Università degli Studi Roma Tre
Dipartimento di Scienze della Formazione

Dottorato in Teoria e Ricerca Educativa
XXXII Ciclo

Anno Accademico 2019/2020

Concreteness fading with Montessori materials to teach mathematical equivalences in primary school

Rosanna Cofone

Docente guida: Prof.ssa Paola Perucchini
Co-tutor: Prof. Benedetto Scoppola
Prof. Giuseppe Bove
Coordinatore: Prof. Luca Diotallevi

Dichiarazione di conformità

Dichiaro che il contenuto e l'organizzazione della tesi è opera originale realizzata da me stessa e non compromette in alcun modo i diritti di terzi. La tesi non è stata presentata per nessun altro titolo o qualifica professionale.

Alcuni dei risultati qui discussi sono stati acettati (a seguido di doppio referaggio cieco) nella segente conferenza internazionale:

18th Biennial EARLI Conference "Rhinking Tomorrow's Education: Learning from the past, in the presente and for the future", Aachen, Germany, August 12-16 2019

General index

Introduction

The learning of mathematics is a wide field of research, both in Italy and abroad, which involves open debates of various kinds. In recent years much importance has been given to the fact that children must acquire knowledge of mathematics not of notional type, allowing them to use the concepts learned even in various troublesome situations, so as in other domains (Hiebert, & Lefevre, 1986; Mattews and Rittle-Johnson, 2009). The research shows the need to learn the procedures necessary to solve problems, in order to acquire the concepts and the laws that govern them. These two types of learning (procedural and conceptual) are both necessary and must be supported by a teaching method giving the necessary relevance to both (Mattews & Rittle-Johnson, 2009; Rittle-Johnson, Siegler & Alibali, 2001; Schneider and Rittle-Johnson, 2011).

Since the difficulties related to a lack of conceptual learning generally occur later than procedural learning, this aspect is not always sufficiently taken into account when teaching. In fact, the teacher often shows only the procedure, but this makes the children lose sight of his previous knowledge, and therefore gives rise to errors and misconceptions (Sbaragli, 2005; Santi & Sbaragli, 2007; Braithwaite & Goldstone, 2015). In primary school, and in particular up to the age of 8, the acquisition of abstract concepts involves some difficulties due to the children's natural cognitive development, which in this phase benefits from a teaching method that starts with the concrete and fade to the abstract (Fyfe, McNeil, Bonjas, 2015). Children up to the age of 8 can get to know the surrounding world through their sensory experience. Therefore, if the mathematics in this phase is thought starting from reality, the child is able to grasp the meaning of the operations he carries out with greater clarity. If, on the contrary, it is thought under an abstract form, there is a risk of mnemonic learning with little understanding of the concept. The benefits and limits of using concrete objects to teach mathematics are discussed in literature. These concrete objects allows the child to use his senses, both the cognitive resources related to the use of his hands and those related to the use of the symbolic system, which seem to be deeply connected. This type of elaboration seems to favour the understanding of concepts in children up to 8 years of age. On the other hand, a strongly contextualized learning like the one that takes place when using concrete material seems to be more difficult to generalize, making difficult the transition to abstraction. Bruner (1966) proposed a three-phase teaching system, called concreteness fading that starts from conrete and fade into abstract. The recent studies on concreteness fading propose to verify if the use of this strategy allows using the benefits of concrete material when teaching mathematics, overcoming the difficulty of the transition to abstraction through a didactic that explicitly links concrete and abstract according to a logical progression (. Research shows that the concreteness fading method used with primary school children seems to benefit the ability to generalize mathematical concepts (Fyfe, McNeil, Son, & Goldstone, 2014). So far, such research has not focused on the type of concrete material to be used in the first of the three phases, which of course is not a minor variable. In fact, the literature suggests that not all concrete material is effective in teaching, and that such material must meet specific characteristics in order to be effective (Laski, Jor'dan, Daoust, and Murray, 2015). The Montessori material, according to the little literature available, meets these characteristics (Lillard, 2016). One of these is the repeated use of concrete material over time. In fact, unlike the learning procedures that involve memory and are fast processes, understanding the concept through the concrete material requires longer times. The time factor is basic, since the child must be able to repeat the sensory experience several times to extract a mathematical concept from the material. This passage allowing the child to get an abstract concept seems to take some time even after the use of the material itself. The literature suggests that our brain reorganizes the experience to build abstract categories and concepts, but this does not happen while using the material or immediately after, but can happen even some days or weeks after the sensory experience. Many studies about the learning of mathematics in primary school children have been conducted by focusing on the teaching of mathematical equivalences, since they are propaedeutic to the learning of algebra as they are based on the knowledge of operations and on the concept of equality. Our research aims at investigating some aspects of the concreteness fading strategy that have not been studied so far, also observing the efficacy of the other two strategies (concrete and abstract). In particular, the study focuses on the eventual effectiveness of the Montessori material compared to other materials, which have been previously tested by the concreteness fading strategy; on the ability to generalize concepts after a teaching session after a certain time; on the benefit of multiple example during the teaching phase. In order to answer to these questions we have used an experimental model, which refers to the research paradigm used by Fyfe, McNeil and Borjas (2015), and we have applied it to two different experiments.

This doctoral thesis is organized in five chapters.

The first chapter is dedicated to a literature review in the field of the learning and teaching of numerical knowledge and the use of symbolic.

The second chapter focus on theaching strategies that use concrete, abstract and concreteness fading to teach equivalence.

In chapter three we present the first Experiment, in which we investigate children’s learning with the three stragies - concrete, abstract and concreteness fading - and compare two different type of concrete materials - Montessori and others.

In the fourth chapter we summarize the second Experiment. The same paradygm of the first experiment was repeated with the addition of multiple examples during the teaching and the transfer test was repeated after two weeks to see the effect of time on children’s learning.

Chapter One

1.1 Numerical knowledge and teaching

The learning of mathematics and the ability to use it in real life contexts is considered one of the Key competences to develop active citizens. The European Council in 2018 adopted the Council Recommendation on Key Competences for Lifelong Learning. It promotes and supports high quality education and lifelong learning.

During the last years, the OECD PISA (Programme for International Student Assessment) surveys have been dealing with detecting students' abilities in the basic areas for the cultural and social development of individuals: reading, science and mathematics. The tests proposed by PISA do not intend to verify the knowledge, but the ability of the students to reason and use concepts and procedures in new and unfamiliar contexts. Thus, we can say that a competent student in mathematics is able to solve real problems in real contexts. In Italy, in recent years there has been a slight improvement of students’ performance in mathematics. In the OECD PISA 2015, the Italian percentage of students at level 2 (considered the minimum level of competence in Mathematics) was 23.3% (OECD = 22.5%). Singapore, the country with the highest average score, had a percentage of 12.4%. Italian students who are at level 2 or above were 76.7% (OECD = 76.6%); in Singapore they were over 90% (OECD 2015, Results of Italian students in science, mathematics and reading, PISA).

At the beginning, in Italy these tests clashed with a teaching strongly linked to a verbal transmission of knowledge and little focused on the ability to apply and transfer knowledge, generating non-positive results in the assessment of skills (OECD, Mathematics performance, PISA, 2003 and 2006). Specifically, the average score for the 2015 cycle was 24 points higher than in 2003 and 28 points compared to 2006, suggesting that the methodological innovations required by the introduction of competence-based education were giving the desired results.

Today, in Italy as in Europe, there is a growing trend towards competency-based education and in many European states is going on an implementation of the European Key Competency Framework. Specifically, in Italy each educational institution issues the certification of skills at the end of primary school, based on a national model (Law 53/2003 and Ministerial Decree 254/2012, Circular No. 3 of 12.02.2015).

PISA's approach to measure learning has given a strong push towards a paradigm shift in teaching. The learner, considered for decades as an empty container to fill, simply absorbing information, is now seen as an active builder of his knowledge. The teaching processes are changing, focusing on “how” more than on “what”; the student-teacher relationship change from a teacher-centred to a learner-centred, with the teacher supporting the learning process. An effective teaching should allow students to solve unfamiliar, complex problems, requiring more competences than simple memorization of procedures (Mevarech & Kramarski, 2014).

Researches and theories about mathematics learning in children have been developed through many decades, debating on the children’s ability to learn numbers, to understand the meaning of counting, and, finally, to understand abstractness. The Constructivist theory (Piaget, 1954) supported the idea that children learn numbers through an interaction with the environment. Children in Piaget’s theory need about 5 years to reach the conservation of numbers, to understand how symbols represent objects and quantities, and to construct abstract concepts. Afterwards, many researches demonstrated that infants can recognize little quantities (Lucangeli, 2003), and 6 months old babies can distinguish between 16 and 8 dots (Halberda & Feigenson, 2008). Numerical cognition is a quality that children hold since their birth, and this skill continuously improves over time (Berteletti, Lucangeli, Piazza, Dehaene & Zorzi, 2010). Thus, when children start to attend school, they already know many things about numbers. Otherwise their intuitive understanding of quantities and the underlying rules, need to be re-constructed to progress in mathematics, because their knowledge is based on counting strategies that are out-dated for school arithmetic.

This “shift”, for Dahaene (1997), could support the overcoming of an intuition-based mathematics for one based on the memory of logical facts.

To build the children’s formal mathematics on their prior knowledge, the teachers should know how the construction of numbers develops in their minds. Gelman and Gallistel, in an important research of 1978, established three implicit principles of counting, guided by innate knowledge: 1) one-to-one correspondence, 2) stable order, 3) cardinality. These principles start to be mastered when the child is about 2 years old, but up to 4 years he is not fully aware of the count (Pesenti, Seron & Van Der Linden 1995; Gelman & Meck, 1983; Wynn, 1990). There are currently two shared theories about the development of the numerical knowledge: the McCloskey's semantic model and the Dahene's triple code model.

The semantic model of McCloskey (McCloskey, Sokol & Goodman, 1986; Campbell & Clark, 1988) is based on three subsystems: understanding, production and calculation. The understanding is both verbal and based on the text (text, Arabic numerals or Roman numerals); the production is similar to the understanding, while the calculation system requires that the operations are memorized and later retrieved or implemented by the calculation procedures learned. Verbal understanding and memory recovery are critical elements according to this model.

The triple code model of Dehaene (1992) is based on the existence of three different codes, which activate and involve three different areas of the brain: processing of the Arabic code (bilateral ventral occipital-temporal areas), verbal coding of numbers (perisilvian areas left), analogical representation of quantities (bilateral intraparietal areas). The three codes communicate with each other. The visual Arabic code is linked to the positional notation of the digits, and its role is to perform operations with multi-digit numbers; the verbal/auditory code allows the arithmetic facts to be numbered and stored in the memory; the analogical representation of the numbers code (preverbal) processes the numbers by treating them as quantities, and uses the numerical comparison, to estimate. Thus, the verbal code is not the only way to compute, but it is necessary to recover the arithmetic facts (i.e. the results of the multiplication tables).

According to this model, the type of input used to present the numbers is irrelevant.

In the Clark and Campbell's theory (1991), the Arabic and verbal inputs use different codes to represent numerical facts. The number representation system is double: language-dependent system (verbal code) + language-independent analogical system (analogical code). When the numbers are presented in the Arabic format, the brain processes them as analogical quantities which are close to each other (giving rise to distance and size effect). Dahaene (1993) showed that there is a link between numbers and space, called SNARC (Spatial-Numerical Association of Response Codes); its existence shows an internal and analogical mental representation of the numerical quantities, which develops on a line going from left to right. The SNARC effect is more evident when comparing Arabic numbers to verbal numbers. Thus, when children learn numbers in a formal context, the teacher should consider that verbal teaching is not enough to provide a deep learning.

The child at school receives teachings that have an impact on the numerical concepts he had previously built. Montessori (1907) and Bruner (1966) both considered the child as a constructor of knowledge. Their theories assigned a central role to the child, to his previous knowledge, interests and abilities. The child is the builder of his knowledge, which must be driven by strong motivation, by the pleasure of discovery and by acting in the environment in a competent and conscious way. His actions in the environment are basic to understand the rules and the laws governing the world. To act means to perceive, to use the body as a sensory mediator between the child and the world, in order to construct concepts, schemes, categorization. Bruner theorized that every field of knowledge could be represented in three ways: a) through a set of actions aimed at achieving a certain result (active representation), b) through a set of summary images or graphs that represent a concept without defining it completely (iconic representation), c) through a set of symbolic or logical propositions, derived from a symbolic system governed by rules or laws for the formation and transformation of propositions (symbolic representation). This progression, for Bruner, describes the course of cognitive development: given this, he argued that an effective teaching should follow the same progression. Bruner experimented a set of concrete mathematical materials (including a balancing scale) on eight-years-old children, and argued that the syntactic intuition on the algorithm corresponded to the perceptive intuition on the concrete manipulatives. The children were building their knowledge through an individual, free work of research supported by both concrete manipulatives and teachers. The children seemed to deeply understand concepts and, additionally, they stored a number of images which helped them to exemplify abstractions. Thus, Bruner demonstrated that a reserve of visual images that make the abstraction achieved as concrete, allows the child to find correspondences and to verify what he is symbolically doing.

In the same way, Montessori (2013a), during her observation of primary school children, noticed that the manipulation of concrete materials induced a series of measured and logical comparisons, which represented a true spontaneous acquisition of knowledge. We shall be writing about Montessori further.

However, the active learning pedagogy has some limits. The teacher could not define his interventions on the basis of the children's interests, even if his role is to motivate and to involve them. Bruner exceeds this limit by hypothesizing that the teacher should awake the interest of the child in order to ensure that he actively participates in a path formulated and led by the teacher himself. The aim should be to promote awareness of the gained experience. We do not limit the children to the activity, but we drive him to learn by research, activating a metacognitive reflection: how I learn, what I learn (Nigris, 2004). Montessori, on the other hand, exceeds this limit by preparing a very structured environment, where children could only use the materials that the teacher already presented to them. The teacher observes and follows the cognitive development of the child by proposing adequate tasks. In this way, the child is free to choose, but in the meantime he follows a highly structured path with very precise objectives. Recently, D'Amore (2007) proposed a didactic of mathematics that starts from the spontaneous, naive and informal knowledge of the child, making him free to express it, and subsequently providing him with the tools to shape mental images and correct models that meet the expected cognitive result. In the Montessori method, this link between prior and formal knowledge is always alive because the child could reiterate the activity until the cognitive conflict is solved.

1.2 Procedural and conceptual knowledge

Many recent studies on mathematics teaching are focused on the most effective teaching practices in order to promote previous skills, integrating them with new and increasingly complex and abstract knowledge. As we said, children develop their mathematical thinking for years before starting to attend school. They solve problems including quantities, and they constantly refine their strategies, using the counting tool without any explicit teaching (Dehaene, 1997). An effective teaching takes into account that the mathematical knowledge involves two different type of knowledge: procedural and conceptual (Hiebert & Lefevre, 1986). The conceptual knowledge, according to Kilpatrick, Swafford, and Findell (2001) is the comprehension of mathematical concepts, operations, and relations. Mattews and Rittle-Johnson (2009) define the conceptual knowledge as an explicit or implicit knowledge of the principles that govern a certain domain and their interrelations, such as that of mathematical concepts and their relations. In contrast, they define the procedural knowledge as the ability to perform solving actions. The conceptual knowledge is refereed to the comprehension of "why" we solve a problem in that specific way, whereas the procedural knowledge is focused on "how". However, they have many different characteristics: the first does not only consist in knowing "why", but also in using general principles, and relations, in allowing flexibly transferable over problem types; on the other hand, the procedural knowledge is knowing "how", but it is also goal directed, tied to routine problems, allowing a quick and efficient problem solving. Bisanz (1999), defined the *procedural learning* as the ability to solve already known problems, using one or more sequences of actions, and the *conceptual knowledge* as the ability to understand the underlying concepts governing a domain and their relations; the *procedural transfer* as the ability to apply known procedures to unfamiliar problems. To measure the conceptual learning, children should solve a problem without knowing the procedure. If children are able to use their prior knowledge in order to find useful concepts, they will solve the problem (Hiebert et al., 1986). To measure the procedural knowledge, as Crooks and Alibali (2014) noticed in a standardised way, is quite easy: indeed, children solve a set of problems and they are evaluated on the number of correct answers or on the type of procedure used to solve the problem.

Giving that these two kinds of knowledge depend on the teaching instructions, Mattews and Rittle-Johnson (2009) define also the conceptual instruction as the "instruction that focuses on domain principles" and the procedural instruction as the "instruction that focuses on step-by-step problem-solving procedures" (p. 3). Thus, the procedural knowledge consists in a sequence of actions that can be used to solve problems (Rittle-Johnson & Siegler, 1998): this kind of knowledge was supported for a long time by the traditional system of mathematics teaching. However, it is not enough to guarantee an effective learning and a number of recent researches showed that teaching based on the memorization of procedures could result in an inadequate understanding of the underlying concepts, with long term errors (Fuson et al., 1997). Many researches are focused on finding effective approaches to promote a teaching method that fosters the construction of strong and effective conceptual structures and their use. The conceptual learning has taken priority over procedural learning, as Star (2005) pointed out. The conceptual knowledge is critical when planning teaching strategies, as shown in Crooks and Alibali's review (2014).

We found different effects from teaching strategies that give greater importance to procedural, conceptual, or both types of learning. As we said, the procedural knowledge is not enough to gain strong understanding of the concepts. Developing both of them improves the performance when solving math problems (Mattews & Rittle-Johnson, 2009; Rittle-Johnson et al. 2001). Thus, if the children's knowledge allows them to solve problems or operations linked by a conceptual aspect to their prior knowledge, then we can consider it as a form of deep learning (Perry, 1991). Hence, mathematics teaching should use strategies that promote a deep understanding of concepts, a good knowledge of the procedures, and of the way to transfer them. According to Perry (1991), children can learn to use procedures through the procedural instruction, but this does not guarantee that they have understood the underlying principles. Therefore, children should be explicitly provided with principles, so that they can figure out their own procedure. Perry (1991) and Alibali (1999), in agreement with Sylva, Bruner & Genova (1976) showed that giving procedural instructions to children may not be facilitating. Children who are provided with examples on how to solve a problem are limited in finding solutions, because they do not need to understand why they are applying that procedure. Moreover, when children must solve new problems similar to those they already know, they may tend to apply a familiar approach, a mental set (McNeil & Alibali, 2000) that their experience suggests as effective, even if the resort to mental sets can cause illogical errors. Considering that learning is defined as “success on similar problems” and transfer is defined as “success on dissimilar, but conceptually related problems” (Chi & Bassok, 1989), then the transfer of knowledge is the key to understand whether or not teaching provided a deep learning. According to Hiebert (1986), the transfer can be facilitated by the teaching instructions, if focused on the procedural approach (how) or the conceptual approach (why).

Mattews and Rittle-Johnson (2009) demonstrated that the conceptual instruction promotes better conceptual knowledge, and it is more effective than the procedural instruction. However, the greatest learning and transfer seems to be obtained when children are given instructions about the correct procedure and explanations of the underlying concepts, supported by students’ self-explanation. Son et al. (2008) showed how generalization for children is strictly connected with the simplicity of the examples provided during the teaching sessions, because only those similarities which are relevant for understanding a certain concept are relevant to young learners. On the contrary, expert learners can infer relevant aspects also from more complex examples.

For Kaminski et al. (2008) teaching mathematical concepts through generic instantiations (symbols) may be more effective for transfer because the concrete context should limit the applicability of the concepts learned.
Recent researches investigated also the individual differences in the way children combine the two types of knowledge in mathematics, demonstrating that some children have more conceptual knowledge, some others more procedural knowledge, and others an equal level of both (Hallett, Nunes, Bryant & Thorpe, 2012). Schneider and Rittle-Johnson (2011) focused on the effects of the children's prior knowledge on the relationship between the two types of knowledge, demonstrating that both the conceptual and procedural knowledge had a stable bidirectional relations, not moderated by the prior knowledge.

