Multiscale techniques

for nonlinear difference equations

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Introduction

Almost every realistic physical system is nonlinear in nature. Nonlinear models can be derived in many areas of physics from the propagation of water waves to the macroscopic theory of supeconductivity and superfluidity and to general relativity. Moreover one can say all *chaos theory* originated from the study of nonlinear dynamics. In fact most of the times those systems present an irregular if not chaotic behavior. Anyway a great number of models of physical situations, named *integrable models* (see later), exhibit features as regularity, stability and predictability of the motion. For example the *Korteweg de Vries equation* (KdV) for a real function $u(x,t)$

$$\partial_t u + \partial_x^3 u = u \partial_x u,$$

arises in the propagation of shallow water surface waves when weakly nonlinear restoring forces are present, of long internal waves in a density stratified ocean, of ion-acoustic waves in a plasma and acoustic waves on a crystal lattice. Another prototypical nonlinear model is represented by the *nonlinear Schrödinger equation* (NLS) for a complex function $u(x,t)$

$$\mathrm{i}\partial_t u + \partial_x^2 u = \pm u|u|^2,$$

In the literature the $(-)$ case is known as the *focusing* NLS equation and the $(+)$ case as the *defocusing* NLS equation. It arises in nonlinear optics in the presence of materials whose dielectric constant increases with the field intensity. In such a situation an electromagnetic beam, which otherwise would broad due to diffraction, can propagate without spreading in nonlinear media and continue focusing. An important feature of the NLS equation, which is closely related to the work presented in this thesis, is also due to its universal character. Generically speaking, most weakly nonlinear, dispersive, energy-preserving systems give rise, in an appropriate limit, to the NLS equation. More in detail, the NLS equation provides a description for the envelope dynamics of a quasi-monocromatic plane wave propagating in a nonlinear weakly dispersive medium when dissipation can be neglected. Variants of the NLS equation offer models for various situations as for example Bose-Einstein condensates.

In more recent years the attention turned to discrete nonlinear systems, i. e. systems described by difference equations. Let's think for a moment to the possible applications in quantum gravity as models for the dynamics of a discrete space-time. In nonlinear optics for example the dynamics of localized pulses in arrays of coupled optical waveguides are described by a discrete version of the continuous NLS equation for a complex function $f_n(t)$

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2\sigma^2} = \pm |f_n|^2 f_n,$$

known as the *discrete nonlinear Schrödinger equation* $(dNLS)$. This equation arises in several situations ranging from the mechanical context for a lattice of coupled anharmonic oscillators to condensed matter physics. Despite of its physical relevance, numerical simulations show for this system situations of irregular dynamics and emergence of chaos [1]. Anyway much of the attractive behind nonlinear discrete systems resides not only in their possible physical applications but also in the domain of numerical calculus as they represent finite difference approximations of the corresponding continuous models.

Beside physical and numerical applications a renewed interest in nonlinear models came from the purely mathematical side since the discovery of the *inverse scattering* or *spectral transform* method. This technique in fact generalizes to the nonlinear domain the *Fourier transform* method thereby allowing to solve, even if for a small portion of nonlinear systems, the Cauchy problem at least in some special classes of solutions. In fact the core of the inverse scattering method is the observation that certain *spectral data*, in one to one correspondence with the given equation, evolve in a simple way contrasted with the generally complicated nonlinear evolution of the system they refer. Moreover, under some conditions, the differential problem can be turned into a purely algebraic one, allowing the explicit construction of the so called *soliton* solutions. For example the previously cited KdV and the continuous NLS equations present solitons solutions. Another feature of the systems solvable by the spectral transform method is that they don't come alone but as members in *hierarchies*. Every system in a hierarchy is indeed solvable by inverse scattering. Moreover every hierarchy of equations represents compatible evolutions for the same function or, otherwise stated, equations in a hierarchy represent *generalized symmetries* of each other. The presence of (point and genarelized) symmetries for a given system offer another instrument to derive explicitly special solutions by means of the so called *symmetry reductions*. Another important feature of those systems is the presence of an infinite number of *conservation laws*. These laws, confining the motion to a restricted area of the phase space, are in some way responsible for the regularity of the dynamics. All these instruments permit the extraction of a great amount of information from the system under study. However we remark that only some special systems lend themselves to a similar analysis. For this reason those systems are termed *integrable systems*. Between them fall also every system which is linear or linearizable by an invertible transformation.

For all the systems for which spectral, symmetry or other algebraic methods are not of great help or at disposal, named *nonintegrable systems*, as the $dNLS$ equation, a great help comes from *perturbative techniques*. In general perturbation theory is a collection of iterative methods for the systematic analysis of the behavior of solutions to differential and difference equations. The general procedure of perturbation theory is to identify a parameter ε such that the solution of the given problem is constructed as a power series of ε around an ε_0 value at which limit the problem becomes soluble, i. e. very often the system reduces to an integrable system. No need to say that perturbation theory results useful even in the case of integrable systems, i. e. in the study of the solutions in all the situations when the spectral problem is not turnable into an algebraic one. On another side, certain features of the dynamics of integrable and nonintegrable systems need not an explicit solution to be enlightened as appear only in specific asymptotic regimes. The description of these regimes is the subject of the so called *reductive* perturbation method [38, 39], a method which reduces the system under study to a more tractable and solvable system. Among reductive techniques we have *multiscale analysis*, a particularly useful method for constructing perturbation series *uniformly approximating* the solution of the problem. In more recent years multiscale reduction techniques proved to be also an excellent integrability test for a large variety of nonlinear partial differential equations [5, 9, 13, 14, 35].

The aim of this thesis is the development of a multiscale reductive perturbation technique for discrete systems, that is systems described by partial difference equations. A guiding principle in such a programme should certainly be the requirement, if one starts from an integrable model, to maintain this integrability property for the reduced models. So, if for an integrable system the reduced equations should always be at all perturbative orders integrable (a member of an integrable hierarchy), for a nonintegrable one the result could be, up to any finite order, either integrable or not. Anyway for a nonintegrable system there should always exist an order at which we obtain a nonintegrable equation. Thus a properly developed multiscale technique should provide us as a by-product, besides approximate solutions to our equations of motion, an integrability test capable in principle to recognize a nonintegrable system, reproducing in this way on the discrete side what multiscale perturbation techniques successfully did on the continuous case. The first attempts to transfer multiscale perturbation techniques to the level of difference equations [2, 26–30] don't succeed in preserving the integrability property of the starting models.

This work is organized as follows:

- **Chapter 2. The continuous and discrete multiscale techniques.**
  In *Section 1* of this chapter we review the classical multiscale perturbative approach as in [4–9, 13, 40] for continuos real dispersive equations. After a general presentation of the method, we will give two illustrative examples of application: the first one is the paradigmatic case of the KdV equation a well known $S-$integrable model and in the second one we will present the reduction of a $C-$integrable *Burgers*-like equation. Both example will be carried out up to the *nonlinear Schrödinger* (NLS) *scale* and we will outline how, at least until this order, one succeed in removing from the obtained reduced equations all the spurious diverging *secular terms*. After that in *Section 2* we will recall the results beyond the NLS order [11–14, 35, 37] necessary to set up an integrability test for partial differential equations based on some *necessary integrability conditions*. There we also formulate the notion of *asymptotic integrability* [13, 14]. The necessary integrability conditions are explicitly reported for different hierarchies of reduced systems, namely the NLS hierarchy and the KdV/*potential* KdV hierarchies. After that we will discuss the problem of the solutions of the linearized equations for both the NLS and potential KdV hierarchies. In *Section 3* we illustrate the application of the method to the case of real dispersive partial difference equations [15, 16]. This extension turns out to be nothing but a slight adaptation of the continuous case it as provides, starting from a partial difference equation, a partial differential one. Hence all the results developed in [13, 14, 35] to test the integrability of differential equations can be used. The results on the so called A_3 integrability conditions (two orders beyond NLS equation, see definition (2.4)) for the NLS hierarchy as well as on the integrability conditions for the KdV/*potential* KdV hierarchies are up to our knowledge new. Also the material presented in *Section 3* is original.

- **Chapter 3. Multiscale reductions of nonlinear discrete systems I: $S-$integrable equations.**
  In *Section 1* of this chapter we run the integrability test for the *lattice potential Korteweg-de Vries* equation ($lpKdV$) which is well known to be an equation integrable by spectral methods. Here we outline how at all orders considered all the integrability conditions are, as expected, satisfied. In *Section 2* we illustrate how the structures entailing the integrability of the $lpKdV$ equation as its *Lax pair* and some of its *generalized symmetries* admits a similar multiscale reduction going respectively to the Lax pair and to the (point and generalized) symmetries of the NLS equation [17]. In *Section 3* we will present the

expansions of the off-centric discretization of the (continuous) KdV equation

$$u_{n,m+1} - u_{n,m-1} = \frac{\alpha}{4}\left(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}\right) - \beta\left(u_{n+1,m}^2 - u_{n,m}^2\right),$$

and in *Section 4* of the symmetric discretization of the KdV equation

$$u_{n,m+1} - u_{n,m-1} = \frac{\alpha}{4}\left(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}\right) - \frac{\beta}{2}\left(u_{n+1,m}^2 - u_{n-1,m}^2\right).$$

We emphasize how at a certain order depending on the equation those systems fail to fullfill the integrability conditions prescribed for that order, thereby showing their non-integrability. In *Section 5* we illustrate how is possible to reduce differential-difference equations applying the multiscale reduction to the integrable *Ablowitz-Ladik* $(A - L)$ discrete NLS equation

$$\mathrm{i}\partial_t f_n(t) + \frac{f_{n+1}(t) - 2f_n(t) + f_{n-1}(t)}{2\sigma^2} = \pm|f_n(t)|^2 \frac{f_{n+1}(t) + f_{n-1}(t)}{2},$$

which, due to its integrability property, always respect all the necessary integrability conditions. In *Section 6* we will apply the integrability test to the previous cited $dNLS$ equation, a nonintegrable discretization of the NLS equation, giving an analytical evidence of its nonintegrability. These two last cases will offer an example of the versatility of the technique, illustrating how one can consider also equations which are not real and/or dispersive [32]. All the results appearing in this chapter are original.

- **Chapter 4. Multiscale reductions of nonlinear discrete systems II: $C-$integrable equations.**
  In *Section 1* of this chapter we will run the integrability test for the *Hietarinta* equation [18]

$$\frac{u_{n,\,m} + e_2}{u_{n,\,m} + e_1} \cdot \frac{u_{n+1,\,m+1} + o_2}{u_{n+1,\,m+1} + o_1} = \frac{u_{n+1,\,m} + e_2}{u_{n+1,\,m} + o_1} \cdot \frac{u_{n,\,m+1} + o_2}{u_{n,\,m+1} + e_1},$$

a well known linearizable system [36], showing that now the reduced equations, for the stated linearizability property, are indeed linear. Applying a similar multiscale expansion on the linearizing transformation, we will show that the *Hietarinta* equation linearizes to its linear part. All the results appearing in this chapter are original.

- **Chapter 5. Conclusions and perspectives.**

2

The continuous and discrete multiscale techniques

2.1 Multiscale expansion of real dispersive partial differential equations

A great variety of mathematical models of concrete physical situations are described by nonlinear partial differential (PDE) or difference equations ($P\Delta E$). Among those equations an important class is represented by differential equations describing the propagation of nonlinear *dispersive waves* when one assumes an idealized situation in which no diffraction (e. g. one dimensional propagation), no losses due to dissipation and no interactions with any source are present. Under such a situation an exceptional part of the models derived share remarkable mathematical properties leading to a concrete possibility from the mathematical side to solve analytically, at least for certain classes of solutions, our equations. Consequently from the physical side one has the possibility to understand their properties such as the regularity of their motion, wave collisions, stability and long-time asymptotics. Systems sharing these mathematical properties can be classified into two major classes. As in [4], we call $C-$*integrable* equations those nonlinear $PDEs$ that can be exactly linearized by an *invertible* transformation of the dependent (sometimes also the independent) variables. On the other hand we call $S-$*integrable* equations those nonlinear $PDEs$ that arise as a compatibility condition of an overdetermined system of linear equations, the so called *Lax pair* of the system. By the *spectral transform* or *inverse scattering* method, the Cauchy problem of an $S-$integrable equation can be solved explicitly in a special class of solutions, named *soliton solutions*, for which one can transform the original differential problem into a purely algebraic one. Soliton solutions exist for $C-$integrable systems too. It turns out that spectral methods allow a linearization of our nonlinear equation into an *integral* (Fredholm, Volterra) one. On the contrary the great majority of the models do not exibit any underlying mathematical structure that can be used as a tool to extract from them their physical properties and this absence allow these systems to behave in an irregular, if not chaotic, way. To express this situation, they are termed *nonintegrable.* To deal with these situations, sometimes extremely hard to analyze, one has to resort to a perturbative approach in order to extract some significant information like, for example, approximate solutions. It is a remarkable fact that the perturbative *multiscale* approach casts a light even into integrable systems, characterizing them in terms of the reduced equations one obtains during the expansion. As a matter of fact this realizes the effective possibility for an algorithmic *integrability test* upon which one can distinguish an integrable system from a nonintegrable one. That possibility is a direct consequence of the following assumption:

- *Integrability is preserved by the perturbative reduction method.*

On the other side, wide occurrence of some reduced models allow us to elevate them to the status of systems of somehow *universal* nature or *model* $PDEs$. Among them we have the NLS equation which, under proper conditions, turn out always to be the first nonlinear reduced model, sometimes called the *weak nonlinearity limit.*

Let us now illustrate in detail the multiscale reduction technique for real dispersive nonlinear differential systems following closely references [4, 13, 35]. Let us consider the class of $PDEs$

$$Du(x,t) = F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots], \tag{2.1}$$

where all the variables (dependent and independent) and constants in the equation are real. The *linear* differential operator D in the lhs can have one of the forms

$$D_{even} \doteq \frac{\partial^2}{\partial t^2} + \sum_{n=0}^{\mathcal{N}} (-1)^n a_n \frac{\partial^{2n}}{\partial x^{2n}}, \tag{2.2a}$$

$$D_{odd} \doteq \frac{\partial}{\partial t} + \sum_{n=0}^{\mathcal{N}} (-1)^n b_n \frac{\partial^{2n+1}}{\partial x^{2n+1}}, \tag{2.2b}$$

with a_n and b_n (real) constants. The function $F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots]$ on the r.h.s. of (2.1) is on the contrary a *nonlinear* real *analytic* function of u and its x and $t-$derivatives. It is straightforward to verify that, given the forms (2.2a), (2.2b), the linear part of (2.1)

$$Du(x,t) = 0, \tag{2.3}$$

is dispersive, admitting as a solution the real travelling wave

$$Ae^{\mathrm{i}[\kappa x - \omega(\kappa)t]} + \mathcal{C.C.} = 2|A| \cos\left[\kappa x - \omega\left(\kappa\right) t + \alpha\right], \tag{2.4}$$

with $\alpha \doteq \left[1 - sgn\left(A\right)\right] \pi/2$. The dispersion relation is respectively given by the following equations

$$\omega^2\left(\kappa\right) = \sum_{n=0}^{\mathcal{N}} a_n \kappa^{2n}, \tag{2.5a}$$

$$\omega\left(\kappa\right) = \sum_{n=0}^{\mathcal{N}} b_n \kappa^{2n+1}. \tag{2.5b}$$

We will require that $\omega\left(\kappa\right)$ is not linear in κ so that the *group velocity* $v\left(\kappa\right) \doteq \frac{d\omega(\kappa)}{d\kappa}$, given respectively by

$$v\left(\kappa\right) = \sum_{n=0}^{\mathcal{N}} n a_n \frac{\kappa^{2n-1}}{\omega\left(\kappa\right)}, \tag{2.6a}$$

$$v\left(\kappa\right) = \sum_{n=0}^{\mathcal{N}} \left(2n+1\right) b_n \kappa^{2n}, \tag{2.6b}$$

is not constant. Moreover in the case (2.5a) we assume that there exists at least one value of κ for which the r.h.s. is positive as $\omega\left(\kappa\right)$ is to be real. Taking into consideration eqs. (2.5), we can rewrite eqs. (2.2) as

$$D_{even} = \frac{\partial^2}{\partial t^2} + \omega^2\left(\mathrm{i}\frac{\partial}{\partial x}\right) = \left[\frac{\partial}{\partial t} + \mathrm{i}\,\omega\left(\mathrm{i}\frac{\partial}{\partial x}\right)\right] \cdot \left[\frac{\partial}{\partial t} - \mathrm{i}\,\omega\left(\mathrm{i}\frac{\partial}{\partial x}\right)\right], \tag{2.7a}$$

$$D_{odd} = \frac{\partial}{\partial t} + \mathrm{i}\,\omega\left(-\mathrm{i}\frac{\partial}{\partial x}\right), \tag{2.7b}$$

where the function $\omega(\kappa)$ may not be polynomial in κ. In fact it is sufficient that $\omega(\kappa)$ is *analytic* (at least *locally* for some κ) so that its definition through the dispersion relation (2.5) can be given by a series expansion (e.g. $\mathcal{N} \to +\infty$).

If $F = 0$, so that our equation is linear, one can write the solution in the form of a wave packet

$$u(x,t) = \int_{-\infty}^{+\infty} d\kappa \mathcal{A}(\kappa)\, e^{\mathrm{i}[\kappa x - \omega(\kappa)t]} + \mathcal{C.C.}, \tag{2.8}$$

which we assume to be *localized* around the wave number κ_0 in the interval $(\kappa_0 - \Delta\kappa_0, \kappa_0 + \Delta\kappa_0)$. We than define a new variable η by the relation $\kappa \doteq \kappa_0 + \eta\Delta\kappa_0$. Then eq. (2.8) can be rewritten as

$$u(x,t;\varepsilon) = u^{(1)}(\xi, t_1, t_2, \ldots; \varepsilon)\, E(x,t) + \mathcal{C.C.}, \tag{2.9}$$

where

$$u^{(1)}(\xi, t_1, t_2, \ldots; \varepsilon) \doteq \varepsilon \int_{-\infty}^{+\infty} d\eta \widetilde{\mathcal{A}}(\eta;\varepsilon)\, e^{\mathrm{i}\left[\kappa_0\eta\xi - \sum_{n=1}^{+\infty}(\kappa_0\eta)^n t_n \omega_n(\kappa_0)\right]}, \tag{2.10}$$

$$\widetilde{\mathcal{A}}(\eta;\varepsilon) \doteq \kappa_0 \mathcal{A}(\kappa_0 + \eta\Delta\kappa_0), \quad E(x,t) \doteq e^{\mathrm{i}[\kappa_0 x - \omega(\kappa_0)t]}, \quad \varepsilon \doteq \frac{\Delta\kappa_0}{\kappa_0}, \tag{2.11}$$

$$\xi \doteq \varepsilon x, \quad t_n \doteq \varepsilon^n t, \quad \omega_n(\kappa) \doteq \frac{1}{n!}\frac{d^n}{d\kappa^n}\omega(\kappa). \tag{2.12}$$

In eq. (2.9) the solution is represented by a monochromatic carrier wave $E(x,t)$ modulated by an amplitude $u^{(1)}(\xi, t_1, t_2, \ldots; \varepsilon)$ which is a function of the *slow-space* variable ξ, of the *slow-times* t_n, $n = 1, 2, \ldots$ and of the quantity ε which will be identified as our *perturbative* parameter. If $\varepsilon \ll 1$, a wave of the type (2.9) wilI be called *quasi-monochromatic*. It is clear that the amplitude $u^{(1)}$ should depend on as many slow-times t_n as the number of nonvanishing coefficients $\omega_n(\kappa)$ in the expansion of $\omega(\kappa)$ in a power series of κ. In the case that $\omega(\kappa)$ is given by a polynomial of degree $\mathcal{N}$ in κ, one will need a maximum of $\mathcal{N}$ slow-variables t_n.

If our equation (2.1) is nonlinear, the nonlinear function F will generate higher order harmonics. So a form like (2.9) is no more valid and we must consider the more general ansatz

$$u(x,t;\varepsilon) \doteq \sum_{\alpha=-\infty}^{+\infty} u^{(\alpha)}(\xi, t_1, t_2, \ldots; \varepsilon)\, E^{\alpha}(x,t), \tag{2.13}$$

where the sum extends over all harmonics and we require $u^{(-\alpha)} = \bar{u}^{(\alpha)}$ in order for the solution to be real. We will also assume that the amplitude $u^{(\alpha)}$s are bounded as $t_n \to \pm\infty$, rapidly decreasing as $\xi \to \pm\infty$ and analytic in ε. Than one can expand the $u^{(\alpha)}$s in power of ε giving the final expression

$$u(x,t;\varepsilon) \doteq \sum_{\alpha=-\infty}^{+\infty}\sum_{n=1}^{+\infty} \varepsilon^n u_n^{(\alpha)}(\xi, t_1, t_2, \ldots)\, E^{\alpha}(x,t), \tag{2.14}$$

with $u_n^{(-\alpha)} = \bar{u}_n^{(\alpha)}$. The starting point $n = 1$ for the ε-expansion is choosen so that the nonlinear contribution to the solution enters as a small perturbation to the linear one. It is important to remark that one can generalize the ansatz (2.14) by defining the slow variables as

$$\xi \doteq \varepsilon^p x, \quad t_n \doteq \varepsilon^{np} t, \quad p > 0, \quad n \geq 1. \tag{2.15}$$

Then the differential operators ∂_x and ∂_t, when acting on a function $u(x,t)$ represented by the expansion (2.14), act as the operators

$$D_x \doteq \partial_x + \varepsilon^p \partial_\xi, \quad D_t \doteq \partial_t + \varepsilon^p \partial_{t_1} + \varepsilon^{2p} \partial_{t_2} + \ldots, \tag{2.16}$$

when one assumes that all the variable involved x, t, ξ, t_1, t_2, $\ldots$ are *independent*. Let us now choose for the linear operator D the odd operator (2.7b) and let us insert (2.14) into (2.1) taking into consideration eq. (2.16). We obtain

$$\sum_{\alpha=-\infty}^{+\infty} E^\alpha(x,t) \sum_{n=1}^{+\infty} \varepsilon^n \left[\sum_{m=1}^{n} D_{n-m}^{(\alpha)} u_m^{(\alpha)} (\xi, t_1, t_2, \ldots) - F_n^{(\alpha)} \right] = 0, \tag{2.17}$$

where the nonlinear functions $F_n^{(\alpha)}$, arising from the expansion in harmonics and in the perturbative parameter of the analytic function $F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots]$, are polynomials of $u_\rho^{(\beta)}$ with $\rho \leq n-1$ and their derivatives with respect to the slow variables. The $F_n^{(\alpha)}$s satisfy the reality conditions $F_n^{(-\alpha)} = \bar{F}_n^{(\alpha)}$. As the quantity $n-m$ must be a multiple of p, the operators $D_{n-m}^{(\alpha)} \doteq D_{\sigma p}^{(\alpha)}$ are defined as

$$D_0^{(\alpha)} \doteq \mathrm{i} \left[\omega \left(\alpha \kappa_0 \right) - \alpha \omega \left(\kappa_0 \right) \right], \tag{2.18a}$$

$$D_{\sigma p}^{(\alpha)} \doteq \partial_{t_\sigma} - (-\mathrm{i})^{\sigma+1} \omega_\sigma \left(\alpha \kappa_0 \right) \partial_\xi^\sigma, \quad \sigma \geq 1. \tag{2.18b}$$

Setting in eq. (2.17) separately to zero every coefficient of $E^\alpha \varepsilon^n$, we have

$$\left[\omega \left(\alpha \kappa_0 \right) - \alpha \omega \left(\kappa_0 \right) \right] u_1^{(\alpha)} = 0, \tag{2.19a}$$

$$\sum_{m=1}^{n} D_{n-m}^{(\alpha)} u_m^{(\alpha)} (\xi, t_1, t_2, \ldots) = F_n^{(\alpha)}, \quad n \geq 2, \quad \alpha \in (-\infty, +\infty), \tag{2.19b}$$

where we took into consideration that from definition (2.14) $F_1^{(\alpha)} = 0$ $\forall \alpha$. Following [13, 35], we can classify the harmonics according to the form of $D_0^{(\alpha)}$. If $D_0^{(\alpha)} = 0$, we say that the α-harmonic is at *resonance*. As we supposed that $\omega(\kappa)$ is odd and nonlinear in κ, the only harmonics that are at resonance $\forall \kappa_0$, denoted as *structural resonances*, are those with $\alpha = 0$, ± 1. On the other hand for some value κ_0 it could happen that $D_0^{(\alpha)} = 0$. In this case we call the α-harmonic an *accidental resonance*. Finally, if $D_0^{(\alpha)} \neq 0$ $\forall \kappa_0$, then the corresponding harmonic is called a *slave harmonic*, the nature of this distiction residing in the following. For a fixed value of α as n varies, if $D_0^{(\alpha)} \neq 0$, equations (2.19) represent a *triangular* system of *algebraic* equations which allow us to express the function $u_n^{(\alpha)}$ in terms of the functions $F_m^{(\alpha)}$ with $m \leq n$. As the $F_m^{(\alpha)}$ are functions of the $u_\rho^{(\beta)}$ with $\rho \leq m-1$, the slave harmonic $u_n^{(\alpha)}$ can be expressed in terms of the $u_\rho^{(\beta)}$ with $\rho \leq n-1$. Iterating the procedure for the various $u_\rho^{(\beta)}$, one sees that a slave harmonic $u_n^{(\alpha)}$ can be finally expressed in terms only of the $u_\rho^{(\beta)}$

with $\rho \leq n-1$ and β the index of a resonant harmonic. Furthermore from (2.19a) one can see that for a slave harmonic we have $u_1^{(\alpha)} = 0$ and iteratively that $u_n^{(\alpha)} = 0$ if $|\alpha| > n$, so that eq. (2.14) and (2.19b) can be rewritten respectively as

$$u(x,t;\varepsilon) = \sum_{n=1}^{+\infty} \sum_{\alpha=-n}^{n} \varepsilon^n u_n^{(\alpha)} \left(\xi, t_1, t_2, \ldots\right) E^{\alpha}(x,t), \tag{2.20}$$

$$\sum_{m=max\{1,\,|\alpha|\}}^{n} D_{n-m}^{(\alpha)} u_m^{(\alpha)} \left(\xi, t_1, t_2, \ldots\right) = F_n^{(\alpha)}, \quad n \geq 2, \quad |\alpha| \leq n, \tag{2.21}$$

where we assumed that the only resonant harmonics are the structural resonances with $\alpha = 0$, ± 1. For these harmonics the systems (2.19) are no more algebraic but become a unique *triangular* system of coupled *differential* equations in the $u_\rho^{(\alpha)}$, $\alpha = 0, \pm 1$ and $\rho \leq n$.

Before passing to some illustrative examples, we note that the expansion on the right hand side of eq. (2.20) is intended to be a *uniformly asymptotic series* for the function $u(x,t)$. This implies that the functions $u_n^{(\alpha)} \left(\xi, t_1, t_2, \ldots\right)$ should always remain bounded in the space of the slow variables. Then, if a function $v\left(t\right)$ satisfies the equation

$$\frac{dv(t)}{dt} + Av(t) = w(t) + s(t), \tag{2.22}$$

where A is a linear operator and the forcing term $s(t)$ solves the homogeneous equation $\frac{ds(t)}{dt} + As(t) = 0$, then the general solution of eq. (2.22) is given by $v(t) = \widetilde{v}(t) + ts(t)$ where $\widetilde{v}(t)$ is the general solution of the equation $\frac{d\widetilde{v}(t)}{dt} + A\widetilde{v}(t) = w(t)$. One can easily see that the so called *resonant* or *secular* term $ts(t)$ produces as $t \to +\infty$ an unbounded motion so that one has to set it separetely to zero, choosing

$$\frac{dv(t)}{dt} + Av(t) = w(t), \quad s(t) = 0. \tag{2.23}$$

So, to preserve the uniform asymptoticity of the series (2.20), one has to properly change the equations so that secular terms do not appear in the solutions.

2.1.1 Examples

1. Let us consider the *Korteweg-de Vries* (KdV) equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = u \frac{\partial u}{\partial x}, \tag{2.24}$$

which, as is well known, is an *S-integrable* equation as it arises as the compatibility of the following overdetermined system of two linear equations for a complex function $\phi\left(x,t\right)$

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{6}u - \lambda\right) \phi(x,t) \doteq \left(L - \lambda\right) \phi(x,t) = 0, \tag{2.25a}$$

$$\left(\frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} - u\frac{\partial}{\partial x} - \frac{1}{2}u_x\right) \phi(x,t) \doteq \left(\frac{\partial}{\partial t} - M\right) \phi(x,t) = 0, \tag{2.25b}$$

where $\lambda \in \mathbb{C}$ is a *spectral parameter* and L and M are two differential operators. Than our equation is equivalent to the operator identity

$$L_t = [M, L] ,$$

between the linear operators L and M when one supposes that the spectral parameter λ is time independent. We note that eq. (2.25b) can be rewritten as

$$\left(\frac{\partial}{\partial t} - \frac{1}{3}u\frac{\partial}{\partial x} + 4\lambda\frac{\partial}{\partial x} + \frac{1}{6}u_x\right)\phi(x,t) = 0.$$

According to eqs. (2.2b) and (2.5b), the dispersion relation is $\omega(\kappa) = -\kappa^3$, so that, when we set $p = 1$ in (2.15), we can introduce only three slow-times t_j, $j = 1, 2, 3$. Consequently from eqs. (2.18) we have

$$D_0^{(\alpha)} = -\mathrm{i}\kappa_0^3\alpha\left(\alpha^2 - 1\right), \tag{2.26a}$$
$$D_1^{(\alpha)} = \partial_{t_1} - 3\left(\alpha\kappa_0\right)^2\partial_\xi, \tag{2.26b}$$
$$D_2^{(\alpha)} = \partial_{t_2} + 3\mathrm{i}\alpha\kappa_0\partial_\xi^2, \tag{2.26c}$$
$$D_3^{(\alpha)} = \partial_{t_3} + \partial_\xi^3. \tag{2.26d}$$

Eq. (2.26a) shows that, if we choose $\kappa_0 \neq 0$, the only resonances are the structural ones with $\alpha = 0, \pm 1$. Let us proceed as in ref. [40] to the multiscale expansion according to eq. (2.21), referring to *Appendix A*, eq. (A.3a) for the functions $F_n^{(\alpha)}$.

i. Order $n = 2$.

- $\alpha = 0$:

$$\partial_{t_1}u_1^{(0)} = 0, \tag{2.27}$$

from which one derives that $u_1^{(0)}$ is independent of t_1;

- $\alpha = 1$:

$$\left[\partial_{t_1} + \mathrm{i}\kappa_0\left(3\mathrm{i}\kappa_0\partial_\xi - u_1^{(0)}\right)\right]u_1^{(1)} = 0. \tag{2.28}$$

Multiplying eq. (2.28) by $\bar{u}_1^{(1)}$ and summing the resulting equation with its complex conjugate, one has

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right)|u_1^{(1)}|^2 = 0, \tag{2.29}$$

which means that $|u_1^{(1)}|^2$ and hence $|u_1^{(1)}|$ depend on the combination $\rho \doteq \xi + 3\kappa_0^2 t_1$. So, defining $u_1^{(1)} \doteq |u_1^{(1)}|e^{\mathrm{i}\theta}$ with $\theta = \theta(\xi, t_1, t_2, t_3)$ and inserting this expression into eq. (2.28), we have

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right)\theta = \kappa_0 u_1^{(0)}. \tag{2.30}$$

Applying the operator ∂_{t_1} to the last equation, it follows that

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right)\partial_{t_1}\theta = 0, \tag{2.31}$$

from which it results that $\theta = A(\rho, t_2, t_3) + B(\xi, t_2, t_3)$, where A and B are arbitrary functions of their arguments. From eq. (2.30) we have

$$\partial_\xi B = -\frac{1}{3\kappa_0}u_1^{(0)}, \tag{2.32}$$

whose solution is

$$B = -\frac{1}{3\kappa_0}\int_{\xi_0}^{\xi} u_1^{(0)}(\xi', t_2, t_3)\, d\xi' + C(t_2, t_3), \tag{2.33}$$

where C is an arbitrary function of its arguments. In conclusion one has

$$u_1^{(1)} = g_1^{(1)}(\rho, t_2, t_3)\, e^{-\frac{\mathrm{i}}{3\kappa_0}\int_{\xi_0}^{\xi} u_1^{(0)}(\xi', t_2, t_3)d\xi'}, \tag{2.34}$$

where all the contribution depending on ρ has been included in the (complex) function $g_1^{(1)}$. ξ_0, by a redefinition of $g_1^{(1)}$, can always be chosen to be a *zero* of $u_1^{(0)}$ (there exist at least one zero because, as $\xi \to \pm\infty$, $u_1^{(0)} \to 0$);

- $\alpha = 2$:

$$u_2^{(2)} = -\frac{1}{6\kappa_0^2}u_1^{(1)2}; \tag{2.35}$$

ii. Order $n = 3$.

- $\alpha = 0$:

$$\partial_{t_1} u_2^{(0)} = \partial_\rho\left(|u_1^{(1)}|^2\right) + \frac{1}{2}\partial_\xi\left(u_1^{(0)2}\right) - \partial_{t_2}u_1^{(0)}. \tag{2.36}$$

The second and the third term in the right hand side of the last formula are secular by eq. (2.27). So one has

$$\partial_{t_1} u_2^{(0)} = \partial_\rho\left(|u_1^{(1)}|^2\right), \tag{2.37}$$

$$\left(\partial_{t_2} - u_1^{(0)}\partial_\xi\right)u_1^{(0)} = 0. \tag{2.38}$$

Applying the operator $\partial_{t_1} - 3\kappa_0^2\partial_\xi$ to the continuity equation (2.37), one derives

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right)\partial_{t_1} u_2^{(0)} = 0, \tag{2.39}$$

from which it follows that $u_2^{(0)} = E\left(\rho, t_2, t_3\right) + F\left(\xi, t_2, t_3\right)$, where E and F are arbitrary functions of their arguments. From eq. (2.37) one has

$$\partial_\rho\left(3\kappa_0^2 E - |u_1^{(1)}|^2\right) = 0, \tag{2.40}$$

which implies

$$E = \frac{|u_1^{(1)}|^2}{3\kappa_0^2} + G\left(t_2, t_3\right), \tag{2.41}$$

where G is an arbitrary function of its arguments. Finally one has

$$u_2^{(0)} = \frac{|u_1^{(1)}|^2}{3\kappa_0^2}, \tag{2.42}$$

where the arbitrary integration function depending on ξ, t_2 and t_3 has been set equal to zero. So $u_2^{(0)}$ depends on ρ too. Eq. (2.38) is the well known *Hopf* equation, the prototypical model describing the *gradient catastrophe* or *wave breaking* phenomenon (in the $\xi - t_2$ plane). To avoid this phenomenon, one should take $u_1^{(0)}$ independent from t_2 and ξ but, as $u_1^{(0)}$ must be rapidly decreasing as $\xi \to \pm\infty$, we should choose $u_1^{(0)} = 0$[1]. In the following however we will choose $u_1^{(0)} \neq 0$;

- $\alpha = 1$:
  Using eqs. (2.35) and (2.42), we get:

$$\left[\partial_{t_1} + \mathrm{i}\kappa_0\left(3\mathrm{i}\kappa_0\partial_\xi - u_1^{(0)}\right)\right] u_2^{(1)} = -\left[\partial_{t_2} + \partial_\xi\left(3\mathrm{i}\kappa_0\partial_\xi - u_1^{(0)}\right) - \frac{\mathrm{i}}{6\kappa_0}|u_1^{(1)}|^2\right] u_1^{(1)}. \tag{2.43}$$

Taking into account the definition (2.34) and a similar one for its higher harmonic $u_2^{(1)}$

$$u_2^{(1)} \doteq g_2^{(1)}\left(\xi, t_1, t_2, t_3\right) e^{-\frac{\mathrm{i}}{3\kappa_0}\int_{\xi_0}^{\xi} u_1^{(0)}(\xi', t_2, t_3)d\xi'}, \tag{2.44}$$

eq. (2.43) becomes

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right) g_2^{(1)} = u_1^{(0)}\left(\frac{\mathrm{i}}{6\kappa_0}u_1^{(0)} - \partial_\rho\right) g_1^{(1)} - \left(\partial_{t_2} + 3\mathrm{i}\kappa_0\partial_\rho^2 - \frac{\mathrm{i}}{6\kappa_0}|g_1^{(1)}|^2\right) g_1^{(1)}. \tag{2.45}$$

[1]This is also consistent with the *linear* evolution equation for $u_1^{(0)}$ one derives as a no-secularity condition at order $n = 4$, $\alpha = 0$, see eq. (2.51).

The last term on the right hand side of the last equation is secular as $g_1^{(1)}$ depends on ρ. So eq. (2.45) splits into

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right) g_2^{(1)} = u_1^{(0)} \left(\frac{\mathrm{i}}{6\kappa_0} u_1^{(0)} - \partial_\rho\right) g_1^{(1)}, \tag{2.46}$$

$$\left(\mathrm{i}\partial_{t_2} - 3\kappa_0\partial_\rho^2 + \frac{1}{6\kappa_0}|g_1^{(1)}|^2\right) g_1^{(1)} = 0. \tag{2.47}$$

Eq. (2.47) is an *integrable* (*defocusing*) NLS equation for the function $g_1^{(1)}$ as the coefficient of its nonlinear term is real. From eq. (2.47) we get the continuity equation

$$\partial_{t_2}|g_1^{(1)}|^2 = -3\mathrm{i}\kappa_0\partial_\rho\left(\bar{g}_1^{(1)}\partial_\rho g_1^{(1)} - \mathcal{C.C.}\right), \tag{2.48}$$

so that $|g_1^{(1)}|^2$ is a density of a conserved quantity. From eq. (2.46) we get

$$\left(\partial_{t_1} - 3\kappa_0^2\partial_\xi\right)\left(g_1^{(1)}\bar{g}_2^{(1)} + \mathcal{C.C.}\right) = -u_1^{(0)}\partial_\rho\left(|g_1^{(1)}|^2\right). \tag{2.49}$$

iii. Order $n = 4$.

- $\alpha = 0$:

$$\partial_{t_1} u_3^{(0)} = \frac{\mathrm{i}}{\kappa_0}\partial_\rho\left(\bar{g}_1^{(1)}\partial_\rho g_1^{(1)} - \mathcal{C.C.}\right) + \partial_\xi\left(g_1^{(1)}\bar{g}_2^{(1)} + \mathcal{C.C.} + \frac{u_1^{(0)}}{3\kappa_0^2}|g_1^{(1)}|^2\right) - \tag{2.50}$$
$$- \left(\partial_{t_3} + \partial_\xi^3\right) u_1^{(0)}.$$

The last term on the right hand side of the last equation is secular as $u_1^{(0)}$ is independent on t_1. Hence eq. (2.50), after one removes the secularity and integrates with respect to t_1 using eq. (2.49), splits into (as usual the arbitrary t_1-independent integration function has been set to zero)

$$\left(\partial_{t_3} + \partial_\xi^3\right) u_1^{(0)} = 0, \tag{2.51}$$

$$u_3^{(0)} = \frac{\mathrm{i}}{3\kappa_0^3}\left(\bar{g}_1^{(1)}\partial_\rho g_1^{(1)} - \mathcal{C.C.}\right) + \frac{1}{3\kappa_0^2}\left(g_1^{(1)}\bar{g}_2^{(1)} + \mathcal{C.C}\right) + \tag{2.52}$$
$$+ \frac{1}{(3\kappa_0^2)^2}\left[2u_1^{(0)}|g_1^{(1)}|^2 + \left(\partial_\xi u_1^{(0)}\right)\int_{\rho_0}^{\rho}|g_1^{(1)}|^2 d\rho'\right].$$

With eq. (2.51) one completes the analysis of the harmonic $u_1^{(0)}$.

The calculations for this example are the first evidence of the general fact that the multiscale reduction of an $S-$integrable equation provides another $S-$integrable model. Moreover the fact that, apart from some gauge terms dependent on $u_1^{(0)}$, the nonlinear equation describing the evolution of $u_1^{(1)}$ at the slow-time t_2 is an NLS equation, is a general fact too when [8,9,13]:

- *The function $F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots]$ in eq. (2.1) contains generic quadratic nonlinearities*;
- *The slow variables scale with ε as in (2.15) with $p = 1$*;
- *The following non-resonance conditions are satisfied*:

$$\omega(2\kappa) - 2\omega(\kappa) \neq 0, \tag{2.53a}$$
$$\omega_1(0) - \omega_1(\kappa) \neq 0, \tag{2.53b}$$
$$\omega_2(\kappa) \neq 0. \tag{2.53c}$$

The presence of a nonzero harmonic $u_1^{(0)}$ would introduce shock wave solutions and nonlocal relations between the various harmonics as in eq. (2.52).

