it is possible to get $e^{i\phi}$ in terms of the coordinates $(\mathbf{q}, \mathbf{p})$, furthermore it is obvious that the equation (2.82) only defines ϕ modulo $\frac{2\pi}{m}$ since the orbit has self intersection as pointed out in the previous subsection. This proves the existence of a generalized Runge Lenz Vector for the Perlick spaces.

2.6 Bertrand spacetimes as intrinsic Kepler / oscillator systems

As pointed out in the previous section, the two Perlick systems can be regarded as a multiparametric M.S. deformation of the Kepler or harmonic systems. Amazingly, this sort of classification of radial M.S. systems in Kepler or Harmonic oscillator potentials survives, in a rather tricky way, also in the non flat cases [59] [39]. To this end, let us introduce the concept of harmonic oscillator and Kepler potential in any spherically symmetric 3-manifold: be $\triangle_g$ the Laplace Beltrami operator defined by the metric (2.25) (2.26). To be more compact let us refer to the Perlick spaces as:

$$g = h(r)^2 dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.83}$$

or equivalently in a conformal frame:

$$g' = f(r)^2(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)), \tag{2.84}$$

it is standard that if $u(r)$ is a function on $\mathcal{M}$ that only depends on the radial coordinate, then its Laplacian is also radial and given by:

$$\triangle_g u(r) = \frac{1}{r^2 h(r)} \frac{d}{dr}\left(\frac{r^2}{h(r)} \frac{du}{dr}\right). \tag{2.85}$$

$$\triangle_{g'} u'(r) = \frac{1}{r^2 f^3(r)} \frac{d}{dr}\left(f(r) r^2 \frac{du}{dr}\right). \tag{2.86}$$

Then the symmetric Green function $u(r)$ is obtained as the solution of the equation $\triangle_g u(r) = 0$ on $\mathcal{M}/0$, namely:

$$u(r) = \int^r \frac{h(r')}{r'^2} dr' \tag{2.87}$$

$$u'(r) = \int^r \frac{1}{f(r) r'^2} dr' \tag{2.88}$$

As the Kepler potential in 3D Euclidean space is simply the radial Green function $u(r)$ of the Laplacian and the harmonic oscillator is its inverse square, it is natural to make the following

Definition 3. The Kepler and the harmonic oscillator potentials in $(\mathcal{M}, g)$ are respectively given by the radial functions

$$V_K(r) = A_1(\int_a^r r'^{-2}h(r')dr' + B_1), \qquad V_O(r) = A_2(\int_a^r r'^{-2}h(r')dr' + B_1)^{-2}, \quad (2.89)$$

Where $a, A_j, B_j (j = 1, 2)$ are constants. This definition is obviously valid in higher dimensions as well.

Theorem 1. *In a type I (resp. type II) Bertrand spacetime, V is the intrinsic Kepler (resp. harmonic oscillator) potential associated with g.*

This theorem can be verified easily by direct computation :
1 $TypeI$:

$$h(r) = \frac{1}{\sqrt{1+Kr^2}} \rightarrow V_K(r) = \int \frac{1}{r^2\sqrt{1+Kr^2}}dr = -\sqrt{\frac{1}{r^2} + K} \quad (2.90)$$

2 $TypeII$:

$$h(r) = \frac{\sqrt{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}}{\sqrt{(1-Dr^2)^2 - Kr^4}} \rightarrow \quad (2.91)$$

$$V_K(r) = \int \frac{\sqrt{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}}{r^2\sqrt{(1-Dr^2)^2 - Kr^4}}dr = \frac{\sqrt{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}}{r}$$

$$V_K(r) \rightarrow V_O = \frac{r^2}{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}$$

2.7 Stackel Transformation

The Bertrand and Perlick analysis seem to show that the class of radial M.S. systems is naturally splitted in two different families. In the first chapter we pointed out the existence of a transformation, named "coupling constant metamorphosis", able to link different quantum exactly solvable systems. Such a transformation was originally defined for classical systems and is known also under the name of Stackel Transformation (S.T.) [23, 56, 57, 58]. The crucial point is that, the Stackel transformation is able to turn a M.S. Hamiltonian to a new M.S. Hamiltonian, then it is quite natural to suppose that this transformation could establish a link between the two M.S. families found by Bertrand and Perlick. Before going on let us introduce how the Stackel Transformation works:
Let us consider the conjugate coordinates and momenta $\mathbf{q}, \mathbf{p} \in \mathbb{R}^N$ with canonical Poisson Bracket $\{q_i, p_j\} = \delta_{i,j}$ and the notation:

$$\mathbf{q}^2 = \sum_{i=1}^{N} q_i^2, \qquad \mathbf{p}^2 = \sum_{i=1}^{N} p_i^2, \qquad |\mathbf{p}| = \sqrt{\mathbf{p}^2} \tag{2.92}$$

Let H be an "initial" Hamiltonian, H_U an "intermediate" one and $\tilde{H}$ the "final" system given by:

$$H = \frac{\mathbf{p}^2}{\mu(\mathbf{q})} + V(\mathbf{q}), \qquad H_U = \frac{\mathbf{p}^2}{\mu(\mathbf{q})} + U(\mathbf{q}), \qquad \tilde{H} = \frac{H}{U} = \frac{\mathbf{p}^2}{\tilde{\mu}(\mathbf{q})} + \tilde{V}(\mathbf{q}) \tag{2.93}$$

such that $\tilde{\mu} = \mu U$ and $\tilde{V} = \frac{V}{U}$. Then let us consider, each *second − order* integral of motion (symmetry) S of H_U. In particular, if S and S_U are written as

$$S = \sum_{i,j=1}^{N} a^{i,j}(\mathbf{q}) p_i p_j + W(\mathbf{q}) = S_0 + W(\mathbf{q}), \qquad S_U = S_0 + W_U(\mathbf{q}) \tag{2.94}$$

then one gets a second-order symmetry of $\tilde{H}$ in the form

$$\tilde{S} = S_0 - W_U \tilde{H}. \tag{2.95}$$

2.8 Intrinsic oscillator as Stackel intrinsic Kepler

In the previous section we have introduced a non trivial transformation which acts on two Hamiltonian systems generating a new Hamiltonian system with the same number of first integrals of the originating ones; because of this property the Stackel transformation can be regarded as an endomorphism on the space of M.S. Hamiltonian systems. Since the Perlick's classification establishes the existence of a closed set of radial M.S. systems, this transformation is expected to split the whole set of Bertrand spacetimes in two different sets of Stackel equivalent systems. As said above the Bertrand space time systems are naturally divided in two sets, respectively by the families of the intrinsic Kepler systems and intrinsic harmonic oscillator systems, therefore it turns out to be interesting to apply the S.T. to the two class of systems:

2.8.1 From the intrinsic Kepler to the intrinsic oscillator

Let us define the two Hamiltonian systems on a conformally flat metric:

$$H = \frac{\mathbf{P}^2}{f(\mathbf{q})^2} + \alpha \tag{2.96}$$

$$H_U = \frac{\mathbf{P}^2}{f(\mathbf{q})^2} + V_K(\mathbf{q}) + \beta, \tag{2.97}$$

by definition of intrinsic Kepler the potential V_K turns out to be linked to the conformal factor $f(\mathbf{q})^2$ by the relation:

$$V_K(r) = \int^r \frac{dr'}{f(r')r'^2}. \tag{2.98}$$

Then let us apply the S.T. to this system:

$$\tilde{H} = \frac{H}{U} = \frac{\mathbf{P}^2}{f(\mathbf{q})^2(V_K(\mathbf{q}) + \beta)} + \frac{\alpha}{V_K(\mathbf{q}) + \beta}, \tag{2.99}$$

this new system has to be M.S. as well, so in order to classify this system in one of the two Perlick's families let us compute the intrinsic Kepler potential relative to the new conformal factor $f(\mathbf{q})^2(V_K(\mathbf{q}) + \beta)$:

$$\tilde{V_K}(r) = \int^r \frac{dr'}{f(r')r'^2\sqrt{V_K(r) + \beta}} \tag{2.100}$$

This integral can be straightforwardly solved integrating by part using the relation (2.98) and it is founded :

$$\tilde{V_K}(r) = 2\sqrt{V_K(r) + \beta} \tag{2.101}$$

it is immediate to verify that the potential part of $\tilde{H}$ turns out to be the intrinsic harmonic oscillator potential :

$$\frac{\alpha}{V_K(\mathbf{q}) + \beta} \propto \frac{1}{\tilde{V_K}^2(\mathbf{q})} \tag{2.102}$$

so it is possible to state the following:

Theorem 2. *The S.T. of any intrinsic Kepler system belong to the family of the intrinsic harmonic oscillator systems.*

2.8.2 From intrinsic harmonic oscillator to ?

The idea is to go through the same path of the last subsection, but starting from an intrinsic oscillator:

$$H = \frac{\mathbf{P}^2}{f(\mathbf{q})^2} + \alpha \tag{2.103}$$

$$H_U = \frac{\mathbf{P}^2}{f(\mathbf{q})^2} + V_{HO}(\mathbf{q}) + \beta = \frac{\mathbf{P}^2}{f(\mathbf{q})^2} + \frac{1}{V_K(\mathbf{q})^2} + \beta, \tag{2.104}$$

such that:

$$V_K(r) = \int^r \frac{dr'}{f(r')r'^2} \tag{2.105}$$

performing the S.T. :

$$\tilde{H} = \frac{V_K(\mathbf{q})^2\mathbf{P}^2}{f(\mathbf{q})^2(1+\beta V_K(\mathbf{q})^2)} + \frac{\alpha V_K(\mathbf{q})^2}{(1+\beta V_K(\mathbf{q})^2)} \tag{2.106}$$

the intrinsic Kepler potential associated to this new metric is :

$$\tilde{V}_K(r) = \int^r \frac{V_K(r')dr'}{r'^2 f(r')\sqrt{1+\beta V_K(r')^2}} = \frac{\sqrt{1+\beta V_K(r)^2}}{\beta}, \qquad \beta \neq 0 \tag{2.107}$$

so when the parameter β is different from zero the S.T. of an intrinsic harmonic oscillator turns out to be an intrinsic harmonic oscillator as well.

$$\frac{\alpha V_K(\mathbf{q})^2}{(1+\beta V_K(\mathbf{q})^2)} = -\frac{\alpha}{\beta^3}\frac{1}{\tilde{V}_K^2(r)} + \frac{\alpha}{\beta} \tag{2.108}$$

The above equation is clearly not defined when the parameter $\beta = 0$ therefore in order to complete the picture we will analyze the Hamiltonian (2.106) with $\beta = 0$

$$\tilde{H} = \frac{V_K(\mathbf{q})^2\mathbf{P}^2}{f(\mathbf{q})^2} + \alpha V_K(\mathbf{q})^2 \tag{2.109}$$

in this case the intrinsic Kepler turns out to be :

$$\tilde{V}_K(r) = \int^r \frac{V_K(r')dr'}{r^2 f(r')} = \frac{V_K(r)^2}{2}, \tag{2.110}$$

namely the same potential of the Hamiltonian (2.109), therefore this system belongs to the intrinsic Kepler systems

2.9 The Bertrand spacetimes as Stackel equivalent systems

So far it has been stressed that the two Bertrand families can be regarded as Kepler type or harmonic oscillator type, then in the section (2.8.1) we have established that any intrinsic Kepler system can be turned into an intrinsic harmonic oscillator system. The aim of this section is to show explicitly that the Stackel transformation applied to the Kepler family of Bertrand spacetimes generates in a very natural way the second family of Bertrand spacetimes. Let us consider the Hamiltonian:

$$H = \beta^2(1+Kr^2)P_r^2 + \frac{1}{r^2}(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}) + \alpha \tag{2.111}$$

then we consider the Hamiltonian H_U

$$H_U = \beta^2(1+Kr^2)P_r^2 + \frac{1}{r^2}(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}) + \sqrt{\frac{1}{r^2}+K} + G \tag{2.112}$$

where the Hamiltonian H_U is the system built from the Bertrand space time I when the potential part $U = \sqrt{\frac{1}{r^2} + K + G}$ is switched on. The next step is applying the S.T. to these couple of Hamiltonians:

$$\tilde{H} = \frac{H}{U} = \beta^2 \frac{(1+Kr^2)}{\sqrt{\frac{1}{r^2}+K}+G} P_r^2 + \frac{1}{r^2(\sqrt{\frac{1}{r^2}+K}+G)}(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}) + \frac{\alpha}{\sqrt{\frac{1}{r^2}+K}+G} \tag{2.113}$$

In order to show that this Hamiltonian is exactly the Bertrand system of type II let us apply the following point canonical transformation:

$$\begin{cases} r'^2 = r^2(\sqrt{\frac{1}{r^2}+K}+G) \to r = \sqrt{\frac{-(1+2Gr'^2)\pm\sqrt{1+4Gr'^2+4Kr'^2}}{2(K-G^2)}} \\ P_r = \frac{1}{\frac{dr}{dr'}} P_{r'} \end{cases} \tag{2.114}$$

$$\tilde{H}(P_{r'}, r') = \tag{2.115}$$

$$\frac{\beta^2(1+4Gr'^2+4Kr'^4)}{2(1+2Gr'^2 \pm \sqrt{1+4Gr'^2 4Kr'^4})} P_{r'}^2 + \frac{1}{r'^2}(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}) +$$

$$+ \frac{2\alpha r'^2}{1+2Gr'^2 \pm \sqrt{1+4Gr'^2+4Kr'^2}}$$

defining the new parameters:

$$G = -\frac{D}{2}, \qquad k = \frac{D^2 - K'}{4} \tag{2.116}$$

$$\tilde{H}(P_{r'}, r') = \tag{2.117}$$

$$\frac{\beta^2((1-Dr'^2)^2 - K'r'^4)}{2(1-Dr'^2 \pm \sqrt{(1-Dr'^2)^2 - K'r'^4})} P_{r'}^2 + \frac{1}{r'^2}(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}) +$$

$$+ \frac{2\alpha r'^2}{1-Dr'^2 \pm \sqrt{(1-Dr'^2)^2 - K'r'^4}}$$

this proves that:

Theorem 3. *The second Bertrand family is the Stackel Transformation of the first family*

Finally, in order to complete our picture let us describe the general intrinsic harmonic oscillator Hamiltonian arranging the present metric in a conformal metric. To this goal it is much easier to compute a new S.T. on a conformal intrinsic Kepler system than to establish a direct change of variables to the Hamiltonian (2.117). Let us apply to the Hamiltonian (2.112) the following change of variables:

$$r = \frac{2}{r'^{-\beta} - Kr'^{\beta}} \tag{2.118}$$

The intrinsic Kepler conformal Hamiltonian turn out to be:

$$H = \frac{r^2(r^{-\beta} - Kr^{\beta})^2}{4}(\mathbf{P}^2) + \alpha \tag{2.119}$$

$$H_U = \frac{r^2(r^{-\beta} - Kr^{\beta})^2}{4}(\mathbf{P}^2) - \frac{r^{-\beta} + Kr^{\beta}}{2} + G \tag{2.120}$$

therefore the S.T. lead to:

$$\frac{H}{U} = -\frac{r^2(r^{-\beta} - Kr^{\beta})^2}{2(r^{-\beta} + Kr^{\beta} - 2G)}(P_r^2 + \frac{L^2}{r^2}) - \frac{2\alpha}{r^{-\beta} + Kr^{\beta} - 2G} \tag{2.121}$$

It is possible to verify the exact equivalence with the Hamiltonian (2.117) performing the following radial point canonical transformation:

$$r = \sqrt[\beta]{-\frac{1}{2Kr'^2} \pm \frac{A(r')}{2Kr'^2} - \frac{1}{\sqrt{2}}\sqrt{\frac{1}{K^2r'^4} + \frac{2G}{K^2r'^2} \mp \frac{4}{KA(r')} \mp \frac{1}{K^2r'^4A(r')} \mp \frac{4G}{K^2r'^2A(r')}}} \tag{2.122}$$

$$A(r') = \sqrt{1 + 4Gr'^2 + 4Kr'^4}$$

$$P_r = \frac{P_{r'}}{\frac{dr}{dr'}}$$

after this canonical transformation and considering the new parameters (2.116) :

$$H = \frac{\beta^2((1 - Dr'^2)^2 - K'r'^4)P_{r'}^2}{2(1 - Dr'^2 \mp \sqrt{(1 - Dr'^2)^2 - K'r'^4})} + \frac{(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta})}{r'^2} +$$

$$+ \frac{2\alpha r'^2}{1 - Dr'^2 \mp \sqrt{(1 - Dr'^2)^2 - K'r'^4}},$$

namely the Hamiltonian (2.117)

Chapter 3

Quantization recipes in literature

3.1 Quantization of classical Hamiltonian systems on non-Euclidean manifolds

In the second chapter we have introduced a family of classical M.S. Hamiltonian systems defined on non-Euclidean manifolds:

$$H = \frac{1}{2} g^{ij}(x) P_i P_j + V(\mathbf{x}) \tag{3.1}$$

If we look at the same problem for a quantum particle, then coordinates and momenta have to be read as noncommuting operators in the Heisenberg algebra:

$$[\hat{P}_j, \hat{P}_k] = 0, \quad \forall j, k$$

$$[\hat{x}_j, \hat{x_k}] = 0, \quad \forall j, k$$

$$[\hat{P}_j, \hat{x}_k] = -i\hbar\delta_{jk}; \quad \delta_{jk} = (1, j = k), (0, j \neq k)$$

In the Euclidean-manifold the metric tensor turns out to be independent of the operator $\hat{x}$ ($g^{ij}(\mathbf{x}) = \delta_{ij}$), therefore there are no problems in performing the substitution $P \to \hat{P}$, $x \to \hat{x}$, and the Hamiltonian becomes a sum of the kinetic energy and potential operator, each of which is composed by commuting operators. The situation changes dramatically as soon as the metric tensor depends on the operator $\hat{x}$: in this case the kinetic energy term turns out to be position dependent causing an obvious ordering ambiguity:

$$g(\hat{\mathbf{x}})^{ij} \hat{P}_i \hat{P}_j \neq \hat{P}_i g(\hat{\mathbf{x}})^{ij} \hat{P}_j \neq \hat{P}_i \hat{P}_j g(\hat{\mathbf{x}})^{ij}$$

therefore any permutation turns out to be different by terms proportional to $\hbar$:

$$\hat{P}_i g^{ij} \hat{P}_j = g^{ij} \hat{P}_i \hat{P}_j - i\hbar \frac{dg^{ij}}{dx^i} \hat{P}_j$$

$$\hat{P}_i \hat{P}_j g^{ij} = g^{ij} \hat{P}_i \hat{P}_j - 2i\hbar \frac{dg_{ij}}{dx^j} \hat{P}_i + \hbar^2 \frac{d^2 g^{ij}}{dx^i dx^j},$$

which disappear in the classical limit $\hbar \to 0$. This means that from an algebraic point of view there exist an infinite number of Hamiltonian operators with the same classical limit:

$$\hat{H}_{\alpha\beta\gamma} = (g^{ij})^\alpha \hat{P}_i (g^{ij})^\beta \hat{P}_j (g^{ij})^\gamma + V(x), \quad \alpha + \beta + \gamma = 1 \tag{3.2}$$

The goal of this chapter is to establish (under certain given criteria) the recipes to quantize a classical system with a position dependent Kinetic energy term.

3.2 Position dependent mass quantization

From a classical point of view the kinetic energy term:

$$\mathcal{T} = g(\mathbf{x})^{ij} P_i P_j$$

can be regarded at the same time as a particle with a constant mass bound to move on a non Euclidean manifold $ds^2 = g_{ij} dx^i dx^j$ or as a particle with a non-constant and anisotropical effective mass on a Euclidean space:

$$m_{ij} = g_{ij}.$$

Anyway, in the particular case of a metric tensor with radial symmetry it is possible to recover an isotropic mass recasting the metric into a conformal metric by performing a change of variable:

$$ds^2 = f(r)^2 dr^2 + r^2 d\Omega^2, \quad r = h(r') \to f(h(r'))^2 \left(\frac{dh}{dr'}\right)^2 dr'^2 + h(r')^2 d\Omega^2$$

therefore imposing:

$$\frac{f(h(r'))}{h(r')} \frac{dh}{dr'} = \frac{1}{r'}$$

we get a conformal metric:

$$ds^2 = \frac{h(r')^2}{r'^2} (dr'^2 + r'^2 d\Omega^2) = \frac{h(|\mathbf{x}|)^2}{(|\mathbf{x}|)^2} (d\mathbf{x}^2)$$

in this case the mass has to be read as:

$$m(\mathbf{x}) = \frac{h(|\mathbf{x}|)^2}{(|\mathbf{x}|)^2} \tag{3.3}$$

Quantum models with a position dependent mass are essential in many condensed matter problems (see for instance [62] - [71] and references therein). Although over the times several different quantization prescriptions for this kind of problems have been proposed, we report the most widely used, namely the symmetrical quantization. Following [72] it is possible to determine the functional form of the Hamiltonian in terms of the canonical operators (X, P) requiring the so called instantaneous Galilean invariance.

3.2.1 Instantaneous Galilean invariance

It is well known that a free-particle has a complete symmetry under the Galilean group, whilst for a particle subjected to a potential the symmetry turns out to be obviously broken; however we can recover a partial symmetry if we introduce the concept of instantaneous Galilean transformations:
for simplicity let us consider the one dimensional case. In classical mechanics, a Galilean transformation at the instant t_0, with velocity v, transforms the position x and the momentum p of a particle with mass m according to:

$$x'(t) = x(t) - v(t - t_0), \tag{3.4}$$

$$p'(t) = p(t) - mv \tag{3.5}$$

An instantaneous Galilean transformation is performed at the instant time $t_0 = t$, thus is defined by:

$$x'(t) = x(t), \tag{3.6}$$

$$p'(t) = p(t) - mv \tag{3.7}$$

In the same way we can define, for a quantum particle, a unitary transformation $U(v)$ implementing the Instantaneous Galilean transformation with velocity v and acting on the canonical pair of operators X and P according to:

$$U(v)XU(v)^{-1} = X, \tag{3.8}$$

$$U(v)PU(v)^{-1} = P - mvI \tag{3.9}$$

The Hamiltonian operator must be such that;

$$V = i[H, X] \tag{3.10}$$

$$U(v)VU(v)^{-1} = V - vI \tag{3.11}$$

By V we have denoted the velocity Operator. Since the transformation $U(v)$ is completely determined by the equations (3.8) (3.9), and since Vdepends on H through

(3.10) then the relation (3.11) determines a constraint for the Hamiltonian. Let us introduce the infinitesimal generator K of instantaneous Galilean transformation through:

$$U(v) = e^{ivK} \tag{3.12}$$

from the (3.8) and (3.9) we get the following relations for the operator K:

$$[K, X] = 0, \tag{3.13}$$

$$[K, P] = imI, \tag{3.14}$$

$$[K, V] = iI, \tag{3.15}$$

now considering the canonical commutation rule:

$$[X, P] = iI,$$

and (3.13) , (3.14) then we get:

$$K = mX$$

up to a trivial constant. From (3.14) and (3.15) we get:

$$[K, P - mV] = 0 \to P - mV = A(X). \tag{3.16}$$

Similarly one can compute straightforwardly the following commutators:

$$[K, H - \frac{mV^2}{2}] = m[X, H] - \frac{m}{2}[K, V^2] = imV - imV = 0 \to H - \frac{m}{2}V^2 = W(X) \tag{3.17}$$

Finally the Hamiltonian takes the form:

$$H = \frac{1}{2m}(P - A(X))^2 + W(X) \tag{3.18}$$

This is, except for a gauge transformation, the usual form of the Schroedinger operator.

3.2.2 Instantaneous Galilean transformations for Position dependent mass systems

Generalizing the construction of the Hamiltonian operator in the case of a position dependent mass is quite a simple matter: the instantaneous Galilean transformation does not modify the position and this makes the transformation quite indifferent to a possible position dependence of the mass. Let us modify the transformation rule for the momentum as follows:

$$U(v)PU(v)^{-1} = P - M(X)v \rightarrow [K, P] = iM(X) \tag{3.19}$$

the (3.13) says that $K = N(x)$; exploiting the canonical commutation rule we get:

$$[K, P] = [N(X), P] = iN'(X) = iM(X) \tag{3.20}$$

Consider now the constraining condition for H, that is, the (3.15). It reads according to (3.10):

$$[N(X), [H, X]] = I \tag{3.21}$$

applying the Jacobi identity it follows that:

$$[X, [H, N(X)]] = I \rightarrow [N(X), H] = i(P - A(X)) \tag{3.22}$$

now it is straightforward to verify that a simple solution for H (for an Hermitian Hamiltonian) is:

$$H = \frac{1}{2}P\frac{1}{M(x)}P \tag{3.23}$$

indeed,

$$[N(X), H] = \frac{1}{2}[N(X), P]\frac{1}{M(X)}P + \frac{1}{2}P\frac{1}{M(X)}[N(X), P] = iP \tag{3.24}$$

the general Hamiltonian thus reads,

$$H = \frac{1}{2}P\frac{1}{M(x)}P + W(x) \tag{3.25}$$

Instead of the (3.23) it is possible to have a different solution of the (3.22) can be also the rather natural one:

$$H_1 = \frac{1}{4}\left(P^2\frac{1}{M(X)} + \frac{1}{M(x)}P^2\right) \tag{3.26}$$

anyway comparing H_0 and H_1 we find:

$$H_1 - H_0 = \frac{1}{4}[P, [P, \frac{1}{M(x)}]] = -\frac{1}{4}(\frac{1}{M(X)})'' = Q(X) \tag{3.27}$$

therefore:

$$H = \frac{1}{2}P\frac{1}{M(x)}P + W(x) = \frac{1}{4}\left(P^2\frac{1}{M(X)} + \frac{1}{M(x)}P^2\right) + W_1(X) \tag{3.28}$$

with relationship:

$$W_1(X) = W(X) + Q(X)$$

These remarks prove that one should not identify a priori H_0 as the purely kinetic energy term of the Hamiltonian nor $W(X)$ as the potential. This result can be used to prove that the most general kinetic Hamiltonian (under the condition we have stated) is:

$$H_{kin} = \frac{1}{4}\left(M^\alpha P M^\beta P M^\gamma + M^\gamma P M^\beta P M^\alpha\right) = \tag{3.29}$$

$$= \frac{1}{2}P\frac{1}{M}P + \frac{1}{2}(\alpha+\gamma+\alpha\gamma)\frac{M'^2}{M^3} - \frac{1}{4}(\alpha+\gamma)\frac{M''}{M^2},$$
$$(\alpha+\beta+\gamma=-1)$$

3.3 Covariant quantization

In the previous section we have considered the quantization of the classical systems introduced in the second chapter considering them as position dependent mass systems. Let us come back to the original point of view of systems defined on non-Euclidean manifold. In a classical context the solution to an Hamiltonian system can be obtained by solving the Hamilton-Jacobi equation, namely a first order partial differential equation:

$$-\frac{\partial S(\mathbf{x},t)}{\partial t} = \frac{1}{2}g^{i,j}(\mathbf{x},t)\left(\frac{\partial S(\mathbf{x},t)}{\partial x_j}\right)\left(\frac{\partial S(\mathbf{x},t)}{\partial x_j}\right). \tag{3.30}$$

This equation involve just the derivative of a scalar function that can be defined simply comparing the value taken by $S(\mathbf{x},t)$ in neighboring points of the manifold, $S(\mathbf{x},t)$ and $S(\mathbf{x}+d\mathbf{x},t)$

$$S' = \lim_{dx\to 0}\frac{S(\mathbf{x}+d\mathbf{x},t) - S(\mathbf{x},t)}{d\mathbf{x}}$$

Therefore it is quite a simple matter to generalize the Hamilton-Jacobi equation from an Euclidean manifold to a non-Euclidean one. The situation changes drastically if we repeat the same derivative operation with a vector field. Let us study for example the derivative of $A^\mu(\mathbf{x})$:

$$\lim_{dx\to 0}\frac{A^\mu(\mathbf{x}+d\mathbf{x},t) - A^\mu(\mathbf{x},t)}{d\mathbf{x}}$$

This operation, as stressed in the second chapter, is no more defined in a general Riemannian manifold, since the operation among vectors are defined just in the tangent vector space, therefore the sum of vectors:

$$A^\mu(\mathbf{x}+d\mathbf{x},t) - A^\mu(\mathbf{x},t)$$

turns out to be meaningless since the two vectors live in the two different spaces $T_{\mathbf{x}}$ and $T_{\mathbf{x}+d\mathbf{x}}$.

