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Electromagnetic Scattering by Cylindrical Objects Buried in a Semi-Infinite Medium with a Rough Surface

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To my father and mother.

To my wife and son, Muhammad Aqeel Arshad.

Abstract

In questo lavoro è stato affrontato lo scattering elettromagnetico di un'onda piana da cilindri sepolti sotto un'interfaccia rugosa. E' stato impostato un problema di tipo bidimensionale, con scatteratori cilindrici di sezione trasversa circolare, collocati in un mezzo semi-infinito con assi paralleli tra loro ed all'interfaccia. La soluzione teorica è stata ricavata in modo analitico mediante Cylindrical Wave Approach (CWA), esprimendo i campi scatterati mediante espansioni modali in onde cilindriche. Tale approccio è stato, inoltre, combinato con il Metodo delle Piccole Perturbazioni (SPM), in merito al trattamento delle rugosità superficiali. I campi ottenuti dall'interazione dell'onda piana incidente e dei campi scatterati con l'interfaccia rugosa sono risultati sovrapposizione di una soluzione di ordine zero, associata ad un problema di scattering da cilindri sepolti sotto un'interfaccia piana, e di una soluzione al primo ordine, relativa alle rugosità superficiali sovrapposte. Nell'ambito di una soluzione con CWA, le onde cilindriche sono state espresse con uno spettro di onde piane; ciò ha consentito di valutare la riflessione e trasmissione dei campi scatterati dai cilindri attraverso l'interfaccia sulle singole onde piane dello spettro. Sono state quindi definite delle Funzioni Cilindriche Riflesse e Trasmesse, da utilizzare per le espansioni modali relative all'interazione del campo scatterato dai cilindri con l'interfaccia. In particolare, per le Funzioni Cilindriche associate alla soluzione al primo ordine, sviluppata in presenza di rugosità superficiali, è stata introdotta una rappresentazione spettrale con integrale doppio. La soluzione teorica del problema è stata ricavata imponendo le condizioni al contorno sulla superficie dei cilindri. E' stato ottenuto un sistema lineare in cui le incognite sono i coefficienti di espansione dei campi scatterati; tale sistema è stato ricavato nei due casi di cilindri perfettamente conduttori e cilindri dielettrici, e per gli stati di polarizzazione TM e TE.

In merito alla valutazione numerica, il metodo teorico è stato implementato per un'interfaccia con profilo sinusoidale. Questo ha consentito di semplificare gli integrali associati alle Funzioni Cilindriche, relativamente ai termini al primo ordine, in integrali singoli. La valutazione delle Funzioni Cilindriche Riflesse e Trasmesse è stata condotta sviluppando un opportuno algoritmo di integrazione, in grado di garantire un buon compromesso tra accuratezza della soluzione e tempi di calcolo contenuti. Sono state utilizzate tecniche di tipo adattivo, in grado di seguire l'andamento oscillante dell'integrando associato alle onde omogenee. In merito al contributo delle onde evanescenti, e' stata considerata l'estensione infinita dello spettro, così da fornire una soluzione al problema sia in campo vicino che in campo lontano. I risultati prodotti sono stati confrontati con risultati di letteratura, mostrando un ottimo accordo. E' inoltre evidenziata la possibilità di utilizzare il metodo per simulare scatteratori cilindrici, dielettrici o perfettamente conduttori, di arbitraria sezione trasversa. La valutazione degli integrali associati alle Funzioni Cilindriche Trasmesse è stata condotta anche con tecniche asintotiche, fornendo una soluzione al problema con validità nella regione di campo lontano. Il confronto, per simulazioni in campo lontano, tra quanto ottenuto con una valutazione accurata degli integrali e la soluzione asintotica ha mostrato un ottimo accordo.

La trattazione teorica è stata in seguito implementata per un'interfaccia con profilo di tipo Gaussiano. Tale perturbazione superficiale è stata trattata in modo periodico, utilizzando come campo di eccitazione un fascio di tipo Gaussiano, al fine di ottenere l'illuminazione di una porzione limitata della superficie. Per il fascio incidente è stata adottata una rappresentazione di tipo spettrale, che ha consentito di ottenere i campi di riflessione e trasmissione da parte dell'interfaccia valutando riflessione e trasmissione su ciascuna onda piana dello spettro. L'utilizzo combinato di CWA e SPM ha condotto anche in questa trattazione alla formulazione di integrali doppi. La definizione di una superficie Gaussiana con andamento periodico ha

tuttavia consentito di superare le difficoltà di valutazione numerica di tali integrali; grazie all'utilizzo di un'espansione in serie di Fourier, gli integrali doppi sono stati trasformati in una sommatoria ed un integrale singolo. La valutazione numerica degli integrali spettrali è stata condotta garantendo criteri di accuratezza, con soluzione del problema di scattering sia in campo vicino che in campo lontano. I risultati numerici sono stati ottenuti innanzitutto per il problema di scattering di un fascio Gaussiano da superficie rugosa con statistica Gaussiana, in assenza di cilindri. In seguito, è stata considerata la soluzione al problema per cilindri sepolti al di sotto dell'interfaccia. In entrambi i casi, l'esecuzione di confronti con risultati di letteratura ha consentito di testare la validità del metodo.

Publications

[C1] M. A. Fiaz, L. Pajewski, C. Ponti, G. Schettini and F. Frezza,“Scattering by a circular cylinder buried beneath a rough surface” XIII International Conference on Ground Penetrating Radar, Lecce, 21-25 June 2010.

[C2] M. A. Fiaz, F. Frezza, L. Pajewski, C. Ponti and G. Schettini,“Scattering by a Dielectric Cylinder Buried under a Rough Surface by the CWA Method” XVIII RINEM, Benevento, 6th September 2010.

[C3] M. A. Fiaz, F.Frezza, L. Pajewski, C. Ponti, G. Schettini and N. Tedeschi,“Recent advances in the Cylindrical Wave Approach for electromagnetic scattering by subsurface targets”European Geosciences Union General Assembly 2011 (EGU2011), Vienna, Austria, 03-08 April 2011.

[C4] M. A. Fiaz, L. Pajewski, C. Ponti, G. Schettini and F. Frezza,“Scattering by cylindrical targets buried in a ground with rough interface”, 6th International Workshop on Advanced Ground Penetrating Radar 2011, Aachen, Germany, 22-24 June 2011.

[C5] M. A. Fiaz, L. Pajewski, C. Ponti, G. Schettini and F. Frezza,“On the scattering by buried objects”, International Conference on Network-Based Information Systems, Tirana, Albania, 7-9 September 2011.