2. Let us consider the $C-$integrable *Burgers-like* equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(3u\frac{\partial u}{\partial x} + u^3\right), \tag{2.54}$$

which, through the *Cole-Hopf* transformation

$$u(x,t) \doteq \frac{\psi_x(x,t)}{\psi(x,t)}, \tag{2.55}$$

linearizes to the equation

$$\frac{\partial \psi}{\partial t} + a\frac{\partial \psi}{\partial x} - \frac{\partial^3 \psi}{\partial x^3} = f(t)\,\psi, \tag{2.56}$$

where $f(t)$ is an arbitrary integration function depending just on t. We note that the Cole-Hopf transformation (2.55) is invertible and its inverse transformation is given by

$$\psi(x,t) = \psi(x_0,t)\, e^{\int_{x_0}^{x} u(x',t)dx'}. \tag{2.57}$$

According to eqs. (2.2b) and (2.5b), the dispersion relation is $\omega(\kappa) = a\kappa + \kappa^3$, so that we need to introduce only three slow-times t_j, $j = 1, 2, 3$. Setting $p = 1$ in (2.15), from eqs. (2.18) we have

$$D_0^{(\alpha)} = \mathrm{i}\kappa_0^3\alpha\left(\alpha^2 - 1\right), \tag{2.58a}$$
$$D_1^{(\alpha)} = \partial_{t_1} + \left[a + 3(\alpha\kappa_0)^2\right]\partial_\xi, \tag{2.58b}$$
$$D_2^{(\alpha)} = \partial_{t_2} - 3\mathrm{i}\alpha\kappa_0\partial_\xi^2, \tag{2.58c}$$
$$D_3^{(\alpha)} = \partial_{t_3} - \partial_\xi^3. \tag{2.58d}$$

Eq. (2.58a) confirms that, if we choose $\kappa_0 \neq 0$, the only resonances are the structural ones with $\alpha = 0, \pm 1$. Let us proceed to the multiscale expansion according to eq. (2.21), referring to *Appendix A* for the corresponding functions $F_n^{(\alpha)}$ written in eqs. (A.3b, A.3c, A.5);

i. Order $n = 2$.

- $\alpha = 0$:

$$\left(\partial_{t_1} + a\partial_\xi\right) u_1^{(0)} = 0, \tag{2.59}$$

which means that $u_1^{(0)}$ depends on the combination $\rho \doteq \xi - at_1$;

- $\alpha = 1$:

$$\left[\partial_{t_1} + a\partial_\xi + 3\kappa_0^2 \left(u_1^{(0)} + \partial_\xi\right)\right] u_1^{(1)} = 0, \tag{2.60}$$

which implies

$$u_1^{(1)} = g_1^{(1)}\left(\sigma, t_2, t_3\right) e^{-\int_{\rho_0}^{\rho} u_1^{(0)}\left(\rho', t_2, t_3\right) d\rho'}, \tag{2.61}$$

where $\sigma \doteq \xi - \left(a + 3\kappa_0^2\right) t_1$;

- $\alpha = 2$:

$$u_2^{(2)} = \frac{\mathrm{i}}{\kappa_0} u_1^{(1)2}; \tag{2.62}$$

ii. Order $n = 3$.

- $\alpha = 0$:

$$\left(\partial_{t_1} + a\partial_\xi\right) u_2^{(0)} = -\partial_{t_2} u_1^{(0)}. \tag{2.63}$$

The right hand side of eq. (2.63) is secular by eq. (2.59) so that one has to rewrite it as the system:

$$\left(\partial_{t_1} + a\partial_\xi\right) u_2^{(0)} = 0, \tag{2.64}$$

$$\partial_{t_2} u_1^{(0)} = 0, \tag{2.65}$$

which means that $u_2^{(0)}$ depends on $\rho \doteq \xi - at_1$ too and that $u_1^{(0)}$ is independent of t_2;

- $\alpha = 1$:
  Using eq. (2.62), we get:

$$\left[\partial_{t_1} + a\partial_\xi + 3\kappa_0^2 \left(u_1^{(0)} + \partial_\xi\right)\right] u_2^{(1)} = -\left[\partial_{t_2} - 3\mathrm{i}\kappa_0 \left(u_1^{(0)} + \partial_\rho\right)^2\right] u_1^{(1)} + \tag{2.66}$$
$$+3\mathrm{i}\kappa_0 u_1^{(1)} \partial_\rho u_1^{(0)} - 3\kappa_0^2 u_2^{(0)} u_1^{(1)}.$$

Eq. (2.66), considering the definition (2.61) and the corresponding one for the harmonic $u_2^{(1)}$

$$u_2^{(1)} \doteq g_2^{(1)} (\xi, t_1, t_2, t_3)\, e^{-\int_{\rho_0}^{\rho} u_1^{(0)}(\rho', t_3) d\rho'}, \tag{2.67}$$

can be rewritten as

$$\left[\partial_{t_1} + \left(a + 3\kappa_0^2\right)\partial_\xi\right] g_2^{(1)} = 3\kappa_0 \left[\mathrm{i}\left(\partial_\rho u_1^{(0)}\right) - \kappa_0 u_2^{(0)}\right] g_1^{(1)} - \left(\partial_{t_2} - 3\mathrm{i}\kappa_0 \partial_\sigma^2\right) g_1^{(1)}. \tag{2.68}$$

The last term on the right hand side of eq. (2.68) is secular as $g_1^{(1)}$ depends on σ so that we finally have

$$\left[\partial_{t_1} + \left(a + 3\kappa_0^2\right)\partial_\xi\right] g_2^{(1)} = 3\kappa_0 \left[\mathrm{i}\left(\partial_\rho u_1^{(0)}\right) - \kappa_0 u_2^{(0)}\right] g_1^{(1)}, \tag{2.69}$$

$$\left(\partial_{t_2} - 3\mathrm{i}\kappa_0 \partial_\sigma^2\right) g_1^{(1)}. \tag{2.70}$$

Equation (2.70) is a *linear* Schrödinger equation for the function $g_1^{(1)}$.

We can see that the linearizability of the starting model reflects itself in the fact that the evolution of $u_1^{(1)}$ at the slow-time t_2 is governed by a linear equation. Other different rescalings with ε of the slow variables will provide *linearizable* equations. If for example [6, 7, 10] one takes the equation

$$u_t - u_{xxx} = 3\left(u_{xx}u + 3u_x^2 + u_x u^3\right) u,$$

which can be exactly linearized, and perform a multiscale reduction choosing $p = 2$ in eq. (2.15), then one obtains the *Eckhauss equation*

$$\left\{\mathrm{i}\partial_{t_2} + 3\kappa_0\partial_\xi^2 + 12\kappa_0\left[|u_1^{(1)}|^4 + \partial_\xi\left(|u_1^{(1)}|^2\right)\right]\right\} u_1^{(1)} = 0,$$

which is another linearizable equation.

2.2 The orders beyond the *NLS* equation and the integrability conditions

In this section we want to emphasize the fundamental role covered by the orders beyond that at which one derives for the harmonic $u_1^{(1)}$ an (integrable) *NLS* equation in setting up an integrability test for nonlinear differential equations. The importance of the following considerations is in the fact that everything we will say remains unchanged even if we consider nonlinear difference systems. We will start supposing that the conditions (2.53), assuring that the amplitude $u_1^{(1)}$ evolves at the slow-time t_2 according to an *NLS* equation, will always be satisfied. The first attempt to go beyond the *NLS* order has been presented by Santini, Degasperis and Manakov in [12] and the authors, starting from $S-$integrable models, through a combination of an asymptotic functional analysis and spectral methods, succeeded in removing all the secular terms from the reduced equations they found order by order. Their findings could be summarized as follows:

- The number of slow-time variables required for the amplitudes $u_n^{(\alpha)}$s indeed coincides with the number of nonvanishing coefficients $\omega_n(\kappa)$ defined in eq. (2.12);
- The amplitude $u_1^{(1)}$ evolves at the slow-times t_n, $n \geq 3$ according to the $n-$th equation of the NLS hierarchy;
- The amplitudes $u_m^{(1)}$, $m \geq 2$ evolve at the slow-times t_n, $n \geq 2$ according to certain *linear, nonhomogeneous* equations supplemented by some *asymptotic conditions* on the functions $u_p^{(1)}$, $p \geq 2$ themselves.

Thus one can concludes that the cancellation at each stage of the perturbation process of all the secular terms from the reduced equations is a sufficient request to uniquely fix the evolution equations followed by every $u_n^{(1)}$, $n \geq 1$ at each slow-time. The result in the second point should be expected as a hierarchy of integrable equations always represent compatible evolutions for a unique function u at different times, or the equations in this hierarchy are *generalized symmetries* of each other. For more details see [11, 37].

Although this procedure provides the most general *necessary and sufficient* conditions to get secularity-free reduced equations, it is not necessary to maintain such a functional approach to develop an integrability test. A *recursive* technique proves to be more suitable. As illustrated in [13, 14, 35] the authors, through a detailed multiscale reduction of the spectral problem associated with an $S-$integrable equation or of the linearizing process associated with a $C-$integrable system, showed the following

Proposition 2.1 *If equation (2.1) is (C or S) integrable, then under a multiscale expansion the functions $u_m^{(1)}$, $m \geq 1$ satisfy the equations*

$$\partial_{t_n} u_1^{(1)} = K_n \left[u_1^{(1)} \right], \tag{2.71a}$$

$$M_n u_j^{(1)} = f_n(j), \quad M_n \doteq \partial_{t_n} - K_n' \left[u_1^{(1)} \right], \tag{2.71b}$$

$\forall\ j,\ n \geq 2$, where $K_n \left[u_1^{(1)} \right]$ is the n-th flow in the nonlinear Schrödinger hierarchy. All the other $u_m^{(\kappa)}$, $\kappa \geq 2$ are expressed in terms of differential monomials of $u_\rho^{(1)}$, $\rho \leq m$.

In the last equations $f_n(j)$ is a nonhomogeneous *nonlinear* forcing term and $K_n'[u]\,v$ is the Frechet derivative of the nonlinear term $K_n[u]$ along the direction v defined by

$$K_n'[u]v \doteq \frac{d}{ds} K_n[u + sv] \;|_{s=0},$$

i. e. the linearization near u of $K_n[u]$ along the direction v. If $K_n[u]$ depends explicitly on x, t, u, u_x, u_{xx}, $\ldots$, $\bar{u}$, $\bar{u}_x$, $\bar{u}_{xx}$, $\ldots$, the explicit expression of $K_n'[u]v$ is

$$K_n'[u]v = \frac{\partial K_n}{\partial u} v + \frac{\partial K_n}{\partial u_x} v_x + \frac{\partial K_n}{\partial u_{xx}} v_{xx} + \ldots + \\ + \frac{\partial K_n}{\partial \bar{u}} \bar{v} + \frac{\partial K_n}{\partial \bar{u}_x} \bar{v}_x + \frac{\partial K_n}{\partial \bar{u}_{xx}} \bar{v}_{xx} + \ldots$$

For future use we note that the operator $K_n'[u]$ is a *linear* operator when it acts on a linear combinations of functions with *real* coefficients. In other words integrability is a *sufficient* condition for the harmonics $u_n^{(1)}$, $n \geq 1$ to satisfy eqs. (2.71). So eqs. (2.71) are a *necessary* condition for integrability. We want to emphasize that eqs. (2.71a) represent a hierarchy of *compatible* evolutions for the same function $u_1^{(1)}$ at different slow-times. Those evolutions are characterized by the commutativity condition

$$[K_r, K_s]_L = 0, \qquad [K_r, K_s]_L \doteq M_r K_s - M_s K_r, \tag{2.72}$$

where $[K_r, K_s]_L$ is called the *Lie commutator*. On the contrary, as we will see, the compatibility of eqs. (2.71b) is not always guaranteed but is subject to a sort of commutativity conditions among their r. h. s. terms $f_n(j)$s. These last commutativity conditions will be the cornerstone of our integrability test.

Let us continue illustrating the results of Degasperis et al. Those authors, following the results of [21], where it was demonstrated that the relations (2.71) implies an infinite number of *asymptotic symmetries* for the *PDE* under investigation, stated the following

Conjecture 2.1 *If a PDE admits a multiscale expansion where the functions $u_m^{(1)}$, $m \geq 1$ satisfy the equations (2.71) $\forall$ j, $n \geq 2$, then the equation is (C or S) integrable.*

In other words the conjecture affirms that the relations (2.71) are a *sufficient* condition for integrability or that integrability is a *necessary* condition to have a multiscale expansion where eqs. (2.71) are satisfied. Following again [13,14,35], we give the definitions

Definition 2.1 *A differential monomial $\rho\left[u_j^{(1)}\right]$, $j \geq 1$ in the functions $u_j^{(1)}$, their complex conjugates and their ξ-derivatives is a monomial of "gauge" 1 if it possesses the transformation property*

$$\rho\left[\widetilde{u}_j^{(1)}\right] = e^{\mathrm{i}\theta}\rho\left[u_j^{(1)}\right], \qquad \widetilde{u}_j^{(1)} \doteq e^{\mathrm{i}\theta}u_j^{(1)};$$

Definition 2.2 *A finite dimensional vector space $\mathcal{P}_n$, $n \geq 2$ is the set of all differential polynomials in the functions $u_j^{(1)}$s, $j \geq 1$, their complex conjugates and their ξ-derivatives of order n in ε and gauge 1 where*

$$\text{order}\left(\partial_\xi^m u_j^{(1)}\right) = \text{order}\left(\partial_\xi^m \bar{u}_j^{(1)}\right) = m + j, \quad m \geq 0;$$

Definition 2.3 *$\mathcal{P}_n(m)$, $m \geq 1$ and $n \geq 2$ is the subspace of $\mathcal{P}_n$ whose elements are differential polynomials in the functions $u_j^{(1)}$s, their complex conjugates and their ξ-derivatives of order n in ε and gauge 1 for $1 \leq j \leq m$.*

- From definition (2.3) one has that $\mathcal{P}_n = \mathcal{P}_n(n-2)$ and moreover one can see that in general $K_n\left[u_1^{(1)}\right] \in \partial_\xi^n u_1^{(1)} \cup \mathcal{P}_{n+1}(1)$ and that $f_n(j) \in \mathcal{P}_{j+n}(j-1)$ where j, $n \geq 2$. The basis monomials of the spaces $\mathcal{P}_n(m)$ can be found in *Appendix B*.

We have the following

Proposition 2.2 *The operators M_m defined in eq. (2.71b) commute among themselves.*

Proof: As eqs. (2.71a) represent a hierarchy of compatible evolutions, the following operator relation is satisfied (see ref. [33], pag. 121),

$$\left(\partial_{t_m} K_n' \left[u_1^{(1)}\right]\right) - \left(\partial_{t_n} K_m' \left[u_1^{(1)}\right]\right) + \left[K_n' \left[u_1^{(1)}\right],\ K_m' \left[u_1^{(1)}\right]\right] = 0. \tag{2.73}$$

Then we have $M_m M_n = \left(\partial_{t_m} - K_m' \left[u_1^{(1)}\right]\right)\left(\partial_{t_n} - K_n' \left[u_1^{(1)}\right]\right) = \partial_{t_m}\partial_{t_n} - \partial_{t_m} K_n' \left[u_1^{(1)}\right] - K_m' \left[u_1^{(1)}\right]\partial_{t_n} + K_m' \left[u_1^{(1)}\right] K_n' \left[u_1^{(1)}\right] = \partial_{t_n}\partial_{t_m} - \left(\partial_{t_m} K_n' \left[u_1^{(1)}\right]\right) - K_n' \left[u_1^{(1)}\right]\partial_{t_m} - K_m' \left[u_1^{(1)}\right]\partial_{t_n} + K_m' \left[u_1^{(1)}\right] K_n' \left[u_1^{(1)}\right] = M_n M_m$, so that $[M_m,\ M_n] = 0$. *Q. E. D.*

Once we fix the index $j \geq 2$ in the set of eqs. (2.71b), this commutativity condition implies the following *compatibility* conditions

$$M_k f_n(j) = M_n f_k(j), \quad \forall\, k, n \geq 2, \tag{2.74}$$

where, as $f_n(j)$ and $f_k(j)$ are functions of the fundamental harmonic up to degree $j-1$, the time derivatives ∂_{t_k}, ∂_{t_n} of those harmonics appearing respectively in M_k and M_n have to be eliminated using the evolution equations (2.71) up to the index $j-1$;

Proposition 2.3 *If for each fixed $j \geq 2$ the equation (2.74) with $k = 2$ and $n = 3$, namely $M_2 f_3(j) = M_3 f_2(j)$, is satisfied, then there exist unique differential polynomials $f_n(j)$ $\forall\, n \geq 4$ such that the flows $M_n u_j^{(1)} = f_n(j)$ commute for any $n \geq 2$.*

Hence among the relations (2.74) only those with $k = 2$ and $n = 2$ have to be tested;

Proposition 2.4 *The homogeneous equation $M_n u = 0$ has no solution u in the vector space $\mathcal{P}_m$, i.e. $Ker(M_n) \cap \mathcal{P}_m = \emptyset$.*

Consequently the multiscale expansion (2.71) is *secularity-free*. Finally by the following definition we express the *degree of integrability* of a given equation:

Definition 2.4 *If the relations (2.74) are satisfied up to the index j, $j \geq 2$, we say that our equation is asymptotically integrable of degree j or A_j integrable.*

Although the theory was developed only in the case of real dispersive PDEs for real functions when the conditions (2.53) are satisfied, one could extend it to include the case of complex PDEs for complex functions and situations where flows in eqs. (2.71a) will not necessarily belong to the NLS hierarchy. This is precisely what we will do when we will consider the multiscale reduction of the *discrete* NLS equation. In this case one has only to separate the modulus and phase of our function and expand them in ε without any expansion in harmonics. As we will see, the leading order of the modulus squared and the phase will evolve respectively as a KdV equation or a *potential* KdV equation and consequently the flows in eqs. (2.71a) will now be respectively those of the KdV hierarchy or those of the potential KdV hierarchy. As for the nonlinear forcing terms in eqs. (2.71b), they will turn out again to belong to a specific finite dimensional polynomial vector space $\mathcal{P}_n$ whose precise definition is given in *Subsections 2.2.2, 2.2.3* and whose basis monomials are listed in *Appendix B.*

2.2.1 Integrability conditions I: NLS hierarchy

In this subsection we will derive the conditions for asymptotic integrability of order n or A_n integrability conditions. To simplify the notation, we will use for $u_j^{(1)}$ the concise form $u(j)$. At first, for future convenience, we list the fluxes $K_n\,[u]$ of the NLS hierarchy for u up to $n=4$:

$$K_1[u] \doteq Au_\xi, \tag{2.75a}$$

$$K_2[u] \doteq -\mathrm{i}\rho_1\left[u_{\xi\xi} + \frac{\rho_2}{\rho_1}|u|^2u\right], \tag{2.75b}$$

$$K_3[u] \doteq B\left[u_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1}|u|^2u_\xi\right], \tag{2.75c}$$

$$K_4[u] \doteq -\mathrm{i}C\left\{u_{\xi\xi\xi\xi} + \frac{\rho_2}{\rho_1}\left[\frac{3\rho_2}{2\rho_1}|u|^4u + 4|u|^2u_{\xi\xi} + 3u_\xi^2\bar{u} + 2|u_\xi|^2u + u^2\bar{u}_{\xi\xi}\right]\right\}, \tag{2.75d}$$

and the corresponding $K_n'[u]v$ up to $n=3$:

$$K_1'[u]v = Av_\xi, \tag{2.76a}$$

$$K_2'[u]v = -\mathrm{i}\rho_1\left\{v_{\xi\xi} + \frac{\rho_2}{\rho_1}\left[u^2\bar{v} + 2|u|^2v\right]\right\}, \tag{2.76b}$$

$$K_3'[u]v = B\left\{v_{\xi\xi\xi} + \frac{3\rho_2}{\rho_1}\left[|u|^2v_\xi + \bar{u}u_\xi v + uu_\xi\bar{v}\right]\right\}, \tag{2.76c}$$

where ρ_1, ρ_2, B and C are arbitrary complex constants.

The A_1 integrability condition is given by the reality of the coefficient ρ_2 of the nonlinear term in the NLS equation. It is obtained commuting the NLS flux $K_2[u]$ with the flux $B\left[u_{\xi\xi\xi} + \tau|u|^2u_\xi + \mu u^2\bar{u}_\xi\right]$ with τ and μ constants. Let's remark again that, if we start from an integrable model, the resulting NLS equation should be integrable as well and, as an integrable equation, it should be a part of an entire hierarchy of equations like (2.71a). This commutativity condition gives, If $\rho_2 \neq 0$,

$$Im\,[\rho_2] = Im\,[B] = Im\,[\rho_1] = 0, \qquad \tau = 3\rho_2/\rho_1, \qquad \mu = 0. \tag{2.77}$$

If $\rho_2 = 0$, it follows $\tau = \mu = 0$ and no conditions on B and ρ_1 althought they will always result real.

The A_2 integrability conditions [13, 14, 35] are obtained choosing $j=2$ in the compatibility conditions (2.74) with $k=2$ and $n=3$

$$M_2 f_3\,(j) = M_3 f_2\,(j)\,. \tag{2.78}$$

In this case we have that $f_2(2) \in \mathcal{P}_4(1)$ and $f_3(2) \in \mathcal{P}_5(1)$ with $\dim(\mathcal{P}_4(1)) = 2$ and $\dim(\mathcal{P}_5(1)) = 5$, so that $f_2(2)$ and $f_3(2)$ will be respectively identified by 2 and 5 complex constants

$$f_2(2) \doteq au_\xi(1)|u(1)|^2 + b\bar{u}_\xi(1)u(1)^2, \tag{2.79a}$$

$$\begin{aligned} f_3(2) \doteq\ & \alpha|u(1)|^4u(1) + \beta|u_\xi(1)|^2u(1) + \gamma u_\xi(1)^2\bar{u}(1) + \\ & +\delta\bar{u}_{\xi\xi}(1)u(1)^2 + \epsilon|u(1)|^2u_{\xi\xi}(1). \end{aligned} \tag{2.79b}$$

In this way, if $\rho_2 \neq 0$, eliminating from eq. (2.78) the derivatives of $u(1)$ with respect to the slow-times t_2 and t_3 using the evolutions (2.71a) with $n = 2$ and $n = 3$ and equating term by term, we obtain the A_2 integrability conditions

$$a = \bar{a}, \quad b = \bar{b}, \tag{2.80}$$

while, if $\rho_2 = 0$, we have no conditions on a and b. So at this stage we have only two integrability conditions expressing the reality of the coefficients a and b. The expression of α, β, α, δ in terms of a and b are:

$$\alpha = \frac{3\mathrm{i}B\rho_2 a}{4\rho_1^2}, \quad \beta = \frac{3\mathrm{i}Bb}{\rho_1}, \quad \gamma = \frac{3\mathrm{i}Ba}{2\rho_1}, \quad \delta = 0, \quad \epsilon = \gamma. \tag{2.81}$$

The A_3 integrability conditions are derived in a similar way setting $j = 3$ in eq. (2.78). In this case we have that $f_2(3) \in \mathcal{P}_5(2)$ and $f_3(3) \in \mathcal{P}_6(2)$ with $\dim(\mathcal{P}_5(2)) = 12$ and $\dim(\mathcal{P}_6(2)) = 26$, so that $f_2(3)$ and $f_3(3)$ will be respectively identified by 12 and 26 complex constants

$$\begin{aligned} f_2(3) \doteq\ & \tau_1|u(1)|^4u(1) + \tau_2|u_\xi(1)|^2u(1) + \tau_3|u(1)|^2u_{\xi\xi}(1) + \tau_4\bar{u}_{\xi\xi}(1)u(1)^2 + \tau_5u_\xi(1)^2\bar{u}(1) + \\ & +\tau_6u_\xi(2)|u(1)|^2 + \tau_7\bar{u}_\xi(2)u(1)^2 + \tau_8u(2)^2\bar{u}(1) + \tau_9|u(2)|^2u(1) + \tau_{10}u(2)u_\xi(1)\bar{u}(1) + \\ & +\tau_{11}u(2)\bar{u}_\xi(1)u(1) + \tau_{12}\bar{u}(2)u_\xi(1)u(1), \end{aligned} \tag{2.82a}$$

$$\begin{aligned} f_3(3) \doteq\ & \gamma_1|u(1)|^4u_\xi(1) + \gamma_2|u(1)|^2u(1)^2\bar{u}_\xi(1) + \gamma_3|u(1)|^2u_{\xi\xi\xi}(1) + \gamma_4u(1)^2\bar{u}_{\xi\xi\xi}(1) + \\ & +\gamma_5|u_\xi(1)|^2u_\xi(1) + \gamma_6\bar{u}_{\xi\xi}(1)u_\xi(1)u(1) + \gamma_7u_{\xi\xi}(1)\bar{u}_\xi(1)u(1) + \gamma_8u_{\xi\xi}(1)u_\xi(1)\bar{u}(1) + \\ & +\gamma_9|u(1)|^4u(2) + \gamma_{10}|u(1)|^2u(1)^2\bar{u}(2) + \gamma_{11}\bar{u}_\xi(1)u(2)^2 + \gamma_{12}u_\xi(1)|u(2)|^2 + \\ & +\gamma_{13}|u_\xi(1)|^2u(2) + \gamma_{14}|u(2)|^2u(2) + \gamma_{15}u_\xi(1)^2\bar{u}(2) + \gamma_{16}|u(1)|^2u_{\xi\xi}(2) + \\ & +\gamma_{17}u(1)^2\bar{u}_{\xi\xi}(2) + \gamma_{18}u(2)\bar{u}_{\xi\xi}(1)u(1) + \gamma_{19}u(2)u_{\xi\xi}(1)\bar{u}(1) + \gamma_{20}\bar{u}(2)u_{\xi\xi}(1)u(1) + \\ & +\gamma_{21}u(2)u_\xi(2)\bar{u}(1) + \gamma_{22}\bar{u}(2)u_\xi(2)u(1) + \gamma_{23}u_\xi(2)u_\xi(1)\bar{u}(1) + \gamma_{24}u_\xi(2)\bar{u}_\xi(1)u(1) + \\ & +\gamma_{25}\bar{u}_\xi(2)u_\xi(1)u(1) + \gamma_{26}\bar{u}_\xi(2)u(2)u(1). \end{aligned} \tag{2.82b}$$

After eliminating from eq. (2.78) with $j = 3$ the derivatives of $u(1)$ with respect to the slow-times t_2 and t_3 using the evolutions (2.71a) respectively with $n = 2$ and $n = 3$ and the same derivatives of $u(2)$ using the evolutions (2.71b) with $n = 2$ and $n = 3$ and equating the remaining term by term, if $\rho_2 \neq 0$, indicating with R_i and I_i the real and imaginary parts of τ_i, $i = 1, \ldots, 12$, we obtain the $A3$ integrability conditions

$$\begin{aligned} & R_1 = -\frac{aI_6}{4\rho_1}, \quad R_3 = \frac{(b-a)I_6}{2\rho_2} - \frac{aI_{12}}{2\rho_2}, \quad R_4 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{aI_{12}}{4\rho_2}, \\ & R_5 = \frac{R_2}{2} + \frac{(a-b)I_6}{4\rho_2} + \frac{(2b-a)I_{12}}{4\rho_2}, \quad R_6 = -\frac{aI_8}{\rho_2}, \quad R_7 = R_{12} + \frac{(a-b)I_8}{\rho_2}, \\ & R_8 = R_9 = 0, \quad R_{10} = R_{12}, \quad R_{11} = R_{12} + \frac{(a-2b)I_8}{\rho_2}, \\ & I_4 = \frac{(b+a)R_{12}}{4\rho_2} + \frac{\rho_1 I_1}{\rho_2} + \frac{I_2 - I_3 - 2I_5}{4} + \frac{\left[2b(a-b) + a^2\right] I_8}{4\rho_2^2}, \quad I_7 = 0, \\ & I_9 = 2I_8, \quad I_{10} = I_{12}, \quad I_{11} = I_6 + I_{12}. \end{aligned} \tag{2.83}$$

Although in [13, 14] it was already reported that these conditions would consist of 15 real equations so that $f_2(3)$ and $f_3(3)$ will be parametrized by $2 \cdot 12 - 15 = 9$ real constants, the precise form of those equations was not given and it appears here for the first time. For completeness we give the expressions of the γ_j, $j = 1, \ldots, 26$ as functions of the τ_i, $i = 1, \ldots, 12$:

$$\gamma_1 = \frac{3B}{8\rho_1^2}\left[-2bR_{12} - 8\rho_1 I_1 + 2(I_2 - 2I_3 - 2I_5)\rho_2 + \mathrm{i}(b-5a)I_6 + \frac{2a^2 I_8}{\rho_2} - 3\mathrm{i}aI_{12}\right],$$
$$\gamma_2 = -\frac{3Ba}{4\rho_1^2}\left[\mathrm{i}I_6 + \frac{(a-2b)I_8}{\rho_2} + \tau_{12}\right], \quad \gamma_3 = \frac{3\mathrm{i}B\tau_3}{2\rho_1}, \quad \gamma_4 = 0, \quad \gamma_5 = \frac{3\mathrm{i}B\tau_2}{2\rho_1},$$
$$\gamma_6 = \frac{3\mathrm{i}B\tau_4}{\rho_1}, \quad \gamma_7 = \gamma_5, \quad \gamma_8 = \gamma_3 + \frac{3\mathrm{i}B\tau_5}{\rho_1}, \quad \gamma_9 = -\frac{3B(\rho_2 I_6 + 3a\mathrm{i}I_8)}{4\rho_1^2},$$
$$\gamma_{10} = \frac{3\mathrm{i}B\rho_2 R_6}{2\rho_1^2}, \quad \gamma_{11} = 0, \quad \gamma_{12} = \frac{3\mathrm{i}B\tau_9}{2\rho_1}, \quad \gamma_{13} = \frac{3\mathrm{i}B\tau_{11}}{2\rho_1}, \quad \gamma_{14} = 0, \quad \gamma_{15} = \frac{3\mathrm{i}B\tau_{12}}{2\rho_1},$$
$$\gamma_{16} = \frac{3\mathrm{i}B\tau_6}{2\rho_1}, \quad \gamma_{17} = \gamma_{18} = 0, \quad \gamma_{19} = \frac{3\mathrm{i}B\tau_{10}}{2\rho_1}, \quad \gamma_{20} = \gamma_{15}, \quad \gamma_{21} = \frac{3\mathrm{i}B\tau_8}{\rho_1},$$
$$\gamma_{22} = \gamma_{12}, \quad \gamma_{23} = \gamma_{16} + \gamma_{19}, \quad \gamma_{24} = \gamma_{13}, \quad \gamma_{25} = \frac{3\mathrm{i}B\tau_7}{\rho_1}, \quad \gamma_{26} = 0.$$

If $\rho_2 = 0$, the A_3 integrability conditions turn out to be:

$$\tau_1 = -\frac{\mathrm{i}}{4\rho_1}\left[b(\tau_{11} - 2\tau_6) + \bar{a}\tau_7\right], \quad \bar{b}\tau_7 = \frac{1}{2}(b-a)(\tau_{11} + \tau_{10} - \tau_6) + \bar{a}\tau_7,$$
$$a\tau_8 = b\tau_8 = 0, \quad a\tau_9 = b\tau_9 = 0, \quad \bar{a}\tau_{12} = a(\tau_{10} - \tau_{11}) + b\tau_6 + \bar{a}\tau_7,$$
$$\left(\bar{b} - \bar{a}\right)\tau_{12} = (b-a)\tau_{10}, \tag{2.84}$$

and the expressions of the γ_j as functions of the τ_i are:

$$\gamma_1 = -\frac{3B}{4\rho_1^2}\left(a\tau_6 - 4\mathrm{i}\rho_1\tau_1 + \bar{b}\tau_{12}\right), \quad \gamma_2 = -\frac{3B}{4\rho_1^2}\left(b\tau_6 + \bar{a}\tau_7\right).$$

The other γ_js are the same as written up above (note that, given the conditions (2.84), from the expressions of γ_9 and γ_{10} one deduces that $\gamma_9 = \gamma_{10} = 0$). Also in this case the conditions given in eqs. (2.84) appear to be new. Their importance resides in the fact that a $C-$integrable equation must satisfy those conditions (in this case eq. (2.75b) is a linear equation).

2.2.2 Integrability conditions II: KdV hierarchy

The first attempt to use the multiple times formalism in connection with the KdV hierarchy can be found in [25], where the authors applied the method to study the propagation of long surface waves in a shallow inviscid fluid. Apart from the different hierarchy of equations followed by the leading order of the field describing the height of the upper surface and the absence of any expansion of the various fields involved in harmonics, the method appears to be in all similar to that previously described.

In our case we will be interested in the situation characterized by:

- The leading order in ε of the field $\varphi(x,t) \in \mathcal{R}$, namely $\varphi^{(1)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right)$ will evolve at the slow-times t_n, $n \geq 2$ according to the KdV hierarchy

$$\partial_{t_n}\varphi^{(1)} = H_n\left[\varphi^{(1)}\right]; \tag{2.85a}$$

- The higher orders in ε of the field $\varphi^{(j)}\left(\xi,\{t_m\}_{m=1}^{\mathcal{N}}\right)$, $j\geq 2$ will evolve at the slow-times t_n, $n\geq 2$ according to the linearized versions ($\varphi^{(1)}$) of the equations (2.85a)

$$M_n\varphi^{(j)}=g_n(j),\quad M_n\doteq\partial_{t_n}-H_n'\left[\varphi^{(1)}\right],\tag{2.85b}$$

  where $g_n(j)$ is a nonhomogeneous *nonlinear* forcing term and $H_n'\left[\varphi\right]\phi$ is the Frechet derivative of the corresponding flow $H_n[\varphi]$ along the direction ϕ.

For future reference, we list the various fluxes $H_n[\varphi]$ of the KdV hierarchy and the corresponding $H_n'\left[\varphi\right]\phi$ up to $n=3$:

$$H_1\left[\varphi\right]\doteq A\varphi_\xi,\tag{2.86a}$$
$$H_2\left[\varphi\right]\doteq\tau_1\left[\varphi_{\xi\xi\xi}+\frac{2\tau_2}{\tau_1}\varphi\varphi_\xi\right],\tag{2.86b}$$
$$H_3\left[\varphi\right]\doteq\lambda\left\{\varphi_{\xi\xi\xi\xi\xi}+\frac{10\tau_2}{3\tau 1}\left[\frac{\tau_2}{\tau_1}\varphi^2\varphi_\xi+2\varphi_\xi\varphi_{\xi\xi}+\varphi\varphi_{\xi\xi\xi}\right]\right\},\tag{2.86c}$$
$$H_1'\left[\varphi\right]\phi=A\phi_\xi,\tag{2.86d}$$
$$H_2'\left[\varphi\right]\phi=\tau_1\left[\phi_{\xi\xi\xi}+\frac{2\tau_2}{\tau_1}\left(\phi\varphi_\xi+\varphi\phi_\xi\right)\right],\tag{2.86e}$$
$$H_3'\left[\varphi\right]\phi=\lambda\left\{\phi_{\xi\xi\xi\xi\xi}+\frac{10\tau_2}{3\tau 1}\left[\varphi\phi_{\xi\xi\xi}+2\varphi_\xi\phi_{\xi\xi}+\left(2\varphi_{\xi\xi}+\frac{\tau_2}{\tau_1}\varphi^2\right)\phi_\xi+ +\left(\frac{2\tau_2}{\tau_1}\varphi\varphi_\xi+\varphi_{\xi\xi\xi}\right)\phi\right]\right\},\tag{2.86f}$$

where τ_1, τ_2, λ are arbitrary real constants. The definitions (2.2), (2.3) of the vector space $\mathcal{P}_n$ and its subspaces now are:

Definition 2.5 *The finite dimensional vector space $\mathcal{P}_n$, $n\geq 2$ is the set of all differential polynomials in the functions $\varphi^{(j)}$s, $j\geq 1$ and their ξ-derivatives of order n in ε where*

$$\text{order}\left(\partial_\xi^m\varphi^{(j)}\right)=m+2j,\quad m\geq 0;$$

Definition 2.6 *$\mathcal{P}_n(m)$, $m\geq 1$ and $n\geq 2$ is the subspace of $\mathcal{P}_n$ whose elements are differential polynomials in the functions $\varphi^{(j)}$s,, $j\geq 1$ and their ξ-derivatives of order n in ε where the index j goes only up to m.*

- One can see that in general $H_n\left[\varphi^{(1)}\right]\in\partial_\xi^{2n-1}\varphi^{(1)}\cup\mathcal{P}_{2n+1}(1)$ and that $g_n(j)\in\mathcal{P}_{2(j+n)-1}($ $j-1)$ where as usual $j,\ n\geq 2$. The basis monomials of the spaces $\mathcal{P}_n(m)$ are given in *Appendix B*.

As every KdV equation $\partial_{t_2}\varphi^{(1)}=H_2\left[\varphi^{(1)}\right]$, where $H_2\left[\varphi^{(1)}\right]$ is given by eq. (2.86b), with real coefficients is an integrable equation, we don't have any A_1 integrability condition.