3.3.1 Differential operators in differentiable manifolds

covariant derivative

The first step, in order to compare two vectors in two different vector spaces, is to introduce the notion of parallel transport: let us consider a curve $\mathcal{C}$ in the manifold $\mathcal{M}$ parametrized by the parameter $\lambda : P \in \mathcal{C} \rightarrow P^i = x^i(\lambda)$, let the tangent vector to $\mathcal{C}$ be $\overline{U} = \frac{d}{d\lambda}$ following the definition (2.5) where $\overline{U}$ can be expressed also as a linear combination of vector basis $\overline{U} = U_i \overline{e}^i ; \overline{e}^i = \frac{d}{dx^i}$. Now let us consider a vector $\overline{V}(\lambda_0 + \epsilon)$ in the tangent vector space $T_{\mathbf{x}(\lambda_0+\epsilon)}$, then we define the parallel transported vector $\overline{V}^*_{\lambda_0+\epsilon}(\lambda_0)$ with respect to the curve $\mathcal{C}$ from the tangent vector space $T_{\mathbf{x}(\lambda_0+\epsilon)}$ to the tangent vector space $T_{\mathbf{x}(\lambda_0)}$ such that the following proportion yields:

$$V^*_{\lambda_0+\epsilon}(\lambda_0)_i : U_i(\lambda_0) = V(\lambda_0 + \epsilon)_i : U_i(\lambda_0 + \epsilon).$$

Thanks to the parallel transport it is now possible to compare two vectors field defined in different vector spaces evaluating them in the same space. We can now define the notion of covariant derivative for a vector field $\overline{V}$ defined everywhere on $\mathcal{C}$ as:

$$\nabla_{\overline{U}} \overline{V}(\lambda_0) = \lim_{\epsilon \rightarrow 0} \frac{\overline{V}^*_{\lambda_0+\epsilon}(\lambda_0) - \overline{V}(\lambda_0)}{\epsilon}. \tag{3.31}$$

This definition can be given also in terms of components: let us define $\mathbf{x} = \mathbf{x}(\lambda_0)$ and $\mathbf{x} + d\mathbf{x} = \mathbf{x}(\lambda_0 + \epsilon)$ so that a vector $A^\mu(\mathbf{x} + d\mathbf{x})$ parallel transported back to $\mathbf{x}$ will be:

$$A^\mu_{\lambda_0+\epsilon}(\mathbf{x}) = A^\mu(\mathbf{x} + d\mathbf{x}) + \delta A^\mu(\mathbf{x} + d\mathbf{x})$$

$\delta A^\mu(\mathbf{x}+d\mathbf{x})$ represents how much the manifold is different from a flat manifold, and moreover it has to be zero whenever $A^\mu(\mathbf{x} + d\mathbf{x})$ or $d\mathbf{x}$ are zero, namely it has to be bilinear both in $A^\mu(\mathbf{x} + d\mathbf{x})$ and $d\mathbf{x}$, therefore at the first order we have:

$$\delta A^\mu(\mathbf{x} + d\mathbf{x}) = \Gamma^\mu_{\nu\rho}(\mathbf{x}) A^\nu(\mathbf{x}) d\mathbf{x}^\rho$$

where it is used the Einstein summation convention. The functions $\Gamma^\mu_{\nu\rho}(\mathbf{x})$ are called Christoffel symbols and they represent the rules of the parallel transport for the manifold. Therefore the vector $A^\mu(\mathbf{x} + d\mathbf{x})$ parallel transported to $\mathbf{x}$ will be:

$$A^\mu(\mathbf{x}+d\mathbf{x}) \rightarrow A^\mu(\mathbf{x}+d\mathbf{x}) + \Gamma^\mu_{\nu\rho}(\mathbf{x}) A^\nu(\mathbf{x}) d\mathbf{x}^\rho = A^\mu(\mathbf{x}) + dA^\mu(\mathbf{x}) + \Gamma^\mu_{\nu\rho}(\mathbf{x}) A^\nu(\mathbf{x}) d\mathbf{x}^\rho + O(d\mathbf{x}^2)$$

so finally we can define the covariant derivative along a component, say, x^σ as :

$$A^\mu{}_{;\sigma} = \lim_{dx^\sigma \rightarrow 0} \frac{A^\mu + dA^\mu + \Gamma^\mu_{\nu\rho}(\mathbf{x}) A^\nu d\mathbf{x}^\rho - A^\mu}{dx^\sigma} = A^\mu_\sigma + \Gamma^\mu_{\nu\sigma} A^\nu \tag{3.32}$$

where:

$$A^{\mu}_{\sigma} = \frac{\partial A^{\mu}}{\partial x^{\sigma}}.$$

In other words the Christoffel symbols can be defined also through the action of the covariant derivative on the basis vectors:

$$\nabla_{\overline{e}_i} \overline{e}_k = \Gamma^{j}_{ki} \overline{e}_j. \tag{3.33}$$

Before closing this brief recall on the covariant derivatives let us consider how this applies on the so called dual space, namely the space of the real, linear functionals $\tilde{\omega}$:

$$(\tilde{\omega}, \alpha_1 \overline{V}_1 + \alpha_2 \overline{V}_2) = \alpha_1(\tilde{\omega}, \overline{V}_1) + \alpha_2(\tilde{\omega}, \overline{V}_2), \quad (\tilde{\omega}, \overline{V}_1), (\tilde{\omega}, \overline{V}_2) \in \mathbb{R}$$

in particular let us consider the function $\tilde{\omega}^l(\overline{V}) = V^l$ where $\overline{V} = V^i \overline{e}_i$ then we see that :

$$(\tilde{\omega}^j, \overline{e}_k) = \delta^j_k \rightarrow \nabla_{\overline{e}_i}(\tilde{\omega}^j, \overline{e}_k) = 0 \rightarrow (\nabla_{\overline{e}_i}\tilde{\omega}^j, \overline{e}_k) = -(\tilde{\omega}^j, \nabla_{\overline{e}_i}\overline{e}_k)$$

and by applying the (3.33) we find:

$$(\nabla_{\overline{e}_i}\tilde{\omega}^j, \overline{e}_k) = -\Gamma^j_{ki} \rightarrow \nabla_{\overline{e}_i}\tilde{\omega}^j = -\Gamma^j_{ki}\tilde{\omega}^k.$$

So hereafter we will regard differently the objects A^{μ} and A_{μ} respectively as components of vectors and dual vectors, in particular the definition of the covariant derivative change from (3.32) to:

$$A_{\mu};\sigma = A_{\mu},\sigma - \Gamma^{\rho}_{\sigma\mu} A_{\rho} \tag{3.34}$$

Christoffel symbols and metric tensor

In the previous section we have introduced the notion of Christoffel symbols, namely the connection coefficients, in order to define the parallel transport of vectors in different tangent spaces. Since our physical systems are defined on a Riemannian-manifold, it is fundamental to establish a rule for the parallel transport once given a metric on the manifold. This is possible requiring the so called compatibility condition between the metric and the parallel transport. In other words, from a physical point of view, this mathematical constraint means that an experimentalist in "free fall" along a curve in the manifold, if parallel transported, cannot be aware of his motion observing the metric in its neghborhood, (namely the "equivalence principle"). Let us introduce a metric in each tangent space defining a scalar product: $g(\overline{u}, \overline{v}) = g_{i,j} u^i v^j$ for each of these spaces, so by definition of parallel transport the scalar product between two constant vectors $\overline{A}, \overline{B}$ has to be invariant respect to this operation:

$$\nabla_{\overline{U}} g(\overline{A}, \overline{B}) = (\nabla_{\overline{U}} g)(\overline{A}, \overline{B}) + g(\nabla_{\overline{U}}\overline{A}, \overline{B}) + g(\overline{A}, \nabla_{\overline{U}}\overline{B}) = 0 \tag{3.35}$$

By ipothesis $\nabla_{\overline{U}}\overline{A} = 0$ and $\nabla_{\overline{U}}\overline{B} = 0$, so the compatibility condition turns out to be $\nabla_{\overline{U}} g = 0$, then applying the (3.34) the compatibility condition turns into:

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\rho}_{\lambda\mu} g_{\rho\nu} - \Gamma^{\rho}_{\lambda\nu} g_{\rho\mu} \equiv g_{\mu\nu,\lambda} - \Gamma_{\nu,\lambda\mu} - \Gamma_{\mu,\lambda\nu} \tag{3.36}$$

the (3.36) gives the constraint that the connection has to respect. This condition is satisfied by the so called Levi-Civita connection.
Let us consider a manifold with no torsion, namely $\nabla_{\overline{e}_i}\overline{e}_k = \nabla_{\overline{e}_k}\overline{e}_i$, this means that the connection is symmetric for the exchange of the indexes $\Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\lambda\nu}$. Now let us rewrite the (3.36) permuting the indexes:

$$g_{\mu\nu,\lambda} = \Gamma_{\nu,\lambda\mu} + \Gamma_{\mu,\lambda\nu}$$

$$g_{\lambda\mu,\nu} = \Gamma_{\mu,\nu\lambda} + \Gamma_{\lambda,\nu\mu}$$

$$g_{\nu\lambda,\mu} = \Gamma_{\lambda,\mu\nu} + \Gamma_{\nu,\mu\lambda}$$

summing the first and the second and subtracting the third it is straightforward to get:

$$\Gamma_{\mu,\lambda\nu} = \frac{1}{2}(g_{\mu\nu,\lambda} + g_{\lambda\mu,\nu} - g_{\lambda\nu,\mu}) \rightarrow \Gamma^{\mu}_{\lambda\nu} = \frac{1}{2} g^{\mu\rho}(\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\lambda\rho} - \partial_\rho g_{\lambda\nu}) \tag{3.37}$$

3.3.2 Covariant differential operators

The generalization of the derivative operation to the non-Euclidean spaces is the first step to generalizing our differential equations to this new spaces. Before going on let us now introduce the generalization of the most frequently used differential operators in such a new context.

• **Divergence operator**

As said previously, in order to generalize an Euclidean manifold to a non-Euclidean one we have to replace the standard derivative with its covariant version so that the divergence of a vector field can be generalized as follows:

$$\nabla \cdot \mathbf{A} = A^{\mu};_{\mu} = \partial_\mu A^\mu + \Gamma^{\mu}_{\mu\lambda} A^\lambda \tag{3.38}$$

following the relation (3.37) we can recast the term $\Gamma^{\mu}_{\mu\lambda}$:

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda}) \tag{3.39}$$

Since $g^{\mu\rho}\partial_\rho g_{\mu\lambda} = g^{\rho\mu}\partial_\mu g_{\rho\lambda} = g^{\mu\rho}\partial_\mu g_{\rho\lambda}$ the equation (3.39) can be simplified in:

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho}\partial_\lambda g_{\rho\mu} \tag{3.40}$$

this relation can be expressed in terms of the determinant of the metric $g = det g_{ij}$: indeed, let us consider the derivative of the determinant g:

$$\partial_\lambda g = \partial_\lambda g_{\rho\mu} a^{\rho\mu}$$

where $a^{\rho\mu}$ is the matrix of the cofactors and this is linked with the inverse matrix $g^{\mu\rho}$ through the relation $g^{\rho\mu} = \frac{a^{\rho\mu}}{g}$ where we have considered the property $a^{\rho\mu} = a^{\mu\rho}$. Therefore $g^{\mu\rho}\partial_\lambda g_{\rho\mu} = \frac{\partial_\lambda g}{g}$:

$$\Gamma^{\mu}_{\mu\lambda} = \frac{\partial_\lambda g}{2g}$$

therefore the equation (3.38) turns into:

$$A^{\mu};_{\mu} = \partial_\mu A^\mu + \frac{\partial_\mu g}{2g} A^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} A^\mu) \tag{3.41}$$

• **Laplacian operator**

once given the generalization of the divergence operator it is now straightforward to get the generalization of the Laplacian operator ∇^2 to non-Euclidean manifold :

$$\nabla^2 \psi = \psi^{;\mu};_{\mu} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \psi^{;\mu}) \tag{3.42}$$

Since ψ is a scalar function its covariant derivative is just a standard one so:

$$\psi;_{\mu} = \psi,_{\mu} \rightarrow \psi^{\ \mu} = g^{\mu\nu} \psi,_{\nu}$$

$$\nabla^2 \psi \rightarrow \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \psi) \tag{3.43}$$

The generalization of the Laplacian operator to non-Euclidean manifold is also known as Laplace-Beltrami operator.

3.3.3 Riemann Christoffel tensor

Before closing this section let us introduce a fundamental object in the study of a differentiable manifold, namely the Riemann-Cristoffel tensor: this tensor characterizes the manifold in each point measuring how much the operation of the parallel transport be path dependent: let us consider an infinitesimal polygon of vertices $x, x + dx, x + dx', x + dx + dx'$ and let us parallel transport a vector, say $\overline{V}$, along two different paths: $x \rightarrow x + dx \rightarrow x + dx + dx'$ and $x \rightarrow x + dx' \rightarrow x + dx' + dx$. The components of the vector V^i in $x + dx$ are :

$$V'^i = V^i - \Gamma^i_{jk}(x) V^j dx^k$$

then in order to transport the vector V' from $x + dx$ to $x + dx + dx'$ we need to know the connection in $x + dx$ namely $\Gamma^i_{jk}(x + dx) = \Gamma^i_{jk}(x) + \partial_m \Gamma^i_{jk} dx^m$ thus the vector $(V'')^i$ in $x + dx + dx'$ will be:

$$(V'')^i = V^i - \Gamma^i_{jk} V^j dx^k - (\Gamma^i_{jk} + \partial_m \Gamma^i_{jk} dx^m)(V^j - \Gamma^j_{lr} V^l dx^r) dx'^k$$

repeating the same operations along the other path $x \to x + dx' \to x + dx' + dx$ we get:

$$(V''')^i = V^i - \Gamma^i_{jk} V^j dx'^k - (\Gamma^i_{jk} + \partial_m \Gamma^i_{jk} dx'^m)(V^j - \Gamma^j_{lr} V^l dx'^r) dx^k$$

let us compare the two vectors:

$$(V''')^i - (V'')^i = (\Gamma^i_{jk}\Gamma^j_{lr} - \Gamma^i_{jr}\Gamma^j_{lk} + \partial_k \Gamma^i_{lr} - \partial_r \Gamma^i_{lk}) V^l dx^k dx'^r = R^i_{klr} V^l dx^k dx'^r$$

R^i_{klr} is the Riemann Christoffel tensor. The Riemann Christoffel tensor, in contrast with the connection Γ^i_{jk}, turns out to be a tensor under a general change of variables like the metric tensor itself:

$$ds^2 = g(x)_{ij} dx^i dx^j = g(x)_{ij} \frac{dx^i}{dx'^r} \frac{dx^j}{dx'^s} dx'^r dx'^s \to g(x')_{rs} = g(x)_{ij} \frac{dx^i}{dx'^r} \frac{dx^j}{dx'^s}$$

so in general the following rule holds for our tensors:

$$A_i(x') = A_l(x) \frac{dx^l}{dx'^i}; \quad A^i(x') = A^l(x) \frac{dx'^i}{dx^l} \tag{3.44}$$

this property turns out to be crucial because it allows us to get quantities which are coordinate independent:

$$V(x')^\mu A(x')_\mu = V(x)^\rho \frac{dx'^\mu}{dx^\rho} A(x)_\sigma \frac{dx^\sigma}{dx'^\mu} = V(x)^\rho A(x)_\sigma \frac{dx^\sigma}{dx^\rho} = V(x)^\rho A(x)_\rho$$

Now let us consider the Riemann Chistoffel tensor with only controvariant indexes $R_{\lambda\mu\nu\rho} = g_{\lambda\sigma} R^\sigma_{\mu\nu\rho}$. Considering also the Levi Civita connection we get:

$$R_{\lambda\mu\nu\rho} = \frac{1}{2}(\partial_\rho \partial_\mu g_{\lambda\nu} + \partial_\nu \partial_\lambda g_{\mu\rho} - \partial_\rho \partial_\lambda g_{\mu\nu} - \partial_\nu \partial_\mu g_{\lambda\rho}) + g_{\eta\sigma}(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\rho} - \Gamma^\eta_{\rho\lambda}\Gamma^\sigma_{\mu\nu})$$

examining this explicit expression we see that this tensor turns out to be symmetric for the exchange of the pairs (λ, ν) and (ν, ρ):

$$R_{\lambda\mu\nu\rho} = R_{\nu\rho\lambda\mu}$$

and antisymmetric for the exchange of the indexes $\lambda\mu$ and also for $\nu\rho$

$$R_{\lambda\mu\nu\rho} = -R_{\mu\lambda\nu\rho} = -R_{\lambda\mu\nu\rho}$$

because of this properties and the symmetry of the metric tensor $g^{ij} = g^{ji}$ the only way to get a contraction different from zero is to do the contraction with respect to $\lambda\nu$ or $\mu\rho$:

$$R_{\mu\rho} = g^{\lambda\nu} R_{\lambda\mu\nu\rho} \tag{3.45}$$

and this tensor is unique. In fact, if we contract the other two indexes:

$$R_{\lambda\nu} = g^{\mu\rho} R_{\lambda\mu\nu\rho} = -g^{\mu\rho} R_{\mu\lambda\nu\rho} = g^{\mu\rho} R_{\mu\lambda\rho\nu}$$

The tensor $R_{\mu\rho}$ is called Ricci tensor.
Finally we can get a scalar function by contracting the Ricci tensor with the metric tensor:

$$R = g^{\lambda\nu} R_{\lambda\nu} \tag{3.46}$$

This function is called scalar curvature and by construction turns out to be an intrinsic property of the space indipendent on the coordinate system chosen, for instance if we have $R = const$ this will be $const$ in any coordinate system.

3.3.4 Laplace Beltrami quantization

We are now ready to generalize the Schroedinger equation from an Euclidean to a non-Euclidean manifold, simply replacing the ordinary derivative with the covariant derivative:

$$\hat{H} = -\frac{\hbar^2}{2}\nabla^2 + V(\mathbf{x}) \rightarrow -\frac{\hbar^2}{2\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) + V(x) \tag{3.47}$$

Namely we identify the quantum kinetic energy operator as the Laplace Beltrami operator, from hereafter we will refer to this quantization as the Laplace Beltrami quantization (LB) (see, for instance [33, 34]).

3.4 Quantization for conformally flat metric

In the section (3.2) we have pointed out that the Bertrand spaces, because of their radial symmetry, can be recasted as conformally flat systems. Aim of this section is to do a deep analysis of the different quantization prescriptions (PDM and L.B.) for systems whose classical Hamiltonian can be written as follows:

$$H(\mathbf{x}, \mathbf{P}) = \frac{1}{2m(|\mathbf{x}|)}|\mathbf{P}|^2 + V(\mathbf{x}), \quad |\mathbf{x}| = \sqrt{\sum_i x_i^2}; \quad |\mathbf{P}| = \sqrt{\sum_i P_i^2}; \quad i = 0, ...N \tag{3.48}$$

As said above this system can be regarded as describing a particle moving on a Riemannian manifold equipped with a metric:

$$ds^2 = g_{ij}dx^i dx^j = m(|\mathbf{x}|)\delta_{ij}dx^i dx^j; \quad i, j = 1, ...N, \quad det(g_{ij}) = g = m(|\mathbf{x}|)^N. \tag{3.49}$$

Let us compute the Laplace Beltrami quantization of this system:

$$\hat{H} = -\frac{\hbar^2}{2m(|\mathbf{x}|)^{\frac{N}{2}}}\partial_i(m(|\mathbf{x}|)^{\frac{N-2}{2}}\partial_i) + V(\mathbf{x}) \tag{3.50}$$

Let us stress that when we deal with $N = 2$ dimensional systems the quantization reduces to the very particular case:

$$\hat{H} = -\frac{\hbar^2}{m(|\mathbf{x}|)}\nabla^2 + V(\mathbf{x}) \tag{3.51}$$

where ∇^2 has to be read as the standard Laplacian operator. In a classical context an Hamiltonian system in a N-dimensional space with a radial symmetry describes always a motion on a two dimensional space (orbital plane), and can be understood also as a consequence of the quasi M.S. of any radial system. The equation (3.50) shows that in quantum mechanics the radial Hamiltonian depends in a non trivial way on the dimension of the system. Since the Darboux classification has been performed explicitly for two dimensional spaces let us define the N-dimensional generalization of the Hamiltonian (3.51):

$$\hat{H}_{sch} = -\frac{\hbar^2}{m(|\mathbf{x}|)}\nabla^2_N + V(x) = -\frac{\hbar^2}{m(|\mathbf{x}|)}\partial^i\partial_i + V(x), \quad i = 1, ...N. \tag{3.52}$$

Hereafter we will refer to this quantization as the Schroedinger or direct quantization.

3.4.1 Differences among Schroedinger, Laplace Beltrami and PDM quantizations

In the previous sections we have proposed three different physical quantizations for a classical system with a position dependent energy kinetic term:

- Schroedinger quantization:

$$\mathcal{T}_{sch} = -\frac{\hbar^2}{m(|\mathbf{x}|)}\partial^i\partial_i = -\frac{1}{m(|\mathbf{x}|)}\mathbf{P}^2 \tag{3.53}$$

- Laplace Beltrami quantization:

$$\mathcal{T}_{LB} = -\frac{\hbar^2}{m(|\mathbf{x}|)^{\frac{N}{2}}}\partial^i(m(|\mathbf{x}|)^{\frac{N-2}{2}}\partial_i) = -\frac{\hbar^2}{m(|\mathbf{x}|)}\partial^i\partial_i - \frac{\hbar^2(N-2)}{2}\frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_i}{|\mathbf{x}|}\partial_i = \tag{3.54}$$

$$= -\frac{1}{m(|\mathbf{x}|)}\mathbf{P}^2 - \frac{i\hbar(N-2)}{2}\frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_iP_i}{|\mathbf{x}|}$$

- Position Dependent Mass quantization:

$$\mathcal{T}_{PDM} = -\hbar^2\partial^i \frac{1}{m(|\mathbf{x}|)}\partial_i = -\frac{\hbar^2}{m(|\mathbf{x}|)}\partial^i\partial_i + \hbar^2 \frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_i}{|\mathbf{x}|}\partial_i = \tag{3.55}$$

$$= \frac{\mathbf{P}^2}{m(|\mathbf{x}|)} + i\hbar \frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_i P_i}{|\mathbf{x}|}$$

These three operators are formally self-adjoints on the standard L^2 space with the following three scalar products:

- Schroedinger type:

$$< \psi|\phi >= \int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch}\phi_{sch} d\mathbf{x} \tag{3.56}$$

- Laplace Beltrami type:

$$< \psi|\phi >= \int_{\mathcal{M}^N} m(\mathbf{x})^{\frac{N}{2}}\overline{\psi}_{LB}\phi_{LB} d\mathbf{x} \tag{3.57}$$

- Position Dependent mass type:

$$< \psi|\phi >= \int_{\mathcal{M}^N} \overline{\psi}_{PDM}\phi_{PDM} d\mathbf{x} \tag{3.58}$$

Since these products differ just for a weight function, let us stress that given a basis for one of the above products, we can get a basis for the others spaces by applying a simple algebraic transformation: Let us suppose to have a basis, say, for the Schroedinger type:

$$\int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch\,i}\psi_{sch\,j} d\mathbf{x} = \delta_{ij} \tag{3.59}$$

then follows:

$$\int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch\,i}\psi_{sch\,j} d\mathbf{x} = \int_{\mathcal{M}^N} (\sqrt{m(\mathbf{x})}\overline{\psi}_{sch\,i})(\sqrt{m(\mathbf{x})}\psi_{sch\,j}) d\mathbf{x} = \tag{3.60}$$

$$= \int_{\mathcal{M}^N} \overline{\psi}_{PDM\,i}\psi_{PDM\,j} d\mathbf{x} = \delta_{ij}, \quad \psi_{PDM\,i} = \sqrt{m(\mathbf{x})}\psi_{sch\,i}$$

or alternatively:

$$\int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch\,i}\psi_{sch\,j} d\mathbf{x} = \int_{\mathcal{M}^N} m(\mathbf{x})^{\frac{N}{2}} (\frac{m(\mathbf{x})^{\frac{1}{2}}}{m(\mathbf{x})^{\frac{N}{4}}}\overline{\psi}_{sch\,i})(\frac{m(\mathbf{x})^{\frac{1}{2}}}{m(\mathbf{x})^{\frac{N}{4}}}\psi_{sch\,i}) d\mathbf{x} = \tag{3.61}$$

$$\int_{\mathcal{M}^N} \overline{\psi_{LB\,i}}\psi_{LB\,j} d\mathbf{x} = \delta_{ij}, \quad \psi_{LB\,i} = m(\mathbf{x})^{\frac{2-N}{4}}\psi_{sch\,i}.$$