[J1] M. A. Fiaz, F. Frezza, L. Pajewski, C. Ponti, and G. Schettini,“Scattering by a circular cylinder buried under a slightly rough surface: the Cylindrical-Wave Approach”, IEEE Transactions on Antennas and Propagation, accepted.

[J2] M. A. Fiaz, F. Frezza, L. Pajewski, C. Ponti and G. Schettini,“Asymptotic solution for the scattered field by cylindrical objects buried beneath a slightly rough surface”, Near Surface Geophysics, submitted.

Introduction

Electromagnetic scattering of a plane wave from buried objects below an interface has been an important research topic since the past several decades owing to practical applications in many domains. Many efforts have been done to develop a solution for the direct scattering problem in different situations. These involve many scattering scenarios containing flat or rough interface, and conducting or dielectric objects. Such topics are of practical interests in the simulation of physical scenarios investigated by *Ground Penetrating Radar* [1]. Remote sensing of the earth internal structure, detection of explosive mines, pipes, tunnels, and buried utilities are some of the common applications. The subject is widely dealt with in the literature, from both a theoretical and a numerical point of view. The method of analysis largely depends upon the application of interest.

From an analytical point of view, dealing with both the circular geometry relevant to the scatterer's cross-section and a surface of different, or irregular, geometry is a hard task. There exist many well-established analytical as well as numerical methods to study the scattering of electromagnetic waves from a rough surface of various heights and statistics. Among the analytical methods, there are two well-known theoretical methods for calculating scattering from rough surfaces without buried objects. One is Small Perturbation Method (SPM) used by Rayleigh and extended by Rice [2]-[5]. The other is based on Kirchhoff approximation [6]-[8]. The applicability of analytical methods is typically limited by their domains of validity for the small or large roughness of the surface. The Kirchhoff approximation is generally assumed to apply to surfaces with radius of curvature large compared with the wavelength. The validity of the Kirchhoff approximation for rough surface scattering using a Gaussian roughness spectrum has been developed in [9]-[10].

Small roughness is mainly coped with by means of the Small Perturbation Method (SPM). The diagram method was presented by Bass and Fuks [11] to study the perturbation method for rough surface scattering. A recursive solution of the fourth and higher order Small Perturbation Method for rough surface scattering has been developed by Demir and Johnson [12]. The perturbation approach is only valid when root mean square height is small compared with the wavelength. The validity of the perturbation approximation for rough surface scattering using a Gaussian roughness spectrum was examined in [13], and the tapered plane wave excitation was firstly applied in the scattering calculation . Small Perturbation Method derived from the Rayleigh hypothesis and extinction theorem, respectively, was validated by Soto *et al.* [14] by comparing the results of SPM with those obtained by using the Kirchhoff approximation, as well as the exact numerical method. The range of validity of the perturbation theory was also discussed by Kim and Stoddart [15]. Some other approximate methods, such as the Phase Perturbation Method (PPM) [16], Small-Slope Approximation (SSA) [17] and the Extended Boundary Condition Method (EBC) [18] etc., are also presented for rough surface scattering.

On the other hand, numerical techniques, such as the Method of Moments (MoM) [19]-[21], the Finite-Difference Time Domain Method (FDTD) [22]-[25], the Finite Element Method (FEM) [26], the Fast Multipole Method (FMM) [27], the Forward-Backward Method (FBM) [28] and the Sparse Matrix Canonical Grid Method (SMCG) [29], etc. have been applied to the problem of electromagnetic scattering from rough surface. They offer the advantage of modeling electromagnetic scattering from rough surfaces with arbitrary roughness. However, they also present several drawbacks and modeling challenges. In MoM, the size of the impedance matrix is determined by the size of the objects, the sample points per wavelength, and the number of scatterers. The computational cost becomes prohibitively high when the problem scenario involves multiple rough surfaces and a cluster of objects. In FDTD, full space discretization is required. Moreover, to resolve small-wavelength geometrical features, a very fine grid spatial discretization is needed. In time domain, for rough surface problem using FDTD, the cell-to-cell phase variation between corresponding points in different unit cells, that arises from a plane-wave illumination source at oblique incidence, makes the implementation very challenging.

Scattering problem in the presence of both a rough surface and a cylindrical scatterer is a more difficult task. Many efforts have been done for the detection of buried objects because of practical applications. Various analytical and numerical contributions are available in literature in the case of buried object below a flat interface. Scattering from a subterranean cylindrical inhomogeneity is solved by Howard [30] with an eigenfunction expansion of a two-dimensional Fredholm integral equation for the scattered field. A previous work done by D'Yakonov [31] has been extended by Ogunade [32], to obtain numerical data for a current line source above a uniform half-space. Mahmoud *et al.* [33] solved the problem using a multipole expansion for the scattered field. Budko [34] developed a Green's function approach, and an effective scattering model for a real buried object is proposed. In [35], Butler *et al.* solved an integral equation for current induced on a conducting cylinder near a planar interface, and various forms of the kernel suitable for an efficient numerical evaluation are proposed. Kobayashi's potential concept [36] has been used to solve the problem of the plane wave scattering by a 2-D cylindrical obstacle in a dielectric half-space by Hongo and Hamamura [37]. Correction to this asymptotic solution using the saddle-point method was done by Naqvi *et al.* [38]. Scattering of electromagnetic waves from a deeply buried circular cylinder is discussed in [39]. A solution to the scattered field by a low contrast buried cylinder which can be considered as a perturbation in the dielectric layer has been obtained in [40] using a perturbation technique. Problem involving PEMC cylinder beneath a dielectric flat surface has also been addressed by Ahmed and Naqvi [41].

Almost all the cited works are making use of integral equations and numerical discretization techniques, as the Method of Moments, to solve the problem of electromagnetic scattering by buried objects. In other works, integrals are solved through asymptotic techniques which give scattered field only in the far-field region. Thus, several hypotheses are made in order to simplify the problem, as deeply buried cylinders or dielectric cylinders with low contrast with respect to the host medium. The case of scattering by an isolated cylinder is mainly dealt with, due to the difficulty in developing multiple-interactions in an analytical model.

However, versatile and accurate models are needed for many practical cases, with a view to characterize realistic scenarios investigated by *Ground Penetrating Radar* (*GPR*). Therefore, an improvement of models for direct scattering is called for. Among the possible solutions, the introduction of rough perturbations in the surface limiting the lower medium undoubtedly allows to model the typical unevenness existing in real surfaces.