1.3 The symbolic language in numerical learning

Mathematics teachers often consider the definition of concepts as the clearest explanation for students. This teaching method strictly connects concepts to procedures, overlapping concepts and procedures during the learning sessions. This type of inaccuracy could generate confusion with respect to the concept of symbol itself.
During their entire lives, humans act and communicate by using symbols, in order to explain their ideas also when referring to objects that are not concretely present in the context. When we refer to generic quantities we use words, as "a lot", or "a little", but when we refer to precise quantities, we need numbers such as "23" or "12.324".
Our brain continually processes symbols to understand the reality. They are represented with different complexity levels (Borghesania & Piazza, 2017): the simplest levels consider symbols as physical entities. Humans, through the use of simple "objects", can describe complex and multilevel realities. Thanks to this ability they generate a rich and stratified knowledge of their meanings, which is commonly called semantic representation. Numbers, together with words, are the most important class of symbols we have. Studies on numerical cognition propose two systems of

representation of numerical cognition in children and adults (Gomez, Piazza, Jobert, Dehaene-Lambertz, Dehaene & Huron, 2015). The first system is called ANS (Approximate number system), it is independent from symbols and from language, it is approximate and based on analogies, and it precedes formal teaching (Odic, Darko & Starr, 2018). We use this system from our first year of life, as we saw in Dehaene, Molko, Cohen and Wilson (2004) and Dehaene (1997), and it can also be found in some animals. This system allows comparing, adding and subtracting quantities (Feigenson, Libertus & Halberda, 2013). It develops abruptly during the first year of life, then more slowly: during pre-adolescence, kids reach the same levels found in adults (Dehaene et al. 1998, Piazza, Pica, Izard, Spelke & Dehaene, 2013).

Conversely, the second system is based on symbols and language and allows accuracy; it is a kind of numerical cognition that lays its basis in formal, scholastic mathematics teaching and, more generally, in education. It is called ENS (Exact Number System) and it develops later than ANS, because its learning must follow the knowledge of a system of symbols. Indeed, its developing lasts for many years, until a formal mathematical competence is achieved (Feigenson et al., 2013; Gomez et al, 2015). Some studies focused on the relationship between non-symbolic knowledge of numbers and formal mathematical knowledge; Gilmore, McCarthy and Spelke (2010) showed that in both infants and older children the ANS ability seems to predict the future capability to use and to understand formal mathematics. Furthermore, it seems that working on the ANS with non-symbolic addition and subtraction, improves the ability to perform symbolic addition and subtraction in a formal way as adults do (Park & Brannon, 2013). Conversely, other studies showed that symbolic number tasks are more effective to predict future skills on formal mathematics (Holloway & Ansari, 2009). Despite this debate, some studies showed that when adults process numbers, they use both ANS and ENS systems (Castronovo & Göbel, 2012); this means that the ANS system is crucial to develop effective methodologies for mathematical teaching. Indeed, the ancient, approximate system for managing quantities is the base for the construction of a formal, cultural based mathematical knowledge (Feigenson et al., 2013). When children represent quantities in their mind they are able, unlike animals, to have knowledge of precise quantities greater than 3 (Berteletti, Lucangeli, Piazza, Dehaene & Zorzi, 2010). This ability allows humans to elaborate large quantities, and brings to the acquisition of a variety of skills during the pre-school and school age. Several studies, in addition, have tried to understand how children and adults mentally represent quantities. What emerged is that numbers are positioned on an ideal line, which in the first years of life is represented as a logarithm, since the distances between numbers are not represented in a proportional way. Later, with the experience, the mental number line becomes a straight line, although it is not entirely clear how this passage takes place (Berteletti et al., 2010). According to Siegler and Opfer (2003) and Siegler and Booth (2004), it is the learning of formal mathematics that plays an important role in this transition phase. Several studies showed that the formal learning of mathematics leads to the ability of manipulating numbers in a precise and accurate way. Hence, in the absence of a formal education, adults show an ability of manipulating quantities similar to that of young children, that is approximate and inaccurate (Castronovo et al. 2012; Pica, Lemer, Izard, & Dehaene, 2004). In order to provide better support to students who learn mathematics, it is important to consider the link between the system for representing approximate quantities (ANS) and the formal mathematics, by improving humans' innate abilities and knowledge.

Even knowing the innate sense of numbers of children, the individual differences in formal learning should be considered. Some students manipulate symbols as mathematical objects, others consider symbols as a procedure to be developed. Those who implicitly understand the general meaning of symbols succeed, but many others are likely to fail. A flexible thinking is needed to apply the concepts formally learned, but many people are more likely to apply procedures. As we saw, applying procedures could be economic in the short-term, but in the long-term flexibility is required to succeed (Gray & Tall, 1993). At school, it is common practice to provide precise definitions of mathematical concepts, focusing on the object and losing the inner process.

The use of concrete materials in primary school teaching could be effective, and we would analyze the benefits of concrete methods in the chapter 2.

Chapter Two

2.1 Concrete representations and abstract concepts

A number of recent studies compared the effectiveness of different teaching strategies in mathematics in primary school showing that the teaching is effective when both conceptual and procedural learning methods are implemented. Thus, both the conceptual and procedural knowledge are required for building generalizable knowledge (Hiebert & Wearne, 1996; Kilpatrick, Swafford and Findell’s, 2001; Crooks & Alibali, 2014).

Over the years, an amount of researches has been conducted to determine the role of concrete representation in student’s conceptual understanding (Montessori, 1917; Bruner, 1966; Piaget, 1970; Sowell, 1989; Martin & Schwartz, 2005; Brown, McNeil & Glenberg, 2009). A definition of “concrete” emerging from McNeil and Fyfe (2012) includes “materials that are grounded in previous perceptual and/or motor experiences and have identifiable correspondences between their form and referents. Abstract materials, in contrast, eliminate detailed perceptual properties and are more arbitrarily linked to referents” (p. 440). Concrete representation may help children to reason and to think about the concept (Sowell, 1989; Burns, 1996; Brown, McNeil & Glenberg 2009; Uttal, 2009). However, many studies showed that the use of concrete representation does not guarantee the transfer of knowledge in a variety of mathematical learning situations (Sowell, 1989; Goswami, 1991; Gentner, Ratterman & Forbus, 1993; Goldstone & Sakamoto, 2003; Sloutsky, Kaminski & Heckler, 2005; McNeil & Jarvin, 2007; Kaminski, Slousky, & Hecker, 2008; Martin, 2009; Uttal, O’Doherty, Newland, Hand, & DeLoache, 2009; Son, Smith, & Goldstone, 2011), since it may divert the children’s attention to some irrelevant perceptual details of the materials (Uttal, Scudder, & DeLoache, 1997; DeLoache, 2000).

The results of this set of researches generated strong resistance in the use of concrete materials by teachers and educators, although neither of the mentioned studies takes into account the principles for an effective use of concrete materials. Indeed, as many authors have reiterated over time (Montessori, 1917; Bruner, 1966; Goldstone & Son, 2005), the use of concrete material must be followed by a progressive shift towards the abstract. Despite these theories, there are few studies that tested the effects of explicit and gradual fading from concrete materials to abstract symbols. McNeil and Fyfe (2012) highlighted the incredible absence of specific studies on the transition from concrete to abstract, proposing in very recent years the testing of the "concreteness fading" method in order to clarify the implications of the methodology used with concrete materials. This method has been specifically tested in mathematical and science domain (Fyfe, McNeil, Son, & Goldstone, 2014) because of the strong presence of prior knowledge associated to these fields of study. Indeed, the use of concreteness fading has been applied only to these domains and it could be interesting to expand this experimentation to others, such as geography, geometry and physics.

Most of the researches on the use of concrete manipulatives suggest that the efficacy of a teaching based on concrete materials is affected by age and prior knowledge. A meta-analysis by Carbonneau, Marley and Selig (2013) analyzed 55 researches, focusing on the effectiveness of a mathematics teaching with manipulatives from kindergartners to college students. The examined studies compared the teaching with manipulatives in mathematics instruction to the teaching with only abstract math symbols. The authors found a significant effect in favor of the use of manipulatives with respect to the use of only abstract symbols. This effects, however, was moderate by instructional and methodological property of the examined studies. A separate analyses was conducted to see the learning outcomes of retention, problem solving, and transfer. It revealed a moderate to large effects on retention and small effects on problem solving and transfer in favor of using manipulatives over abstract math symbols. Empirical evidence appears to confirm that young children benefit more than teenagers and adults from concrete manipulatives for conceptual understanding

(Goldstone & Sakamoto 2003; Goldstone & Son 2005; Koedinger, Alibali & Nathan, 2008; Petersen & McNeil, 2013; Braithwaite & Goldstone, 2015).

A significant study by Kaminski et al. (2008), on the contrary, argued that concepts learned with generic and abstract examples allow to generalize mathematical concepts if they have been taught with “the use of generic instantiations” (p. 455). This study, however, even if much quoted in the debate on abstract vs concrete, should be used with caution if referring to the use of manipulatives for teaching to primary school children. Indeed, it was conducted on undergraduate college student, testing their transfer ability on a mathematical task after instructed them in two conditions: generic-abstract vs contextualized-concrete. Furthermore, the concrete instruction was made through graphic images and not by concrete objects. “The elements were three images of measuring cups containing varying levels of liquid. The same mathematical rules were presented in slices of pizza or tennis balls in a container, rather than portions of a measuring cup of liquid” (p. 454). Giving that methodology, it is difficult to connect these results to those referred to primary school children taught with concrete manipulatives. Children instructed with concrete manipulatives learn through a perceptual exercise which effectiveness, as we have seen, is linked to age.

The cited Carbonneau et al. meta-analysis, reported that the use of manipulatives is affected by many variables: the perceptual richness of the object, the level of guidance given by teachers, and the level of previous knowledge and development status of students. Perceptually rich manipulatives can result in a more complex memory recovery process due to the distracting elements, preventing an effective transit to abstraction (Son, Smith, & Goldstone, 2008). According with previous studies, Laski, Jor’dan, Daoust, and Murray (2015) proposed four principles for maximizing effectiveness of manipulatives: (1) use a manipulative consistently, over a long period of time; (2) begin with highly transparent concrete representations and move to more abstract representations over time; (3) avoid manipulatives that resemble everyday objects or have distracting irrelevant features; (4) explicitly explain the relation between the manipulatives and the math concept.

2.2 Concrete manipulatives and the case of Montessori

The use of manipulatives for children’s learning was studied for decades, since Montessori (1917), Piaget (1952) and Bruner (1996) theorized that babies and children had not an innate capacity to think to abstract object. Children during their development acquire concept through the manipulation of real objects they found in the environments. Montessori was the first to develop many concrete materials in order to teach mathematical concepts. As we saw, she specifically designed concrete objects to make children able to make links, to infer; one of the most known is the place value material (Lillard & Else-Quest, 2006). Thus, for Montessori, concrete materials helps the children to find connections between the environment and the laws governing it, and learn through active work. Recent researchers confirmed Montessori’s theory, especially in mathematics, providing evidence to a better learning for children who attended Montessori program (Lillard & Else-Quest, 2006; Lillard, 2012; Dohrmann, Nishida, Gartner, Kerzner Lipsky, Grimm, 2007).

The study by Laski et al. (2015) largely confirm how Montessori method represents a concrete example of the application of these principles in practice. According to Laski et al. (2015), however, even though the material is well structured children must be helped to generalize the concept. From a research carried out between children of Montessori schools and not, on the knowledge and understanding of the base 10, the children of Montessori kindergarten showed a better performance than the non-Montessori. However, this positive gap decrease during first and second primary school classes.

Scoppola (2011), reads Montessori materials in the light of recent discoveries in neuroscience. Cognitive tests and PET (electroencephalogram and positron emission tomography) show us how the brain represents mathematical concepts and what are the involved brain areas. The brain perceives the approximate quantities with the area of perception of the forms. This area is close to the one that supervises the hands movements. But the area involved with the symbolic-linguistic is very far from the perception area. It means that, to understand mathematics our brain activate two different, far areas. It seem that children's brain are often not yet coordinated in their long-distance connections. This implies that the understanding of mathematics is relegated above all to the symbolic linguistic area, and the use and manipulation of symbols. Children often struggle to understand the concrete meaning of symbols and this leads them to math panic. PET shows that children who easily understand mathematics are the same that use symbolic and perceptive areas in parallel. On the contrary, the children who barely understand mathematics use only the symbolic area.

Montessori's concrete materials bring the two brain areas in contact, stimulating both of them. Her materials start from a stimulation of the perceptive area, by providing geometrical and arithmetic aspects through an object, to arrive gradually to the construction of symbols and specific vocabulary. Concrete Montessori materials allow the children to use the both brain areas needed for a deep understanding of mathematical concepts, by providing a perceptive experience together with the symbolic notions. Many recent studies confirm that the sensory-motor experience as the first vehicle of mathematical ideas allows children to understand the symbolic system and the concepts (DeLoache, 2004; Fyfe & McNeil 2009; de Hevia, Vallar & Girelli, 2008; Gentner & Markman, 1997).

Maria Montessori spent decades perfecting the development material used in her schools. Likewise, she considered essential to present to children only objects with specific characteristics (shapes and colors that have a specific purpose), not commonly used so as not to divert the attention from the concept they were conveying. Learning mathematics and geometry through the involvement of the senses was one of the first fundamental discoveries of Montessori in structuring the development material. There are many materials of Montessori schools that perform the function of conveying generalizable concepts such as height, length, weight, etc. The possibility for the child to move in the environment, exploring the material with the senses even for very long times and repeatedly for days, weeks, months, allows him to explore not only the material but the concept itself, in order to draw conclusions from it, comparable with those of his peers and the teacher. The Montessori material is designed to give children not only sensory knowledge (tactile and visual) but also to provide operational schemes and figurative patterns. The same material can be used at three years for sensory knowledge, subsequently to learn a procedure, and finally to proceed towards the next and more complex step of abstraction. The phases through which learning occurs in the Montessori method environments follow children’s development to allow spontaneous theories to emerge. Montessori believed that the child's brain is not a sponge, but an already structured organ that only learns what resonates with its previous knowledge. Indeed, Montessori observed that children independently chose materials that resonated with their previous knowledge. She affirmed that the child, unknowingly, choose the most suitable development material for him, and stay with it for the time necessary to "convince himself" of a given concept. If the change of conceptions takes place at school, according to Hatano and Inagaki (1984) it is "induced", while if it happens thanks to the enrichment of the experience it is "spontaneous". According to Montessori, this change can occur spontaneously even at school through the experience of the child with the environment and educational material. However, Montessori development materials should be used according to the educational principles she proposed, to be really effective. Lillard (2016), summarized nine fundamental principles of Montessori education: 1) movement and cognition are closely entwined, and movement can enhance thinking and learning; 2) learning and well-being are improved when people have a sense of control over their lives; 3) the ability to direct one’s attention in a sustained and concentrated way fosters an array of positive developments and is itself trainable; 4) people learn better when they are interested in what they are learning; 5) tying extrinsic rewards to an activity, like money for reading or high grades for tests, negatively impacts motivation to engage in that activity when the reward is withdrawn; 6) collaborative arrangements can be very conducive to learning; 7) learning situated in meaningful contexts is often deeper and richer than learning in abstract contexts; 8) particular forms of adult interaction are associated with more optimal child outcomes; 9) order in the environment is beneficial to children (p. 28).

We have little research on Montessori education, not even focused on the benefits of the manipulatives she developed. Rathunde and Csikszentmihaly (2005) compared Montessori and not-Montessori middle school students, focusing on social climate and motivation, reporting that Montessori students had higher intrinsic motivation than the others. Besançon and Lubart (2008) focused on creativity in primary school, reporting that Montessori students had an higher level of creativity with respect to not-Montessori ones. The longitudinal study of Laski, Vasilyeva and Shiffman (2016) tested children in Montessori and not-Montessori school on the learning of place value. Their results showed that Montessori children in kindergarten had better performance than others children but this advantage was not confirmed over time. On the other hand, the Mix, Smith, Stockton, Cheng and Barterian study (2016), similar to the Laski et al. (2016) study, showed that Montessori children performed similarly to others in kindergarten, but they had significantly better performance at the end of the second grade. As Lillard (2016) reported, however, research on Montessori should be improved, because too little is still known about Montessori education benefits.

2.3 Concreteness Fading

Concreteness Fading is a method of teaching explored by researchers in recent years (for a complete review see Fyfe & Nathan, 2018). The advantages and disadvantages of concrete and abstract representations suggested that the two teaching methods could be integrated, to take advantage of both. Some children may fail to link spontaneously the concept learned with concrete materials to related symbols (Uttal, 2003), so it is recommended to teachers to explicitly provide this link. To establish this connection, and promote both conceptual learning and transfer abilities, teaching should provide multiple representations (Ainsworth, 1999; McNeil & Jarvin 2007) linked to each other (Nathan, 2012; Goldstone & Son, 2005, 2009; Fyfe et al. 2018).

The concreteness fading instructional approach is based on Bruner’s theory (1966) who claimed that children should be helped to develop a set of mental frames to pass from concrete thinking to conceptually and abstract thinking. Bruner argued that this progression supports the cognitive development of children, that begins with a concrete relationship with the environment and move on to rules and concepts. Bruner’s theory showed that the most effective way to teach follows three progressive phases. During the first, enactive phase, concept is introduced by a physical model; during the second, iconic phase, concept is re-introduced in a graphic representation linked to the physical model; during the third, symbolic phase, the concept is showed through the symbolic representation. According to Fyfe and Nathan (2018) we can define Concreteness Fading as a “three-step progression by which a concrete representation of a concept is explicitly faded into a generic, idealized representation of that same concept” (p. 9).

Benefits of concreteness fading are well explained by Fyfe, McNeil and Borjas (2015), that summarized three crucial aspects:

1) the abstract representations could be ambiguous for some children, and the concrete objects could help to interpret them. “If the concrete materials precede the abstract materials, the learner can successfully interpret the ambiguous abstract materials in terms of the already understood concrete context. This process may underlie children’s improved performance on symbolic equations when they are preceded by equations constructed from concrete manipulatives” (Fyfe et al., 2015, p. 12);

2) action and perception allow children to represent experiences in a symbolic way. “… these embodied experiences are linked to the abstract symbols, facilitating the mapping between abstract concepts and perceptual processes” (Fyfe et al. 2015, p. 13);

3) as theorized by Bruner (1966), learners who use concrete materials have a variety of mental images that can be retrieved when the symbolic system does not seem to have a clear meaning. “The stored images provide learners with an accessible, back-up representation that can be used when the abstract symbols are detached from their

referent … Concreteness fading not only encourages teachers to focus on both concrete and abstract understanding but also provides learners with abstractions explicitly linked to a stock of images" (Fyfe et al., 2015, p. 13).