The crucial point is that transformations on the wave function induce a similarity transformation for the operators:

$$\int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch}\hat{\mathcal{T}}_{sch}\phi_{sch}d\mathbf{x} = \int_{\mathcal{M}^N} \overline{\psi}_{PDM}\left(\sqrt{m(\mathbf{x})}\hat{\mathcal{T}}_{sch}\frac{1}{\sqrt{m(\mathbf{x})}}\right)\phi_{PDM}d\mathbf{x} \quad (3.62)$$

$$\int_{\mathcal{M}^N} m(\mathbf{x})\overline{\psi}_{sch}\hat{\mathcal{T}}_{sch}\phi_{sch}d\mathbf{x} = \int_{\mathcal{M}^N} m(\mathbf{x})^{\frac{N}{2}}\overline{\psi}_{LB}\left(m(\mathbf{x})^{\frac{2-N}{4}}\hat{\mathcal{T}}_{sch}m(\mathbf{x})^{\frac{N-2}{4}}\right)\phi_{LB}d\mathbf{x} \quad (3.63)$$

explicitly:

$$\sqrt{m(\mathbf{x})}\hat{\mathcal{T}}_{sch}\frac{1}{\sqrt{m(\mathbf{x})}} = -\hbar^2\sqrt{m(\mathbf{x})}\frac{1}{m(\mathbf{x})}\partial^i\partial_i\frac{1}{\sqrt{m(\mathbf{x})}} = \quad (3.64)$$

$$= \frac{\hbar^2}{m(|\mathbf{x}|)}\partial^i\partial_i + \hbar^2\frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_i}{|\mathbf{x}|}\partial_i - \hbar^2\left(\frac{3}{4}\frac{m(|\mathbf{x}|)'^2}{m(|\mathbf{x}|)^3} - \frac{1}{2}\frac{m(|\mathbf{x}|)''}{m(|\mathbf{x}|)^2} + \frac{(1-N)m(|\mathbf{x}|)'}{2|\mathbf{x}|m(|\mathbf{x}|)^2}\right) =$$

$$= \hat{\mathcal{T}}_{PDM} + V_{PDM}, \quad V_{PDM} = -\hbar^2\left(\frac{3}{4}\frac{m(|\mathbf{x}|)'^2}{m(|\mathbf{x}|)^3} - \frac{1}{2}\frac{m(|\mathbf{x}|)''}{m(|\mathbf{x}|)^2} + \frac{(1-N)m(|\mathbf{x}|)'}{2|\mathbf{x}|m(|\mathbf{x}|)^2}\right)$$

considering the Laplace-Beltrami quantization:

$$m(|\mathbf{x}|)^{\frac{2-N}{4}}\hat{\mathcal{T}}_{sch}m(|\mathbf{x}|)^{\frac{N-2}{4}} = -\hbar^2 m(|\mathbf{x}|)^{\frac{2-N}{4}}\frac{1}{m(|\mathbf{x}|)}\partial^i\partial_i m(|\mathbf{x}|)^{\frac{N-2}{4}} = \quad (3.65)$$

$$= -\hbar^2\frac{1}{m(|\mathbf{x}|)}\partial^i\partial_i - \hbar^2\left(\frac{N-2}{2}\right)\frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{x_i}{|\mathbf{x}|}\partial_i +$$
$$-\hbar^2\frac{N-2}{4}\left(\frac{N-6}{4}\frac{m(|\mathbf{x}|)'^2}{m(|\mathbf{x}|)^3} + \frac{m(|\mathbf{x}|)''}{m(|\mathbf{x}|)^2} + \frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{N-1}{|\mathbf{x}|}\right)$$

$$= \hat{\mathcal{T}}_{LB} + V_{LB}, \quad V_{LB} = -\hbar^2\frac{N-2}{4}\left(\frac{N-6}{4}\frac{m(|\mathbf{x}|)'^2}{m(|\mathbf{x}|)^3} + \frac{m(|\mathbf{x}|)''}{m(|\mathbf{x}|)^2} + \frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{N-1}{|\mathbf{x}|}\right)$$

This means that the three quantization prescriptions considered give quantum kinetic terms which are similarity equivalent up to a quantum potential term. It is interesting to do some remarks in regards to the quantum potential terms (V_{PDM}, V_{LB}):
REMARK1
The potential term V_{PDM} turns out to be exactly one of those given by Levy Leblond in his paper [72] when $N = 1$ and in general :

$$\sqrt{m(|\mathbf{x}|)}\hat{\mathcal{T}}_{sch}\frac{1}{\sqrt{m(|\mathbf{x}|)}} = \frac{1}{\sqrt{m(|\mathbf{x}|)}}|\mathbf{P}|^2\frac{1}{\sqrt{m(|\mathbf{x}|)}}$$

Namely the Position dependent Hamiltonian (3.29) when the parameters are $\alpha = -\frac{1}{2}$; $\beta = 0$; $\gamma = -\frac{1}{2}$

REMARK2

The potential term V_{LB} turns out to be proportional to the scalar curvature R of the system with metric $g_{ij} = m(|\mathbf{x}|)\delta_{ij}$:

$$R = (1-N)\left(\frac{N-6}{4}\frac{m(|\mathbf{x}|)'^2}{m(|\mathbf{x}|)^3} + \frac{m(|\mathbf{x}|)''}{m(|\mathbf{x}|)^2} + \frac{m(|\mathbf{x}|)'}{m(|\mathbf{x}|)^2}\frac{N-1}{|\mathbf{x}|}\right) \rightarrow V_{LB} = \hbar^2\frac{N-2}{4(N-1)}R$$

according to the pioneering paper by Paneitz in 1983 in which was firstly established the connection between LB operators and scalar curvature associated with two different conformally flat Riemannian manifold [73]. Moreover the Laplace Beltrami quantization with the addition of a potential term proportional to the scalar curvature of the manifold is in full agreement with many prescriptions used in the analysis of scalar field theories in General Relativity or when dealing with quantization on arbitrary Riemannian manifolds [75] [76] [77]. Some analogies are present also in condensed matter physics in the study of particles which are bound to move in a non-flat space by a delta potential, also in this case the hamiltonian has a part proportional to the Laplace Beltrami operators and potential parts connected with the scalar curvature of the manifold [74].

Chapter 4

Quantum Bertrand systems

4.1 Introduction

Until now we have given a survey of the so called exactly solvable systems both in quantum and classical mechanics. This analysis has been achieved following two completely different point of views.
In quantum mechanics we have considered a general basis of orthogonal polynomials on a suitable Hilbert space and then we have obtained, through a number of algebraic transformations, the class of the so called shape invariant systems. The shape invariant systems are exactly solvable 1-dimensional quantum systems.
In classical mechanics we have faced the classification of the exactly solvable systems introducing the definition of Maximal superintegrability:
As showed in the second chapter the Maximal superintegrability gives $2N - 1$ conserved quantities for an Hamiltonian system whose trajectory is defined on a $2N$ dimensional phase space, therefore these $2N - 1$ constraints determine the trajectory of a system and this fact plays a fundamental role in the exact solvability of the system. Since the 1-dimensional systems are by definition M.S. we have presented the classification, under certain given prescriptions, of the 2-dimensional M.S. systems. In particular, the generalized Bertrand theorem states that, requiring radial symmetry, there are only two multiparametric families of systems with the M.S. property. So far we have seen that shape invariant systems and M.S. systems seem to be linked by a "*fil rouge*" :
• Both shape invariant systems and M.S. systems are a rare class of systems respectively in quantum and classical mechanics.
• They can be solved explicitly by using algebraic methods
• The solution can be given in terms of the standard trascendental functions and not just by "quadrature".
The next step of this thesis is to study the relationship existing between the class of the M.S. systems in classical mechanics and the shape invariant systems in quantum mechanics, following the outline. Since there are some ambiguities in the quantization of the classical systems on a non-Euclidean manifold, as stated in the

third chapter, we will begin analyzing the quantum systems. As we have introduced the shape invariant systems, as one-dimensional systems so the second step consists in looking at the one dimensional Schroedinger equation as the radial equation of an higher dimensional space. The third step is to perform the classical limit and to compare such systems with the radial M.S. systems classified in the second chapter.

4.2 From a shape invariant system to a M.S. system

To begin with let us recall the shape invariant systems introduced in the first chapter. We have seen that the orthogonal polynomials give two classes of exactly solvable quantum problems connected by a coupling constant metamorphosis:

Jacobi system:

$$\hat{H} = -\partial_x^2 + \frac{1}{8}\left(\frac{4\alpha^2 - 1}{1 - \sin(x)} + \frac{4\beta^2 + 1}{1 + \sin(x)}\right) \tag{4.1}$$

and one of its Stackel equivalent, namely the Rosen-Morse system:

$$\hat{H} = -\partial_x^2 - \frac{l(l+1)}{\cosh(x)^2} + 2\mu\tanh(x). \tag{4.2}$$

Let us focus on the Stackel equivalent systems: if we change the variable x to $kx + \frac{i\pi}{2}$ we get another shape invariant system, since the shape invariant condition is invariant under shift and rescaling of the independent variable. After the above transfomation, the Hamiltonian (4.2) becomes:

$$\hat{H} = -\frac{1}{k^2}\partial_x^2 + \frac{l(l+1)}{\sinh(kx)^2} + 2\mu\coth(kx) \tag{4.3}$$

whose eigenfunction equation is, in general, given by:

$$\left(-\frac{1}{k^2}\partial_x^2 + \frac{l(l+1)}{\sinh(kx)^2} + 2\mu\coth(kx)\right)\psi_{n,l,\mu} = E(n,l,\mu)\psi_{n,l,\mu} \tag{4.4}$$

or equivalently:

$$\left(-\partial_x^2 + k^2\frac{l(l+1)}{\sinh(kx)^2} - 2\mu k\coth(kx)\right)\psi_{n,l,\frac{-\mu}{k}} = k^2 E(n,l,\frac{-\mu}{k})\psi_{n,l,\frac{-\mu}{k}}. \tag{4.5}$$

This system is universally known as the generalized Kepler problem, namely the Kepler problem on a space of constant curvature [54], [79] -[83]. As said above this new quantum system has to be a shape invariant system. We can see explicitly this property factorizing the Hamiltonian operator through the two ladder operators:

$$A_l = i\partial_x - iW_l'(x) = i\partial_x + i\frac{\mu}{l+1} - i(l+1)k\coth(kx) \tag{4.6}$$

$$A_l^\dagger = i\partial_x + iW_l'(x) = i\partial_x - i\frac{\mu}{l+1} + i(l+1)k\coth(kx) \tag{4.7}$$

where the function $W_l(x)$ is the so called prepotential function:

$$W_l(x) = -\frac{\mu}{l+1}x + (l+1)\ln(\sinh(kx)), \quad (l+1)^2 < \frac{\mu}{k}, l \geq 0 \tag{4.8}$$

now we can recast the Hamiltonian operator of the equation (4.5) as follows:

$$A_l^\dagger A_l = \hat{H}_l - \epsilon_l \tag{4.9}$$

$$\hat{H}_l = -\partial_x^2 + k^2 \frac{l(l+1)}{\sinh(kx)^2} - 2\mu k \coth(kx)$$

$$\epsilon_l = -\left(\frac{\mu^2}{(l+1)^2} + k^2(l+1)\right)$$

the shape invariant property is easily verified by intertwining the ladder operators $\hat{A}_l^\dagger$ and $\hat{A}_l$:

$$\hat{A}_l\hat{A}_l^\dagger = \hat{H}_{l+1} - \epsilon_l = \hat{A}_{l+1}^\dagger \hat{A}_{l+1} + \epsilon_{l+1} - \epsilon_l. \tag{4.10}$$

Therefore, as expected, the Hamiltonian operator (4.5) is exactly solvable by the application of the shape invariant property. In particular the fundamental state turns out to be:

$$\hat{A}_l e^{W_l(x)} = 0 \rightarrow \psi_{0,l}(x) = e^{W_l(x)} = e^{-\frac{\mu x}{l+1}}(\sinh(kx))^{l+1}$$

according to the strategy introduced in the first chapter we get the eigenvalue equation for the hamiltonian $\hat{H}_l$:

$$\hat{H}_l\psi_{n,l} = -\left(\frac{\mu^2}{(l+n+1)^2} + k^2(l+n+1)^2\right)\psi_{n,l} \tag{4.11}$$

while the eigenfunctions can be generated by the application of the ladder operators:

$$\psi_{n,l}(x) = \prod_{i=0}^{n-1} \hat{A}_{l+i}^\dagger \psi_{0,l+n}(x) \tag{4.12}$$

Since we have a multiparametric Hamiltonian operator $\hat{H}_{\mu,l,k}$ let us define some constraint on these parameters in order to determine the space on which $\hat{H}_{\mu,l,k}$ turns out to be self-adjoint:

$$\mu, l, k \in \mathbb{R}, \quad \mu > 0, \quad k > 0, \quad \frac{\mu}{k} > (l + N_{max} + 1)^2$$

so we can finally state that $\hat{H}_{\mu,l,k}$ is formally self-adjoint on the standard L^2 space with product:

$$< \psi|\phi >= \int_0^\infty \overline{\psi}(x)\phi(x)dx \tag{4.13}$$

finally we stress that in this space the set of eigenfunctions (4.12) is now a *finite* set of normalizable eigenfunctions:

$$\psi_{n,l}(x) = \prod_{i=0}^{n-1} \hat{A}^{\dagger}_{l+i}\psi_{0,l+n}(x), \quad n \leq N_{max}, \quad (l + N_{max})^2 = [\frac{\mu}{k}] \tag{4.14}$$

4.2.1 Quantum hamiltonian with a degenerate spectrum

In the previous subsection we have presented an exactly solvable quantum 1-dimensional system whose spectrum is:

$$E_{n,l,k,\mu} = -\left(\frac{\mu^2}{(l+n+1)^2} + k^2(l+n+1)^2\right) \tag{4.15}$$

in this spectrum l, μ, k have to be read as fixed parameters while n is the quantum number. The crucial point is that this spectrum becomes a *degenerate* spectrum whenever the parameter l is read as a quantum number with integer values and not just as parameter. In this case the degeneracy entails that the Hamiltonian has different eigenfunctions with the same energy eigenvalue, moreover this feature is indicative of the presence of the maximal superintegrability for the quantum system:

$$\hat{H}\psi_{n,N-n} = -\left(\frac{\mu^2}{(N+1)^2} + k^2(N+1)^2\right)\psi_{n,N-n}; \quad \forall n \leq N, |m| \leq N-n. \tag{4.16}$$

We stress that the form of the spectrum suggests how to modify the 1-dim hamiltonian in order to get a new n-dimensional Hamiltonian with a degenerate spectrum; this is a peculiarity of the quantum systems in contrast with the classical ones, in which the solution of the 1-dimensional motion does not say anything about the extension to M.S. systems. Since the Hamiltonian depends on $l(l+1)$ we have to replace this parameter with the operator whose spectrum is $l(l+1)$, namely the angular momentum operator:

$$\hat{L}^2 Y_{l,m}(\theta,\phi) = -\left(\partial_\theta^2 + \cot(\theta)\partial_\theta + \frac{1}{\sin(\theta)^2}\partial_\phi^2\right) Y_{l,m}(\theta,\phi) = l(l+1)Y_{l,m}(\theta,\phi) \tag{4.17}$$

where $Y_{l,m}(\theta,\phi)$ are the standard spherical harmonic functions, that defines an orthogonal basis on the sphere $\mathbb{S}^2$:

$$\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overline{Y}_{l,m} Y_{l',m'} d\Omega = \delta_{l,l'}\delta_{m,m'}, \quad d\Omega = \sin\theta d\theta d\phi, \quad l,m \in \mathbb{N} \tag{4.18}$$

$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{2\pi(l+m)!}} P_l^m(\cos(\theta)) e^{im\phi} \tag{4.19}$$

with the $P_l^m(\cos(\theta))$ the standard Legendre polynomials.
The eigenvalue equation (4.17) helps us to recast the (4.5) 1-dimensional Hamiltonian equation in an higher dimensional Hamiltonian operator:

$$\left(-\partial_r^2 + k^2 \frac{\hat{L}^2}{\sinh(kr)^2} + 2\mu k \coth(kr)\right) \psi_{n,l}(r) Y_{l,m}(\theta,\phi) = \tag{4.20}$$
$$= -\left(\frac{\mu^2}{(l+n+1)^2} + k^2(l+n+1)^2\right) \psi_{n,l}(r) Y_{l,m}(\theta,\phi).$$

where we have replaced x by r to make clearer that the original 1-dimensional system represents the radial part of an higher dimensional system.

$$\hat{H}(r,\theta,\phi) = \left(-\partial_r^2 + k^2 \frac{\hat{L}^2}{\sinh(kr)^2} + 2\mu k \coth(kr)\right) \tag{4.21}$$

with eigenfunctions:

$$\Phi(r,\theta,\phi)_{n,l,m} = \psi_{n,l}(r) Y_{l,m}(\theta,\phi) \tag{4.22}$$

wich represent an orthogonal basis with the scalar product:

$$\int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \overline{\Phi}_{n,l,m} \Phi_{n',l',m'} dr \sin(\theta) d\theta d\phi \propto \delta_{n,n'} \delta_{l,l'} \delta_{m,m'} \tag{4.23}$$

Following our outline the next step is to compare the Hamiltonian (4.21) with one of the Hamiltonians belonging to the classical Perlick classification. Since the centrifugal part of the Perlick Hamiltonians is proportional to $\frac{L^2}{r^2}$ let us recast the centrifugal kinetic energy operator as follows:

$$\frac{k^2 \hat{L}^2}{\sinh(kr)^2} \to \frac{\hat{L}^2}{r'^2} \tag{4.24}$$

this induces the change of variables: $r' = \frac{\sinh(kr)}{k}$. This substitution turns the Hamiltonian (4.21) in:

$$\hat{H}(r',\theta,\phi) = -(1+k^2 r'^2)\partial_{r'}^2 - k^2 r' \partial_{r'} + \frac{\hat{L}^2}{r'^2} - 2\mu\sqrt{\frac{1}{r'^2} + k^2}. \tag{4.25}$$

Let us reintroduce the Planck constant in order to perform the classical limit $\hbar \to 0$.

$$\left(-(1+k^2 r'^2)\partial_{r'}^2 - k^2 r' \partial_{r'} + \frac{\hat{L}^2}{r'^2} - 2\mu\sqrt{\frac{1}{r'^2} + k^2}\right) \Phi_{n,l,m} =$$
$$= -\left(\frac{\mu^2}{(l+n+1)^2} + k^2(l+n+1)^2\right) \Phi_{n,l,m}$$

we multiply both sides by $\frac{\hbar^2}{2}$:

$$\left(-\frac{\hbar^2}{2}(1+k^2r'^2)\partial^2_{r'}-\frac{\hbar^2}{2}k^2r'\partial_{r'}+\frac{\hbar^2\hat{L}^2}{2r'^2}-\hbar^2\mu\sqrt{\frac{1}{r'^2}+k^2}\right)\Phi_{n,l,m}=$$
$$=-\frac{\hbar^2}{2}\left(\frac{\mu^2}{(l+n+1)^2}+k^2(l+n+1)^2\right)\Phi_{n,l,m}$$

and let us rescale the coupling constant as $\mu \to \frac{\mu}{\hbar^2}$. This leads to the eigenvalue equation:

$$\left(-\frac{\hbar^2}{2}(1+k^2r'^2)\partial^2_{r'}-\frac{\hbar^2}{2}k^2r'\partial_{r'}+\frac{\hbar^2\hat{L}^2}{2r'^2}-\mu\sqrt{\frac{1}{r'^2}+k^2}\right)\Phi_{n,l,m}=$$
$$=-\left(\frac{\mu^2}{2\hbar^2(l+n+1)^2}+\frac{\hbar^2k^2}{2}(l+n+1)^2\right)\Phi_{n,l,m}$$

Let us consider now the operators:

$$\begin{aligned}\hat{P}_{r'} &= -i\hbar\partial_{r'}\\ \hat{P}_{\theta} &= -i\hbar\partial_{\theta}\\ \hat{P}_{\phi} &= -i\hbar\partial_{\phi}\end{aligned} \tag{4.26}$$

the Hamiltonian can be recasted as:

$$\frac{1}{2}(1+k^2r'^2)\hat{P}^2_{r'}-i\hbar\frac{k^2}{2}\hat{P}_{r'}+\frac{1}{2r'^2}\left(\hat{P}^2_{\theta}-i\hbar\cot(\theta)\hat{P}_{\theta}+\frac{1}{\sin(\theta)^2}\hat{P}^2_{\phi}\right)-\mu\sqrt{\frac{1}{r'^2}+k^2} \tag{4.27}$$

Performing the classical limit $\hbar \to 0$ we get the classical Hamiltonian:

$$H(r',\theta,\phi,P_{r'},P_{\theta},P_{\phi})=\frac{1}{2}(1+k^2r'^2)P^2_{r'}+\frac{1}{2r'^2}\left(P^2_{\theta}+\frac{1}{\sin(\theta)^2}P^2_{\phi}\right)-\mu\sqrt{\frac{1}{r'^2}+k^2} \tag{4.28}$$

As expected we find one of the classical radial Hamiltonian in the Perlick classification, namely the particular case of the family I when $\beta = 1$ and $K = k^2$ that corresponds to the motion on a space of constant curvature whose metric tensor is:

$$ds^2=\frac{dr^2}{1+k^2r'^2}+r^2(d\theta^2+\sin(\theta)^2d\phi^2) \tag{4.29}$$

In this case we have considered $k \in \mathbb{R}$, therefore we have $K > 0$ and this amounts to consider an hyperbolic space; anyway the case on the sphere $K < 0$ can be reobtained quite straightforwardly replacing $k \to ik$ and modifying in a suitable way the scalar product for the wave functions. The main difference between the hyperbolic case and the spherical case is that in the hyperbolic case considered above the number of bound states turns out to be a finite set in contrast with the spherical case where this set turns out to be a denumerably infinite set.

Laplace Beltrami quantization of system (4.21)

Let us came back to the quantum problem (4.25). The Hamiltonian operator has a degenerate spectrum and its classical limit is a maximally superintegrable system. The next step is to understand which quantization prescription, among those proposed in the third chapter, better describes the quantization of the Hamiltonian (4.28). Notice that, after the change of variable (4.24), the scalar product turns out to be changed in:

$$< \Phi_{n,l,m}|\Phi_{n',l',m'} >= \int_{r'=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overline{\Phi}_{n,l,m}\Phi_{n',l',m'}\frac{dr'}{\sqrt{1+k^2r'^2}}\sin(\theta)d\theta d\phi \quad (4.30)$$

Let us consider now the Laplace Beltrami operator relative to the metric (4.29):

$$\hat{T}_{L.B.} = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{i,j}\partial_j) \quad (4.31)$$

this is a Sturm Liouville operator whenever the scalar product is equipped with the weight function (considering the metric (4.29)):

$$w(r',\theta,\phi) = \sqrt{g} = \frac{r'^2\sin(\theta)}{\sqrt{1+k^2r'^2}} \quad (4.32)$$

$$< \tilde{\Phi}|\tilde{\Psi} >= \int_{r'=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \overline{\Phi}(r',\theta,\phi)\Psi(r',\theta,\phi)\sqrt{g}dr'd\theta d\phi \quad (4.33)$$

Following the strategy showed in the third chapter it is possible to change the weight function in the scalar product (4.30) through a gauge transformation:

$$< \Phi_{n,l,m}|\Phi_{n',l',m'} >= \int_{r'=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\overline{\Phi}_{n,l,m}}{r'}\frac{\Phi_{n',l',m'}}{r'}\frac{r'^2dr'}{\sqrt{1+k^2r'^2}}\sin(\theta)d\theta d\phi \quad (4.34)$$

therefore let us introduce the new eigenfunctions :

$$\tilde{\Phi}_{n,l,m} = \frac{\Phi_{n,l,m}}{r'} \quad (4.35)$$

and the new operator:

$$\tilde{\hat{H}}(r',\theta,\phi) = \frac{1}{r'}\hat{H}(r',\theta,\phi)r' = -\frac{\hbar^2}{2\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) - \mu\sqrt{\frac{1}{r'^2}+k^2} - \frac{\hbar^2k^2}{2} \quad (4.36)$$

or equivalently:

$$\hat{H}'_{L.B.} = \tilde{\hat{H}}(r',\theta,\phi) + \frac{\hbar^2k^2}{2} = -\frac{\hbar^2}{2\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j) - \mu\sqrt{\frac{1}{r'^2}+k^2} \quad (4.37)$$

this new operator has the spectrum:

$$E_{n,l} = -\frac{\mu^2}{2\hbar^2(n+l+1)^2} - \frac{\hbar^2 k^2}{2}(n+l+1)^2 + \frac{\hbar^2 k^2}{2} \tag{4.38}$$

Before closing the section let us rewrite the Hamiltonian (4.37) in a conformally flat reference frame in order to be coherent with the results of the third chapter.
Let us recall the metric of the Perlick I system is:

$$ds^2 = \frac{dr^2}{\beta^2(1+k^2r^2)} + r^2 d\Omega^2, \quad r \Rightarrow \frac{2}{r^{-\beta} - k^2 r^{\beta}} \tag{4.39}$$

$$\Rightarrow ds^2 = \frac{4}{r^2(r^{-\beta} - k^2 r^{\beta})^2}(dr^2 + r^2 d\Omega^2).$$

Since we are dealing with the particular case $\beta = 1$ let us apply the transformation:

$$r \Rightarrow \frac{2r}{1-k^2r^2} \tag{4.40}$$

In this new coordinate system the Hamiltonian (4.37) (with $\hbar = 1$)turns out to be:

$$\hat{H}_{L.B.} = -\frac{1}{8}(1-k^2r^2)^2\left(\partial_r^2 + \frac{2k^2 r}{1-k^2r^2}\partial_r + \frac{2}{r}\partial_r - \frac{\hat{L}^2}{r^2}\right) - \frac{\mu}{2}\left(\frac{1}{r} + k^2 r\right) \tag{4.41}$$

The above Hamiltonian is of course still the LB quantization of the generalized Kepler in the new coordinate system, but let us stress that the transformation is not defined in $r = \frac{1}{k}$ and in fact the new radial variable turns out to be defined in the new domain $\mathcal{D}$:

$$\mathcal{D} = r \in [0, \frac{1}{k}) \tag{4.42}$$

consequently the new scalar product is:

$$< \tilde{\Phi}_{n,l,m}|\tilde{\Phi}_{n',l',m'} >= \int_{r=0}^{\frac{1}{k}}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \bar{\tilde{\Phi}}_{n,l,m}\tilde{\Phi}_{n',l',m'}\frac{8r^2 dr}{(1-k^2r^2)^3}\sin(\theta)d\theta d\phi \tag{4.43}$$

In order to complete the picture let us see how this Hamiltonian change in the "Schroedinger" and in the "Position dependent mass" quantization through the similarity transformation introduced in the third chapter:
• Position dependent mass quantization:

$$\hat{H}_{PDM} = \frac{1}{(1-k^2r^2)^{\frac{3}{2}}}\hat{H}_{LB}(1-k^2r^2)^{\frac{3}{2}} = \tag{4.44}$$

$$= -\frac{1}{8}(1-k^2r^2)^2\left(\partial_r^2 + \frac{2-6k^2r^2}{r(1-k^2r^2)}\partial_r - \frac{\hat{L}^2}{r^2}\right) - \frac{k^2(9-6k^2r^2)}{8} - \frac{\mu}{2}\left(\frac{1}{r} + k^2 r\right)$$

whose eigenfunctions are:

$$\tilde{\Phi}_{PDM}(r,\theta,\phi)_{n,l,m} = \frac{1}{(1-k^2r^2)^{\frac{3}{2}}}\tilde{\Phi}_{LB}(r,\theta,\phi)_{n,l,m} \tag{4.45}$$