The analysis of scattering pattern from a buried object beneath a rough surface with sinusoidal profile has been dealt with by Cottis and Kanellopoulos in [42], using Green's function theory together with Extended Boundary Condition Method. An analytical solution based on spectral plane-wave representation of the fields is employed by Lawrence and Sarabandi [43]. Spectral integrals are evaluated with asymptotic techniques, and numerical results are given in far field, in terms of Radar Cross Section. Electromagnetic scattering by a partially-buried perfectly-conducting cylinder at the dielectric rough interface is presented in [44], where Transverse Magnetic (TM) polarization has been addressed. In [45], surfaces with arbitrary and localized roughnesses are considered, and the problem is solved with an integral equations/Method of Moments approach. A geometry with a cylinder buried in layered media with rough interfaces is proposed in [46]. Extended Boundary Condition Method and T-matrix algorithm are applied to construct reflection and transmission matrices of arbitrary rough interfaces as well as of an isolated single cylinder, respectively. Results are given in the far-field, in terms of bistatic scattering coefficient.

In this work, an analytical solution, based on the *Cylindrical Wave Approach*

(CWA) developed in [47] and [48] , is proposed to the scattering by a set of dielectric or perfectly conducting cylinders placed below a rough surface. The method employs the concept of plane-wave spectrum of a cylindrical wave [49]. Thus, reflection and transmission of cylindrical waves are obtained just evaluating reflection and transmission on each plane wave of the spectrum. Reflected and Transmitted Cylindrical Functions are defined, which generalize the ones given in [48] to the case of surface roughnesses. In particular, since Small Perturbation Method is employed, the involved fields are the sum of fields relevant to a flat surface, i.e. unperturbed terms, and first order terms taking into account the superimposed roughness. Spectral integrals associated to cylindrical waves are evaluated in an accurate way, and results can be obtained both in near and far field regions.

In chapter 1, the theoretical method developed to solve electromagnetic scattering by perfectly-conducting and dielectric cylinders below a rough surface is described. The integration alghorithm implemented for the solution of the spectral integrals is presented in chapter 2. Numerical results obtained from the numerical implementation for the cases of both perfectly-conducting and dielectric objects below a rough surface with sinusoidal profile are presented in chapter 3. Checks confirming the validity of the approach are also given. In chapter 4, scattering from buried objects below a rough surface with Gaussian roughness spectrum is presented, where a Gaussian beam has been considered as an incident field to limit the amount of incident energy on the rough surface. Numerical implementation of the theory developed in chapter 4 is implemented in chapter 5, and results have been given in chapter 6. Finally, some concluding remarks about the work are reported.

Chapter 1

Scattering from cylinders buried below a rough surface

In this chapter, an analytical solution, based on the *Cylindrical Wave Approach* (*CWA*) developed in [48], is proposed to the scattering by dielectric or perfectly-conducting cylinders placed below a rough surface. The method employs the concept of plane-wave spectrum of a cylindrical wave [49].

Cylinders are of infinite extension and their axes are parallel to the interface, therefore a 2-D problem has to be solved. The involved media are assumed to be linear, isotropic, homogeneous, and lossless. Two states of linear polarization for the incident field are considered, since any other state of polarization can be expressed by the superimposition of:

- TM Polarization - The magnetic field is transverse to the axes of the cylindrical scatterers, and the electric field is directed along the cylinders axes. The propagation vector lies in the plane orthogonal to the cylinders axes.

- TE Polarization - The electric field is transverse to the axes of the cylindrical scatterers, with the magnetic field lying in the orthogonal plane.

Note that this is the usual convention for cylinder scattering problems but it is opposite to the usual convention for rough surface scattering. The electromagnetic field is represented by means of a scalar function $V(\xi, \zeta)$, which in each medium is standing for a total field, i.e. the electric or magnetic field component parallel to the y-axis, in case of $\mathrm{T}M^{(y)}$ or $\mathrm{T}E^{(y)}$ polarization, respectively. Throughout all the analysis, a time dependence $e^{-i\omega t}$ is omitted, where ω is the angular frequency. The total field $V(\xi, \zeta)$ is the superposition of the following field contributions, due to the interaction of the incident plane-wave with the rough surface and the cylinders:

- $V_{\mathrm{i}}(\xi, \zeta)$: plane-wave incident field;
- $V_{\mathrm{r}}(\xi, \zeta)$: plane-wave reflected field, due to the reflection in medium 0 of the incident plane-wave V_{i} by the interface;
- $V_{\mathrm{t}}(\xi, \zeta)$: plane-wave transmitted field, due to the transmission in medium 1 of the incident plane-wave V_{i} by the interface;
- $V_{\mathrm{s}}(\xi, \zeta)$: field scattered by the cylinders in medium 1;
- $V_{\mathrm{sr}}(\xi, \zeta)$: scattered-reflected field, due to reflection of V_{s} in medium 1;
- $V_{\mathrm{st}}(\xi, \zeta)$: scattered-transmitted field, due to the transmission of V_{s} in medium 0;
- $V_{\mathrm{cp}}(\xi, \zeta)$: field transmitted inside the p-*th* cylinder, in case of dielectric scatterers.

The reflected and transmitted fields from rough surface in the absence of any buried scatterer are calculated in Section 1.3 using first order Small Perturbation Method. The fields propagating in the medium 1 are expressed into an expansion of cylindrical functions taking into account the circular geometry of the scatterer's cross-section. For the transmitted field by rough surface, an expansion in terms of Bessel Functions J_{m} is used, while the addition theorem of Hankel functions [63] is employed for the scattered field V_{s}. The scattered-reflected and transmitted fields are written into a modal expansion with unknown coefficients of *Reflected* and *Transmitted Cylindrical Functions*, respectively, where *Perturbed Reflected* and *Transmitted Cylindrical Functions* are defined in order to take into account of rough deviations together with

Unperturbed Reflected and *Transmitted Cylindrical Functions* already developed in [48], relevant to a flat interface. This expansion in terms of cylindrical waves is the most appropriate for scatterer of circular cross section. Since properties of reflection and transmission from a interface are are known only for plane waves, the concept of plane-wave spectrum of a cylindrical wave is used to tackle the two geometries simultaneously.