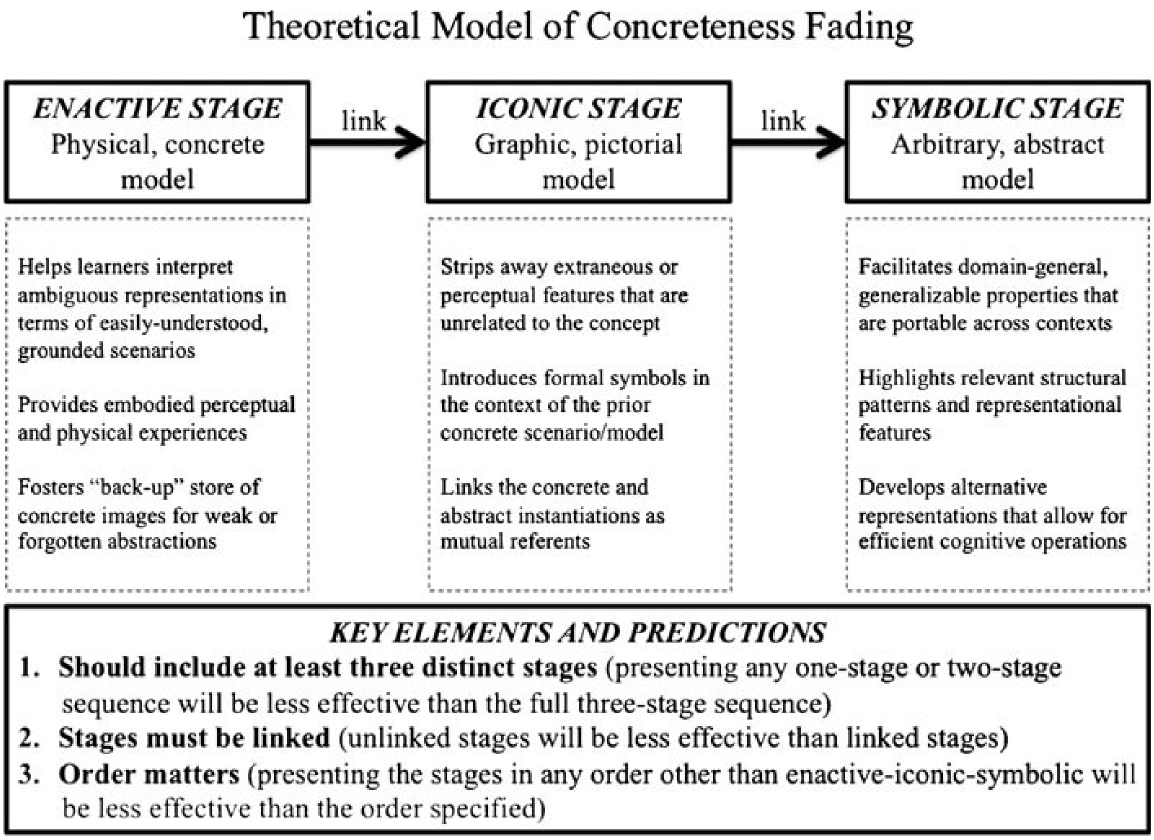

Fig. 1. *Theoretical model of concreteness fading. Fyfe, McNeil, Son, Goldstone (2014).*

Some studies also demonstrated that the order of the progression matters; thus, it is more effective for learners to start from a concrete object and proceed through a more generic and abstract representation, instead of the reverse progression (Fyfe, McNeil, & Borjas, 2015; Ottmar, 2017). However, this method of teaching is affected by many aspects, such as the age of learners, their prior knowledge, the learning time, the study domain (Tapola et al. 2013; Vershaffel, 2016; Fyfe, McNeil, Son, & Goldstone, 2014).

One aspect that has not been investigated so far is how the object used in the first phase of concreteness fading affects learning (Brown, McNeil, & Glemberg, 2009). Fyfe and colleagues (2015) tested if the concreteness fading could help children to extend their knowledge beyond a simple instructed procedure. In their first experiment, they instructed 64 2nd grade children with low prior knowledge on equivalence by assigning them to one out of four learning conditions: concrete materials (children received instructions accompanied by the use of concrete materials, first by sharing stickers among two puppets and then by putting the same quantity of some objects on a scale), abstract (children received instructions to solve six abstract, symbolic math equivalence problems written on paper), concreteness fading (children received instructions in three formats: first with concrete materials, then with "fading" worksheets, and finally with abstract exercises), introduction to concreteness (children received instructions as in the concreteness fading condition, but in reverse order. Children were taught first through abstract, symbolic problems, then through worksheets, and finally through concrete materials). All children solved six exercises of equivalence. They found that children in concreteness fading condition showed significant better transfer than children in the other conditions. In their second experiment, it was also examined the possibility that concreteness fading may result in a better transfer because starting the teaching phase with concrete materials could make the children more attentive during the explanation. The second experiment was conducted with 22 1st and 2nd grade children, assigned to one out of two conditions: concreteness fading (children received instruction as in experiment 1) and play-to-abstract (children were shown concrete materials and worksheets, but instructions were given only during the abstract phase). It was found again that children in concreteness fading condition had better results in the transfer of knowledge.

Finally, the third experiment showed that also children with high prior knowledge can benefit from concreteness fading, by testing 50 2nd and 3rd grade children, with the same design and procedure of experiment 1, but involving also an advanced problem-solving procedure.

Fyfe and McNeil demonstrated that primary school children in concreteness fading condition, both with low and high prior knowledge, benefit from a teaching method that starts with concrete representation and fade into abstract, symbolic representation. Even considering that the authors focused on the role of the three-step teaching method, they do not explicitly explained how they chose the concrete materials used.

Chapter Three

The first study

3.1 Introduction

As we discussed in previous chapters, learning mathematics in primary school is essential for the develop of future skills. The debate on the most effective teaching strategy is still open as the discussion on the use of concrete materials. As we have seen, both teaching through concrete material and through the symbolic system have strengths and weaknesses (Brown et al. 2009; Carbonneau et al., 2013; Goldstone & Son, 2005; Gray & Tall, 1993; Kaminski et al. 2008). A teaching strategy that combine the positive aspects of both is identifiable in the concreteness fading. Consequently, it is critical to understand the most suitable type of concrete material and the most effective use of it, in order to improve children's transfer of knowledge. We identified in the literature some basic aspects that the concrete material should have to be effective, and the Montessori materials of mathematics meet these criteria (Laski et al., 2015).

Consistently with this theoretical framework, in the first study we aim to extend prior studies by examining the three teaching strategies: concrete, concreteness fading, and abstract, and to compare two different types of concrete materials. To verify the role of these two aspects we choose to use the experimental methodology, and we measure the transfer abilities by administering a transfer test that allow us to examine both.

Thus, the current study investigate: 1) the role of the type of concrete materials (Montessori versus other materials), 2) the role of the teaching strategy (concreteness fading, concrete and abstract) in learning of mathematics.

We used the research paradigm by Fyfe and colleagues (2015), which experimented three teaching conditions to introduce equivalences. Children learn through six reflexive equivalence exercises, driven by the researcher. The three teaching strategy differ as follows: children in *Concrete* condition were provided with examples with concrete materials, children in *Abstract* condition were provided with symbolic examples, children in *Concreteness Fading* condition were provided with concrete materials, iconic worksheet, and symbolic worksheet (in this order of progression). Although the cited researches examined the transfer ability by comparing these three strategies, our experiment add a comparison of the concrete materials used in Concrete and Concreteness Fading condition. Children in Concrete condition with the Montessori materials solved the exercises by using number rods and abacuses, children in Concrete condition with Other materials solved the exercises by using puppets and a scale as in Fyfe et al. (2015). Following the teaching phase, all children completed a symbolic test to evaluate the transfer ability acquired.

We focus on equivalence, as in many previous studies (Rittle-Johnson, 2006; Perry, 1991; Fyfe, McNeil & Borjas, 2015; McNeil & Alibali, 2000; Falkner, Levi & Carpenter, 1999), since it is useful in showing both the conceptual understanding and the ability to apply it to similar exercises, as the applying of the procedure with no conceptual understanding of equivalence produces errors.

Concerning the Montessori materials, in their meta-analysis Carbonneau et al. (2013), highlighted three variables to be considered in experimenting the use of concrete materials: the perceptual richness of the object, the level of guidance given by teachers, and the level of previous knowledge and development status of students.

With respect to the first variable, we used a) Montessori materials of number rods and abacuses that, as largely reported, do not offer any perceptive distractions; b) Fyfe et al. (2015) materials, in order to partially replicate their study. Fyfe and colleagues used: 1) the puppets of the monkey and the frog, that could offer perceptual distractions, and that are commonly used by children for playing; 2) the balance scale, that could be considered a didactic object with low distraction.

With respect to the second variable, Carbonneau and Marley (2015) demonstrated that an high level of instructional guidance improve children learning. We offered an high level of instructional guidance in our experiment, replicating the script experimented by Fyfe et al. (2015) by offering to the children continue feedback, the possibility to repeat the exercise twice, and giving them a positive feedback at the end of each exercise. This teaching approach was the same for all the children regardless of the condition.

With respect to the third variable, we pre-tested the children in order to examine only the ones with no prior knowledge on mathematics equivalence.

As said, the Montessori manipulatives we chose are number rods and abacuses. Number rods are one of the basic sensorial materials in Montessori curriculum. “The sensorial materials in Montessori are designed to introduce mathematical concepts. The transition from sensorial to math materials is a simple step: a new set of rods is introduced, just like the Red Rods, except on the Number Rods each 10 cm unit is painted alternately red and blue… The teacher shows the child how to count the units on each rod, arriving at the cardinal number with which it is identified, and to name the rods, “one”, “two”, “three”, and so on, while touching each rod… To take the child from a concrete understanding of number, based on the length of the rods, into this abstract realm, the teacher shows the child how to place Sandpaper Number cards beside each rod” (Lillard, 2016, pp. 62-63).

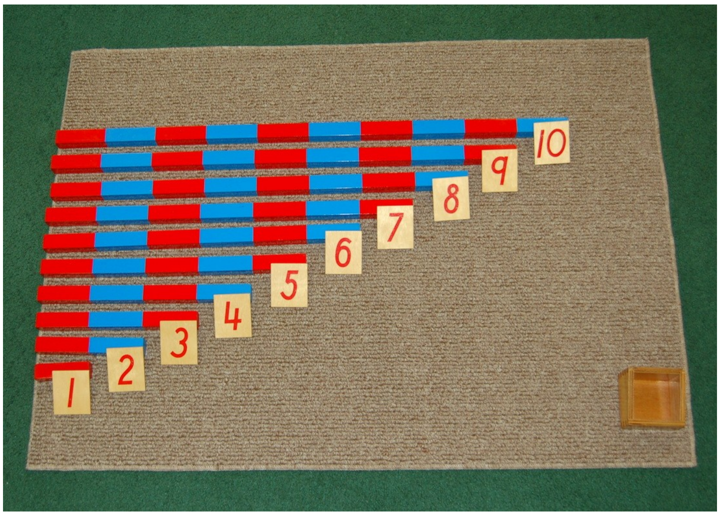

Fig. 2. *Montessori Number Rods*

The Small Beads Frame in the Montessori curriculum is one of the activities that follow the Golden Beads of the Decimal System, on which children in primary school could count and make operations such as addition and subtraction. Montessori (2013b) describes the Beads Frames as follows: “This material is made by a frame that is similar to the common abacuses. The Small Beads Frame extends only to the units of thousands, the Large Beads Frame to the units of millions… The beads on the Frame had symbolic value. Each green bead indicates a unit, just as the looses beads of the Decimal System material, each blue bead has the value of a dozen, like the bar of ten. Each red bead has the same value as a square of a hundred beads, and finally each green bead belonging to the fourth thread has the value of a cube of a thousand beads. … In this way, it is the *position* that makes the value, instead of the quantity” (p. 121-125). The child counts by moving the beads, in order to see what happens when he moves on the 10.

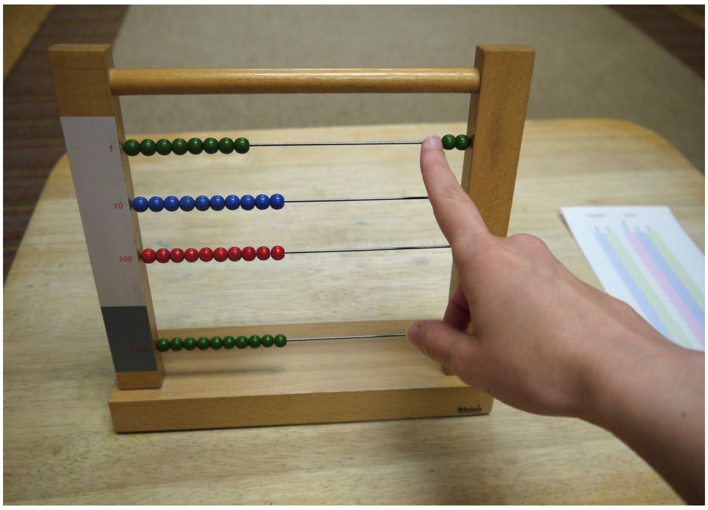

Fig. 3. *Montessori Small Bead Frame*

As we saw, the Montessori abacus is made of four lines: units, tens, hundreds, unit of thousands. In our experiment we used only the unit line. Indeed, the child had to operate on small numbers. He had to work on the two sides of an equivalence by moving the beads and adding or subtracting the requested quantity.

The materials we used had been chosen because they seemed suitable to teach equivalence. However, it is crucial to highlight that we presented them to children in a single short teaching session, so we used Montessori materials such as didactic materials, not as development materials. Montessori mathematics materials, indeed, are designed in a special progression, in order to drive the children from a more concrete to a less concrete representation of the concept. The number rods are a start material, thought to teach the numbers from 1 to 10 and their relations. After a long work, during years, the children in Montessori classroom arrive to use the Small Bead Frame, and they operate with big numbers. It is a long route that show its effectiveness if developed through all its stages.

We did not work in a Montessori environment and children could not use materials for a prolonged time. Thus, we can not claim to have used the Montessori method, but only to have used, in the context of our research design, some materials deemed suitable for the purpose. Rather, we used for the first time Montessori mathematics materials through the concreteness fading strategy.

Thus, the current study extend prior studies by examining the benefits of the Montessori materials (Laski et al., 2015; Laski et al., 2016; Fresco, 2000; Lillard, 2012; Lillard, 2016; Rathunde & Csikszentmihalyi, 2005) both in concrete and concreteness fading strategy and verifying if concreteness fading confirms its effectiveness (Fyfe & Nathan, 2019; Ottmar & Landy, 2017; Fyfe et al., 2015; Fyfe et al., 2014; McNeil et al., 2012).

We hypothesize that Montessori concrete materials, designed with the specific intention of conveying the concept without distracting elements and with the greatest possible level of conceptual clarity and transparency, could be more effective than other objects.

Moreover, we hypothesize that concreteness fading strategy, which is not typically used in the Montessori method, may provide children with help in moving form concrete to abstract, if in the enactive phase the material used is the Montessori one.

Specifically, we test the following predictions:

1. The Montessori concrete material has more positive effects on transfer of knowledge than other materials (concrete materials effect on transfer).

2. The Concreteness Fading is a more effective strategy in the transfer of knowledge than the Concrete and the Abstract (strategy effect on transfer).

Moreover, we analyze the effect of strategy and material on the teaching phase (material and strategy effect on learning). The teaching phase data allow us to investigate the learning differences across conditions. Even if our hypotheses concerned transfer ability, the differences observed during the teaching phase may show the immediate response of the children understanding of the teacher's explanation.

On the other hand, the results of the transfer test show us if learning has been effective in terms of the ability to use it in different contexts and generalize.

3.2 Method

Participants

Participants in this study include 181 second-grade children from two public elementary schools near Rome, in Italy. Before the study started, written parental consent was received. We excluded from this sample children with disability or learning difficulties, since they could not solve the exercises without specific support.

Only 167 meet the criteria and therefore were included in the study, because they could not solve any of the four equivalence exercise of the pretest. Since the subject of the equivalence is unknown to second-class children, only 10 children answered correctly to more than one pretest exercise, as expected. Thus, the final sample was of 167 Italian children (78 boys, 89 girls) born between September 2009 and March 2011, so that at the time of the study they were about 7 years old (M = 7 years 8 months, range = 7-9 years).

Design

The study consisted of two phases: a teaching phase and a transfer test phase. For the teaching phase children were randomly distributed among one out of five conditions ensuring gender and age homogeneity: Concreteness Fading with Montessori material (n = 29), Concreteness Fading with Other materials (n = 29), Concrete with Montessori material (n = 28), Concrete with Other materials (n = 29), Abstract (n = 52).

Both phases had a total duration of 35 minutes. There were no significant differences between groups in terms of age $F_{(4,161)}$= .142, p = .96, or gender X^2 (4, N=161)= 2.51, p= .64.

Instruments

The instruction, pretest and post-test are included in Appendix.

Pretest: the pretest was the same already used by Fyfe and colleagues (2015). It included four mathematical equivalences: two exercises with two addends on the left side of equal sign and two on the right side; two exercises with three addends on the left side of equal sign and two on the right side. All items were scored on a 0/1 scale, 0 if correct and 1 if incorrect. The score obtained was calculated on the basis of the number of mistakes, between 0 and 4.

Teaching phase: all children solved the same six equivalence exercises as in Fyfe and colleagues (2015). All the exercises had four addends, they were reflexive (i.e. 2+1=2+1), with a blank space on the fourth place. Children had two possibilities to solve each item. All items are scored on a 0/2 scale, 0 if correct, 1 for one mistake, 2 for two mistakes. If children could not solve the item after two attempts, they were given the solution. The score obtained was calculated on the basis of the number of mistakes, with a value between 0 and 12.

Post-test: at the end of the teaching phase, children solved a transfer test in order to verify the transfer of learning. Many previous studies (Alibali, 1999; Fyfe et al., 2012; Mattews et al., 2009; Perry, 1991; Fyfe, 2015) used this kind of transfer test. It included four equivalence exercises and a multiple-choice problem. The test was more complex than the exercises submitted during the teaching phase, but such that they could be solved by logical reasoning and by applying the knowledge acquired during the teaching phase. The exercises had three addends instead of two; the position of the blank space on the third place instead of the fourth; the presence/absence of a repeated number on the right and left side. All items were scored on a 0/1 scale, 0 if incorrect and 1 if correct, so that the final score corresponded to the number of correct answers (dependent variable), therefore a value between 0 and 5.

Materials

We used different materials depending on the teaching condition.

Montessori - Concrete (see Fig. 4). Two series of Montessori small number rods and two Montessori small abacuses.

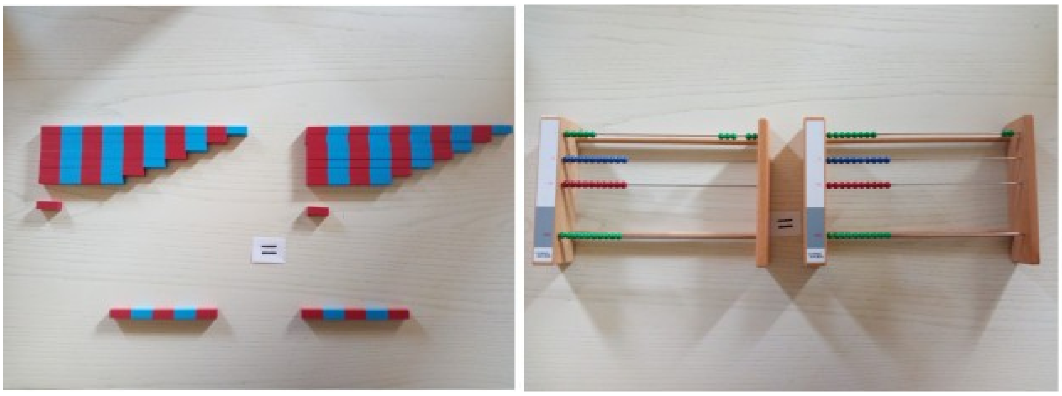

Fig. 4. *Montessori materials used in concrete condition of teaching. Number Rods and Small Bead Frame.*

Other Materials - Concrete (see Fig. 5). Two puppets (a monkey and a frog) with a box of stickers and a balance scale with eighteen small plastic bears.

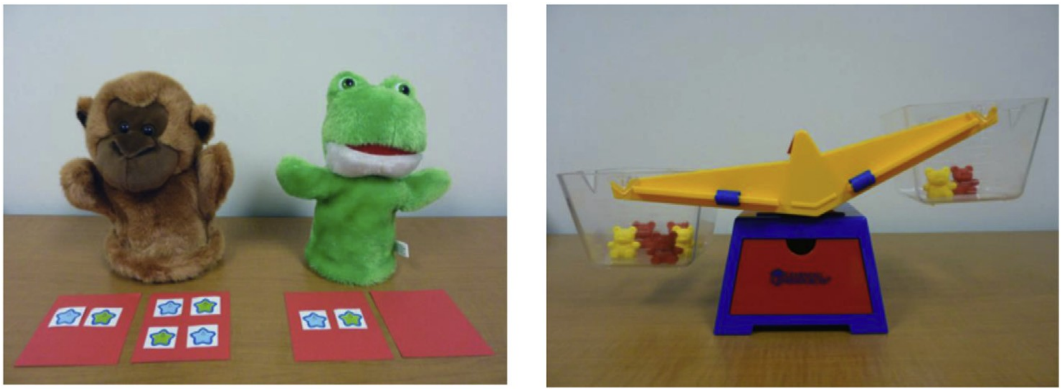

Fig. 5. *Other materials used in concrete condition of teaching. Puppets and scale.*

Montessori - Concreteness Fading (see Fig. 6). Two series of Montessori small number rods and two Montessori small abacuses (enactive phase); a worksheet where images of concrete material and numbers are placed side-by-side (iconic phase); a worksheet with numbers only (symbolic phase).