• Schroedinger quantization:

$$\hat{H}'_{sch} = \frac{1}{\sqrt{1-k^2r^2}}\hat{H}_{L.B.}\sqrt{1-k^2r^2} = (1-k^2r^2)\hat{H}_{PDM}\frac{1}{(1-k^2r^2)} = \tag{4.46}$$

$$= -\frac{1}{8}(1-k^2r^2)^2\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat{L}^2}{r^2}\right) - \frac{\mu}{2}\left(\frac{1}{r} + k^2r\right) + \frac{3}{8}k^2$$

or equivalently:

$$\hat{H}_{sch} = \hat{H}'_{sch} - \frac{3}{8}k^2 = -\frac{1}{8}(1-k^2r^2)^2\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat{L}^2}{r^2}\right) - \frac{\mu}{2}\left(\frac{1}{r} + k^2r\right) \tag{4.47}$$

whose spectrum and eigenfunctions are:

$$E_{sch\ n,l} = -\frac{\mu^2}{2(n+l+1)^2} - \frac{k^2}{2}(n+l+1)^2 + \frac{k^2}{8} \tag{4.48}$$

$$\tilde{\Phi}_{sch}(r,\theta,\phi)_{n,l,m} = \frac{1}{\sqrt{(1-k^2r^2)}}\tilde{\Phi}_{LB}(r,\theta,\phi)_{n,l,m} \tag{4.49}$$

Since this Hamiltonian operator derives directly, through algebraic manipulations, from the differential operator wich defines the Jacobi orthogonal polynomials $P_{n,\alpha,\beta}$, let us summarize all the algebraic steps we have carried out displaying the general eigenfunctions written in terms of $P_{n,\alpha,\beta}$ [78]:

$$\tilde{\Phi}_{sch}(r,\theta,\phi)_{n,l,m} = \psi_{n,l}(r)Y_{l,m}(\theta,\phi) = \tag{4.50}$$

$$\frac{e^{-\frac{2\mu\tanh^{-1}(kr)}{k(n+l+1)}}r^{n+l}}{(1-k^2r^2)^{n+l+\frac{1}{2}}}P\left(n, \frac{\mu - k(n+l+1)^2}{k(n+l+1)}, -\frac{\mu + k(n+l+1)^2}{k(n+l+1)}\right)\left(\frac{1+k^2r^2}{2kr}\right)Y_{l,m}(\theta,\phi)$$

with the integrability constraint :

$$n < N_{max}, \quad (N_{max} + l + \frac{3}{2})^2 = [\frac{\mu}{k}]$$

4.3 Quantum Bertrand systems of type I

In the previous sections we have shown how the orthogonal polynomials and therefore the shape invariant systems, because of their exact solvability, can help us to find M.S. systems both in classical and quantum mechanics.
The aim of this section is to show that an exact solution can be found also for the whole family of Perlick I through the generalization of the particular case ($\beta = 1$) that we solved explicitly in the previous section, strenghtening the relations existing between M.S. systems and exactly solvable systems.
To begin with, let us point out that the general metric $\beta \neq 1$ and the particular case $\beta = 1$ differ just by a rescaling of the angular part, anyway this difference is highly nontrivial since it makes the intrinsic scalar curvature radial dependent (2.43) [59]:

$$ds^2 = \frac{4\beta^2}{r^2(r^{-\beta} - k^2 r^\beta)^2}(dr^2 + r^2 d\Omega^2), \;\; r' = r^\beta \Rightarrow \frac{4}{(1-k^2 r'^2)^2}(dr'^2 + r'^2 \beta^2 d\Omega^2) \tag{4.51}$$

$$ds^2 = \frac{4}{(1-k^2r^2)^2}(dr^2 + r^2 d\Omega^2); \beta = 1$$

Let us now consider the metric (4.51); since the three quantizations are linked by similarity transformations let us compute the simplest one, namely the so called direct or Schroedinger quantization:

$$\hat{H}_{sch\beta} = -\frac{r^2(r^{-\beta} - k^2 r^\beta)^2}{8\beta^2}\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{\hat{L}^2}{r^2}\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^\beta) \tag{4.52}$$

Where we are considering the domain $\tilde{\mathcal{D}}$

$$r \in [0, \frac{1}{k^{\frac{1}{\beta}}}), \tag{4.53}$$

in order to have a positive definite mass $m(r) = \frac{4\beta^2}{r^2(r^{-\beta} - k^2 r^\beta)}$.
Because of the spherical symmetry the exact solvability of the system depends on the radial equation, therefore let us go back to the 1-dimensional problem:

$$\hat{H}_{sch\beta}(r) = -\frac{r^2(r^{-\beta} - k^2 r^\beta)^2}{8\beta^2}\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{l(l+1)}{r^2}\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^\beta) \tag{4.54}$$

According to the above considerations let us perform the substitution (4.51)

$$r = r'^a, \quad a = \frac{1}{\beta},$$

where the variable r' is now defined again in the domain $\mathcal{D}$ (4.42), and the Hamiltonian operator becomes:

$$\hat{H}_{sch\beta}(r') = -\frac{(1-k^2r'^2)^2}{8}\left(\partial^2_{r'} + \frac{1+a}{r'}\partial_{r'} - \frac{a^2l(l+1)}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'} + k^2r'\right) \quad (4.55)$$

Let us compare this differential operator with the radial part of the operator (4.47):

$$\hat{H}_{sch}(r') = -\frac{(1-k^2r'^2)^2}{8}\left(\partial^2_{r'} + \frac{2}{r'}\partial_{r'} - \frac{l(l+1)}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'} + k^2r'\right). \quad (4.56)$$

As expected by the consideration (4.51) these two radial Hamiltonians are very similar: the potential part is exactly the same, the centrifugal part, according to the (4.51), turns out to be rescaled by a factor a^2, but we find a difference on the differential part $\frac{2}{r}\partial_r \neq \frac{1+a}{r}\partial_r$ if $(a \neq 1)$.
The idea is managing to make equal also the differential part in order to use the solution of the operator (4.55). We can achieve this result performing a similarity transformation:

$$\hat{H}'_{sch\beta}(r') = r'^{\frac{a-1}{2}}\hat{H}_{sch\beta}(r')r'^{\frac{1-a}{2}} = \quad (4.57)$$

$$-\frac{(1-k^2r'^2)^2}{8}\left(\partial^2_{r'} + \frac{2}{r'}\partial_{r'} - \frac{a^2l(l+1) - \frac{1-a^2}{4}}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'} + k^2r'\right)$$

The crucial point is that the operator (4.56) regarded as a 1-dimensional operator turns out to be exactly solvable for all values of the parameter l (we gave to the parameter l the domain $l \in \mathbb{N}$, after we upgraded the system from a 1-dimensional to a 3 dimensional system), so in order to avoid confusion let us introduce the exactly solvable 1-dimensional Hamiltonian:

$$\hat{\mathcal{H}}_{q,\mu} = -\frac{(1-k^2r'^2)^2}{8}\left(\partial^2_{r'} + \frac{2}{r'}\partial_{r'} - \frac{q(q+1)}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'} + k^2r'\right) \quad (4.58)$$

whose spectrum depends on the parameters q, μ:

$$\hat{\mathcal{H}}_{q,\mu}\psi_{n,q,\mu} = \left(-\frac{\mu^2}{2(n+q+1)^2} - \frac{k^2}{2}(n+q+1)^2 + \frac{k^2}{8}\right)\psi_{n,q,\mu}$$

In particular let us set $q = al + \frac{a-1}{2}$:

$$\hat{\mathcal{H}}_{al+\frac{a-1}{2},\mu} = -\frac{(1-k^2r'^2)^2}{8}\left(\partial^2_{r'} + \frac{2}{r'}\partial_{r'} - \frac{a^2l(l+1) - \frac{1-a^2}{4}}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'} + k^2r'\right) =$$

$$= \hat{H}'_{sch\beta}(r')$$

This means that the Hamiltonian $\hat{H}'_{sch\beta}(r')$ is exactly solvable as well as the particular case in which we considered $\beta = 1$. Let us now draw our attention on the spectrum of $\hat{H}'_{sch\beta}(r')$:

$$E_{n,l} = -\frac{\mu^2}{2(n + al + \frac{a+1}{2})^2} - \frac{k^2}{2}(n + al + \frac{a+1}{2})^2 + \frac{k^2}{8}, \quad n, l \in \mathbb{N} \tag{4.59}$$

In chapter 2 we stressed that the Perlick systems admit periodical motion only for $\beta \in \mathbb{Q}$, this condition entails the same domain for the parameter $a = \frac{1}{\beta} \Rightarrow a \in \mathbb{Q}$, so let us consider $a = \frac{m_1}{m_2}$, $m_1, m_2 \in \mathbb{N}$ and two different wave functions, say, $\psi_{n',l'}$ and $\psi_{n,l}$ such that $n' = n - sm_1$ and $l' = l + sm_2$ then $\forall sm_1 \in \mathbb{N}$ as $n' > 0$ we have two wave functions with different quantum numbers but with the same energy eigenvalue. This fact entails that the direct (or Schroedinger) quantization of the M.S. Perlick system I yields again a quantum system whose spectrum shows the accidental degeneracy as expected for a M.S. quantum system. Yet the M.S. turns out to be associated with the exact solvability of the system both in classical and quantum mechanics, in this case the wave function can be expressed regarding to the solution $\psi_{n,l}(r)$ of the system (4.50):

$$< \psi_{n,al+\frac{a-1}{2}}(r')Y_{l,m}|\hat{H}'_{sch\beta}(r')|Y_{l,m}\psi_{n,al+\frac{a-1}{2}}(r') >= \tag{4.60}$$

$$=< \psi_{n,al+\frac{a-1}{2}}(r')r'^{\frac{1-a}{2}}Y_{l,m}|\hat{H}_{sch\beta}(r')|Y_{l,m}r'^{\frac{1-a}{2}}\psi_{n,al+\frac{a-1}{2}}(r') >=$$

$$=< \psi_{n,al+\frac{a-1}{2}}(r^\beta)r^{\frac{\beta-1}{2}}Y_{l,m}|\hat{H}_{sch\beta}(r)|Y_{l,m}r^{\frac{\beta-1}{2}}\psi_{n,al+\frac{a-1}{2}}(r^\beta) > .$$ [1]

This defines the eigenfunctions for the direct quantization of the Perlick systems of the family I:

$$\Phi_{sch\beta}(r)_{n,l,m} = r^{\frac{\beta-1}{2}}\psi_{n,al+\frac{a-1}{2}}(r^\beta)Y_{l,m}(\theta, \phi)$$

4.4 Generalization of Bertrand metric to a N-dimensional hyperspherical space

So far, analyzing the M.S. systems on spaces of non-constant curvature, we have followed an historical order: first we introduced the Darboux spaces (2-dimensional spaces), then we introduced the Bertrand or Perlick spaces (3-dimensional spaces) now the next step is represented by trying to generalize the M.S. property to N-dimensional spaces which, analogously to the Darboux and Perlick spaces, are characterized by the hyperspherical symmetry.

[1] Notice that the weight functions involved in the three scalar products 4.60 are different from each other.

Let us introduce the N hyperspherical coordinate which are formed by a radial-type one $r \in \mathbb{R}^+$ and $N-1$ angles θ_j such that $\theta_k \in [0, 2\pi)$ for $K < N-1$ and $\theta_{N-1} \in [0, \pi)$. They can be put easily in correspondence with the cartesian coordinates $(x_1, x_2, ...x_N)$:

$$x_j = r\cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k, \quad 1 \leq j < N, \quad x_N = r\prod_{k=1}^{N-1} \sin\theta_k. \tag{4.61}$$

Now let us introduce the quantum operators:

$$\hat{p}_r = -i\partial_r, \quad \hat{p}_{\theta_j} = -i\partial_{\theta_j}$$

it is useful to establish a correspondence with the cartesian ones:

$$\hat{p}_j = -i\partial_j = \prod_{k=1}^{j-1} \sin\theta_k \cos\theta_j \hat{p}_r + \frac{\cos\theta_j}{r} \sum_{l=1}^{j-1} \frac{\prod_{k=l+1}^{j-1} \sin\theta_k}{\prod_{m=1}^{l-1} \sin\theta_m} \cos\theta_l \hat{p}_{\theta_l} - \frac{\sin\theta_j}{r\prod_{k=1}^{j-1} \sin\theta_k} \hat{p}_{\theta_j},$$

$$\hat{p}_N = -i\partial_j = \prod_{k=1}^{N-1} \sin\theta_k \hat{p}_r + \frac{1}{r} \sum_{l=1}^{N-1} \frac{\prod_{k=l+1}^{N-1} \sin\theta_k}{\prod_{m=1}^{l-1} \sin\theta_m} \cos\theta_l \hat{p}_{\theta_l}.$$

Hence we obtain that

$$\sum_{j=1}^{N} x_j^2 = r^2, \quad \sum_{j=1}^{N} \hat{p}_j^2 = \frac{1}{r^{N-1}} \hat{p}_r r^{N-1} \hat{p}_r + \frac{\hat{L}^2}{r^2} = \hat{p}_r^2 - i\frac{N-1}{r}\hat{p}_r + \frac{\hat{L}^2}{r^2}, \tag{4.62}$$

where $\hat{L}^2$ is the square of the total angular momentum given by

$$\hat{L}^2 = \sum_{j=1}^{N-1} \left(\prod_{k=1}^{j-1} \frac{1}{\sin^2\theta_k} \right) \frac{1}{(\sin\theta_j)^{N-1-j}} \hat{p}_{\theta_j} (\sin\theta_j)^{N-1-j} \hat{p}_{\theta_j}.$$

we are now ready to define the N-dimensional Perlick space of type I :

$$ds^2 = \frac{4\beta^2}{r^2(r^{-\beta} + r^{\beta})^2}(dr^2 + r^2 d\Omega_{N-1}^2), \quad d\Omega_{N-1}^2 = \sum_{i=1}^{N-1} d\theta_i^2 \prod_{j=1}^{i-1} \sin^2\theta_j, \tag{4.63}$$

therefore the N-dimensional generalization of the Hamiltonian (4.52) turns out to be:

$$\hat{H}_{sch\beta}^{N} = -\frac{r^2(r^{-\beta} - k^2 r^{\beta})^2}{8\beta^2} \left(\partial_r^2 + \frac{N-1}{r}\partial_r - \frac{\hat{L}^2}{r^2} \right) - \frac{\mu}{2} \left(r^{-\beta} + k^2 r^{\beta} \right) \tag{4.64}$$

The N-dimensional Hamiltonian can be reduced to a 1-dimensional problem by factorizing the wave function in the usual radial and angular components:

$$\Phi^{(N)}_{sch\beta}(r,\theta_j)_{n,l,l_1\ldots l_{N-2}} = \psi(r)_{n,l} Y(\theta_j)_{l,l_1,\ldots l_{N-2}}$$

$Y(\theta_j)$ are the hyperspherical harmonics functions, namely the eigenfunctions of $\hat{L}^2$ which satisfy the eigenvalue equation given by:

$$\hat{L}^2 Y(\theta_j)_{l,l_1,\ldots l_{N-2}} = l(l+N-2) Y(\theta_j)_{l,l_1,\ldots l_{N-2}} \tag{4.65}$$

this factorization leads to the radial Hamiltonian:

$$\hat{H}^N_{sch\beta}(r) = -\frac{r^2(r^{-\beta}-k^2r^{\beta})^2}{8\beta^2}\left(\partial_r^2 + \frac{N-1}{r}\partial_r - \frac{l(l+N-2)}{r^2}\right) - \frac{\mu}{2}\left(r^{-\beta}+k^2r^{\beta}\right) \tag{4.66}$$

now let us apply, analogously to the 3-dimensional case, the substitution $r' = r^{\beta}$:

$$\hat{H}^N_{sch\beta}(r') = -\frac{(1-k^2r'^2)^2}{8}\left(\partial_{r'}^2 + \frac{1+a(N-2)}{r'}\partial_{r'} - \frac{a^2l(l+N-2)}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'}+k^2r'\right) \tag{4.67}$$

following exactly the same strategy as for the 3-dimensional case, let us apply a similarity transformation in order to compare the differential operator (4.67) with the exactly solvable operator (4.58):

$$r'^{\frac{-1+a(N-2)}{2}} \hat{H}^N_{sch\beta}(r') r'^{\frac{1-a(N-2)}{2}} = \tag{4.68}$$

$$= -\frac{(1-k^2r'^2)^2}{8}\left(\partial_{r'}^2 + \frac{2}{r'}\partial_{r'} - \frac{a^2l(l+N-2)+\frac{-1+a^2(N-2)^2}{4}}{r'^2}\right) - \frac{\mu}{2}\left(\frac{1}{r'}+k^2r'\right)$$

Analogously to the 3-dimensional case the operator $\hat{H}^N_{sch\beta}(r')$ turns out to be the operator (4.58) $\hat{\mathcal{H}}_{q,\mu}$ when the parameter q is set to $q = al + \frac{a(N-2)-1}{2}$. Therefore from the eigenfunctions of $\hat{\mathcal{H}}_{q,\mu}$ $\psi_{q,\mu}$ we determine the solution of $\hat{H}^N_{sch\beta}(r')$:

$$\hat{H}^N_{sch\beta}(r')\psi_{al+\frac{a(N-2)-1}{2},\mu}(r') = \tag{4.69}$$

$$= \frac{-\mu^2}{2(n+al+\frac{a(N-2)+1}{2})^2} - \frac{k^2}{2}(n+al+\frac{a(N-2)+1}{2})^2 + \frac{k^2}{8}, \quad n,l \in \mathbb{N}$$

From the eigenvalue equation (4.69) we determine the eigenfunctions of the N-dimensional Perlick I system $\hat{H}^N_{sch\beta}$:

$$\Phi^N_{sch\beta}(r,\theta_j) = r^{\frac{\beta-(N-2)}{2}} \psi_{n,al+\frac{a(N-2)-1}{2}}(r^{\beta}) Y_l(\theta_j) \tag{4.70}$$

4.4.1 Laplace-Beltrami and PDM quantization

According to the analysis presented in the third chapter we can link the exact solution of the so called "Schroedinger" quantization to other quantization prescriptions of the same classical problem through similarity transformations:

• **Position Dependent Mass quantization**
Considering the position dependent mass quantization of the classical Perlick I system in cartesian coordinates $\partial_i = \frac{\partial}{\partial_{x_i}}$, $r = \sqrt{\sum_i x_i^2}$ we get:

$$\hat{H}^N_{PDM\beta} = -\frac{1}{2m(r)}\left(\partial_i^2 - \frac{m'(r)}{m(r)}\frac{x_i}{r}\partial_i\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^{\beta}) \tag{4.71}$$

where the position dependent mass turns out to be $m(r) = \frac{4}{r^2(r^{-\beta} - k^2 r^{\beta})^2}$, which is correctly positive in the domain $\tilde{\mathcal{D}}$ (4.53).

• **Laplace Beltrami quantization** Let us consider now the geometrical point of view or the Laplace Beltrami quantization, then the quantization of the Perlick I system turns out to be:

$$\hat{H}^N_{LB\beta} = -\frac{1}{2m(r)}\left(\partial_i^2 + \frac{N-2}{2}\frac{m'(r)}{m(r)}\frac{x_i}{r}\partial_i\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^{\beta}) \tag{4.72}$$

Remark
If the position dependent mass or the Laplace Beltrami quantization are considered then the operators $\hat{H}^N_{PDM\beta}$, $\hat{H}^N_{LB\beta}$ cannot be reduced through the algebraic manipulations introduced in the first chapter to the differential operators which define the classical orthogonal polynomials. We can overcome this problem if we consider the same Hamiltonian operators with a quantum potential correction:

$$\hat{H}^N_{gPDM\beta} = -\frac{1}{2m(r)}\left(\partial_i^2 - \frac{m'(r)}{m(r)}\frac{x_i}{r}\partial_i\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^{\beta}) + \tag{4.73}$$

$$-\frac{1}{2}\left(\frac{3}{4}\frac{m(r)'^2}{m(r)^3} - \frac{m(r)''}{2m(r)^2} + \frac{(1-N)m(r)'}{2rm(r)^2}\right)$$

$$\hat{H}^N_{gLB\beta} = -\frac{1}{2m(r)}\left(\partial_i^2 + \frac{N-2}{2}\frac{m'(r)}{m(r)}\frac{x_i}{r}\partial_i\right) - \frac{\mu}{2}(r^{-\beta} + k^2 r^{\beta}) + \tag{4.74}$$

$$+\frac{N-2}{8(N-1)}R(r)$$

provided that the scalar curvature $R(r)$ of the system turns out to be:

$$R(r) = (1-N)\left(\frac{m(r)''}{m(r)} + \frac{N-6}{4}\frac{m(r)'^2}{m(r)^3} + \frac{m(r)'}{m(r)^2}\frac{N-1}{r}\right)$$

As pointed out in the third chapter the systems $\hat{H}^N_{gPDM\beta}$ and $\hat{H}^N_{gLB\beta}$ can be transformed in the system $\hat{H}^N_{sch\beta}$ through a similarity transformation:

$$\hat{H}^N_{gPDM\beta} = \sqrt{m(r)}\hat{H}^N_{sch\beta}\frac{1}{\sqrt{m(r)}}, \quad \Phi^N_{PDM\beta}(r,\theta_j)_{n,l} = \sqrt{m(r)}\Phi^N_{sch\beta}(r,\theta_j)_{n,l}$$

$$\hat{H}^N_{gLB\beta} = m(r)^{\frac{2-N}{4}}\hat{H}^N_{sch\beta}m(r)^{\frac{N-2}{4}}, \quad \Phi^N_{PDM\beta}(r,\theta_j)_{n,l} = m(r)^{\frac{2-N}{4}}\Phi^N_{sch\beta}(r,\theta_j)_{n,l}$$

this transformation keeps the spectrum itself:

$$\begin{aligned}
\hat{H}^N_{sch\beta}\Phi^N_{sch\beta}(r,\theta_j)_{n,l} &= E_{n,l}\Phi^N_{sch\beta}(r,\theta_j)_{n,l} \\
\left(\sqrt{m(r)}\hat{H}^N_{sch\beta}\frac{1}{\sqrt{m(r)}}\right)\left(\sqrt{m(r)}\Phi^N_{sch\beta}(r,\theta_j)_{n,l}\right) &= E_{n,l}\left(\sqrt{m(r)}\Phi^N_{sch\beta}(r,\theta_j)_{n,l}\right) \\
\hat{H}^N_{gPDM\beta}\Phi^N_{PDM\beta}(r,\theta_j)_{n,l} &= E_{n,l}\Phi^N_{PDM\beta}(r,\theta_j)_{n,l}
\end{aligned}$$

$$\begin{aligned}
\left(m(r)^{\frac{2-N}{4}}\hat{H}^N_{sch\beta}m(r)^{\frac{N-2}{4}}\right)\left(m(r)^{\frac{2-N}{4}}\phi^N_{sch\beta}(r,\theta_j)_{n,l}\right) &= E_{n,l}\left(m(r)^{\frac{2-N}{4}}\phi^N_{sch\beta}(r,\theta_j)_{n,l}\right) \\
\hat{H}^N_{gLB\beta}\Phi^N_{LB\beta}(r,\theta_j)_{n,l} &= E_{n,l}\Phi^N_{LB\beta}(r,\theta_j)_{n,l}
\end{aligned}$$

These considerations make both $\hat{H}^N_{gPDM\beta}$ and $\hat{H}^N_{gLB\beta}$ two exactly solvable quantizations of the Perlick system I, in the two different contexts of position dependent mass and curved space.

4.5 Quantum Bertrand system of type II

In the second chapter we classified the systems belonging to the second Bertrand family as the Stackel or coupling constant metamorphosis of the systems belonging to the first family; moreover in the first chapter we showed how, by applying the coupling constant metamorphosis to the exactly solvable quantum systems, one could generate new quantum systems with the same integrability properties as the initial one. Since we have an exact solution for the quantum Bertrand systems of type I then we have all the ingredients to generate the exactly solvable quantization of the Bertrand system II. To begin with let us define $\Phi^N_{n,l,\mu} = \Phi^N_{sch\beta}$, so that it turns out to be solution of the eigenvalue problem:

$$\left(-\frac{1}{2\beta^2 m(r)}\nabla^2_N - \mu\left(\frac{r^{-\beta}}{2} + \frac{k^2 r^\beta}{2} + G\right) + \alpha\right)\Phi^N_{n,l,\mu} = E_{\nu,\mu}\Phi^N_{n,l,\mu} \qquad (4.75)$$

$$m(r) = \frac{4}{r^2(r^{-\beta} - k^2 r^\beta)^2}, \quad \nabla^2_N = \sum_i \partial_i^2, \quad E_{\nu,\mu} = \frac{-\mu^2}{2\nu^2} - \frac{k^2}{2}\nu^2 + \frac{k^2}{8} - \mu G + \alpha,$$

$$\nu = n + \frac{l}{\beta} + \frac{a(N-2)+1}{2}$$

Namely we get the eigenvalue equation for the operator (4.64) where we have added the two constant terms μG and α. Following the coupling constant metamorphosis let us recast the differential equation (4.75) as follows:

$$\begin{aligned}\left(-\frac{r^2(r^{-\beta}-k^2r^{\beta})^2}{4\beta^2(r^{-\beta}+k^2r^{\beta}+2G)}\nabla_N^2+\frac{2\alpha}{r^{-\beta}+k^2r^{\beta}+2G}\right)\Phi^N_{n,l,\mu} &= \\ =\left(\frac{2E_{\nu,\mu}}{r^{-\beta}+k^2r^{\beta}+2G}+\mu\right)\Phi^N_{n,l,\mu}\end{aligned} \tag{4.76}$$

As showed in the first chapter when dealing with the coupling constant metamorphosis we can consider μ a parameter that can be turned into a function of the quantum number ν, so let us solve the equation in the variable μ:

$$E_{\nu,\mu}=0, \rightarrow \mu(k^2,\alpha,G)_\nu = -G\nu^2 \pm \sqrt{(G^2-k^2)\nu^4+(\frac{k^2}{4}+2\alpha)\nu^2} \equiv \tilde{E}_\nu \tag{4.77}$$

This defines the spectrum of the new Hamiltonian $\hat{H}_{II}$:

$$\hat{H}_{II}\Phi^N_{n,l,\mu} = \left(-\frac{r^2(r^{-\beta}-k^2r^{\beta})^2}{4\beta^2(r^{-\beta}+k^2r^{\beta}+2G)}\nabla_N^2+\frac{2\alpha}{r^{-\beta}+k^2r^{\beta}+2G}\right)\Phi^N_{n,l,\mu} = \tilde{E}_\nu\Phi^N_{n,l,\mu}. \tag{4.78}$$

Because of the many parameters α, G, β, k the system turns out to be very general. The choice of the right solution in the equation (4.77) depends on the parameters and must be determined so that the eigenfunctions $\Phi^N_{n,l,\mu}$ be L^2 namely normalizable; a deeper analysis on the space of these parameters is still in progress. However this doesn't change the main result of the thesis, namely the fundamental fact that because of their maximal superintegrability,the classical Bertrand systems have associated a quantum hamiltonian whose eigenfunctions can be described in terms of the classical orthogonal polynomials and its spectrum shows the so called accidental degeneracy; indeed the accidental degeneracy is still present in the family II since the spectrum $\tilde{E}_\nu$ depends just on the quantum number ν and for $\beta \in \mathbb{Q}$ we can have different values of n and l which produce the same ν as showed explicitly in the previous section.
Moreover we point out that $\hat{H}_{II}$ turns out to be exactly the direct or Schroedinger quantization of the classical system (2.121) once one makes the following replacements:

$$K = k^2, \quad \alpha \rightarrow -2\alpha\beta^2, \quad H \rightarrow \frac{H}{2\beta^2}, \quad G \rightarrow -G$$

Therefore analogously to what happens for the systems of family I, the exactly solvable quantization turns out to be the direct quantization and by similarity transformation we can get the position dependent mass or the geometrical quantization by adding some quantum potentials which correspond respectively to the Levy Leblond potential type [72] and a function proportional to the scalar curvature of the system.