1.1 Problem description

The geometry of the scattering problem is depicted in Figure 1.1: N circular cross-section cylinders are arbitrarily buried beneath a rough surface, in a semi-infinite medium (medium 1) with permittivity $\varepsilon_1 = \varepsilon_0 n_1^2$, which is assumed to be linear, isotropic, homogeneous and lossless. A monochromatic plane-wave, propagating in

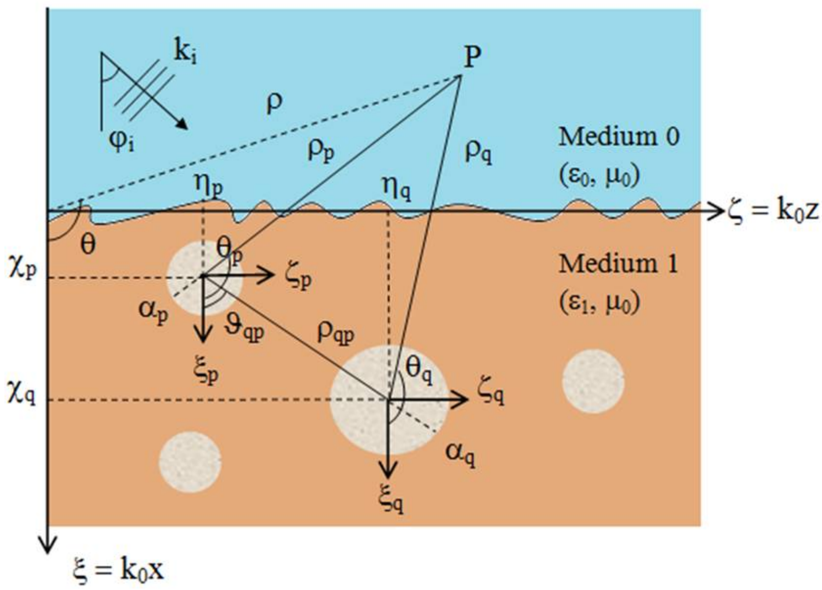


Figure 1.1: Geometry of the problem.

medium 0 (permittivity ε_0), with wave vector lying in the xz plane, impinges on the rough interface. Normalized coordinates $\xi = k_0 x$ and $\zeta = k_0 z$, being $k_0 = \omega/c$ the vacuum wave number, are employed. The cylinder's axes are parallel to the y-axis, and the whole structure infinitely extends along the y-direction, so that the problem is two-dimensional.

A Main Reference Frame (MRF) (0, ξ, ζ), with normalized coordinates $\xi = k_0 x$ and $\zeta = k_0 z$ is introduced. A second Reference Frame (RF_p) (0, ξ_p, ζ_p) centered on the p-*th* cylinder is considered, both in rectangular (ξ_p,ζ_p) and polar (ρ_p,θ_p) coordinates, where $\xi_\mathrm{p} = k_0 x_\mathrm{p}$, $\zeta_\mathrm{p} = k_0 z_\mathrm{p}$, $\rho_\mathrm{p} = k_0 r_\mathrm{p}$. The relation between coordinates in Reference Frame (MRF) and (RF_p) is defined as:

$$\begin{cases} \xi_\mathrm{p} = \xi - \chi_\mathrm{p} \\ \zeta_\mathrm{p} = \zeta - \eta_\mathrm{p} \end{cases} \tag{1.1.1}$$

Both perfectly-conducting or dielectric cylinders are taken into account, with normalized radius $\alpha_\mathrm{p} = k_0 a_\mathrm{p}$ and center in RF_p in $(\chi_\mathrm{p}, \eta_\mathrm{p})$, being $\mathrm{p} = 1, ..., N$. Dielectric scatterers have permittivity $\varepsilon_\mathrm{c} = \varepsilon_0 n_\mathrm{cp}^2$.

1.2 Fields description

In each medium, the scalar function $V(\xi, \zeta)$ represents the field component parallel to the cylinder's axes, which is the sum of all the contributions following the interaction of the incident field with the rough surface and the circular scatterers.
In medium 0,

$$V^{(0)} = V_\mathrm{i}(\xi, \zeta) + V_\mathrm{r}(\xi, \zeta) + V_\mathrm{st}(\xi, \zeta)$$

In medium 1,

$$V^{(1)} = V_\mathrm{t}(\xi, \zeta) + V_\mathrm{s}(\xi, \zeta) + V_\mathrm{sr}(\xi, \zeta)$$

In case of dielectric cylinders, we have an additional field $V_\mathrm{cp}(\xi, \zeta)$ transmitted inside the cylinders. The decomposition of the total field is shown in Figure 1.2.

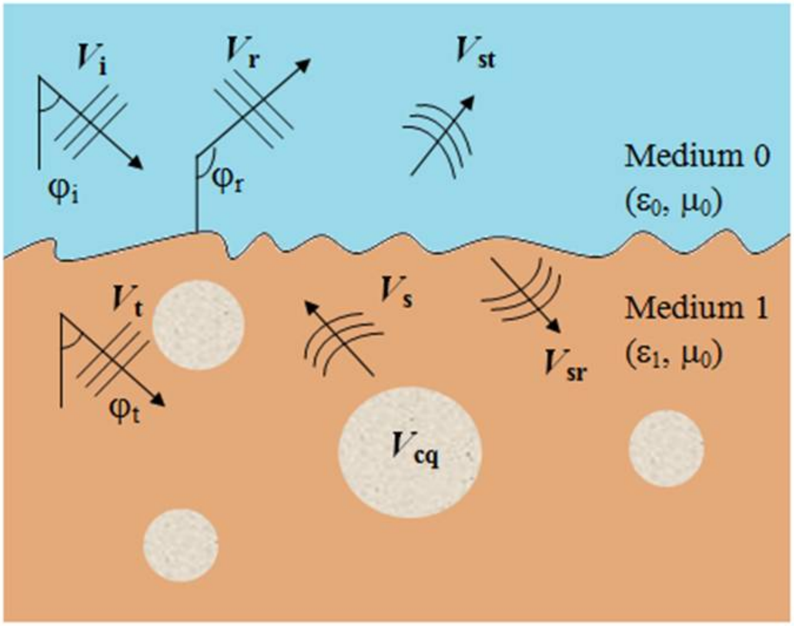


Figure 1.2: Decomposition of the total field.