The A_2 integrability conditions are obtained choosing $j=2$ in the compatibility conditions (2.74) with $k=2$ and $n=3$ and $f_n(j)$ replaced by $g_n(j)$

$$M_2 g_3\,(j) = M_3 g_2\,(j)\,. \tag{2.87}$$

In this case $g_2(2) \in \mathcal{P}_7(1)$ and $g_3(2) \in \mathcal{P}_9(1)$ with $\dim(\mathcal{P}_7(1)) = 3$ and $\dim(\mathcal{P}_9(1)) = 7$, so that $g_2(2)$ and $g_3(2)$ will be respectively identified by 3 and 7 real constants

$$g_2(2) \doteq a\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi} + b\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi} + c\varphi^{(1)2}\varphi^{(1)}_{\xi}, \tag{2.88a}$$

$$\begin{aligned} g_3(2) \doteq\; & \alpha\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi} + \beta\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi} + \gamma\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi\xi\xi} + \delta\varphi^{(1)3}_{\xi} + \\ & +\epsilon\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi} + \pi\varphi^{(1)2}\varphi^{(1)}_{\xi\xi\xi} + \sigma\varphi^{(1)3}\varphi^{(1)}_{\xi}. \end{aligned} \tag{2.88b}$$

In this way, eliminating from eq. (2.87) the derivatives of $\varphi^{(1)}$ with respect to the slow-times t_2 and t_3 using the evolutions (2.85a) with $n = 2$ and $n = 3$ and equating term by term, we obtain *no* A_2 integrability conditions, i. e. eq. (2.87) can be always satisfied $\forall a$, b and c. The expressions of α, β, γ, δ, ϵ, π and σ as functions of a, b and c are given by

$$\begin{aligned} &\alpha = \frac{5\lambda\,(2a+b)}{3\tau_1}, \quad \beta = \frac{5\lambda\,(a+b)}{3\tau_1}, \quad \gamma = \frac{5\lambda b}{3\tau_1}, \\ &\delta = \frac{5\lambda\,[3c\tau_1 + (3a-b)\,\tau_2]}{9\tau_1^2}, \quad \epsilon = 2\,(\delta+\pi)\,, \quad \pi = \frac{5\lambda\,(3c\tau_1 + 5b\tau_2)}{9\tau_1^2}, \\ &\sigma = \frac{10\lambda\tau_2\,(9c\tau_1 - b\tau_2)}{27\tau_1^3}. \end{aligned} \tag{2.89}$$

The A_3 integrability conditions are derived in a similar way setting $j = 3$ in eq. (2.87). In this case we have that $g_2(3) \in \mathcal{P}_9(2)$ and $g_3(3) \in \mathcal{P}_{11}(2)$ with $\dim(\mathcal{P}_9(2)) = 14$ and $\dim(\mathcal{P}_{11}(2)) = 31$, so that $g_2(3)$ and $g_3(3)$ will be identified by 14 and 31 real constants

$$\begin{aligned} g_2(3) \doteq\; & \mu_1\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi} + \mu_2\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi} + \mu_3\varphi^{(1)}_{\xi\xi\xi\xi\xi}\varphi^{(1)} + \\ & +\mu_4\varphi^{(1)3}_{\xi} + \mu_5\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi}\varphi^{(1)} + \mu_6\varphi^{(1)}_{\xi\xi\xi}\varphi^{(1)2} + \\ & +\mu_7\varphi^{(1)}_{\xi}\varphi^{(1)3} + \mu_8\varphi^{(2)}_{\xi\xi\xi}\varphi^{(1)} + \mu_9\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi\xi} + \\ & +\mu_{10}\varphi^{(1)}_{\xi\xi}\varphi^{(2)}_{\xi} + \mu_{11}\varphi^{(1)}_{\xi\xi\xi}\varphi^{(2)} + \mu_{12}\varphi^{(2)}_{\xi}\varphi^{(1)2} + \\ & +\mu_{13}\varphi^{(1)}_{\xi}\varphi^{(1)}\varphi^{(2)} + \mu_{14}\varphi^{(2)}_{\xi}\varphi^{(2)}, \end{aligned} \tag{2.90a}$$

$$
\begin{aligned}
g_3(3) \doteq\ & \delta_1 \varphi^{(1)}_{\xi\xi\xi} \varphi^{(1)}_{\xi\xi\xi\xi} + \delta_2 \varphi^{(1)}_{\xi\xi} \varphi^{(1)}_{\xi\xi\xi\xi\xi} + \delta_3 \varphi^{(1)}_{\xi} \varphi^{(1)}_{\xi\xi\xi\xi\xi\xi} + \\
& + \delta_4 \varphi^{(1)}_{\xi\xi\xi\xi\xi\xi\xi} \varphi^{(1)} + \delta_5 \varphi^{(1)}_{\xi} \varphi^{(1)2}_{\xi\xi} + \delta_6 \varphi^{(1)2}_{\xi} \varphi^{(1)}_{\xi\xi\xi} + \\
& + \delta_7 \varphi^{(1)}_{\xi\xi} \varphi^{(1)}_{\xi\xi\xi} \varphi^{(1)} + \delta_8 \varphi^{(1)}_{\xi} \varphi^{(1)}_{\xi\xi\xi\xi} \varphi^{(1)} + \\
& + \delta_9 \varphi^{(1)}_{\xi\xi\xi\xi\xi} \varphi^{(1)2} + \delta_{10} \varphi^{(1)}_{\xi\xi\xi} \varphi^{(1)3} + \delta_{11} \varphi^{(1)}_{\xi} \varphi^{(1)}_{\xi\xi} \varphi^{(1)2} + \\
& + \delta_{12} \varphi^{(1)3}_{\xi} \varphi^{(1)} + \delta_{13} \varphi^{(1)}_{\xi} \varphi^{(1)4} + \delta_{14} \varphi^{(2)}_{\xi\xi\xi\xi\xi} \varphi^{(1)} + \\
& + \delta_{15} \varphi^{(2)}_{\xi\xi\xi\xi} \varphi^{(1)}_{\xi} + \delta_{16} \varphi^{(2)}_{\xi\xi\xi} \varphi^{(1)}_{\xi\xi} + \\
& + \delta_{17} \varphi^{(2)}_{\xi\xi} \varphi^{(1)}_{\xi\xi\xi} + \delta_{18} \varphi^{(2)}_{\xi} \varphi^{(1)}_{\xi\xi\xi\xi} + \delta_{19} \varphi^{(1)}_{\xi\xi\xi\xi\xi} \varphi^{(2)} + \\
& + \delta_{20} \varphi^{(2)}_{\xi\xi\xi} \varphi^{(1)2} + \delta_{21} \varphi^{(2)}_{\xi\xi} \varphi^{(1)}_{\xi} \varphi^{(1)} + \\
& + \delta_{22} \varphi^{(2)}_{\xi} \varphi^{(1)2}_{\xi} + \delta_{23} \varphi^{(2)}_{\xi} \varphi^{(1)}_{\xi\xi} \varphi^{(1)} + \\
& + \delta_{24} \varphi^{(1)}_{\xi} \varphi^{(1)}_{\xi\xi} \varphi^{(2)} + \delta_{25} \varphi^{(1)}_{\xi\xi\xi} \varphi^{(1)} \varphi^{(2)} + \delta_{26} \varphi^{(2)}_{\xi} \varphi^{(1)3} + \\
& + \delta_{27} \varphi^{(1)}_{\xi} \varphi^{(1)2} \varphi^{(2)} + \delta_{28} \varphi^{(2)}_{\xi} \varphi^{(1)} \varphi^{(2)} + \delta_{29} \varphi^{(1)}_{\xi} \varphi^{(2)2} + \\
& + \delta_{30} \varphi^{(2)}_{\xi\xi\xi} \varphi^{(2)} + \delta_{31} \varphi^{(2)}_{\xi} \varphi^{(2)}_{\xi\xi} .
\end{aligned}
\tag{2.90b}
$$

If $\tau_2 \neq 0$, eliminating in eq. (2.87) with $j = 3$ the derivatives of $\varphi^{(1)}$ with respect to the slow-times t_2 and t_3 using the evolutions (2.85a) with $n = 2$ and $n = 3$ and the same derivatives of $\varphi^{(2)}$ by evolutions (2.85b) with $n = 2$ and $n = 3$ and equating the rest term by term, we have the following A_3 integrability conditions for the coefficients μ_j, $j = 1, \ldots, 14$:

$$
\begin{aligned}
\mu_7 =\ & \frac{\left[9\theta_1 \left(5\theta_2 + 12\theta_3\right) \tau_1 - \left(45\theta_2^2 + 88\theta_2\theta_3 + 12\theta_3^2\right) \tau_2\right] \mu_{14}}{54\tau_1^2\tau_2} + \\
& + \frac{\left[\left(3\theta_2 - 8\theta_3\right) \tau_2 - 3\theta_1\tau_1\right] \mu_{10}}{9\tau_1^2} + \frac{\left[\left(8\theta_2 + 42\theta_3\right) \tau_2 - 18\theta_1\tau_1\right] \mu_9}{27\tau_1^2} + \\
& + \frac{\left(9\mu_5 + 8\mu_6 - 24\mu_4\right) \tau_2}{9\tau_1} - \frac{2\left(12\mu_1 - 30\mu_2 + 85\mu_3\right) \tau_2^2}{27\tau_1^2}, \\
\mu_8 =\ & \frac{\theta_2\mu_{14}}{2\tau_2}, \quad \mu_{11} = \mu_{10} - \mu_9 + \frac{\theta_2\mu_{14}}{2\tau_2}, \quad \mu_{12} = \frac{3\theta_1\mu_{14}}{2\tau_2}, \quad \mu_{13} = \frac{3\theta_1\mu_{14}}{\tau_2}.
\end{aligned}
\tag{2.91a}
$$

If $\tau_2 = 0$, the linearizable case, and $\theta_1 \neq 0$, we have

$$
\begin{aligned}
& \mu_7 = \frac{(5\theta_2 + 12\theta_3)\mu_{12} - 3\theta_1(\mu_{10} + 2\mu_9)}{9\tau_1}, \quad \mu_8 = \frac{\theta_2\mu_{12}}{3\theta_1}, \\
& \mu_{11} = \mu_{10} - \mu_9 + \frac{\theta_2\mu_{12}}{3\theta_1}, \quad \mu_{13} = 2\mu_{12}, \quad \mu_{14} = 0,
\end{aligned}
\tag{2.91b}
$$

If $\tau_2 = \theta_1 = 0$, and $\theta_2 \neq 0$, we have

$$
\mu_7 = \frac{(\theta_2^2 - 7\theta_2\theta_3 + 2\theta_3^2)\mu_{14}}{9\tau_1^2}, \quad \mu_{12} = \frac{(\theta_3 - 2\theta_2)\mu_{14}}{3\tau_1}, \quad \mu_{13} = \frac{(2\theta_3 - 3\theta_2)\mu_{14}}{3\tau_1}, \tag{2.91c}
$$

and finally, if $\tau_2 = \theta_1 = \theta_2 = 0$, we have

$$\mu_7 = \frac{\left[3\left(2\mu_{12}+\mu_{13}\right)\tau_1 - 2\theta_3\mu_{14}\right]\theta_3}{9\tau_1^2}. \tag{2.91d}$$

The expressions of the coefficients δ_κ, $\kappa = 1, \ldots, 31$ of $g_3(3)$ as functions of the coefficients μ_j, $j = 1, \ldots, 14$ is given in *Appendix C.*

2.2.3 Integrability conditions III: potential KdV hierarchy

Here we discuss the case of the potential KdV hierarchy, closely related to the KdV one. It arises when one has a set of functions $\phi^{(j)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right)$, $j \geq 1$, which are supposed go to a constant while all their ξ-derivatives go to zero as $\xi \to \pm\infty$ and are related to the previous functions $\varphi^{(j)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right)$, $j \geq 1$ by the the relation $\varphi^{(j)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right) = \partial_\xi \phi^{(j)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right)$, $j \geq 1$. Under such positions the function $\phi^{(1)}\left(\xi, \{t_m\}_{m=1}^{\mathcal{N}}\right)$ follows the *potential KdV* hierarchy of equations at the slow-times t_n, $n \geq 2$

$$\partial_{t_n}\phi^{(1)} = K_n\left[\phi^{(1)}\right], \quad K_n\left[\phi^{(1)}\right] \doteq \int H_n\left[\phi_\xi^{(1)}\right] d\xi. \tag{2.92}$$

The various $K_n[\phi]$ up to $n = 4$ are given by:

$$K_1[\phi] = A\phi_\xi, \tag{2.93a}$$

$$K_2[\phi] = \tau_1\left[\phi_{\xi\xi\xi} + \frac{\tau_2}{\tau_1}\phi_\xi^2\right], \tag{2.93b}$$

$$K_3[\phi] = \lambda\left[\phi_{\xi\xi\xi\xi\xi} + \frac{5\tau_2}{3\tau_1}\left(\frac{2\tau_2}{3\tau_1}\phi_\xi^3 + \phi_{\xi\xi}^2 + 2\phi_\xi\phi_{\xi\xi\xi}\right)\right], \tag{2.93c}$$

$$K_4[\phi] = \chi\left\{\phi_{\xi\xi\xi\xi\xi\xi\xi} + \frac{7\tau_2}{\tau_1}\left[\frac{2}{3}\phi_\xi\phi_{\xi\xi\xi\xi\xi} + \frac{10\tau_2}{9\tau_1}\left(\phi_\xi\phi_{\xi\xi}^2 + \phi_\xi^2\phi_{\xi\xi\xi}\right) + \right.\right. \tag{2.93d}$$
$$\left.\left. + \frac{5\tau_2^2}{27\tau_1^2}\phi_\xi^4 + \frac{4}{3}\phi_{\xi\xi}\phi_{\xi\xi\xi\xi} + \phi_{\xi\xi\xi}^2\right]\right\},$$

where χ is a real constant and the arbitrary ξ−independent integration functions have been set to zero to match the asymptotic conditions on the $\phi^{(j)}$s as $\xi \to \pm\infty$. Let us stress that the flows (2.93) are completely *local* despite the presence of an integral in their definition (2.92). On the contrary the evolutions of the functions $\phi^{(j)}$, $j \geq 2$ according to the slow-times t_n, $n \geq 2$

$$N_n\phi^{(j)} = f_n(j), \quad N_n \doteq \partial_{t_n} - K_n'\left[\phi^{(1)}\right], \quad f_n(j) \doteq \int g_n(j) d\xi, \tag{2.94}$$

where $K_n'[\phi]\zeta$ is the Frechet derivative of the corresponding flow $K_n[\phi]$ along the direction ζ, are *not always local.* This depends from the fact that, when we replace in $g_n(j)$ the various $\varphi^{(\kappa)}$ up to $\kappa = j-1$ with $\partial_\xi\phi^{(\kappa)}$ and perform the integration given in eq. (2.94) to find $f_n(j)$, the resulting expression could in general involve integrals. We postpone to the end of the subsection the discussion of the conditions of locality of the resulting equations. Suppose for a moment that they are indeed local. In this case the nonlinear forcing terms $f_n(j)$ belong to some subspace of the polynomial vector space $\mathcal{P}_m$, $m \geq 2$. These spaces, taking into account the definitions (2.5), (2.6), are defined in terms of the functions $\phi^{(j)}$ by

Definition 2.7 *The finite dimensional vector space $\mathcal{P}_n$, $n \geq 2$ is the set of all differential polynomials in the ξ-derivatives of the functions $\phi^{(j)}$s, $j \geq 1$ of order n in ε where*

$$\text{order}\left(\partial_\xi^m \phi^{(j)}\right) = m + 2j - 1, \quad m \geq 1.$$

Definition 2.8 *$\mathcal{P}_n(m)$, $m \geq 1$ and $n \geq 2$ is the subspace of $\mathcal{P}_n$ whose elements are differential polynomials in the ξ-derivatives of the functions $\phi^{(j)}$, $j \geq 1$ of order n in ε where the index j goes only up to m.*

- One can see that in general $K_n\left[\phi^{(1)}\right] \in \partial_\xi^{2n-1}\phi^{(1)} \cup \mathcal{P}_{2n}(1)$ and that $f_n(j) \in \mathcal{P}_{2(j+n-1)}(j-1)$ where as usual $j,\ n \geq 2$.

Let us recall a few $K_n'[\phi]\,\zeta$ for n up to 3:

$$K_1'[\phi]\,\zeta = A\zeta_\xi, \tag{2.95a}$$
$$K_2'[\phi]\,\zeta = \tau_1\zeta_{\xi\xi\xi} + 2\tau_2\phi_\xi\zeta_\xi, \tag{2.95b}$$
$$K_3'[\phi]\,\zeta = \lambda\left\{\zeta_{\xi\xi\xi\xi\xi} + \frac{10\tau_2}{3\tau_1}\left[\phi_\xi\zeta_{\xi\xi\xi} + \phi_{\xi\xi}\zeta_{\xi\xi} + \left(\frac{\tau_2}{\tau_1}\phi_\xi^2 + \phi_{\xi\xi\xi}\right)\zeta_\xi\right]\right\}. \tag{2.95c}$$

As every *potential KdV* equation $\partial_{t_2}\phi^{(1)} = K_2\left[\phi^{(1)}\right]$, where $K_2\left[\phi^{(1)}\right]$ is given in eq. (2.93b), with real coefficients is integrable, there are no A_1 integrability conditions.

The A_2 integrability conditions are obtained choosing $j = 2$ in the compatibility conditions

$$M_2 f_3(j) = M_3 f_2(j). \tag{2.96}$$

In this case we have that $f_2(2) \in \mathcal{P}_6(1)$ and $f_3(2) \in \mathcal{P}_8(1)$ with $\dim(\mathcal{P}_6(1)) = 3$ and $\dim(\mathcal{P}_8(1)) = 6$, so that $f_2(2)$ and $f_3(2)$ will be respectively identified by 3 and 6 real constants

$$f_2(2) \doteq \theta_1\phi_\xi^{(1)3} + \theta_2\phi_\xi^{(1)}\phi_{\xi\xi\xi}^{(1)} + \theta_3\phi_{\xi\xi}^{(1)2}, \tag{2.97a}$$
$$\begin{aligned} f_3(2) \doteq{}& \xi_1\phi_\xi^{(1)}\phi_{\xi\xi}^{(1)2} + \xi_2\phi_\xi^{(1)}\phi_{\xi\xi\xi\xi\xi}^{(1)} + \xi_3\phi_{\xi\xi}^{(1)}\phi_{\xi\xi\xi\xi}^{(1)} + \\ &+\xi_4\phi_\xi^{(1)4} + \xi_5\phi_\xi^{(1)2}\phi_{\xi\xi\xi}^{(1)} + \xi_6\phi_{\xi\xi\xi}^{(1)2}. \end{aligned} \tag{2.97b}$$

In this way, eliminating from eq. (2.96) the derivatives of $\phi^{(1)}$ with respect to the slow-times t_2 and t_3 using the evolutions (2.92) respectively with $n = 2$ and $n = 3$ and equating term by term, we obtain, as in the corresponding case for the *KdV* hierarchy, *no* A_2 integrability conditions for the coefficients θ_i, $i = 1, \ldots, 3$, i. e. eq. (2.96) can always be satisfied $\forall\theta_1$, θ_2 and θ_3. The expressions of the ξ_j, $j = 1, \ldots, 6$ as functions of the θ_i, $i = 1, \ldots, 3$ are given by:

$$\xi_1 = \frac{5\lambda\left(9\theta_1\tau_1 + 2\theta_2\tau_2 + 6\theta_3\tau_2\right)}{9\tau_1^2}, \qquad \xi_2 = \frac{5\lambda\theta_2}{3\tau_1}, \qquad \xi_3 = \frac{5\lambda\left(\theta_2 + 2\theta_3\right)}{3\tau_1},$$
$$\xi_4 = \frac{5\lambda\tau_2\left(27\theta_1\tau_1 - \theta_2\tau_2\right)}{54\tau_1^3}, \qquad \xi_5 = \frac{5\lambda\left(9\theta_1\tau_1 + 5\theta_2\tau_2\right)}{9\tau_1^2}, \qquad \xi_6 = \frac{5\lambda\left(\theta_2 + \theta_3\right)}{3\tau_1}. \tag{2.98}$$

To close this subsection, let us investigate the conditions under which the equations (2.94) are completely local. In this respect we will give a series of straightforward propositions.

Proposition 2.5 *The integration of the term $g_2(2)$ given in eq. (2.88a) gives a local expression for the term $f_2(2)$ given in eq. (2.97a) $\forall a$, b and c. We have*

$$\theta_1 = \frac{c}{3}, \qquad \theta_2 = b, \qquad \theta_3 = \frac{(a-b)}{2};$$

Proposition 2.6 *The integration of the term $g_3(2)$ given in eq. (2.88b) gives a local expression for the term $f_3(2)$ given in eq. (2.97b) iff*

$$\epsilon = 2\,(\delta + \pi)\,. \tag{2.99}$$

We get

$$\xi_1 = \delta, \qquad \xi_2 = \gamma, \qquad \xi_3 = (\beta - \gamma)\,, \qquad \xi_4 = \frac{\sigma}{4}, \qquad \xi_5 = \pi, \qquad \xi_6 = \frac{(\alpha - \beta + \gamma)}{2}.$$

If the coefficients of $g_3(2)$ are given by the expressions (2.89), then the above conditions (2.99) are automatically satisfied;

Proposition 2.7 *The integration of the term $g_2(3)$ given in eq. (2.90a) gives a local expression for the term $f_2(3)$ iff*

$$\mu_5 = 2\,(\mu_4 + \mu_6)\,, \qquad \mu_8 = \mu_9 - \mu_{10} + \mu_{11}, \qquad \mu_{13} = 2\mu_{12}. \tag{2.100}$$

The A_3 integrability conditions (2.91) for the coefficients μ_j, $j = 1, \ldots, 14$, do not imply that eqs. (2.100), are automatically satisfied;

Proposition 2.8 *The integration of the term $g_3(3)$ given in eq. (2.90b) provides a local expression for the term $f_3(3)$ iff*

$$\begin{aligned}
&\delta_7 = 2\delta_5 - 4\delta_6 + 5\delta_8 - 10\delta_9, \qquad \delta_{11} = 3\delta_{10} + \delta_{12},\\
&\delta_{14} = \delta_{15} - \delta_{16} + \delta_{17} - \delta_{18} + \delta_{19}, \qquad \delta_{23} = \delta_{21} - 2\delta_{20} + \delta_{25},\\
&\delta_{24} = 2\,(2\delta_{20} - \delta_{21} + \delta_{22}) + \delta_{25}, \qquad \delta_{27} = 3\delta_{26}, \qquad \delta_{28} = 2\delta_{29}.
\end{aligned} \tag{2.101}$$

If the coefficients δ_κ, $\kappa = 1, \ldots, 31$ are given by eqs. (C.1), where the coefficients μ_j, $j = 1, \ldots, 14$ satisfy the A_3 integrability conditions (2.91) and the conditions (2.100), then the conditions (2.101) are automatically satisfied.

2.2.4 Solutions of the linearized equations I: NLS hierarchy

The NLS equation given in eqs. (2.71a) with $n = 2$ and (2.75b),

$$\mathrm{i}u_{t_2}(1) = \rho_1 u_{\xi\xi}(1) + \rho_2 |u(1)|^2 u(1),$$

provided that the coefficient ρ_2 is real, represents an $S-$integrable evolution equation for the complex function $u(1)(\xi, t_2)$, in the sense that it arises [1] as the compatibility condition of the following overdetermined system of two matrix linear partial differential equations for the vector function $\boldsymbol{\psi}(\xi, t_2) \doteq \left(\psi^{(1)}(\xi, t_2), \psi^{(2)}(\xi, t_2)\right)^T$

$$\boldsymbol{\psi}_\xi = \begin{pmatrix} -\mathrm{i}\lambda & \eta u(1) \\ -\epsilon\eta\bar{u}(1) & \mathrm{i}\lambda \end{pmatrix} \boldsymbol{\psi} \doteq L\boldsymbol{\psi}, \tag{2.102a}$$

$$\boldsymbol{\psi}_{t_2} = \rho_1 \begin{pmatrix} 2\mathrm{i}\lambda^2 - \mathrm{i}\epsilon\eta^2|u(1)|^2 & -2\lambda\eta u(1) - \mathrm{i}\eta u_\xi(1) \\ 2\epsilon\lambda\eta\bar{u}(1) - \mathrm{i}\epsilon\eta\bar{u}_\xi(1) & -2\mathrm{i}\lambda^2 + \mathrm{i}\epsilon\eta^2|u(1)|^2 \end{pmatrix} \boldsymbol{\psi} \doteq M\boldsymbol{\psi}, \tag{2.102b}$$

where $\epsilon \doteq \mathrm{sgn.}\dfrac{\rho_2}{\rho_1}$, $\eta \doteq \left|\dfrac{\rho_2}{2\rho_1}\right|^{1/2}$ and $\lambda \in \mathbb{C}$ is the spectral parameter. Hence our equation is equivalent to the following matrix equation

$$L_{t_2} - M_\xi + [L, M] = 0.$$

As it has been said, eqs. (2.71), as n varies, represent the NLS hierarchy of compatible evolutions for the same function $u(1)(\xi, t_1, t_2, \ldots)$ at the different slow-times. Each equation in this hierarchy is an $S-$integrable evolution emerging as a compatibility condition between eq. (2.102a) and a suitable t_n evolution for the spectral function $\boldsymbol{\psi}(\xi, t_1, t_2, \ldots)$. The common *one-soliton* solution of the *focusing* hierarchy, i. e. $\epsilon = 1$, up to $n = 4$ is given by

$$u(1) = \frac{\kappa}{\eta}\mathrm{sech}\left\{\kappa\left[\xi + At_1 - 2\beta\rho_1 t_2 - \left(3\beta^2 - \kappa^2\right)Bt_3 - 4\left(\kappa^2 - \beta^2\right)\beta C t_4 + \xi_0\right]\right\} \tag{2.103}$$
$$e^{-\mathrm{i}\left[\beta\xi + \beta A t_1 - \left(\beta^2 - \kappa^2\right)\rho_1 t_2 - \left(\beta^2 - 3\kappa^2\right)\beta B t_3 - \left(6\kappa^2\beta^2 - \kappa^4 - \beta^4\right)C t_4 + \theta_0\right]},$$

where ξ_0, θ_0, β and κ are four real constants, ξ_0 and θ_0 are arbitrary and $2\lambda \doteq \beta + \mathrm{i}\kappa$. On the other hand the general solution bounded at infinity of the linearized Schrödinger equation given in (2.71b) with $n = 2$ and (2.76b)

$$u_{t_2}(2) + \mathrm{i}\rho_1 u_{\xi\xi}(2) + \mathrm{i}\rho_2\left[u(1)^2\bar{u}(2) + 2|u(1)|^2 u(2)\right] = a u_\xi(1)|u(1)|^2 + b\bar{u}_\xi(1)u(1)^2,$$

is given by the sum of the general integral bounded at infinity of the homogeneous equation and a particular solution of the nonhomogeneous equation. The general integral bounded at infinity of the homogeneous equations is given [20] by a linear superposition of the squares of the spectral functions corresponding to the discrete eigenvalues λ_m, i. e. $\psi^{(j)}(\lambda_m) \doteq \psi^{(j)}_m$, $j = 1, 2$, and of those corresponding to the continuous ones $\lambda(\rho)$, i. e. $\psi^{(j)}(\lambda(\rho)) \doteq \psi^{(j)}(\rho)$, $j = 1, 2$

$$\mathcal{S}_{homo.} = \sum_{m=1}^{N}\left(c_m\psi_m^{(1)2} - \bar{c}_m\bar{\psi}_m^{(2)2}\right) + \int_\rho\left[c(\rho)\,\psi^{(1)2}(\rho) - \bar{c}(\rho)\,\bar{\psi}^{(2)2}(\rho)\right]d\rho. \tag{2.104}$$

A particular solution of the nonhomogeneous equation is given by

$$\mathcal{S}_{part.} = -\mathrm{i}\frac{a}{2\rho_1}u(1)\int_{\xi^0}^{\xi}|u(1)(\xi')|^2 d\xi' + \mathrm{i}\frac{b-a}{2\rho_2}u_\xi(1), \tag{2.105}$$

where ξ^0 is a real constant. As one can see, in this particular solution enter the integrals of the conserved densities of the Schrödinger equation as $|u(1)|^2$ obeys the following *continuity equation*

$$\left(|u(1)|^2\right)_{t_2} = -\mathrm{i}\rho_1 \left(\bar{u}(1)u_\xi(1) - \mathcal{C.C.}\right)_\xi$$

and $u_\xi(1) = \int_{-\infty}^{\xi} u_{\xi'\xi'}(1)\,(\xi')\,d\xi'$.

2.2.5 Solutions of the linearized equations II: potential KdV hierarchy

The potential KdV equation given in eqs. (2.92) wit $n = 2$ and (2.93b)

$$\phi^{(1)}_{t_2} = \tau_1 \phi^{(1)}_{\xi\xi\xi} + \tau_2 \phi^{(1)2}_{\xi},$$

where τ_1, τ_2 are two real constants, represents an $S-$integrable evolution equation for the real function $\phi^{(1)}$, in the sense that it arises as the compatibility condition of the following overdetermined system of two linear equations for a complex function $\psi\,(x,t)$

$$\left(\frac{\partial^2}{\partial \xi^2} + \frac{\tau_2}{3\tau_1}\phi^{(1)}_\xi - \eta\right)\psi(\xi,t_2) \doteq (L-\eta)\,\psi(\xi,t_2) = 0, \tag{2.106a}$$

$$\left(\frac{\partial}{\partial t_2} - 4\tau_1\frac{\partial^3}{\partial \xi^3} - 2\tau_2\phi^{(1)}_\xi\frac{\partial}{\partial \xi} - \tau_2\phi^{(1)}_{\xi\xi}\right)\psi(\xi,t_2) \doteq \left(\frac{\partial}{\partial t_2} - M\right)\psi(\xi,t_2) = 0, \tag{2.106b}$$

where $\eta \in \mathbb{C}$ is a *spectral parameter* and L and M are two differential operators. Than our equation is equivalent to the operator identity

$$L_t = [M, L]\,,$$

between the linear operators L and M when one supposes that the spectral parameter λ is time independent. As it has been said, eqs. (2.92), as n varies, represent the potential KdV hierarchy of compatible evolutions for the same function $\phi^{(1)}\,(\xi, t_1, t_2, \ldots)$ at the different slow-times. Each equation in this hierarchy is an $S-$integrable evolution emerging as a compatibility condition between eq. (2.106a) and a suitable t_n evolution for the spectral function $\psi\,(\xi, t_1, t_2, \ldots)$. The common *one-soliton* solution of this hierarchy up to $n = 4$ is given by

$$\phi^{(1)} = -\frac{12\kappa\tau_1}{\tau_2}\frac{1}{1 + e^{2\kappa(\xi + At_1 + 4\kappa^2\tau_1 t_2 + 16\kappa^4\lambda t_3 + 64\kappa^6\chi t_4 + \xi_0)}}, \tag{2.107}$$

where ξ_0 and κ are two real constants, ξ_0 is arbitrary and $\eta = \kappa^2$. Differentiating the solution (2.107) once with respect to ξ, we get the common *one-soliton* solution of the KdV hierarchy (2.85a) up to $n = 4$

$$\varphi^{(1)} = \frac{6\tau_1\kappa^2}{\tau_2}\,\mathrm{sech}^2\left[\kappa\left(\xi + At_1 + 4\kappa^2\tau_1 t_2 + 16\kappa^4\lambda t_3 + 64\kappa^6\chi t_4 + \xi_0\right)\right]. \tag{2.108}$$

On the other hand the general solution bounded at infinity of the linearized potential KdV equation given in (2.94) with $n = 2$ and (2.95b)

$$\phi^{(2)}_{t_2} - \tau_1\phi^{(2)}_{\xi\xi\xi} - 2\tau_2\phi^{(1)}_\xi\phi^{(2)}_\xi = \theta_1\phi^{(1)3}_\xi + \theta_2\phi^{(1)}_\xi\phi^{(1)}_{\xi\xi\xi} + \theta_3\phi^{(1)2}_{\xi\xi},$$

is given by the sum of the general integral bounded at infinity of the homogeneous equation and a particular solution of the nonhomogeneous equation. The general integral bounded at infinity of the homogeneous equations is given [22] by a linear superposition of the squares of the spectral functions corresponding to the discrete eigenvalues η_m, i. e. $\psi(\eta_m) \doteq \psi_m$ and of those corresponding to the continuous ones $\eta(\rho)$, i. e. $\psi(\eta(\rho)) \doteq \psi(\rho)$

$$\mathcal{S}_{homo.} = \sum_{m=1}^{N} c_m \psi_m^2 + \int_\rho c(\rho)\, \psi^2(\rho)\, d\rho. \tag{2.109}$$

A particular solution of the nonhomogeneous equation is given by

$$\mathcal{S}_{part.} = -\frac{\theta_2}{3\tau_1}\phi^{(1)}\phi^{(1)}_\xi + \frac{1}{2}\left(\frac{\theta_2}{\tau_1} - \frac{3\theta_1}{\tau_2}\right)\int_{-\infty}^{\xi} \phi^{(1)2}_{\xi'} d\xi' + \frac{1}{2\tau_2}\left(\theta_3 + \frac{\theta_2}{2} - \frac{9\tau_1\theta_1}{2\tau_2}\right)\phi^{(1)}_{\xi\xi}. \tag{2.110}$$

As one can see, as before in this particular solution enter the integrals of the conserved densities of the potential KdV equation as $\phi^{(1)2}_\xi$ obeys the following *continuity equation*

$$\left(\phi^{(1)2}_\xi\right)_{t_2} = \left(2\tau_1\phi^{(1)}_\xi\phi^{(1)}_{\xi\xi\xi} - \tau_1\phi^{(1)2}_{\xi\xi} + \frac{4\tau_2}{3}\phi^{(1)3}_\xi\right)_\xi$$

and $\phi^{(1)}_\xi = \int_{-\infty}^{\xi}\phi^{(1)}_{\xi'\xi'}d\xi'$, $\phi^{(1)}_{\xi\xi} = \int_{-\infty}^{\xi}\phi^{(1)}_{\xi'\xi'\xi'}d\xi'$.

2.3 Multiscale expansion of real dispersive partial difference equations

2.3.1 From shifts to derivatives

We now illustrate all the ingredients of the discrete reductive perurbation technique as given in [15, 16]. Consider at first a function $u_n : \mathbb{Z} \to \mathbb{R}$ depending on a discrete index $n \in \mathbb{Z}$ and let us suppose that:

- The dependence of u_n on n is realized through the *slow variable* $n_1 \doteq \varepsilon n \in \mathbb{R}$, $\varepsilon \in \mathbb{R}$, $0 < \varepsilon \ll 1$, that is to say $u_n \doteq u(n_1)$;
- The variable n_1 can vary in a (full measure) region so that $u(n_1)$ is analytical (Taylor series expandibility) and that region contains the point $\widetilde{n}_1 \doteq \varepsilon\widetilde{n}$, $\widetilde{n} \in \aleph$;
- The radius of convergence of the Taylor series starting at $\widetilde{n}_1$ is wide enough to include as an *inner point* the point $\widetilde{n}_1 + \varepsilon$.

Under these hypotheses one can write the action of the shift operator T_n such that $T_n u_n \doteq u_{n+1} = u(n_1 + \varepsilon)$ around $\widetilde{n}_1$ as

$$T_n u(\widetilde{n}_1) = u(\widetilde{n}_1) + \varepsilon u^{(1)}(\widetilde{n}_1) + \frac{\varepsilon^2}{2}u^{(2)}(\widetilde{n}_1) + ... + \frac{\varepsilon^i}{i!}u^{(i)}(\widetilde{n}_1) + ... = \sum_{i=0}^{+\infty}\frac{\varepsilon^i}{i!}u^{(i)}(\widetilde{n}_1), \tag{2.111}$$

where $u^{(i)}(\widetilde{n}_1) \doteq d^i u(n_1)/dn_1^i|_{n_1=\widetilde{n}_1} \doteq d^i_{n_1}u(\widetilde{n}_1)$, being d_{n_1} the total derivative operator. The last expression suggests the following formal expansion for the differential operator T_n:

$$T_n = \sum_{i=0}^{+\infty} \frac{\varepsilon^i}{i!} d_{n_1}^i \doteq e^{\varepsilon d_{n_1}}, \tag{2.112}$$

valid only when the previous conditions, which assure that the series in eq. (2.112) is converging, are satisfied. Let us introduce more complicated dependencies of u_n on n. For example one can assume a simultaneous dependence on the *fast variable* n and on the slow variable n_1 or $u_n \doteq u(n, n_1)$. The action of the *total* shift operator T_n will now be given by $T_n u_n \doteq u_{n+1} = u(n+1, n_1+\varepsilon)$ so that we can write

$$T_n \doteq \mathcal{T}_n \mathcal{T}_{n_1}^{(\varepsilon)}, \tag{2.113}$$

where the *partial* shift operators $\mathcal{T}_n$ and $\mathcal{T}_{n_1}^{(\varepsilon)}$ are defined respectively by $\mathcal{T}_n u(n, n_1) = u(n+1, n_1)$ and $\mathcal{T}_{n_1}^{(\varepsilon)} u(n, n_1) = u(n, n_1+\varepsilon)$. Here $\mathcal{T}_{n_1}^{(\varepsilon)}$ is given by an expansion similar to eq. (2.112). The dependence of u_n on n can be easily extended to the case of one fast variable n and K slow variables $n_j \doteq \varepsilon_j n$, $\varepsilon_j \in \mathbb{R}$, $1 \le j \le K$ each of them being defined by its own parameter ε_j. The action of the total shift operator T_n will now be given in terms of the partial shifts $\mathcal{T}_n$, $\mathcal{T}_{n_j}$:

$$T_n \doteq \mathcal{T}_n \prod_{j=1}^{K} \mathcal{T}_{n_j}^{(\varepsilon_j)}. \tag{2.114}$$

Let us pass now to consider the nonlinear partial difference equation

$$F\left[u_{\{n+i\}_{i=-\mathcal{N}^{(-)}}^{\mathcal{N}^{(+)}}, \{m+j\}_{j=-\mathcal{M}^{(-)}}^{\mathcal{M}^{(+)}}}\right] = 0, \qquad \mathcal{N}^{(\pm)},\ \mathcal{M}^{(\pm)} \ge 0, \tag{2.115}$$

for a function $u_{n,m} : \mathbb{Z}^2 \to \mathbb{R}$ which now depends on two indexes n and $m \in \mathbb{Z}$ which for future convenience we will term respectively as *space* and *time* variables. As indicated, in eq. (2.115) there appear some m and n-shifts from $m - \mathcal{M}^{(-)}$ up to $m + \mathcal{M}^{(+)}$ and from $n - \mathcal{N}^{(-)}$ up to $n + \mathcal{N}^{(+)}$. Let us suppose that

- The dependence of $u_{n,m}$ on n and m is realized through the K_n *slow-space* variables $n_i \doteq \varepsilon_{n_i} n \in \mathbb{R}$ and K_m *slow-time* variables $m_j \doteq \varepsilon_{m_j} m \in \mathbb{R}$ with ε_{n_i} and $\varepsilon_{m_j} \in \mathbb{R}$, $1 \le i \le K_n$, $1 \le j \le K_m$ besides a simultaneous dependence on the *fast-space* variable n and on the *fast-time* variable m;
- The slow-space and slow-time variables can vary in a (full measure) $K_n + K_m$-dimensional region so that $u\left(n, m, \{n_i\}_{i=1}^{K_n}, \{m_j\}_{j=1}^{K_m}\right)$ is analytical (Taylor series expandibility) and that region contains the point $\left(\{\widetilde{n}_i \doteq \varepsilon_{n_i}\widetilde{n}\}_{i=1}^{K_n},\ \{\widetilde{m}_j \doteq \varepsilon_{m_j}\widetilde{m}\}_{j=1}^{K_m}\right)$, $\widetilde{n}$, $\widetilde{m} \in \aleph$;
- The radius of convergence of the Taylor series starting at $(\widetilde{n}, \widetilde{m}) \doteq \Big(\widetilde{n}, \widetilde{m}, \{\widetilde{n}_i \doteq \varepsilon_{n_i}\widetilde{n}\}_{i=1}^{K_n}$, $\{\widetilde{m}_j \doteq \varepsilon_{m_j}\widetilde{m}\}_{j=1}^{K_m}\Big)$ is wide enough to include as *inner points* all the points of the form $(\widetilde{n}+\alpha, \widetilde{m}+\beta)$ with $-\mathcal{N}^{(-)} \le \alpha \le \mathcal{N}^{(+)}$ and $-\mathcal{M}^{(-)} \le \beta \le \mathcal{M}^{(+)}$ which effectively appear in the difference equation (2.115).

Under such hypotheses all the n and m shifts of the function $u_{n,m}$, which are involved in the difference equation (2.115), admit a series representation around the point $(\widetilde{n}, \widetilde{m})$. In the following will make the following choices:

$$\varepsilon_{n_i} \doteq N_i \varepsilon^i, \qquad 1 \le i \le K_n, \qquad \varepsilon_{m_j} \doteq M_j \varepsilon^j, \qquad 1 \le j \le K_m,$$

where the various constants N_i, M_j and ε are all real numbers and we will assume $K_n = 1$ and $K_m = K$ (eventually $K = +\infty$) so that

$$T_n = \mathcal{T}_n \mathcal{T}_{n_1}^{(\varepsilon_{n_1})} = \mathcal{T}_n \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_n^{(j)}, \qquad \mathcal{A}_n^{(j)} \doteq \frac{N_1^j}{j!} \partial_{n_1}^j, \tag{2.116a}$$

$$T_m = \mathcal{T}_m \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\varepsilon_{m_j})} = \mathcal{T}_m \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_m^{(j)}, \tag{2.116b}$$

$$T_n T_m = \mathcal{T}_n \mathcal{T}_m \mathcal{T}_{n_1}^{(\varepsilon_{n_1})} \prod_{j=1}^{K} \mathcal{T}_{m_j}^{(\varepsilon_{m_j})} = \mathcal{T}_n \mathcal{T}_m \sum_{j=0}^{+\infty} \varepsilon^j \mathcal{A}_{n,m}^{(j)}, \tag{2.116c}$$

where the operators $\mathcal{A}_n^{(j)}$, $\mathcal{A}_m^{(j)}$, $\mathcal{A}_{n,m}^{(j)}$ up to $j = 4$ are given in Table (2.1).

Finally, in complete analogy with eq. (2.20), we will assume for the function $u(n, m, n_1, \{m_j\}_{j=1}^{K}, \varepsilon)$ a double expansion in harmonics and in the perturbative parameter ε

$$u\left(n, m, n_1, \{m_j\}_{j=1}^{K}, \varepsilon\right) = \sum_{\gamma=1}^{+\infty} \sum_{\alpha=-\gamma}^{\gamma} \varepsilon^\gamma u_\gamma^{(\alpha)} \left(n_1, \{m_j\}_{j=1}^{K}\right) E_{n,m}^{\alpha}, \tag{2.117}$$
$$E_{n,m} \doteq e^{i[\kappa n - \omega(\kappa) m]}, \qquad u_\gamma^{(-\alpha)} = \bar{u}_\gamma^{(\alpha)}$$

where the index γ is chosen ≥ 1 in order to let any nonlinear part of eq. (2.115) to enter as a perturbation in the multiscale expansion. The expansions of the n and m-shifts of the function $u_{n,m}$ as well as of the nonlinear monomials present in eq. (2.115) will be given in *Appendix D*. It should be clear that a Taylor expansion in ε near $\varepsilon = 0$ should also be considered for every parameter present in eq. (2.115).