Chapter 5

Quantum Darboux III system

5.1 Darboux III quantum system

The present thesis has been devoted to the classification of quantum radial systems whose eigenfunctions and its spectrum can be obtained by algebraic methods. This classification consists of two multiparametric families which we have named as Perlick I (Kepler Type) and Perlick II (Oscillator Type). Now for the sake of concreteness let us analyze in detail a particular case of this family, namely the subcase of the family II when $K = 0$ and $\beta = 2$: This system was introduced for the first time in [86] as a particular case of the so called 3-dimensional multifold Kepler systems introduced in the second chapter, and then deeply analyzed by our group in a series of papers [88, 39, 101], in particular the content of this chapter is based on [100]. This system presents many peculiar characteristics of the M.S. quantum systems, other than which we have already mentioned like a degenerate spectrum and the possibility of solving it in a N dimensional space. Namely we can solve it in at least two different coordinate systems, and furthermore we can write explicitly the $2N - 1$ constants of the motion that make this systems a quantum M.S. system defined on a manifold with non-constant scalar curvature and whose particular case $N = 2$ correspond to the Darboux system of type III. The N-dimensional generalization of the Darboux III system is defined as the Riemannian metric space:

$$g_{ij} = (1 + \lambda \mathbf{q}^2)\delta_{ij}; \quad i, j = 0, ...N \tag{5.1}$$

whose scalar curvature turns out to be:

$$R(\mathbf{q}) = -\lambda \frac{(N-1)(2N + 3(N-2)\lambda \mathbf{q}^2)}{(1 + \lambda \mathbf{q}^2)^3} \tag{5.2}$$

Following the assumptions $K = 0$ and $\beta = 2$ for the general Bertrand II system the M.S. classical Hamiltonian associated to the metric space (5.1) turns out to be:

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \frac{\mathbf{p}^2 + \omega^2 \mathbf{q}^2}{2(1 + \lambda \mathbf{q}^2)} \tag{5.3}$$

Theorem 1

(i) The Hamiltonian $\mathcal{H}$ (5.3) is endowed with the following constants of the motion.

• $(2N-3)$ angular momentum integrals:

$$\mathcal{C}^m = \sum_{1\le i<j\le m} (q_i p_j - q_j p_i)^2, \quad \mathcal{C}_{(m)} = \sum_{N-m\le i<j\le m} (q_i p_j - q_j p_i)^2, \tag{5.4}$$

where $m = 2, ...N$ and $\mathcal{C}^{(N)} = \mathcal{C}_{(N)}$.

• N^2 integrals which form the ND curved Fradkin tensor [53] :

$$I_{ij} = p_i p_j - (2\lambda\mathcal{H}(\mathbf{q},\mathbf{p}) - \omega^2) q_i q_j, \tag{5.5}$$

where $i, j = 1, ..., N$ and such that $\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} I_{ii}$.

(ii) Each of the three sets $\{\mathcal{H}, \mathcal{C}^{(m)}\}, \{\mathcal{H}, \mathcal{C}_{(m)}\}(m = 2, ..., N)$ and $\{I_{ii}\}(i = 1, ..., N)$ is formed by N functionally indipendent functions in involution.

(iii) The set $\{\mathcal{H}, \mathcal{C}^{(m)}, \mathcal{C}_{(m)}, I_{ii}\}$ for$m = 2, ..., N$ with a fixed index i is constituted by $2N-1$ functionally independent functions.

Notice that the first set of $2n-3$ integrals (5.4) is the same for any central potential on any spherically symmetric space [87] since it is provided by an underlying $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry (also by an $\mathfrak{s}O(N)$ symmetry), while the second one (5.5) comes from the specific oscillator potential that we consider here. The latter, in fact, correspond to a curved analog of the Fradkin tensor of integrals of motion [53] for the isotropic harmonic oscillator. We also recall that the Hamiltonian (5.3) together with both sets of integrals of (5.4) and (5.5) can alternatively be obtained [88] from the free Euclidean motion by means of a Stackel transform or coupling constant metamorphosis as seen in the Thesis and explicitly in regard to the Darboux III system in [21] - [23](and references therein). Thus ,in general, the latter integrals (5.5 do not exist for a generic central potential so that, in principle, the M.S. property is not ensured at all. From this view point the ND nonlinear oscillator Hamiltonian $\mathcal{H}$ (5.3) can be regarded as the "closest neighbour" (with nonconstant curvature) to the isotropic harmonic oscillator system with ($\lambda = 0$) as both share the same M.S. property. In fact, the real parameter λ behaves as a "deformation" parameter governing the nonlinear behaviour of $\mathcal{H}$, and this parameter is deeply related to the variable curvature of the underling Darboux space.

5.1.1 Expressions in terms of hyperspherical coordinates in phase space

The above results can also be expressed in terms of hyperspherical coordinates r, θ_j, and canonical momenta p_r, p_{θ_j}, $(j = 1, \ldots, N-1)$. The N hyperspherical coordinates are formed by a radial-type one $r = |\mathbf{q}| \in \mathbb{R}^+$ and $N-1$ angles θ_j such that $\theta_k \in [0, 2\pi)$ for $k < N-1$ and $\theta_{N-1} \in [0, \pi)$. These are defined by

$$q_j = r\cos\theta_j \prod_{k=1}^{j-1} \sin\theta_k, \quad 1 \le j < N, \qquad q_N = r\prod_{k=1}^{N-1}\sin\theta_k, \tag{5.6}$$

where hereafter any product $\prod_l^m$ such that $l > m$ is assumed to be equal to 1. The metric (5.1) now adopts the form

$$\mathrm{d}s^2 = (1 + \lambda r^2)(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \tag{5.7}$$

where $\mathrm{d}\Omega^2$ is the metric on the unit $(N-1)$D sphere $\mathbb{S}^{N-1}$

$$\mathrm{d}\Omega^2 = \sum_{j=1}^{N-1} \mathrm{d}\theta_j^2 \prod_{k=1}^{j-1} \sin^2\theta_k.$$

The relations between $\mathbf{p}$ and p_r, p_{θ_j} read $(1 \le j < N)$ [87]:

$$\begin{aligned} p_j &= \prod_{k=1}^{j-1} \sin\theta_k \cos\theta_j \, p_r + \frac{\cos\theta_j}{r} \sum_{l=1}^{j-1} \frac{\prod_{k=l+1}^{j-1}\sin\theta_k}{\prod_{m=1}^{l-1}\sin\theta_m} \cos\theta_l \, p_{\theta_l} - \frac{\sin\theta_j}{r\prod_{k=1}^{j-1}\sin\theta_k}\, p_{\theta_j}, \\ p_N &= \prod_{k=1}^{N-1} \sin\theta_k \, p_r + \frac{1}{r} \sum_{l=1}^{N-1} \frac{\prod_{k=l+1}^{N-1}\sin\theta_k}{\prod_{m=1}^{l-1}\sin\theta_m} \cos\theta_l \, p_{\theta_l}, \end{aligned} \tag{5.8}$$

where from now on any sum $\sum_l^m$ such that $l > m$ is assumed to be zero. From (5.8) we obtain that

$$\mathbf{p}^2 = p_r^2 + r^{-2}\mathbf{L}^2, \tag{5.9}$$

where $\mathbf{L}^2$ is the total angular momentum given by

$$\mathbf{L}^2 = \sum_{j=1}^{N-1} p_{\theta_j}^2 \prod_{k=1}^{j-1} \frac{1}{\sin^2\theta_k}. \tag{5.10}$$

By introducing (5.6) and (5.8) in the Hamiltonian (5.3) we find

$$\mathcal{H}(r, p_r) = \frac{p_r^2 + r^{-2}\mathbf{L}^2}{2(1+\lambda r^2)} + \frac{\omega^2 r^2}{2(1+\lambda r^2)} = \mathcal{T}(r, p_r) + \mathcal{U}(r). \tag{5.11}$$

The integrals of motion $C_{(m)}$ (5.4) adopt a compact form (the remaining $C^{(m)}$ and I_{ij} have more cumbersome expressions):

$$C_{(m)} = \sum_{j=N-m+1}^{N-1} p_{\theta_j}^2 \prod_{k=N-m+1}^{j-1} \frac{1}{\sin^2\theta_k}, \quad m = 2, \ldots, N;$$

and $C_{(N)} = \mathbf{L}^2$, which is just the second-order Casimir of the $\mathfrak{so}(N)$-symmetry algebra of a central potential.

Furthermore, the complete integrability determined by the set of N functions $\{\mathcal{H}, C_{(m)}\}$ $(m = 2, \ldots, N)$ leads to a separable set of N equations, since each of them depends on a unique pair of canonical variables. These are the $N-1$ angular equations

$$\begin{aligned} &C_{(2)}(\theta_{N-1}, p_{\theta_{N-1}}) = p^2_{\theta_{N-1}}, \\ &C_{(k)}(\theta_{N-k+1}, p_{\theta_{N-k+1}}) = p^2_{\theta_{N-k+1}} + \frac{C_{(k-1)}}{\sin^2\theta_{N-k+1}}, \qquad k = 3, \ldots, N-1, \\ &C_{(N)}(\theta_1, p_{\theta_1}) = p^2_{\theta_1} + \frac{C_{(N-1)}}{\sin^2\theta_1} \equiv \mathbf{L}^2, \end{aligned} \tag{5.12}$$

together with the single radial equation corresponding to the 1D Hamiltonian (5.11).

5.2 The Darboux space and the classical effective potential

The underlying manifold of the classical Hamiltonian (5.3) is the ND Darboux space with metric (5.1), whose kinetic energy corresponds to the geodesic motion on the complete Riemannian manifold $\mathcal{M}^N = (\mathbb{R}^N, g)$ with

$$g_{ij} := (1 + \lambda\mathbf{q}^2)\,\delta_{ij}, \tag{5.13}$$

and provided that $\lambda > 0$. The scalar curvature $R(r) \equiv R(|\mathbf{q}|)$ (5.2) coming from this metric is always a *negative increasing* function such that $\lim_{r\to\infty} R = 0$ and it has a minimum at the origin

$$R(0) = -2\lambda N(N-1),$$

which is exactly the scalar curvature of the ND *hyperbolic space* with negative constant sectional curvature equal to -2λ (see figure 5.1). Recall that the four Darboux surfaces are the only 2D spaces of nonconstant curvature whose geodesic motion is (quadratically) MS, therefore they are the "closest" ones to the classical Riemannian spaces of constant curvature [26, 33].
As far as the nonlinear radial oscillator potential $\mathcal{U}(r)$ (5.11) is concerned, we find that it is a *positive increasing* function of r, such that

$$\mathcal{U}(r) = \frac{\omega^2 r^2}{2(1+\lambda r^2)}, \qquad \mathcal{U}(0) = 0, \qquad \lim_{r\to\infty} \mathcal{U}(r) = \frac{\omega^2}{2\lambda}. \tag{5.14}$$

This potential is shown in figure 5.2 for several values of λ. Consequently, in contrast with the (Euclidean) isotropic harmonic oscillator, $\mathcal{U}(r)$ yields a nonlinear behavior governed by λ, which means that the oscillator potential has the asymptotic maximum $\omega^2/(2\lambda)$.
Nevertheless, since the underlying manifold $\mathcal{M}^N$ is not flat, the interplay between the oscillator potential $\mathcal{U}(r)$ and the kinetic energy term is rather subtle. For this reason, the complete classical system can be better understood by introducing a

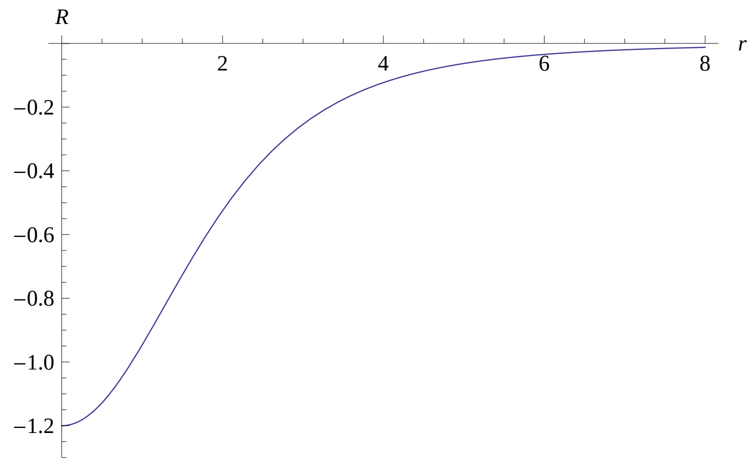

Figure 5.1: Shape of the scalar curvature (5.2) of the Darboux space where $r = |\mathbf{q}|$ for $N = 3$ and $\lambda = 0.1$. The minimum is always located at the origin, and its value in this case is $R(0) = -1.2$.

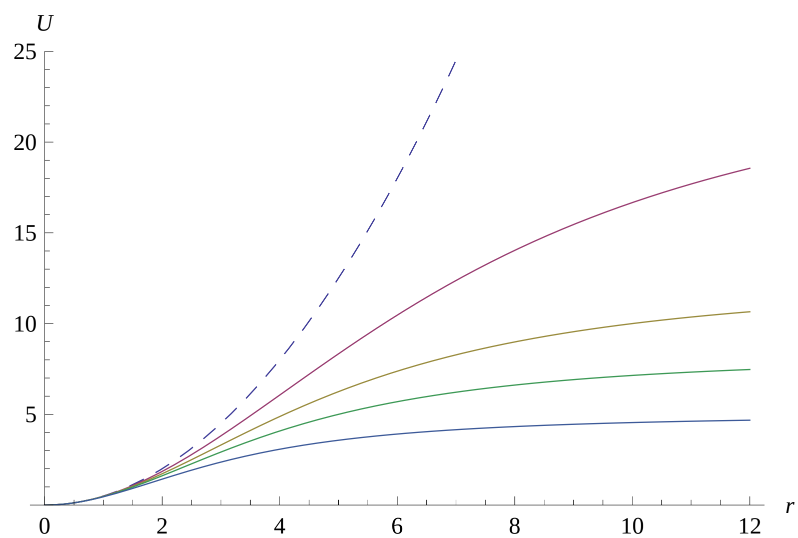

Figure 5.2: The nonlinear oscillator potential (5.14) with $\omega = 1$ for $\lambda = \{0, 0.02, 0.04, 0.06, 0.1\}$ starting from the upper dashed line corresponding to the isotropic harmonic oscillator with $\lambda = 0$. The limit $r \to \infty$ gives $\{+\infty, 25, 12.5, 8.33, 5\}$, respectively.

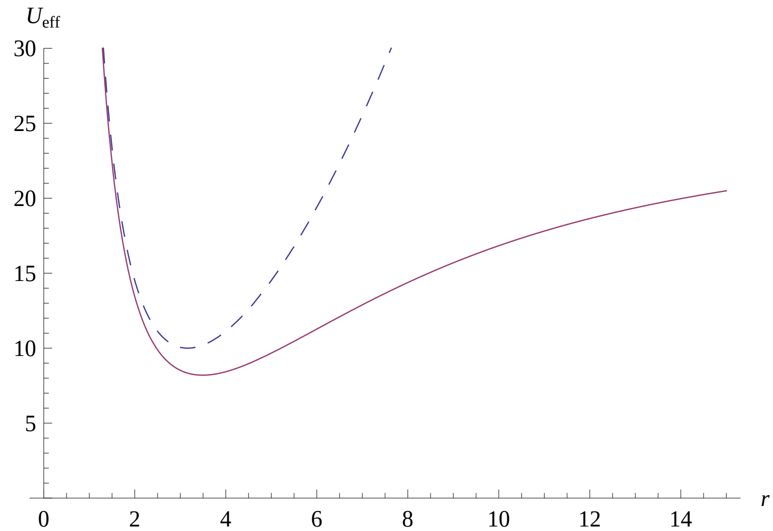

Figure 5.3: The classical effective nonlinear oscillator potential (5.16) for $\lambda = 0.02$, $c_N = 100$ and $\omega = 1$. The minimum of the potential is located at $r_{\min} = 3.49$ with $\mathcal{U}_{\rm eff}(r_{\min}) = 8.2$ and $\mathcal{U}_{\rm eff}(\infty) = 25$. The dashed line corresponds to the effective potential of the harmonic oscillator with $\lambda = 0$ with minimum $\mathcal{U}_{\rm eff}(r_{\min}) = 10$ at $r_{\min} = 3.16$.

classical effective potential. This can be achieved by applying the 1D canonical transformation defined by

$$P(r, p_r) = \frac{p_r}{\sqrt{1+\lambda r^2}}, \qquad Q(r) = \frac{1}{2} r\sqrt{1+\lambda r^2} + \frac{\operatorname{arcsinh}(\sqrt{\lambda} r)}{2\sqrt{\lambda}}, \tag{5.15}$$

(where the new canonical variables fulfill $\{Q, P\} = 1$), to the radial Hamiltonian (5.11). Notice that $Q(r)$ has a unique (continuously differentiable) inverse $r(Q)$, on the whole positive semiline, that is, both $r, Q \in [0, \infty)$ and $\mathrm{d}Q(r) = \sqrt{1+\lambda r^2}\mathrm{d}r$. In this way, we obtain that

$$\mathcal{H}(Q, P) = \frac{1}{2}P^2 + \mathcal{U}_{\rm eff}(Q), \qquad \mathcal{U}_{\rm eff}(Q(r)) = \frac{c_N}{2(1+\lambda r^2)r^2} + \frac{\omega^2 r^2}{2(1+\lambda r^2)}, \tag{5.16}$$

where the constant $c_N \geq 0$ is the value of the integral of motion corresponding to the square of the total angular momentum $C_{(N)} \equiv \mathbf{L}^2$ (5.12). Hence the classical system can be described as a particle on a 1D flat space under the effective potential $\mathcal{U}_{\rm eff}(Q(r))$, which is represented in figure 5.3.

The analysis of $\mathcal{U}_{\rm eff}$ shows that this is always positive and it has a minimum located at $r_{\min}$ such that

$$r^2_{\min} = \frac{\lambda c_N + \sqrt{\lambda^2 c_N^2 + \omega^2 c_N}}{\omega^2}, \qquad \mathcal{U}_{\rm eff}(Q(r_{\min})) = -\lambda c_N + \sqrt{\lambda^2 c_N^2 + \omega^2 c_N}. \tag{5.17}$$

Therefore, $r_{\min}$ and $\mathcal{U}_{\text{eff}}(Q(r_{\min}))$ are, in this order, greater and smaller than those corresponding to the isotropic harmonic oscillator:

$$\lambda = 0: \quad r_{\min}^2 = \frac{\sqrt{c_N}}{\omega}, \qquad \mathcal{U}_{\text{eff}}(Q(r_{\min})) = \omega\sqrt{c_N}. \tag{5.18}$$

Moreover $\mathcal{U}_{\text{eff}}$ has two representative limits:

$$\lim_{r\to 0} \mathcal{U}_{\text{eff}}(Q(r)) = +\infty, \qquad \lim_{r\to\infty} \mathcal{U}_{\text{eff}}(Q(r)) = \frac{\omega^2}{2\lambda}, \tag{5.19}$$

the latter being the same of (5.14). Thus, this effective potential is hydrogen-like and one should expect that its quantum counterpart should have both bounded and unbounded states. Now we are ready to solve such a quantum problem in full detail.

5.3 Superintegrable quantizations of the Darboux III oscillator

Let us consider the quantum position and momenta operators, $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$, with Lie brackets and differential representation given by

$$[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}, \qquad \hat{q}_i = q_i, \qquad \hat{p}_i = -\mathrm{i}\hbar\frac{\partial}{\partial q_i}. \tag{5.20}$$

Hereafter we will use the standard notation

$$\nabla = \left(\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_N}\right), \qquad \Delta = \nabla^2 = \frac{\partial^2}{\partial^2 q_1} + \dots + \frac{\partial^2}{\partial^2 q_N}.$$

Let us consider the "direct" quantization approach intruduced in the third chapter in regard to the classical Hamiltonian 5.1

Theorem 4. *Let $\hat{\mathcal{H}}$ be the quantum Hamiltonian given by*

$$\hat{\mathcal{H}} = \frac{1}{2(1+\lambda\hat{\mathbf{q}}^2)}\,\hat{\mathbf{p}}^2 + \frac{\omega^2\hat{\mathbf{q}}^2}{2(1+\lambda\hat{\mathbf{q}}^2)} = \frac{1}{2(1+\lambda\mathbf{q}^2)}\big(-\hbar^2\Delta + \omega^2\mathbf{q}^2\big). \tag{5.21}$$

Then:
(i) $\hat{\mathcal{H}}$ commutes with the following observables:
• *The $(2N-3)$ quantum angular momentum operators,*

$$\hat{C}^{(m)} = \sum_{1\le i<j\le m} (\hat{q}_i\hat{p}_j - \hat{q}_j\hat{p}_i)^2, \qquad \hat{C}_{(m)} = \sum_{N-m<i<j\le N} (\hat{q}_i\hat{p}_j - \hat{q}_j\hat{p}_i)^2, \tag{5.22}$$

where $m = 2, \dots, N$ and $\hat{C}^{(N)} = \hat{C}_{(N)}$.
• *The N^2 operators defining the ND quantum Fradkin tensor, given by*

$$\hat{I}_{ij} = \hat{p}_i\hat{p}_j - 2\lambda\hat{q}_i\hat{q}_j\mathcal{H}(\hat{\mathbf{q}}, \hat{\mathbf{p}}) + \omega^2\hat{q}_i\hat{q}_j, \tag{5.23}$$

where $i,j=1,\dots,N$ and such that $\hat{\mathcal{H}}=\frac{1}{2}\sum_{i=1}^{N}\hat{I}_{ii}$.
(ii) Each of the three sets $\{\hat{\mathcal{H}},\hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}},\hat{C}_{(m)}\}$ ($m=2,\dots,N$) and $\{\hat{I}_{ii}\}$ ($i=1,\dots,N$) is formed by N algebraically independent commuting observables.
(iii) The set $\{\hat{\mathcal{H}},\hat{C}^{(m)},\hat{C}_{(m)},\hat{I}_{ii}\}$ for $m=2,\dots,N$ with a fixed index i is formed by $2N-1$ algebraically independent observables.
(iv) $\hat{\mathcal{H}}$ is formally self-adjoint on the L^2 Hilbert space defined by the scalar product

$$\langle\Psi|\Phi\rangle=\int_{\mathcal{M}^N}\overline{\Psi(\mathbf{q})}\Phi(\mathbf{q})(1+\lambda\mathbf{q}^2)\mathrm{d}\mathbf{q}. \tag{5.24}$$

Proof. Some points of this statement can be straightforwardly proven through the coalgebra symmetry [89, 90, 91] of the quantum Hamiltonian (5.21). Let us consider the $\mathfrak{sl}(2,\mathbb{R})$ Lie coalgebra in the basis $\{J_\pm,J_3\}$ with commutation rules, Casimir invariant and (nondeformed) coproduct given by

$$[J_3,J_+]=2\mathrm{i}\hbar J_+,\qquad [J_3,J_-]=-2\mathrm{i}\hbar J_-,\qquad [J_-,J_+]=4\mathrm{i}\hbar J_3, \tag{5.25}$$

$$\mathcal{C}=\frac{1}{2}(J_+J_-+J_-J_+)-J_3^2, \tag{5.26}$$

$$\Delta(J_l)=J_l\otimes 1+1\otimes J_l,\qquad l=+,-,3. \tag{5.27}$$

An N-particle realization of $\mathfrak{sl}(2,\mathbb{R})$ reads

$$J_+=\hat{\mathbf{p}}^2,\qquad J_-=\hat{\mathbf{q}}^2,\qquad J_3=\frac{1}{2}(\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})=\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}-\frac{1}{2}\mathrm{i}\hbar N. \tag{5.28}$$

Therefore, $\hat{\mathcal{H}}$ (5.21) has an $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry since it can be written as

$$\hat{\mathcal{H}}=\frac{1}{2(1+\lambda J_-)}\,J_++\frac{\omega^2 J_-}{2(1+\lambda J_-)}. \tag{5.29}$$

Hence, by construction, $\hat{\mathcal{H}}$ commutes with the $(2N-3)$ observables $\hat{C}^{(m)}$ and $\hat{C}_{(m)}$ ($m=2,\dots,N$) (5.22) which come from the "left" and "right" m-th coproducts [90, 91] of the invariant (5.26), respectively, up to an additive constant $\hbar^2 m(m-4)/4$. Furthermore, the coalgebra approach also ensures that these are algebraically independent and that each set $\{\hat{\mathcal{H}},\hat{C}^{(m)}\}$ and $\{\hat{\mathcal{H}},\hat{C}_{(m)}\}$ is formed by N commuting observables (to be more precise, they are polynomially independent as operators in a Jordan algebra).

Next, by direct computations it can be proven that the N^2 observables $\hat{I}_{ij}$ (5.23) commute with $\hat{\mathcal{H}}$, and that the N (diagonal) observables $\hat{I}_{ii}$ ($i=1,\dots,N$) commute amongst themselves as well; it is obvious that the latter $\hat{I}_{ii}$ are algebraically independent. Finally, it is also clear that any single $\hat{I}_{ii}$ is algebraically independent with respect to the set of $2N-2$ observables $\{\hat{\mathcal{H}},\hat{C}^{(m)},\hat{C}_{(m)}\}$ (as it is when $\lambda=0$) $\square$.

We stress that, as a byproduct of the above proof, any quantum Hamiltonian defined as a function of (5.28),

$$\hat{\mathcal{H}}=\hat{\mathcal{H}}(J_+,J_-,J_3)=\hat{\mathcal{H}}(\hat{\mathbf{p}}^2,\hat{\mathbf{q}}^2,\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}-\mathrm{i}\hbar N/2), \tag{5.30}$$

is endowed with the same $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry. This shows that this is quasi-MS [87, 90, 91], that is, it commutes, at least, with the $(2N-3)$ observables $\hat{C}^{(m)}$ and $\hat{C}_{(m)}$. In this respect, we remark that what makes the quantum Darboux III oscillator (5.29) very special, is the existence of a quantum Fradkin tensor formed by the "additional" symmetries $\hat{I}_{ij}$. This algebraic property implies that the system is MS and, as we shall see, that its discrete energy spectrum is maximally degenerate.