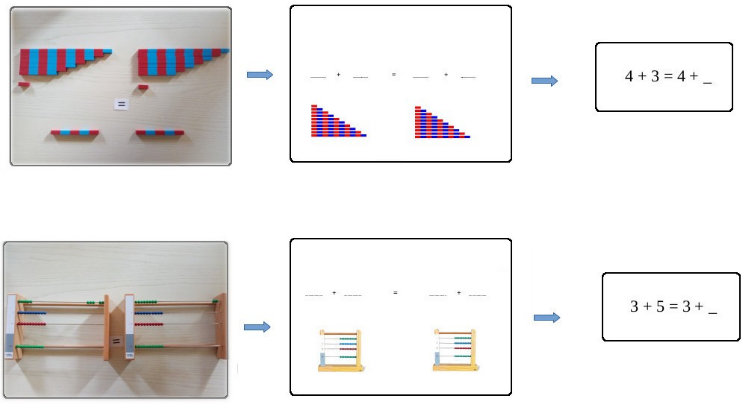

Fig. 6. *Progression of materials used in concreteness fading condition with Montessori materials.*

Other Materials - Concreteness Fading (see Fig. 7). Two puppets (a monkey and a frog) with a box of stickers and a balance scale with eighteen small plastic bears (enactive phase); a worksheet where images of concrete material and numbers are placed side by side (iconic phase); a worksheet with numbers only (symbolic phase).

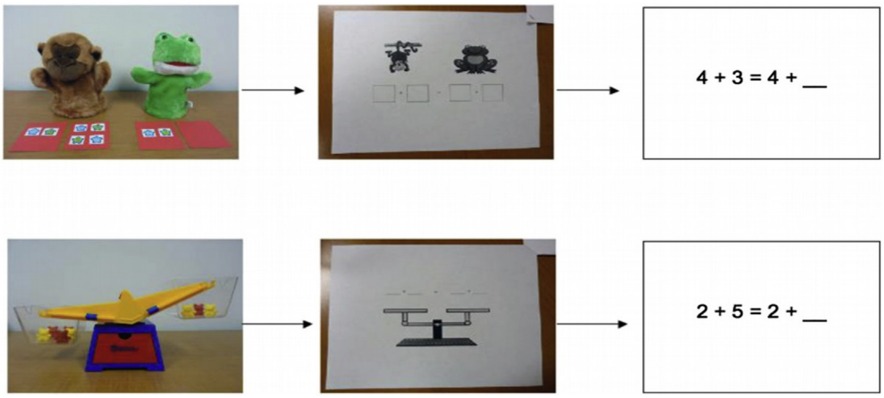

Fig. 7. *Progression of materials used in concreteness fading condition with Other materials.*

Abstract. A worksheet with the exercises in symbolic form.

Procedure

Pretest: all children completed the pretest on February 2017, during four consecutive days, in the morning. The pretest was concurrently solved by children in each classroom, who did not receive any explanations about the equivalence. The pretest had not a standard time, since children delivered the exercises when completed.

Teaching phase: the teaching phase took place from March to May 2017. All children received instructions individually, inside a quiet classroom of the school. Regardless of the condition, all children received the same amount of information. Thus, the difference between the conditions was the concreteness or abstractness of the format; the instructions were consistent across all the conditions. Each child participated to a single session of the teaching phase. If the child answered correctly, he received a positive feedback; if the child made a mistake, it was provided with an explanation, so that he could try to answer again. If the child was still wrong, the correct answer was provided. If the child answered correctly, he proceeded to the next exercise.

Montessori Concrete:

The children were told that each rod corresponded to a quantity, from 1 to 10. Then the concept of equal was presented showing a note with the symbol and explaining that in the equivalence the quantities on the right and left of the equal had to be the same. We taught the first exercise (4 + 3 = 4 + _) by asking the child to position the rods 4 and 3 on the left side of the equal symbol, and the 4 rod on the right side. The child was asked what quantity should be added on the right of the equal in order to obtain two equal quantities on both sides. The children who worked in this condition carried out the first three exercises with this procedure.

We submitted the following three exercises with the abacuses.

We taught the fourth exercise with the abacus (5 + 4 = 5 + _) by explaining to the child that there was an abacus on the right and one on the left of the equal, and that we were going to use only the units beads of each abacus. Subsequently we showed the use of the abacus, explaining that the desired quantity is obtained by moving the beads from left to right of the thread. Hence, we asked the child to move 5 beads on the left abacus, and then 4 beads. Then, he was asked to move 5 beads on the right abacus. The child was asked what quantity should be added to the right of the equal in order to obtain two equal quantities on both sides. The children who worked in this condition carried out the last three exercises with this procedure.

Other materials Concrete:

The materials used by Fyfe and colleagues (2015) were two puppets and a balance scale. The concept of equal was presented by showing a note with the symbol and explaining that in equivalence the quantities on the right and left of the equal had to be the same. We taught the first exercise (4 + 3 = 4 + _) with the two puppets: the Monkey and the Frog. Each puppet had two small red squares of paper in front of it, to collect stickers. The child was asked to share the stickers between the two puppets, giving to both of them the same quantity. We asked the child to place 4 stickers in the monkey's collector, and then 3 stickers. Then, the child placed 4 stickers in the frog's collector. The child was asked what quantity should be added to the right of the equal in order to obtain the same quantities on both sides. The children who worked in this condition carried out the first three exercises with this procedure.

We presented the following three exercises with a balance scale with two empty bins, and small plastic bears to be placed on the two bins. We taught the first exercise with the balance scale (5 + 4 = 5 + _) explaining to the child that there were two bins, one on the right and one on the left of the balance, and that we wanted to have the same quantity of bears on each bin, in order to make the scale balance. The child was asked to put 5 bears on the left bin, and then 4 bears; then, the child placed 5 bears on the right bin. The child was asked what quantity had to be added to the right in order to to obtain two equal quantities on both sides. The children who worked in this condition carried out the last three exercises with this procedure.

Concreteness Fading (Montessori and Other materials):

The teaching phase in concreteness Fading condition, as we saw, toke place in three phases: enactive, iconic, and symbolic.

The first phase (enactive) exactly follows the procedure used with the Concrete groups, and it concerned the first exercise with the number rods and the fourth exercise with the abacuses for the Montessori condition; the first exercise with the puppets and the fourth with the balance scale for the Other Materials condition.

In the second phase (iconic) a worksheet was used immediately after the first exercise with the concrete material, so the child was explicitly told that it was the same exercise he had just done, but with the graphic reproduction of the material used. The child was asked to write the numbers while imagining to use concrete materials, so he wrote 3 and 5 on the left side of the equal, then 3 on the left side. Finally, he was asked which quantity he had to add on the right side to have the same quantities on both sides. The feedback we gave followed the same script used in the concrete phase. We used the worksheet for the second and the fifth exercise.

For the third phase (symbolic), a paper with an equivalence exercise was provided to the child, with one blank space on the fourth place. We asked the child to remember both the exercise just done first with concrete manipulatives and then with iconic worksheet. This time, the child had only numbers, so we pointed to the numbers with the finger (4+2=4+_), in order to show him that on the left side of the equal we had a four and a two, and on the right side we just had a four, and then we asked him to find the right quantity so to have the same quantities on the both sides. The feedback, also in this phase, followed the script used in the two previous phases. We used the paper with symbolic exercises for the third and sixth exercises.

The concreteness fading procedure consists in linking the three phases; indeed, the children are always invited to remember the previous phases while solving the exercises, in order to better underline the tight connection between concrete materials and symbols.

Abstract:

The teaching began by showing the child the equal sign, and then by pointing with the finger the left and the right side of the equivalence. Thus, he was asked to observe the quantities on the left, and to identify the missing quantity on the right to obtain two equal quantities on the two sides of the equivalence. The child received the feedback with the same script used for the others conditions. All six exercises were solved in this way.

Transfer Test: immediately after the teaching phase every child completed the Transfer Test. The child had to solve all the exercises on his own. Thus, during the testing feedback no further explanations were provided. There was not time limit to complete the test. Once the test was completed, the child was taken back to the classroom and the whole procedure was restarted with another child.

3.3 Results

3.3.1 Transfer Test

The transfer test showed low results for the complete sample, as shown in Table 1, which was foreseeable because it was designed to accurately set the transfer ability. Therefore, it was composed by elements not explained during the instructions phase, but deducible from what had already been learned through the logical reasoning.

Table 1. Transfer test score by condition (Concreteness Fading, Concrete, Abstract) and Materials (Other materials, Montessori)

Strategy	Materials	N	Mean	Std. Deviation	Min	Max
Concreteness Fading	Other Material	29	2.21	1.634	0	5
	Montessori Material	29	2.59	1.701	0	5
	Total	58	2.40	1.667	0	5
Concrete	Other Material	29	.62	1.147	0	4
	Montessori Material	28	.43	.742	0	3
	Total	57	.52	.944	0	4
Abstract		52	1.42	1.719	0	5
	Other Material	58	1.41	1.390	0	5
	Montessori Material	57	1.51	1.221	0	5
	Total	115	1.46	1.305	0	5
Total		167	1.46	1.667	0	5

No effect of the gender on transfer test performance occurred $F_{(1,157)}=2.55$, $p= .11$, $\mu^2 = .01$.

To test our first prediction that Montessori materials has more positive effects on transfer than other materials, we performed a 2 (materials*:* Montessori, Other materials) X 2 (strategy: Concrete, Concreteness Fading) ANOVA tests with the correct (out of 5) transfer test score as dependent variable. The Abstract group has not been analyzed in this analysis, because we were verifying the effect of the type of concrete materials used. Thus, the analyzed sample was made up of 115 children. The four groups were well matched in in terms of gender $X^2_{(3,\ N=\ 115)}$= 5.57, p= .13 and age $F_{(3,111)}$= .67, p=.56.

As shown in Table 2, there was a main effect of the Strategy on transfer test score. Conversely, despite our first prediction, results show no effect of the material on the transfer test score and no interaction effect between Material and Strategy.

Table 2. Analysis of variance (Transfer Test Score) by Material and Strategy

Source	Type III Sum of squares	*df*	Mean Square	*F*	Sig.	Partial Eta squared
Corrected model	103.166[a]	3	34.389	18.398	.000	.332
Intercept	245.276	1	245.276	131.222	.000	.542
Material	.252	1	.252	.135	.714	.001
Strategy	100.719	1	100.719	53.885	.000	.327
Material * Strategy	2.346	1	2.346	1.255	.265	.011
Error	207.478	111	1.869			
Total	559.000	115				
Corrected total	310.643	114				

a. R Squared = .332 (Adjusted R Squared = .314)

To test our second prediction, that concreteness fading could be a more effective strategy than concrete and abstract, the three teaching strategies were compared (Concrete, Concreteness Fading, Abstract) to see the effects on transfer. Thus, we analyzed the entire sample. The three groups were well matched in in terms of gender $X^2_{(2,\ N=\ 167)}$= 1.59, p= .45 and age $F_{(2,164)}$= .11, p=.9. An one-way ANOVA was carried out with the Strategy as independent variable and the Transfer Test score as dependent variable.

Consistent with our prediction, results showed a main effect of the Strategy as shown in Table 3. Post-hoc comparison with the Tukey HSD showed a significant difference between Concreteness Fading and Concrete, with a significantly higher score of the Concreteness Fading group (p<.001); between Concreteness Fading and Abstract, with a significantly higher score of the Concreteness Fading group (p=.002); between

Concrete and Abstract, with a significantly higher score of the Abstract group (p=.005).

Table 3. Analysis of variance (Transfer Test Score) by Strategy (Concreteness Fading, Concrete, Abstract)

	Sum of squares	df	Mean Square	F	Sig.	Partial Eta Squared
Between groups	100.631	2	50.316	22,872	.000	.22
Within groups	360.782	164	2.200			
Total	461.413	166				

3.3.2 Teaching phase

During the teaching phase, each child could make a maximum of 2 errors per exercise, so the number of total errors could range between 0 and 12.

As showed in Table 4, the children made a few errors across the six exercises. This is not surprising, because the exercises proposed were simple, and the children received a feedback every time they gave an answer.

Table 4. Number of errors committed during the teaching phase by Strategy (Concreteness Fading, Concrete, Abstract) and Material (other Materials, Montessori)

Strategy	Materials	N	Mean	Std. Deviation	Min	Max
Concreteness Fading	Other Material	29	.93	1.193	0	4
	Montessori Material	29	.66	.974	0	3
	Total	58	.79	1.083	0	4
Abstract		52	1.06	1.697	0	6
Concrete	Other Material	29	.21	.412	0	1
	Montessori Material	28	.07	.262	0	1
	Total	57	.14	.337	0	1
Total		167	.65	1.217	0	6

In order to analyze the effect of strategy and materials we performed a 2 (materials*:* Montessori, Other materials) X 2 (strategy: Concrete, Concreteness Fading) ANOVA analysis with the number of errors (out of 12) as the dependent variable. The sample was made by 115 children. The ANOVA showed a significant main effect of the Strategy but no effect either of the Material or the interaction between Strategy and Material (see Table 5).

Table 5. Analysis of variance (Number of errors committed during the teaching phase) by Strategy and Material

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	13.614[a]	3	4.538	6.897	.000	.157
Intercept	24.981	1	24.981	37.970	.000	.255
Material	1.216	1	1.216	1.848	.177	.016
Strategy	12.292	1	12.292	18.683	.000	.144
Material* Strategy	.142	1	.142	.215	.644	.002
Error	73.030	111	.658			
Total	112.000	115				
Corrected Total	86.643	114				

a. R Squared = .157 (Adjusted R Squared = .134)

We carried out a one-way ANOVA with Strategy (Concreteness Fading, Concrete, Abstract) as independent variable, and the number of errors as dependent variable. We analyzed the entire sample. The ANOVA showed that there was a significant main effect of the Strategy (see Table 6). Tukey HSD post-hoc revealed a significant difference between Concreteness Fading and Concrete ($p= .008$), with a significant higher number of errors committed by the children in Concreteness Fading Group; and between Abstract and Concrete with a significant higher number of errors committed by the children in the Abstract group ($p< .001$). No significant differences were found between the number of errors committed by the children in Concreteness Fading and

Abstract groups (p=45). As expected, the children in Concrete condition had the better performance across the three groups.

Table 6. Analysis of variance (Number of errors committed during the teaching phase) by Strategy

	Sum of Squares	df	Mean Square	F	Sig.
Between groups	24.635	2	12.317	9.131	.000
Within groups	221.221	164	1.349		
Total	245.856	166			

To predict the relationship between material, strategy and number of errors committed during the teaching phase and the transfer test score, we did a stepwise multiple regression analyses with Material as independent variable, later adding the independent variable of Strategy, and then adding the independent variable of Number of errors committed during the teaching phase. In each estimated model the variable Material was not significant. Consequently, the final model adopted is shown in Table 7. The model explains the 26% of the variance. As expected, both strategy and number of errors committed in the teaching phase are significant predictors.

Table 7. Multivariate Analysis of variance

Source	B	Std. Err.	Beta	t
(Constant)	1.737	.225		7,704
Concreteness Fading Strategy	.895	-.278	.256	3.224
Concrete Strategy	-1.169	-.291	-.334	-4.012
Number of errors during teaching	-.297	.097	-.217	-3.054

3.4 Discussion and conclusion

This study was designed to verify if the Montessori conrete materials has more positive effects on transfer than other materials and if the Concreteness Fading is a more effective strategy for transfer than the Concrete and the Abstract. These findings suggest, despite our first prediction, that in a single session of instruction the Montessori materials do not promote a better transfer of the knowledge compared to the other materials experimented by Fyfe et al. (2015). Children in the Montessori group showed no significant differences in transfer abilities with respect to the children in the Other materials group. Our first prediction was suggested from the idea that the concrete materials are effective when they respects some parameters, as exposed by Laski et al. (2015). One of these parameters requires that the object is not a toy or an object of common use. In this regard, the puppets used by Fyfe et al. (2015) seemed not to be adequate to introduce to the children the concept of equivalence, as they are commonly used toys and not educational materials. On the other hand, given the results, it can be assumed that although they were not didactic materials, they were suitable to teach the concept of equivalence as they allowed for an equitable distribution of quantities between the two puppets. Similarly, although the scale is an instrument used in the educational field (and widely used in the American schools), the plastic bears were again commonly used toys, but even in this case we can think that the scale was appropriate for teaching equivalences, since it shows how to control errors (if the quantities are equal the scale is balanced, otherwise not).

Beyond the type of material used, which in the context of a single teaching session does not make a difference, we found interesting differences with respect to the way how the material is used when teaching.

Indeed, the teaching strategy used had an effect on children performance during the transfer test, as in our second prediction. Based on our results, confirming the previous study by Fyfe et al. (2015), the concreteness fading strategy promotes better performance in the transfer of equivalence knowledge and helps children to transfer the knowledge acquired regardless of the specific concrete material used during the teaching phase. On the contrary, the concrete strategy does not seem to help the transfer of knowledge, regardless of the specific concrete material used during the teaching phase. The children in concrete condition, furthermore, achieved the lowest results, even worse than the children in abstract condition. Despite the positive performance obtained by the children in concrete condition during the teaching phase, they faced great difficulty in solving the transfer test. Indeed, the children in the concrete condition had significant low outcomes than children in the concreteness Fading and Abstract condition, which did not differ from each other.

We could infer that the children in concrete conditions obtained an effective learning of the equivalence during the learning phase. This is consistent with the previous study that stated the advantage provided by concrete materials to the children during their first stages of learning new concepts, due to their contextualized and perceptive approach (Baranes, Perry & Stigler, 1989; Kotovsky, Hayes & Simon, 1985). On the contrary, a low number of errors during the teaching phase will be associated to an increased probability of committing errors in the transfer test. Children in concrete condition showed a gap in outcomes between teaching phase (high) and transfer test (low), meaning that the children were able to solve equivalence with manipulatives, but not to link their knowledge to symbols during the transfer test. This result, apparently contradictory, could be explained considering that the transfer test was submitted immediately after the teaching phase. The equivalence in the symbolic form has been solved during the teaching phase by both the children in abstract and concreteness fading condition, while the children in the concrete group has never worked with abstract symbols. Children in concrete conditions, when tested, have been subjected to exercises that required a quick passage from the concept to the symbol. In order to interpret these results it is very important to consider also that one of the efficacy parameters is that the use of the material must be long lasting. Our study, instead, deals with a single teaching session, immediately followed by the transfer test. What emerges therefore is that a single teaching session with concrete material may not be effective if the goal is the fast transfer of knowledge.

Furthermore, we observed that the children in abstract condition, which carried out the teaching session in a very similar manner to the transfer test, did not have the best performance neither in the teaching phase nor in the transfer test. This is in contrast with Kaminski et al. (2008) findings, that showed a better transfer ability in students provided with generic and symbolic instruction. We could infer that 2nd grade children may not benefit from a teaching based on symbolic explanation. It is surprising to see that teaching with symbolic or eventually graphic explanation is commonly used in Italian schools, although it seems not to be the most effective way at all. The regression analysis of the role of the material, the strategy and the learning during the teaching phase on the transfer test, showed how it is predicted primary by the strategy (i.e. Concreteness Fading) and secondary by the learning during the teaching phase (i.e. number of errors). Thus, we had further confirmation of the effect of the teaching strategy on equivalence learning. The role of the learning during the teaching phase could be interpreted by considering the individual differences between children, due to their previous knowledge, their calculation skills and speed or deductive skills.