2.3.2 From derivatives to shifts

As our multiscale approach produces from a given partial difference equation a partial differential equation for one of the amplitudes $u_\gamma^{(\alpha)}$, one could wonder if it would be possible at least formally, starting from the obtained partial differential equation, to write down a partial difference equation inverting the expression

$$\mathcal{T}_{n_1} = e^{\partial_{n_1}} \doteq \sum_{i=0}^{+\infty} \frac{1}{i!} \partial_{n_1}^i,$$

where $\mathcal{T}_{n_1} u_{n,m} \doteq u\left(n, m, n_1 + 1, \{m_j\}_{j=1}^{K}\right)$, and similarly for $\mathcal{T}_m$. In fact one formally could write

$$\partial_{n_1} = ln \mathcal{T}_{n_1} = ln\left(1 + \Delta_{n_1}^{(+)}\right) \doteq \sum_{i=1}^{+\infty} \frac{(-1)^{i-1}}{i} \Delta_{n_1}^{(+)i}, \tag{2.118}$$

where $\Delta_{n_1}^{(+)} \doteq \mathcal{T}_{n_1} - 1$ is just the first *forward* difference operator with respect to the slow-variable n_1. Note that this is just one of the possible inversion formulae for the operator $\mathcal{T}_{n_1}$. For example it can also written in terms of the first *backward* difference operator $\Delta_{n_1}^{(-)} \doteq 1 - \mathcal{T}_{n_1}^{-1}$ as

$$\partial_{n_1} = -ln\mathcal{T}_{n_1}^{-1} = -ln\left(1 - \Delta_{n_1}^{(-)}\right) \doteq \sum_{i=1}^{+\infty} \frac{1}{i} \Delta_{n_1}^{(-)i}, \tag{2.119}$$

or in terms of the first *symmetric* difference operator $\Delta_{n_1}^{(s)} \doteq \left(\mathcal{T}_{n_1} - \mathcal{T}_{n_1}^{-1}\right)/2$ as

$$\partial_{n_1} = sinh^{-1} \Delta_{n_1}^{(s)} \doteq \sum_{i=1}^{+\infty} \frac{P_{i-1}(0)}{i} \Delta_{n_1}^{(s)i}, \tag{2.120}$$

where $P_i(0)$ is the i-th *Legendre* polynomial evaluated in $x = 0$. Now we have:

Definition 2.9 *A function u_n is a slow-varying function of order l with respect to the index n iff $\Delta_n^{l+1} u_n = 0$.*

Hence one can see that the ∂_{n_1} operator, which is given by formal series which in general contain infinite powers of the Δ_{n_1}, when acting on slow-varying functions of order l, reduces to polynomials in the Δ_{n_1} of order at most l. In [27], choosing $l = 2$ for the indexes n_1 and m_1 and $l = 1$ for m_2, it was shown that the integrable *lattice potential KdV* equation [31] reduces to a completely discrete and local nonlinear Schrödinger equation ($dNLS$) which has been proved to be not integrable by singularity confinement and algebraic entropy. Consequently, if one wants to pass from derivatives to shifts, one ends up in general with a *nonlocal* partial difference equation in the slow variables n_κ and m_δ. In particular, choosing

$$\varepsilon_{n_i} \doteq \frac{1}{\rho_i}, \qquad \rho_i \in \aleph, \qquad 1 \le i \le K_n,$$
$$\varepsilon_{m_j} \doteq \frac{1}{\theta_j}, \qquad \theta_j \in \aleph, \qquad 1 \le j \le K_m,$$

the slow-space variable n_κ will vary on a one dimensional lattice indexed by rational numbers with spacing $1/\rho_\kappa$ and the slow-time variable m_δ will vary on a one dimensional lattice indexed by rational numbers with spacing $1/\theta_\delta$. Hence, extracting from the previous lattice respectively the ρ_κ and θ_δ sublattices characterized by unit spacing, the obtained nonlocal partial difference equation describes the evolution of the amplitude $u_\gamma^{(\alpha)}$ on the $\rho_\kappa \cdot \theta_\delta$ two dimensional sublattices obtained from the composition of the previous one dimensional sublattices. In each of these sublattices, a unit shift in the direction n_κ will correspond to a ρ_κ units shift of the fast variable n and a unit shift in the direction m_δ will correspond to a θ_δ units shift of the fast variable m.

Table 2.1: *Operators* $\mathcal{A}_n^{(j)}$, $\mathcal{A}_m^{(j)}$ *and* $\mathcal{A}_{n,m}^{(j)}$ *in eqs.* (2.116)

	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$
$\mathcal{A}_n^{(j)}$	1	$N_1\partial_{n_1}$	$\frac{N_1^2}{2}\partial_{n_1}^2$	$\frac{N_1^3}{6}\partial_{n_1}^3$	$\frac{N_1^4}{24}\partial_{n_1}^4$
$\mathcal{A}_m^{(j)}$	1	$M_1\partial_{m_1}$	$\frac{M_1^2}{2}\partial_{m_1}^2 + M_2\partial_{m_2}$	$\frac{M_1^3}{6}\partial_{m_1}^3 + M_1M_2\partial_{m_1}\partial_{m_2} + M_3\partial_{m_3}$	$\frac{M_1^4}{24}\partial_{m_1}^4 + \frac{M_1^2M_2}{2}\partial_{m_1}^2\partial_{m_2} + \frac{M_2^2}{2}\partial_{m_2}^2 +$ $+M_1M_3\partial_{m_1}\partial_{m_3} + M_4\partial_{m_4}$
$\mathcal{A}_{n,m}^{(j)}$	1	$\mathcal{A}_n^{(1)} + \mathcal{A}_m^{(1)}$	$\mathcal{A}_n^{(2)} + \mathcal{A}_m^{(2)} +$ $+N_1M_1\partial_{n_1}\partial_{m_1}$	$\mathcal{A}_n^{(3)} + \mathcal{A}_m^{(3)} + N_1M_2\partial_{n_1}\partial_{m_2} +$ $+\frac{M_1N_1^2}{2}\partial_{n_1}^2\partial_{m_1} + \frac{N_1M_1^2}{2}\partial_{n_1}\partial_{m_1}^2$	$\mathcal{A}_n^{(4)} + \mathcal{A}_m^{(4)} +$ $+\frac{M_1^3N_1}{6}\partial_{m_1}^3\partial_{n_1} + \frac{N_1^3M_1}{6}\partial_{n_1}^3\partial_{m_1} +$ $+N_1M_1M_2\partial_{n_1}\partial_{m_1}\partial_{m_2} + \frac{N_1^2M_2}{2}\partial_{n_1}^2\partial_{m_2} +$ $+\frac{N_1^2M_1^2}{4}\partial_{n_1}^2\partial_{m_1}^2 + N_1M_3\partial_{n_1}\partial_{m_3}$

3

Multiscale reduction of discretizations of $S-$integrable nonlinear PDEs

We present here some examples of reduction of nonlinear partial difference equations which arises as discretizations of $S-$integrable partial differential equations. This $S-$integrability property or *spectral transform* integrability is the consequence of the fact that the considered nonlinear equations emerges as a compatibility condition of an overdetermined system of two *linear* equations, the so called *Lax pair* of the system. Some discretizations succeed in preserving that integrability property, in that some of the partial difference equations obtained have their own Lax pairs, while other discretizations fail. Usually the integrable discretizations are non trivial as shifts will appear also in the nonlinear terms. The aim of the following examples is to explain how the multiscale technique, developed together with all the integrability conditions in the last chapter, can be used as an integrability test, effectively proving if a particular discretization is not integrable. We will always start performing the multiscale reduction on the known integrable discretization and then we will reduce other discretizations whose integrability is not a priori known.

3.1 Multiscale reduction of the lattice potential KdV equation ($lpKdV$)

Let us consider the $lpKdV$ equation [34]

$$(p - q + u_{n,m+1} - u_{n+1,m}) (p + q + u_{n,m} - u_{n+1,m+1}) - \left(p^2 - q^2\right) = 0, \tag{3.1}$$

where p, q are two different real parameters. The above equation is probably the best known completely discrete integrable nonlinear equation which involves just four points which lay on a square. It represents the nonlinear superposition formula for the (continuous) KdV equation and emerges as the compatibility of the following pair of scalar linear partial difference equations [31]

$$\varphi_{n+2,m} = (2p - u_{n+2,m} + u_{n,m}) \varphi_{n+1,m} + \lambda \varphi_{n,m}, \tag{3.2a}$$
$$\varphi_{n,m+2} = (2q - u_{n,m+2} + u_{n,m}) \varphi_{n,m+1} + \left(\lambda + p^2 - q^2\right) \varphi_{n,m}, \tag{3.2b}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter. By defining $\mu \doteq p - q$ and $\zeta \doteq p + q$, eq. (3.1) can be rewritten as

$$\mu (u_{n+1,m+1} - u_{n,m}) + \zeta (u_{n+1,m} - u_{n,m+1}) = (u_{n+1,m} - u_{n,m+1}) (u_{n+1,m+1} - u_{n,m}). \tag{3.3}$$

The l.h.s. of eq. (3.3) represents the linear part, $\mathcal{P}_l$, of our equation while the r.h.s. represents the nonlinear part, $\mathcal{P}_{nl}$. If we assume for the solution $u_{n,m}$ the form given in eq. (2.117), the linear part takes the form

$$\mathcal{P}_l = \sum_{\gamma=1}^{+\infty} \varepsilon^\gamma \sum_{\alpha=-\gamma}^{\gamma} E^\alpha \sum_{j=max\{1,\,|\alpha|\}}^{\gamma} \mathcal{L}_j u_j^{(\alpha)}, \tag{3.4}$$

$$\mathcal{L}_j \doteq \mu\left(e^{\mathrm{i}\alpha(\kappa-\omega)}\mathcal{A}_{n,m}^{(\gamma-j)} - \delta_{j,\gamma}\right) + \zeta\left(e^{\mathrm{i}\alpha\kappa}\mathcal{A}_n^{(\gamma-j)} - e^{-\mathrm{i}\alpha\omega}\mathcal{A}_m^{(\gamma-j)}\right),$$

where the symbol $\delta_{j,\gamma}$ represents the Kronecker delta. $\mathcal{P}_{nl}$, is given in *Appendix D*, in particular eqs. (D.4b, D.4c) and their variants while the operators $\mathcal{A}_n^{(j)}$, $\mathcal{A}_m^{(j)}$, $\mathcal{A}_{n,m}^{(j)}$ up to $j = 4$ are given in Table (2.1). Let us proceed to the multiscale expansion [15] of eq. (3.3) in the same way as we did in the previous chapter for the KdV.

i. Order $\gamma = 1$.

- $\alpha = 0$:
  At this order the equation (3.3) is automatically satisfied;
- $\alpha = 1$:

$$\left[\mu\left(e^{\mathrm{i}(\kappa-\omega)} - 1\right) + \zeta\left(e^{\mathrm{i}\kappa} - e^{-\mathrm{i}\omega}\right)\right] u_1^{(1)} = 0. \tag{3.5}$$

If one wants $u_1^{(1)} \neq 0$, one obtains the *dispersion relation*

$$e^{-\mathrm{i}\omega} = \frac{\mu - \zeta e^{\mathrm{i}\kappa}}{\mu e^{\mathrm{i}\kappa} - \zeta}, \tag{3.6}$$

which, solved, gives

$$\omega(\kappa) = 2\arctan\left(\frac{\zeta+\mu}{\mu-\zeta}\tan\frac{\kappa}{2}\right); \tag{3.7}$$

ii. Order $\gamma = 2$.

- $\alpha = 1$:

$$\left[e^{\mathrm{i}\kappa}\left(\mu e^{-\mathrm{i}\omega} + \zeta\right) N_1 \partial_{n_1} + e^{-\mathrm{i}\omega}\left(\mu e^{\mathrm{i}\kappa} - \zeta\right) M_1 \partial_{m_1}\right] u_1^{(1)} = 0. \tag{3.8}$$

If we define

$$N_1 \doteq \epsilon S e^{-\mathrm{i}\omega}\left(\mu e^{\mathrm{i}\kappa} - \zeta\right), \qquad M_1 \doteq -S e^{\mathrm{i}\kappa}\left(\mu e^{-\mathrm{i}\omega} + \zeta\right), \qquad S \in \mathbb{C}, \qquad \epsilon = \pm 1, \tag{3.9}$$

taking into consideration eq. (3.6), eq. (3.8) becomes

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_1^{(1)} = 0, \tag{3.10}$$

which is solved by

$$u_1^{(1)}\left(n_1,\{m_j\}_{j=1}^K\right)=u_1^{(1)}\left(n_2,\{m_j\}_{j=2}^K\right),\qquad n_2\doteq n_1+\epsilon m_1\qquad \epsilon=-\frac{N_1}{M_1}\omega_1 \quad (3.11)$$

(the quantities ω_n are defined in eq. (2.12)). The complex constant $S\doteq re^{\mathrm{i}\theta}$, $r>0$, is to be choosen so that $\theta=-\arctan\left[\zeta\sin\kappa/\left(\zeta\cos\kappa-\mu\right)\right]$ in such a way that N_1 and M_1 are indeed real numbers, which, taking into account the dispersion relation (3.6), can be rewritten as

$$N_1=\epsilon S\left(\mu-\zeta e^{\mathrm{i}\kappa}\right),\qquad M_1=Se^{\mathrm{i}\kappa}\frac{\zeta^2-\mu^2}{\mu e^{\mathrm{i}\kappa}-\zeta}; \quad (3.12)$$

- $\alpha=0$:

$$\left[(\mu+\zeta)N_1\partial_{n_1}+(\mu-\zeta)M_1\partial_{m_1}\right]u_1^{(0)}=2\left(e^{-\mathrm{i}\omega}-e^{\mathrm{i}\kappa}+\mathcal{C.C.}\right)|u_1^{(1)}|^2. \quad (3.13)$$

Introducing the variable $\widetilde{n}_2\doteq n_1-\epsilon m_1$ and taking into account eq. (3.6), the last equation can be rewritten as

$$\begin{aligned}\left\{\left[\epsilon(\mu+\zeta)N_1+(\mu-\zeta)M_1\right]\partial_{n_2}+\left[\epsilon(\mu+\zeta)N_1-(\mu-\zeta)M_1\right]\partial_{\widetilde{n}_2}\right\}u_1^{(0)}&= \quad (3.14)\\ =-\frac{8\epsilon\mu\zeta\sin^2\kappa}{\mu^2+\zeta^2-2\mu\zeta\cos\kappa}|u_1^{(1)}|^2.&\end{aligned}$$

Acting on eq. (3.14) with $\partial_{\widetilde{n}_2}$, we obtain

$$\left\{\left[\epsilon(\mu+\zeta)N_1+(\mu-\zeta)M_1\right]\partial_{n_2}+\left[\epsilon(\mu+\zeta)N_1-(\mu-\zeta)M_1\right]\partial_{\widetilde{n}_2}\right\}\partial_{\widetilde{n}_2}u_1^{(0)}=0, \quad (3.15)$$

whose solution, taking into account eqs. (3.12), is

$$u_1^{(0)}=F\left(\sigma n_2-\widetilde{n}_2\right)+G\left(n_2\right),\qquad \sigma\doteq 1+\frac{(\mu-\zeta)^2}{\mu\zeta\left(1-\cos\kappa\right)}, \quad (3.16)$$

where F and G are arbitrary functions of their arguments (we have omitted their explicit dependence on the other slow-times). Inserting eq. (3.16) into eq. (3.14), we have that

$$\partial_{n_2}G=\alpha_1|u_1^{(1)}|^2,\qquad \alpha_1\doteq-\frac{4\left(1+\cos\kappa\right)}{N_1\left(\mu+\zeta\right)}. \quad (3.17)$$

We will choose $F\left(\sigma n_2-\widetilde{n}_2\right)=0$ in eq. (3.16), so that $u_1^{(0)}$ will be a function just of n_2 and

$$\partial_{n_2}u_1^{(0)}=\alpha_1|u_1^{(1)}|^2; \quad (3.18)$$

- $\alpha = 2$:

  Using the dispersion relation (3.6), we find

$$u_2^{(2)} = \alpha_2 u_1^{(1)2}, \qquad \alpha_2 \doteq \frac{1 + e^{\mathrm{i}\kappa}}{\left(1 - e^{\mathrm{i}\kappa}\right)(\mu + \zeta)}; \tag{3.19}$$

iii. Order $\gamma = 3$.

- $\alpha = 1$:

  Taking into account eqs. (3.6, 3.12, 3.18, 3.19) and the fact that both $u_1^{(0)}$ and $u_1^{(1)}$ depend on n_2, we have

$$(\partial_{n_1} - \epsilon \partial_{m_1}) u_2^{(1)} = \mathcal{N}_1 \left(u_1^{(1)}\right),$$

  where $\mathcal{N}_1 \left(u_1^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, it is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$(\partial_{n_1} - \epsilon \partial_{m_1}) u_2^{(1)} = 0, \tag{3.20a}$$

$$\mathrm{i}\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)} |u_1^{(1)}|^2, \qquad -\frac{2\rho_2}{\rho_1} = \alpha_1^2, \tag{3.20b}$$

$$\rho_1 \doteq \frac{\mu \zeta M_1^2 \sin \kappa}{M_2 \left(\mu^2 - \zeta^2\right)} = -\frac{N_1^2}{M_2} \omega_2, \tag{3.20c}$$

$$\rho_2 \doteq \frac{8 \zeta \mu (\zeta - \mu)(1 + \cos \kappa)^2 \sin \kappa}{M_2 (\mu + \zeta)\left(\zeta^2 + \mu^2 - 2\mu\zeta \cos \kappa\right)^2}.$$

  Equation (3.20a) tells us that $u_2^{(1)}$ depends on n_2.

- *Equation (3.20b), whose coefficients are defined in (3.20c), is an integrable (continuous, defocusing) nonlinear Schrödinger equation, its integrability arising from the manifest reality of its coefficients. This proves the* A_1 *asymptotic integrability of the lpKdV equation.*

  From the above *NLS* equation one derives the *continuity equation*

$$\partial_{m_2} d^{(1)} = \rho_1 \partial_{n_2} J_2^{(1)}, \qquad d^{(1)} \doteq |u_1^{(1)}|^2, \qquad J_2^{(1)} \doteq -\mathrm{i}\left(\bar{u}_1^{(1)} \partial_{n_2} u_1^{(1)} - \mathcal{C.C.}\right), \tag{3.21}$$

  where we used the symbols $d^{(1)}$ and $J_2^{(1)}$ to indicate that those quantities represent respectively a density of a conserved quantity and a current density. Differentiating by m_2 eq. (3.18), using the continuity equation (3.21) and integrating with respect to n_2 taking equal to zero the arbitrary n_2-independent integration function (all the $u_n^{(\alpha)}$s go to zero as $n_2 \to \pm\infty$), we have the evolution

$$\partial_{m_2} u_1^{(0)} = \alpha_1 \rho_1 J_2^1, \tag{3.22}$$

to be used later;

- $\alpha = 0$:
Taking into consideration eqs. (3.6, 3.12, 3.18, 3.22) and the fact that $u_1^{(0)}$ and $u_1^{(1)}$ depend on n_2 and choosing $u_2^{(0)}$ dependent on n_2 too, we obtain

$$\partial_{n_2} u_2^{(0)} = d^{(2)}, \qquad d^{(2)} \doteq \alpha_1 \left(u_1^{(1)} \bar{u}_2^{(1)} + \mathcal{C.C.} \right) + \rho_3 J_2^{(1)}, \qquad \rho_3 \doteq \frac{2\sin(\kappa)}{(\mu+\zeta)}, \tag{3.23}$$

where we used the notation $d^{(2)}$ to indicate that this expression represents another density of a conserved quantity;

- $\alpha = 2$:
Taking into consideration eqs. (3.6, 3.12, 3.19) and the fact that both $u_1^{(1)}$ and $u_2^{(2)}$ depend on n_2, we have

$$u_3^{(2)} = u_1^{(1)} \left[\alpha_3 \partial_{n_2} u_1^{(1)} + 2\alpha_2 u_2^{(1)} \right], \qquad \alpha_3 \doteq \frac{2\epsilon S e^{i\kappa} \left(\mu - \zeta e^{i\kappa} \right)}{\left(e^{i\kappa} - 1 \right)^2 (\mu+\zeta)} = \frac{2iN_1 \alpha_2}{(\mu+\zeta)\rho_3}; \tag{3.24}$$

- $\alpha = 3$:
Using eqs. (3.6, 3.19), we obtain

$$u_3^{(3)} = \alpha_2^2 u_1^{(1)3}; \tag{3.25}$$

iv. Order $\gamma = 4$.

- $\alpha = 1$:
Taking into account eqs. (3.6, 3.12, 3.18, 3.19, 3.20b, 3.22, 3.23, 3.24), that $u_1^{(0)}$, $u_2^{(0)}$, $u_1^{(1)}$, $u_2^{(1)}$ and $u_2^{(2)}$ depend on n_2 and that (see *Section 2.2* and *Subsection 2.2.1*) the amplitude $u_1^{(1)}$ evolves at the slow-time m_3 according to the *complex modified KdV equation (cmKdV)*

$$\partial_{m_3} u_1^{(1)} - B \left(\partial_{n_2}^3 u_1^{(1)} + \frac{3\rho_2}{\rho_1} |u_1^{(1)}|^2 \partial_{n_2} u_1^{(1)} \right) = 0, \tag{3.26}$$

we have

$$\left(\partial_{n_1} - \epsilon \partial_{m_1} \right) u_3^{(1)} = \mathcal{N}_2 \left(u_1^{(1)}, u_2^{(1)} \right).$$

$\mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate and linear in $u_2^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, it is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$(\partial_{n_1} - \epsilon\partial_{m_1})\, u_3^{(1)} = 0, \tag{3.27a}$$

$$\partial_{m_2} u_2^{(1)} - K_2'\left[u_1^{(1)}\right] u_2^{(1)} = \mathcal{N}_3\left(u_1^{(1)}\right), \tag{3.27b}$$

$$K_2'\left[u_1^{(1)}\right] u_2^{(1)} \doteq -\mathrm{i}\rho_1\left[\partial_{n_2}^2 u_2^{(1)} + \frac{\rho_2}{\rho_1}\left(u_1^{(1)2}\bar{u}_2^{(1)} + 2|u_1^{(1)}|^2 u_2^{(1)}\right)\right].$$

The first relation tells us that $u_3^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right) = 0$, $\mathcal{N}_3\left(u_1^{(1)}\right)$ is another nonlinear function involving $u_1^{(1)}$ and its complex conjugate only and $K_2'\left[u_1^{(1)}\right] u_2^{(1)}$ is the Frechet derivative of the *NLS* flux $K_2\left[u_1^{(1)}\right]$ (see *Section 2.2.1*). Now the term $\mathcal{N}_3\left(u_1^{(1)}\right)$ depends from the free real constant B. Choosing the coefficient B so as to eliminate any dependence of the resulting equation on $\partial_{n_2}^3 u_1^{(1)}$, we obtain

$$\partial_{m_2} u_2^{(1)} - K_2'\left[u_1^{(1)}\right] u_2^{(1)} = b u_1^{(1)2}\partial_{n_2}\bar{u}_1^{(1)} + a|u_1^{(1)}|^2\partial_{n_2} u_1^{(1)}, \tag{3.28a}$$

$$a \doteq -N_1\rho_2\cot\kappa, \qquad b \doteq a\frac{2-\cos\kappa}{\cos\kappa}, \qquad b = a - 2\rho_1\rho_3\alpha_1 \tag{3.28b}$$

$$B = \frac{\epsilon\mu\zeta M_1^3}{3M_3\left(\zeta^2-\mu^2\right)^2}\left[\left(\mu^2+\zeta^2+2\mu\zeta\cos\kappa\right)\cos\kappa - 4\mu\zeta\right] = \tag{3.28c}$$

$$= \frac{N_1^3}{M_3}\omega_3.$$

The elimination of any term of the form $\partial_{n_2}^3 u_1^{(1)}$ from the r. h. s. of eq. (3.28a) is justified from following proposition:

Proposition 3.1 *If a function $q(x, t_r, t_s)$ evolves according to the equation $\partial_{t_r} q - K_r[q] = 0$ and if $K_s[q]$ is such that $[K_r, K_s]_L = 0$ (cfr. eq. (2.72)), then the term $\partial_{t_s} q - K_s[q]$ is secular for the equation $(\partial_{t_r} - K_r'[q])\,\phi(x, t_r) = f_r(x, t_r, q, q_x, \ldots)$, where $f_r(x, t_r, q, q_x, \ldots)$ is a generic forcing term and $\phi(x, t_r)$ a generic function of its arguments.*

Proof: It is sufficient to show that $\partial_{t_s} q - K_s[q]$ solves the homogeneous equation. In fact we have: $(\partial_{t_r} - K_r'[q])(\partial_{t_s} q - K_s[q]) = \partial_{t_r}(\partial_{t_s} q - K_s[q]) - K_r'[q](\partial_{t_s} q - K_s[q]) = \partial_{t_s}\partial_{t_r} q - \widehat{\partial}_{t_r} K_s[q] - K_s'[q]\,\partial_{t_r} q - K_r'[q]\,\partial_{t_s} q + K_r'[q]\,K_s = \partial_{t_s}\partial_{t_r} q - \widehat{\partial}_{t_r} K_s[q] - K_s'[q]\,K_r - \partial_{t_s} K_r[q] + \widehat{\partial}_{t_s} K_r[q] + K_r'[q]\,K_s = \partial_{t_s}(\partial_{t_r} q - K_r[q]) - [K_r, K_s]_L = 0.$ *Q. E. D.*

The notation $\widehat{\partial}_{t_j}$ indicates differentiation with respect to a possible *explicit* dependence of the various $K_m[q]$ on the slow-times. From the proof is clear that, when the various $K_m[q]$ don't exibit an explicit dependence on the slow-times, the two terms $\partial_{t_s} q$ and $K_s(q)$ are indeed *separately secular*[1]. If the r. h. s. of eq. (3.28a) contains a term of

[1]For completeness we give also the following proposition:

the form $\partial^3_{n_2} u_1^{(1)}$, it is always possible to evidence in it a term of the form $K_3\left[u_1^{(1)}\right]$, the flux of the *cmKdV* equation (3.26), which, from the above proposition is secular.

- *The coefficients of the r. h. s. of equation (3.28a) given by the relations (3.28b), obviously satisfy both the two A_2 integrability conditions (2.80). This proves the A_2 asymptotic integrability of the lpKdV equation.*

From equations (3.26) and (3.28a, 3.20b) we get respectively the continuity equations

$$\partial_{m_3} d^{(1)} = B \partial_{n_2} J_3^{(1)}, \tag{3.29a}$$

$$J_3^{(1)} \doteq \left(\frac{3\rho_2}{2\rho_1} |u_1^{(1)}|^4 + u_1^{(1)} \partial^2_{n_2} \bar{u}_1^{(1)} - |\partial_{n_2} u_1^{(1)}|^2 + \bar{u}_1^{(1)} \partial^2_{n_2} u_1^{(1)} \right), \tag{3.29b}$$

$$\partial_{m_2} d^{(2)} = \partial_{n_2} J_2^{(2)}, \tag{3.29c}$$

$$\begin{aligned} J_2^{(2)} \doteq & -\rho_1\rho_3 \left(\bar{u}_1^{(1)} \partial^2_{n_2} u_1^{(1)} - 2|\partial_{n_2} u_1^{(1)}|^2 + u_1^{(1)} \partial^2_{n_2} \bar{u}_1^{(1)} \right) + \\ & + \left[(a+b) \frac{\alpha_1}{2} - \rho_2\rho_3 \right] |u_1^{(1)}|^4 + i\rho_1\alpha_1 \left(u_1^{(1)} \partial_{n_2} \bar{u}_2^{(1)} + \right. \\ & \left. + u_2^{(1)} \partial_{n_2} \bar{u}_1^{(1)} - \mathcal{C.C.} \right), \end{aligned} \tag{3.29d}$$

which, combined respectively with (3.18) and (3.23), give the relations (as usual the arbitrary n_2-independent integration functions have been set to zero to match the asymptotic conditions on the $u_n^{(\alpha)}$)

$$\partial_{m_3} u_1^{(0)} = \alpha_1 B J_3^{(1)}, \qquad \partial_{m_2} u_2^{(0)} = J_2^{(2)}, \tag{3.30}$$

that will prove to be essential in the prosecution of the expansion. As before with $J_3^{(1)}$ and $J_2^{(2)}$ we have indicated the current densities related respectively to the densities $d^{(1)}$ and $d^{(2)}$;

- $\alpha = 0$:
Taking into account eqs. (3.6, 3.12, 3.18, 3.19, 3.20b, 3.22, 3.23, 3.30) and that $u_1^{(0)}$, $u_2^{(0)}$, $u_1^{(1)}$, $u_2^{(1)}$ depend on n_2 and choosing $u_3^{(0)}$ dependent on n_2 too, we have

$$\partial_{n_2} u_3^{(0)} = d^{(3)}, \tag{3.31}$$

$$\begin{aligned} d^{(3)} \doteq & -\mathrm{i}\rho_3 \left(\bar{u}_2^{(1)} \partial_{n_2} u_1^{(1)} + \bar{u}_1^{(1)} \partial_{n_2} u_2^{(1)} - \mathcal{C.C.} \right) + \\ & + \alpha_1 \left(u_1^{(1)} \bar{u}_3^{(1)} + \bar{u}_1^{(1)} u_3^{(1)} + |u_2^{(1)}|^2 \right) + \frac{3M_1}{2(\alpha-\beta)} \alpha_1^2 |u_1^{(1)}|^4 + \\ & + \frac{M_1^2}{12} \alpha_1 \left(\bar{u}_1^{(1)} \partial^2_{n_2} u_1^{(1)} + u_1^{(1)} \partial^2_{n_2} \bar{u}_1^{(1)} + \frac{4\sin^2(\kappa/2) - 1}{\cos^2(\kappa/2)} |\partial_{n_2} u_1^{(1)}|^2 \right); \end{aligned}$$

Proposition 3.2 *If a function $q(x, t_r, t_s)$ evolves according to the equation $M_r q = f_r$ and the two differential equations $M_r q = f_r$, $M_s q = f_s$ where $[K_r, K_s]_L = 0$ represent compatible evolutions, the term $M_s q - f_s$ is secular for the equation $M_r \phi(x, t_r) = g_r(x, t_r, q, q_x, \ldots)$, where $g_r(x, t_r, q, q_x, \ldots)$ is a generic forcing term and $\phi(x, t_r)$ a generic function of its arguments.*

Proof: It is sufficient to show that $M_s q - f_s$ solves the homogeneous equation. In fact we have: $M_r[M_s q - f_s] = M_r[M_s q] - M_r f_s = M_s[M_r q] - M_r f_s = M_s f_r - M_r f_s = 0.$ *Q. E. D.*

- $\alpha = 2$:

  Taking into account eqs. (3.6, 3.12, 3.18, 3.19, 3.20b, 3.24, 3.25) and that $u_1^{(0)}$, $u_1^{(1)}$, $u_2^{(1)}$, $u_2^{(2)}$ and $u_3^{(2)}$ depend on n_2, we have

$$u_4^{(2)} = 2\alpha_2^3 \left(1 + 4\sin^2(\kappa/2)\right) |u_1^{(1)}|^2 u_1^{(1)2} + \frac{\alpha_3^2}{2\alpha_2} \left[\left(\partial_{n_2} u_1^{(1)}\right)^2 + u_1^{(1)} \partial_{n_2}^2 u_1^{(1)} \cos\kappa \right] + \\ +\alpha_2 \left(2u_1^{(1)} u_3^{(1)} + u_2^{(1)2}\right) + \alpha_3 \partial_{n_2} \left(u_2^{(1)} u_1^{(1)}\right); \qquad (3.32)$$

- $\alpha = 3$:

  Taking into account eqs. (3.6, 3.12, 3.19, 3.24, 3.25) and that $u_1^{(1)}$, $u_2^{(2)}$ and $u_3^{(3)}$ depend on n_2, we have

$$u_4^{(3)} = \alpha_2 \left[3\alpha_2 u_2^{(1)} + 2\alpha_3 \left(\partial_{n_2} u_1^{(1)}\right)\right] u_1^{(1)2}; \qquad (3.33)$$

- $\alpha = 4$:

  Taking into account eqs. (3.6, 3.19, 3.25) we obtain

$$u_4^{(4)} = \alpha_2^3 u_1^{(1)4}; \qquad (3.34)$$

v. Order $\gamma = 5$.

- $\alpha = 1$:

  Taking into account eqs. (3.6, 3.12, 3.18, 3.19, 3.20b, 3.22, 3.23, 3.24, 3.25, 3.26, 3.28a, 3.28b, 3.28c, 3.29b, 3.29d, 3.30, 3.31, 3.32), that $u_1^{(0)}$, $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$, $u_2^{(1)}$, $u_3^{(1)}$, $u_2^{(2)}$, $u_3^{(2)}$ depend on n_2 and that (see *Section 2.2* and *Subsection 2.2.1*),

$$\partial_{m_3} u_2^{(1)} - K_3' \left[u_1^{(1)}\right] u_2^{(1)} = f_3(2), \qquad (3.35)$$

$$\partial_{m_4} u_1^{(1)} + \mathrm{i}C \left\{ \partial_{n_2}^4 u_1^{(1)} + \frac{\rho_2}{\rho_1} \left[\frac{3\rho_2}{2\rho_1} |u_1^{(1)}|^4 u_1^{(1)} + 4|u_1^{(1)}|^2 \partial_{n_2}^2 u_1^{(1)} + \right.\right. \qquad (3.36) \\ \left.\left. +3\bar{u}_1^{(1)} \left(\partial_{n_2} u_1^{(1)}\right)^2 + 2|\partial_{n_2} u_1^{(1)}|^2 u_1^{(1)} + u_1^{(1)2} \partial_{n_2}^2 \bar{u}_1^{(1)} \right] \right\} = 0,$$

  where $K_3'[u]\,v$ is given in eq. (2.76c) and $f_3(2)$ in eqs. (2.79b, 2.81), we obtain

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_4^{(1)} = \mathcal{N}_4 \left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right).$$

  $\mathcal{N}_4 \left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right)$ is a function linear in $u_3^{(1)}$ and its complex conjugate and nonlinear in $u_1^{(1)}$ and $u_2^{(1)}$ and their complex conjugates. As the r.h.s. of the last equation depends on n_2, it is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$(\partial_{n_1} - \epsilon\partial_{m_1})\, u_4^{(1)} = 0, \tag{3.37a}$$
$$\partial_{m_2} u_3^{(1)} - K_2^{'}\left[u_1^{(1)}\right] u_3^{(1)} = \mathcal{N}_5\left(u_1^{(1)}, u_2^{(1)}\right). \tag{3.37b}$$

The first relation tells us that $u_4^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_4\left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right) = 0$, $\mathcal{N}_5\left(u_1^{(1)}, u_2^{(1)}\right)$ is another nonlinear function involving $u_1^{(1)}$, $u_2^{(1)}$ and their complex conjugates. Now the term $\mathcal{N}_5\left(u_1^{(1)}, u_2^{(1)}\right)$ contains the free real constant C which is chosen so as to eliminate any dependence of the resulting equation on $\partial_{n_2}^4 u_1^{(1)}$. From *Proposition 3.1*, the presence of this term can always introduce a dependence on the secular term $K_4\left[u_1^{(1)}\right]$, the flux of the equation (3.36). So we obtain

$$\partial_{m_2} u_3^{(1)} - K_2^{'}\left[u_1^{(1)}\right] u_3^{(1)} = f_2(3), \tag{3.38a}$$
$$C = \frac{\mu\zeta M_1^4\left[\mu^4 - 20\mu^2\zeta^2 + \zeta^4 + 8\mu\zeta\left(\mu^2+\zeta^2\right)\cos\kappa + 2\mu^2\zeta^2\cos(2\kappa)\right]}{12 M_4\left(\mu^2-\zeta^2\right)^3}\cdot$$
$$\cdot \sin\kappa = \frac{N_1^4}{M_4}\omega_4, \tag{3.38b}$$

where the forcing term $f_2(3)$ is given by eq. (2.82a).

- *The term $f_2(3)$ appearing in equation (3.38a) obviously has all its coefficients that satisfy all the fifteen A_3 integrability conditions (2.83). This proves the A_3 asymptotic integrability of the lpKdV equation.*

Due to the fact that the twelve complex coefficients of $f_2(3)$ respect the A_3 integrability conditions, they can all be generated giving a convenient nine-dimensional real basis. We choose τ_1, τ_2, I_3, I_5, R_6 and τ_{12} (see eq. (2.82a)) and report in the following their expressions:

$$\tau_1 = -\mathrm{i}\rho_2\frac{\left(-23 + 16\cos\kappa + \cos(2\kappa)\right)\cot^2(\kappa/2)}{2\left(\mu+\zeta\right)^2},$$
$$\tau_2 = -\mathrm{i}N_1^2\rho_2\frac{\left(29 - 24\cos\kappa + 7\cos(2\kappa)\right)}{12\sin^2\kappa}, \qquad I_3 = -\frac{1}{6}N_1^2\left(1 + 3\csc^2\kappa\right)\rho_2,$$
$$I_5 = -\frac{1}{4}N_1^2\left(1+2\csc^2\kappa\right)\rho_2, \qquad R_6 = \tau_{12} = a. \tag{3.39}$$

From equations (3.36), (3.35, 3.26) and (3.38a, 3.28a, 3.20b) we get the continuity equations

$$\partial_{m_4} d^{(1)} = C\partial_{n_2} J_4^{(1)}, \tag{3.40a}$$

$$J_4^{(1)} \doteq -i\left[\frac{3\rho_2}{\rho_1}\bar{u}_1^{(1)}|u_1^{(1)}|^2\partial_{n_2}u_1^{(1)} + \left(\partial_{n_2}u_1^{(1)}\right)\partial_{n_2}^2\bar{u}_1^{(1)} + \bar{u}_1^{(1)}\partial_{n_2}^3 u_1^{(1)} - \mathcal{C.C.}\right], \tag{3.40b}$$

$$\partial_{m_3} d^{(2)} = B\partial_{n_2} J_3^{(2)}, \tag{3.40c}$$

$$\begin{aligned} J_3^{(2)} \doteq& \frac{3}{\rho_1}\left[\alpha_1\rho_2\left(\bar{u}_1^{(1)}u_2^{(1)} + \mathcal{C.C.}\right) - \left(\frac{a\alpha_1}{2} - \rho_2\rho_3\right)J_2^{(1)}\right]|u_1^{(1)}|^2 + \\ &+\alpha_1\left[\bar{u}_2^{(1)}\partial_{n_2}^2 u_1^{(1)} + \bar{u}_1^{(1)}\partial_{n_2}^2 u_2^{(1)} - \left(\partial_{n_2}\bar{u}_1^{(1)}\right)\partial_{n_2}u_2^{(1)} + \mathcal{C.C.}\right] + \\ &+i\rho_3\left[u_1^{(1)}\partial_{n_2}^3\bar{u}_1^{(1)} + 2\left(\partial_{n_2}\bar{u}_1^{(1)}\right)\partial_{n_2}^2 u_1^{(1)} - \mathcal{C.C.}\right], \end{aligned} \tag{3.40d}$$

$$\partial_{m_2} d^{(3)} = \partial_{n_2} J_2^{(3)}, \tag{3.40e}$$

$$\begin{aligned} J_2^{(3)} \doteq& \mathrm{i}\alpha_1\rho_1\left(u_1^{(1)}\partial_{n_2}\bar{u}_3^{(1)} + u_3^{(1)}\partial_{n_2}\bar{u}_1^{(1)} + u_2^{(1)}\partial_{n_2}\bar{u}_2^{(1)} - \mathcal{C.C.}\right) + \\ &+\rho_1\rho_3\left[2\left(\partial_{n_2}\bar{u}_1^{(1)}\right)\partial_{n_2}u_2^{(1)} - u_2^{(1)}\partial_{n_2}^2\bar{u}_1^{(1)} - \bar{u}_1^{(1)}\partial_{n_2}^2 u_2^{(1)} + \mathcal{C.C.}\right] + \\ &+\frac{\mathrm{i}\rho_3}{3\alpha_1^2}\left(\alpha_1\rho_1\rho_3 - 2a\right)\left(u_1^{(1)}\partial_{n_2}^3\bar{u}_1^{(1)} - \mathcal{C.C.}\right) + \\ &+\frac{\mathrm{i}\rho_3}{3\alpha_1^2}\left(\alpha_1\rho_1\rho_3 + 4a\right)\left[\left(\partial_{n_2}^2\bar{u}_1^{(1)}\right)\partial_{n_2}u_1^{(1)} - \mathcal{C.C.}\right] - \\ &-\alpha_1\left(\alpha_1\rho_1\rho_3 - 2a\right)\left(\bar{u}_1^{(1)}u_2^{(1)} + \mathcal{C.C.}\right)|u_1^{(1)}|^2 - \\ &-\left(\frac{a^2}{\alpha_1\rho_1} - 6a\rho_3 + \frac{7}{2}\alpha_1\rho_1\rho_3^2\right)|u_1^{(1)}|^2 J_2^{(1)}, \end{aligned} \tag{3.40f}$$

which, combined respectively with (3.18), (3.23) and (3.31), give the relations (as usual the arbitrary n_2-independent integration functions have been set to zero to match the asymptotic conditions on the $u_n^{(\alpha)}$s)

$$\partial_{m_4} u_1^{(0)} = \alpha_1 C J_4^{(1)}, \qquad \partial_{m_3} u_2^{(0)} = B J_3^{(2)}, \qquad \partial_{m_2} u_3^{(0)} = J_2^{(3)}. \tag{3.41}$$

In other words $d^{(1)}$ is a density of a quantity conserved by the entire NLS hierarchy (2.71a), $d^{(2)}$ is a density of a quantity conserved by the entire hierarchy of systems of two differential equations (2.71) obtained when $j = 2$ as n varies and $d^{(3)}$ is a density of a quantity conserved by the entire hierarchy of systems of three differential equations (2.71) obtained when $j = 2$, 3 as n varies.