1.3 Evaluation of reflected and transmitted fields by rough surface

In this Section, electromagnetic scattering of a plane wave from a rough surface using first order Small Perturbation Method (SPM) is discussed. Small Perturbation Method for rough surface scattering, originally derived by Rice [2], has been extensively applied to problems in optics, remote sensing, and propagation. It yields the Bragg scatter phenomenon [4] of rough-surface scattering when only first-order terms are considered. The Small Perturbation Method makes use of the Rayleigh hypothesis [2] to express the reflected and transmitted fields into upward- and downward-going waves, respectively. Typical uses of the theory involve only the first- or second-order scattered fields, owing to increasing complexity of the SPM equations as order is increased. A first order perturbation solution can be applied when the phase difference due to the height variation is much smaller than 2π [61], and the slope of the rough perturbation is much smaller than unity.

1.3.1 TM polarization

Since we are considering TM polarization, the scalar function $V(\xi,\zeta)$ corresponds to the field E_y. The incident plane-wave field, propagating in medium 0, impinges on the interface forming an angle $\varphi_{\rm i}$ with the positive direction of the x-axis. The propagation wave vector $\mathbf{k}^{\rm i} = k^{\rm i}_{0\|}\hat{z} + k^{\rm i}_{0\perp}\hat{x}$ is related to angle $\varphi_{\rm i}$ through the following expressions:

$$\begin{cases} k^{\rm i}_{0\|} = k_0 \sin\varphi_{\rm i} \\ k^{\rm i}_{0\perp} = k_0 \cos\varphi_{\rm i} \end{cases} \tag{1.3.1}$$

where $k^{\rm i}_{0\|}$ and $k^{\rm i}_{0\perp}$ are the parallel and orthogonal components of the wave vector, respectively. In the following, the symbols $\perp$ and $\|$ will always be associated to the orthogonal and parallel components of a generic vector with respect to the planar interfaces. The incident field of complex amplitude V_0 is given by

$$V_{\rm i}(x,z) = V_0 e^{i(k^{\rm i}_{\perp}x + k^{\rm i}_{\|}z)} \tag{1.3.2}$$

For the sake of simplicity, normalized coordinates $\xi = k_0 x$ and $\zeta = k_0 z$, being $k_0 = \omega/c$ the vacuum wave number, are employed. So, the expression of the incident field in dimensionless coordinates is the following

$$V_{\rm i}(\xi,\zeta) = V_0 e^{in_0(n^{\rm i}_{\perp}\xi + n^{\rm i}_{\|}\zeta)} \tag{1.3.3}$$

where $n^{\rm i}_{\|}$ and $n^{\rm i}_{\perp}$ are the parallel and orthogonal components of the unit propagation vector, and they are given by:

$$\begin{cases} n^{\rm i}_{\|} = \sin\varphi^{\rm i} \\ n^{\rm i}_{\perp} = \cos\varphi^{\rm i} \end{cases} \tag{1.3.4}$$

The surface perturbation can be defined by the function $g(\zeta)$, and the relevant Fourier transform $G(n_{\|})$ is

$$G(n_{\|}) = \int_{-\infty}^{\infty} g(\zeta) e^{-in_{\|}\zeta} d\zeta \tag{1.3.5}$$

Mathematically, the conditions of validity of Small Perturbation Method are given by

$$|g(\zeta)\cos\varphi_{\mathrm{i}}| << 1$$

and

$$|\partial g(\zeta)/\partial\zeta| << 1$$

By making use of the Rayleigh hypothesis [2], the reflected field in medium 0 can be expressed as:

$$V_{\mathrm{r}}(\xi,\zeta) = \frac{1}{2\pi}V_0\int_{-\infty}^{\infty} V_{\mathrm{r}}(n_{\parallel})e^{in_0(-n_{\perp}\xi+n_{\parallel}\zeta)}\,dn_{\parallel} \tag{1.3.6}$$

where $n_{\perp} = \sqrt{1-(n_{\parallel})^2}$

The transmitted field in medium 1 can be written as:

$$V_{\mathrm{t}}(\xi,\zeta) = \frac{1}{2\pi}V_0\int_{-\infty}^{\infty} V_{\mathrm{t}}(n_{\parallel})e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\xi+n_{\parallel}\zeta\right]}\,dn_{\parallel} \tag{1.3.7}$$

The H-field is derived from the E-field by the normalized curl

$$\mathbf{H} = \frac{k}{i\omega\mu}\tilde{\nabla}\times\mathbf{E} = \frac{1}{iZ}\tilde{\nabla}\times\mathbf{E} \tag{1.3.8}$$

The curl of $\mathbf{E} = E_y\hat{y}$ has the following expression

$$\tilde{\nabla}\times\mathbf{E} = -\frac{\partial}{\partial\zeta}E_y\hat{\xi} + \frac{\partial}{\partial\xi}E_y\hat{\zeta} \tag{1.3.9}$$

Boundary conditions can be expressed as

$$V_{\mathrm{i}}(\xi,\zeta) + V_{\mathrm{r}}(\xi,\zeta) = V_{\mathrm{t}}(\xi,\zeta) \tag{1.3.10}$$

$$\frac{\partial}{\partial\xi}V_{\mathrm{i}}(\xi,\zeta) + \frac{\partial}{\partial\xi}V_{\mathrm{r}}(\xi,\zeta) = \frac{\partial}{\partial\xi}V_{\mathrm{t}}(\xi,\zeta) \tag{1.3.11}$$

In the spectral domain, the scattered field is written as perturbation series:

$$V_{\mathrm{r}}(n_{\parallel}) = \left[V_{\mathrm{r}}^0(n_{\parallel}) + V_{\mathrm{r}}^1(n_{\parallel}) + V_{\mathrm{r}}^2(n_{\parallel}) + ...\right] \tag{1.3.12}$$

and the transmitted field is written as:

$$V_{\mathrm{t}}(n_\parallel) = \left[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + V_{\mathrm{t}}^2(n_\parallel) + ...\right] \tag{1.3.13}$$

For electric field, we impose boundary condition (1.3.10) at $\xi = g(\zeta)$ together with a series expansion of the exponential functions assuming that $n_\perp g(\zeta) \ll 1$

$$\begin{aligned} &2\pi e^{in_0 n_\parallel^{\mathrm{i}}\zeta}\left[1 + in_0 n_\perp^{\mathrm{i}} g(\zeta) + ...\right] \\ &+\int_{-\infty}^{\infty} e^{in_0 n_\parallel \zeta}\left[1 - in_0 n_\perp g(\zeta) + ...\right]\left[V_{\mathrm{r}}^0(n_\parallel) + V_{\mathrm{r}}^1(n_\parallel) + ...\right] dn_\parallel \\ &= \int_{-\infty}^{\infty} e^{in_1 n_\parallel \zeta}\left[1 + in_1\sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2} g(\zeta) + ...\right]\left[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + ...\right] dn_\parallel \end{aligned} \tag{1.3.14}$$