Thus, we can conclude that for a single teaching session, whose objective is to obtain both conceptual learning and transfer ability, the most effective strategy could be concreteness fading. Therefore, the role of time remains an open question, since it could be possible that if children could use the material for a longer time, they may be able to reach a deep understanding which could be transferred in a symbolic form. Also, it could be supposed that submitting the transfer test after a longer time, instead of immediately after the teaching session, may allow children to process the acquired knowledge and to link it to the other symbolic knowledge already possessed. McNeil and Fyfe (2012) suggested that testing the children immediately after the teaching phase may be a limitation for two reasons. First, students are not evaluated immediately after the teaching phase when attending school. On average, teachers evaluations occurs weeks or months after the lesson; thus, it may be misleading to assess the post-test only after the teaching phase, in order to verify the better teaching strategy. Second, the McGaugh (2000) review of the literature on memory consolidation shows that little is yet known about processes involved in consolidation, although it is clear that consolidation occurs through several hours or longer after learning, in order to generate our lifelong memories. Consolidation of learning is the increasing of skills and the stabilization of memories (Robertson, Pascual-Leone, &

Miall, 2004). The skill improvement occurs “off-line”, and it do not need further pysical practice since it is ofter related to sleep. As exposed by Robertson et al., performance in a perceptual discrimination task increases by 15-20% after a night's sleep. However, the authors pointed out the possibility that off-line learning could be only time-dependent. Empirical evidence in Gomez, Bootzin and Nadel (2006) appear to confirm that sleeping is implicated in significant changing in children's memories organization, promoting flexibility and generalization in learning. Processes hidden in the new memories, thus, persist in a fragile state and consolidate over time. Therefore, a short time (or no time) between teaching phase and post-test may not allow the consolidation of learning.

Chapter Four

The second study

4.1 Introduction

The first study showed that the type of material used in teaching equivalence did not affect the children's transfer ability, which was, indeed, affected by the teaching strategy: children in the Concreteness Fading condition had the better transfer, while those in the Concrete condition had lower transfer ability. It could be that that their low results were due to the short time span between teaching and testing. To test the transfer ability of the children immediately after the teaching phase could result in an advantage for the children that were taught with generic examples, faster to process. It could be inferred that symbolic, abstract examples, like the ones proposed by Kaminski et al., (2008) may be more effective in a immediate test, unlike a learning with concrete manipulatives. Learning with concrete, besides, may need longer time to be reorganized into abstract, general concepts. To suitably test the ability of generalization of concepts acquired by children with concrete manipulatives, researchers should consider the time as a significant variable.

However, another finding of the first study was that the transfer test scores were very low for the complete sample. It is possible that this happened because children were offered only one type of equivalence example, generating a low understanding of the concept of equal. Multiple representations involve the possibility to provide children a variety of examples in order to deeply understand the concept underlying the procedure. The choice of examples through which the teacher proposes a concept to the children significantly influences learning. If the representations the teacher provides are always the same, it follows that the child accepts them as the only possible representation of the concept itself. This type of approach can cause, especially in mathematics and geometry, the establishment of so-called "avoidable" misconceptions (Sbaragli, 2005). This type of misconceptions is the result of an overlap between the example proposed by the teacher and the concept to be taught. As we have seen, this type of learning also causes limitations in the ability to generalize the concepts to other areas and domains. "Misconceptions have been divided into two big categories: "avoidable"and “unavoidable"; the first do not depend directly on the teacher didactic transposition, whereas the second dependence is exactly on the didactic choices " (Santi & Sbaragli, 2007). D’amore (2003) affirms that “the student does not know that he is learning signs that stand for concepts and that he should instead learn concepts; if the teacher has never reflected on this point, he will believe that the student is learning concepts, while he is actually learning only to make use of signs". Therefore, the teacher’s critical reflection about the type of examples he proposes is crucial for the success of learning, especially in mathematics. Indeed, mathematics is made up of abstract, general concepts. As we have seen, in learning concepts children benefit greatly from the use of concrete examples. At the same time, however, such examples can generate both generalization difficulties and the birth of real misconceptions.

Therefore, teaching must take these aspects into account to offer solutions aimed at achieving the learning objective and overcoming these problems.

According to Duval (2006) to avoid the formation of misconceptions due to the didactic intervention, the teacher should provide the child with various and different representations of the concept, and avoid ambiguous or misleading representations. Often, the teachers are not aware of the effects that a teaching based on the uniqueness of examples and representations can generate (Santi & Sbaragli, 2007). As we have seen, the concrete example or the use of concrete material, to make the concept tangible carries within it a series of unnecessary information that can distract the child. So, just as the choice of the concrete material must take into account the distracting elements, in the same way the examples chosen by the teacher to show a concept should be chosen carefully.

Teaching allows the restructuring of "unavoidable" misconceptions, so it should work to ensure that avoidable ones are not created or reduced (Mason, 2013).
The child is able to overcome by himself the confusion generated by the characteristics of the model/example provided, as long as he can compare it with other models that allow him to eliminate all irrelevant aspects. The characteristics that the child could find in the various and several models/examples, have a general character and allow him to understand the underlying concept. This approach also allows the child to reach the concept through a classification and synthesis operation, which Montessori indicated as necessary for the construction of a deep knowledge.
If the didactic intervention occurs through the presentation of a series of examples, all bearing the same characteristics, although different examples appear, it is actually the repetition of the same model that the child tends to identify as the concept itself.
In the case of the concept of the equal sign applied to the fractions, Santi and Sbaragli (2006) showed that primary school teachers possess a deep-rooted misconception with respect to the fraction considered exclusively as a whole divided into equal parts. The “equal” term is used naively by teachers, facilitating the onset of misconceptions by children. This type of misconception has been studied by Fandiño Pinilla (2005) who states: "In doing so, fractions are not referred to specific properties such as length, numerosity, surface extension, volume, ... of a given integer, but to the congruence of the parts". The same phenomenon occurs when in the teaching of geometry we always proceed using the same figures, positioned in the same way in space, constraining the characteristics of the object to the occupied position. In this case the concrete object or its graphic representations can "exasperate the reference to characteristics related to the perception that cause deformation" (Sbaragli, 2005).
Byrd, McNeil, Chesney and Matthews (2015) showed that children's misconceptions of the equal sign can predict future algebraic difficulties. Researchers suggest to teachers to assess equal sign interpretations in primary school, in order to avoid entrenched misconception later. Children's experience support the idea that equal is the result of operations (e.g. 4+5=_), so they do not experience other views, as the

relational one (2+3=2+_). Fyfe, McNeil and Borjas (2014) and Rittle-Johnson (2006) focuses on the fact that children in primary school misinterpret the equal sign, as it could only means “find the total”, and they miss the correct answer of an equivalence.

The interpretation that children consolidate during their primary school years translate into a rigid and unaware use of the equal sign (McNeil & Alibali, 2005). This makes it difficult to carry out equivalence and to understand more complex algebraic concepts requested after the primary school.

On the basis of the above considerations and literature, the aim of the current study is to better understand if: 1) the use of multiple examples could translate in a better learning, regardless of the teaching strategy used; 2) a longer time after the teaching phase allows to increase transfer ability of children who learned with concrete strategy and to maintain stable the transfer ability of children who learned with concreteness fading strategy.

Thus, the current study extend prior studies by examining the benefits of the use of multiple examples during the teaching phase, and the role of time in the transfer of knowledge acquired with concrete and concreteness fading strategies. We focused on the two teaching strategies that used concrete materials (concreteness fading and concrete), by excluding the abstract strategy, already examined in our Experiment 1.

During the teaching sessions of our previous study we used the same six reflexive examples (i.e. 2+3=2+_), with two addends on each side, used by Fyfe et al. (2015). On the contrary, in the transfer test, the exercises had a non-reflexive form, three addends on the left side and a different position of the blank space. In order to better understand the concept and to have a better performance during the the transfer test, children may need to see various examples, i.e. non-reflexive exercises (i.e. 2+3=3+_) or different blank space position (i.e. 2+_=2+3). Thus, in the second study we used six reflexive and six non-reflexive exercises during the teaching phase.

Concerning the effect of time on the transfer ability, we examined the outcomes immediately after the teaching phase and after two weeks. During the two weeks between the first and the second test, children did not receive any additional information about equivalence, indeed equivalence is not a 2nd grade topic in the Italian schools.

Specifically, we tested the following predictions:

1) children who learn math equivalence benefit from a teaching method that provides multiple examples;

2) the effect of the teaching strategy is not stable over time: children who use concrete materials may have higher outcomes if tested two weeks after the teaching session.

4.2 Method

Participants

Participants in this study include 81 second-grade children from a public elementary school near Rome, in Italy. Before the study started, written parental consent was received. We excluded from the sample children with disability or learning difficulties, since they could not solve the exercises without specific support. Only 75 meet criteria were included in the study, because they could not solve any of the four equivalence exercises of the pretest. Since equivalence are not known to 2nd grade children, as expected we found that only 6 children answered correctly to more than one pretest exercise. Thus, the final sample was of 75 Italian children (33 boys, 42 girls), born between September 2010 and April 2012, so that, at the time of the study they were about 7 years old (M = 7 years 7 months, range = 6-8 years).

Design

The experiment consisted of three phases: a teaching phase, a transfer test phase in Time1 and a Transfer Test phase in Time2. For the teaching phase children were randomly distributed to one of four conditions ensuring gender and age homogeneity: Concrete with reflexive exercises (n=18), Concrete with non-reflexive exercises (n=18), Concreteness fading with reflexive exercises (n=19), Concreteness fading with non-reflexive exercises (n=20). There were no significant difference between groups in terms of age $F_{(3, 71)}$= 1.20, p = .31, and gender $X^2_{(3, N=75)}$ = .03, p = .99.

Teaching phase and Transfer Test in Time 1 were one after the other. The Transfer Test in Time 2 was provided to all children two weeks after the teaching phase and the Transfer Test in Time 1.

Instruments

The instructions, pretest and post-test were the same used in Experiment 1.

The same post-test was repeated at two stages (T1 and T2).

Teaching phase: all children in the Reflexive groups solved the same six reflexive equivalence exercises as in Experiment 1 (i.e. 2+1=2+1). All children in the Non-Reflexive groups solved six equivalence exercises (see Appendix) similar to the reflexive ones, but with the addends not repeated in the same order (i.e. 3+5=5+_). The children had two possibilities to solve each item. All items were scored on a 0/2 scale, 0 if correct, 1 for one error, 2 for two errors. If children could not solve an item twice, they were given the solution. The score obtained was the sum of the total number of errors, with a value between 0 and 12.

Materials

We used only Montessori materials, for both Concrete and Concreteness Fading groups. The Montessori materials were the same used in Experiment 1.

Procedure

Pretest: all children completed the pretest on October 2018 while staying in their classrooms. The format and the procedure were the same of Experiment 1.

Teaching phase: the teaching phase took place from October to December 2018. The children received the instructions in a quiet classroom of the school. Regardless of the condition, all the children received the same amount of information. Thus, the difference between the conditions was the concreteness or abstractness of the format and the type of exercises: reflexive or not. The children participate individually to the teaching phase, in a single session of 35 minutes. If the child answered correctly, he received positive feedback; if the child made a mistake, he was provided with an explanation, so that he could try to answer again. If the child was still wrong, the correct answer was provided. If the child answered correctly, he could go to the following exercise.

The children were divided into four groups. The procedure differed by method and type of exercises, as follow.

Concrete:

The children of this group did the same work of Experiment 1, with the Montessori materials.

Concreteness Fading:

The teaching phase in concreteness fading condition took place in three steps: enactive, iconic, and symbolic, as in Experiment 1.

Reflexive Exercises:

The children received the same exercises used in Experiment 1.

Not Reflexive Exercises:

Children received the same instructions of Experiment 1, but the six exercises were modified in order to provide a variety of examples. The first and third exercises were the same of Experiment 1. The second exercise had the addends in a not reflexive position. The fourth exercise was reflexive but with the blank at the second place. The fifth exercise had addends in a not reflexive position and the blank at the first place.

Transfer Test in Time 1: immediately after the teaching phase every child completed the Transfer Test. The child solved all the exercises without any help. Thus, during the test, feedback or further explanation were not provided. There was not a time limit to solve the exercises. Once the test was completed the child was taken back to the classroom and the whole procedure was restarted with another child.

Transfer Test in Time 2: the Transfer Test in Time 2 occurred approximately two weeks after the teaching phase. It was identical to the Transfer Test completed in Time 1. During the two weeks the children did not receive any intervention. All the classroom solved the transfer test at same time. There was not a time limit to complete the test.

4.3 Results

As for the first study, we analyzed both the data collected during the teaching phase and the transfer test phase.

4.3.1 Transfer Tests

As shown in Table 8, the transfer test showed moderate results for the complete sample both in T1 and in T2. Nevertheless, these scores were higher if compared with the scores of Experiment 1. We did not find any main effect of the gender in the transfer performance in Time1, $F_{(1,73)} = .27$, $p = .59$, $\mu^2 = .004$, neither in Time2, $F_{(1,73)} = 2.13$, $p = .14$, $\mu^2 = .02$.

Table 8. Score on the transfer test by Time (T1, T2), Strategy (Concreteness Fading, Concrete) and Exercise (Reflexive, Non-Reflexive)

	Strategy	Exercise	Mean	Std. Deviation	N
TT1	Concreteness Fading	Reflexive	3.67	1.879	18
		Non-Reflexive	4.44	1.464	18
		Total	4.06	1.706	36
	Concrete	Reflexive	2.63	2.060	19
		Non-Reflexive	1.10	1.651	20
		Total	1.85	1.994	39
	Total	Concreteness Fading	3.14	2.016	37
		Concrete	2.68	2.291	38
		Total	2.91	2.157	75
TT2	Concreteness Fading	Reflexive	2.33	1.572	18
		Non-Reflexive	2.83	1.855	18
		Total	2.58	1.713	36
	Concrete	Reflexive	2.47	1.429	19
		Non-Reflexive	1.70	1.129	20
		Total	2.08	1.326	39
	Total	Concreteness Fading	2.41	1.481	37
		Concrete	2.24	1.601	38
		Total	2.32	1.535	75
T		Total	2.61	1.846	75

In order to test our first prediction, that children who learn math equivalence benefit from a teaching method that provides multiple examples, we carried out a 2 (Concreteness Fading Strategy, Concrete Strategy) X 2 (Reflexive Exercise, Not Reflexive Exercise) X 2 (Repeated measure: Time1, Time2) multivariate analyses of the variance (MANOVAs) with Strategy and Exercise as between subject factors and Time as within subject factors. The number of correct answers (out of 5) of Transfer Test was the dependent variable.

Table 9. Multivariate analysis of variance. Tests of within subjects (Time) effects

Source		Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Time	Sphericity Assumed	14.648	1	14.648	6.388	.014	.083
	Greenhouse-Geisser	14.648	1.000	14.648	6.388	.014	.083
	Huynh-Feldt	14.648	1.000	14.648	6.388	.014	.083
	Lower Bound	14.648	1.000	14.648	6.388	.014	.083
Time * Strategy	Sphericity Assumed	26.828	1	26.828	11.700	.001	.141
	Greenhouse-Geisser	26.828	1.000	26.828	11.700	.001	.141
	Huynh-Feldt	26.828	1.000	26.828	11.700	.001	.141
	Lower Bound	26.828	1.000	26.828	11.700	.001	.141
Time * Exercise	Sphericity Assumed	.539	1	.539	.235	.629	.003
	Greenhouse-Geisser	.539	1.000	.539	.235	.629	.003
	Huynh-Feldt	.539	1.000	.539	.235	.629	.003
	Lower Bound	.539	1.000	.539	.235	.629	.003
Time * Strategy * Exercise	Sfericità presunta	2.509	1	2.509	1.094	.299	.015
	Greenhouse-Geisser	2.509	1.000	2.509	1.094	.299	.015
	Huynh-Feldt	2.509	1.000	2.509	1.094	.299	.015
	Lower Bound	2.509	1.000	2.509	1.094	.299	.015
Error(Time)	Sphericity Assumed	162.802	71	2.293			
	Greenhouse-Geisser	162.802	71.00	2.293			
	Huynh-Feldt	162.802	71.00	2.293			
	Limite inferiore	162.802	71.00	2.293			

Table 10. Multivariate analysis of variance. Tests of between subjects (Strategy, Exercise) effects

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Intercept	524.838	1	524.838	333.752	.000	.825
Strategy	33.760	1	33.760	21.469	.000	.232
Exercise	1.235	1	1.235	.785	.379	.011
Strategy * Exercise	15.016	1	15.016	9.549	.003	.119
Error	111.650	71	1.573			

The MANOVA revealed significant Time X Strategy interaction effect $F_{(1,71)}$=11.70, p= .001, μ^2=.141, and significant Strategy X Exercise interaction effect (see Table 9 and 10).

On the other hand, MANOVA did not reveal Time X Strategy X Exercise significant interaction effect, neither Exercise X Time significant interaction effect. Statistically significant result were obtained for the simple main effects of Time and of Strategy, but not for the main effect of Exercise.

4.3.2 Teaching Phase

During the teaching phase, each child could commit a maximum of 2 errors per exercise, so the number of total errors could range between 0 and 12.

As shown in Table 11, the children made few errors across the six exercises, with respect to our Experiment 1.

Table 11. Number of errors committed during the teaching phase by Strategy (Concreteness Fading, Concrete) and Exercise (Reflexive, Non-Reflexive)

		N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean Lower Bound	95% Confidence Interval for Mean Upper Bound	Min	Max
Concreteness Fading	flexive	18	.44	1.247	.294	-.18	1.06	0	5
	Non-Reflexive	18	1.22	1.801	.424	.33	2.12	0	6
	Total	36	.83	1.524	.359	-.18	2.12	0	6
Concrete	Reflexive	19	1.05	1.840	.422	.17	1.94	0	6
	Non-Reflexive	20	.70	1.129	.252	.17	1.23	0	3
	Total	39	.87	1.484	.337	.17	1.94	0	6
Total		75	.85	1.531	.177	.50	1.21	0	6

In order to find differences in the children performance due to the teaching condition we conducted an one-way ANOVA with the Strategy (Concreteness Fading, Concrete) as the independent variable and the total number of errors as the the dependent variable. As shown in Table 12 , the ANOVA analysis do not showed any significant main effect of the Strategy, so in the teaching phase children in Concreteness Fading and Concrete conditions did the same number of errors.

With respect to our first study, the sample in this second experiment was smaller, so we did not perform any logistic regression to predict the effect of the number of errors during the teaching phase on the transfer test score.

Table 12. Analysis of variance (Number of errors committed during the teaching phase) by Strategy

	Sum of Squares	df	Mean Square	F	Sig.
Between groups	.028	1	.028	.012	.914
Within groups	173.359	73	2.375		
Total	173.387	74			

4.4 Discussion and conclusion

With this study we would verify that children who learn math equivalence benefit from a teaching method that provides multiple examples. Children who learn math equivalence could benefit from a variety of examples during the teaching session: our first prediction was confirmed for children in concreteness fading condition, but not for those in concrete condition. Thus, children who learned through the concreteness fading strategy, could get benefit from a variety of examples because they worked, during two stages out of three, with paper and pen (enactive and symbolic phases), with the opportunity to see the different position of numbers and blanks. The transfer test they solved was similar to the exercises completed during the teaching phase, even if more complex. On the contrary, the children who worked with concrete manipulatives manipulated quantities of objects instead of numbers, could have not realized, during the teaching phase, that the numbers and the blank positions were different through the examples. It may be possible that expliciting the differences between the exercises proposed could help children in the concrete group to focus on them. Indeed, children in concrete condition may need a clear explanation of the type of examples proposed, in order to see and recognize differences and analogies, and to better understand the relevant changes between reflexive and not-reflexive equivalences.

This study aims also to verify if the effect of the teaching strategy is stable over time: children who use concrete materials may have higher outcomes if tested two weeks after the teaching session. Results confirmed our second prediction, children in concrete condition increased their performance between Time 1 and Time 2. On the contrary, children in concreteness fading condition had similar performance in Time 1 and Time 2, meaning that their learning was stable over time. These findings provide evidence in favor of the concreteness fading strategy, showing that children's learning is deep and not affected by time. The outcomes of the children in concreteness fading condition, after a single teaching session, could be considered high both in Time 1 and

Time 2. Furthermore, confirming our findings of Experiment 1, children in concrete condition had very low performance in Time 1, probably due to the difficulty in solving a symbolic test immediately after a teaching session based only on concrete manipulatives. In Time 2, the performance of those children improved significantly, and this is consistent with our prediction that learning with concrete manipulatives needs more time to transfer concepts from concrete to abstract and to memorize them. Children in concrete condition did not work with concrete materials between Time 1 and Time 2, confirming the possibility that the consolidation process occurs after some time, especially if learning is obtained through concrete training.

The current study contribute to the debate on the use of concrete manipulatives, by adding interesting result. First, it confirm the effectiveness of concreteness fading strategy for primary school children, despite the type of material used in the enactive phase. Second, it highlights the importance of time in the teaching with concrete materials, showing that evaluate the transfer ability of children in that teaching condition immediately after the teaching phase could return a unreliable result.