We will stop here the multiscale analysis of the $lpKdV$ equation and we will pass to the multiscale analysis of the spectral problem (3.2) of the $lpKdV$ equation.

3.2 Multiscale reduction of the lattice potential KdV spectral problem

In this section, taking into account ref. [17] and following the analogous calculation for differential equations as presented in [40], we want to show how a proper multiscale expansion of the $lpKdV$ spectral problem produces the spectral problem of the NLS equation (3.20b). First of all we will perform on the spectral problem (3.2) a gauge transformation to reduce it into a more convenient form. Transforming the wave function $\varphi_{n,m}$ according to the rule

$$\varphi_{n,m} \doteq h_n(\lambda)\, g_{n,m}(\{u_{n,m}\})\, \psi_{n,m},$$

where by the notation $\{u_{n,m}\}$ we indicate indicate a dependence on $u_{n,m}$ and its shifted values, it is possible to tranform eq. (3.2a) into

$$\psi_{n-1,m} + a_{n,m}(\{u_{n,m}\})\, \psi_{n+1,m} = \delta(\lambda)\, \psi_{n,m}. \tag{3.42}$$

In fact, it it sufficient to take the function $g_{n,m}$ satisfying the linear difference equation

$$g_{n,m} = \frac{(u_{n+2,m} - u_{n,m} - 2p)}{\theta} g_{n+1,m}, \tag{3.43}$$

where θ is an arbitrary complex constant. The solution of the previous linear equation is given by

$$g_{n,m} = g_{n_0,m} \prod_{\beta=n_0}^{n-1} w_{\beta,m}, \qquad n \geq n_0 + 1, \tag{3.44a}$$

$$g_{n,m} = g_{n_0,m} \prod_{\beta=n}^{n_0-1} w_{\beta,m}^{-1}, \qquad n \leq n_0 - 1, \tag{3.44b}$$

$$w_{n,m} \doteq \frac{\theta}{(u_{n+2,m} - u_{n,m} - 2p)},$$

where $g_{n_0,m}$ is the arbitrary $m-$dependent initial condition at $n = n_0$. If for the function $g_{n,m}$ an asymptotic behavior is prescribed, the previous solution has to be changed. In fact, as $u_{n,m}$ tends to a constant as $n \to \pm\infty$, as $n \to \pm\infty$ eq. (3.43) reduces to

$$g_{n,m} = -\frac{2p}{\theta} g_{n+1,m},$$

whose solution is

$$g_{n,m} = \left(-\frac{\theta}{2p}\right)^n.$$

Hence setting $g_{n,m} \doteq (-\theta/2p)^n\, \widetilde{g}_{n,m}$ with $\widetilde{g}_{n,m} \to 1$ as $n \to \pm\infty$, $\widetilde{g}_{n,m}$, now satisfies with an $n-$independent asymptotic condition an equation similar to (3.43) but with θ replaced by $-2p$. Now one can take the expression (3.44a) with $n_0 \to -\infty$ and $g_{n_0,m} = 1$. Finally we have

$$g_{n,m} = \left(-\frac{\theta}{2p}\right)^n \prod_{\beta=-\infty}^{n-1} \widetilde{w}_{\beta}, \tag{3.45}$$

$$\widetilde{w}_{n,m} \doteq -\frac{2p}{(u_{n+2,m} - u_{n,m} - 2p)}.$$

A natural choice would be to set $\theta = -2p$. Moreover the function $h_n(\lambda)$ is defined by

$$h_n(\lambda) \doteq \pi(\lambda)(-1)^n(-\lambda)^{n/2},$$

with $\pi(\lambda)$ an arbitrary function of λ. Hence the new wave function $\psi_{n,m}$ satisfies eq. (3.42) where

$$a_{n,m} = \frac{\theta^2}{(u_{n+2} - u_{n,m} - 2p)(u_{n+1,m} - u_{n-1,m} - 2p)}, \qquad \delta = \frac{\theta}{(-\lambda)^{1/2}}. \tag{3.46}$$

At the same time eq. (3.2b) transforms into

$$\psi_{n,m+2} = \frac{(2q - u_{n,m+2} + u_{n,m})\, g_{n,m+1}}{g_{n,m+2}}\psi_{n,m+1} + \frac{\left(\lambda + p^2 - q^2\right) g_{n,m}}{g_{n,m+2}}\psi_{n,m}.$$

Let us do the multiscale expansion of eq. (3.42). We expand the field $u_{n,m}$ according to the formula (2.117) while the wave function $\psi_{n,m}$ and the spectral parameter δ are given by

$$\psi(n, m, n_1, \{m_j\}_{j=1}^K, \varepsilon) = \sum_{\gamma=0}^{+\infty} \varepsilon^\gamma \sum_{\alpha_{odd}=-(2\gamma+1)}^{2\gamma+1} \psi_\gamma^{(\alpha)}\left(n_2, \{m_j\}_{j=2}^K\right) E_{n,m}^{\alpha/2}, \tag{3.47a}$$

$$\psi_\gamma^{(-\alpha)} = \bar{\psi}_\gamma^{(\alpha)}, \qquad \delta(\varepsilon) = \sum_{\gamma=0}^{+\infty} \varepsilon^\gamma \delta_\gamma. \tag{3.47b}$$

i. Order $\gamma = 0$.

- $\alpha = 1$:

$$\theta = \pm 2p, \qquad \delta_0 = 2\cos(\kappa/2); \tag{3.48}$$

ii. Order $\gamma = 1$.

- $\alpha = \pm 1$:

$$N_1 \partial_{n_2}\psi_0^{(1)} + \frac{2u_1^{(1)}}{p}\cos^2(\kappa/2)\,\bar{\psi}_0^{(1)} = -\frac{\mathrm{i}\delta_1}{2\sin(\kappa/2)}\psi_0^{(1)}, \tag{3.49a}$$

$$N_1 \partial_{n_2}\bar{\psi}_0^{(1)} + \frac{2\bar{u}_1^{(1)}}{p}\cos^2(\kappa/2)\,\psi_0^{(1)} = \frac{\mathrm{i}\delta_1}{2\sin(\kappa/2)}\bar{\psi}_0^{(1)}; \tag{3.49b}$$

- $\alpha = 3$:

$$\psi_1^{(3)} = \frac{e^{\mathrm{i}\kappa} + 1}{2p\left(e^{-\mathrm{i}\kappa} - 1\right)} u_1^{(1)} \psi_0^{(1)}. \tag{3.50}$$

- *Eqs. (3.49) represent the space part of the Zakharov-Shabat spectral problem of the NLS equation (3.20b) for the function* $u_1^{(1)}$.

In the next section we will perform the multiscale analysis on a different discretization of the KdV equation.

3.3 Multiscale analysis of the off-centric discretization of the KdV equation

Let us consider the trivial discretization of the KdV equation

$$u_{n,m+1} - u_{n,m-1} = \frac{\alpha}{4}\left(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}\right) - \beta\left(u_{n+1,m}^2 - u_{n,m}^2\right), \quad (3.51)$$

which presents an off-centric discretization of the nonlinear part, and perform on it a similar multiscale analysis as in the case of the $lpKdV$ equation.

i. Order $\gamma = 1$.

- $\alpha = 0$:
  At this order eq. (3.51) is automatically satisfied;
- $\alpha = 1$:
  At this order we find the dispersion relation

$$\sin\omega = \alpha \sin^3\kappa. \quad (3.52)$$

In the following we will use the dispersion relation by expressing α in terms of κ and ω;

ii. Order $\gamma = 2$.

- $\alpha = 0$:

$$\partial_{m_1} u_1^{(0)} = 0, \quad (3.53)$$

so that $u_1^{(0)}$ is independent of m_1;

- $\alpha = 1$:
  Using eq. (3.52), we obtain

$$\left(3N_1 \cot\kappa \sin\omega \partial_{n_1} + M_1 \cos\omega \partial_{m_1}\right) u_1^{(1)} = -\mathrm{i}\beta \left[\sin\kappa + \mathrm{i}\left(1 - \cos\kappa\right)\right] u_1^{(0)} u_1^{(1)}. \quad (3.54)$$

Choosing

$$u_1^{(0)} = 0, \qquad N_1 = \epsilon S \sin\kappa \cos\omega, \qquad M_1 = -3S \cos\kappa \sin\omega, \quad (3.55)$$

where S is an arbitrary real constant and $\epsilon = \pm 1$, eq. (3.54) becomes

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_1^{(1)} = 0, \qquad \epsilon = -\frac{N_1}{M_1}\omega_1, \quad (3.56)$$

so that $u_1^{(1)}$ is a function of $n_2 \doteq n_1 + \epsilon m_1$;

- $\alpha = 2$:
  Using eq. (3.52), we obtain

$$u_2^{(2)} = \alpha_2 u_1^{(1)2}, \qquad \alpha_2 \doteq -\frac{\beta e^{i\kappa} \sin\kappa \csc\omega}{2\left(4\cos^3\kappa - \cos\omega\right)}; \tag{3.57}$$

iii. Order $\gamma = 3$.

- $\alpha = 0$:
  Using all the three relations (3.55), supposing $u_2^{(0)}$ dependent on n_2 and integrating once with respect to n_2 setting to zero the arbitrary n_2-independent integration function to satisfy the asymptotic decrease of all the $u_n^{(\alpha)}$s as $n_2 \to \pm\infty$, we get

$$u_2^{(0)} = \alpha_1 |u_1^{(1)}|^2, \qquad \alpha_1 \doteq \frac{1}{3}\beta \cot\omega \tan\kappa = -\frac{N_1}{M_1}\beta\epsilon; \tag{3.58}$$

- $\alpha = 1$:
  Using all the three relations (3.55), and eqs. (3.52, 3.57, 3.58), we obtain

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right) u_2^{(1)} = \mathcal{N}_1\left(u_1^{(1)}\right),$$

where $\mathcal{N}_1\left(u_1^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, this side is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right) u_2^{(1)} = 0, \tag{3.59a}$$

$$i\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)} |u_1^{(1)}|^2, \tag{3.59b}$$

$$\rho_1 \doteq -\frac{3S^2}{4M_2}\left[2 + 3\cos(2\kappa) - \cos(2\omega)\right]\tan\omega = -\frac{N_1^2}{M_2}\omega_2, \tag{3.59c}$$

$$\rho_2 \doteq \beta^2\left(1 - \cos\kappa\right)\left(1 + e^{i\kappa}\right)\frac{8\cos^2\kappa - 2\sec\kappa\cos\omega - 3e^{i\kappa}\sec\omega}{2M_2\sin\omega\left(12\cos^3\kappa - 3\cos\omega\right)}.$$

The first relation says that $u_2^{(1)}$ depends on n_2 too while the second one is an NLS equation giving the evolution of $u_1^{(1)}$ according to the slow-time m_2.

- *As ρ_2 is a complex number, the A_1 integrability condition in (2.77) is not respected and the obtained NLS equation is not integrable. Hence our starting model (3.51) is* **not integrable**.

3.4 Multiscale analysis of the symmetric discretization of the KdV equation

Let us consider the discretization of the KdV equation

$$u_{n,m+1} - u_{n,m-1} = \frac{\alpha}{4}\left(u_{n+3,m} - 3u_{n+1,m} + 3u_{n-1,m} - u_{n-3,m}\right) - \frac{\beta}{2}\left(u_{n+1,m}^2 - u_{n-1,m}^2\right), \quad (3.60)$$

which is similar to that of the previous section in all but the discretization of the nonlinear part which now presents symmetric shifts. Let us perform on it a multiscale analysis.

i. Order $\gamma = 1$.

- $\alpha = 0$:
  At this order eq. (3.60) is automatically satisfied;
- $\alpha = 1$:
  At this order we find the dispersion relation

$$\sin\omega = \alpha \sin^3\kappa. \quad (3.61)$$

  As before in the following we will use the dispersion relation by expressing α in terms of κ and ω;

ii. Order $\gamma = 2$.

- $\alpha = 0$:

$$\partial_{m_1} u_1^{(0)} = 0, \quad (3.62)$$

  so that $u_1^{(0)}$ is independent of m_1;
- $\alpha = 1$:
  Using eq. (3.61), we obtain

$$\left(3N_1 \cot\kappa \sin\omega \partial_{n_1} + M_1 \cos\omega \partial_{m_1}\right) u_1^{(1)} = -\mathrm{i}\beta \sin\kappa u_1^{(0)} u_1^{(1)}. \quad (3.63)$$

  Choosing

$$u_1^{(0)} = 0, \qquad N_1 = \epsilon S \sin\kappa \cos\omega, \qquad M_1 = -3S\cos\kappa \sin\omega, \quad (3.64)$$

  where S is an arbitrary real constant and $\epsilon = \pm 1$, eq. (3.63) becomes

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_1^{(1)} = 0, \qquad \epsilon = -\frac{N_1}{M_1}\omega_1, \quad (3.65)$$

  so that $u_1^{(1)}$ is a function of $n_2 \doteq n_1 + \epsilon m_1$;

- $\alpha = 2$:

  Using eq. (3.61), we obtain

$$u_2^{(2)} = \alpha_2 u_1^{(1)2}, \qquad \alpha_2 \doteq -\frac{\beta \sin(2\kappa)\csc\omega}{4\left(4\cos^3\kappa - \cos\omega\right)}; \tag{3.66}$$

iii. Order $\gamma = 3$.

- $\alpha = 0$:

  Using all the three relations (3.64), supposing $u_2^{(0)}$ dependent on n_2 and integrating once with respect to n_2, setting to zero the arbitrary n_2-independent integration function to satisfy the asymptotic decrease of all the $u_n^{(\alpha)}$ as $n_2 \to \pm\infty$, we get

$$u_2^{(0)} = \alpha_1 |u_1^{(1)}|^2, \qquad \alpha_1 \doteq \frac{1}{3}\beta \cot\omega \tan\kappa = -\frac{N_1}{M_1}\beta\epsilon; \tag{3.67}$$

- $\alpha = 1$:

  Using all the three relations (3.64), eqs. (3.61, 3.66, 3.67) and the dependence of $u_1^{(1)}$ on n_2, we obtain

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right) u_2^{(1)} = \mathcal{N}_1\left(u_1^{(1)}\right),$$

  where $\mathcal{N}_1\left(u_1^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, this side is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right) u_2^{(1)} = 0, \tag{3.68a}$$

$$\mathrm{i}\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_2}^2 u_1^{(1)} + \rho_2 u_1^{(1)} |u_1^{(1)}|^2, \tag{3.68b}$$

$$\rho_1 \doteq -\frac{3S^2}{4M_2}\left[2 + 3\cos(2\kappa) - \cos(2\omega)\right]\tan\omega = -\frac{N_1^2}{M_2}\omega_2, \tag{3.68c}$$

$$\rho_2 \doteq -\frac{\beta^2\left[5 + 3\cos(2\kappa) - 16\cos^3\kappa\cos\omega + 2\cos(2\omega)\right]\sin\kappa\tan\kappa\csc(2\omega)}{6M_2\left(4\cos^3 k - \cos\omega\right)} = \\ = \frac{(\alpha_1 + \alpha_2)\beta\sin\kappa\sec\omega}{M_2}.$$

  The first relation says that $u_2^{(1)}$ depends on n_2 too while the second one is an NLS equation giving the evolution of $u_1^{(1)}$ according to the slow-time m_2.

- *As ρ_2 is a real number, the A_1 integrability condition in (2.77) is satisfied and the obtained NLS equation is integrable. Hence our starting model (3.60) is A_1-integrable*;

- $\alpha = 2$:
  Using all the three relations (3.64), eqs. (3.61, 3.66) and that $u_1^{(1)}$ depends on n_2, we get

$$u_3^{(2)} = \alpha_3 u_1^{(1)} \partial_{n_2} u_1^{(1)} + 2\alpha_2 u_1^{(1)} u_2^{(1)}, \tag{3.69}$$
$$\alpha_3 \doteq -\frac{\mathrm{i} S\epsilon\beta \sin\kappa \left\{ \left[1 + 16\cos^3\kappa\cos\omega - 2\cos(2\omega)\right]\cos(2\kappa) - 3\cos(2\omega) \right\}}{4\left(4\cos^3\kappa - \cos\omega\right)^2 \sin\omega};$$

- $\alpha = 3$:
  Using eqs. (3.61, 3.66), we get

$$u_3^{(3)} = \alpha_2 \alpha_4 u_1^{(1)3}, \qquad \alpha_4 \doteq -\frac{\beta\sin(3\kappa)}{\left\{ [1 + 2\cos(2\kappa)]^3 + 4\sin^2\omega - 3 \right\}\sin\omega}; \tag{3.70}$$

iv. Order $\gamma = 4$.

- $\alpha = 0$:
  Using all the three relations (3.64), relations (3.67, 3.68b), supposing $u_3^{(0)}$ dependent on n_2 and integrating once with respect to n_2, setting to zero the arbitrary n_2-independent integration function to satisfy the asymptotic decrease of all the $u_n^{(\alpha)}$ as $n_2 \to \pm\infty$, we get

$$u_3^{(0)} = \alpha_1 \left(u_1^{(1)} \bar{u}_2^{(1)} + \mathcal{C.C.} \right) + \rho_3 J_2^{(1)}, \tag{3.71}$$
$$J_2^{(1)} \doteq -\mathrm{i} \left(\bar{u}_1^{(1)} \partial_{n_2} u_1^{(1)} - \mathcal{C.C.} \right),$$
$$\rho_3 \doteq -\frac{S\epsilon\beta\,[2 + 3\cos(2\kappa) - \cos(2\omega)]\csc\omega\sec\kappa\tan\kappa}{12} = -\frac{\epsilon M_2 \alpha_1 \rho_1}{M_1};$$

- $\alpha = 1$:
  Using all the three relations (3.64), eqs. (3.61, 3.66, 3.67, 3.68b, 3.69, 3.71), the dependence of $u_1^{(1)}$, $u_2^{(1)}$ on n_2 and that (see *Section 2.2* and *Subsection 2.2.1*) the amplitude $u_1^{(1)}$ evolves at the slow-time m_3 according to the *cmKdV* equation

$$\partial_{m_3} u_1^{(1)} - B\left(\partial_{n_2}^3 u_1^{(1)} + \frac{3\rho_2}{\rho_1} |u_1^{(1)}|^2 \partial_{n_2} u_1^{(1)} \right) = 0, \tag{3.72}$$

  we obtain

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right) u_3^{(1)} = \mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right).$$

  $\mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate and linear in $u_2^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, this side is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$(\partial_{n_1} - \epsilon\partial_{m_1})\, u_3^{(1)} = 0, \tag{3.73a}$$
$$\partial_{m_2} u_2^{(1)} - K_2^{'}\left[u_1^{(1)}\right] u_2^{(1)} = \mathcal{N}_3\left(u_1^{(1)}\right), \tag{3.73b}$$
$$K_2'\left[u_1^{(1)}\right] u_2^{(1)} \doteq -\mathrm{i}\rho_1\left[\partial_{n_2}^2 u_2^{(1)} + \frac{\rho_2}{\rho_1}\left(u_1^{(1)2}\bar{u}_2^{(1)} + 2|u_1^{(1)}|^2 u_2^{(1)}\right)\right].$$

The first relation tells us that $u_3^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_2\left(u_1^{(1)}, u_2^{(1)}\right) = 0$, $\mathcal{N}_3\left(u_1^{(1)}\right)$ is another nonlinear function involving $u_1^{(1)}$ and its complex conjugate only and $K_2'\left[u_1^{(1)}\right] u_2^{(1)}$ is the Frechet derivative of the *NLS* flux $K_2\left[u_1^{(1)}\right]$ (see *Section 2.2.1*). Now the term $\mathcal{N}_3\left(u_1^{(1)}\right)$ contains the free real constant B. Choosing the coefficient B so as to eliminate any dependence of the resulting equation on $\partial_{n_2}^3 u_1^{(1)}$, we obtain

$$\partial_{m_2} u_2^{(1)} - K_2'\left[u_1^{(1)}\right] u_2^{(1)} = b u_1^{(1)2}\partial_{n_2}\bar{u}_1^{(1)} + a|u_1^{(1)}|^2\partial_{n_2} u_1^{(1)}, \tag{3.74a}$$
$$\begin{aligned} a &\doteq -\frac{3BM_3\rho_2}{M_2\rho_1} + 6N_1\rho_2\cot\kappa\tan^2\omega + \\ &+\frac{\beta\sec\omega\left[2N_1(\alpha_1+\alpha_2)\cos\kappa + (\mathrm{i}\alpha_3+\rho_3)\sin\kappa\right]}{M_2}, \\ b &\doteq \frac{\beta\sec\omega\left[-N_1(\alpha_1+\alpha_2)\cos\kappa + \rho_3\sin\kappa\right] - 3N_1M_2\rho_2\cot\kappa\tan^2\omega}{M_2}, \end{aligned} \tag{3.74b}$$
$$\begin{aligned} B &\doteq \frac{\epsilon S^3\left\{21 + 18\cos(2\kappa)\left[2-\cos(2\omega)\right] - 32\cos(2\omega) + \cos(4\omega)\right\}}{8M_3}\cdot \\ &\cdot\cos\kappa\sec\omega\tan\omega = \frac{N_1^3}{M_3}\omega_3. \end{aligned} \tag{3.74c}$$

- *The coefficients of the r. h. s. of equation (3.74a) given by the relations (3.74b), respect the two A_2 integrability conditions (2.80). This proves the A_2 asymptotic integrability of the symmetrically discretized KdV equation*;

- $\alpha = 2$:

  Using eqs. (3.61, 3.64, 3.66, 3.67, 3.68b, 3.69, 3.70) and the dependence of $u_1^{(1)}$, $u_2^{(1)}$ on n_2, we get

$$\begin{aligned} u_4^{(2)} &= 2\alpha_2^2\left(\alpha_1+\alpha_4+\alpha_5\rho_2\right)|u_1^{(1)}|^2u_1^{(1)2}+\left[\alpha_3\delta+\alpha_2\left(\pi+N_1^2\right)\right]\left(\partial_{n_2}u_1^{(1)}\right)^2+\\ &+\left[\alpha_3\delta+\alpha_2\left(\pi+N_1^2+2\rho_1\alpha_2\alpha_5\right)\right]u_1^{(1)}\partial_{n_2}^2u_1^{(1)}+\alpha_2\left(2u_1^{(1)}u_3^{(1)}+u_2^{(1)2}\right)+\\ &+\alpha_3\partial_{n_2}\left(u_2^{(1)}u_1^{(1)}\right), \qquad \alpha_5 \doteq -\frac{2M_2\cos(2\omega)\csc(2\kappa)}{\beta}, \qquad (3.75)\\ \delta &\doteq \frac{3\mathrm{i}\epsilon S\cos\kappa\left[4\cos\kappa\cos\omega\cos(2\kappa)-\cos(2\omega)\right]}{2\left(4\cos^3\kappa-\cos\omega\right)},\\ \pi &\doteq \frac{3S^2\cos\kappa\cos\omega\left[\cos\omega+3\cos\omega\cos(4\kappa)+6\cos\kappa\sin^2\omega\right]}{2\left(4\cos^3\kappa-\cos\omega\right)}; \end{aligned}$$

- $\alpha = 3$:
  Using eqs. (3.61, 3.64, 3.66, 3.69, 3.70) and the dependence of $u_1^{(1)}$ on n_2, we get

$$\begin{aligned} u_4^{(3)} &= \alpha_4\left[3\alpha_2u_2^{(1)}+\left(\alpha_2F+\alpha_3\right)\left(\partial_{n_2}u_1^{(1)}\right)\right]u_1^{(1)2}, \qquad (3.76)\\ F &\doteq \frac{24\mathrm{i}\epsilon S\cos^3\kappa\cos\omega\left[4\cos^3(2\kappa)-\cos(2\omega)\right]}{\left[1+2\cos(2\kappa)\right]\left[3+6\cos(2\kappa)+3\cos(4\kappa)+\cos(6\kappa)-\cos(2\omega)\right]}; \end{aligned}$$

- $\alpha = 4$:
  Using eqs. (3.61, 3.66, 3.70), we get

$$\begin{aligned} u_4^{(4)} &= G\alpha_2\left(\alpha_2+2\alpha_4\right)u_1^{(1)4}, \qquad (3.77)\\ G &\doteq \frac{\beta\csc\omega\sin(4\kappa)}{4\left\{-4\left[\cos\kappa+\cos(3\kappa)\right]^3+\cos\omega+\cos(3\omega)\right\}}; \end{aligned}$$

v. Order $\gamma = 5$.

- $\alpha = 0$:
  Using relations (3.64, 3.66, 3.67, 3.68b, 3.71, 3.72, 3.74a), taking into account the dependence of $u_1^{(1)}$ on n_2 and supposing $u_4^{(0)}$ dependent on n_2 and integrating once with respect to n_2 setting to zero the arbitrary n_2-independent integration function to satisfy the asymptotic decrease of all the $u_n^{(\alpha)}$s as $n_2 \to \pm\infty$, we get

$$\begin{aligned} u_4^{(0)} &= -\mathrm{i}\rho_3\left(\bar{u}_2^{(1)}\partial_{n_2}u_1^{(1)}+\bar{u}_1^{(1)}\partial_{n_2}u_2^{(1)}-\mathcal{C.C.}\right)+ \qquad (3.78)\\ &+\alpha_1\left(\bar{u}_1^{(1)}u_3^{(1)}+u_1^{(1)}\bar{u}_3^{(1)}+|u_2^{(1)}|^2\right)+f|u_1^{(1)}|^4+g|\delta_{n_2}u_1^{(1)}|^2+\\ &+h\left(u_1^{(1)}\partial_{n_2}^2\bar{u}_1^{(1)}+\mathcal{C.C.}\right),\\ f &\doteq -\frac{\epsilon}{M_1}\left\{\left[(a+b)\frac{\alpha_1}{2}-\rho_2\rho_3\right]M_2+\frac{N_1\beta\left(\alpha_1^2+2\alpha_2^2\right)}{2}+\frac{3BM_3\alpha_1\rho_2}{2\rho_1}\right\},\\ g &\doteq -\frac{-3BM_3\alpha_1+N_1^3\left(\beta-6\alpha\alpha_1\right)+M_1^3\alpha_1\epsilon+6M_2\rho_1\rho_3}{3M_1\epsilon},\\ h &\doteq -\frac{6BM_3\alpha_1+N_1^3\left(\beta-6\alpha\alpha_1\right)+M_1^3\alpha_1\epsilon-6M_2\rho_1\rho_3}{6M_1\epsilon}; \end{aligned}$$

- $\alpha = 1$:

  Using eqs. (3.61, 3.64, 3.66, 3.67, 3.68b, 3.69, 3.70, 3.71, 3.72, 3.74a, 3.75, 3.78), the dependence of $u_1^{(1)}$, $u_2^{(1)}$, $u_3^{(1)}$ on n_2 and that (see *Section 2.2* and *Subsection 2.2.1*),

$$\partial_{m_3} u_2^{(1)} - K_3' \left[u_1^{(1)}\right] u_2^{(1)} = f_3(2) \tag{3.79}$$

$$\partial_{m_4} u_1^{(1)} + \mathrm{i}C \left\{ \partial_{n_2}^4 u_1^{(1)} + \frac{\rho_2}{\rho_1} \left[\frac{3\rho_2}{2\rho_1} |u_1^{(1)}|^4 u_1^{(1)} + 4|u_1^{(1)}|^2 \partial_{n_2}^2 u_1^{(1)} + \right.\right. \tag{3.80}$$
$$\left.\left. +3\bar{u}_1^{(1)} \left(\partial_{n_2} u_1^{(1)}\right)^2 + 2|\partial_{n_2} u_1^{(1)}|^2 u_1^{(1)} + u_1^{(1)2} \partial_{n_2}^2 \bar{u}_1^{(1)} \right]\right\} = 0,$$

where $K_3'[u]\,v$ is given in eq. (2.76c) and $f_3(2)$ in eqs. (2.79b, 2.81), we obtain

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_4^{(1)} = \mathcal{N}_4 \left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right).$$

$\mathcal{N}_4\left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right)$ is function linear in $u_3^{(1)}$ and its complex conjugate and nonlinear in $u_1^{(1)}$ and $u_2^{(1)}$ and their complex conjugates. As the r.h.s. of the last equation depends on n_2, this side is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_4^{(1)} = 0, \tag{3.81a}$$

$$\partial_{m_2} u_3^{(1)} - K_2' \left[u_1^{(1)}\right] u_3^{(1)} = \mathcal{N}_5 \left(u_1^{(1)}, u_2^{(1)}\right). \tag{3.81b}$$

The first relation tells us that $u_4^{(1)}$ itself depends on n_2. In the second relation, which comes directly from $\mathcal{N}_4\left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right) = 0$, $\mathcal{N}_5\left(u_1^{(1)}, u_2^{(1)}\right)$ is another nonlinear function involving $u_1^{(1)}$, $u_2^{(1)}$ and their complex conjugates. Now the term $\mathcal{N}_5\left(u_1^{(1)}, u_2^{(1)}\right)$ contains the free real constant C which is chosen so as to eliminate any dependence of the resulting equation on $\partial_{n_2}^4 u_1^{(1)}$, as from *Proposition 3.1*, the presence of this term can always introduce a dependence on the secular term $K_4\left[u_1^{(1)}\right]$, the flux of the equation (3.80). We obtain

$$\partial_{m_2} u_3^{(1)} - K_2' \left[u_1^{(1)}\right] u_3^{(1)} = f_2(3), \tag{3.82a}$$

$$C \doteq \frac{S^4 \{-3\left[404 + 549\cos(2\kappa) + 126\cos(4\kappa)\right]\cos(2\omega)}{128 M_4} +$$
$$+\frac{3\left[73 + 78\cos(2\kappa) + 9\cos(4\kappa)\right]\cos(4\omega) - \left[2 + \cos(2\kappa)\right]\cos(6\omega)}{128 M_4} +$$
$$+\frac{997 + 1358\cos(2\kappa) + 405\cos(4\kappa)\}\sec^2\omega\tan\omega}{128 M_4} = \frac{N_1^4}{M_4}\omega_4, \tag{3.82b}$$

where the forcing term $f_2(3)$ is given by eq. (2.82a). The real and imaginary parts of the coefficients τ_i, $i = 1, \ldots, 12$ of $f_2(3)$ are given by

$$
\begin{aligned}
R_1 &= 0,\\
I_1 &= -\frac{2\beta\rho_1 \sin\kappa\left[12N_1\alpha_2^2\alpha_5\rho_1\sec\omega + (3BM_3\alpha_1 - 2M_2\rho_1\rho_3)\csc\omega\tan\kappa\right]}{12N_1M_2\rho_1^2}\rho_2 -\\
&\quad -\frac{3aBM_3}{4M_2\rho_1^2}\rho_2 + \left(\frac{3CM_4}{2M_2\rho_1^2} - \frac{1}{2}M_2\tan\omega\right)\rho_2^2 - \frac{\beta\alpha_2^2\left(2\alpha_1 + 3\alpha_4\right)\sin\kappa\sec\omega}{M_2} -\\
&\quad -\frac{\beta\left[(a+b)\,M_2\alpha_1 + N_1\left(\alpha_1^2 + 2\alpha_2^2\right)\beta\right]\csc\omega\tan\kappa\sin\kappa}{6N_1M_2},\\
R_2 &= 0,\\
I_2 &= \frac{2\left[CM_4 + 9N_1^2M_2\rho_1\left(1 - \csc^2\kappa + \cot^2\kappa\sec^2\omega\right) - M_2^2\rho_1^2\tan\omega\right]}{M_2\rho_1}\rho_2 -\\
&\quad -\frac{9BM_3N_1\cot\kappa\tan^2\omega}{M_2\rho_1}\rho_2 - 3\left[\frac{bBM_3}{M_2\rho_1} + (a+2b)\,N_1\cot\kappa\tan^2\omega\right] -\\
&\quad -\frac{N_1^2\csc\omega\sin\kappa\tan\kappa}{9M_2}\beta^2 - \left(\frac{N_1\cos\kappa\sec\omega}{M_2} + \frac{2\rho_1\csc\omega\sin\kappa\tan\kappa}{3N_1}\right)\beta\rho_3 +\\
&\quad +\frac{N_1\sec\omega\left(\mathrm{i}\alpha_3\cos\kappa - 2N_1\alpha_2\sin\kappa\right)}{M_2}\beta + \frac{BM_3\csc\omega\sin\kappa\tan\kappa}{3N_1M_2}\beta\alpha_1 +\\
&\quad +\frac{2\csc\kappa\sec\kappa + 3\sec\omega\left(3\cos\kappa\cot\kappa\tan^2\omega - 2\sin\kappa\right)}{3M_2}N_1^2\beta\alpha_1,\\
R_3 &= 0,\\
I_3 &= -\frac{N_1^2\csc\omega\sin\kappa\tan\kappa}{18M_2}\beta^2 - \frac{BM_3\csc\omega\sin\kappa\tan\kappa}{3N_1M_2}\beta\alpha_1 +\\
&\quad +\frac{2\csc\kappa\sec\kappa + 3\sec\omega\left(3\cos\kappa\cot\kappa\tan^2\omega - 2\sin\kappa\right)}{6M_2}N_1^2\beta\alpha_1 +\\
&\quad +\frac{\beta\sec\omega\left\{N_1\left(\mathrm{i}\alpha_3 + \rho_3\right)\cos\kappa - \left[\alpha_3\delta + \alpha_2\left(2N_1^2 + \pi + 2\alpha_2\alpha_5\rho_1\right)\right]\sin\kappa\right\}}{M_2} +\\
&\quad +\frac{\beta\rho_1\rho_3\csc\omega\sin\kappa\tan\kappa}{3N_1} - 3a\left[\frac{BM_3}{2M_2\rho_1} + N_1\cot\kappa\tan^2\omega\right] +\\
&\quad +\frac{4CM_4 - 2M_2^2\rho_1^2\tan\omega + 9N_1\left(N_1M_2\rho_1\cot\kappa - BM_3\right)\cot\kappa\tan^2\omega}{M_2\rho_1}\rho_2,\\
R_4 &= 0,\\
I_4 &= -\frac{N_1^2\csc\omega\sin\kappa\tan\kappa}{18M_2}\beta^2 + \frac{3}{2}N_1\left(2b + 3N_1\rho_2\cot\kappa\right)\cot\kappa\tan^2\omega +\\
&\quad +\frac{CM_4\rho_2}{M_2\rho_1} - \frac{BM_3\csc\omega\sin\kappa\tan\kappa}{3N_1M_2}\beta\alpha_1 + \frac{\beta\rho_1\rho_3\csc\omega\sin\kappa\tan\kappa}{3N_1} +\\
&\quad +\frac{2\csc\kappa\sec\kappa + 3\sec\omega\left(3\cos\kappa\cot\kappa\tan^2\omega - \sin\kappa\right)}{6M_2}N_1^2\beta\alpha_1 -\\
&\quad -\frac{N_1\left(2\rho_3\cos\kappa + N_1\alpha_2\sin\kappa\right)\sec\omega}{2M_2}\beta,
\end{aligned}
$$

$$
\begin{aligned}
&R_5 = 0,\\
&I_5 = \frac{(\mathrm{i}N_1 \cos\kappa - \delta \sin\kappa)\sec\omega}{M_2}\alpha_3 - \frac{\left[N_1^2(\alpha_1 + 2\alpha_2) + \alpha_2\pi\right]\sin\kappa\sec\omega}{M_2}\beta +\\
&\quad + \frac{N_1\rho_3\cos\kappa\sec\omega}{M_2}\beta - M_2\rho_2\rho_1\tan\omega + 3N_1(3N_1\rho_2\cot\kappa - a)\cot\kappa\tan^2\omega +\\
&\quad + \frac{3\left(2CM_4\rho_2 - 6BM_3N_1\rho_2\cot\kappa\tan^2\omega - aBM_3\right)}{2M_2\rho_1},\\
&\tau_6 = \tau_7 = a, \qquad \tau_8 = -\mathrm{i}\rho_2, \qquad \tau_9 = -2\mathrm{i}\rho_2,\\
&\tau_{10} = a, \qquad \tau_{11} = 2b, \qquad \tau_{12} = a.
\end{aligned} \tag{3.83}
$$

- *The coefficients given in eqs. (3.83) respect* **only fourteen out of the fifteen** A_3 *integrability conditions (2.83) (the one involving* I_4 *is not satisfied). This proves that the symmetrically discretized* KdV *equation is* **not integrable**.

3.5 Multiscale analysis of differential-difference equations I: reduction of the Ablowitz-Ladik NLS equation

Let us consider the integrable $A-L$ discrete NLS equation [1] for the *complex* function $f_n(t)$

$$
\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2\sigma^2} = \epsilon |f_n|^2 \frac{f_{n+1} + f_{n-1}}{2}, \tag{3.84}
$$

where $\epsilon \doteq \pm 1$. Equation (3.84) is an $S-$integrable system as it arises as the compatibility condition of the following overdetermined system of two matrix linear partial difference equations for the vector function $\mathbf{v}_n(t) \doteq \left(v_n^{(1)}(t), v_n^{(2)}(t)\right)^T$

$$
\mathbf{v}_{n+1} = \begin{pmatrix} z & \sigma f_n \\ \epsilon\sigma \bar{f}_n & z^{-1} \end{pmatrix} \mathbf{v}_n \doteq A_n \mathbf{v}_n, \tag{3.85a}
$$

$$
\begin{aligned}
2\mathrm{i}\partial_t \mathbf{v}_n &= \begin{pmatrix} \epsilon f_n \bar{f}_{n-1} - \left(z - z^{-1}\right)^2 / \left(2\sigma^2\right) & \left(z^{-1} f_{n-1} - z f_n\right)/\sigma \\ \epsilon\left(z^{-1}\bar{f}_n - z\bar{f}_{n-1}\right)/\sigma & \left(z - z^{-1}\right)^2/\left(2\sigma^2\right) - \epsilon \bar{f}_n f_{n-1} \end{pmatrix} \mathbf{v}_n \doteq \\
&\doteq 2\mathrm{i} B_n \mathbf{v}_n,
\end{aligned} \tag{3.85b}
$$

where $z \in \mathbb{C}$ is the spectral parameter. Hence our equation is equivalent to the following matrix equation

$$
A_{n,t} = B_{n+1}A_n - A_n B_n.
$$

If we set $f_n(t) \doteq \nu_n(t)^{1/2} e^{\mathrm{i}\phi_n(t)}$, where $\nu_n(t)$ and $\phi_n(t)$ are both real functions, eq. (3.84), separating real and imaginary parts, turns into the following system of two real nonlinear differential-difference equations

$$
\partial_t \nu_n + \left(\frac{1}{\sigma^2} - \epsilon\nu_n\right)\left(\delta_+^{1/2}\sin\beta_+ + \delta_-^{1/2}\sin\beta_-\right) = 0, \tag{3.86a}
$$

$$
\partial_t \phi_n + \frac{1}{\sigma^2} - \frac{1}{2}\left(\frac{1}{\sigma^2} - \epsilon\nu_n\right)\left(\gamma_+^{1/2}\cos\beta_+ + \gamma_-^{1/2}\cos\beta_-\right) = 0, \tag{3.86b}
$$

where $\beta_\pm \doteq \phi_{n\pm1}(t)-\phi_n(t)$, $\gamma_\pm \doteq \nu_n(t)^{-1}\nu_{n\pm1}(t)$, $\delta_\pm \doteq \nu_n(t)\nu_{n\pm1}(t)$. We expand the functions $\nu_n(t)$ and $\phi_n(t)$ as

$$\nu\left(n,t,n_1,\{t_j\}_{j=1}^K,\varepsilon\right) = 1+\sum_{\kappa=1}^{+\infty}\varepsilon^{2\kappa}\nu^{(\kappa)}\left(n_1,\{t_j\}_{j=1}^K\right), \tag{3.87a}$$

$$\phi\left(n,t,n_1,\{t_j\}_{j=1}^K,\varepsilon\right) = -\epsilon t+\sum_{\kappa=1}^{+\infty}\varepsilon^{2\kappa-1}\phi^{(\kappa)}\left(n_1,\{t_j\}_{j=1}^K\right), \tag{3.87b}$$

where $n_1 \doteq N_1\varepsilon n$, $t_j \doteq \alpha_j\varepsilon^{2j-1}t$ and N_1, α_j are real constants, $1 \le j \le K$ (eventually $K=+\infty$). The expansions of the n-shifts and of the $t-$derivatives of the functions $\nu_n(t)$ and $\phi_n(t)$ as well as of the nonlinear monomials present in eqs. (3.86) are given in *Appendix D*, see especially eqs. (D.6c, D.6d, D.11). The orders $\kappa = 0$, 1 are trivially satisfied.

i. Order $\kappa = 2$.

$$\nu^{(1)} = -\epsilon\alpha_1\partial_{t_1}\phi^{(1)}; \tag{3.88}$$

ii. Order $\kappa = 3$.

$$\alpha_1\partial_{t_1}\nu^{(1)}+N_1^2\left(\frac{1}{\sigma^2}-\epsilon\right)\partial_{n_1}^2\phi^{(1)} = 0,$$

which, after inserting in it the expression (3.88), becomes

$$\left(\partial_{t_1}^2-c^2\partial_{n_1}^2\right)\phi^{(1)} = 0, \qquad c \doteq \pm\frac{N_1}{\sigma\alpha_1}\left(\epsilon-\sigma^2\right)^{1/2},$$

We see that one has to ensure that $\left(\epsilon-\sigma^2\right) > 0$ from which $\epsilon = 1$ and $-1 < \sigma < 1$ ($\sigma \neq 0$). Hence we will eliminate σ^2 using the relation

$$\sigma^2 = \frac{1}{1+\left(\frac{c\alpha_1}{N_1}\right)^2}.$$

If one desires to study the continuum limit, in order to get a finite limit for c as $\sigma \to 0$, one should set $N_1 = \alpha_1\sigma$ so that $c = \pm\left(1-\sigma^2\right)^{1/2}$ and $N_1 = \pm\alpha_1\left(1-c^2\right)^{1/2}$. We choose $\phi^{(1)}$ depending on $\xi \doteq n_1-ct_1$ so that

$$\left(\partial_{t_1}+c\partial_\xi\right)\phi^{(1)} = 0, \tag{3.89}$$

from which, using (3.88),

$$\nu^{(1)} = \alpha_1 c\partial_\xi\phi^{(1)}, \tag{3.90}$$

so that $\nu^{(1)}$ itself depends on ξ;

iii. Order $\kappa = 4$.