From equation(1.3.11), we have the following boundary condition for H-field on the rough interface

$$\begin{aligned} &2\pi(n_0 n_\perp^{\mathrm{i}}) e^{in_0 n_\parallel^{\mathrm{i}}\zeta}\left[1 + in_0 n_\perp^{\mathrm{i}} g(\zeta) + ...\right] \\ &-\int_{-\infty}^{\infty} (n_0 n_\perp) e^{in_0 n_\parallel \zeta}\left[1 - in_0 n_\perp g(\zeta) + ...\right]\left[V_{\mathrm{r}}^0(n_\parallel) + V_{\mathrm{r}}^1(n_\parallel) + ...\right] dn_\parallel \\ &= \int_{-\infty}^{\infty}\left[n_1\sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2}\right] e^{in_1 n_\parallel \zeta}\left[1 + in_1\sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2} g(\zeta) + ...\right] \\ &\quad \times\left[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + ...\right] dn_\parallel \end{aligned} \tag{1.3.15}$$

Balancing the equations (1.3.14) and (1.3.15) to the zero-*th* order gives

$$2\pi e^{in_0 n_\parallel^{\mathrm{i}}\zeta} + \int_{-\infty}^{\infty} e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^0(n_\parallel)\, dn_\parallel = \int_{-\infty}^{\infty} e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^0(n_\parallel)\, dn_\parallel \tag{1.3.16}$$

and

$$
\begin{aligned}
&2\pi n_0 n_\perp^{\mathrm{i}} e^{in_0 n_\parallel^{\mathrm{i}}\zeta} - \int_{-\infty}^{\infty} (n_0 n_\perp) e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^0(n_\parallel)\, dn_\parallel \\
&= \int_{-\infty}^{\infty} \left[n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2} \right] e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^0(n_\parallel)\, dn_\parallel
\end{aligned} \tag{1.3.17}
$$

The Fourier transform of equation (1.3.16) is

$$
2\pi\delta(n_0 n_\parallel - n_0 n_\parallel^{\mathrm{i}}) + V_{\mathrm{r}}^0(n_\parallel) = V_{\mathrm{t}}^0(n_\parallel) \tag{1.3.18}
$$

Putting equation (1.3.18) in equation (1.3.17), we get

$$
\begin{aligned}
&2\pi \left[n_0 n_\perp^{\mathrm{i}} - n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2} \right] e^{in_0 n_\parallel^{\mathrm{i}}\zeta} \\
&= \int_{-\infty}^{\infty} \left[n_0 n_\perp + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2} \right] e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^0(n_\parallel)\, dn_\parallel
\end{aligned} \tag{1.3.19}
$$

$$
V_{\mathrm{r}}^0(n_\parallel) = \left[\frac{n_0 n_\perp^{\mathrm{i}} - n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2}}{n_0 n_\perp^{\mathrm{i}} + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2}} \right] 2\pi\delta(n_0 n_\parallel - n_0 n_\parallel^{\mathrm{i}}) \tag{1.3.20}
$$

From equation (1.3.18), we have

$$
V_{\mathrm{t}}^0(n_\parallel) = \left[\frac{2 n_0 n_\perp^{\mathrm{i}}}{n_0 n_\perp^{\mathrm{i}} + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2}} \right] 2\pi\delta(n_0 n_\parallel - n_0 n_\parallel^{\mathrm{i}}) \tag{1.3.21}
$$

Balancing the equations (1.3.14) and (1.3.15) to the first-order leads to

$$
\begin{aligned}
&i2\pi n_0 n_\perp^{\mathrm{i}} g(\zeta) e^{in_0 n_\parallel^{\mathrm{i}}\zeta} - i \int_{-\infty}^{\infty} n_0 n_\perp g(\zeta) e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^0(n_\parallel) dn_\parallel + \int_{-\infty}^{\infty} e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^1(n_\parallel) dn_\parallel \\
&= i \int_{-\infty}^{\infty} \left[n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2} \right] g(\zeta) e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^0(n_\parallel) dn_\parallel + \int_{-\infty}^{\infty} e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^1(n_\parallel) dn_\parallel
\end{aligned} \tag{1.3.22}
$$

and

$$
\begin{aligned}
&i2\pi(n_0 n_\perp^{\mathrm{i}})^2 g(\zeta) e^{in_0 n_\parallel^{\mathrm{i}}\zeta} + i\int\limits_{-\infty}^{\infty} (n_0 n_\perp)^2 g(\zeta) e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^0(n_\parallel)\, dn_\parallel \\
&-\int\limits_{-\infty}^{\infty} (n_0 n_\perp) e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^1(n_\parallel)\, dn_\parallel = i\int\limits_{-\infty}^{\infty} [n_1^2 - (n_0 n_\parallel)^2] g(\zeta) e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^0(n_\parallel) dn_\parallel \\
&+\int\limits_{-\infty}^{\infty} \left[n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2}\, \right] e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^1(n_\parallel)\, dn_\parallel
\end{aligned}
\tag{1.3.23}
$$

Putting the values of $V_{\mathrm{r}}^0(n_\parallel)$ and $V_{\mathrm{t}}^0(n_\parallel)$ in equation (1.3.22), we have

$$
\int\limits_{-\infty}^{\infty} dn_\parallel e^{in_0 n_\parallel \zeta} V_{\mathrm{r}}^1(n_\parallel) = \int\limits_{-\infty}^{\infty} dn_\parallel e^{in_1 n_\parallel \zeta} V_{\mathrm{t}}^1(n_\parallel) \tag{1.3.24}
$$

Therefore, it is

$$
V_{\mathrm{r}}^1(n_\parallel) = V_{\mathrm{t}}^1(n_\parallel) \tag{1.3.25}
$$

Using equation (1.3.25) along with the values of $V_{\mathrm{r}}^0(n_\parallel)$ and $V_{\mathrm{t}}^0(n_\parallel)$ in equation (1.3.23), we get

$$
\begin{aligned}
\frac{i(n_0^2 - n_1^2) 2 n_0 n_\perp^{\mathrm{i}}}{\left[n_0 n_\perp^{\mathrm{i}} + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\, \right]} g(\zeta) e^{in_0 n_\parallel^{\mathrm{i}} \zeta} &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} dn_\parallel V_{\mathrm{r}}^1(n_\parallel) e^{in_0 n_\parallel \zeta} \\
&\quad \times \left[n_0 n_\perp + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2}\, \right]
\end{aligned}
\tag{1.3.26}
$$

$$
V_{\mathrm{r}}^1(n_\parallel) = \frac{i(n_0^2 - n_1^2) 2 n_0 n_\perp^{\mathrm{i}} G(n_0 n_\parallel - n_0 n_\parallel^{\mathrm{i}})}{\left[n_0 n_\perp^{\mathrm{i}} + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\, \right] \left[n_0 \sqrt{1 - (n_\parallel)^2} + n_1 \sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2}\, \right]} \tag{1.3.27}
$$