Conclusions

In these studies, we valued the effect of the concrete manipulatives, concreteness fading and symbolic methods on the children's ability to transfer their knowledge after a short teaching session. In our first experiment, we thought equivalences to children in five different conditions (concrete with Montessori materials, concrete with other materials, concreteness fading with Montessori materials, concreteness fading with other materials, abstract), and tested them at the end of the teaching session. In our second experiment, we thought equivalences to children in four different conditions (concrete with Montessori materials and reflexive exercises, concrete with Montessori materials and not reflexive exercises, concreteness fading with Montessori materials and reflexive exercises, concreteness fading with Montessori materials and not reflexive exercises), testing them both at the end of the teaching session and after two weeks.

Overall, all children had moderate performance during each teaching phase followed by constant feedback. However, for all children the transfer is difficult because it requires understanding the underlying concept from the procedure thought. Furthermore, the results between the teaching phase and the transfer test are discordant. During the teaching phase the concrete group had good performance, while the abstract and concreteness fading groups had similar performances. Significantly, in the performance of the transfer carried out immediately after the test, the concreteness fading is the most adequate strategy for understanding the concepts and making children able to apply them to other contexts. On the contrary, children in concrete condition showed difficulties in transferring the knowledge they had just learned. Even children in abstract condition could not easily transfer their knowledge, despite having learned the equivalences by using the same system of symbols proposed during the teaching phase.

Specifically, the results of both studies suggest that children benefit from a teaching strategy that starts with concrete elements and fades into the abstract. A teaching method based on what was theorized by Bruner, starting from a concrete object that must be followed by an explicit link to the symbolic and abstract form of the concept, allows understanding the concept and transferring it. Namely, we confirm the results obtained by Fyfe and colleague (2015) on the effectiveness of concreteness fading strategy, its benefit for both learning and transferring. Our results, indeed, support the need to extend studies on this strategy of teaching, by verifing also the stability over time of the learning obtained by learning equivalence with concreteness fading.

Our first study also show that, as theorized by Kaminski (2008), a teaching method based only on the use of concrete material may cause transfer difficulties, because it is probably too contextualized, while a teaching method based only on symbolic elements is more effective in facilitating the transfer. Although, despite Kaminski (2008) findings, the symbolic method does not seem to be the best teaching method. Indeed, the children of the abstract group had worse results compared to those of the concreteness fading group showing that, at this age, a symbolic teaching method could generate errors due to the repetition of a procedure without understanding the underlying concept (McNeil & Alibali, 2005; Fyfe et al., 2015), therefore effectively preventing the ability to use what has been learned in different contexts. Consistently, Fyfe et al. (2005) found that the children of the abstract group make the mistake of adding up all the addends, typical mistake of those who apply a procedure without contextualizing it, in a mechanical way (Rittle-Johnson, 2006). If the symbolic explanation occurs after the concrete and iconic one, it could be interpreted by the child with reference to a family context, according to Goldstone and Son (2005), the symbolic explanation takes on meaning and clarity within a fading process. Thus, the symbols can be understood by the children of primary school, and deeply understood so to be transferred if linked to concrete knowledge. Indeed, both Bruner and Montessori argue that the concrete material allows the child to materialize the abstract concept, by forming a mental image to which he can also refer later. According to Montessori (2013a, 2013b), a child of school age and of the first years of primary school needs to know the concept and the underlying rules through a sensory experience, which is the means through which he learns in this phase. The child's mind later develops the ability to think in an abstract way and to form mental categories. Therefore, despite the child cannot explain the concept with the direct use of symbols, he can "rediscover" the knowledge acquired through the manipulation of concrete material from his memory and use it to reformulate the concept in a symbolic and abstract form. In this way, according to Montessori, the child is able to constantly construct abstract knowledge on the basis of laws that he has profoundly internalized, avoiding the use of mechanically acquired and meaningless procedures (Alibali, 1999; McNeil & Alibali, 2000). Furthermore, this way of learning allows connecting concepts that can help an effective transfer, as seen in Experiments 1 and 2 by the Concreteness Fading groups.

These results also confirm that a teaching method based on procedures, if sided by deep attention to the real understanding of the concept, is more effective than a procedural teaching only (Hiebert & Lefevre, 1986). According to Fyfe et al. (2015), our results in Experiment 1 also suggest that if the transfer takes place immediately after the teaching session, the concreteness fading strategy is more effective than teaching with symbols alone. On the other hand, in Experiment 2 our results suggest that after some time even the learning session through the concrete strategy alone allows reaching results comparable to those of the concreteness fading. Thus, it can be said that both teaching that uses only concrete material, as well as teaching that proceeds along the three phases of fading, bring more effective results in terms of understanding the concept and transfer ability that the teaching with symbols only.

The result of our Experiment 1 also demonstraded that the Montessori materials, if used in a single teaching session, are not more effective than other concrete materials. Probably, the difference between different type of concrete materials should be indagate during a longer time of teaching, with respect to the specific characteristic that effective concrete materials should have, as in Laski et al. (2015). Additionally, Montessori materials, thought to be used by children in a specific progression, may not be effective if used without the needed knowledge of the other linked materials.

Concrete manipulatives examined by Fyfe and colleagues (2015) and compared in our first study with Montessori’s, showed their effectiveness in equivalence teaching despite they did not met the criteria indicated by Laski et al. (2015).

The use of multiple examples, in our Experiment 2, did not showed effectiveness regardless of teaching strategy, and it could be due to the short teaching time. Indeed, children in concrete condition did not recognize the difference between the two type of examples, and children in concreteness fading, that showed a better performance while using non-reflexive exercise, may benefit by multiple representation if provided for many times.

Despite the positive contributions of this study, several questions remain.

First of all, in our second experiment we did not test children in T2 who had worked with the abstract strategy, as we wanted to verify if there was a difference between concreteness fading and concrete after some time. It remains to be seen whether the children’s performance in the abstract group can change in T2.

Secondly, our experiments did not take into account all the differences in the children’s prior knowledge. In line with the previous studies of Fyfe and colleagues (2015), children able to solve equivalences were excluded a priori, but we did not take into account the children's previous knowledge of additions and subtractions, which are necessary for solving exercises such as those proposed in our experiments.

Third, we evaluated the Montessori materials by comparing them with other materials, and we do not obtained significantly different results. It is necessary, however, to reiterate that the Montessori material is not intended to be used as educational material but as development one. This means that the material must be available to the child who chooses it freely, uses it for as long as it deems necessary and repeats the exercise as often as he wants. This allows him to focus on the experience and to make connections with his previous knowledge. The materials of the Montessori environment are often closely related, allowing the child to think about the concept through different types of experiences; finally, the Montessori material involves selfcorrection and not necessarily the teacher's feedback (Hoenneger Fresco, 2000). It is therefore necessary to specify that in our experiments the Montessori materials have been used as teaching materials as they have been de-contextualized and used for a short period, under the control and with the explanations of the teacher. All this does not allow us to discuss the results of the general comparison between the Montessori material and other materials, but only to compare the didactic use of this material in a single teaching session, respectively with the strategy of concreteness fading or concrete. Although in our Experiment 2, where we only used Montessori material, children who worked with the concrete strategy after two weeks had comparable performances to those of children who worked with concreteness fading, further studies are needed to understand if the Montessori material used for a longer period with both strategies bring to better results if used alone (concrete) or with the iconic link to symbols (concreteness fading).

Despite these limitations, our study provides valuable information about the ability to transfer knowledge in mathematics. It can be added to the few studies dealing with concreteness fading, and it provides interesting remarks about the material to be used during the first phase (enactive). It shows how concreteness fading works both for learning and for transfer, and combines the positive aspects of all the methods discussed for a long time: concrete, iconic, symbolic. At the same time, it offers an interesting view on the ability of children who learn with concreteness fading to maintain their transfer skills even after some time. As for the use of concrete material, the second study focuses on how the time factor must always be taken into great consideration when evaluating the effectiveness of the concrete material in children’s learning and transfering.

Appendix 1 - English

Screening exercises (Pretest)

The following four exercises were used in the screening phase.

1) 1 + 5 = _ + 2
2) 7 + 2+ 4 = _ + 4
3) 2 + 7 = 6 + _
4) 3 +5 + 6 = 3 + _

Teaching phase exercises

The following six exercises were used in the teaching phase of Experiment 1, and in teaching phase of Experiment 2 for the REFLEXIVE condition.

1) 4 + 3 = 4 + _

2) 3 + 5 = 3 + _

3) 2 + 4 = 2 + _

4) 5 + 4 = 5 + _

5) 3 + 2 = 3 + _

6) 2 + 5 = 2 + _

The following six exercises were used in the teaching phase of Experiment 2 for the NOT- REFLEXIVE condition.

1) 4 + 3 = 4 + _

2) 3 + 5 = 5 + _ (reflexive with numbers in different position)

3) 2 + 4 = 2 + _

4) 5 + _ = 5 + 4 (reflexive with blank in the second place)

5) _ + 2 = 2 + 3 (numbers in different position and blank in the first place)

6) 2 +5 = 2 + _

Transfer test exercises

The following exercises has been used as Transfer Test

1) 4 + 8 + 9 = 4 + _

2) 3 + 9 + 5 = 7 + _

3) 9 +2 + 7 = _ + 7

4) 6 + 4 + 8 = _ + 3

5) The boys want to have the same number of stars as the girls. Sara has 5 stars, and Giulia has 4 stars. Matteo has 7 stars, Francesco has some stars, too. How many stars does Francesco need for the boys and girls to have the same? Circle the answer that represents the above story.

a) 5 + _ = 4 + 7

b) 5 + 4 + 7 = _

c) 5 + 4 = 7 + _

d) 5 + 4 + 7 + 2 = _

Script used for pre-test

The child is invited to leave the class. The researcher greets him and asks him some questions: “Hi Andrea, how are you? .... Do you already know the classroom we are going to? .... Have you also done kindergarten in this school? " Once we reached the classroom, we sit down and the 4 exercises are presented to the child.

Scripts used for instruction

Script for abstract context

We're going to solve a Math problem together. (Show to the child the equivalence exercise [e.g. 4+3=4+_]). This is the left side of the problem (circle the left side with the finger) and this is the right side of the exercise (circle the right side with the finger).

In the concreteness fading condition, the instructions of the exercises in symbolic form came right after children had worked on exercises in the worksheet contexts, so this sentence was replaced with: “Now I want you to think about the game we did before, it helps you to solve this exercise.”

I want you to figure out how much you should put on the right side (point to the blank), so that the right side (circle the whole right side) will have the same quantity as the left side (circle the whole left side).

If the child is correct: That's correct. Go ahead and write a 3 in the blank.

If the child is incorrect: Good try, but that won't make them equal. The left side has 7 (point to the 4 while counting one, two, three, four and then point to the 3 while counting five, six, seven).

The right side only has 4 (point to the 4 while counting one, two, three, four). How much does the right side need to have 7? If the child is correct: That's correct. Go ahead and write a 3 in the blank. If the child is still incorrect: Actually the right side needs 3 more. Write a 3 in the blank.

Let's check to make sure they're equal. How many does the left side have? (the child should say 7) How many does the right side have? (the child should say 7) Great! You figured it out. Both sides have 7! Let's do the next exercise.

Script for concrete puppet-sharing context

We're going to play a game in which you’ll share some of these stickers with Monkey and Frog. I'll tell you how many stickers to give them, and you put the stickers on their sticker collectors.

First, give to the Monkey 4 stickers (point to Monkey's first sticker collector). Good! Now give to the Monkey 3 more stickers (point to Monkey's second sticker collector). OK! Give to the Frog 4 stickers (point to Frog's first sticker collector). Great! Monkey and Frog want to have an equal amount of stickers (Stress the word equal and gesture towards each character with hands up like a scale). So, I want you to figure out how much stickers you should give to the Frog (point to the Frog's second sticker collector), so the Frog will have the same amount of stickers as the Monkey (point to the Monkey's sticker collectors). If the child is correct: That's correct. Go ahead and give to the Frog 3 stickers.

If the child is incorrect: Good try, but that won't make them equal. The Monkey has 7 stickers (point to each of Monkey's stickers as you count them aloud). The Frog only has 4 stickers (point to each of the Frog's stickers as you count aloud). How much does Frog need so he can have 7 stickers?

If the child is correct: That's correct. Go ahead and give to the Frog 3 stickers. If the child is still incorrect: Actually the Frog needs 3 more stickers. Give to the Frog 3 stickers. Let's check to make sure they're equal. How many stickers does Monkey have? (The child should say 7.) How many stickers does the Frog have? (The child should say 7.) Great! You figured it out. Monkey and Frog both have 7 stickers!

Script for puppet-sharing worksheet

Now we're going to play the same game on paper. (Show the worksheet to the child). This stands for the Monkey (point to the monkey) and this stands for the Frog (point to the frog). Here are their sticker collectors (point to the four boxes). I'm going to tell you how many stickers you should give them, and you will write the number in the blanks, OK? First, give to the Monkey 3 stickers (point to the first blank on the left, wait for the child to write 3). Good! Give to the Monkey 5 more stickers (point to the second blank on the left, wait for the child to write 5).

Now give to the Frog 3 stickers (point to the first blank on the right, wait for the child to write 3). Great! The Monkey and the Frog want to have an equal amount of stickers (stress the word equal and gesture towards each character with hands up like a scale). So, I want you to figure out how many more stickers you should give to the Frog (point to the last blank), so the Frog will have the same amount of stickers as the Monkey. If the child is correct: That's correct. Go ahead and write 5. If the child is incorrect: Good try, but that won't make them equal. The Monkey has 8 stickers (point to each of the two numbers he wrote as you count them aloud). The Frog only has 3 stickers (point to the number he wrote as you count it aloud). How many more does Frog need so he can have 8 stickers? If the child is correct: That's correct. Go ahead and write 5. If the child is still incorrect: actually the Frog needs 5 more stickers. Write a 5. Let's check to make sure they're equal. How many stickers does the Monkey have? (The child should say 8.) How many stickers does the Frog have? (The child should say 8.) Great! You figured it out. The Monkey and the Frog both have 8 stickers!

Script for concrete balance scale context

We're going to play a new game with this scale and these bears. You are going to use the bears to make the scale balance. I'll tell you how many bears to put on each side, OK? First, put 5 bears on the left side (point to the left side of the scale). Good job! Now put 4 more bears on the left side (point to the left side of the scale). OK! Put 5 bears on the right side (point to the right side of the scale). Great! We want the scale to balance (stress the word balance and gesture towards each side of the scale with hands up like a scale). So, I want you to figure out how many more bears you need to put on the right side (point to the right side), to make the right side (point to the right side) the same amount as the left side (point to the left side of the scale). If the child is correct: That's correct. Go ahead and put 4 bears on the right side. If the child is incorrect: good try, but that won't make the scale balance. The left side has 9 bears (point to each one of the bears as you count them aloud). The right side only has 5 bears (point to each one of the bears as you count them aloud). How many more bears does the right side (point to right side of the scale) need to have 9 bears? If the child is correct: That's correct. Go ahead and put 4 bears on the right side. If the child is still incorrect: actually it needs 4 more bears. Put 4 bears on the right side. Let's check to make sure it's balanced. How many bears does the left side have? (The child should say 9.) How many bears does the right side have? (The child should say 9.) Great! You figured it out. Both sides have 9 bears!

Script for balance scale worksheet

Now we're going to play the same game on paper. (Show the worksheet to the child). This stands for the left side of the scale (point to left side) and this stands for the right side of the scale (point to right side). I'm going to tell you how many bears to put on each side, and you will write the number that I tell you in the blanks, OK? First, put 3 bears on the left (point to the first blank on the left, wait for child to write 3). Good! Now put 2 more bears on the left (point to the second blank on the left, wait for child to write 2). Put 3 bears on the right side (point to the first blank on the right, wait for child to write 3). Great! We want the scale to balance (stress the word balance and gesture towards each side of the scale with hands up like a scale). So, I want you to figure out how many more bears you need to put on the right side (point to the second blank on the right side), to make the right side (circle the right side) the same amount as the left side (circle the left side). If the child is correct: That's correct. Go ahead and write 2. If the child is incorrect: Good try, but that won't make the scale balance. The left side has 5 bears (point to each number he wrote as you count them aloud). The right side only has 3 bears (point to the number he wrote as you count it aloud). How many more bears does the right side (point to right side) need to have 5 bears?

If the child is correct: That's correct. Go ahead and write 2. If the child is still incorrect: Actually it needs 2 bears. Write a 2. Let's check to make sure it's balanced. How many bears does the left side have? (The child should say 5.) How many bears does the right side have? (The child should say 5.) Great! You figured it out. Both sides have 5 bears!

Script for concrete Montessori number rods context

We're going to play a new game with the rods that are in this box. You see? This is one, this is two ... up to ten (touching the rods marking the color change). Try to count this (wait for the child to finish counting). After pointing out to the child that each rod has a value between 1 and 10, indicate the note with the equal sign on the table. Do you know this sign? (The child should say that it is equal. If he does not say it, he is reminded.) Exactly, it is the equal. We must put on this side (indicate with the finger to the left of the equal), the same amount that we put on this side (indicate with the finger to the right of the equal). They must be equal (underline the word equal). Now take the 4 rod (wait for the child to count). Well. Put it here (indicate the space on the left). Now take 3 rod, and put it close to 4. Great! Now, put a 4 rod on this side (indicating the right side of the equal). Tell me, what rod do you think you should put here (indicating the right side of the equal) to have the same amount that you have here? (indicating the left side of the equal). If the child's answer is correct: Right. Go ahead and put a 3 rod on the right side.

If the child's answer is incorrect: but that won't make them equal. This is 7 (indicate the red/blue colors of the rods as if you are counting them). Here there is only 4 (indicating the red/blue colors of the rods as if you are counting them). What rod should we put here to have the same quantity? If the child's answer is correct: Right. Go ahead and put a 3 rod. If the child's answer is still incorrect: actually, a 3 rod is needed. Take a 3 rod and put it here. Let's check to make sure they are equal. How much there is in this side? (the child should say 7) And in the other? (the child should say 7) Great! You figured it out. Both sides have 7 rods!

Script for number rods worksheet

Now we're going to play the same game on paper. (Show the worksheet to the child). This stands for a group of number rods (point to left side) and this stands for the other group of the rods (point to right side). I'm going to tell you how many rods to put on each side, and you will write the number that I tell you in the blanks, OK? First, put 3 rod on the left (point to the first blank on the left, wait for child to write 3). Good! Now put the 2 rod on the left (point to the second blank on the left, wait for child to write 2). Put the 3 rod on the right side (point to the first blank on the right, wait for child to write 3). Great! We want the two sides to be equal (stress the word equal and gesture towards each side with hands up like a scale). So, I want you to figure out which rod you need to put on the right side (point to the second blank on the right side), to make the right side (circle the right side) the same amount as the left side (circle the left side). If the child is correct: That's correct. Go ahead and write 2. If the child is incorrect: Good try, but that won't make them equal. The left side has 5 (point to each number he wrote as you count them aloud). The right side only has 3 (point to the number he wrote as you count it aloud). Which rod does the right side (point to right side) need to have 5?

If the child is correct: That's correct. Go ahead and write 2. If the child is still incorrect: Actually it needs a 2 rod. Write a 2. Let's check to make sure they are equal. How many does the left side have? (the child should say 5.) How many does the right side have? (the child should say 5.) Great! You figured it out. Both sides have 5!

Script for concrete Montessori abacuses context

We're going to play a new game with these abacuses. You see? On this line we have the units, and we are going to count them. Look, this is five (show to the child how to move the beads from left to right while counting: 1, 2, 3, 4, 5). Now, choose a number and count it with the beads (hand the frame to the child and wait for him to finish counting). After the child understands how to operate on the frame, indicate the note with the equal sign on the table. Do you know this sign? (The child should say that it is equal. If he does not say it, he is reminded.) Exactly, it is the equal. We must put on this side (indicate with the finger to the left of the equal), the same amount that we put on this side (indicate with the finger to the right of the equal). They must be equal (underline the word equal).