Using (3.90), we find

$$\nu^{(2)} = -\alpha_1 \partial_{t_1} \phi^{(2)} - \alpha_2 \partial_{t_2} \phi^{(1)} + \frac{(c\alpha_1)^2}{2} \left[\frac{c\alpha_1}{2} \partial_\xi^3 \phi^{(1)} - \left(\partial_\xi \phi^{(1)} \right)^2 \right]; \tag{3.91}$$

iv. Order $\kappa = 5$.

$$\alpha_1 \partial_{t_1} \nu^{(2)} + \alpha_2 \partial_{t_2} \nu^{(1)} + (c\alpha_1)^2 \left[\partial_{n_1}^2 \phi^{(2)} + \frac{N_1^2}{12} \partial_\xi^4 \phi^{(1)} + \left(\partial_\xi \phi^{(1)} \right) \left(\partial_\xi \nu^{(1)} \right) \right] + \\ + \left[(c\alpha_1)^2 - N_1^2 \right] \nu^{(1)} \partial_\xi^2 \phi^{(1)} = 0. \tag{3.92}$$

Taking into account eqs. (3.89, 3.90, 3.91), eq. (3.92) becomes

$$\alpha_1 \left(\partial_{t_1}^2 - c^2 \partial_{n_1}^2 \right) \phi^{(2)} = 2c\partial_\xi \left[\alpha_2 \partial_{t_2} \phi^{(1)} - (c\alpha_1)^2 \rho_1 \partial_\xi^3 \phi^{(1)} + 6c\alpha_1 \rho_1 \left(\partial_\xi \phi^{(1)} \right)^2 \right], \\ \rho_1 \doteq \frac{1}{8} \left(c\alpha_1 - \frac{N_1^2}{3c\alpha_1} \right). \tag{3.93}$$

The right hand side of eq. (3.93) is secular so we have to require that

$$\left(\partial_{t_1}^2 - c^2 \partial_{n_1}^2 \right) \phi^{(2)} = 0, \tag{3.94a}$$

$$\alpha_2 \partial_{t_2} \phi^{(1)} - c\alpha_1 \rho_1 \left[c\alpha_1 \partial_\xi^3 \phi^{(1)} - 6 \left(\partial_\xi \phi^{(1)} \right)^2 \right] = 0, \tag{3.94b}$$

where we choose $\phi^{(2)}$ and as a consequence $\nu^{(2)}$ depending on ξ. The arbitrary $\xi-$independent integration function has been set equal to zero to met the asymptotic conditions on all the functions $\phi^{(j)}$ and $\nu^{(i)}$ and their $\xi-$derivatives as $\xi \to \pm\infty$. As we can see in eqs. (2.92) with $n = 2$ and (2.93b), eq. (3.94b) is a potential KdV equation for $\phi^{(1)}$ at the slow-time t_2 with

$$\tau_1 \doteq (c\alpha_1)^2 \rho_1 / \alpha_2, \qquad \tau_2 \doteq -6c\alpha_1 \rho_1 / \alpha_2. \tag{3.95}$$

From eq. (3.94b), differentiating once with respect to ξ and using (3.90), we have

$$\alpha_2 \partial_{t_2} \nu^{(1)} - (c\alpha_1)^2 \rho_1 \partial_\xi^3 \nu^{(1)} + 12 \rho_1 \nu^{(1)} \partial_\xi \nu^{(1)} = 0,$$

which is a KdV equation for $\nu^{(1)}$ at the slow-time t_2. Taking into account eq. (3.94b) and that $\phi^{(2)}$ depends on ξ, eq. (3.91) becomes

$$\nu^{(2)} = c\alpha_1 \partial_\xi \phi^{(2)} + (c\alpha_1)^2 \upsilon_1 \partial_\xi^3 \phi^{(1)} + \frac{1}{4} \left[(c\alpha_1)^2 - N_1^2 \right] \left(\partial_\xi \phi^{(1)} \right)^2, \tag{3.96}$$

$$\upsilon_1 \doteq \frac{1}{8} \left(c\alpha_1 + \frac{N_1^2}{3c\alpha_1} \right);$$

v. Order $\kappa = 6$.

Taking into account eqs. (3.90, 3.91, 3.94b), we have

$$\begin{aligned}
\nu^{(3)} &= -\alpha_3 \partial_{t_3} \phi^{(1)} - \alpha_2 \partial_{t_2} \phi^{(2)} - \alpha_1 \partial_{t_1} \phi^{(3)} + \frac{(c\alpha_1)^3}{4} \partial_\xi^3 \phi^{(2)} + \\
&+ (c\alpha_1)^2 \left[\rho_2 \left(\partial_\xi \phi^{(1)} \right) \left(\partial_\xi^3 \phi^{(1)} \right) + c\alpha_1 \rho_3 \partial_\xi^5 \phi^{(1)} - \left(\partial_\xi \phi^{(1)} \right) \left(\partial_\xi \phi^{(2)} \right) \right] + \\
&+ \frac{c\alpha_1 N_1^2}{2} \left[\left(\partial_\xi \phi^{(1)} \right)^3 - \frac{c\alpha_1}{2} \left(\partial_\xi^2 \phi^{(1)} \right)^2 \right], \\
&\rho_2 \doteq -\frac{1}{24} \left[3 (c\alpha_1)^2 + 13 N_1^2 \right], \qquad \rho_3 \doteq \frac{1}{32} \left[(c\alpha_1)^2 + N_1^2 \right];
\end{aligned} \tag{3.97}$$

vi. Order $\kappa = 7$.

Taking into account eqs. (3.90, 3.91, 3.94b, 3.97), that $\phi^{(1)}$ and $\phi^{(2)}$ depend on ξ and the fact that $\phi^{(1)}$ evolves at the slow-time t_3 according to the equation given by (2.92) with $n = 3$, (2.93c, 3.95), we have

$$\begin{aligned}
\alpha_1 \left(\partial_{t_1}^2 - c^2 \partial_{n_1}^2 \right) \phi^{(3)} = 2c\alpha_2 \partial_\xi \Big\{ & \partial_{t_2} \phi^{(2)} - \tau_1 \partial_\xi^3 \phi^{(2)} - 2\tau_2 \left(\partial_\xi \phi^{(1)} \right) \left(\partial_\xi \phi^{(2)} \right) - \\
& - \theta_1 \left(\partial_\xi \phi^{(1)} \right)^3 - \theta_2 \left(\partial_\xi \phi^{(1)} \right) \left(\partial_\xi^3 \phi^{(1)} \right) - \theta_3 \left(\partial_\xi^2 \phi^{(1)} \right)^2 + \\
& + \frac{1}{\alpha_2} \left(\alpha_3 \lambda - c\alpha_1 \rho_4 \right) \partial_\xi^5 \phi^{(1)} \Big\},
\end{aligned} \tag{3.98}$$

$$\begin{aligned}
&\rho_4 \doteq \frac{1}{1920} \left[-N_1^4 + 30 (N_1 c\alpha_1)^2 + 15 (c\alpha_1)^4 \right], \\
&\theta_1 \doteq -\rho_1 \left[N_1^2 + 5 (c\alpha_1)^2 \right] / (2\alpha_2), \qquad \theta_2 \doteq -c\alpha_1 \rho_1 \tau_2, \qquad \theta_3 \doteq 12 \tau_1 \upsilon_1.
\end{aligned}$$

As the r. h. s. of eq. (3.98) is in the kernel of the operator in the l. h. s., the r. h. s. is secular. In order to remove this secularity, both members have to be set equal to zero and the free constant λ has to be chosen so that to eliminate any dependence on the term $\partial_\xi^5 \phi^{(1)}$ as from *Proposition 3.1* the presence of this term can always introduce a dependence on the secular term $K_3 \left[\phi^{(1)} \right]$, the flux of the equation (2.93c). We get

$$\left(\partial_{t_1}^2 - c^2 \partial_{n_1}^2 \right) \phi^{(3)} = 0, \tag{3.99a}$$

$$\partial_{t_2} \phi^{(2)} - K_2' \left[\phi^{(1)} \right] \phi^{(2)} = f_2(2), \qquad \lambda = \frac{\rho_4 c\alpha_1}{\alpha_3}, \tag{3.99b}$$

where $K_2' [\phi] \zeta$ is given in eq. (2.95b) and $f_2(2)$ in eq. (2.97a). We choose $\phi^{(3)}$ and as a consequence $\nu^{(3)}$ depending on ξ and the arbitrary $\xi-$independent integration function has been set equal to zero to met the asymptotic conditions on all the functions $\phi^{(j)}$ and $\nu^{(i)}$ and their $\xi-$derivatives as $\xi \to \pm\infty$. As we saw in *Subsection 2.2.3*, We have no A_2 integrability conditions for the coefficients θ_i, $i = 1, \ldots, 3$ of $f_2(2)$ in the case of the potential *KdV* hierarchy. Taking into account eq. (2.92) with $n = 3$, (2.93c, 3.113), eq. (3.99b) and that $\phi^{(3)}$ depends on ξ, eq. (3.97) becomes

$$
\begin{aligned}
\nu^{(3)} &= c\alpha_1 \partial_\xi \phi^{(3)} + (c\alpha_1)^2 \upsilon_1 \partial_\xi^3 \phi^{(2)} + \frac{1}{2}\left[(c\alpha_1)^2 - N_1^2\right]\left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi \phi^{(2)}\right) - \\
&\quad - \frac{c\alpha_1 N_1^2}{6}\left(\partial_\xi \phi^{(1)}\right)^3 + \upsilon_2\left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi^3 \phi^{(1)}\right) + \upsilon_3\left(\partial_\xi^2 \phi^{(1)}\right)^2 + \\
&\quad + c\alpha_1 \upsilon_4 \partial_\xi^5 \phi^{(1)}, \qquad \upsilon_2 \doteq -\frac{1}{48}\left[N_1^4 + 8\left(c\alpha_1 N_1\right)^2 + 3\left(c\alpha_1\right)^4\right], \\
\upsilon_3 &\doteq \frac{1}{64}\left[N_1^2 + (c\alpha_1)^2\right]\left[N_1^2 - 7\left(c\alpha_1\right)^2\right], \\
\upsilon_4 &\doteq \frac{1}{1920}\left[N_1^4 + 30\left(c\alpha_1 N_1\right)^2 + 45\left(c\alpha_1\right)^4\right];
\end{aligned}
\tag{3.100}
$$

vii. Order $\kappa = 8$.

Taking into account eq. (2.92) with $n = 3$, (2.93c, 3.95) and eqs. (3.90, 3.91, 3.94b, 3.97, 3.99b):

$$
\begin{aligned}
\nu^{(4)} &= -\alpha_4 \partial_{t_4}\phi^{(1)} - \alpha_3 \partial_{t_3}\phi^{(2)} - \alpha_2 \partial_{t_2}\phi^{(3)} - \alpha_1 \partial_{t_1}\phi^{(4)} + N_1^2 \rho_5 \left(\partial_\xi \phi^{(1)}\right)^4 + \\
&\quad + c\alpha_1 \rho_6 \left(\partial_\xi \phi^{(1)}\right)^2 \left(\partial_\xi^3 \phi^{(1)}\right) + N_1^2 c\alpha_1 \rho_7 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi^2 \phi^{(1)}\right)^2 + \\
&\quad + (c\alpha_1)^2 \rho_8 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi^5 \phi^{(1)}\right) + (c\alpha_1)^2 \rho_9 \left(\partial_\xi^3 \phi^{(1)}\right)^2 + \\
&\quad + (c\alpha_1)^2 \rho_{10} \left(\partial_\xi^2 \phi^{(1)}\right)\left(\partial_\xi^4 \phi^{(1)}\right) + (c\alpha_1)^3 \rho_{11}\left(\partial_\xi^7 \phi^{(1)}\right) + \\
&\quad + \frac{3c\alpha_1 N_1^2}{2}\left(\partial_\xi \phi^{(1)}\right)^2 \left(\partial_\xi \phi^{(2)}\right) + (c\alpha_1)^2 \rho_2 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi^3 \phi^{(2)}\right) - \\
&\quad - \frac{(N_1 c\alpha_1)^2}{2}\left(\partial_\xi^2 \phi^{(1)}\right)\left(\partial_\xi^2 \phi^{(2)}\right) + (c\alpha_1)^2 \rho_2 \left(\partial_\xi^3 \phi^{(1)}\right)\left(\partial_\xi \phi^{(2)}\right) - \\
&\quad - \frac{(c\alpha_1)^2}{2}\left(\partial_\xi \phi^{(2)}\right)^2 + (c\alpha_1)^3 \rho_3 \left(\partial_\xi^5 \phi^{(2)}\right) - (c\alpha_1)^2 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi \phi^{(3)}\right) + \\
&\quad + \frac{(c\alpha_1)^3}{4}\left(\partial_\xi^3 \phi^{(3)}\right), \qquad \rho_5 \doteq -\frac{1}{24}\left[3N_1^2 - 4(c\alpha_1)^2\right], \\
&\quad \rho_6 \doteq \frac{1}{16}\left[6N_1^4 + (N_1 c\alpha_1)^2 + (c\alpha_1)^4\right], \qquad \rho_7 \doteq \frac{1}{4}\left[N_1^2 - (c\alpha_1)^2\right], \\
&\quad \rho_8 \doteq -\frac{1}{960}\left[53N_1^4 + 90(N_1 c\alpha_1)^2 + 45(c\alpha_1)^4\right], \\
&\quad \rho_9 \doteq -\frac{1}{1152}\left[61N_1^4 + 150(N_1 c\alpha_1)^2 + 117(c\alpha_1)^4\right], \\
&\quad \rho_{10} \doteq -\frac{1}{384}\left[25N_1^4 + 54(N_1 c\alpha_1)^2 + 45(c\alpha_1)^4\right], \\
&\quad \rho_{11} \doteq \frac{1}{7680}\left[13N_1^4 + 50(N_1 c\alpha_1)^2 + 45(c\alpha_1)^4\right];
\end{aligned}
\tag{3.101}
$$

viii. Order $\kappa = 9$.

Taking into account eqs. (2.92) with $n = 3, 4$, (2.93c, 2.93d, 3.95), eq. (2.94) with $j = 2$, $n = 3$, (2.95c, 2.97b, 2.98, 3.95), eqs. (3.90, 3.91, 3.94b, 3.97, 3.99b, 3.101), that $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ depend on ξ and separating to remove the secular terms, we get

$$\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(4)} = 0, \tag{3.102a}$$

$$\begin{aligned}\partial_{t_2}\phi^{(3)} - K_2'\left[\phi^{(1)}\right]\phi^{(3)} &= 3\theta_1\left(\partial_\xi\phi^{(1)}\right)^2\left(\partial_\xi\phi^{(2)}\right) + \\ &+\theta_2\left[\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^3\phi^{(2)}\right) + \left(\partial_\xi^3\phi^{(1)}\right)\left(\partial_\xi\phi^{(2)}\right)\right] + \\ &+2\theta_3\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(2)}\right) + \tau_2\left(\partial_\xi\phi^{(2)}\right)^2 + \\ &+\frac{2\left(N_1\rho_1\right)^2}{\alpha_2}\left(\partial_\xi\phi^{(1)}\right)^4 + \frac{c\alpha_1 N_1^2}{40}\theta_1\left(\partial_\xi^3\phi^{(1)}\right)^2 - \\ &-\frac{\upsilon_4\tau_2}{3}\left[4\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^4\phi^{(1)}\right) + \left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^5\phi^{(1)}\right)\right] + \\ &+\frac{\rho_1}{\alpha_2}\left[\rho_{12}\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(1)}\right)^2 + \rho_{13}\left(\partial_\xi\phi^{(1)}\right)^2\left(\partial_\xi^3\phi^{(1)}\right)\right],\end{aligned} \tag{3.102b}$$

$$\begin{aligned}\chi &= \frac{c\alpha_1\left[-N_1^6 + 273N_1^4(c\alpha_1)^2 + 525N_1^2(c\alpha_1)^4 + 315(c\alpha_1)^6\right]}{322560\alpha_4}, \\ \rho_{12} &\doteq \frac{1}{96}\left[11N_1^4 - 78(N_1c\alpha_1)^2 - 9(c\alpha_1)^4\right], \\ \rho_{13} &\doteq -\frac{1}{24}\left[N_1^4 + 18(N_1c\alpha_1)^2 + 9(c\alpha_1)^4\right],\end{aligned}$$

where $K_2'[\phi]\zeta$ is given in eq. (2.95b). We choose $\phi^{(4)}$ and as a consequence $\nu^{(4)}$ depending on ξ and the arbitrary $\xi-$independent integration function has been set equal to zero to met the asymptotic conditions on all the functions $\phi^{(j)}$ and $\nu^{(i)}$ and their $\xi-$derivatives as $\xi \to \pm\infty$. The free constant χ is fixed so that to eliminate any dependence on the term $\partial_\xi^7\phi^{(1)}$ as from *Proposition 3.1* the presence of this term can always introduce a dependence on the secular term $K_4\left[\phi^{(1)}\right]$, the flux of the equation (2.93d). If one differentiates by ξ eq. (3.102b), we obtain an equation for $\partial_\xi\phi^{(3)}$ of the form of eq. (2.85b) with $j = 3$, $n = 2$, (2.86e, 2.90a).

- *The coefficients of the r. h. s. of this equation obviously respect all the A_3 integrability conditions (2.91a) for the KdV hierarchy. This proves the A_3 asymptotic integrability of the $A-L$ discrete NLS equation.*

Taking into account eqs. (2.92) with $n = 4$, (2.93d, 3.95), eq. (2.94) with $j = 2$, $n = 3$, (2.95c, 2.97b, 2.98, 3.95), eqs. (3.102b, 2.95b, 3.95) and that $\phi^{(4)}$ depends on ξ, eq. (3.101) becomes

$$\begin{aligned}
\nu^{(4)} &= c\alpha_1\partial_\xi\phi^{(4)} + (c\alpha_1)^2\,\upsilon_1\partial_\xi^3\phi^{(3)} - 2\rho_7\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi\phi^{(3)}\right) + c\alpha_1\upsilon_4\partial_\xi^5\phi^{(2)} + \\
&+\upsilon_2\left[\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^3\phi^{(2)}\right) + \left(\partial_\xi^3\phi^{(1)}\right)\left(\partial_\xi\phi^{(2)}\right)\right] - \rho_7\left(\partial_\xi\phi^{(2)}\right)^2 - \\
&-\frac{c\alpha_1 N_1^2}{2}\left(\partial_\xi\phi^{(1)}\right)^2\left(\partial_\xi\phi^{(2)}\right) + 2\upsilon_3\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(2)}\right) + \frac{N_1^2}{12}\rho_4\left(\partial_\xi\phi^{(1)}\right)^4 + \\
&+c\alpha_1\left[\upsilon_5\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(1)}\right)^2 + \upsilon_6\left(\partial_\xi\phi^{(1)}\right)^2\left(\partial_\xi^3\phi^{(1)}\right) + \upsilon_7\partial_\xi^7\phi^{(1)}\right] + \\
&+\upsilon_8\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^4\phi^{(1)}\right) + \upsilon_9\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^5\phi^{(1)}\right) + \upsilon_{10}\left(\partial_\xi^3\phi^{(1)}\right)^2, \\
&\upsilon_5 \doteq \frac{7}{64}\left[N_1^2 + (c\alpha_1)^2\right]^2, \qquad \upsilon_6 \doteq \frac{1}{96}\left[7N_1^4 + 6\,(c\alpha_1 N_1)^2 + 3\,(c\alpha_1)^4\right], \\
&\upsilon_7 \doteq \frac{1}{322560}\left[N_1^6 + 273N_1^4\,(c\alpha_1)^2 + 1575N_1^2\,(c\alpha_1)^4 + 1575\,(c\alpha_1)^6\right], \\
&\upsilon_8 \doteq \frac{1}{1536}\left[N_1^2 + (c\alpha_1)^2\right]\left[N_1^4 - 46\,(c\alpha_1 N_1)^2 - 207\,(c\alpha_1)^4\right], \\
&\upsilon_9 \doteq -\frac{1}{3840}\left[N_1^6 + 89N_1^4\,(c\alpha_1)^2 + 255N_1^2\,(c\alpha_1)^4 + 135\,(c\alpha_1)^6\right], \\
&\upsilon_{10} \doteq -\frac{1}{2304}\left[N_1^6 + 53N_1^4\,(c\alpha_1)^2 + 267N_1^2\,(c\alpha_1)^4 + 207\,(c\alpha_1)^6\right].
\end{aligned} \tag{3.103}$$

3.6 Multiscale analysis of differential-difference equations II: reduction of the *dNLS*

Let us consider the *dNLS* equation

$$\mathrm{i}\partial_t f_n + \frac{f_{n+1} - 2f_n + f_{n-1}}{2\sigma^2} = \epsilon|f_n|^2 f_n, \tag{3.104}$$

where $\epsilon \doteq \pm 1$. If we set $f_n(t) \doteq \nu_n(t)^{1/2}e^{\mathrm{i}\phi_n(t)}$, where $\nu_n(t)$ and $\phi_n(t)$ are both real functions, eq. (3.104), separating real and imaginary parts, turns into the following system of two real nonlinear differential-difference equations

$$\partial_t\nu_n + \frac{1}{\sigma^2}\left(\delta_+^{1/2}\sin\beta_+ + \delta_-^{1/2}\sin\beta_-\right) = 0, \tag{3.105a}$$

$$\partial_t\phi_n + \frac{1}{\sigma^2} - \frac{1}{2\sigma^2}\left(\gamma_+^{1/2}\cos\beta_+ + \gamma_-^{1/2}\cos\beta_-\right) + \varepsilon\nu_n = 0, \tag{3.105b}$$

where $\beta_\pm \doteq \phi_{n\pm1}(t) - \phi_n(t)$, $\gamma_\pm \doteq \nu_n(t)^{-1}\nu_{n\pm1}(t)$, $\delta_\pm \doteq \nu_n(t)\nu_{n\pm1}(t)$. We expand the functions $\nu_n(t)$ and $\phi_n(t)$ as in eqs. (3.87). The orders $\kappa = 0$, 1 are trivially satisfied.

i. Order $\kappa = 2$.

$$\nu^{(1)} = -\epsilon\alpha_1\partial_{t_1}\phi^{(1)}; \tag{3.106}$$

ii. Order $\kappa = 3$.

$$\alpha_1\partial_{t_1}\nu^{(1)} + \left(\frac{N_1}{\sigma}\right)^2\partial_{n_1}^2\phi^{(1)} = 0,$$

which, after inserting in it the expression (3.106), becomes

$$\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(1)} = 0, \qquad c \doteq \pm\frac{N_1}{\sigma\alpha_1}(\epsilon)^{1/2},$$

As before we have to choose $\epsilon = 1$. Hence we will eliminate σ^2 using the relation

$$\sigma^2 = \left(\frac{N_1}{c\alpha_1}\right)^2.$$

If one desires to study the continuum limit, in order to get a finite limit for c as $\sigma \to 0$, one should set $N_1 = \alpha_1\sigma$ so that $c = \pm 1$. We choose $\phi^{(1)}$ depending on $\xi \doteq n_1 - ct_1$ so that

$$\left(\partial_{t_1} + c\partial_\xi\right)\phi^{(1)} = 0, \tag{3.107}$$

from which, using (3.106),

$$\nu^{(1)} = \alpha_1 c\partial_\xi\phi^{(1)}, \tag{3.108}$$

so that $\nu^{(1)}$ itself depends on ξ;

iii. Order $\kappa = 4$.

Using (3.108), we find

$$\nu^{(2)} = -\alpha_1\partial_{t_1}\phi^{(2)} - \alpha_2\partial_{t_2}\phi^{(1)} + \frac{(c\alpha_1)^2}{2}\left[\frac{c\alpha_1}{2}\partial_\xi^3\phi^{(1)} - \left(\partial_\xi\phi^{(1)}\right)^2\right]; \tag{3.109}$$

iv. Order $\kappa = 5$.

$$\begin{aligned} &\alpha_1\partial_{t_1}\nu^{(2)} + \alpha_2\partial_{t_2}\nu^{(1)} + \\ &+ (c\alpha_1)^2\left[\partial_{n_1}^2\phi^{(2)} + \frac{N_1^2}{12}\partial_\xi^4\phi^{(1)} + \left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi\nu^{(1)}\right) + \nu^{(1)}\partial_\xi^2\phi^{(1)}\right] = 0. \end{aligned} \tag{3.110}$$

Taking into account eqs. (3.107, 3.108, 3.109), eq. (3.110) becomes

$$\alpha_1\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(2)} = 2c\partial_\xi\left[\alpha_2\partial_{t_2}\phi^{(1)} - (c\alpha_1)^2\rho_1\partial_\xi^3\phi^{(1)} + \frac{3(c\alpha_1)^2}{4}\left(\partial_\xi\phi^{(1)}\right)^2\right],$$

$$\rho_1 \doteq \frac{1}{8}\left(c\alpha_1 - \frac{N_1^2}{3c\alpha_1}\right). \tag{3.111}$$

The right hand side of eq. (3.111) is secular so we have to require that

$$\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(2)} = 0, \tag{3.112a}$$

$$\alpha_2\partial_{t_2}\phi^{(1)} - (c\alpha_1)^2\left[\rho_1\partial_\xi^3\phi^{(1)} - \frac{3}{4}\left(\partial_\xi\phi^{(1)}\right)^2\right] = 0, \tag{3.112b}$$

where we choose $\phi^{(2)}$ and as a consequence $\nu^{(2)}$ depending on ξ. The arbitrary $\xi-$independent integration function has been set equal to zero to met the asymptotic conditions on all the functions $\phi^{(j)}$ and $\nu^{(i)}$ and their $\xi-$derivatives as $\xi \to \pm\infty$. As we can see in eqs. (2.92) with $n = 2$ and (2.93b), eq. (3.112b) is a potential KdV equation for $\phi^{(1)}$ at the slow-time t_2 with

$$\tau_1 \doteq (c\alpha_1)^2 \rho_1/\alpha_2, \qquad \tau_2 \doteq -3\,(c\alpha_1)^2 \,/\, (4\alpha_2)\,. \tag{3.113}$$

From eq. (3.112b), differentiating once with respect to ξ and using (3.108), we have

$$\alpha_2 \partial_{t_2} \nu^{(1)} - (c\alpha_1)^2 \rho_1 \partial_\xi^3 \nu^{(1)} + \frac{3c\alpha_1}{2} \nu^{(1)} \partial_\xi \nu^{(1)} = 0,$$

which is a KdV equation for $\nu^{(1)}$ at the slow-time t_2. Taking into account eq. (3.112b) and that $\phi^{(2)}$ depends on ξ, eq. (3.109) becomes

$$\begin{aligned} \nu^{(2)} &= c\alpha_1 \partial_\xi \phi^{(2)} + (c\alpha_1)^2\, \upsilon_1 \partial_\xi^3 \phi^{(1)} + \frac{(c\alpha_1)^2}{4} \left(\partial_\xi \phi^{(1)}\right)^2, \\ \upsilon_1 &\doteq \frac{1}{8}\left(c\alpha_1 + \frac{N_1^2}{3c\alpha_1}\right); \end{aligned} \tag{3.114}$$

v. Order $\kappa = 6$.

Taking into account eqs. (3.108, 3.109, 3.112b), we have

$$\begin{aligned} \nu^{(3)} &= -\alpha_3 \partial_{t_3} \phi^{(1)} - \alpha_2 \partial_{t_2} \phi^{(2)} - \alpha_1 \partial_{t_1} \phi^{(3)} + \frac{(c\alpha_1)^3}{4} \partial_\xi^3 \phi^{(2)} - \\ &-(c\alpha_1)^2 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi \phi^{(2)}\right) - \frac{(c\alpha_1)^2 N_1^2}{8}\left(\partial_\xi^2 \phi^{(1)}\right)^2 + \\ &+(c\alpha_1)^2 \rho_2 \left(\partial_\xi \phi^{(1)}\right)\left(\partial_\xi^3 \phi^{(1)}\right) + (c\alpha_1)^3 \rho_3 \partial_\xi^5 \phi^{(1)}, \\ \rho_2 &\doteq -\frac{1}{24}\left[3(c\alpha_1)^2 + 4N_1^2\right], \qquad \rho_3 \doteq \frac{1}{32}\left[(c\alpha_1)^2 + N_1^2\right]; \end{aligned} \tag{3.115}$$

vi. Order $\kappa = 7$.

Taking into account eqs. (3.108, 3.109, 3.112b, 3.115), that $\phi^{(1)}$ and $\phi^{(2)}$ depend on ξ and the fact that $\phi^{(1)}$ evolves at the slow-time t_3 according to the equation given by (2.92) with $n = 3$, (2.93c, 3.113), we have

$$\alpha_1\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(3)} = 2c\alpha_2\partial_\xi\left\{\partial_{t_2}\phi^{(2)} - \tau_1\partial_\xi^3\phi^{(2)} - 2\tau_2\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi\phi^{(2)}\right) - \right.$$
$$-\theta_1\left(\partial_\xi\phi^{(1)}\right)^3 - \theta_2\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^3\phi^{(1)}\right) - \theta_3\left(\partial_\xi^2\phi^{(1)}\right)^2 +$$
$$\left. + \frac{1}{\alpha_2}\left(\alpha_3\lambda - c\alpha_1\rho_4\right)\partial_\xi^5\phi^{(1)}\right\}, \tag{3.116}$$
$$\rho_4 \doteq \frac{1}{1920}\left[-N_1^4 + 30(N_1c\alpha_1)^2 + 15(c\alpha_1)^4\right],$$
$$\theta_1 \doteq -\frac{c\alpha_1\left[-4N_1^6 + 15(c\alpha_1)^2N_1^4 + 234(c\alpha_1)^4N_1^2 + 135(c\alpha_1)^6\right]}{48\alpha_2\left[N_1^2 - 3(c\alpha_1)^2\right]^2},$$
$$\theta_2 \doteq -\frac{(c\alpha_1)^2\left[N_1^2 + (c\alpha_1)^2\right]\left[5N_1^2 + 9(c\alpha_1)^2\right]}{32\alpha_2\left[N_1^2 - 3(c\alpha_1)^2\right]},$$
$$\theta_3 \doteq -\frac{(c\alpha_1)^2\left[N_1^4 + 2(c\alpha_1N_1)^2 + 9(c\alpha_1)^4\right]}{16\alpha_2\left[N_1^2 - 3(c\alpha_1)^2\right]}.$$

As the r. h. s. of eq. (3.116) is in the kernel of the operator in the l. h. s., the r. h. s. is secular. In order to remove this secularity, both members have to be set equal to zero and the free constant λ has to be chosen so that to eliminate any dependence on the term $\partial_\xi^5\phi^{(1)}$ as from *Proposition 3.1* the presence of this term can always introduce a dependence on the secular term $K_3\left[\phi^{(1)}\right]$, the flux of the equation (2.93c). We get

$$\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(3)} = 0, \tag{3.117a}$$
$$\partial_{t_2}\phi^{(2)} - K_2'\left[\phi^{(1)}\right]\phi^{(2)} = f_2\left(2\right), \qquad \lambda = \frac{\rho_4 c\alpha_1}{\alpha_3}, \tag{3.117b}$$

where $K_2'\left[\phi\right]\zeta$ is given in eq. (2.95b) and $f_2\left(2\right)$ in eq. (2.97a). We choose $\phi^{(3)}$ and as a consequence $\nu^{(3)}$ depending on ξ and the arbitrary $\xi-$independent integration function has been set equal to zero to met the asymptotic conditions on all the functions $\phi^{(j)}$ and $\nu^{(i)}$ and their $\xi-$derivatives as $\xi \to \pm\infty$. As we saw in *Subsection 2.2.3*, We have no A_2 integrability conditions for the coefficients θ_i, $i = 1, \ldots, 3$ of $f_2\left(2\right)$ in the case of the potential *KdV* hierarchy. Taking into account eq. (2.92) with $n = 3$, (2.93c, 3.113), eq. (3.117b) and that $\phi^{(3)}$ depends on ξ, eq. (3.115) becomes

$$\nu^{(3)} = c\alpha_1\partial_\xi\phi^{(3)} + \left(c\alpha_1\right)^2\upsilon_1\partial_\xi^3\phi^{(2)} + \frac{\left(c\alpha_1\right)^2}{2}\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi\phi^{(2)}\right) - \tag{3.118}$$
$$-\frac{c\alpha_1N_1^2}{12}\left(\partial_\xi\phi^{(1)}\right)^3 - \frac{\left(c\alpha_1\right)^3}{2}\rho_1\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^3\phi^{(1)}\right) +$$
$$+\left(c\alpha_1\right)^2\upsilon_2\left(\partial_\xi^2\phi^{(1)}\right)^2 + c\alpha_1\upsilon_3\partial_\xi^5\phi^{(1)},$$
$$\upsilon_2 \doteq -\frac{1}{64}\left[3N_1^2 + 7\left(c\alpha_1\right)^2\right], \qquad \upsilon_3 \doteq \frac{1}{1920}\left[N_1^4 + 30\left(c\alpha_1N_1\right)^2 + 45\left(c\alpha_1\right)^4\right];$$

vii. Order $\kappa = 8$.

Taking into account eqs. (3.108, 3.109, 3.112b, 3.115, 3.117b) and the fact that $\phi^{(1)}$ evolves at the slow-time t_3 according to the equation given by (2.92) with $n = 3$, (2.93c, 3.113), we have

$$\nu^{(4)} = -\alpha_4\partial_{t_4}\phi^{(1)} - \alpha_3\partial_{t_3}\phi^{(2)} - \alpha_2\partial_{t_2}\phi^{(3)} - \alpha_1\partial_{t_1}\phi^{(4)} + \tag{3.119}$$
$$+(c\alpha_1)^3\rho_5\left(\partial_\xi\phi^{(1)}\right)^2\left(\partial_\xi^3\phi^{(1)}\right) - \frac{3N_1^2(c\alpha_1)^3}{8}\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(1)}\right)^2 +$$
$$+(c\alpha_1)^2\rho_6\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^5\phi^{(1)}\right) + (c\alpha_1)^2\rho_7\left(\partial_\xi^3\phi^{(1)}\right)^2 +$$
$$+(c\alpha_1)^2\rho_8\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^4\phi^{(1)}\right) + (c\alpha_1)^3\rho_9\left(\partial_\xi^7\phi^{(1)}\right) +$$
$$+(c\alpha_1)^2\rho_2\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi^3\phi^{(2)}\right) - \frac{(c\alpha_1)^2N_1^2}{4}\left(\partial_\xi^2\phi^{(1)}\right)\left(\partial_\xi^2\phi^{(2)}\right) +$$
$$+(c\alpha_1)^2\rho_2\left(\partial_\xi^3\phi^{(1)}\right)\left(\partial_\xi\phi^{(2)}\right) - \frac{(c\alpha_1)^2}{2}\left(\partial_\xi\phi^{(2)}\right)^2 + (c\alpha_1)^3\rho_3\left(\partial_\xi^5\phi^{(2)}\right) -$$
$$-(c\alpha_1)^2\left(\partial_\xi\phi^{(1)}\right)\left(\partial_\xi\phi^{(3)}\right) + \frac{(c\alpha_1)^3}{4}\left(\partial_\xi^3\phi^{(3)}\right) + \frac{N_1^2(c\alpha_1)^2}{24}\left(\partial_\xi\phi^{(1)}\right)^4,$$
$$\rho_5 \doteq \frac{1}{16}\left[-3N_1^2 + (c\alpha_1)^2\right],$$
$$\rho_6 \doteq -\frac{1}{960}\left[8N_1^4 + 15(c\alpha_1 N_1)^2 + 45(c\alpha_1)^4\right],$$
$$\rho_7 \doteq -\frac{1}{1152}\left[16N_1^4 + 33(c\alpha_1 N_1)^2 + 117(c\alpha_1)^4\right],$$
$$\rho_8 \doteq -\frac{1}{384}\left[8N_1^4 + 9(c\alpha_1 N_1)^2 + 45(c\alpha_1)^4\right],$$
$$\rho_9 \doteq \frac{1}{7680}\left[13N_1^4 + 50(c\alpha_1 N_1)^2 + 45(c\alpha_1)^4\right];$$

viii. Order $\kappa = 9$.