5.3.1 The Laplace–Beltrami quantization

Let us consider now the Laplace Beltrami quantization as introduced in chapter three:

$$\hat{\mathcal{T}}_{\mathrm{LB}}(\hat{\mathbf{q}},\hat{\mathbf{p}}) = -\frac{\hbar^2}{2}\Delta_{\mathrm{LB}}, \quad \Delta_{\mathrm{LB}} = \sum_{i,j=1}^{N} \frac{1}{\sqrt{g}}\partial_i \sqrt{g} g^{ij}\partial_j,$$

If we apply such LB quantization to the Hamiltonian (5.3) with metric tensor (5.1) we get

$$\begin{aligned}\hat{\mathcal{H}}_{\mathrm{LB}} &= -\frac{\hbar^2}{2}\Delta_{\mathrm{LB}} + \frac{\omega^2\mathbf{q}^2}{2(1+\lambda\mathbf{q}^2)} \\ &= -\frac{\hbar^2}{2(1+\lambda\mathbf{q}^2)}\Delta - \frac{\hbar^2\lambda(N-2)}{2(1+\lambda\mathbf{q}^2)^2}(\mathbf{q}\cdot\nabla) + \frac{\omega^2\mathbf{q}^2}{2(1+\lambda\mathbf{q}^2)}.\end{aligned}$$

Then, $\hat{\mathcal{H}}$ (5.21) and $\hat{\mathcal{H}}_{\mathrm{LB}}$ only coincide in the case $N=2$ (as it should be for any spherically symmetric space [23]) and for $N>2$ they differ by a momentum-dependent potential, namely:

$$\hat{\mathcal{H}}_{\mathrm{LB}} = \hat{\mathcal{H}} + \mathcal{U}_1, \qquad \mathcal{U}_1(\hat{\mathbf{q}},\hat{\mathbf{p}}) = -\mathrm{i}\frac{\hbar\lambda(N-2)}{2(1+\lambda\hat{\mathbf{q}}^2)^2}(\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}),$$

where we have introduced the quantum variables (5.20). Notice that $\mathcal{U}_1$ is linear in $\hbar$, so this term does not have any classical analog. This situation reminds what happens in the context of the so-called *quasi-exactly solvable* quantum models [92]. On the other hand, although the Hamiltonian $\hat{\mathcal{H}}_{\mathrm{LB}}$ commutes with the operators (5.22) (the quantum correction $\mathcal{U}_1$ preserves the $\mathfrak{sl}(2,\mathbb{R})$ coalgebra symmetry (5.25)–(5.28)), in this case there is no hint about the existence of an additional symmetry of the type (5.23).
Nevertheless, it is possible to find a "superintegrable" LB quantization (in the sense that it does preserve the MS property) by adding a second potential term to $\hat{\mathcal{H}}$ (besides $\mathcal{U}_1$) which makes the Laplace Beltrami Hamiltonian similarity equivalent to the Hamiltonian $\hat{\mathcal{H}}$, thus conveying N^2 additional integrals of the type (5.23) together with the separability property in terms of the N "diagonal" ones. If we define:

$$\hat{\mathcal{H}}_{\mathrm{TLB}} = \hat{\mathcal{H}}_{\mathrm{LB}} + \mathcal{U}_2 = \hat{\mathcal{H}} + \mathcal{U}_1 + \mathcal{U}_2, \tag{5.31}$$

$$\mathcal{U}_2(\mathbf{q}) = -\frac{\hbar^2\lambda(N-2)}{8(1+\lambda\mathbf{q}^2)^3}\left(2N+3\lambda\mathbf{q}^2(N-2)\right) = \frac{\hbar^2(N-2)}{8(N-1)}\,R(\mathbf{q}).$$

then this satisfies:

$$\hat{\mathcal{H}}_{\mathrm{TLB}} = \mathrm{e}^{f}\hat{\mathcal{H}}\mathrm{e}^{-f}. \tag{5.32}$$

where $f(\mathbf{q}) = \frac{2-N}{4}\ln(1+\lambda\mathbf{q}^2)$.

Hence, as a direct consequence, all the symmetries of $\hat{\mathcal{H}}$ give rise to those corresponding to $\hat{\mathcal{H}}_{\mathrm{TLB}}$:

$$\hat{X}_{\mathrm{TLB}} = \mathrm{e}^{f}\hat{X}\mathrm{e}^{-f}, \qquad \hat{X} = \{\hat{C}^{(m)}, \hat{C}_{(m)}, \hat{I}_{ij}\}, \qquad [\hat{\mathcal{H}}_{\mathrm{TLB}}, \hat{X}_{\mathrm{TLB}}] = 0. \tag{5.33}$$

Therefore, by taking into account Theorem 2 and the equations (5.32) and (5.33) we find that $\hat{\mathcal{H}}_{\mathrm{TLB}}$ is, in fact, a quantum MS Hamiltonian.

Theorem 5. *Let $\hat{\mathcal{H}}_{\mathrm{TLB}}$ be the quantum Hamiltonian given by*

$$\begin{aligned}\hat{\mathcal{H}}_{\mathrm{TLB}} &= \frac{1}{2(1+\lambda\hat{\mathbf{q}}^2)}\,\hat{\mathbf{p}}^2 + \frac{\omega^2\hat{\mathbf{q}}^2}{2(1+\lambda\hat{\mathbf{q}}^2)} - \mathrm{i}\frac{\hbar\lambda(N-2)}{2(1+\lambda\hat{\mathbf{q}}^2)^2}(\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}) \\ &\quad - \frac{\hbar^2\lambda(N-2)}{8(1+\lambda\hat{\mathbf{q}}^2)^3}\left(2N+3\lambda\hat{\mathbf{q}}^2(N-2)\right) \\ &= -\frac{\hbar^2}{2}\Delta_{\mathrm{LB}} + \frac{\omega^2\mathbf{q}^2}{2(1+\lambda\mathbf{q}^2)} - \frac{\hbar^2\lambda(N-2)}{8(1+\lambda\mathbf{q}^2)^3}\left(2N+3\lambda\mathbf{q}^2(N-2)\right).\end{aligned} \tag{5.34}$$

Then:
(i) $\hat{\mathcal{H}}_{\mathrm{TLB}}$ commutes with the same observables (5.22), that is, $\hat{C}^{(m)}_{\mathrm{TLB}} = \hat{C}^{(m)}$ and $\hat{C}_{\mathrm{TLB},(m)} = \hat{C}_{(m)}$, as well as with the N^2 Fradkin operators given by

$$\begin{aligned}\hat{I}_{\mathrm{TLB},ij} &= \hat{p}_i\hat{p}_j - (N-2)\frac{\mathrm{i}\hbar\lambda}{2(1+\lambda\hat{\mathbf{q}}^2)}\left(\hat{q}_i\hat{p}_j + \hat{q}_j\hat{p}_i\right) + \frac{(N-2)\hbar^2\lambda^2\hat{q}_i\hat{q}_j}{(1+\lambda\hat{\mathbf{q}}^2)^2}\left(1-\frac{N-2}{4}\right) \\ &\quad - \frac{(N-2)\hbar^2\lambda}{2(1+\lambda\hat{\mathbf{q}}^2)}\,\delta_{ij} - 2\lambda\hat{q}_i\hat{q}_j\hat{\mathcal{H}}_{\mathrm{TLB}}(\hat{\mathbf{q}},\hat{\mathbf{p}}) + \omega^2\hat{q}_i\hat{q}_j,\end{aligned} \tag{5.35}$$

with $i,j = 1,\ldots,N$ and such that $\hat{\mathcal{H}}_{\mathrm{TLB}} = \frac{1}{2}\sum_{i=1}^{N}\hat{I}_{\mathrm{TLB},ii}$.
(ii) Each of the three sets $\{\hat{\mathcal{H}}_{\mathrm{TLB}}, \hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}_{\mathrm{TLB}}, \hat{C}_{(m)}\}$ $(m = 2,\ldots,N)$ and $\{\hat{I}_{\mathrm{TLB},ii}\}$ $(i = 1,\ldots,N)$ is formed by N algebraically independent commuting observables.
(iii) The set $\{\hat{\mathcal{H}}_{\mathrm{TLB}}, \hat{C}^{(m)}, \hat{C}_{(m)}, \hat{I}_{\mathrm{TLB},ii}\}$ for $m = 2,\ldots,N$ with a fixed index i is formed by $2N-1$ algebraically independent observables.
(iv) $\hat{\mathcal{H}}_{\mathrm{TLB}}$ is formally self-adjoint on the space $L^2(\mathcal{M}^N)$ associated with the underlying Darboux space, defined by

$$\langle\Psi|\Phi\rangle_{\mathrm{TLB}} = \int_{\mathcal{M}^N}\overline{\Psi(\mathbf{q})}\,\Phi(\mathbf{q})\,(1+\lambda\mathbf{q}^2)^{N/2}\,\mathrm{d}\mathbf{q}. \tag{5.36}$$

5.3.2 A position-dependent mass quantization

Let us going on the analysis considering the quantization of the classical Hamiltonian (5.3) regarding the system as a position dependent mass system:
The Kinetic energy term turns out to be:

$$\hat{\mathcal{T}}_{\rm PDM}(\hat{\mathbf{q}},\hat{\mathbf{p}}) = \frac{1}{2}\,\hat{\mathbf{p}}\cdot\frac{1}{(1+\lambda\hat{\mathbf{q}}^2)}\hat{\mathbf{p}} = -\frac{\hbar^2}{2}\nabla\cdot\frac{1}{(1+\lambda\mathbf{q}^2)}\nabla.$$

Then, by adding the oscillator potential and ordering terms in the kinetic term, we obtain the following PDM quantization of the Hamiltonian (5.3):

$$\begin{aligned}\hat{\mathcal{H}}_{\rm PDM} &= \hat{\mathcal{T}}_{\rm PDM}(\hat{\mathbf{q}},\hat{\mathbf{p}}) + \mathcal{U}(\hat{\mathbf{q}})\\ &= -\frac{\hbar^2}{2(1+\lambda\mathbf{q}^2)}\Delta + \frac{\hbar^2\lambda}{(1+\lambda\mathbf{q}^2)^2}\,(\mathbf{q}\cdot\nabla) + \frac{\omega^2\mathbf{q}^2}{2(1+\lambda\mathbf{q}^2)}.\end{aligned}$$

Similarly to the LB quantization, the MS property can be explicitly restored through a similarity transformation and this process will require to add another central potential to the initial $\hat{\mathcal{H}}_{\rm PDM}$.

Explicitly, if we apply $\hat{\mathcal{H}}_{\rm PDM}$ to the product $\exp(v(\mathbf{q}))\Psi(\mathbf{q})$ and define

$$v(\mathbf{q}) = \frac{1}{2}\ln(1+\lambda\mathbf{q}^2),$$

then we get the following similarity transformation between $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}_{\rm PDM}$:

$$\begin{aligned}&\hat{\mathcal{H}}_{\rm PDM}\mathrm{e}^{v}\Psi = \mathrm{e}^{v}(\hat{\mathcal{H}} - \mathcal{V}_2(\mathbf{q}))\Psi, \qquad \hat{\mathcal{H}}_{\rm PDM} = \mathrm{e}^{v}\hat{\mathcal{H}}\mathrm{e}^{-v} - \mathcal{V}_2,\\ &\mathcal{V}_2(\mathbf{q}) = \frac{\hbar^2\lambda}{2(1+\lambda\mathbf{q}^2)^3}\left(N + \lambda\mathbf{q}^2(N-3)\right).\end{aligned}$$

Hence, in contrast with the LB quantization, now both $v(\mathbf{q})$ and $\mathcal{V}_2(\mathbf{q})$ are nontrivial for any dimension N (including $N = 2$). In this way, we define the following "transformed-PDM" Hamiltonian,

$$\hat{\mathcal{H}}_{\rm TPDM} = \hat{\mathcal{H}}_{\rm PDM} + \mathcal{V}_2 = \hat{\mathcal{H}} + \mathcal{V}_1 + \mathcal{V}_2, \qquad \hat{\mathcal{H}}_{\rm TPDM} = \mathrm{e}^{v}\hat{\mathcal{H}}\mathrm{e}^{-v}, \tag{5.37}$$

whose symmetries are thus obtained from those of $\hat{\mathcal{H}}$ as

$$\hat{X}_{\rm TPDM} = \mathrm{e}^{v}\hat{X}\mathrm{e}^{-v}, \qquad \hat{X} = \{\hat{C}^{(m)}, \hat{C}_{(m)}, \hat{I}_{ij}\}, \qquad [\hat{\mathcal{H}}_{\rm TPDM}, \hat{X}_{\rm TPDM}] = 0. \tag{5.38}$$

The MS property of the Hamiltonian $\hat{H}_{\rm TPDM}$ is summarized in the following statement.

Theorem 6. *Let $\hat{\mathcal{H}}_{\rm TPDM}$ be the quantum Hamiltonian defined by*

$$\hat{\mathcal{H}}_{\rm TPDM} = \frac{1}{2(1+\lambda\hat{\mathbf{q}}^2)}\,\hat{\mathbf{p}}^2 + \frac{\omega^2\hat{\mathbf{q}}^2}{2(1+\lambda\hat{\mathbf{q}}^2)} + \frac{\mathrm{i}\hbar\lambda}{(1+\lambda\hat{\mathbf{q}}^2)^2}(\hat{\mathbf{q}}\cdot\hat{\mathbf{p}}) + \frac{\hbar^2\lambda\left(N + \lambda\mathbf{q}^2(N-3)\right)}{2(1+\lambda\mathbf{q}^2)^3}. \tag{5.39}$$

Then:
(i) $\hat{\mathcal{H}}_{\rm TPDM}$ *commutes with the observables (5.22) as well as with* $(i, j = 1, \dots, N)$

$$\hat{I}_{\mathrm{TPDM},ij} = \hat{p}_i\hat{p}_j + \frac{\mathrm{i}\hbar\lambda}{(1+\lambda\hat{\mathbf{q}}^2)}\,(\hat{q}_i\hat{p}_j + \hat{q}_j\hat{p}_i) + \frac{\hbar^2\lambda}{(1+\lambda\hat{\mathbf{q}}^2)}\left(\delta_{ij} - \frac{3\lambda\hat{q}_i\hat{q}_j}{(1+\lambda\hat{\mathbf{q}}^2)}\right)$$
$$-2\lambda\hat{q}_i\hat{q}_j\hat{\mathcal{H}}_{\rm TPDM}(\hat{\mathbf{q}},\hat{\mathbf{p}}) + \omega^2\hat{q}_i\hat{q}_j,$$

which form a quantum Fradkin tensor and verify that $\hat{\mathcal{H}}_{\rm TPDM} = \frac{1}{2}\sum_{i=1}^N \hat{I}_{\mathrm{TPDM},ii}$.
(ii) Each of the three sets $\{\hat{\mathcal{H}}_{\rm TPDM}, \hat{C}^{(m)}\}$, $\{\hat{\mathcal{H}}_{\rm TPDM}, \hat{C}_{(m)}\}$ $(m = 2, \dots, N)$ *and* $\{\hat{I}_{\mathrm{TPDM},ii}\}$ $(i = 1, \dots, N)$ *is formed by* N *algebraically independent commuting observables.*
(iii) The set $\{\hat{\mathcal{H}}_{\rm TPDM}, \hat{C}^{(m)}, \hat{C}_{(m)}, \hat{I}_{\mathrm{TPDM},ii}\}$ *for* $m = 2, \dots, N$ *with a fixed index* i *is formed by* $2N-1$ *algebraically independent observables.*
(iv) $\hat{\mathcal{H}}_{\rm TPDM}$ *is formally self-adjoint on the standard* L^2 *space with product*

$$\langle\Psi|\Phi\rangle_{\rm TPDM} = \int_{\mathcal{M}^N} \overline{\Psi(\mathbf{q})}\,\Phi(\mathbf{q})\,\mathrm{d}\mathbf{q}.$$

Finally, we remark that by combining the similarity transformations (5.32) and (5.37) we obtain the relationship between $\hat{\mathcal{H}}_{\rm TLB}$ and $\hat{\mathcal{H}}_{\rm TPDM}$:

$$\hat{\mathcal{H}}_{\rm TPDM} = \mathrm{e}^{v-f}\hat{\mathcal{H}}_{\rm TLB}\mathrm{e}^{-(v-f)} = (1+\lambda\mathbf{q}^2)^{N/4}\hat{\mathcal{H}}_{\rm TLB}(1+\lambda\mathbf{q}^2)^{-N/4}.$$

5.4 Radial Schroedinger equations

In this section we obtain the 1D radial Schroedinger equation coming from each of the above three ND quantum Hamiltonians by, firstly, introducing hyperspherical coordinates and, secondly, by making use of the observables $\hat{C}_{(m)}$ (5.22) that encode the full spherical symmetry of the three systems.

Let us introduce the map from the initial quantum operators (5.20) to the quantum hyperspherical ones $\hat{r}$, $\hat{\theta}_j$, $\hat{p}_r$, $\hat{p}_{\theta_j}$ $(j = 1.\dots, N-1)$ with Lie brackets and differential representation given by

$$[\hat{r},\hat{p}_r] = \mathrm{i}\hbar, \qquad [\hat{r},\hat{p}_{\theta_j}] = 0, \qquad [\hat{\theta}_j,\hat{p}_r] = 0, \qquad [\hat{\theta}_j,\hat{p}_{\theta_k}] = \mathrm{i}\hbar\delta_{jk},$$
$$\hat{r} = r, \qquad \hat{p}_r = -\mathrm{i}\hbar\frac{\partial}{\partial r}, \qquad \hat{\theta}_j = \theta_j, \qquad \hat{p}_{\theta_k} = -\mathrm{i}\hbar\frac{\partial}{\partial\theta_j}. \tag{5.40}$$

Here we point out that the "radial and phase operators" that we have just introduced are nothing but formal multiplicative operators on the angular variables, whose "canonical" transformation rules with respect to the Cartesian ones are:

$$\hat{q}_j = \hat{r}\cos\hat{\theta}_j \prod_{k=1}^{j-1}\sin\hat{\theta}_k, \quad 1\le j<N; \qquad \hat{q}_N = \hat{r}\prod_{k=1}^{N-1}\sin\hat{\theta}_k,$$

$$\hat{p}_j = \prod_{k=1}^{j-1}\sin\hat{\theta}_k\cos\hat{\theta}_j\,\hat{p}_r + \frac{\cos\hat{\theta}_j}{\hat{r}}\sum_{l=1}^{j-1}\frac{\prod_{k=l+1}^{j-1}\sin\hat{\theta}_k}{\prod_{m=1}^{l-1}\sin\hat{\theta}_m}\cos\hat{\theta}_l\,\hat{p}_{\theta_l} - \frac{\sin\hat{\theta}_j}{\hat{r}\prod_{k=1}^{j-1}\sin\hat{\theta}_k}\,\hat{p}_{\theta_j},$$

$$\hat{p}_N = \prod_{k=1}^{N-1}\sin\hat{\theta}_k\,\hat{p}_r + \frac{1}{\hat{r}}\sum_{l=1}^{N-1}\frac{\prod_{k=l+1}^{N-1}\sin\hat{\theta}_k}{\prod_{m=1}^{l-1}\sin\hat{\theta}_m}\cos\hat{\theta}_l\,\hat{p}_{\theta_l}.$$

Hence we obtain that

$$\hat{\mathbf{q}}^2 = \hat{r}^2, \quad \hat{\mathbf{p}}^2 = \frac{1}{\hat{r}^{N-1}}\,\hat{p}_r\,\hat{r}^{N-1}\,\hat{p}_r + \frac{\hat{\mathbf{L}}^2}{\hat{r}^2} = \hat{p}_r^2 - \mathrm{i}\hbar\frac{(N-1)}{\hat{r}}\,\hat{p}_r + \frac{\hat{\mathbf{L}}^2}{\hat{r}^2}, \quad \hat{\mathbf{q}}\cdot\hat{\mathbf{p}} = \hat{r}\hat{p}_r, \tag{5.41}$$

where $\hat{\mathbf{L}}^2$ is the square of the total quantum angular momentum given by

$$\hat{\mathbf{L}}^2 = \sum_{j=1}^{N-1}\left(\prod_{k=1}^{j-1}\frac{1}{\sin^2\hat{\theta}_k}\right)\frac{1}{(\sin\hat{\theta}_j)^{N-1-j}}\,\hat{p}_{\theta_j}(\sin\hat{\theta}_j)^{N-1-j}\,\hat{p}_{\theta_j}.$$

Notice that the expressions (5.41) provide a 1D (radial) representation of the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra (5.25) by introducing them in (5.28).

The $N-1$ commuting observables $\hat{C}_{(m)}$ (5.22) turn out to be ($m = 2,\dots,N$)

$$\hat{C}_{(m)} = \sum_{j=N-m+1}^{N-1}\left(\prod_{k=N-m+1}^{j-1}\frac{1}{\sin^2\hat{\theta}_k}\right)\frac{1}{(\sin\hat{\theta}_j)^{N-1-j}}\,\hat{p}_{\theta_j}(\sin\hat{\theta}_j)^{N-1-j}\,\hat{p}_{\theta_j},$$

with $\hat{C}_{(N)} = \hat{\mathbf{L}}^2$. Thus we obtain a set of $N-1$ angular equations ($k = 3,\dots,N-1$):

$$\begin{aligned}
&\hat{C}_{(2)}(\hat{\theta}_{N-1},\hat{p}_{\theta_{N-1}}) = \hat{p}^2_{\theta_{N-1}},\\
&\hat{C}_{(k)}(\hat{\theta}_{N-k+1},\hat{p}_{\theta_{N-k+1}}) = \frac{1}{(\sin\hat{\theta}_{N-k+1})^{k-2}}\,\hat{p}_{\theta_{N-k+1}}(\sin\hat{\theta}_{N-k+1})^{k-2}\hat{p}_{\theta_{N-k+1}} + \frac{\hat{C}_{(k-1)}}{\sin^2\hat{\theta}_{N-k+1}},\\
&\hat{C}_{(N)}(\hat{\theta}_1,\hat{p}_{\theta_1}) = \frac{1}{(\sin\hat{\theta}_1)^{N-2}}\,\hat{p}_{\theta_1}(\sin\hat{\theta}_1)^{N-2}\,\hat{p}_{\theta_1} + \frac{\hat{C}_{(N-1)}}{\sin^2\hat{\theta}_1} \equiv \hat{\mathbf{L}}^2,
\end{aligned} \tag{5.42}$$

which are worth to be compared with (5.12). Therefore the quantum radial Hamiltonian corresponding to (5.21) is obtained in the form

$$\hat{\mathcal{H}}(\hat{r},\hat{p}_r\ ) = \frac{1}{2(1+\lambda\hat{r}^2)}\left(\frac{1}{\hat{r}^{N-1}}\,\hat{p}_r\,\hat{r}^{N-1}\,\hat{p}_r + \frac{\hat{\mathbf{L}}^2}{\hat{r}^2} + \omega^2\hat{r}^2\right). \tag{5.43}$$

After reordering terms and introducing the differential operators (5.40) in the Hamiltonian (5.43) we arrive at the following Schroedinger equation, $\hat{\mathcal{H}}\Psi = E\Psi$,

$$\frac{1}{2(1+\lambda r^2)}\left(-\hbar^2\partial_r^2 - \frac{\hbar^2(N-1)}{r}\,\partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} + \omega^2 r^2\right)\Psi(r,\boldsymbol{\theta}) = E\Psi(r,\boldsymbol{\theta}), \tag{5.44}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{N-1})$. Next we factorize the wave function in the usual radial and angular components and consider the separability provided by the first integrals $\hat{C}_{(m)}$ (5.42) with eigenvalue equations given by

$$\Psi(r, \boldsymbol{\theta}) = \Phi(r) Y(\boldsymbol{\theta}), \qquad \hat{C}_{(m)} \Psi = c_m \Psi, \quad m = 2, \ldots, N. \tag{5.45}$$

Consequently, we obtain that $Y(\boldsymbol{\theta})$ solves completely the angular part and such hyperspherical harmonics verify

$$\hat{C}_{(N)} Y(\boldsymbol{\theta}) = \hat{\mathbf{L}}^2 Y(\boldsymbol{\theta}) = \hbar^2 l(l+N-2) Y(\boldsymbol{\theta}), \quad l = 0, 1, 2 \ldots \tag{5.46}$$

where l is the angular quantum number. By taking into account the angular equations (5.42), we find that the eigenvalues c_m of the operators $\hat{C}_{(m)}$ are related to the $N-1$ quantum numbers of the angular observables as

$$c_k \leftrightarrow l_{k-1}, \quad k = 2, \ldots, N-1, \qquad c_N \leftrightarrow l,$$

that is,

$$Y(\boldsymbol{\theta}) \equiv Y^{c_N}_{c_{N-1},..,c_2}(\theta_1, \theta_2, ..., \theta_{N-1}) \equiv Y^{l}_{l_{N-2},..,l_1}(\theta_1, \theta_2, ..., \theta_{N-1}).$$

Hence the radial Schroedinger equation provided by $\hat{\mathcal{H}}$ is

$$\frac{1}{2(1+\lambda r^2)} \left(-\hbar^2 \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(N-1)}{r} \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2} \right) + \omega^2 r^2 \right) \Phi(r) = E \Phi(r). \tag{5.47}$$

In the same way, the 1D radial Hamiltonian operators coming from the transformed LB (5.34) and PDM (5.39) quantizations are found to be

$$\begin{aligned} \hat{\mathcal{H}}_{\mathrm{TLB}} &= -\frac{\hbar^2}{2(1+\lambda r^2)} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{N-1}{r} + \frac{\lambda(N-2)r}{1+\lambda r^2} \right) \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2} \right) \\ &\quad + \frac{\omega^2 r^2}{2(1+\lambda r^2)} - \frac{\hbar^2 \lambda (N-2)}{8(1+\lambda r^2)^3} \left(2N + 3\lambda r^2 (N-2) \right), \qquad (5.48) \\ \hat{\mathcal{H}}_{\mathrm{TPDM}} &= -\frac{\hbar^2}{2(1+\lambda r^2)} \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \left(\frac{N-1}{r} - \frac{2\lambda r}{1+\lambda r^2} \right) \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l(l+N-2)}{r^2} \right) \\ &\quad + \frac{\omega^2 r^2}{2(1+\lambda r^2)} + \frac{\hbar^2 \lambda \left(N + \lambda r^2 (N-3) \right)}{2(1+\lambda r^2)^3}. \qquad (5.49) \end{aligned}$$

Recall that the three radial Hamiltonians $\hat{\mathcal{H}}$, $\hat{\mathcal{H}}_{\mathrm{TLB}}$ and $\hat{\mathcal{H}}_{\mathrm{TPDM}}$, are related through the similarity transformations as

$$\begin{aligned} &\hat{\mathcal{H}}_{\mathrm{TLB}} = (1+\lambda r^2)^{(2-N)/4} \hat{\mathcal{H}} (1+\lambda r^2)^{(N-2)/4}, \\ &\hat{\mathcal{H}}_{\mathrm{TPDM}} = (1+\lambda r^2)^{1/2} \hat{\mathcal{H}} (1+\lambda r^2)^{-1/2}, \\ &\hat{\mathcal{H}}_{\mathrm{TPDM}} = (1+\lambda r^2)^{N/4} \hat{\mathcal{H}}_{\mathrm{TLB}} (1+\lambda r^2)^{-N/4}. \end{aligned}$$

Therefore the three corresponding radial Schroedinger equations share the same energy spectrum and have different but equivalent radial wave functions:

$$\begin{aligned}
&\hat{\mathcal{H}}\Phi(r) = E\Phi(r), \quad \hat{\mathcal{H}}_{\rm TLB}\Phi_{\rm TLB}(r) = E\Phi_{\rm TLB}(r), \quad \hat{\mathcal{H}}_{\rm TPDM}\Phi_{\rm TPDM}(r) = E\Phi_{\rm TPDM}(r),\\
&\Phi_{\rm TLB}(r) = (1+\lambda r^2)^{(2-N)/4}\Phi(r), \quad \Phi_{\rm TPDM}(r) = (1+\lambda r^2)^{1/2}\Phi(r),\\
&\Phi_{\rm TPDM}(r) = (1+\lambda r^2)^{N/4}\Phi_{\rm TLB}(r).
\end{aligned} \tag{5.50}$$

5.5 Spectrum and eigenfunctions

In this section we shall compute, in a rigorous manner, the (continuous and discrete) spectrum and eigenfunctions of the quantum nonlinear oscillator by using the quantum Hamiltonian $\hat{\mathcal{H}}_{\rm TLB}$ (5.39) characterized in Theorem 3. Recall that both quantizations share the same spectrum but they have different radial wave functions which are related through the similarity transformation (5.50).