Now, move 4 unit beads on the left abacus (wait until the child has counted and moved the beads). How many beads did you move? (the child will say 4) Good! Now we have to put 3 more beads, move them close to the 4. Perfect!

Now, we shall do the same on the right frame (indicating the right side of the equal). Move 4 beads on the abacus. Good!

Tell me, how many more beads do you think you have to move (indicating the abacus on the right of the equal) to have the same amount that you have here? (indicating the abacus on the left of the equal).

If the child's answer is correct: Right. Go ahead and move 3 beads.

If the child's answer is wrong: Good try, but they do not become the same. Here there are 7 beads (indicate each of the beads as if you count them aloud). Hence, there are only 4 beads (indicate each of the beads as if you count them aloud). How many beads should we move here to have 7?

If the child's answer is correct: Right. Go ahead and move 3 beads.

If the child's answer is still wrong: actually, you need 3 more beads. Move 3 more beads.

Let's check to make sure they are equal. How many does the left side have? (the child should say 7.) How many does the right side have? (the child should say 7.) Great! You figured it out. Both sides have 7!

Script for abacuses worksheet

Now we're going to play the same game on paper. (Show the worksheet to the child).

This stands for an abacus (point to left side) and this stands for the other abacus (point to right side). I'm going to tell you how many beads to move on each side, and you will write the number that I tell you in the blanks, OK? First, move 3 beads on the left abacus (point to the first blank on the left, wait for the child to write 3). Good! Now move 5 more beads on the left abacus (point to the second blank on the left, wait for the child to write 5). Move 3 beads on the right side (point to the first blank on the right, wait for child to write 3). Great! We want the two sides to be equal (stress the word equal and gesture towards each side with hands up like a scale). So, I want you to figure out how many beads you need to move on the right abacus (point to the second blank on the right side), to make the right side (circle the right side) the same amount as the left side (circle the left side).

If the child is correct: That's correct. Go ahead and write 5.

If the child is incorrect: Good try, but that won't make them equal. The left side has 8 beads (point to each number he wrote as you count them aloud). The right side only has 3 beads (point to the number he wrote as you count it aloud). How many beads does the right side (point to right side) need to have 8?

If the child is correct: That's correct. Go ahead and write 5.

If the child is still incorrect: Actually, it needs 5 beads. Write a 5.

Let's check to make sure they are equal. How many does the left side have? (the child should say 8)

How many does the right side have? (the child should say 8)

Great! You figured it out. Both sides have 8!

Appendix 2 – Italian

Below, we report the procedure in Italian, as it was actually used in our studies 1 and 2.

Procedura per gli esercizi presentati in forma astratta

Adesso risolveremo insieme un esercizio di matematica. (Mostra al bambino l'esercizio di equivalenza matematica (es. 4 + 3 = 4 + _)). Questo è il lato sinistro dell'esercizio (cerchia il lato sinistro con il dito) e questo è il lato destro dell'esercizio (cerchia il lato destro con il dito). Vorrei che scoprissi quanti devi metterne in più sul lato destro (indica lo spazio vuoto), in modo che il lato destro (cerchia a destra) abbia la stessa quantità del lato sinistro (cerchia a sinistra).

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi un 3 nello spazio vuoto.

Se la risposta del bambino è errata: Buon tentativo, ma così non diventano uguali. Il lato sinistro ha 7 (indica il 4 mentre conti uno, due, tre, quattro e poi indica il 3 mentre conti cinque, sei, sette).

Il lato destro ha solo 4 (indica il 4 mentre conti uno, due, tre quattro). Quanti ne mancano al lato destro per diventare 7?

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi un 3 nello spazio vuoto.

Se la risposta del bambino è ancora errata: in realtà il lato destro ha bisogno di altri 3.

Scrivi un 3 nello spazio vuoto.

Proviamo a verificare che siano uguali. Quanti sono a sinistra? (Il bambino dovrebbe dire 7.) Quanti sono a destra? (Il bambino dovrebbe dire 7.) Ottimo! Hai capito. Entrambe le parti hanno 7!

Facciamo l'esercizio successivo.

Procedura per gli esercizi presentati in forma astratta – terza fase del concreteness fading

Adesso vorrei che tu pensassi a quel gioco (riferendosi al materiale concreto e alle schede utilizzate in precedenza) per aiutarti a risolvere questo esercizio di matematica. Mostra al bambino l'esercizio di equivalenza matematica (es. 4 + 3 = 4 + _). Questo è il lato sinistro dell'esercizio (cerchia il lato sinistro con il dito) e questo è il lato destro dell'esercizio (cerchia il lato destro con il dito). Vorrei che scoprissi quanti devi metterne in più sul lato destro (indica lo spazio vuoto), in modo che il lato destro (cerchia a destra) abbia la stessa quantità del lato sinistro (cerchia a sinistra).

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi un 3 nello spazio vuoto.

Se la risposta del bambino è errata: Buon tentativo, ma così non diventano uguali. Il lato sinistro ha 7 (indica il 4 mentre conti uno, due, tre, quattro e poi indica il 3 mentre conti cinque, sei, sette).

Il lato destro ha solo 4 (indica il 4 mentre conti uno, due, tre quattro). Quanti ne mancano al lato destro per diventare 7?

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi un 3 nello spazio vuoto.

Se la risposta del bambino è ancora errata: in realtà il lato destro ha bisogno di altri 3.

Scrivi un 3 nello spazio vuoto.

Proviamo a verificare che siano uguali. Quanti sono a sinistra? (Il bambino dovrebbe dire 7). Quanti sono a destra? (Il bambino dovrebbe dire 7). Ottimo! Hai capito. Entrambe le parti hanno 7!

Facciamo l'esercizio successivo.

Concreto con le marionette

Adesso faremo un gioco in cui distribuisci alcuni di questi adesivi alla scimmia e alla rana. Ti dirò quanti adesivi dare loro, e tu li metterai sui loro raccoglitori.

Innanzitutto, dai 4 adesivi alla scimmia (indica il primo spazio del raccoglitore della scimmia). Bene! Ora dai alla scimmia altri 3 adesivi (indica il secondo spazio del raccoglitore della scimmia). Ok! Dai 4 adesivi alla rana (indica il primo spazio del raccoglitore della rana). Perfetto!

La scimmia e la rana vogliono avere una quantità uguale di adesivi (sottolinea la parola uguale e indica verso ogni personaggio muovendo le mani come una bilancia, come a mostrare la quantità uguale). Adesso, vorrei che tu mi dicessi quanti altri adesivi bisogna dare alla rana (indica il secondo spazio del raccoglitore della rana), in modo che la rana abbia la stessa quantità di adesivi della scimmia (indica il raccoglitore della scimmia).

Se la risposta del bambino è <u>*corretta*</u>: Giusto. Vai avanti e dai 3 adesivi alla rana.

Se la risposta del bambino è <u>*errata*</u>*:* Buon tentativo, ma così non diventano uguali. La scimmia ha 7 adesivi (indica ciascuno degli adesivi della scimmia come se li contassi ad alta voce). La rana ha solo 4 adesivi (indica ciascuno degli adesivi della rana come se li contassi ad alta voce). Di quanti altri adesivi ha bisogno la rana in modo che possa averne 7?

Se la risposta del bambino è <u>*corretta*</u>*:* Giusto. Vai avanti e dai 3 adesivi alla rana.

Se la risposta del bambino è ancora <u>*errata*</u>*:* in realtà la rana ha bisogno di altri 3 adesivi. Dai 3 adesivi alla rana.

Cerchiamo di verificare che siano uguali. Quanti adesivi ha la scimmia? (Il bambino dovrebbe dire 7.) Quanti adesivi ha la rana? (Il bambino dovrebbe dire 7.) Ottimo! Hai capito. Scimmia e rana hanno entrambi 7 adesivi!

Scheda delle marionette

Ora stiamo facendo lo stesso gioco, ma su carta. (Mostra al bambino la scheda di lavoro).

Questo rappresenta la scimmia (indica la scimmia) e questo la rana (indica la rana). Ecco i loro raccoglitori di adesivi (indica i quattro spazi bianchi).

Ti dirò quanti adesivi dare loro, e scriverai il numero sulla linea, OK?

Intanto, dai 3 adesivi alla scimmia (indica il primo spazio bianco del lato sinistro, aspetta che il bambino abbia scritto 3). Bene!

Dai alla scimmia altri 5 adesivi (indica il secondo spazio bianco del lato sinistro, aspetta che il bambino scriva 5).

Adesso dai 3 adesivi alla rana (indica il primo spazio bianco del lato destro, aspetta che il bambino abbia scritto 3). Perfetto!

La scimmia e la rana vogliono avere una quantità uguale di adesivi (sottolinea la parola uguale e indica verso ogni personaggio muovendo le mani come una bilancia, come a mostrare la quantità uguale).

Adesso, vorrei che tu mi dicessi quanti altri adesivi bisogna dare alla rana (indica il secondo spazio bianco del lato destro) in modo che la rana abbia la stessa quantità di adesivi della scimmia (cerchia col dito il lato sinistro).

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e dai 5 adesivi alla rana.

Se la risposta del bambino è <u>errata</u>: Buon tentativo, ma così non diventano uguali. La scimmia ha 8 adesivi (indica i due numeri che il bambino ha scritto a sinistra). La rana ha solo 3 adesivi (indica il numero che il bambino ha scritto a destra).

Di quanti altri adesivi ha bisogno la rana in modo che possa averne 8?

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e dai 5 adesivi alla rana, scrivi 5.

Se la risposta del bambino è <u>errata</u>: in realtà la rana ha bisogno di altri 5 adesivi. Dai 5 adesivi alla rana, scrivi 5.

Verifichiamo che siano uguali. Quanti adesivi ha la scimmia? (Il bambino dovrebbe dire 8.) Quanti adesivi ha la rana? (Il bambino dovrebbe dire 8.) Ottimo! Hai capito. Scimmia e rana hanno entrambe 8 adesivi!

Concreto con bilancia

Adesso faremo un gioco nuovo con questa bilancia e questi orsi.

Utilizzerai gli orsi per mettere in equilibrio la bilancia. Ti dirò quanti orsi mettere su ogni lato, ok?

Innanzitutto, metti 5 orsi sul lato sinistro (indica il lato sinistro della bilancia). Ben fatto!

Ora metti altri 4 orsi sul lato sinistro (indica il lato sinistro della bilancia). Ok!

Metti 5 orsi sul lato destro (indica il lato destro della bilancia). Perfetto!

Vogliamo che la bilancia sia in equilibrio (sottolinea la parola equilibrio e rivolgiti con le mani ad entrambi i lati della bilancia a rendere il senso delle quantità uguali).

Ora vorrei che mi dicessi quanti orsi in più devi mettere sul lato destro (indica il lato destro), per avere a destra (indica il lato a destra) la stessa quantità del lato sinistro (indica il lato sinistro della bilancia).

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e metti 4 orsi nel lato destro.

Se la risposta del bambino è <u>errata</u>: Buon tentativo, ma così non sarà in equilibrio. Il lato sinistro ha 9 orsi (indica ciascuno degli orsi come se li contassi ad alta voce). Il lato destro ha solo 5 orsi (indica ciascuno degli orsi come se li contassi ad alta voce). Di quanti orsi in più ha bisogno il lato destro (indica il lato destro della bilancia) per avere 9 orsi?

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e metti 4 orsi nel lato destro.

Se la risposta del bambino è *errata*: in realtà servono altri 4 orsi. Metti 4 orsi nel lato destro.

Verifichiamo che sia equilibrata. Quanti orsi ci sono a sinistra? (Il bambino dovrebbe dire 9.) Quanti orsi ci sono a destra? (Il bambino dovrebbe dire 9.) Ottimo! Hai capito. Entrambi i lati hanno 9 orsi!

Scheda della bilancia

Ora faremo lo stesso gioco, ma su carta. (Mostra al bambino la scheda di lavoro).

Questo indica il lato sinistro della bilancia (indica a sinistra) e questo rappresenta il lato destro della bilancia (indica a destra). Adesso ti dirò quanti orsi mettere da ogni lato, e tu lo farai scrivendo il numero che ti dico negli spazi vuoti, ok?

Innanzitutto, metti 3 orsi a sinistra (indica la prima casella a sinistra, aspetta che il bambino scriva 3). Bene!

Ora metti altri 2 orsi a sinistra (indica la seconda casella a sinistra, aspetta che il bambino scriva 2).

Metti 3 orsi sul lato destro (indica la prima casella a destra, aspetta che il bambino scriva 3). Perfetto!

Vogliamo che la bilancia sia equilibrata (sottolinea la parola equilibrata e rivolgiti con le mani ad entrambi i lati della bilancia a rendere il senso delle quantità uguali).

Ora vorrei che mi dicessi quanti orsi in più devi mettere sul lato destro (indica il lato destro), per avere a destra (indica il lato a destra) la stessa quantità del lato sinistro (indica il lato sinistro della bilancia).

Se la risposta del bambino è *corretta:* Giusto. Vai avanti e metti 2 orsi nel lato destro, scrivi 2.

Se la risposta del bambino è *errata:* Buon tentativo, ma così non sarà in equilibrio. Il lato sinistro ha 5 orsi (indica i due numeri che il bambino ha scritto a sinistra). Il lato destro ha solo 3 orsi (indica il numero che il bambino ha scritto a destra). Di quanti orsi in più ha bisogno il lato destro (indica il lato destro) per avere 5 orsi?

Se la risposta del bambino è corretta: Giusto. Vai avanti e metti 2 orsi nel lato destro, scrivi 2.

Se la risposta del bambino è errata: in realtà servono altri 2 orsi. Metti 2 orsi nel lato destro, scrivi 2.

Verifichiamo che siano uguali. Quanti orsi ci sono a sinistra? (Il bambino dovrebbe dire 5.) Quanti orsi ci sono a destra? (Il bambino dovrebbe dire 5.) Ottimo! Hai capito. Entrambi i lati hanno 5 orsi!

Concreto con le aste della numerazione Montessori

Adesso faremo un lavoro con le aste della numerazione che sono in questa scatola. Vedi? Questo è uno, questo è due ... fino a dieci (toccando le aste marcando il cambio di colore). Prova a contare questa (porgi al bambino un'asta maggiore di cinque. Aspetta che il bambino abbia terminato di contare). Dopo aver fatto osservare al bambino che ogni asta ha un valore compreso tra 1 e 10, indica il bigliettino col segno uguale. Conosci questo segno? (Il bambino dovrebbe dire che è uguale. Se non lo dice, gli viene ricordato.) Esatto, è uguale. Noi dobbiamo mettere da questa parte (indica con la mano lo spazio a sinistra dell'uguale), la stessa quantità che mettiamo da questa parte (indica con la mano lo spazio a destra dell'uguale). Devono essere uguali (sottolinea la parola uguale).

Ora prendi l'asta 4. (Aspetta che il bambino abbia contato). Bene. Mettila qui (indica lo spazio a sinistra). Adesso prendi l'asta 3, e mettila vicino al 4. Perfetto!

Adesso, metti un'asta 4 da questa parte (indica il lato a destra dell'uguale).

Dimmi, secondo te cosa dovresti mettere qui (indico il lato a destra dell'uguale) per avere la stessa quantità che hai di qua? (indica il lato a sinistra dell'uguale).

Se la risposta del bambino è corretta: Giusto. Vai avanti e metti un'asta 3.

Se la risposta del bambino è <u>errata</u>: Buon tentativo, ma così non diventano uguali. Di qua ci c’è 7 (indica i passaggi rosso/blu dell’asta come se contassi ad alta voce). Di qua c’è solo 4 (indica i passaggi rosso/blu dell’asta come se contassi ad alta voce). Che asta dobbiamo mettere per avere le due parti uguali?

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e metti un’asta 3.

Se la risposta del bambino è ancora <u>errata</u>: in realtà c'è bisogno dell’asta 3. Prendi l’asta 3 e mettila qui.

Verifichiamo che siano uguali. Quanto c’è da questa parte? (Il bambino dovrebbe dire 7.) E da quest'altra? (Il bambino dovrebbe dire 7.) Ottimo! Hai capito. Da entrambi i lati dell'uguale c’è la stessa quantità.

Scheda delle aste della numerazione

Ora faremo lo stesso lavoro, ma su carta. (Mostra al bambino la scheda).

Questo rappresenta una serie di aste della numerazione (indica il disegno delle aste) e questo l'altra serie (indica l'altro disegno delle aste). Qui sopra bisogna scrivere il numero. Io ti dico quale asta prendere, e tu scrivi il numero nello spazio, ok?

Prendi l’asta 4 (indica le aste del lato sinistro, aspetta che il bambino abbia scritto 4). Bene!

Adesso, metti l’asta 3 da questa parte (indica il lato sinistro, aspetta che il bambino abbia scritto 3).

Ora, prendi l’asta 4 e mettila a destra (indica il disegno delle aste di destra, aspetta che il bambino abbia scritto 4). Perfetto!

A destra e sinistra dell'uguale vogliamo avere una quantità uguale (sottolinea la parola uguale e indica entrambi i lati muovendo le mani come una bilancia, come a mostrare la quantità uguale).

Adesso, vorrei che tu mi dicessi quale asta dobbiamo mettere qui (indica lo spazio vuoto a destra) in modo che i due lati abbiano la stessa quantità.

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi 3.

Se la risposta del bambino è errata: Buon tentativo, ma così non diventano uguali. A sinistra abbiamo 7 (indica i due numeri scritti dal bambino). A destra abbiamo solo 4 (indica il numero scritto dal bambino).

Di quale asta abbiamo bisogno in modo che a destra ce ne siano 7?

Se la risposta del bambino è corretta: Giusto. Vai avanti e scrivi 3.

Se la risposta del bambino è errata: In realtà c'è bisogno dell’asta 3, scrivi 3.

Verifichiamo che siano uguali. Quanto c’è a sinistra? (Il bambino dovrebbe dire 7.) Quanto c’è a destra? (Il bambino dovrebbe dire 7) Ottimo! Hai capito. Entrambi i lati hanno 7!

Concreto con abaco Montessori (telaio piccolo)

Adesso faremo un lavoro con questi telai. Vedi? Su questa fila ci sono le unità, e noi conteremo queste.

Guarda, questo è cinque (mostra al bambino come spostare le perle delle unità da sinistra a destra per contare, 1,2,3,4,5). Scegli tu un numero e conta sul telaio (porgi al bambino il telaio e aspetta che abbia terminato di contare). Dopo che il bambino ha compreso il funzionamento del telaio, indica il bigliettino col segno uguale. Conosci questo segno? (il bambino dovrebbe dire che è uguale. Se non lo dice, gli viene ricordato.) Esatto, è uguale. Noi dobbiamo spostare da questa parte (indica con la mano lo spazio a sinistra dell'uguale), la stessa quantità che spostiamo da questa parte (indica con la mano lo spazio a destra dell'uguale).

Devono essere uguali (sottolinea la parola uguale).

Ora sposta sul telaio di sinistra 4 perle. (Aspetta che il bambino abbia contato e spostato le perle). Quante perle hai spostato? (il bambino dirà 4) Bene! Adesso ne dobbiamo mettere altre 3, spostale vicino al 4. Perfetto!

Adesso, lo facciamo sul telaio a destra (indica il lato a destra dell'uguale). Sposta 4 perle sul telaio. Bene!

Dimmi, secondo te quante bisogna spostarne ancora (indico il telaio a destra dell'uguale) per avere la stessa quantità che hai di qua? (indica il telaio a sinistra dell'uguale).

Se la risposta del bambino è corretta: Giusto. Vai avanti e sposta 3 perle.

Se la risposta del bambino è errata: Buon tentativo, ma così non diventano uguali. Di qua ci sono 7 perle (indica ciascuna delle perle come se le contassi ad alta voce). Di qua ci sono solo 4 perle (indica ciascuna delle perle come se le contassi ad alta voce). Quante perle dobbiamo mettere per avere le due parti uguali?

Se la risposta del bambino è corretta: Giusto. Vai avanti e sposta 3 perle.

Se la risposta del bambino è ancora errata: in realtà c'è bisogno di altre 3 perle. Sposta altre 3 perle.

Verifichiamo che siano uguali. Quante perle ci sono da questa parte? (Il bambino dovrebbe dire 7). E da quest'altra? (Il bambino dovrebbe dire 7). Ottimo! Hai capito. Su entrambi i telai ci sono 7 perle.