Taking into account eqs. (2.92) with $n = 3$, 4, (2.93c, 2.93d, 3.113), eq. (2.94) with $j = 2$, $n = 3$, (2.95c, 2.97b, 2.98, 3.113), eqs. (3.108, 3.109, 3.112b, 3.115, 3.117b, 3.119), that $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ depend on ξ and separating to remove the secular terms, we get

$$\left(\partial_{t_1}^2 - c^2\partial_{n_1}^2\right)\phi^{(4)} = 0, \tag{3.120a}$$
$$\partial_{t_2}\varphi^{(3)} - H_2'\left[\varphi^{(1)}\right]\varphi^{(3)} = g_2(3), \tag{3.120b}$$
$$\chi = \frac{c\alpha_1\left[-N_1^6 + 273N_1^4(c\alpha_1)^2 + 525N_1^2(c\alpha_1)^4 + 315(c\alpha_1)^6\right]}{322560\alpha_4},$$

where $\varphi^{(j)} \doteq \partial_\xi\phi^{(j)}$, $j = 1, \ldots, 3$, $H_2'[\varphi]\phi$ is given in eq. (2.86e) and $g_2(3)$ in eq. (2.90a). We choose $\phi^{(4)}$ and as a consequence $\nu^{(4)}$ depending on ξ. We had to pass from the potential KdV hierarchy to the KdV hierarchy as, integrating once with respect to ξ, one cannot obtain a purely differential evolution equation for $\phi^{(3)}$ at the slow time t_2 but we get an integro-differential equation (see the corresponding situations in the right hand sides of the eqs. (3.111, 3.116) where a ∂_ξ operator has been put in evidence). This depend from the fact that the first of the three conditions (2.100) is not satisfied by the coefficients μ_j, $j = 1, \ldots, 14$ of the forcing term $g_2(3)$ (see later). On the contrary the $A - L$ discrete NLS equation satisfied these conditions. The free constant χ is fixed so that to eliminate any dependence on the term $\partial_\xi^7\phi^{(1)}$ as from *Proposition 3.1* the presence of this term can always introduce a dependence on the secular term $K_4\left[\phi^{(1)}\right]$, the flux of the equation (2.93d). Here follow the coefficients μ_j, $j = 1, \ldots, 14$ of the forcing term $g_2(3)$

$$\mu_1 \doteq -\frac{(c\alpha_1)^2\left[N_1^8+64(c\alpha_1)^2N_1^6+242(c\alpha_1N_1)^4-40(c\alpha_1)^6N_1^2-27(c\alpha_1)^8\right]}{128\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^2},$$
$$\mu_2 \doteq -\frac{(c\alpha_1)^2\left[15N_1^8+440N_1^6(c\alpha_1)^2+1710(c\alpha_1N_1)^4-144N_1^2(c\alpha_1)^6-405(c\alpha_1)^8\right]}{1536\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^2},$$
$$\mu_3 \doteq -\frac{(c\alpha_1)^2\left[13N_1^4+30(c\alpha_1N_1)^2-15(c\alpha_1)^4\right]\left[3N_1^4+62(c\alpha_1N_1)^2+27(c\alpha_1)^4\right]}{7680\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^2},$$
$$\mu_4 \doteq \frac{(c\alpha_1)\left[4N_1^{10}-69N_1^8(c\alpha_1)^2-1668N_1^6(c\alpha_1)^4-15606N_1^4(c\alpha_1)^6-10440N_1^2(c\alpha_1)^8+243(c\alpha_1)^{10}\right]}{768\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^3},$$
$$\mu_5 \doteq \frac{(c\alpha_1)\left[4N_1^{10}-3N_1^8(c\alpha_1)^2-1580N_1^6(c\alpha_1)^4-10026N_1^4(c\alpha_1)^6-5232N_1^2(c\alpha_1)^8+405(c\alpha_1)^{10}\right]}{128\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^3},$$
$$\mu_6 \doteq \frac{(c\alpha_1)\left[N_1^{10}+18N_1^8(c\alpha_1)^2-741N_1^6(c\alpha_1)^4-3753N_1^4(c\alpha_1)^6-1152N_1^2(c\alpha_1)^8+243(c\alpha_1)^{10}\right]}{192\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^3},$$
$$\mu_7 \doteq \frac{(c\alpha_1N_1)^2\left[7N_1^8+105N_1^6(c\alpha_1)^2-423(c\alpha_1N_1)^4-15309N_1^2(c\alpha_1)^6-8964(c\alpha_1)^8\right]}{48\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^4},$$
$$\mu_8=\mu_{11} \doteq -\frac{(c\alpha_1)^2\left[5N_1^4+14(c\alpha_1N_1)^2+9(c\alpha_1)^4\right]}{32\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]},$$
$$\mu_9=\mu_{10} \doteq -\frac{(c\alpha_1)^2\left[9N_1^4+22(c\alpha_1N_1)^2+45(c\alpha_1)^4\right]}{32\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]},$$
$$2\mu_{12}=\mu_{13} \doteq -\frac{(c\alpha_1)\left[-4N_1^6+15N_1^4(c\alpha_1)^2+234N_1^2(c\alpha_1)^4+135(c\alpha_1)^6\right]}{8\alpha_2\left[N_1^2-3(c\alpha_1)^2\right]^2}, \quad \mu_{14}\doteq 2\tau_2.$$

- *The coefficients of $g_2(3)$ respect* **only four out of the five** *A_3 integrability conditions (2.91a) for the KdV hierarchy (the one involving μ_7 is not satisfied). This proves that the dNLS equation is* **not integrable**.

On the contrary the integrability condition involving μ_7 results satisfied only in the continuous limit $\sigma \to 0$ (integrable NLS equation) and in the particular case where $450(c\alpha_1)^8 + 345N_1^2(c\alpha_1)^6 - 1413(c\alpha_1N_1)^4 - 557N_1^6(c\alpha_1)^2 - 17N_1^8 = 0$.

4

Multiscale reduction of $C-$integrable nonlinear $P\Delta E$s

We present here some examples of reduction of $C-$integrable nonlinear partial difference equations. The $C-$integrability property is a consequence of the fact that the considered nonlinear equations can be transformed into linear ones by means of some invertible transformation of the dependent (sometimes also of the independent) variables. The aim of the following examples is to show that under a multiscale reduction, contrary to the case of $S-$integrable $P\Delta E$s, now the amplitude $u_1^{(1)}$ follows a hierarchy of *linear* equations. This property together with all the integrability conditions developed in *Chapter 2* opens the way to an integrability test for discretizations of $C-$integrable PDEs. In this way one can in principle prove if a discretization fails in preserving this $C-$integrability property.

4.1 Multiscale analysis of the *Hietarinta* equation

Let us consider the *Hietarinta* equation [18]

$$\frac{u_{n,m}+e_2}{u_{n,m}+e_1}\cdot\frac{u_{n+1,m+1}+o_2}{u_{n+1,m+1}+o_1}=\frac{u_{n+1,m}+e_2}{u_{n+1,m}+o_1}\cdot\frac{u_{n,m+1}+o_2}{u_{n,m+1}+e_1}. \tag{4.1}$$

For future convenience we transform the parameters as $o_i \to 1/o_i$, $e_i \to 1/e_i$, $i=1,2$ and introduce the parameters $\mu \doteq o_2-o_1$, $\zeta \doteq o_2-e_1$. Eq. (4.1) then becomes

$$P_l = P_{nl}^{(2)} + P_{nl}^{(3)}, \tag{4.2}$$

where

$$\begin{aligned}
P_l &\doteq (e_2-e_1)\,u_{n,m}+(o_1-e_2)\,u_{n+1,m}+\mu u_{n+1,m+1}-\zeta u_{n,m+1},\\
P_{nl}^{(2)} &\doteq (e_2-o_2)\left(o_1u_{n+1,m}u_{n+1,m+1}-e_1u_{n,m}u_{n,m+1}\right)+\\
&\quad+(\mu-\zeta)\left(e_2u_{n,m}u_{n+1,m}-o_2u_{n,m+1}u_{n+1,m+1}\right)+\\
&\quad+(o_1e_1-o_2e_2)\left(u_{n,m}u_{n+1,m+1}-u_{n+1,m}u_{n,m+1}\right),\\
P_{nl}^{(3)} &\doteq -e_2o_1\zeta u_{n,m}u_{n+1,m}u_{n+1,m+1}+e_1o_2\,(o_1-e_2)\,u_{n,m}u_{n,m+1}u_{n+1,m+1}+\\
&\quad+e_1e_2\mu u_{n,m}u_{n+1,m}u_{n,m+1}+o_1o_2\,(e_2-e_1)\,u_{n+1,m}u_{n,m+1}u_{n+1,m+1},
\end{aligned}$$

where P_l indicates the linear part, $P_{nl}^{(2)}$ the quadratic part and $P_{nl}^{(3)}$ the cubic part. Defining a new function $Q_{n,m}$ according to the *invertible* transformation

$$\frac{Q_{n+1,m}}{Q_{n,m}} \doteq \frac{o_2-\mu}{o_2}\left[\frac{1+o_2u_{n,m}}{1+(o_2-\mu)\,u_{n,m}}+\frac{\zeta}{\mu-\zeta}\right], \tag{4.3}$$

the eq. (4.2) linearizes to the equation [36]

$$Q_{n+1,m+1} - AQ_{n,m+1} - Q_{n+1,m} + (A-B)\,Q_{n,m} = 0, \tag{4.4}$$
$$A \doteq -\frac{\zeta\,(\mu-o_2)}{o_2\,(\mu-\zeta)}, \qquad B \doteq \frac{(\mu-o_2)\,(o_2-e_2)}{o_2\,(o_2-e_2-\mu)}.$$

We will expand eq. (4.2) according to eq. (2.117). One can transform eq. (4.4) to a simpler form considering that $u_{n,m} \to 0$ as $\varepsilon \to 0$, asymptotic behavior which is compatible with eq. (4.2). As $\varepsilon \to 0$, definition (4.3) becomes

$$Q_{n+1,m} - \lambda Q_{n,m} = 0, \qquad \lambda \doteq \frac{\mu(o_2-\mu)}{o_2\,(\mu-\zeta)},$$

which gives, as $\varepsilon \to 0$, $Q_{n,m} = \lambda^n f_m$, with f_m an arbitrary function of m. So we put $Q_{n,m} \doteq \lambda^n f_{n,m}$, with $f_{n,m} \to f_m$ when $\varepsilon \to 0$, and substitute this expression into eq. (4.4) obtaining

$$\mu f_{n+1,m+1} - \zeta f_{n,m+1} - \mu f_{n+1,m} + \frac{\mu\,(e_2-e_1)}{e_2-o_1} f_{n,m} = 0. \tag{4.5}$$

As $\varepsilon \to 0$, eq. (4.5) becomes

$$f_{m+1} - \frac{\mu}{e_2-o_1} f_m = 0,$$

which, solved, gives $f_m = \left[\mu/\left(e_2-o_1\right)\right]^m \rho$, ρ a constant. Consequently we define $f_{n,m} \doteq \left[\mu/\left(e_2-o_1\right)\right]^m \rho\,(1+r_{n,m})$, with $r_{n,m} \to 0$ when $\varepsilon \to 0$, and substitute into eq. (4.5) obtaining

$$\mu r_{n+1,m+1} - \zeta r_{n,m+1} - (e_2-o_1)\,r_{n+1,m} + (e_2-e_1)\,r_{n,m} = 0. \tag{4.6}$$

The complete transformation between eq. (4.4) and eq. (4.6) reads

$$Q_{n,m} \doteq \rho \left[\frac{\mu\,(o_2-\mu)}{o_2\,(\mu-\zeta)}\right]^n \cdot \left(\frac{\mu}{e_2-o_1}\right)^m (1+r_{n,m}), \tag{4.7}$$

and remains valid even if we don't prescribe any asymptotic behavior for $u_{n,m}$ (and consequently for $Q_{n,m}$ and $r_{n,m}$). Hence the transformation from eq. (4.2) to eq. (4.6) reads

$$\frac{1+r_{n+1,m}}{1+r_{n,m}} = \frac{\mu-\zeta}{\mu}\left[\frac{1+o_2 u_{n,m}}{1+(o_2-\mu)\,u_{n,m}} + \frac{\zeta}{\mu-\zeta}\right]. \tag{4.8}$$

The form of eq. (4.6) will result particularly useful for the future. One could say that eqs. (4.6, 4.8) are the Lax pair of eq. (4.2). Let us begin the multiscale expansion of eq. (4.2) referring to *Appendix D* for useful formulas.

i. Order $\gamma = 1$.

$$\left[e_1 - e_2 + \zeta e^{-\mathrm{i}\alpha\omega} + (e_2-o_1)\,e^{\mathrm{i}\alpha\kappa} - \mu e^{\mathrm{i}\alpha(\kappa-\omega)}\right] u_1^{(\alpha)} = 0, \tag{4.9}$$

which, if $\alpha = 0$, is identically solved $\forall\ u_1^{(0)}$ and, if $\alpha = 1$ and $u_1^{(1)} \neq 0$, gives the dispersion relation,

$$e^{-\mathrm{i}\omega} = \frac{\left[e_1 - e_2 + (e_2 - o_1)\, e^{\mathrm{i}\kappa}\right]}{(\mu e^{\mathrm{i}\kappa} - \zeta)},$$

from which

$$\begin{aligned}
e^{Im(\omega)} \cos\left[Re(\omega)\right] &= \frac{(e_2 - o_1)\,(\mu - \zeta\cos\kappa) - (e_2 - e_1)\,(\mu\cos\kappa - \zeta)}{\zeta^2 + \mu^2 - 2\mu\zeta\cos\kappa},\\
e^{Im(\omega)} \sin\left[Re(\omega)\right] &= \frac{(\mu + \beta)(o_2 - e_2)\sin\kappa}{\zeta^2 + \mu^2 - 2\mu\zeta\cos\kappa)},\\
e^{2Im(\omega)} &= \frac{(e_2 - e_1)^2 + (e_2 - o_1)^2 - 2(e_2 - e_1)(e_2 - o_1)\cos\kappa}{\zeta^2 + \mu^2 - 2\mu\zeta\cos\kappa}.
\end{aligned}$$

In order to have a real dispersion relation, from the last relation we have that one has to require that

$$o_1 + e_1 = o_2 + e_2, \tag{4.10}$$

so that

$$e^{-\mathrm{i}\omega} = \frac{\mu - \zeta e^{\mathrm{i}\kappa}}{\mu e^{\mathrm{i}\kappa} - \zeta}, \tag{4.11}$$

which, solved, gives

$$\omega\,(\kappa) = 2\arctan\left(\frac{\zeta + \mu}{\mu - \zeta}\tan\frac{\kappa}{2}\right). \tag{4.12}$$

This dispersion relation is just the dispersion relation of the $lpKdV$ equation given in eqs. (3.6, 3.7). Taking into account eq. (4.10), the linear, quadratic and cubic parts of eq. (4.2) simplifie to

$$P_l = \mu(u_{n+1,m+1} - u_{n,m}) + \zeta(u_{n+1,m} - u_{n,m+1}), \tag{4.13a}$$

$$\begin{aligned}
P_{nl}^{(2)} &= (\mu + \zeta)\,(e_1 u_{n,m} u_{n,m+1} - o_1 u_{n+1,m} u_{n+1,m+1}) + \\
&+(\mu - \zeta)\,(e_2 u_{n,m} u_{n+1,m} - o_2 u_{n,m+1} u_{n+1,m+1}) + \\
&+\mu\zeta(u_{n,m} u_{n+1,m+1} - u_{n+1,m} u_{n,m+1}),
\end{aligned} \tag{4.13b}$$

$$\begin{aligned}
P_{nl}^{(3)} &= \zeta u_{n,m} u_{n+1,m+1}\,(e_1 o_2 u_{n,m+1} - e_2 o_1 u_{n+1,m}) + \\
&+\mu u_{n+1,m} u_{n,m+1}\,(e_1 e_2 u_{n,m} - o_1 o_2 u_{n+1,m+1})\,.
\end{aligned} \tag{4.13c}$$

As one can see, eq. (4.13a) is just the linear part of the $lpKdV$ equation (3.3). If $|\alpha| \geq 2$, eq. (4.9) implies that $u_1^{(\alpha)} = 0$;

ii. Order $\gamma = 2$.

- $\alpha = 0$:

$$\left[N_1\left(\mu+\zeta\right)\partial_{n_1} + M_1\left(\mu-\zeta\right)\partial_{m_1}\right]u_1^{(0)} = 0, \tag{4.14}$$

which gives

$$u_1^{(0)}\left(n_1, \{m_j\}_{j=1}^K\right) = F\left(\rho, \{m_j\}_{j=2}^K\right), \qquad \rho \doteq n_1 - \frac{N_1\left(\mu+\zeta\right)}{M_1\left(\mu-\zeta\right)}m_1, \tag{4.15}$$

where F is an arbitrary function of its arguments;

- $\alpha = 1$:

$$\begin{aligned}\left[e^{\mathrm{i}\kappa}\left(\mu e^{-\mathrm{i}\omega}+\zeta\right)N_1\partial_{n_1} + e^{-\mathrm{i}\omega}\left(\mu e^{\mathrm{i}\kappa}-\zeta\right)M_1\partial_{m_1}\right]u_1^{(1)} &= \\ = -\frac{\mu\zeta\left(\mu+\zeta\right)\left(1-e^{\mathrm{i}\kappa}\right)^2}{\mu e^{\mathrm{i}\kappa}-\zeta}u_1^{(0)}u_1^{(1)},\end{aligned} \tag{4.16}$$

which, taking into account eq. (4.11) and solved, gives

$$\begin{aligned}u_1^{(1)} &= H\left(n_1 + \frac{1}{\sigma}m_1, \{m_j\}_{j=2}^K\right)e^{\frac{\zeta-\mu}{N_1}\int_{\rho_0}^{\rho}u_1^{(0)}(\rho')d\rho'}, \\ \sigma &\doteq -\frac{M_1\left(\mu^2+\zeta^2-2\mu\zeta\cos\kappa\right)}{N_1\left(\mu^2-\zeta^2\right)},\end{aligned} \tag{4.17}$$

where H is an arbitrary function of its argument and ρ_0 is a (real) constant. In the case one choses $u_1^{(0)} = 0$, defining

$$N_1 \doteq \epsilon S e^{-\mathrm{i}\omega}\left(\mu e^{\mathrm{i}\kappa}-\zeta\right), \qquad M_1 \doteq -S e^{\mathrm{i}\kappa}\left(\mu e^{-\mathrm{i}\omega}+\zeta\right), \qquad S \in \mathbb{C}, \qquad \epsilon = \pm 1, \tag{4.18}$$

eq. (4.16) becomes

$$\left(\partial_{n_1} - \epsilon\partial_{m_1}\right)u_1^{(1)} = 0, \tag{4.19}$$

which is solved by

$$u_1^{(1)}\left(n_1, \{m_j\}_{j=1}^K\right) = u_1^{(1)}\left(n_2, \{m_j\}_{j=2}^K\right), \qquad n_2 \doteq n_1 + \epsilon m_1, \qquad \epsilon = -\frac{N_1}{M_1}\omega_1, \tag{4.20}$$

the quantities ω_n being defined in eq. (2.12). The complex constant $S \doteq re^{\mathrm{i}\theta}$, $r > 0$, is to be choosen so that $\theta = -\arctan\left[\zeta\sin\kappa/\left(\zeta\cos\kappa-\mu\right)\right]$ in such a way that N_1 and M_1 are indeed real numbers, which, taking into account the dispersion relation (4.11), can be rewritten as

$$N_1 = \epsilon S\left(\mu - \zeta e^{\mathrm{i}\kappa}\right), \qquad M_1 = S e^{\mathrm{i}\kappa}\frac{\zeta^2-\mu^2}{\mu e^{\mathrm{i}\kappa}-\zeta}; \tag{4.21}$$

- $\alpha = 2$:
  Taking into account the dispersion relation (4.11), we find

$$u_2^{(2)} = \left(o_2 + \frac{\mu e^{\mathrm{i}\kappa} - \zeta}{1 - e^{\mathrm{i}\kappa}} \right) u_1^{(1)2}; \tag{4.22}$$

iii. Order $\gamma = 3$.

- $\alpha = 0$:
  Taking into account eqs. (4.11, 4.17) and requiring no secular terms (see eq. (4.15)), we find

$$\left[\partial_{n_1} + \frac{M_1 (\mu - \zeta)}{N_1 (\mu + \zeta)} \partial_{m_1} \right] u_2^{(0)} = \tag{4.23a}$$
$$= \frac{4\mu\zeta (2o_2 - \zeta - \mu) \sin^2 (\kappa/2)}{\mu^2 + \zeta^2 - 2\mu\zeta \cos\kappa} e^{\frac{2(\zeta-\mu)}{N_1} \int_{\rho_0}^{\rho} u_1^{(0)}(\rho') d\rho'} \partial_{n_1} |H|^2,$$
$$\partial_{m_2} u_1^{(0)} = 0, \tag{4.23b}$$

so that

$$u_2^{(0)} = G\left(\rho, \{m_j\}_{j=2}^{K}\right) + (2o_2 - \zeta - \mu)\, |u_1^{(1)}|^2, \tag{4.24}$$

where G is an arbitrary function of its arguments which we choose to be zero. In the case one choses $u_1^{(0)} = 0$, the equation for $u_2^{(0)}$ could be simply solved introducing the variable $\widetilde{n}_2 \doteq n_1 - \epsilon m_1$. In this case, if $G = 0$, $u_2^{(0)}$ depends on n_2;

- $\alpha = 1$:
  Setting $u_1^{(0)} = 0$, taking into account eqs. (4.11, 4.21, 4.22, 4.24) and that $u_1^{(1)}$ depends on n_2, we have

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_2^{(1)} = \mathcal{N}_1 \left(u_1^{(1)}\right),$$

where $\mathcal{N}_1 \left(u_1^{(1)}\right)$ is a nonlinear function in $u_1^{(1)}$ and its complex conjugate. As the r.h.s. of the last equation depends on n_2, it is in the kernel of the linear operator on the l.h.s. and consequently it is a *secular* term. In order to remove this secularity, we have to demand that both the r.h.s. and the l.h.s. be equal to zero. We obtain

$$\left(\partial_{n_1} - \epsilon \partial_{m_1}\right) u_2^{(1)} = 0, \tag{4.25a}$$
$$\mathrm{i}\partial_{m_2} u_1^{(1)} = \rho_1 \partial_{n_2}^2 u_1^{(1)}, \qquad \rho_1 \doteq \frac{\mu\zeta M_1^2 \sin\kappa}{M_2 (\mu^2 - \zeta^2)} = -\frac{N_1^2}{M_2} \omega_2. \tag{4.25b}$$

Equation (4.25a) tells us that $u_2^{(1)}$ depends on n_2.

- *Equation (4.25b) is a* **linear** *Schrödinger equation.*

4.2 Reduction of the Lax pair for the *Hietarinta* equation

One could say that a Lax pair for a $C-$integrable equation is formed by

- The linearizing transformation;
- The equation to which it linearizes.

In this respect the Lax pair of the *Hietarinta* equation is formed by eqs. (4.6, 4.8). Now we apply a multiscale reduction to this Lax pair as we did for the *lpKdV* spectral problem. The multiscale procedure requires that the parameters respect the constraint (4.10), so that the Lax pair simplifies to

$$\frac{1+r_{n+1,m}}{1+r_{n,m}} = \frac{\mu-\zeta}{\mu}\left[\frac{1+o_2 u_{n,m}}{1+(o_2-\mu)\,u_{n,m}}+\frac{\zeta}{\mu-\zeta}\right], \tag{4.26a}$$

$$\mu\left(r_{n+1,m+1}-r_{n,m}\right)+\zeta\left(r_{n+1,m}-r_{n,m+1}\right)=0. \tag{4.26b}$$

One can see from eq. (4.26b) that in this reduction the Hietarinta equation linearizes to its linear part eq. (4.13a) which is the same linear part of the *lpKdV* equation (3.3). We begin expanding eq. (4.26b), choosinf for $r_{n,m}$ an expansion of the form given in eq. (2.117) with the slow-variables scaling with ε in the usual way. The coefficients N_1 and M_j, $j \geq 1$, are chosen to be the same as in the case of the *Hietarinta* equation.

i. Order $\gamma = 1$.

$$\left[\mu(e^{\mathrm{i}\alpha(\kappa-\omega)}-1)+\zeta(e^{\mathrm{i}\alpha\kappa}-e^{-\mathrm{i}\alpha\omega})\right] r_1^{(\alpha)}=0, \tag{4.27}$$

which, if $\alpha = 0$, is identically solved $\forall\ r_1^{(0)}$ and, if $\alpha = 1$ and $r_1^{(1)} \neq 0$, gives an analogous dispersion relation than that of the Hietarinta equation, eq. (4.11). If $|\alpha| \geq 2$, all other $r_1^{(\alpha)} = 0$;

ii. Order $\gamma = 2$.

- $\alpha = 0$:

$$\left[N_1(\mu+\zeta)\partial_{n_1}+M_1(\mu-\zeta)\partial_{m_1}\right] r_1^{(0)}=0. \tag{4.28}$$

  As a consequence, taking into account equations (4.21), $r_1^{(0)}$ depends on ρ defined in (4.15);

- $\alpha = 1$:

$$\left[e^{\mathrm{i}\kappa}\left(\mu e^{-\mathrm{i}\omega}+\zeta\right)N_1\partial_{n_1}+e^{-\mathrm{i}\omega}\left(\mu e^{\mathrm{i}\kappa}-\zeta\right)M_1\partial_{m_1}\right] r_1^{(1)}=0, \tag{4.29}$$

  which, taking into account the definitions (4.18), becomes

$$\left(\partial_{n_1}-\epsilon\partial_{m_1}\right) r_1^{(1)}=0, \tag{4.30}$$

  so that $r_1^{(1)}$ depends on n_2 defined in (4.20);

- $\alpha = 2$:

$$\left[\mu\left(e^{2\mathrm{i}(\kappa-\omega)}-1\right)-\zeta\left(e^{-2\mathrm{i}\omega}-e^{2\mathrm{i}\kappa}\right)\right]r_2^{(2)}=0, \tag{4.31}$$

which implies that $r_2^{(2)} = 0$;

iii. Order $\gamma = 3$.

- $\alpha = 0$:
Taking into account eq. (4.28) and requiring no secular terms, we obtain

$$[N_1(\mu+\zeta)\partial_{n_1}+M_1(\mu-\zeta)\partial_{m_1}]\,r_2^{(0)}=0, \tag{4.32a}$$
$$\partial_{m_2}r_1^{(0)}=0. \tag{4.32b}$$

As a consequence, taking into account eq. (4.21), $r_2^{(0)}$ depends on ρ too and $r_1^{(0)}$ is independent on m_2;

- $\alpha = 1$:
Taking into account eqs. (4.11, 4.21), that $r_1^{(1)}$ depends on n_2 and requiring no secular terms, we obtain

$$(\partial_{n_1}-\epsilon\partial_{m_1})\,r_2^{(1)}=0, \tag{4.33a}$$
$$\mathrm{i}\partial_{m_2}r_1^{(1)}=\rho_1\partial_{n_2}^2 r_1^{(1)}, \qquad \rho_1 \doteq \frac{\mu\zeta M_1^2\sin\kappa}{M_2\left(\mu^2-\zeta^2\right)}=-\frac{N_1^2}{M_2}\omega_2, \tag{4.33b}$$

so that $r_2^{(1)}$ depends on n_2 and $r_1^{(1)}$ evolves at the slow-time m_2 according to the same linear Schrödinger equation of $u_1^{(1)}$, eq. (4.25b).

Now we apply a multiscale reduction to eq. (4.26a). Expanding the l. h. s. of eq. (4.26a) near $r_{n,m} = 0$ and the r. h. s. near $u_{n,m} = 0$, we obtain

$$\begin{aligned}(r_{n+1,m}-r_{n,m})\left(1-r_{n,m}+r_{n,m}^2-r_{n,m}^3+\mathcal{O}\left(r_{n,m}^4\right)\right)=\\ =(\mu-\zeta)\left[1-(o_2-\mu)u_{n,m}+(o_2-\mu)^2u_{n,m}^2+\mathcal{O}\left(u_{n,m}^3\right)\right]u_{n,m}.\end{aligned} \tag{4.34}$$

Now we insert the expansions of $r_{n,m}$ and $u_{n,m}$ into eq. (4.34). We obtain the following identifications.

i. Order $\gamma = 1$.

$$r_1^{(\alpha)}(e^{\mathrm{i}\alpha\kappa}-1)=(\mu-\zeta)u_1^{(\alpha)}.$$

- $\alpha = 0$:

$$u_1^{(0)}=0, \tag{4.35}$$

a result compatible with our previous choice;

- $\alpha = 1$:

$$r_1^{(1)} \left(e^{\mathrm{i}\kappa} - 1\right) = (\mu - \zeta)\, u_1^{(1)}; \tag{4.36}$$

ii. Order $\gamma = 2$.

- $\alpha = 0$:
  Taking into account that $u_1^{(0)} = 0$, eq. (4.36) and setting $r_1^{(0)} = 0$, we have

$$2\,(1 - \cos\kappa)\, \frac{\zeta + \mu - 2o_2}{\zeta - \mu} |r_1^{(1)}|^2 = (\mu - \zeta)\, u_2^{(0)}; \tag{4.37}$$

- $\alpha = 1$:
  Taking into account that $u_1^{(0)} = 0$ and setting $r_1^{(0)} = 0$, we have

$$N_1 e^{\mathrm{i}\kappa} \partial_{n_1} r_1^{(1)} + \left(e^{\mathrm{i}\kappa} - 1\right) r_2^{(1)} = (\mu - \zeta)\, u_2^{(1)}; \tag{4.38}$$

- $\alpha = 2$:
  Taking into account that $r_2^{(2)} = 0$ and eq. (4.36), we obtain

$$\left(e^{\mathrm{i}\kappa} - 1\right) \left[\frac{o_2 - \mu}{\mu - \zeta} \left(e^{\mathrm{i}\kappa} - 1\right) - 1\right] r_1^{(1)2} = (\mu - \zeta)\, u_2^{(2)}. \tag{4.39}$$

- *If one substitutes the expression eq. (4.36) into (4.30), (4.33b), (4.37) and (4.39), recovers eqs. (4.19), (4.25b), (4.24) with $G = 0$ and (4.22); applying the operator $\partial_{n_1} - \epsilon \partial_{m_1}$ to (4.38) and taking into account eqs. (4.30, 4.33a), we obtain eq. (4.25a).*

5

Conclusions and perspectives

In this work we showed how it is possible to develop an integrability test for nonlinear partial difference equations by using perturbative multiscale reductions techniques. The key ingredient of this approach is to suppose an *analytic* dependence on the slow-variables of the solutions of our nonlinear systems. This property, once the shifts with respect to the slow-variables are expressed in terms of derivatives with respect to the same variables, allow one to use all the machinery of multiscale techniques developed for partial differential equations in ref. [4–9, 11–14, 35, 40]. Thus the reduction process, starting from a partial difference equation, produces partial differential equations. In this way it was possible to give an explicit analytical evidence of nonintegrability for some discretizations of well known continuous $S-$integrable models. Among them we mention the (defocusing) $dNLS$ equation, whose importance emerges in several physical contexts, for which we previously had only numerical evidences of situations of irregular, if not chaotic, dynamics. The multiscale perturbative reductions were performed on $C-$integrable systems too. All the results obtained are in exact parallelism with those previously known for partial differential equations. For example the multiscale reduction of the $lpKdV$ spectral problem gives the one corresponding to the (continuous) NLS equation exactly as the (continuous) KdV spectral problem does [40]. The method can be applied to a large variety of nonlinear systems. In this respect we developed explicitly the cases of the NLS, KdV and potential KdV hierarchies along with the relative integrability conditions till the A_3 level included. Those integrability conditions are another original contribution to the theory of perturbative multiscale techniques. Anyway we are aware there are some points that deserve a further analysis. Among them we list:

- *To develop an integrability test using perturbative multiscale reductions techniques* ***not*** *relying on the assumption of an analytical dependence on the slow-variable.* This would enable one to reduce a partial difference system to another partial difference system, thus opening the way to the discovery of new discrete integrable models;
- *To improve the method including other possible hierarchies for the reduced systems with all the relative integrability conditions.* For example one could consider dissipative models for which the hierarchy of reference is the *Burgers* one;
- *To improve the method including maps, i. e. ordinary difference equations*;
- *To investigate other scaling with ε of the slow-variables.* For example, starting from a $C-$integrable model, it should be possible to obtain an *Eckhauss* equation;
- *Give an explicit evidence of nonintegrability for the focusing $dNLS$ equation.*

Appendix A

Nonlinear terms in $F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots]$ and derivatives of $u\,(x,t)$

In this appendix we will give the expressions of the expansion of the space and time derivatives of the function $u\,(x,t;\varepsilon)$ as well as the general expression of the expansion of the nonlinear quadratic and cubic monomials of $F[u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots]$ in eq. (2.1). Those expressions will prove to be very useful especially when one tries to apply the multiscale technique by hand. From eq. (2.20) one has

$$\partial_x^s u = \sum_{\rho=0}^{s} \sum_{n=\rho+1}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-\rho)}^{n-\rho} E^\alpha C_{s,\,\rho} \left(\mathrm{i}\kappa_0\alpha\right)^{s-\rho} \partial_\xi^\rho u_{n-\rho}^{(\alpha)}, \tag{A.1a}$$

$$u_t = -\mathrm{i}\omega\left(\kappa_0\right) \sum_{n=1}^{+\infty} \varepsilon^n \sum_{\alpha=-n}^{n} E^\alpha \alpha u_n^{(\alpha)} + \sum_{n=2}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=max\{1,\,|\alpha|\}}^{n-1} \partial_{t_{n-j}} u_j^{(\alpha)}. \tag{A.1b}$$

From eq. (A.1a) one derives

$$\begin{aligned}\left(\partial_x^r u\right)\left(\partial_x^s u\right) = \sum_{\rho=0}^{s+r} \sum_{n=\rho+2}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-\rho)}^{n-\rho} E^\alpha \sum_{j=\rho+1}^{n-1} \sum_{\pi=max\{-(j-\rho),\,\alpha-(n-j)\}}^{min\{j-\rho,\,\alpha+(n-j)\}} \sum_{\gamma=max\{0,\,\rho-r\}}^{min\{\rho,\,s\}} \\ C_{s,\,\gamma} C_{r,\,\rho-\gamma} \left(\mathrm{i}\kappa_0\pi\right)^{s-\gamma} \left[\mathrm{i}\kappa_0\left(\alpha-\pi\right)\right]^{r-(\rho-\gamma)} \left(\partial_\xi^\gamma u_{j-\rho}^{(\pi)}\right) \left(\partial_\xi^{\rho-\gamma} u_{n-j}^{(\alpha-\pi)}\right).\end{aligned} \tag{A.2}$$

where $C_{i,\,j}$ represents the *binomial coefficient* $C_{i,\,j} = \dfrac{i!}{j!(i-j)!}$. From eq. (A.2), we get

$$\begin{aligned}uu_x = \mathrm{i}\kappa_0 \sum_{n=2}^{+\infty} \varepsilon^n \sum_{\alpha=-n}^{n} E^\alpha \sum_{j=1}^{n-1} \sum_{\pi=max\{-j,\,\alpha-(n-j)\}}^{min\{j,\,\alpha+(n-j)\}} \pi u_j^{(\pi)} u_{n-j}^{(\alpha-\pi)} + \\ + \sum_{n=3}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=2}^{n-1} \sum_{\pi=max\{-(j-1),\,\alpha-(n-j)\}}^{min\{j-1,\,\alpha+(n-j)\}} \left(\partial_\xi u_{j-1}^{(\pi)}\right) u_{n-j}^{(\alpha-\pi)},\end{aligned} \tag{A.3a}$$

$$\begin{aligned}uu_{xx} = -\kappa_0^2 \sum_{n=2}^{+\infty} \varepsilon^n \sum_{\alpha=-n}^{n} E^\alpha \sum_{j=1}^{n-1} \sum_{\pi=max\{-j,\,\alpha-(n-j)\}}^{min\{j,\,\alpha+(n-j)\}} \pi^2 u_j^{(\pi)} u_{n-j}^{(\alpha-\pi)} + \\ +2\mathrm{i}\kappa_0 \sum_{n=3}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=2}^{n-1} \sum_{\pi=max\{-(j-1),\,\alpha-(n-j)\}}^{min\{j-1,\,\alpha+(n-j)\}} \pi \left(\partial_\xi u_{j-1}^{(\pi)}\right) u_{n-j}^{(\alpha-\pi)} + \\ + \sum_{n=4}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-2)}^{n-2} E^\alpha \sum_{j=3}^{n-1} \sum_{\pi=max\{-(j-2),\,\alpha-(n-j)\}}^{min\{j-2,\,\alpha+(n-j)\}} \left(\partial_\xi^2 u_{j-2}^{(\pi)}\right) u_{n-j}^{(\alpha-\pi)},\end{aligned} \tag{A.3b}$$

$$u_x^2 = -\kappa_0^2 \sum_{n=2}^{+\infty} \varepsilon^n \sum_{\alpha=-n}^{n} E^\alpha \sum_{j=1}^{n-1} \sum_{\pi=max\{-j,\, \alpha-(n-j)\}}^{min\{j,\, \alpha+(n-j)\}} \pi\,(\alpha-\pi)\, u_j^{(\pi)} u_{n-j}^{(\alpha-\pi)} + \tag{A.3c}$$
$$+\mathrm{i}\kappa_0 \sum_{n=3}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=2}^{n-1} \sum_{\pi=max\{-(j-1),\, \alpha-(n-j)\}}^{min\{j-1,\, \alpha+(n-j)\}} \pi u_{j-1}^{(\pi)} \left(\partial_\xi u_{n-j}^{(\alpha-\pi)}\right) +$$
$$+\mathrm{i}\kappa_0 \sum_{n=3}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=2}^{n-1} \sum_{\pi=max\{-(j-1),\, \alpha-(n-j)\}}^{min\{j-1,\, \alpha+(n-j)\}} (\alpha-\pi) \left(\partial_\xi u_{j-1}^{(\pi)}\right) u_{n-j}^{(\alpha-\pi)} +$$
$$+ \sum_{n=4}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-2)}^{n-2} E^\alpha \sum_{j=3}^{n-1} \sum_{\pi=max\{-(j-2),\, \alpha-(n-j)\}}^{min\{j-2,\, \alpha+(n-j)\}} \left(\partial_\xi u_{j-2}^{(\pi)}\right) \left(\partial_\xi u_{n-j}^{(\alpha-\pi)}\right).$$

From eqs. (A.1a) and (A.2) we have

$$\left(\partial_x^r u\right)\left(\partial_x^s u\right)\left(\partial_x^t u\right) = \sum_{\rho=0}^{r+s+t} \sum_{n=\rho+3}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-\rho)}^{n-\rho} E^\alpha \sum_{j=\rho+2}^{n-1} \sum_{\pi=max\{-(j-\rho),\, \alpha-(n-j)\}}^{min\{j-\rho,\, \alpha+(n-j)\}} \tag{A.4}$$
$$\sum_{\delta=\rho+1}^{j-1} \sum_{z=max\{-(\delta-\rho),\, \pi-(j-\delta)\}}^{min\{\delta-\rho,\, \pi+(j-\delta)\}} \sum_{\gamma=max\{0,\, \rho-t\}}^{min\{\rho,\, s+r\}} \sum_{\chi=max\{0,\, \gamma-r\}}^{min\{\gamma,\, s\}}$$
$$C_{s,\,\chi} C_{r,\,\gamma-\chi} C_{t,\,\rho-\gamma} \left(\mathrm{i}\kappa_0 z\right)^{s-\chi} \left[\mathrm{i}\kappa_0 \left(\pi - z\right)\right]^{r-(\gamma-\chi)}$$
$$\left[\mathrm{i}\kappa_0 \left(\alpha-\pi\right)\right]^{t-(\rho-\gamma)} \left(\partial_\xi^\chi u_{\delta-\rho}^{(z)}\right) \left(\partial_\xi^{\gamma-\chi} u_{j-\delta}^{(\pi-z)}\right) \left(\partial_\xi^{\rho-\gamma} u_{n-j}^{(\alpha-\pi)}\right),$$

and, from eq. (A.4), it results that

$$u^2 u_x = \mathrm{i}\kappa_0 \sum_{n=3}^{+\infty} \varepsilon^n \sum_{\alpha=-n}^{n} E^\alpha \sum_{j=2}^{n-1} \sum_{\rho=max\{-j,\, \alpha-(n-j)\}}^{min\{j,\, \alpha+(n-j)\}} \sum_{\delta=1}^{j-1} \sum_{\pi=max\{-\delta,\, \rho-(j-\delta)\}}^{min\{\delta,\, \rho+(j-\delta)\}} \tag{A.5}$$
$$\pi u_\delta^{(\pi)} u_{j-\delta}^{(\rho-\pi)} u_{n-j}^{(\alpha-\rho)} + \sum_{n=4}^{+\infty} \varepsilon^n \sum_{\alpha=-(n-1)}^{n-1} E^\alpha \sum_{j=3}^{n-1} \sum_{\rho=max\{-(j-1),\, \alpha-(n-j)\}}^{min\{j-1,\, \alpha+(n-j)\}} \sum_{\delta=2}^{j-1}$$
$$\sum_{\pi=max\{-(\delta-1),\, \rho-(j-\delta)\}}^{min\{\delta-1,\, \rho+(j-\delta)\}} \left(\partial_\xi u_{\delta-1}^{(\pi)}\right) u_{j-\delta}^{(\rho-\pi)} u_{n-j}^{(\alpha-\rho)}.$$

Appendix B

Basis monomials of the vector spaces $\mathcal{P}_n(m)$

In this appendix we present the following tables where we list the basis monomials of the subspaces $\mathcal{P}_n(m)$ needed in the calculation of the integrability conditions. The first table refers to the situation when the flows in eqs. (2.71a) belong to the NLS hierarchy and the second one when they belongs to the KdV/*potential* KdV hierarchies. A genereic element of $\mathcal{P}_n(m)$ will then be a a polynomial with complex coefficients resulting from the linear combination of those basis elements[1].

[1] In all the tables it is intended that, to have all the basis monomials in a particular $\mathcal{P}_n(m)$, to those listed next to it one must add those in $\mathcal{P}_n(m-1)$. For the *potential* KdV hierarchy is $\varphi^{(j)} \doteq \phi_\xi^{(j)}$.