5.5.1 Continuous spectrum

Since $\mathcal{M}^N$ is a complete manifold and the potential is continuous and bounded, it is standard that $\hat{\mathcal{H}}_{\rm TLB}$ is essentially self-adjoint on the space $C_0^\infty(\mathbb{R}^N)$ of smooth functions of compact support. It should be remarked that one cannot immediately determine the continuous spectrum of $\hat{\mathcal{H}}_{\rm TLB}$ from asymptotics of the potential: in a complete Riemannian manifold, even the spectrum of the LB operator can be extremely difficult to analyze; e.g., it can be either purely continuous (as in Euclidean space), purely discrete [93] or consist of both a continuous part and eigenvalues, possibly embedded in the continuous spectrum [94].

In fact, to compute the continuous spectrum of $\hat{\mathcal{H}}_{\rm TLB}$ it is convenient to take advantage of the spherical symmetry to decompose

$$L^2(\mathcal{M}^N) = \bigoplus_{l\in\mathbb{N}} L^2(\mathbb{R}^+, {\rm d}\nu)\otimes\mathcal{Y}_l\,, \tag{5.51}$$

where ${\rm d}\nu(r) = r^{N-1}(1+\lambda r^2)^{N/2}{\rm d}r$ and $\mathcal{Y}_l$ is the finite-dimensional space of (generalized) spherical harmonics, defined by

$$\mathcal{Y}_l := \left\{Y\in L^2(\mathbb{S}^{N-1}) : \Delta_{\mathbb{S}^{N-1}}Y = -l(l+N-2)Y\right\},$$

where $\mathbb{N}$ stands for the set of nonnegative integers and $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplacian on the $(N-1)$D sphere $\mathbb{S}^{N-1}$ (or minus the angular momentum operator). This decomposition is tantamount to setting

$$\Psi_{\rm TLB}(\mathbf{q}) = \sum_{l\in\mathbb{N}} Y_l(\boldsymbol{\theta})\,\Phi_{{\rm TLB},l}(r)\,,$$

with $\boldsymbol{\theta} = \mathbf{q}/r\in\mathbb{S}^{N-1}$, $r=|\mathbf{q}|$ and $Y_l\in\mathcal{Y}_l$.

As $\hat{\mathcal{H}}_{\mathrm{TLB}}$ is spherically symmetric, the decomposition (5.51) allows us to write $\hat{\mathcal{H}}_{\mathrm{TLB}}$ as the direct sum of operators

$$\hat{\mathcal{H}}_{\mathrm{TLB}} = \bigoplus_{l\in\mathbb{N}} \hat{H}_{\mathrm{TLB},l} \otimes \mathrm{id}_{\mathcal{Y}_l}\,, \tag{5.52}$$

with each $\hat{H}_{\mathrm{TLB},l}$ standing for the Friedrichs extension of the differential operator on $L^2(\mathbb{R}^+, \mathrm{d}\nu)$; namely

$$\begin{aligned} 2\hat{H}_{\mathrm{TLB},l} = &-\frac{\hbar^2}{r^{N-1}(1+\lambda r^2)}\frac{\mathrm{d}}{\mathrm{d}r}r^{N-1}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\hbar^2\lambda(N-2)r}{(1+\lambda r^2)^2}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\hbar^2 l(l+N-2)}{r^2(1+\lambda r^2)} \\ &+ \frac{\omega^2 r^2}{1+\lambda r^2} - \frac{\hbar^2\lambda(N-2)}{4(1+\lambda r^2)^3}\left(2N + 3\lambda r^2(N-2)\right). \end{aligned}$$

The continuous spectrum of $\hat{\mathcal{H}}_{\mathrm{TLB}}$ is most easily dealt with using this decomposition. Indeed, from (5.52) it is apparent that

$$\mathrm{spec}(\hat{\mathcal{H}}_{\mathrm{TLB}}) = \overline{\bigcup_{l\in\mathbb{N}} \mathrm{spec}(\hat{H}_{\mathrm{TLB},l})}\,.$$

To understand the spectrum of $\hat{H}_{\mathrm{TLB},l}$ we proceed to compute and analyze its associated quantum effective potential $\hat{\mathcal{U}}_{\mathrm{eff},l}$. For this purpose we apply the same change of variable $Q = Q(r)$ (5.15) used in the classical case, together with a change of the radial wave function $\Phi_{\mathrm{TLB},l}(r) \mapsto u(Q(r))$. We require that these transformations map the Schroedinger equation $\hat{H}_{\mathrm{TLB},l}\Phi_{\mathrm{TLB},l} = E\Phi_{\mathrm{TLB},l}$ into

$$\left(-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}Q^2} + \hat{\mathcal{U}}_{\mathrm{eff},l}(Q)\right)u(Q) = Eu(Q). \tag{5.53}$$

This is achieved by setting

$$\Phi_{\mathrm{TLB},l}(r) = \frac{r^{(1-N)/2}}{(1+\lambda r^2)^{(N-1)/4}}\,u(r) \tag{5.54}$$

in the radial Schroedinger equation, thus yielding

$$\hat{\mathcal{U}}_{\mathrm{eff},l}(r) = \frac{1}{2(1+\lambda r^2)}\left(\frac{\hbar^2\left(8(1+\lambda r^2)-5\right)}{4r^2(1+\lambda r^2)^2} + \frac{\hbar^2}{r^2}\left(l(l+N-2) + \frac{N(N-4)}{4}\right) + \omega^2 r^2\right). \tag{5.55}$$

The behavior of $\hat{\mathcal{U}}_{\mathrm{eff},l}$ is rather similar to that of the classical effective potential (5.16) (see figure 5.4), that is, $\hat{\mathcal{U}}_{\mathrm{eff},l}$ is a positive function with a unique minimum, whose expression is rather cumbersome and which for the harmonic oscillator reduces to

$$\begin{aligned} \lambda = 0: \qquad & r_{\min}^2 = \hbar\frac{\sqrt{l(l+N-2)+(N-1)(N-3)/4}}{\omega}, \\ & \hat{\mathcal{U}}_{\mathrm{eff},l}(r_{\min}) = \hbar\omega\sqrt{l(l+N-2)+(N-1)(N-3)/4}. \end{aligned} \tag{5.56}$$

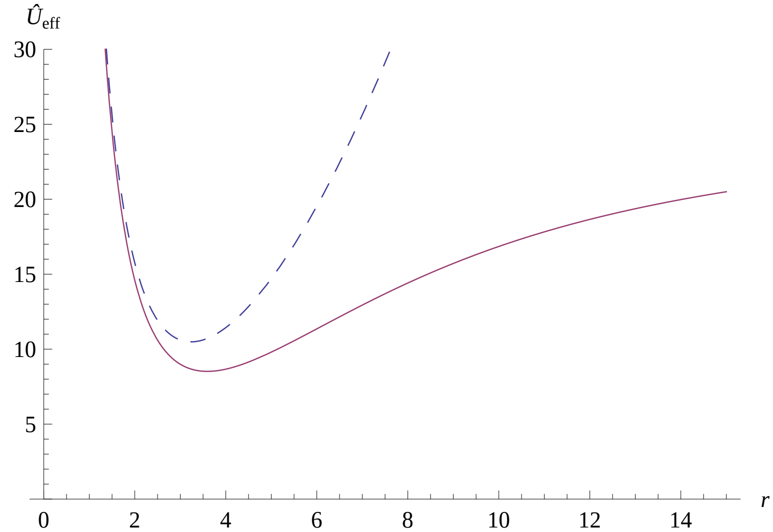

Figure 5.4: The quantum effective nonlinear oscillator potential (5.55) for $N = 3$, $\lambda = 0.02$, $l = 10$ and $\hbar = \omega = 1$. The minimum of the potential is located at $r_{\rm min} = 3.59$ with $\hat{\mathcal{U}}_{{\rm eff},l}(r_{\rm min}) = 8.52$ and $\hat{\mathcal{U}}_{{\rm eff},l}(\infty) = 25$. The dashed line corresponds to the quantum effective potential of the isotropic oscillator with $\lambda = 0$ with minimum $\hat{\mathcal{U}}_{{\rm eff},l}(r_{\rm min}) = 10.49$ at $r_{\rm min} = 3.24$.

Similarly to the classical system, the values of $r_{\rm min}$ and $\hat{\mathcal{U}}_{{\rm eff},l}(r_{\rm min})$ are respectively greater and smaller than those corresponding to the quantum harmonic oscillator (5.56), but $\hat{\mathcal{U}}_{{\rm eff},l}$ has the same asymptotic behaviour, namely,

$$\lim_{r\to 0} \hat{\mathcal{U}}_{{\rm eff},l}(r) = +\infty, \qquad \lim_{r\to\infty} \hat{\mathcal{U}}_{{\rm eff},l}(r) = \frac{\omega^2}{2\lambda}. \tag{5.57}$$

We remark that there is a single exceptional particular case for $l = 0$ and $N = 2$ for which $\hat{\mathcal{U}}_{{\rm eff},l}$ reads

$$\hat{\mathcal{U}}_{{\rm eff},l}(r) = \frac{1}{2(1+\lambda r^2)} \left(\frac{-\hbar^2 \left(1 + 4\lambda^2 r^4\right)}{4r^2(1+\lambda r^2)^2} + \omega^2 r^2 \right).$$

Thus $\lim_{r\to 0} \hat{\mathcal{U}}_{{\rm eff},l} = -\infty$ and $\lim_{r\to\infty} \hat{\mathcal{U}}_{{\rm eff},l} = \omega^2/(2\lambda)$, so $\hat{\mathcal{U}}_{{\rm eff},l}$ has no minimum and can take both negative and positive values.

Since we have just related the nonnegative, self-adjoint second-order differential operator on the half-line $\hat{H}_{{\rm TLB},l}$ to (5.53), standard results in spectral theory [95, Theorem XIII.7.66] ensure that the eigenvalues of $\hat{H}_{{\rm TLB},l}$ are contained in $(0, E_\infty)$ and its continuous spectrum is absolutely continuous and given by $[E_\infty, \infty)$, where we have set

$$E_\infty = \lim_{r\to\infty} \hat{\mathcal{U}}_{{\rm eff},l} = \frac{\omega^2}{2\lambda}\,.$$

Altogether, this guarantees that the continuous spectrum of $\hat{\mathcal{H}}_{\text{TLB}}$ is

$$\operatorname{spec}_{\text{cont}}(\hat{\mathcal{H}}_{\text{TLB}}) = \big[\omega^2/(2\lambda), \infty\big)\,,$$

and that there are no embedded eigenvalues.

5.5.2 Discrete spectrum and eigenfunctions

Let us now compute the eigenvalues and eigenfunctions of $\hat{\mathcal{H}}_{\text{TLB}}$. To begin with, let us denote by $\psi_n(q)$ the nth eigenfunction of the 1D harmonic oscillator which satisfies

$$\frac{1}{2}\left(-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}q^2} + \omega^2 q^2\right)\psi_n(q) = \hbar\omega\left(n + \frac{1}{2}\right)\psi_n(q)\,.$$

The explicit expression of ψ_n in terms of Hermite polynomials is

$$\psi_n(q) = \exp\left(-\frac{\omega}{2\hbar}\,q^2\right)\,H_n\left(\sqrt{\frac{\omega}{\hbar}}\,q\right), \tag{5.58}$$

up to a normalization constant.

Due to the relationship between the Schroedinger and LB quantizations (5.32) we have that $\Psi_{\text{TLB}}(\mathbf{q}) = (1+\lambda\mathbf{q}^2)^{(2-N)/4}\Psi(\mathbf{q})$ and the eigenvalue equation

$$\hat{\mathcal{H}}_{\text{TLB}}\Psi_{\text{TLB}}(\mathbf{q}) = E\Psi_{\text{TLB}}(\mathbf{q})$$

can also be written as (see (5.21))

$$(-\hbar^2\Delta + \Omega^2\mathbf{q}^2)\Psi(\mathbf{q}) = 2E\Psi(\mathbf{q})\,, \tag{5.59}$$

where

$$\Omega = \sqrt{\omega^2 - 2\lambda E}. \tag{5.60}$$

Since $\hat{\mathcal{H}}_{\text{TLB}}$ has no embedded eigenvalues (as shown in the previous subsection), one can safely assume that $\omega^2 - 2\lambda E > 0$. The condition $\Psi_{\text{TLB}} \in L^2(\mathcal{M}^N)$ translates, according to (5.24), as

$$\int |\Psi(\mathbf{q})|^2(1+\lambda\mathbf{q}^2)\,\mathrm{d}\mathbf{q} < \infty\,;$$

in particular, Ψ is square-integrable with respect to the Lebesgue measure. Therefore, by the standard theory of the harmonic oscillator, there must exist some $n \in \mathbb{N}$ such that

$$E = \hbar\Omega\left(n + \frac{N}{2}\right).$$

Substituting the formula for Ω, taking squares and isolating E, one readily finds that any eigenvalue of $\hat{\mathcal{H}}_{\text{TLB}}$ must be of the form

$$\begin{aligned} E_n &= -\lambda\hbar^2\left(n+\frac{N}{2}\right)^2 + \hbar\left(n+\frac{N}{2}\right)\sqrt{\hbar^2\lambda^2\left(n+\frac{N}{2}\right)^2 + \omega^2} \\ &= \lambda\hbar^2\left(n+\frac{N}{2}\right)^2\left(\sqrt{1+\frac{\omega^2}{\hbar^2\lambda^2(n+\frac{N}{2})^2}} - 1\right). \end{aligned} \tag{5.61}$$

Conversely, one can prove that E_n is an eigenvalue of $\hat{\mathcal{H}}_{\mathrm{TLB}}$ for any $n \in \mathbb{N}$. This is easily seen by taking any partition $(n_i)_{i=1}^N \subset \mathbb{N}$ such that $n_1 + \cdots + n_N = n$ and noticing that, by (5.58) and (5.59),

$$\Psi_{\mathrm{TLB}}(\mathbf{q}) = (1 + \lambda \mathbf{q}^2)^{(2-N)/4} \prod_{i=1}^{N} \exp\{-\beta^2 q_i^2/2\} H_{n_i}(\beta q_i), \quad \beta = \sqrt{\frac{\Omega}{\hbar}}, \tag{5.62}$$

is an $L^2(\mathcal{M}^N)$ solution of the equation $\hat{\mathcal{H}}_{\mathrm{TLB}} \Psi_{\mathrm{TLB}} = E_n \Psi_{\mathrm{TLB}}$.

Together with the result of the previous subsection, this proves the following

Theorem 7. *Let $\hat{\mathcal{H}}_{\mathrm{TLB}}$ be the quantum Hamiltonian (5.39). Then:*
(i) The continuous spectrum of $\hat{\mathcal{H}}_{\mathrm{TLB}}$ is given by $[\frac{\omega^2}{2\lambda}, \infty)$. Moreover, there are no embedded eigenvalues and its singular spectrum is empty.
(ii) $\hat{\mathcal{H}}_{\mathrm{TLB}}$ has an infinite number of eigenvalues, all of which are contained in $(0, \frac{\omega^2}{2\lambda})$. Their only accumulation point is $\frac{\omega^2}{2\lambda}$, that is, the bottom of the continuous spectrum.
(iii) All the eigenvalues of $\hat{\mathcal{H}}_{\mathrm{TLB}}$ are of the form (5.61)*, and Ψ_{TLB} is eigenfunction of $\hat{\mathcal{H}}_{\mathrm{TLB}}$ with eigenvalue E_n if and only if it is given by a linear combination of the functions* (5.62) *with $n_i \in \mathbb{N}$ and $n_1 + \cdots + n_N = n$.*

Therefore the bound states of this system satisfy

$$E_\infty = \lim_{n\to\infty} E_n = \frac{\omega^2}{2\lambda}, \qquad \lim_{n\to\infty} (E_{n+1} - E_n) = 0.$$

Such a discrete spectrum is depicted in figure 5.5 for several values of λ.

5.6 Discrete spectrum in explicit radial form and comparison with the general solution of Bertrand family

Let us now compute the eigenfunctions of $\mathcal{H}_{TLB}$ in an explicit radial form. Analogously to what we did previously let us recall the radial solution of the isotropic harmonic oscillator:

$$(-\frac{\hbar^2}{2}(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{\hat{\mathbf{L}}^2}{r^2}) + \frac{1}{2}\omega^2 r^2)\Psi(r,\theta) = \hbar\omega(2n + l + \frac{N}{2})\Psi(r,\theta) \tag{5.63}$$

Let us consider again the relationship (5.59), then we get the same spectrum, but written in different quantum numbers n, l we get:

$$E_n = -\lambda\hbar^2 \left(2n + l + \frac{N}{2}\right)^2 + \hbar \left(2n + l + \frac{N}{2}\right) \sqrt{\hbar^2\lambda^2 \left(2n + l + \frac{N}{2}\right)^2 + \omega^2} \tag{5.64}$$

$$\Psi(r,\theta)_{n,l} = r^l e^{-\frac{\omega r^2}{2\hbar}} L_k^{l+\frac{N-2}{2}} \left(\frac{\omega r^2}{\hbar}\right) Y(\theta)_l \tag{5.65}$$

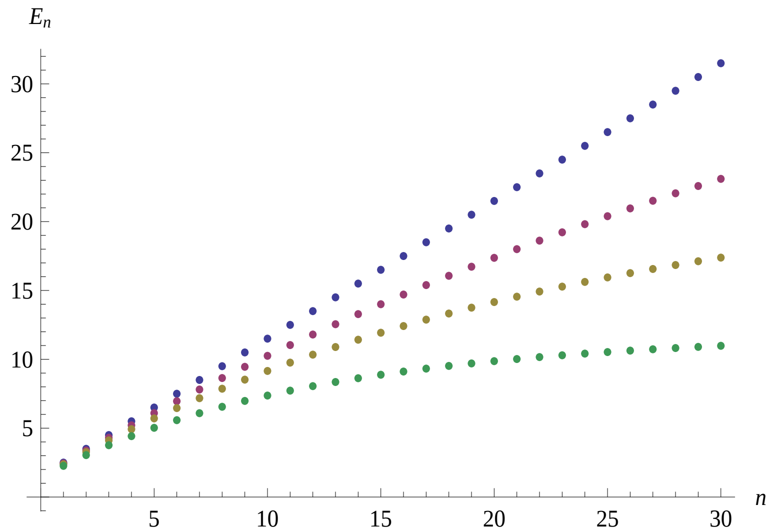

Figure 5.5: The discrete spectrum (5.61) for $0 \leq n \leq 25$, $N = 3$, $\hbar = \omega = 1$ and $\lambda = \{0, 0.01, 0.02, 0.04\}$ starting from the upper dot line corresponding to the isotropic harmonic oscillator with $\lambda = 0$; in the same order, $E_0 = \{1.5, 1.48, 1.46, 1.41\}$ and $E_\infty = \{\infty, 50, 25, 12.5\}$.

Finally let us compare this spectrum with the spectrum obtained for the general Hamiltonian (4.77) :

$$\left(-\frac{r^2(r^{-\beta} - k^2 r^\beta)^2}{4\beta^2(r^{-\beta} + k^2 r^\beta + 2G)}\left(\partial_r^2 + \frac{n-1}{r}\partial_r + \frac{\hat{\mathbf{L}}^2}{r^2}\right) + \frac{2\alpha}{r^{-\beta} + k^2 r^\beta + 2G}\right)\Psi(r,\theta)_{n,l} = \tag{5.66}$$

$$= E_{n,l}\Psi(r,\theta)_{n,l}$$

$$\nu = n + \frac{l}{\beta} + \frac{\frac{N-2}{\beta} + 1}{2}$$

$$E_{n,l} = -G\nu^2 \pm \sqrt{(G^2 - k^2)\nu^4 + (\frac{k^2}{4} + 2\alpha)\nu^2}$$

Let us set the parameters of the general Bertrand Hamiltonian as $\beta = 2, k = 0, \alpha = \frac{\omega^2}{32}, G = \frac{\lambda}{2}$ and select the positive root . The equation (5.66) turns into:

$$-\frac{1}{16(1+\lambda r^2)}\left(\partial_r^2 + \frac{N-1}{r}\partial_r + \frac{\hat{\mathbf{L}}^2}{r^2} + \omega^2 r^2\right)\Psi(r,\theta)_{n,l} = \frac{1}{8}\mathcal{H}\Psi(r,\theta)_{n,l} = \tag{5.67}$$

$$\left(-\frac{\lambda}{2}(n+\frac{l}{2}+\frac{N}{4})^2+\sqrt{\frac{\lambda^2}{4}(n+\frac{l}{2}+\frac{N}{4})^4+\frac{\omega^2}{16}(n+\frac{l}{2}+\frac{N}{4})^2}\right)\Psi(r,\theta)_{n,l}$$

$$\to \mathcal{H}\Psi(r,\theta)_{n,l}=\left(-\lambda(2n+l+\frac{N}{2})^2+\sqrt{\lambda^2(2n+l+\frac{N}{2})^4+\omega^2(2n+l+\frac{N}{2})^2}\right)\Psi(r,\theta)_{n,l}$$

This is in full agreement with the eigenvalues obtained in (5.64)

5.7 Concluding remarks

To summarize this last chapter we have analyzed in detail one of the quantum exactly solvable superintegrable systems presented in the thesis, in particular it can be understood as a simultaneous "analytic" $\lambda-$deformation of both the usual isotropic oscillator potential and the underlying space on which the dynamics is defined. In the particular case of the N-dimensional Darboux III system, it turns out that, other than the exactly solvability of the system, we have also the explicit expression for the additional constant of the motion which make this system Maximally Superintegrable. It is worth stressing that such an explicit solution could be of interest from the physical viewpoint, since a parabolic effective-mass function has been proposed in [96, 97] in order to describe realistic quantum wells formed by semiconductor heterostructures. Finally, we recall that the real parameter $\lambda = \frac{1}{k}$ was restricted in [98] to take a positive value. However, the M.S. of the classical Hamiltonian does hold for negative λ as well. Nevertheless, the underlying space and the oscillator potential change dramaticallty when $\lambda < 0$ (see [88]), and the corresponding quantum problem is currently under investigation by making use of the techniques presented in this thesis.

Chapter 6

Conclusions

In this thesis we have analyzed a class of maximally superintegrable systems with radial symmetry on Non-Euclidean manifolds following two distinct footpaths:
In the first chapter we have introduced a class of 1-dimensional exactly solvable quantum systems known as shape invariant systems. This class of systems can be solved by applying algebraic techniques and their eigenfunctions can be described in terms of orthogonal polynomials; this is not accidental and in fact we pointed out that such systems can be found directly starting from the second order differential equations defining the orthogonal polynomials and then modifying them through algebraic operations; such as point canonical transformations, gauge transformations, and the coupling constant metamorphosis (which to the best of my knowledge it is used here for the first time to link explicitly different classes of shape invariant systems).
In the second chapter we changed the scale of the physical problems analyzed (from quantum to classical mechanics). From a classical point of view to solve exactly a system means to know the trajectory in the phase space; to this aim we introduced a class of systems whose trajectory can be determined without solving explicitly the equations of motion, namely the Maximally Superintegrable systems. For this class of systems the trajectory can be determined by solving a system of algebraic equations obtained by knowing the values of its $2N-1$ constants of motion. One of the main consequences of the Maximal Superintegrability for a Hamiltonian system is that any bounded motion turns out to be periodic; this statement is indeed the starting point of the second part of our analysis: in fact we turned the problem as considering all those systems whose all bounded motions turn out to be periodic: exploiting the Perlick classification of all the radial spacetimes whose timelike geodesics are closed we found two multiparametric families of radial Hamiltonians describing particles moving on a non-Euclidean manifold and characterized by the Maximal Superintegrability property. Also in this case the coupling constant metamorphosis (CCM) plays an important role, in fact we have pointed out that the two Perlick Hamiltonians are linked by the CCM.
If a classical exactly solvable system has an exactly solvable quantum mechanical

version as well then there must exist a link between the shape invariant systems and the M.S. systems. In the third chapter we have emphasized that there not exists a unique way to quantize a classical system whilst it is very well known that, in our context, the classical limit is well defined for any quantum system. In the light of the above considerations we have started the fourth chapter by considering a 1-dimensional shape invariant system then we up-graded this system as the radial part of an higher dimensional quantum Hamiltonian and finally we have performed the classical limit finding one of the Perlick systems. This strategy led us to find a solution for one of the principal problems tackled in the present thesis, namely the quantization of the Perlick Hamiltonians. Concluding let us summarize the principal achievements of this thesis:

•1 The quantization which preserves the exact solvability and yields a spectrum with accidental degeneracy is what we have defined as direct or Schroedinger quantization.

•2 Other exactly solvable quantizations are also possible but in any other case we have to introduce a quantum "correction" potential (whose coupling constant is proportional to the Planck constant namely classically invisible). It is important to stress that these quantum corrections are nontrivial and in particular if we consider the "most physical" quantizations of the Bertrand Hamiltonians, namely the position dependent mass and the geometrical quantization, then these corrections turn out to be respectively the Levy Le Blond potential and a function proportional to the scalar curvature of the system (to the best of my knowledge it is the first time that these corrections are considered for integrability requirements).

•3 Another important point is that both the exact solvability and the accidental degeneracy are independent on the dimension of the space in which the radial motion is embedded.

•4 At the end we can also state that the conjecture that links the concepts of Maximal superinegrability and of exact solvability [84], [85] is also confirmed by our results. Moreover, personally speaking, I find quite remarkable that the Bertrand spaces could be obtained alternatively as "classical limits" of the second order differential equations defining the clasical orthogonal polynomials. This fact seems to establish a strong relationship among orthogonal polynomials and the family of Maximally superintagrable systems.