Scheda dell'abaco

Ora faremo lo stesso lavoro, ma su carta. (Mostra al bambino la scheda).

Questo rappresenta un abaco (indica il disegno dell’abaco) e questo l'altro abaco (indica l'altro). Qui sopra bisogna scrivere il numero.

Io ti dico quante perle si spostano, e tu scrivi il numero nello spazio, OK?

Sposta 3 perle su questo abaco (indica il primo spazio del lato sinistro, aspetta che il bambino abbia scritto 3). Bene!

Adesso, spostane altre 5 da questa parte (indica il secondo spazio del lato sinistro, aspetta che il bambino abbia scritto 5).

Ora, dobbiamo spostarne 3 dall'altra parte dell'uguale (indica il primo spazio bianco di destra, aspetta che il bambino abbia scritto 3). Perfetto!

A destra e sinistra dell'uguale vogliamo avere una quantità uguale di perle (sottolinea la parola uguale e indica entrambi i lati muovendo le mani come una bilancia, come a mostrare la quantità uguale).

Adesso, vorrei che tu mi dicessi quante perle dobbiamo spostare qui (mostra il secondo spazio bianco a destra) in modo che i due lati abbiano la stessa quantità di perle.

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e scrivi 5.

Se la risposta del bambino è <u>errata</u>: Buon tentativo, ma così non diventano uguali. A sinistra abbiamo 8 perle (indica i due numeri scritti dal bambino). A destra abbiamo solo 3 perle (indica il numero scritto dal bambino).

Di quante altre perle abbiamo bisogno in modo che a destra ce ne siano 8?

Se la risposta del bambino è <u>corretta</u>: Giusto. Vai avanti e scrivi 5.

Se la risposta del bambino è <u>errata</u>: in realtà c'è bisogno di altre 5 perle, scrivi 5.

Verifichiamo che siano uguali. Quante perle ci sono a sinistra? (Il bambino dovrebbe dire 8.) Quante perle ci sono a destra? (Il bambino dovrebbe dire 8.) Ottimo! Hai capito. Entrambi i lati hanno 8 perle!

References

Ainsworth, S. (1999). The functions of multiple representations. *Computers & education*, *33*(2-3), 131-152.

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental psychology,* 35(1), 127.

Baranes, R., Perry, M., & Stigler, J. W. (1989). Activation of real-world knowledge in the solution of word problems. *Cognition and Instruction*, 6(4), 287-318.

Barth, H., La Mont, K., Lipton, J., Dehaene, S., Kanwisher, N., & Spelke, E. (2006). Non-symbolic arithmetic in adults and young children. *Cognition*, *98*(3), 199-222.

Berteletti, I., Lucangeli, D., Piazza, M., Dehaene, S., & Zorzi, M. (2010). Numerical estimation in preschoolers. *Developmental psychology*, *46*(2), 545.

Besançon, M., & Lubart, T. (2008). Differences in the development of creative competencies in children schooled in diverse learning environments. *Learning and individual differences*, *18*(4), 381-389.

Bisanz (1999), The development of mathematical cognition: arithmetic. *Journal of experimental child pshycology*, 153-156

Borghesani, V., & Piazza, M. (2017). The neuro-cognitive representations of symbols: The case of concrete words. *Neuropsychologia*, *105*, 4-17.

Braithwaite, D. W., & Goldstone, R. L. (2015). Effects of variation and prior knowledge on abstract concept learning. *Cognition and Instruction*, *33*(3), 226-256.

Brown, M. C., McNeil, N. M., & Glenberg, A. M. (2009). Using concreteness in education: Real problems, potential solutions. *Child Development Perspectives*, *3*(3), 160-164.

Bruner, J. S. (1966). Toward a theory of instruction (Vol. 59). *Harvard University*

Press.

Burns, M. (1996). How to make the most of math manipulatives. *Instructor*, 105(7), 45–51.

Byrd, C. E., McNeil, N. M., Chesney, D. L., & Matthews, P. G. (2015). A specific misconception of the equal sign acts as a barrier to children's learning of early algebra. *Learning and Individual Differences*, 38, 61-67.

Carbonneau, K. J., & Marley, S. C. (2015). Instructional guidance and realism of manipulatives influence preschool children's mathematics learning. The Journal of Experimental Education, 83(4), 495-513.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380.

Castronovo, J., & Göbel, S. M. (2012). Impact of high mathematics education on the number sense. *PloS one*, *7*(4), e33832.

Chi, M. T., Bassok, M., Lewis, M. W., Reimann, P., & Glaser, R. (1989). Self-explanations: How students study and use examples in learning to solve problems. *Cognitive science*, 13(2), 145-182.

Campbell, J. I., & Clark, J. M. (1988). An encoding-complex view of cognitive number processing: Comment on McCloskey, Sokol, and Goodman (1986).

Clark, J. M., & Campbell, J. I. (1991). Integrated versus modular theories of number skills and acalculia. *Brain and Cognition*, *17*(2), 204-239.

D'Amore, B. (2003). Le basi filosofiche, pedagogiche, epistemologiche e concettuali della Didattica della Matematica. Pitagora.

D'Amore, B., & Pinilla, M. I. F. (2007). Le didattiche disciplinari. *Edizioni Erickson.*

Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of Experimental Child Psychology*, 113(4), 469-486.

de Hevia, M. D., Vallar, G., & Girelli, L. (2008). Visualizing numbers in the mind's eye: The role of visuo-spatial processes in numerical abilities. *Neuroscience & Biobehavioral Reviews*, *32*(8), 1361-1372.

Dehaene, 1997, The number sense. How the mind creates mathematics, *Oxford University Press*, New York.

Dehaene, S., Molko, N., Cohen, L., & Wilson, A. J. (2004). Arithmetic and the brain. *Current opinion in neurobiology, 14(2), 218-224.*

DeLoache, J. S. (2004). Becoming symbol-minded. *Trends in cognitive sciences*, *8*(2), 66-70.

Deloche, G., & Willmes, K. (2000). Cognitive neuropsychological models of adult calculation and number processing: the role of the surface format of numbers. *European child & adolescent psychiatry,* 9(2), S27.

Dohrmann, K. R., Nishida, T. K., Gartner, A., Lipsky, D. K., & Grimm, K. J. (2007). High school outcomes for students in a public Montessori program. *Journal of Research in Childhood Education*, *22*(2), 205-217.

Donovan, A. M., & Fyfe, E. (2019). Connecting concrete objects and abstract symbols promotes children's mathematics learning.

Duval R. (2006). Trasformazioni di rappresentazioni semiotiche e prassi di pensiero in matematica. *La matematica e la sua didattica*, 4, 585-619.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. Teaching children mathematics, 6(4), 232.

Pinilla, M. I. F. (2005). Le frazioni: aspetti concettuali e didattici. Pitagora.

Feigenson, L., Libertus, M. E., & Halberda, J. (2013). Links Between the Intuitive Sense of Number and Formal Mathematics Ability. *Child development perspectives*, *7*(2), 74–79. doi:10.1111/cdep.12019

Fresco, G. H. (2000). Montessori: perché no. *Una pedagogia per la crescita, Il Leone Verde, (238-239).*

Fuson, K. C., Wearne, D., Hiebert, J. C., Murray, H. G., Human, P. G., Olivier, A. I. & Fennema, E. (1997). Children's conceptual structures for multidigit numbers and methods of multidigit addition and subtraction. *Journal for research in mathematics education*, 130-162.

Fyfe, E. R., & Nathan, M. J. (2019). Making "concreteness fading" more concrete as a theory of instruction for promoting transfer. *Educational Review,* 71(4), 403-422.

Fyfe, E. R., Matthews, P. G., Amsel, E., McEldoon, K. L., & McNeil, N. M. (2018). Assessing formal knowledge of math equivalence among algebra and pre-algebra students. *Journal of educational psychology*, *110*(1), 87.

Fyfe, E. R., McNeil, N. M., & Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. *Learning and Instruction, 35,* 104-120.

Fyfe, E. R., McNeil, N. M., Son, J. Y., & Goldstone, R. L. (2014). Concreteness fading in mathematics and science instruction: A systematic review. *Educational psychology review*, *26*(1), 9-25.

Gelman, R., & Gallistel, C. R. (1986). *The child's understanding of number*. Harvard University Press.

Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, *13*(3), 343-359.

Gentner, D., & Markman, A. B. (1997). Structure mapping in analogy and similarity. *American psychologist*, *52*(1), 45.

Gentner, D., Ratterman, M., & Forbus, K. (1993). The Roles of Similarity in Transfer: Separating Retrievability from Inferential Soundness. *Cognitive Psychology*, 25, 524-575.

Gilmore, C. K., McCarthy, S. E., & Spelke, E. S. (2010). Non-symbolic arithmetic abilities and mathematics achievement in the first year of formal schooling. *Cognition*, *115*(3), 394-406.

Goldstone, R. L., & Sakamoto, Y. (2003). The transfer of abstract principles governing complex adaptive systems. *Cognitive psychology*, *46*(4), 414-466.

Goldstone, R. L., & Son, J. Y. (2005). The transfer of scientific principles using concrete and idealized simulations. *The Journal of the learning sciences*, *14*(1), 69-110.

Goldstone, R. L., Landy, D. H., & Son, J. Y. (2010). The education of perception. *Topics in Cognitive Science*, *2*(2), 265-284.

Gomez, A., Piazza, M., Jobert, A., Dehaene-Lambertz, G., Dehaene, S. & Huron, C. (2015) Mathematical difficulties in developmental coordination disorder: Symbolic and nonsymbolic number processing. *Research in Developmental Disabilities*, 43–44:167–78.

Gómez, R. L., Bootzin, R. R., & Nadel, L. (2006). Naps promote abstraction in language-learning infants. *Psychological science*, *17*(8), 670-674.

Goswami, U. (1991). Analogical reasoning: What develops? A review of research and theory. *Child development*, *62*(1), 1-22.

Gray, E., & Tall, D. (1993). Success and failure in mathematics: the flexible meaning of symbols as process and concept. *Mathematics teaching*, *142*(6-10).

Hatano, G., & Inagaki, K. (1984). Two courses of expertise. *Research and clinical center for child development Annual Report*, *6*, 27-36.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. *Conceptual and procedural knowledge: The case of mathematics*, *2*, 1-27.

Hiebert, J., & Wearne, D. (1996). Instruction, understanding, and skill in multidigit addition and subtraction. *Cognition and instruction*, *14*(3), 251-283.

Holloway, I. D., & Ansari, D. (2009). Mapping numerical magnitudes onto symbols: The numerical distance effect and individual differences in children’s mathematics achievement. *Journal of experimental child psychology*, *103*(1), 17-29.

Kaminski, J. A., Sloutsky, V. M., & Heckler, A. F. (2008). The advantage of abstract examples in learning math. *Science*, *320*(5875), 454-455.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). The strands of mathematical proficiency. *Adding it up: Helping children learn mathematics*, 115-155.

Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade offs between grounded and abstract representations: Evidence from algebra problem solving. *Cognitive Science*, *32*(2), 366-397.

Kotovsky, K., Hayes, J. R., & Simon, H. A. (1985). Why are some problems hard? Evidence from Tower of Hanoi. *Cognitive psychology, 17(2), 248-294.*

Laski, E. V., Jor'dan, J. R., Daoust, C., & Murray, A. K. (2015). What makes mathematics manipulatives effective? Lessons from cognitive science and Montessori education. *SAGE Open*, *5*(2).

Laski, E. V., Vasilyeva, M., & Schiffman, J. (2016). Longitudinal Comparison of Place-Value and Arithmetic Knowledge in Montessori and Non-Montessori Students. *Journal of Montessori Research*, *2*(1), 1-15.

Lillard, A. S. (2012). Preschool children's development in classic Montessori, supplemented Montessori, and conventional programs. *Journal of School Psychology*, *50*(3), 379-401.

Lillard, A. S. (2016). Montessori: The science behind the genius. *Oxford University Press.*

Lillard, A., & Else-Quest, N. (2006). Evaluating Montessori education. *Science*, *313*(5795), 1893-1894.

Lucangeli, D., De Candia, C., & Poli, S. (2003). L'intelligenza numerica. Vol 3. Abilità cognitive e metacognitive nella costruzione della conoscenza numerica dagli 8 agli 11 anni. *Erickson Edizioni.*

Martin, T. (2009). A theory of physically distributed learning: How external environments and internal states interact in mathematics learning. *Child Development Perspectives*, *3*(3), 140-144.

Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: Adapting and reinterpreting physical environments in the development of fraction concepts. *Cognitive science*, *29*(4), 587-625.

Mason, L. (2013). *Psicologia dell'apprendimento e dell'istruzione*. Il Mulino.

Matthews, P., & Rittle-Johnson, B. (2009). In pursuit of knowledge: Comparing self-explanations, concepts, and procedures as pedagogical tools. *Journal of experimental child psychology*, 104(1), 1-21.

Mevarech & Kramarski, 2014, Critical Maths for Innovative Societies:The Role of Metacognitive Pedagogies, *OECD Publishing*, Paris

McCloskey, M., Sokol, S. M., & Goodman, R. A. (1986). Cognitive processes in verbal-number production: Inferences from the performance of brain-damaged subjects. Journal of Experimental Psychology: General, 115(4), 307.

McGaugh, J. L. (2000). Memory--a century of consolidation. *Science*, *287*(5451), 248-251.

Mix, K. S., Prather, R. W., Smith, L. B., & Stockton, J. D. (2014). Young children's interpretation of multidigit number names: From emerging competence to mastery. *Child Development*, *85*(3), 1306-1319.

McNeil, N. M., & Alibali, M. W. (2000). Learning mathematics from procedural instruction: Externally imposed goals influence what is learned. *Journal of Educational Psychology*, 92(4), 734.

McNeil, N.M., & Alibali, M.W. (2005). Why won't you change your mind? Knowledge of operational patterns hinders learning and performance on equations. *Child Development*, 76, 883–899

McNeil, N. M., & Fyfe, E. R. (2012). “Concreteness fading” promotes transfer of mathematical knowledge. *Learning and Instruction*, *22*(6), 440-448.

McNeil, N., & Jarvin, L. (2007). When theories don't add up: disentangling he manipulatives debate. *Theory into practice*, *46*(4), 309-316.

Montessori, M. (1917). *The Advanced Montessori Method.* Frederick A. Stokes

Company.

Montessori, M. (2013a). *The Montessori method*. Transaction publishers.

Montessori, M. (2013b). *Psicoaritmetica*. Edizioni Opera Nazionale Montessori.

Nathan, M. 2012. “Rethinking Formalisms in Formal Education.” *Educational Psychologist,* 47 (2), 125–148.

Nigris, E. (2004). Didattica generale: Ambienti di apprendimento. Guerini scientifica.

Crooks, N. M., & Alibali, M. W. (2014). Defining and measuring conceptual knowledge in mathematics. *Developmental Review,* 34(4), 344-377.

Odic, D., & Starr, A. (2018). An introduction to the approximate number system. *Child development perspectives*, 12(4), 223-229.

Ottmar, E., & Landy. D. (2017). Concreteness Fading of Algebraic Instruction: Effects on Learning. *Journal of the Learning Sciences* 26 (1): 51–78.

Peeters, D., Degrande, T., Ebersbach, M., Verschaffel, L., & Luwel, K. (2016). Children’s use of number line estimation strategies. *European Journal of Psychology of Education*, *31*(2), 117-134.

Perry, M. (1991). Learning and transfer: Instructional conditions and conceptual change. *Cognitive Development*, 6, 449-468.

Pesenti, M., Seron, X., & Van Der Linden, M. (1994). Selective impairment as evidence for mental organisation of arithmetical facts: BB, a case of preserved subtraction?. *Cortex*, *30*(4), 661-671.

Petersen, L. A., & McNeil, N. M. (2013). Effects of perceptually rich manipulatives on preschoolers' counting performance: Established knowledge counts. *Child development*, *84*(3), 1020-1033.

Piaget, J. (1954). The construction of reality in the child (M. Cook, Trans.). New York, NY, US: Basic Books.

Piaget, J. (1970). Science of education and the psychology of the child. Trans. D. Coltman.

Piaget, J. (1952). The child's conception of number. London: Routledge & Kegan Paul.

Piazza, M., Pica, P., Izard, V., Spelke, E. S. & Dehaene, S. (2013) Education enhances the acuity of the nonverbal approximate number system. *Psychological Science*, 24:1037–43.

Pica, P., Lemer, C., Izard, V., & Dehaene, S. (2004). Exact and approximate arithmetic in an Amazonian indigene group. *Science*, *306*(5695), 499-503.

Rathunde, K., & Csikszentmihalyi, M. (2005). Middle school students’ motivation and quality of experience: A comparison of Montessori and traditional school environments. American journal of education, 111(3), 341-371.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of educational psychology*, *93*(2), 346.

Rittle-Johnson, Promoting transfer: Effects of self-explanation and direct instruction, *Child Development*, Vol 77, 1, 1-15, 2006

Robertson, E. M., Pascual-Leone, A., & Miall, R. C. (2004). Current concepts in procedural consolidation. *Nature Reviews Neuroscience*, *5*(7), 576.

Santi, G., & Sbaragli, S. (2007). Semiotic representations," avoidable" and" unavoidable" misconceptions. *La matematica e la sua didattica*, *21*(1), 105-110.

Sbaragli S. (2005). Misconcezioni “inevitabili” e misconcezioni “evitabili”. *La matematica e la sua didattica*, 1. 57-71.

Scheiter, K., Gerjets, P., & Schuh, J. (2010). The acquisition of problem-solving skills in mathematics: How animations can aid understanding of structural problem features and solution procedures. *Instructional Science*, *38*(5), 487-502.

Schneider, M., Rittle-Johnson, B., & Star, J. R. (2011). Relations among conceptual knowledge, procedural knowledge, and procedural flexibility in two samples differing in prior knowledge. *Developmental Psychology*, *47*(6), 1525-1538.

Scoppola B., Lessons of Maria Montessori. In, *Annali di storia dell'educazione*, 18(413-434)

Sidney, P. G., Hattikudur, S., & Alibali, M. W. (2015). How do contrasting cases and self-explanation promote learning? Evidence from fraction division. *Learning and Instruction*, *40*, 29-38.

Siegler, R. S., & Booth, J. L. (2004). Development of numerical estimation in young children. *Child development*, *75*(2), 428-444.

Siegler, R. S., & Opfer, J. E. (2003). The development of numerical estimation: Evidence for multiple representations of numerical quantity. *Psychological science*, *14*(3), 237-250.

Sloutsky, V. M., Kaminski, J. A., & Heckler, A. F. (2005). The advantage of simple symbols for learning and transfer. *Psychonomic bulletin & review*, *12*(3), 508-513.

Son, J. Y., Smith, L. B., & Goldstone, R. L. (2008). Simplicity and generalization: Short-cutting abstraction in children's object categorizations. *Cognition*, *108*(3), 626-638.

Son, J. Y., Smith, L. B., & Goldstone, R. L. (2011). Connecting instances to promote children's relational reasoning. *Journal of experimental child psychology*, *108*(2), 260-277.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics instruction. *Journal for research in mathematics education*, 498-505.

Star, J. R. (2005). Reconceptualizing procedural knowledge. *Journal for research in mathematics education*, 404-411.

Stickgold, R., Hobson, J. A., Fosse, R., & Fosse, M. (2001). Sleep, learning, and dreams: off-line memory reprocessing. *Science*, *294*(5544), 1052-1057.

Tapola, A., Veermans, M., & Niemivirta, M. (2013). Predictors and outcomes of situational interest during a science learning task. *Instructional Science*, *41*(6), 1047-1064.

Uttal, D. H. (2003). On the relation between play and symbolic thought: The case of mathematics manipulatives. *Contemporary perspectives in early childhood*, 97-114.

Uttal, D. H., O'Doherty, K., Newland, R., Hand, L. L., & DeLoache, J. (2009). Dual representation and the linking of concrete and symbolic representations. *Child Development Perspectives*, *3*(3), 156-159.

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of applied developmental psychology*, *18*(1), 37-54.

Wynn, K. (1990). Children's understanding of counting. *Cognition*, *36*(2), 155-193.