Table B.1: *Vector spaces for the NLS hierarchy* ($u_n^{(1)} \doteq u(n)$)

$\mathcal{P}_2(1)$:	$\emptyset$				
$\mathcal{P}_3(1)$:	$u(1)\|u(1)\|^2$				
$\mathcal{P}_4(1)$:	$u_\xi(1)\|u(1)\|^2$	$\bar{u}_\xi(1)u(1)^2$			
$\mathcal{P}_4(2)$:	$u(2)\|u(1)\|^2$	$\bar{u}(2)u(1)^2$			
$\mathcal{P}_5(1)$:	$\|u(1)\|^4u(1)$	$\|u_\xi(1)\|^2u(1)$	$u_\xi(1)^2\bar{u}(1)$	$\bar{u}_{\xi\xi}(1)u(1)^2$	$\|u(1)\|^2u_{\xi\xi}(1)$
$\mathcal{P}_5(2)$:	$u(2)^2\bar{u}(1)$	$\|u(2)\|^2u(1)$	$u_\xi(2)\|u(1)\|^2$	$\bar{u}_\xi(2)u(1)^2$	$u(2)u_\xi(1)\bar{u}(1)$
	$u(2)\bar{u}_\xi(1)u(1)$	$\bar{u}(2)u_\xi(1)u(1)$			
$\mathcal{P}_5(3)$:	$\|u(1)\|^2u(3)$	$\bar{u}(3)u(1)^2$			
$\mathcal{P}_6(1)$:	$\|u(1)\|^4u_\xi(1)$	$\|u(1)\|^2u(1)^2\bar{u}_\xi(1)$	$\|u(1)\|^2u_{\xi\xi\xi}(1)$	$u(1)^2\bar{u}_{\xi\xi\xi}(1)$	$\|u_\xi(1)\|^2u_\xi(1)$
	$\bar{u}_{\xi\xi}(1)u_\xi(1)u(1)$	$u_{\xi\xi}(1)\bar{u}_\xi(1)u(1)$	$u_{\xi\xi}(1)u_\xi(1)\bar{u}(1)$		
$\mathcal{P}_6(2)$:	$\|u(1)\|^4u(2)$	$\|u(1)\|^2u(1)^2\bar{u}(2)$	$\bar{u}_\xi(1)u(2)^2$	$u_\xi(1)\|u(2)\|^2$	$\|u_\xi(1)\|^2u(2)$
	$\|u(2)\|^2u(2)$	$u_\xi(1)^2\bar{u}(2)$	$\|u(1)\|^2u_{\xi\xi}(2)$	$u(1)^2\bar{u}_{\xi\xi}(2)$	$u(2)\bar{u}_{\xi\xi}(1)u(1)$
	$u(2)u_{\xi\xi}(1)\bar{u}(1)$	$\bar{u}(2)u_{\xi\xi}(1)u(1)$	$u(2)u_\xi(2)\bar{u}(1)$	$\bar{u}(2)u_\xi(2)u(1)$	$u_\xi(2)u_\xi(1)\bar{u}(1)$
	$u_\xi(2)\bar{u}_\xi(1)u(1)$	$\bar{u}_\xi(2)u_\xi(1)u(1)$	$\bar{u}_\xi(2)u(2)u(1)$		
$\mathcal{P}_6(3)$:	$u(3)u(2)\bar{u}(1)$	$u(3)\bar{u}(2)u(1)$	$u(3)u_\xi(1)\bar{u}(1)$	$u(3)\bar{u}_\xi(1)u(1)$	$\bar{u}(3)u(2)u(1)$
	$\bar{u}(3)u_\xi(1)u(1)$	$u_\xi(3)\|u(1)\|^2$	$\bar{u}_\xi(3)u(1)^2$		
$\mathcal{P}_6(4)$:	$u(4)\|u(1)\|^2$	$\bar{u}(4)u(1)^2$			

Table B.2: *Vector spaces for the KdV/potential KdV hierarchies*

$\mathcal{P}_2(1) = \mathcal{P}_3(1)$:	$\emptyset$					
$\mathcal{P}_4(1) = \mathcal{P}_4(2)$:	$\varphi^{(1)2}$					
$\mathcal{P}_5(1) = \mathcal{P}_5(2)$:	$\varphi^{(1)}\varphi^{(1)}_{\xi}$					
$\mathcal{P}_6(1)$:	$\varphi^{(1)2}_{\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)3}$			
$\mathcal{P}_6(2) = \mathcal{P}_6(3)$:	$\varphi^{(2)}\varphi^{(1)}$					
$\mathcal{P}_7(1)$:	$\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi}$			
$\mathcal{P}_7(2) = \mathcal{P}_7(3)$:	$\varphi^{(2)}_{\xi}\varphi^{(1)}$	$\varphi^{(2)}\varphi^{(1)}_{\xi}$				
$\mathcal{P}_8(1)$:	$\varphi^{(1)2}_{\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)2}_{\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)4}$
$\mathcal{P}_8(2)$:	$\varphi^{(2)}_{\xi\xi}\varphi^{(1)}$	$\varphi^{(2)}_{\xi}\varphi^{(1)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(2)}\varphi^{(1)2}$	$\varphi^{(2)2}$	
$\mathcal{P}_8(3) = \mathcal{P}_8(4)$:	$\varphi^{(3)}\varphi^{(1)}$					
$\mathcal{P}_9(1)$:	$\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi\xi\xi}$	$\varphi^{(1)3}_{\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi\xi\xi}$
	$\varphi^{(1)3}\varphi^{(1)}_{\xi}$					
$\mathcal{P}_9(2)$:	$\varphi^{(1)}\varphi^{(2)}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}\varphi^{(1)}_{\xi}$
	$\varphi^{(2)}\varphi^{(2)}_{\xi}$					
$\mathcal{P}_9(3) = \mathcal{P}_9(4)$:	$\varphi^{(1)}\varphi^{(3)}_{\xi}$	$\varphi^{(3)}\varphi^{(1)}_{\xi}$				
$\mathcal{P}_{10}(1)$:	$\varphi^{(1)2}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi\xi\xi\xi}$	$\varphi^{(1)2}_{\xi}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)}\varphi^{(1)2}_{\xi\xi}$
	$\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)3}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)2}_{\xi}$	$\varphi^{(1)5}$	

Table B.2: *Vector spaces for the KdV/potential KdV hierarchies (continued)*

$\mathcal{P}_{10}(2)$:	$\varphi^{(1)}\varphi^{(2)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(2)}_{\xi\xi}$	$\varphi^{(1)}_{\xi\xi\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(2)}_{\xi\xi}$
	$\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)2}_{\xi}$	$\varphi^{(1)}\varphi^{(2)}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)3}\varphi^{(2)}$	$\varphi^{(1)}\varphi^{(2)2}$	$\varphi^{(2)}\varphi^{(2)}_{\xi\xi}$
	$\varphi^{(2)2}_{\xi}$					
$\mathcal{P}_{10}(3)$:	$\varphi^{(1)}\varphi^{(3)}_{\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(3)}_{\xi}$	$\varphi^{(3)}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)2}\varphi^{(3)}$		
$\mathcal{P}_{10}(4)=\mathcal{P}_{10}(5)$:	$\varphi^{(1)}\varphi^{(4)}$					
$\mathcal{P}_{11}(1)$:	$\varphi^{(1)}_{\xi\xi\xi}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi\xi\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(1)2}_{\xi\xi}$	$\varphi^{(1)2}_{\xi}\varphi^{(1)}_{\xi\xi\xi}$
	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi\xi\xi\xi\xi}$	$\varphi^{(1)3}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)}\varphi^{(1)3}_{\xi}$
	$\varphi^{(1)4}\varphi^{(1)}_{\xi}$					
$\mathcal{P}_{11}(2)$:	$\varphi^{(1)}\varphi^{(2)}_{\xi\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi\xi\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(2)}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi\xi\xi}\varphi^{(2)}_{\xi\xi}$	$\varphi^{(1)}_{\xi\xi\xi\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}_{\xi\xi\xi\xi\xi}$
	$\varphi^{(1)2}\varphi^{(2)}_{\xi\xi\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi}\varphi^{(2)}_{\xi\xi}$	$\varphi^{(1)2}_{\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(1)}\varphi^{(1)}_{\xi\xi}\varphi^{(2)}_{\xi}$	$\varphi^{(2)}\varphi^{(1)}_{\xi}\varphi^{(1)}_{\xi\xi}$	$\varphi^{(1)}\varphi^{(2)}\varphi^{(1)}_{\xi\xi\xi}$
	$\varphi^{(1)3}\varphi^{(2)}_{\xi}$	$\varphi^{(1)2}\varphi^{(2)}\varphi^{(1)}_{\xi}$	$\varphi^{(1)}\varphi^{(2)}\varphi^{(2)}_{\xi}$	$\varphi^{(2)2}\varphi^{(1)}_{\xi}$	$\varphi^{(2)}\varphi^{(2)}_{\xi\xi\xi}$	$\varphi^{(2)}_{\xi}\varphi^{(2)}_{\xi\xi}$
$\mathcal{P}_{11}(3)$:	$\varphi^{(1)}\varphi^{(3)}_{\xi\xi\xi}$	$\varphi^{(1)}_{\xi}\varphi^{(3)}_{\xi\xi}$	$\varphi^{(1)}_{\xi\xi}\varphi^{(3)}_{\xi}$	$\varphi^{(3)}\varphi^{(1)}_{\xi\xi\xi}$	$\varphi^{(1)2}\varphi^{(3)}_{\xi}$	$\varphi^{(1)}\varphi^{(3)}\varphi^{(1)}_{\xi}$
$\mathcal{P}_{11}(4)=\mathcal{P}_{11}(5)$:	$\varphi^{(1)}\varphi^{(4)}_{\xi}$	$\varphi^{(4)}\varphi^{(1)}_{\xi}$				

Appendix C

Coefficients of the differential polynomial $g_3(3)$

In this appendix we present the list containing the coefficients of the polynomial forcing term $g_3(3)$ eq. (2.90b) in the evolution equation of $\varphi^{(3)}$ at the slow-time t_3 (eq. (2.85b), $n = j = 3$, KdV hierarchy).

$$\delta_1 = \frac{5\lambda(2\mu_1 + \mu_2)}{3\tau_1}, \quad \delta_2 = \frac{5\lambda(\mu_1 + \mu_2 + \mu_3)}{3\tau_1}, \quad \delta_3 = \frac{5\lambda(\mu_2 + \mu_3)}{3\tau_1}, \quad \delta_4 = \frac{5\lambda\mu_3}{3\tau_1},$$

$$\delta_5 = \frac{5\lambda[(\theta_2 + 2\theta_3)(\mu_{10} + \mu_9) + 3(3\mu_4 + 2\mu_5)\tau_1 + (7\mu_1 + 5\mu_2 + \mu_3)\tau_2]}{9\tau_1^2},$$

$$\delta_6 = \frac{5\lambda[2\theta_2\mu_9 + (\mu_{11} + 2\mu_9)\theta_3 + 3(3\mu_4 + \mu_5 + \mu_6)\tau_1 + 2(\mu_1 + 5\mu_2 - \mu_3)\tau_2]}{9\tau_1^2},$$

$$\delta_7 = \frac{5\lambda[6\theta_3\mu_8 + (\mu_{10} + \mu_{11} + 4\mu_8)\theta_2 + 6(\mu_5 + \mu_6)\tau_1 + 2(3\mu_1 + 10\mu_3)\tau_2]}{9\tau_1^2},$$

$$\delta_8 = \frac{5\lambda[2\theta_3\mu_8 + (3\mu_8 + \mu_9)\theta_2 + 3(\mu_5 + 2\mu_6)\tau_1 + 6(\mu_2 + 2\mu_3)\tau_2]}{9\tau_1^2},$$

$$\delta_9 = \frac{5\lambda(\theta_2\mu_8 + 3\mu_6\tau_1 + 5\mu_3\tau_2)}{9\tau_1^2},$$

$$\delta_{10} = \frac{5\lambda\{9\tau_1[\theta_2\mu_{12} + (\mu_{11} + 3\mu_8)\theta_1 + 3\mu_7\tau_1] + [-2(3\theta_2 + \theta_3)\mu_8 + \theta_2\mu_9 + 48\mu_6\tau_1]\tau_2 - 10\mu_3\tau_2^2\}}{81\tau_1^3},$$

$$\delta_{11} = \frac{5\lambda[\theta_2^2\mu_{14} + 3(\mu_{12} + \mu_{13})\theta_2\tau_1 + 9(\mu_{10} + 6\mu_8 + \mu_9)\theta_1\tau_1 + 6(\theta_3\mu_{12} + 9\mu_7\tau_1)\tau_1]}{27\tau_1^3} +$$
$$+ \frac{5\lambda\tau_2\{(\mu_{10} - 9\mu_8 - \mu_9)\theta_2 - 6[\theta_3\mu_8 - (3\mu_5 + 4\mu_6)\tau_1] - 20\mu_3\tau_2\}}{27\tau_1^3},$$

$$\delta_{12} = \frac{5\lambda\{-6(\theta_2^2 - \theta_2\theta_3 - \theta_3^2)\mu_{14} + 9[-2(\theta_2 + \theta_3)\mu_{12} + 2\theta_3\mu_{13} + (\mu_{11} + 18\mu_8 + 21\mu_9)\theta_1]\tau_1\}}{243\tau_1^3} +$$
$$+ \frac{10\lambda[(3\mu_{10} - 3\mu_{11} + 27\mu_8 - 16\mu_9)\theta_2 + (6\mu_{10} + 6\mu_{11} + 14\mu_8 - 15\mu_9)\theta_3]\tau_2}{243\tau_1^3} +$$
$$+ \frac{10\lambda[135\mu_7\tau_1^2 + 2(6\mu_1 - 15\mu_2 + 20\mu_3)\tau_2^2 + 3(39\mu_4 + 9\mu_5 - 13\mu_6)\tau_1\tau_2]}{243\tau_1^3},$$

$$\delta_{13} = \frac{45\lambda\theta_1\tau_1\{18\theta_3\mu_{14} + 81(2\mu_{12} + \mu_{13})\tau_1 + 4[9\mu_{10} + \mu_{11} - 6(6\mu_8 + \mu_9)]\tau_2\}}{5832\tau_1^4} +$$
$$+ \frac{30\lambda\tau_2\{-4(\theta_2^2 - \theta_2\theta_3 - \theta_3^2)\mu_{14} - 3[4(\theta_2 + \theta_3)\mu_{12} + (3\theta_2 + 5\theta_3)\mu_{13}]\tau_1 + 666\mu_7\tau_1^2\}}{5832\tau_1^4} +$$
$$+ \frac{20\lambda\tau_2^2[2(6\mu_{10} + 6\mu_{11} + 14\mu_8 - 15\mu_9)\theta_3 + (6\mu_{10} - 6\mu_{11} + 81\mu_8 - 5\mu_9)\theta_2]}{5832\tau_1^4} +$$
$$+ \frac{20\lambda\tau_2^2[3(24\mu_4 - 9\mu_5 - 26\mu_6)\tau_1 + 4(6\mu_1 - 15\mu_2 + 20\mu_3)\tau_2]}{5832\tau_1^4},$$

$$\delta_{14} = \frac{5\lambda\mu_8}{3\tau_1}, \quad \delta_{15} = \frac{5\lambda(\mu_8+\mu_9)}{3\tau_1}, \quad \delta_{16} = \frac{5\lambda(\mu_8+\mu_9+\mu_{10})}{3\tau_1}, \quad \delta_{17} = \frac{5\lambda(\mu_9+\mu_{10}+\mu_{11})}{3\tau_1},$$
$$\delta_{18} = \frac{5\lambda(\mu_{10}+\mu_{11})}{3\tau_1}, \quad \delta_{19} = \frac{5\lambda\mu_{11}}{3\tau_1}, \quad \delta_{20} = \frac{5\lambda(3\mu_{12}\tau_1+5\mu_8\tau_2)}{9\tau_1^2},$$
$$\delta_{21} = \frac{5\lambda[3(2\mu_{12}+\mu_{13})\tau_1+2(4\mu_8+3\mu_9)\tau_2]}{9\tau_1^2},$$
$$\delta_{22} = \frac{5\lambda[\theta_3\mu_{14}+3(\mu_{12}+\mu_{13})\tau_1+2(\mu_{10}-\mu_8+3\mu_9)\tau_2]}{9\tau_1^2},$$
$$\delta_{23} = \frac{5\lambda[\theta_2\mu_{14}+3(2\mu_{12}+\mu_{13})\tau_1+6(\mu_{10}+\mu_8)\tau_2]}{9\tau_1^2},$$
$$\delta_{24} = \frac{5\lambda[(\theta_2+2\theta_3)\mu_{14}+6\mu_{13}\tau_1+2(\mu_{10}+4\mu_{11}+\mu_9)\tau_2]}{9\tau_1^2},$$
$$\delta_{25} = \frac{5\lambda[\theta_2\mu_{14}+3\mu_{13}\tau_1+2(3\mu_{11}+\mu_8)\tau_2]}{9\tau_1^2}, \quad \delta_{26} = \frac{5\lambda[3\theta_1\mu_{14}\tau_1-2(\mu_8\tau_2-8\mu_{12}\tau_1)\tau_2]}{27\tau_1^3},$$
$$\delta_{27} = \frac{5\lambda\{9\theta_1\mu_{14}\tau_1+[\theta_2\mu_{14}+6(2\mu_{12}+3\mu_{13})\tau_1-8\mu_8\tau_2]\tau_2\}}{27\tau_1^3}, \quad \delta_{28} = 2\delta_{29},$$
$$\delta_{29} = \frac{5\lambda\mu_{14}\tau_2}{3\tau_1^2}, \quad \delta_{30} = \frac{5\lambda\mu_{14}}{3\tau_1}, \quad \delta_{31} = 2\delta_{30}. \tag{C.1}$$

Appendix D

Nonlinear discrete monomials, shifts and derivatives of $u_{n,m}$, $\nu_n(t)$, $\phi_n(t)$

In this appendix we give at first the expressions of the expansion of the n and m-shifts of the function $u_{n,m}(\varepsilon)$ in eq. (2.117) as well as the general expressions of the expansions of the nonlinear quadratic and cubic monomials in eq. (2.115). These expressions are useful especially when one tries to apply the multiscale technique by hand. From eqs. (2.116, 2.117) one has

$$u_{n+1,\,m} = \sum_{\gamma=1}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=max\{1,\,|\alpha|\}}^{\gamma} \left(\mathcal{A}_n^{(\gamma-j)} u_j^{(\alpha)}\right) e^{i\alpha\kappa}, \tag{D.1}$$

$$u_{n+2,\,m} = \sum_{\gamma=1}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=max\{1,\,|\alpha|\}}^{\gamma} \left(\mathcal{A}_n^{(\gamma-j)} \sum_{\beta=max\{1,\,|\alpha|\}}^{j} \mathcal{A}_n^{(j-\beta)} u_\beta^{(\alpha)}\right) e^{2i\alpha\kappa}, \tag{D.2}$$

$$u_{n+3,\,m} = \sum_{\gamma=1}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=max\{1,\,|\alpha|\}}^{\gamma} \left(\mathcal{A}_n^{(\gamma-j)} \sum_{\beta=max\{1,\,|\alpha|\}}^{j} \mathcal{A}_n^{(j-\beta)} \right. \tag{D.3}$$
$$\left. \sum_{\rho=max\{1,\,|\alpha|\}}^{\beta} \mathcal{A}_n^{(\beta-\rho)} u_\rho^{(\alpha)}\right) e^{3i\alpha\kappa}.$$

The quadratic terms are obtained from eq. (2.117) and eq. (D.1). For example

$$u_{n,\,m}^2 = \sum_{\gamma=2}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=1}^{\gamma-1} \sum_{\pi=max\{-j,\,\alpha-(\gamma-j)\}}^{min\{j,\,\alpha+(\gamma-j)\}} u_j^{(\pi)} u_{\gamma-j}^{(\alpha-\pi)}, \tag{D.4a}$$

$$u_{n,\,m} \cdot u_{n+1,\,m} = \sum_{\gamma=2}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=1}^{\gamma-1} \sum_{\beta=max\{1,\,|\alpha|-(\gamma-j)\}}^{j} \tag{D.4b}$$
$$\sum_{\pi=max\{-\beta,\,\alpha-(\gamma-j)\}}^{min\{\beta,\,\alpha+(\gamma-j)\}} \left(\mathcal{A}_n^{(j-\beta)} u_\beta^{(\pi)}\right) u_{\gamma-j}^{(\alpha-\pi)} e^{i\pi\kappa},$$

$$u_{n+1,\,m} \cdot u_{n,\,m+1} = \sum_{\gamma=2}^{+\infty} \varepsilon^{\gamma} \sum_{\alpha=-\gamma}^{\gamma} E^{\alpha} \sum_{j=1}^{\gamma-1} \sum_{\beta=max\{1,\,|\alpha|-(\gamma-j)\}}^{j} \sum_{\lambda=max\{1,\,|\alpha|-\beta\}}^{\gamma-j} \tag{D.4c}$$
$$\sum_{\pi=max\{-\beta,\,\alpha-\lambda\}}^{min\{\beta,\,\alpha+\lambda\}} \left(\mathcal{A}_n^{(j-\beta)} u_\beta^{(\pi)}\right) \left(\mathcal{A}_m^{(\gamma-j-\lambda)} u_\lambda^{(\alpha-\pi)}\right) e^{i\pi\kappa} e^{-i(\alpha-\pi)\omega}.$$

The cubic terms are obtained from eq. (2.117) and eqs. (D.1, D.4). For example

$$u^3_{n,\,m} = \sum_{\gamma=3}^{+\infty} \varepsilon^\gamma \sum_{\alpha=-\gamma}^{\gamma} E^\alpha \sum_{\xi=2}^{\gamma-1}\sum_{j=1}^{\xi-1} \sum_{\sigma=\alpha-(\gamma-\xi)}^{\alpha+(\gamma-\xi)} \sum_{\pi=max\{-j,\,\sigma-(\xi-j)\}}^{min\{j,\,\sigma+(\xi-j)\}} u_j^{(\pi)} u_{\xi-j}^{(\sigma-\pi)} u_{\gamma-\xi}^{(\alpha-\sigma)}, \tag{D.5a}$$

$$u^2_{n,\,m}\cdot u_{n+1,\,m} = \sum_{\gamma=3}^{+\infty} \varepsilon^\gamma \sum_{\alpha=-\gamma}^{\gamma} E^\alpha \sum_{\xi=2}^{\gamma-1}\sum_{j=1}^{\xi-1}\sum_{\beta=1}^{j} \sum_{\sigma=\alpha-(\gamma-\xi)}^{\alpha+(\gamma-\xi)} \sum_{\pi=max\{-\beta,\,\sigma-(\xi-j)\}}^{min\{\beta,\,\sigma+(\xi-j)\}} \left(\mathcal{A}_n^{(j-\beta)} u_\beta^{(\pi)}\right) u_{\xi-j}^{(\sigma-\pi)} u_{\gamma-\xi}^{(\alpha-\sigma)} e^{i\pi\kappa}, \tag{D.5b}$$

$$u_{n,\,m}\cdot u_{n+1,\,m}\cdot u_{n,\,m+1} = \sum_{\gamma=3}^{+\infty} \varepsilon^\gamma \sum_{\alpha=-\gamma}^{\gamma} E^\alpha \sum_{\xi=2}^{\gamma-1}\sum_{j=1}^{\xi-1}\sum_{\beta=1}^{j}\sum_{\lambda=1}^{\xi-j} \sum_{\sigma=\alpha-(\gamma-\xi)}^{\alpha+(\gamma-\xi)} \sum_{\pi=max\{-\beta,\,\sigma-\lambda\}}^{min\{\beta,\,\sigma+\lambda\}} \left(\mathcal{A}_n^{(j-\beta)} u_\beta^{(\pi)}\right) \left(\mathcal{A}_m^{(\xi-j-\lambda)} u_\lambda^{(\sigma-\pi)}\right) u_{\gamma-\xi}^{(\alpha-\sigma)} e^{i\pi\kappa} e^{-i(\sigma-\pi)\omega}, \tag{D.5c}$$

$$u_{n+1,\,m}\cdot u_{n,\,m+1}\cdot u_{n+1,\,m+1} = \sum_{\gamma=3}^{+\infty} \varepsilon^\gamma \sum_{\alpha=-\gamma}^{\gamma} E^\alpha \sum_{\xi=2}^{\gamma-1}\sum_{j=1}^{\xi-1}\sum_{\beta=1}^{j}\sum_{\lambda=1}^{\xi-j}\sum_{\rho=1}^{\gamma-\xi} \sum_{\sigma=\alpha-\rho}^{\alpha+\rho} \sum_{\pi=max\{-\beta,\,\sigma-\lambda\}}^{min\{\beta,\,\sigma+\lambda\}} \left(\mathcal{A}_n^{(j-\beta)} u_\beta^{(\pi)}\right) \left(\mathcal{A}_m^{(\xi-j-\lambda)} u_\lambda^{(\sigma-\pi)}\right) \left(\mathcal{A}_{n,\,m}^{(\gamma-\xi-\rho)} u_\rho^{(\alpha-\sigma)}\right) e^{i\pi\kappa} e^{-i(\sigma-\pi)\omega} e^{i(\alpha-\sigma)(\kappa-\omega)}. \tag{D.5d}$$

If in the above formulae one wants to replace say an n-shift with an m (or with a positive shift in both n and m) one, one has simply to replace the corresponding $\mathcal{A}_n^{(j)}$ with $\mathcal{A}_m^{(j)}$ ($\mathcal{A}_{n,\,m}^{(j)}$) and every factor $e^{\mathrm{i}\rho\kappa}$, except that in E^α, with $e^{-\mathrm{i}\rho\omega}$ ($e^{\mathrm{i}\rho(\kappa-\omega)}$). Moreover, if one wants to repalce an n-shift (m) with the corresponding negative one, one ha simply to replace the corresponding $\mathcal{A}_n^{(j)}$ ($\mathcal{A}_m^{(j)}$) with $\mathcal{A}_{-n}^{(j)}(\mathcal{A}_{-m}^{(j)})$ and every factor $e^{\mathrm{i}\rho\kappa}$ ($e^{-\mathrm{i}\rho\omega}$), except that in E^α, with its complex conjugate. The operators $\mathcal{A}_{-n}^{(j)}$ ($\mathcal{A}_{-m}^{(j)}$) are obtainded from the corresponding $\mathcal{A}_n^{(j)}$ ($\mathcal{A}_m^{(j)}$) by replacing any $\partial^j_{n_\rho}$ with $(-1)^j\partial^j_{n_\rho}$ (any $\partial^j_{m_\rho}$ with $(-1)^j\partial^j_{m_\rho}$).

We now give the expressions of the expansion of the n-shifts and $t-$derivatives of the functions $\nu_n(t)$, $\phi_n(t)$ in eqs. (3.87) as well as the general expressions of the expansions of the linear and nonlinear monomials $\beta_\pm$, $\gamma_\pm$ and $\delta_\pm$ appearing in eqs. (3.86). From eqs. (2.116a, 3.87) one has

$$\nu_{n+1} = 1 + \sum_{\kappa=2}^{+\infty}\sum_{j=1}^{[\kappa/2]} \varepsilon^\kappa \left(\mathcal{A}_n^{(\kappa-2j)} \nu^{(j)}\right), \tag{D.6a}$$

$$\phi_{n+1} = -\epsilon t + \sum_{\kappa=1}^{+\infty}\sum_{j=1}^{[(\kappa+1)/2]} \varepsilon^\kappa \left(\mathcal{A}_n^{(\kappa-2j+1)} \phi^{(j)}\right), \tag{D.6b}$$

$$\partial_t \nu_n = \sum_{\kappa=2}^{+\infty}\sum_{j=1}^{\kappa-1} \alpha_j \varepsilon^{2\kappa-1} \partial_{t_j} \nu^{(\kappa-j)}, \tag{D.6c}$$

$$\partial_t \phi_n = -\epsilon + \sum_{\kappa=1}^{+\infty}\sum_{j=1}^{\kappa} \alpha_j \varepsilon^{2\kappa} \partial_{t_j} \phi^{(\kappa-j+1)}, \tag{D.6d}$$

with $[\rho]$ standing for the entire part of ρ. We made use also of the fact that the operator ∂_t, when operates on the functions $\nu_n(t)$, $\phi_n(t)$, acts as the operator

$$D_t \doteq \partial_t + \sum_{j=1}^{+\infty} \alpha_j \varepsilon^{2j-1} \partial_{t_j},$$

when one assumes that all the variables n_1, t, t_j, $1 \leq j$, are *independent.* Setting

$$\nu_n(t)^{-1} \doteq \sum_{\kappa=0}^{+\infty} \varepsilon^{2\kappa} \theta^{(\kappa)} \left(n_1, \{t_j\}_{j=1}^{+\infty} \right), \tag{D.7}$$

we have that the functions $\theta^{(\kappa)}$ are given in terms of the functions $\nu^{(\kappa)}$, $\kappa \geq 0$ ($\nu^{(0)} = 1$) by the following recursion relation

$$\theta^{(0)} = \nu^{(0)-1}, \qquad \theta^{(\kappa)} = -\nu^{(0)-1} \sum_{j=1}^{\kappa} \nu^{(j)} \theta^{(\kappa-j)}, \qquad \kappa \geq 1. \tag{D.8}$$

Here follow a few of the functions $\theta^{(\kappa)}$ for $0 \leq \kappa \leq 4$:

$$\theta^{(0)} = 1, \tag{D.9a}$$
$$\theta^{(1)} = -\nu^{(1)}, \tag{D.9b}$$
$$\theta^{(2)} = \nu^{(1)2} - \nu^{(2)}, \tag{D.9c}$$
$$\theta^{(3)} = -\nu^{(1)3} + 2\nu^{(1)}\nu^{(2)} - \nu^{(3)}, \tag{D.9d}$$
$$\theta^{(4)} = \nu^{(1)4} - 3\nu^{(1)2}\nu^{(2)} + 2\nu^{(1)}\nu^{(3)} + \nu^{(2)2} - \nu^{(4)}. \tag{D.9e}$$

From eqs. (3.87, D.6a, D.6b, D.7) we get

$$\beta_+ = \sum_{\kappa=1}^{+\infty} \varepsilon^{2\kappa-1} \left[\sum_{j=1}^{\kappa} \left(\mathcal{A}_n^{(2\kappa-2j)} \phi^{(j)} \right) - \phi^{(\kappa)} \right] + \sum_{\kappa=1}^{+\infty} \sum_{j=1}^{\kappa} \varepsilon^{2\kappa} \left(\mathcal{A}_n^{(2\kappa-2j+1)} \phi^{(j)} \right) \doteq$$
$$\doteq 1 + \sum_{\kappa=2}^{+\infty} \varepsilon^{\kappa} \beta_+^{(\kappa)}, \tag{D.10a}$$

$$\gamma_+ = 1 + \sum_{\kappa=1}^{+\infty} \varepsilon^{2\kappa} \left[\theta^{(\kappa)} + \sum_{j=1}^{\kappa} \left(\mathcal{A}_n^{(2\kappa-2j)} \nu^{(j)} \right) \right] + \sum_{\kappa=1}^{+\infty} \sum_{j=1}^{\kappa} \varepsilon^{2\kappa+1} \left(\mathcal{A}_n^{(2\kappa-2j+1)} \nu^{(j)} \right) +$$
$$+ \sum_{\kappa=4}^{+\infty} \sum_{j=1}^{[\kappa/2-1]} \sum_{\rho=1}^{[\kappa/2-j]} \varepsilon^{\kappa} \left(\mathcal{A}_n^{(\kappa-2j-2\rho)} \nu^{(\rho)} \right) \theta^{(j)} \doteq 1 + \sum_{\kappa=3}^{+\infty} \varepsilon^{\kappa} \gamma_+^{(\kappa)}, \tag{D.10b}$$

$$\delta_+ = 1 + \sum_{\kappa=1}^{+\infty} \varepsilon^{2\kappa} \left[\nu^{(\kappa)} + \sum_{j=1}^{\kappa} \left(\mathcal{A}_n^{(2\kappa-2j)} \nu^{(j)} \right) \right] + \sum_{\kappa=1}^{+\infty} \sum_{j=1}^{\kappa} \varepsilon^{2\kappa+1} \left(\mathcal{A}_n^{(2\kappa-2j+1)} \nu^{(j)} \right) +$$
$$+ \sum_{\kappa=4}^{+\infty} \sum_{j=1}^{[\kappa/2-1]} \sum_{\rho=1}^{[\kappa/2-j]} \varepsilon^{\kappa} \left(\mathcal{A}_n^{(\kappa-2j-2\rho)} \nu^{(\rho)} \right) \nu^{(j)} \doteq 1 + \sum_{\kappa=2}^{+\infty} \varepsilon^{\kappa} \delta_+^{(\kappa)}. \tag{D.10c}$$

Some of the coefficients $\beta_+^{(\kappa)}$, $\gamma_+^{(\kappa)}$ and $\delta_+^{(\kappa)}$ for $2 \leq \kappa \leq 9$ are given in the Table (D.1). Consequently one has

$$\begin{aligned} \sin\beta_+ = \beta_+^{(2)}\varepsilon^2 + \beta_+^{(3)}\varepsilon^3 + \beta_+^{(4)}\varepsilon^4 + \beta_+^{(5)}\varepsilon^5 + \\ + \left(\beta_+^{(6)} - \frac{1}{6}\beta_+^{(2)3}\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7), \end{aligned} \tag{D.11a}$$

$$\begin{aligned} \cos\beta_+ = 1 - \frac{\beta_+^{(2)2}}{2}\varepsilon^4 - \beta_+^{(2)}\beta_+^{(3)}\varepsilon^5 - \\ -\frac{1}{2}\left(\beta_+^{(3)2} + 2\beta_+^{(2)}\beta_+^{(4)}\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7), \end{aligned} \tag{D.11b}$$

$$\begin{aligned} \gamma_+^{1/2} = 1 + \frac{\gamma_+^{(3)}}{2}\varepsilon^3 + \frac{\gamma_+^{(4)}}{2}\varepsilon^4 + \frac{\gamma_+^{(5)}}{2}\varepsilon^5 + \\ +\frac{1}{8}\left(4\gamma_+^{(6)} - \gamma_+^{(3)2}\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7), \end{aligned} \tag{D.11c}$$

$$\begin{aligned} \delta_+^{1/2} = 1 + \frac{\delta_+^{(2)}}{2}\varepsilon^2 + \frac{\delta_+^{(3)}}{2}\varepsilon^3 + \frac{1}{8}\left(4\delta_+^{(4)} - \delta_+^{(2)2}\right)\varepsilon^4 + \\ +\frac{1}{4}\left(2\delta_+^{(5)} - \delta_+^{(2)}\delta_+^{(3)}\right)\varepsilon^5 + \\ +\frac{1}{16}\left(\delta_+^{(2)3} - 2\delta_+^{(3)2} - 4\delta_+^{(2)}\delta_+^{(4)} + 8\delta_+^{(6)}\right)\varepsilon^6 + \mathcal{O}(\varepsilon^7). \end{aligned} \tag{D.11d}$$

The coefficients $\beta_-^{(\kappa)}$, $\gamma_-^{(\kappa)}$ and $\delta_-^{(\kappa)}$ for β_-, γ_- and δ_- are obtained replacing $\mathcal{A}_n^{(\kappa)}$ with $\mathcal{A}_{-n}^{(\kappa)}$.

Table D.1: *Coefficients* $\beta_+^{(\kappa)}$, $\gamma_+^{(\kappa)}$ *and* $\delta_+^{(\kappa)}$

	$\kappa = 2$	$\kappa = 3$	$\kappa = 4$	$\kappa = 5$
$\beta_+^{(\kappa)}$	$\mathcal{A}_n^{(1)}\phi^{(1)}$	$\mathcal{A}_n^{(2)}\phi^{(1)}$	$\mathcal{A}_n^{(3)}\phi^{(1)} + \mathcal{A}_n^{(1)}\phi^{(2)}$	$\mathcal{A}_n^{(4)}\phi^{(1)} + \mathcal{A}_n^{(2)}\phi^{(2)}$
$\gamma_+^{(\kappa)}$	0	$\mathcal{A}_n^{(1)}\nu^{(1)}$	$\mathcal{A}_n^{(2)}\nu^{(1)}$	$\mathcal{A}_n^{(3)}\nu^{(1)} + \mathcal{A}_n^{(1)}\nu^{(2)} - \nu^{(1)}\mathcal{A}_n^{(1)}\nu^{(1)}$
$\delta_+^{(\kappa)}$	$2\nu^{(1)}$	$\mathcal{A}_n^{(1)}\nu^{(1)}$	$2\nu^{(2)} + \mathcal{A}_n^{(2)}\nu^{(1)} + \nu^{(1)2}$	$\mathcal{A}_n^{(3)}\nu^{(1)} + \mathcal{A}_n^{(1)}\nu^{(2)} + \nu^{(1)}\mathcal{A}_n^{(1)}\nu^{(1)}$

Table D.1: *Coefficients* $\beta_+^{(\kappa)}$, $\gamma_+^{(\kappa)}$ *and* $\delta_+^{(\kappa)}$ *(continued)*

	$\kappa = 6$	$\kappa = 7$
$\beta_+^{(\kappa)}$	$\mathcal{A}_n^{(5)}\phi^{(1)} + \mathcal{A}_n^{(3)}\phi^{(2)} + \mathcal{A}_n^{(1)}\phi^{(3)}$	$\mathcal{A}_n^{(6)}\phi^{(1)} + \mathcal{A}_n^{(4)}\phi^{(2)} + \mathcal{A}_n^{(2)}\phi^{(3)}$
$\gamma_+^{(\kappa)}$	$\mathcal{A}_n^{(4)}\nu^{(1)} + \mathcal{A}_n^{(2)}\nu^{(2)} - \nu^{(1)}\mathcal{A}_n^{(2)}\nu^{(1)}$	$\mathcal{A}_n^{(5)}\nu^{(1)} + \mathcal{A}_n^{(3)}\nu^{(2)} + \mathcal{A}_n^{(1)}\nu^{(3)} -$ $-\nu^{(1)}\mathcal{A}_n^{(3)}\nu^{(1)} - \left(\mathcal{A}_n^{(1)}\nu^{(1)}\nu^{(2)}\right) + \nu^{(1)2}\mathcal{A}_n^{(1)}\nu^{(1)}$
$\delta_+^{(\kappa)}$	$2\nu^{(3)} + 2\nu^{(1)}\nu^{(2)} + \mathcal{A}_n^{(4)}\nu^{(1)} +$ $+\mathcal{A}_n^{(2)}\nu^{(2)} + \nu^{(1)}\mathcal{A}_n^{(2)}\nu^{(1)}$	$\mathcal{A}_n^{(5)}\nu^{(1)} + \mathcal{A}_n^{(3)}\nu^{(2)} + \mathcal{A}_n^{(1)}\nu^{(3)} +$ $+\nu^{(1)}\mathcal{A}_n^{(3)}\nu^{(1)} + \left(\mathcal{A}_n^{(1)}\nu^{(1)}\nu^{(2)}\right)$
	$\kappa = 8$	$\kappa = 9$
$\beta_+^{(\kappa)}$	$\mathcal{A}_n^{(7)}\phi^{(1)} + \mathcal{A}_n^{(5)}\phi^{(2)} +$ $+\mathcal{A}_n^{(3)}\phi^{(3)} + \mathcal{A}_n^{(1)}\phi^{(4)}$	$\mathcal{A}_n^{(8)}\phi^{(1)} + \mathcal{A}_n^{(6)}\phi^{(2)} +$ $+\mathcal{A}_n^{(4)}\phi^{(3)} + \mathcal{A}_n^{(2)}\phi^{(4)}$
$\gamma_+^{(\kappa)}$	$\mathcal{A}_n^{(6)}\nu^{(1)} + \mathcal{A}_n^{(4)}\nu^{(2)} + \mathcal{A}_n^{(2)}\nu^{(3)} +$ $+\left(\nu^{(1)2} - \nu^{(2)}\right)\mathcal{A}_n^{(2)}\nu^{(1)} -$ $-\left(\mathcal{A}_n^{(4)}\nu^{(1)} + \mathcal{A}_n^{(2)}\nu^{(2)}\right)\nu^{(1)}$	$\mathcal{A}_n^{(7)}\nu^{(1)} + \mathcal{A}_n^{(5)}\nu^{(2)} + \mathcal{A}_n^{(3)}\nu^{(3)} + \mathcal{A}_n^{(1)}\nu^{(4)} -$ $-\left(\mathcal{A}_n^{(5)}\nu^{(1)} + \mathcal{A}_n^{(3)}\nu^{(2)} + \mathcal{A}_n^{(1)}\nu^{(3)}\right)\nu^{(1)} +$ $+\left(\nu^{(1)2} - \nu^{(2)}\right)\left(\mathcal{A}_n^{(3)}\nu^{(1)} + \mathcal{A}_n^{(1)}\nu^{(2)}\right) +$ $+\left(2\nu^{(1)}\nu^{(2)} - \nu^{(1)3} - \nu^{(3)}\right)\mathcal{A}_n^{(1)}\nu^{(1)}$

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