From equation(1.3.25), we get

$$V_{\mathrm{t}}^{1}(n_{\|}) = \frac{i(n_0^2 - n_1^2)2n_0 n_{\perp}^{\mathrm{i}} G(n_0 n_{\|} - n_0 n_{\|}^{\mathrm{i}})}{\left[n_0 n_{\perp}^{\mathrm{i}} + n_1\sqrt{1 - \left(\frac{n_0 n_{\|}^{\mathrm{i}}}{n_1}\right)^2}\,\right]\left[n_0\sqrt{1-(n_{\|})^2} + n_1\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\,\right]} \tag{1.3.28}$$

Putting equations (1.3.27) and (1.3.20) in equation (1.3.6), the reflected field can be written as a sum of zero-*th* and first-order fields, which are the contribution relevant to flat interface and rough interface, respectively

$$V_{\mathrm{r}}(\xi,\zeta) = \Gamma_{01}^{\mathrm{TM}}(n_{\|}^{\mathrm{i}})V_0 e^{in_0(-n_{\perp}^{\mathrm{i}}\xi + n_{\|}^{\mathrm{i}}\zeta)} + V_0\frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_{01}^{\mathrm{TM}}(n_{\|})e^{in_0(-n_{\perp}\xi + n_{\|}\zeta)}dn_{\|} \tag{1.3.29}$$

where $\Gamma_{01}^{\mathrm{TM}}(n_{\|}^{\mathrm{i}})$ is the Fresnel reflection coefficient relevant to the zero-*th* order term and it is given by

$$\Gamma_{01}^{\mathrm{TM}}(n_{\|}^{\mathrm{i}}) = \frac{n_0 n_{\perp}^{\mathrm{i}} - n_1 n_{\perp}^{\mathrm{t}}}{n_0 n_{\perp}^{\mathrm{i}} + n_1 n_{\perp}^{\mathrm{t}}} \tag{1.3.30}$$

with

$$n_{\perp}^{\mathrm{t}} = \sqrt{1 - \left(\frac{n_0 n_{\|}^{\mathrm{i}}}{n_1}\right)^2}$$

The reflection coefficient $\gamma_{01}^{\mathrm{TM}}(n_{\|})$ is applied to define the first-order term of the reflected field

$$\gamma_{01}^{\mathrm{TM}}(n_{\|}) = \frac{(n_0^2 - n_1^2)2n_0 n_{\perp}^{\mathrm{i}} iG(n_{\|} - n_{\|}^{\mathrm{i}})}{(n_0 n_{\perp}^{\mathrm{i}} + n_1 n_{\perp}^{\mathrm{t}})\left[n_0\sqrt{1-n_{\|}^2} + n_1\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\,\right]} \tag{1.3.31}$$

The expressions for the reflected field V_{r} can be written in a more compact way

$$V_{\mathrm{r}}(\xi,\zeta) = V_0\left[\Gamma_{01}^{\mathrm{TM}}(n_{\|}^{\mathrm{i}})e^{in_0(-n_{\perp}^{\mathrm{i}}\xi + n_{\|}^{\mathrm{i}}\zeta)} + \Delta A_{\mathrm{r}}^{\mathrm{TM}}\right] \tag{1.3.32}$$

where the term $\Delta A_{\mathrm{r}}^{\mathrm{TM}}$ in equation (1.3.32) denotes a small quantity due to a small surface perturbation

$$\Delta A_{\mathrm{r}}^{\mathrm{TM}} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\gamma_{01}^{\mathrm{TM}}(n_{\|})e^{in_0(-n_{\perp}\xi + n_{\|}\zeta)}dn_{\|} \tag{1.3.33}$$

Putting equations (1.3.28) and (1.3.21) in equation (1.3.7), the transmitted field $V_{\rm t}$ can be defined

$$\begin{aligned} V_{\rm t}(\xi,\zeta) &= T_{01}^{\rm TM}(n_{\parallel}^{\rm i})V_0 e^{in_1(n_{\perp}^{\rm t}\xi+n_{\parallel}^{\rm t}\zeta)} \\ &+ V_0\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\tau_{01}^{\rm TM}(n_{\parallel})e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\xi+n_{\parallel}\zeta\right]}dn_{\parallel} \end{aligned} \quad (1.3.34)$$

where $T_{01}^{\rm TM}(n_{\parallel}^{\rm i})$ and $\tau_{01}^{\rm TM}(n_{\parallel})$ are coefficients for unperturbed and perturbed transmitted fields, respectively. They are given by

$$T_{01}^{\rm T\mathit{M}}(n_{\parallel}^{\rm i}) = \frac{2n_0 n_{\perp}^{\rm i}}{n_0 n_{\perp}^{\rm i}+n_1 n_{\perp}^{\rm t}} \quad (1.3.35)$$

and

$$\tau_{01}^{\rm TM}(n_{\parallel}) \equiv \gamma_{01}^{\rm TM}(n_{\parallel}) \quad (1.3.36)$$

Introducing equation (1.1.1) in equation (1.3.34), the transmitted field $V_{\rm t}$ can be written in a point of coordinates (ξ,ζ) of (MRF) as a function of the coordinates $(\xi_{\rm p},\zeta_{\rm p})$ in $(RF_{\rm p})$

$$\begin{aligned} V_{\rm t}(\xi,\zeta) &= T_{01}^{\rm TM}(n_{\parallel}^{\rm i})V_0 e^{in_1\left[n_{\perp}^{\rm t}(\xi_{\rm p}+\chi_{\rm p})+n_{\parallel}^{\rm t}(\zeta_{\rm p}+\eta_{\rm p})\right]} \\ &+ V_0\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\tau_{01}^{\rm TM}(n_{\parallel})e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}(\xi_{\rm p}+\chi_{\rm p})+n_{\parallel}(\zeta_{\rm p}+\eta_{\rm p})\right]}dn_{\parallel} \end{aligned} \quad (1.3.37)$$