Before concluding let us make some remarks about the open questions that should be tackled after this thesis. We have obtained an exact solution for all the quantum Bertrand systems and all these quantum systems show degeneracy in the spectrum, this is a very strong clue about the maximal superintegrability of these quantum systems. However the possibility of obtaining an exact expression for the quantum extra integrals of motion is, at the moment, restricted to a class of particular cases in which they are at most quadratic in momenta, namely when the parameter β is set to be ($\beta = 1, 2$); on the other hand when a general β is considered then the additional constants of the motion are no more quadratical, but in general are nontrivial functions of the momentum also for the classical case [51], and this makes their quantization a hard work.

Appendix A

Approximation of planetary motion on a Schwarzschild metric through Bertrand spaces

So far we have presented the Bertrand Hamiltonians as systems defined on non-Euclidean manifold characterized by the relevant facts of being exactly solvable, M.S. and pliable because of the number of free parameters contained in it. Let us consider the General Hamiltonian II:

$$H = \frac{r^2(r^{-\beta} - k^2 r^{\beta})^2}{2(r^{-\beta} + k^2 r^{\beta} - 2G)}\mathbf{P}^2 - \frac{2\alpha}{(r^{-\beta} + k^2 r^{\beta} - 2G)} \tag{A.1}$$

Let us consider the subcase $k = 0, \beta = 1, \alpha = \frac{d}{4\lambda}, G = -\frac{1}{2\lambda}$, now we are ready to define the new Hamiltonian H_T:

$$H_T = \frac{2}{\lambda}H + \frac{d}{\lambda} = \frac{1}{1+\frac{\lambda}{r}}\left(\mathbf{P}^2 + \frac{d}{r}\right) \tag{A.2}$$

H_T is by construction M.S. and it is known as TAUB NUT system (see for instance [49]), therefore there exist $2N-1$ constants of the motion I_i such that:

$$\{I_i, H_T\} = 0, \quad i = 1, ..., 2N-1$$

Without losing generality let us consider the bidimensional space which represents the plane of the motion:

$$H_T(r, P_r, \phi, P_\phi) = \frac{2}{\lambda}H + \frac{d}{\lambda} = \frac{1}{1+\frac{\lambda}{r}}\left(P_r^2 + \frac{P_\phi^2}{r^2} + \frac{d}{r}\right) \tag{A.3}$$

Let us going on considering a Hamiltonian non-quadratical in momentum variables $\tilde{H}$ such that:

$$\tilde{H} = f(H_T) \tag{A.4}$$

$\tilde{H}$ is M.S. as well since has the same symmetries of H_T

$$\{f(H_T), I_i\} = f'(H_T)\{H_t, I_i\} = 0, \quad \forall i = 1, 2, 3 \tag{A.5}$$

Let us define the Hamilton equations for $\tilde{H}$:

$$\begin{aligned} \dot{r} &= f'(H_T)2g(r)P_r \\ \dot{\phi} &= f'(H_T)2g(r)L \\ \dot{P}_r &= -f'(H_T)\left(g(r)'(P_r^2 + \frac{L^2}{r^2} + \frac{d}{r}) - g(r)\left(\frac{2L^2}{r^3} + \frac{d}{r^2}\right)\right) \end{aligned} \tag{A.6}$$

where $g(r) = \frac{1}{1+\frac{\lambda}{r}}$ and $P_\phi = L$.

As stated in the chapter 2 it is possible to solve explicitly the orbit equation for any Bertrand system, in particular the solution is characterized (because of the M.S. of the system) by the property:

$$\begin{aligned} r(t+T) &= r(t) \\ \phi(t+T) &= \phi(t) + 2\pi \end{aligned}$$

where T is the period of the motion, independent of the value of L and of H.

Now let us break the M.S. adding a monopole term of the type $\frac{b}{r^2}$ defining the new Hamiltonian H_{Tb}:

$$H_{Tb} = \frac{1}{1+\frac{\lambda}{r}}\left(P_r^2 + \frac{L^2}{r^2} + \frac{b}{r^2} + \frac{d}{r}\right) \tag{A.7}$$

The Hamilton equations associated to $\hat{H}_b = f(H_{Tb})$ turn out to be:

$$\begin{aligned} \dot{r} &= f'(H_{Tb})2g(r)P_r \\ \dot{\phi} &= f'(H_{Tb})2g(r)L \\ \dot{P}_r &= -f'(H_{Tb})\left(g(r)'(P_r^2 + \frac{L^2}{r^2} + \frac{b}{r^2} + \frac{d}{r}) - g(r)\left(\frac{2L^2}{r^3} + \frac{2b}{r^3} + \frac{d}{r^2}\right)\right) \end{aligned} \tag{A.8}$$

these equation of motions are formally equal to the (A.6) once it is defined $L' = \sqrt{L^2 + b}$ and the new function $\phi'(t) = \frac{L'}{L}\phi(t)$:

$$\begin{aligned} \dot{r} &= f'(H_{Tb})2g(r)P_r \\ \dot{\phi}' &= f'(H_{Tb})2g(r)L' \\ \dot{P}_r &= -f'(H_{Tb})\left(g(r)'(P_r^2 + \frac{L'^2}{r^2} + \frac{d}{r}) - g(r)\left(\frac{2L'^2}{r^3} + \frac{d}{r^2}\right)\right) \end{aligned} \tag{A.9}$$

The main consequence of the addition of the term $\frac{b}{r^2}$ is that the motion is no more periodic in $\phi(t)$:

$$\begin{aligned} \phi'(t+T) &= \phi'(t) + 2\pi \\ \frac{L'}{L}\phi(t+T) &= \frac{L'}{L}\phi(t) + 2\pi \rightarrow \phi(t+T) = \phi(t) + 2\pi\frac{L}{L'} = \phi(t) + 2\pi + \delta, \end{aligned} \tag{A.10}$$

$$\delta = 2\pi\left(\frac{L}{L'} - 1\right).$$

This means that there is a precession of an angle δ due to the presence of the monopole term $\frac{b}{r^2}$

A.0.1 Planetary motion on a Schwarzschild metric

Let us consider the Schwarzschild metric on a conformal reference frame:

$$ds^2 = \frac{\left(1-\frac{a}{r}\right)^2}{\left(1+\frac{a}{r}\right)^2}c^2dt^2 - \left(1+\frac{a}{r}\right)^4(dx^2+dy^2+dz^2), \quad a = \frac{GM}{2c^2} \tag{A.11}$$

A particle of mass m moving on such a space it is described by the following Hamilton Jacobi equation:

$$g^{ij}\partial_i S \partial_j S = m^2c^2 \tag{A.12}$$

if we consider : $S(\mathbf{x},t) = -Et + \tilde{S}(\mathbf{x})$ than we obtain:

$$\frac{E^2}{c^2}\frac{\left(1+\frac{a}{r}\right)^2}{\left(1-\frac{a}{r}\right)^2} = \frac{1}{\left(1+\frac{a}{r}\right)^4}\mathbf{P}^2 + m^2c^2 \tag{A.13}$$

or alternatively:

$$\frac{E^2}{m^2c^4} = \frac{\left(1-\frac{a}{r}\right)^2}{\left(1+\frac{a}{r}\right)^6}\frac{\mathbf{P}^2}{m^2c^2} + \frac{\left(1-\frac{a}{r}\right)^2}{\left(1+\frac{a}{r}\right)^2} \tag{A.14}$$

now let us consider $\frac{\mathbf{P}^2}{m^2c^2} = O(\epsilon)$ and $\frac{a}{r} = O(\epsilon)$, where $\epsilon << 1$. Let us expand the equation (A.14) up to $O(\epsilon^3)$:

$$\frac{E^2}{m^2c^4} = \left(1 - \frac{8a}{r}\right)\frac{\mathbf{P}^2}{m^2c^2} + \left(1 - \frac{4a}{r} + \frac{8a^2}{r^2}\right) + O(\epsilon^3) \tag{A.15}$$

it is straightforward to recover the classical results:
order zero:

$$\frac{E_0^2}{m^2c^4} = 1 \rightarrow E_0 = mc^2$$

first order:

$$\frac{2E_0E_1}{m^2c^4} = \frac{\mathbf{P}^2}{m^2c^2} - \frac{4a}{r} \to E_1 = \frac{\mathbf{P}^2}{2m} - \frac{GMm}{r}$$

Let us consider $\frac{H_{Tb}}{m^2c^2}$:

$$\frac{H_{Tb}}{m^2c^2} = \frac{1}{1+\frac{\lambda}{r}}\left(\frac{\mathbf{P}^2}{m^2c^2} + \frac{d}{m^2c^2r} + \frac{b}{m^2c^2r^2}\right) \tag{A.16}$$

Let us suppose $\frac{d}{m^2c^2r} = O(\epsilon)$, $\frac{\lambda}{r} = O(\epsilon)$, $\frac{b}{m^2c^2r^2} = O(\epsilon^2)$ and let us expand, analogously (A.15), in order of ϵ:

$$\frac{H_{Tb}}{m^2c^2} = \left(1 - \frac{\lambda}{r}\right)\frac{\mathbf{P}^2}{m^2c^2} + \left(\frac{d}{m^2c^2r} + \frac{b-\lambda d}{m^2c^2r^2}\right) + O(\epsilon^3) \tag{A.17}$$

compare (A.17) with (A.15) we obtain for the parameters λ, b, d the following relations:

$$\begin{aligned} \lambda &= 8a; \\ d &= -4am^2c^2; \\ b &= -24a^2m^2c^2 \end{aligned}$$

These results entail that we can approximate up to $O(\epsilon^3)$ the equation (A.14) as follows:

$$\frac{E^2}{m^2c^4} = \frac{H_{Tb}}{m^2c^2} + 1 + O(\epsilon^3) \to E \approx \sqrt{c^2H_{Tb} + m^2c^4} \tag{A.18}$$

furthermore this system is characterized by a precession of the pericentrum of the orbit:

$$\delta = 2\pi\left(\frac{1}{\sqrt{1+\frac{b}{L^2}}} - 1\right) \approx 2\pi\left(1 - \frac{b}{2L^2} - 1\right) = \frac{6\pi G^2M^2m^2}{L^2c^2} \tag{A.19}$$

in perfect agreement with the standard results of the general relativity (see for instance [102] section 101 pag 330). Therefore we can regard at the Hamiltonian system $H = \sqrt{c^2H_{Tb} + m^2c^4}$ as an exactly solvable model describing the planetary motion on a central field.

Bibliography

[1] V.I. Arnold, *Mathematical methods of classical mechanics*, 2nd edition, Springer (1989);

[2] P.M. Morse, *Diatomic molecules according to the wave mechanics II Vibrational levels*, Phys. Rev. 34 (1929) 57

[3] E. Witten, *Dynamical breaking of supersymmetry* Nucl. Phys. B 188 (1981) 513

[4] F. Cooper, B. Freedman, *Aspects of supersymmetric quantum mechanics* Ann. Phys. 146 (1983) 262

[5] L.E. Gedenshtein, *Derivation of exact spectra of the Schroedinger equation by means of supersymmetry* JETP Lett. 38 (1983) 356

[6] F. Cooper, A. Khare, R. Musto, A. Wipf, *Supersymmetry and the Dirac equation* Ann. Phys. 187 (1988) 1

[7] D. Gomez Ullate, N. Kamran, R. Milson,*An extension of Bochners problem: Exceptional invariant subspaces* J. Appox theory 162 (2010) 987-1006

[8] C. Quesne,*Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry* J. Phys. A: Math. Theor. 41 (2008) 392001

[9] S. Odake, R. Sasaki, *Infinitely many shape invariant potentials and new orthogonal polynomials* Phys. Lett. B 679 (2009) 414-417

[10] S. Odake, R. Sasaki,*Another set of infinitely many exceptional (X) Laguerre polynomials* Phys. Lett. B 684 (2010) 173-176

[11] V.G. Bagrov, B.F. Samsonov, *Darboux transformation, factorization, and supersymmetry in one dimensional quantum mechanics* Theoret. and Math. Phys. 104 (1995) 1051

[12] M.M. Crum *Associated Sturm Liouville systems* Quart. J. Math. Oxford ser.(2) 6 (1955) 121-127 physics/9908019

[13] L. Infield, T. E. Hull, *The factorization method* Rev. Mod. Phys. 23 (1951) 21-68

[14] F. Cooper, A. Khare, U. Sukhatme *Supersymmetry and quantum mechanics* Physics Reports 251 (1995) 267-385

[15] A. Khare and U. Sukhatme, *New exactly solvable Hamiltonians: Shape invariance and self similarity* Jour. Phys. A 26 (1993) L901.

[16] D. Barclay, R. Dutt, A. Gangopadhyaya, A. Khare, A. Pagnamenta and U. Sukhatme, *New shape-Invariant potentials in supersymmetric quantum mechanics* Phys. Rev. A 48 (1993) 2786.

[17] M.Faux, D. Spector *A BPS interpretation of shape invariance* J. Phys. A: Math. Gen. 37 (2004) 10397-10407

[18] Ismail M E H (2005) *Classical and quantum orthogonal polynomials in one variable* Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge

[19] Koekoek R and Swarttouw R F (1996) *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue* arXiv:mat.CA/9602214

[20] S. Bochner *Uber Sturm-Liouvillesche polynomsysteme* Math. Zeit. 29 (1929) 730-736

[21] J. Hietarinta, B. Grammaticos, B. Dorizzi, *Coupling-Constant Metamorphosis and Duality between Integrable Hamiltonian Systems*A. Ramani, Phys. Rev. Lett. 53 (1984) 1707

[22] Daskaloyannis C., Tanoudis Y., *Classification of the quantum two-dimensional superintegrable systems with quadratic integrals and the Stackel transforms* Phys. Atom. Nuclei 71 (2008), 853861

[23] E. G. Kalnins, W. Miller Jr, S. Post,*Coupling constant metamorphosis and Nth-order symmetries in classical and quantum mechanics* J. Phys. A: Math. Theor. 43 (2010) 035202

[24] S. Karigiannis *QUASI-EXACTLY SOLVABLE SYSTEMS IN QUANTUM MECHANICS* http://www.math.uwaterloo.ca/ karigian/papers/quasi.pdf

[25] Bernard Schutz, *Geometrical Methods of mathematical physics*, Cambridge University Press, Cambridge (1999);

[26] G.Koenigs, in: G. Darboux *Lecons sur la théorie générale des surfaces*, vol4 , Chelsea New York (1972) p. 368;

[27] J.Bertrand, *C.R. Acad. Sci. Paris.*, 77 (1873);

[28] J. Drach,*Sur l'integration logique des équations de la dynamique à deux variables. Forces conservatives Intégrales cubiques mouvement dans le plan* Compt. Rend. Acad. Sci. Paris 200,22 (1935);

[29] J. Fris, V. Mandrosov, Ya.A. Smorodinsky, M. Hulir, P.Winternitz, *On higher symmetries in quantum mechanics*, Phys. Lett. 16, 354 (1965);

[30] N.W. Evans *Superintegrability in classical mechanics* Phys. Rev. A 41 No 10 (1990) 5666

[31] M. F. Rañada, *Superintegrable n=2 systems, quadratic constants of motion, and potentials of Drach*, J. Math. Phys. 38 (1997) 4165

[32] E.G.Kalnins J.M.Kress P.Winternitz , *Superintegrability in a two dimensional space of nonconstant curvature*, Volume 43, Number 2, J. Math. Phys. (2002);

[33] E.G.Kalnins J.M.Kress W.Miller P.Winternitz, *Superintegrable systems in Darboux Spaces*, Volume 44 Numero 12, J. Math. Physics (2003) 5811;

[34] R.S. Strichartz, *Analysis of the Laplacian on the complete Riemannian manifold* J. Funct. Anal. 52 (1983) 48.

[35] E.G. Kalnins, W. Miller, Jr, G.S. Pogosyan, G.C. Williams,*Superintegrability in three-dimensional Euclidean space* J. Math. Phys. 40, 708 (1999)

[36] E.G. Kalnins, W. Miller, Jr, G.S. Pogosyan,*Superintegrability and associated polynomial solutions: Euclidean space and the sphere in two dimensions* J. Math. Phys. 37, 6439 (1996)

[37] Iwai T.,Katayama N. *Two classes of dynamical systems all of whose bounded trajectories are closed* , J. Math. Phys. 35, 2914-2933 (1994)

[38] Iwai T.,Katayama N. *Multifold Kepler systems - Dynamical systems all whose bounded trajectories are closed* , J. Math. Phys. 36, 1790 (1995)

[39] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, *New superintegrablemodels with position-dependent mass from Bertrands Theorem on curved spaces* J. Phys. Conf. Ser. 284:012011,2011

[40] D. Zwanziger, *Exactly soluble nonrelativistic model of particles with both electric and magnetic charges* Phys. Rev. 176 (1968) 1480

[41] H.V. McInstosh, A. Cisneros, *degeneracy in the presence of a magnetic monopole*J. Math. Phys. 11 (1970) 896

[42] N.S. Manton, *A remark on the scattering of BPS monopoles*, Phys. Lett. B 110 (1982) 54

[43] M.F. Atiyah, N.J. Hitchin,*Low energy scattering of non-Abelian monopoles* Phys. Lett. A 107 (1985) 21

[44] G.W. Gibbons, N.S. Manton, *Classical and quantum dynamics of BPS monopoles* Nucl. Phys. B 274 (1986) 183

[45] L.G. Feher, P.A. Horvathy,*Dynamical symmetry of monopole scatering* Phys. Lett. B 183 (1987) 182

[46] G.W. Gibbons, P.J. Ruback, *The hidden symmetries of multi centre metrics* Commun. Math. Phys. 115 (1988) 267

[47] D. Bini, C. Cherubini, R.T. Jantzen,*Circular Holonomy in the TAUB NUT spacetime* Class. Quant. Grav. 19 (2002) 5481

[48] D. Bini, C. Cherubini, R.T. Jantzen, B. Mashhoon,*Gravitomagnetism in the Kerr-Newman-TAUB-NUT spacetimes* Class. Quant. Grav. 20 (2003) 457

[49] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, *Superintegrable Oscillator and Kepler Systems on Spaces of Nonconstant Curvature via the Stackel Transform* SIGMA 7: 048, (2011)

[50] Perlick V., *Bertrand Spacetimes* Class. Quant. Grav. 9, 1009-1021 (1992)

[51] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco *Hamiltonian systems admitting a Runge-Lenz vector and an optimal extension of Bertrand's theorem to curved manifolds* , Commun. Math. Phys. 290 (2009) 1033

[52] J.F. Cariñena, M. F. Rañadam, M. Santander, M. Senthilvelan *A non-linear oscillator with quasi-harmonic behaviour: two- and n-dimensional oscillators*, Nonlinearity 17, 1941 (2004)

[53] D.M. Fradkin, *Three-Dimensional Isotropic Harmonic Oscillator and* SU_3, Amer. J. Phys. 33 (1965) 207.

[54] P. W. Higgs, *Dynamical symmetries in a spherical geometry* J. Phys. A: Math. Gen. 12 (1979) 309

[55] A. Ballesteros, F.J. Herranz, *Universal integrals for superintegrable systems on N-dimensional spaces of constant curvature* J. Phys. A: Math. Theor. 40 (2007) F51-F59

[56] Sergyeyev A., Blaszak M., *Generalized Stackel transform and reciprocal transformations for finitedimensional integrable systems*, J. Phys. A: Math. Theor. 41 (2008), 105205, 20 pages, arXiv:0706.1473.

[57] E. G. Kalnins, J. M. Kress, W. Miller Jr, *Second order superintegrable systems in conformally flat spaces. II. The classical two-dimensional Stackel transform* J. Math. Phys. 46 (2005) 053510

[58] E. G. Kalnins, J. M. Kress, W. Miller Jr,*Second order superintegrable systems in conformally flat spaces. IV. The classical 3D Stackel transform and 3D classification theory*J. Math. Phys. 47 (2006) 043514

[59] A. Ballestereos, A. Enciso, F. J. Herranz, O. Ragnisco, *Bertrand Spacetimes as Kepler/Oscillator potentials* Class. Quantum Grav. 25 (2008) 165005(93)

[60] Leach, P.G.L. , Flessas, G.P. *Generalisations of the Laplace-Runge-Lenz vector* J. Nonlin. Math. Phys. 10, 340-423 (2003)

[61] Fradkin, D.M.: *Existence of the dynamic symmetries O_4 and SU_3 for all classical central potential problems* Prog. Theor. Phys. 37, 798-812 (1967)

[62] Gritsev V. V., Kurochkin Yu. A., *Model of excitations in quantum dots based on quantum mechanics in space of constant curvature* , Phys. Rev. B 64 (2001), 035308

[63] Quesne C., Tkachuk V.M., *Deformed algebras, position-dependent effective masses and curved spaces: an exactly solvable Coulomb problem* J. Phys. A: Math. Gen. 37 (2004), 4267

[64] Bagchi B., Quesne C., Roychoudury R., *Pseudo-Hermitian versus Hermitian position-dependent-mass Hamiltonians in a perturbative framework* J. Phys. A: Math. Gen. 39 (2006)

[65] C. Quesne,*First-order intertwining operators and position-dependent mass Schrodinger equations in d dimensions* Ann. Phys. 321 (2006) 1221

[66] G.Bastard *Wave mechanics applied to semiconductor heterostructures* Les editions de physique, Paris, 1988

[67] P.Harrison *Quantum wells, wires and dots* Wiley, New York, 2009.

[68] O.Von Roos, *Position-dependent effective masses in semiconductor theory* Phys. Rev. B 27 (1983) 7547.

[69] Cruz y Cruz S., Negro J., Nieto L.M., *Classical and quantum position dependent mass harmonic oscillators* , Phys. Lett. A 369 (2007), 400

[70] Mazharimousavi S., Mustafa O., *Flatland position dependent mass: polar coordinates separability and exact solvability* SIGMA 6 (2010), 088

[71] Tanaka T., *N-fold supersymmetry in quantum systems with position dependent mass*, J. Phys. A: Math. Gen. 39 (2006), 219

[72] J.M. Levy-Leblond *Position-dependent effective mass and Galilean invariance*, Phys. Rev. A 52 (1995) 1845

[73] S.M. Paneitz *A Quartic Conformally Covariant Differential Operator for Arbitrary Pseudo-Riemannian Manifolds* SIGMA 4 (2008) 36

[74] R.C.T. da Costa *Quantum mechanics of a constraint particle* Phys. Rev. A No4 Vol 23 (1981)

[75] R. M. Wald, *General Relativity* The University of Chicago Press, Chicago (1984)

[76] N.P. Landsman, *Mathematical Topics between classical and quantum mechanics*, Springer New York (1998)

[77] Z.J. Liui M. Qian.*Gauge invariant quantizationon Riemannian-manifolds* Trans. Amer. Math. Soc. 331 (1992) 321.

[78] O. Ragnisco, D. Riglioni *A family of exactly solvable radial quantum systems on space of non-constant curvature with accidental degeneracy in the spectrum* SIGMA 6 (2010), 097

[79] Schroedinger E., *A method of determining quantum mechanical eigenvalues and eigenfunctions*, Proc. Roy. Irish Acad. Sect. A 46 (1940), 916.

Schroedinger E., *Further studies on solving eigenvalue problems by factorization*, Proc. Roy. Irish Acad. Sect. A 46 (1941), 183206.

[80] Infield L., Schild A., *A note on the Kepler problem in a space of constant negative curvature*, Phys. Rev. 67 (1945), 121122.

[81] Leemon H.I.,*Dynamical symmetries in a spherical geometry*. II, J. Phys. A: Math. Gen. 12 (1979), 489501.

[82] Kalnins E.G., Miller W. Jr., Pogosyan G.S., *Coulomb-oscillator duality in spaces of constant curvature*, J. Math. Phys. 41 (2000), 26292657, quant-ph/9906055.

[83] Cariñena J.F., Rañada M.F., Santander M., *Central potentials on spaces of constant curvature: the Kepler problem on the two-dimensional sphere S2 and hyperbolic plane H2*, J. Math. Phys. 46 (2005), 052702, 25 pages, math-ph/0504016.

[84] P.Tempesta, A. Turbiner, P. Winternitz, *Exact solvability of superintegrable systems* J. Math. Phys. 42 No9 (2001) 4248

[85] A.P. Fordy *Quantum Super-Integrable systems as exactly solvable models* SIGMA 3 (2007) 025

[86] T. Iway, Y. Uwano, N. Katayama, *Quantization of the multifold Kepler system* J. Math Phys.37 (1996) 608

[87] A.Ballesteros, A.Enciso, F.J. Herranz, O.Ragnisco,*Superintegrability on N-dimensional curved spaces: Central potentials, centrifugal terms and monopoles* Ann. Phys. 324 (2009) 1219

[88] A. Ballesteros, A. Enciso, F.J. Herranz, O.Ragnisco, D.Riglioni Int.*On two superintegrable nonlinear oscillators in N dimensions* J. Theor. Phys., (2011) doi: 10.1007/s10773-011-0750-x

[89] A.Ballesteros, O. Ragnisco,*A systematic construction of completely integrable Hamiltonians from coalgebras* J. Phys. A: Math. Gen. 31 (1998) 3791

[90] O.Ragnisco, A. Ballesteros, F.J. Herranz, F. Musso,*Quantum Deformations and Superintegrable Motions on Spaces with Variable Curvature* SIGMA 3 (2007) 026

[91] A. Ballesteros, A. Blasco, F.J. Herranz, F.Musso, O. Ragnisco,*(Super)Integrability from coalgebra symmetry: formalism and applications* J. Phys.: Conf. Ser. 175 (2009) 012004

[92] M.A. Shifman, A.V. Turbiner,*QUANTAL PROBLEMS WITH PARTIAL ALGEBRAIZATION OF THE SPECTRUM* Commun. Math. Phys. 126 (1989) 347

[93] H. Donnelly, P. Li,*Pure point spectrum and negative curvature for noncompact manifolds* Duke Math. J. 46 (1979) 497

[94] H. Donnelly,*Eigenvalues embedded in the continuum for negatively curved manifolds* Mich. Math. J. 28 (1981) 53

[95] N. Dunford, J.T. Schwartz, *Linear Operators II*, Wiley, New York, (1988)

[96] R. Koc, M. Koca, G. Sahinoglu *Scattering in abrupt heterostructures using a position dependent mass Hamiltonian* , Eur. Phys. J. B 48 (2005) 583

[97] A.G.M. Schmidt,*Wave-packet revival for the Schrdinger equation with position-dependent mass* Phys. Lett. A 353 (2006) 459

[98] A. Ballesteros, A. Enciso, F.J. Herranz, O.Ragnisco,*A maximally superintegrable system on an n-dimensional space of nonconstant curvature* Phys. D 237 (2008) 505.

[99] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, *N-dimensional Sl_2-coalgebra spaces with non-constant curvature*, Phys. Lett. B 652 (2007) 376

[100] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, *Quantum mechanics on space of nonconstant curvature: The oscillator problem and superintegrability*, Ann. Phys. 326 (2011) 2053

[101] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, D. Riglioni, *A new exactly solvable quantum model in N dimensions*, Phys. Lett. A 375 (2011) 1431

[102] L.D. Landau, E.M. Lifshitz *The classical theory of fields, Course of theoretical physics Vol.2* , Butterworth-Heinemann 1994