Using the plane wave expansion into Bessel functions, the transmitted field is expressed in polar coordinates. Thus, the final solution to the problem, which is obtained imposing boundary conditions on the cylinder's surfaces can be obtained in an easier way

$$V_{\rm t}(\xi_{\rm p},\zeta_{\rm p}) = V_0\sum_{\ell=-\infty}^{\infty} i^{\ell}J_{\ell}(n_1\rho_{\rm p})e^{i\ell\theta_{\rm p}}\left[A_{\rm t}^{\rm TM}+\Delta B_{\rm t}^{\rm TM}\right] \quad (1.3.38)$$

where

$$A_{\rm t}^{\rm TM} = T_{01}^{\rm TM}(n_{\parallel}^{\rm i})e^{-i\ell\varphi_{\rm t}}e^{in_1(n_{\perp}^{\rm t}\chi_{\rm p}+n_{\parallel}^{\rm t}\eta_{\rm p})} \quad (1.3.39)$$

and

$$\Delta B_{\mathrm{t}}^{\mathrm{TM}} = \int_{-\infty}^{\infty} \tau_{01}^{\mathrm{TM}}(n_\parallel) e^{-i\ell\phi_{\mathrm{t}}} e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_\parallel}{n_1}\right)^2}\chi_{\mathrm{P}}+n_\parallel \eta_{\mathrm{P}}\right]} dn_\parallel \tag{1.3.40}$$

being

$$\phi_{\mathrm{t}} = \tan^{-1}\left[\frac{n_\parallel}{\sqrt{1-\left(\frac{n_0 n_\parallel}{n_1}\right)^2}}\right] \tag{1.3.41}$$

1.3.2 TE Polarization

Since we are considering TE polarization, the scalar function $V_{\mathrm{i}}(\xi, \zeta)$ corresponds to the magnetic field H_y.

The E-field is derived by the H-field from the normalized curl

$$\mathbf{E} = \frac{ik}{\omega\epsilon}\tilde{\nabla} \times \mathbf{H} = iZ\tilde{\nabla} \times \mathbf{H} \tag{1.3.42}$$

The curl of $\mathbf{H} = H_y \hat{y}$ has the following expression

$$\tilde{\nabla} \times \mathbf{H} = -\frac{\partial}{\partial\zeta} H_y \hat{\xi} + \frac{\partial}{\partial\xi} H_y \hat{\zeta} \tag{1.3.43}$$

Boundary conditions on the interface specify that tangential electric and magnetic fields must be continuous:

$$V_{\mathrm{i}}(\xi, \zeta) + V_{\mathrm{r}}(\xi, \zeta) = V_{\mathrm{t}}(\xi, \zeta) \tag{1.3.44}$$

$$\begin{aligned} &-\frac{\partial g(\zeta)}{\partial\zeta}\left[\frac{\partial}{\partial\zeta}V_{\mathrm{i}}(\xi,\zeta) + \frac{\partial}{\partial\zeta}V_{\mathrm{r}}(\xi,\zeta)\right] + \left[\frac{\partial}{\partial\xi}V_{\mathrm{i}}(\xi,\zeta) + \frac{\partial}{\partial\xi}V_{\mathrm{r}}(\xi,\zeta)\right] \\ &= -\frac{\partial g(\zeta)}{\partial\zeta}\frac{\partial}{\partial\zeta}V_{\mathrm{t}}(\xi,\zeta) + \frac{\partial}{\partial\xi}V_{\mathrm{t}}(\xi,\zeta) \end{aligned} \tag{1.3.45}$$

For H-field, we impose boundary condition (1.3.44) at $\xi = g(\zeta)$

$$\begin{aligned}
&2\pi e^{in_0 n_\parallel^{\mathrm{i}}\zeta}\Big[1 + in_0 n_\perp^{\mathrm{i}} g(\zeta) + ...\Big] \\
&+\int_{-\infty}^{\infty} e^{in_0 n_\parallel \zeta}\Big[1 - in_0 n_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n_\parallel) + V_{\mathrm{r}}^1(n_\parallel) + ...\Big]\,dn_\parallel \\
&= \int_{-\infty}^{\infty} e^{in_1 n_\parallel \zeta}\Big[1 + in_1\sqrt{1 - \Big(\frac{n_0 n_\parallel}{n_1}\Big)^2}\, g(\zeta) + ...\Big]\Big[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + ...\Big] dn_\parallel
\end{aligned} \tag{1.3.46}$$

From equation(1.3.45), we have the following boundary condition for E-field on the rough interface

$$\begin{aligned}
&\Big(\frac{n_\perp^{\mathrm{i}}}{n_0}\Big) e^{in_0 n_\parallel^{\mathrm{i}}\zeta}\Big[1 + in_0 n_\perp^{\mathrm{i}} g(\zeta) + ...\Big] - \frac{\partial g(\zeta)}{\partial \zeta}\Big(\frac{n_\parallel^{\mathrm{i}}}{n_0}\Big) e^{in_0 n_\parallel^{\mathrm{i}}\zeta}\Big[1 + in_0 n_\perp^{\mathrm{i}} g(\zeta) + ...\Big] \\
&-\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{n_\perp}{n_0}\Big) e^{in_0 n_\parallel\zeta}\Big[1 - in_0 n_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n_\parallel) + V_{\mathrm{r}}^1(n_\parallel) + ...\Big]\,dn_\parallel \\
&-\frac{\partial g(\zeta)}{\partial \zeta}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{n_\parallel}{n_0}\Big) e^{in_0 n_\parallel\zeta}\Big[1 - in_0 n_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n_\parallel) + V_{\mathrm{r}}^1(n_\parallel) + ...\Big]\,dn_\parallel \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\Bigg[\frac{1}{n_1}\sqrt{1 - \Big(\frac{n_0 n_\parallel}{n_1}\Big)^2}\,\Bigg] e^{in_1 n_\parallel\zeta}\Big[1 + in_1\sqrt{1 - \Big(\frac{n_0 n_\parallel}{n_1}\Big)^2}\, g(\zeta) + ...\Big] \\
&\qquad \times\Big[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + ...\Big]\,dn_\parallel \\
&-\frac{\partial g(\zeta)}{\partial \zeta}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\frac{n_\parallel}{n_1}\Big) e^{in_1 n_\parallel\zeta}\Big[1 + in_1\sqrt{1 - \Big(\frac{n_0 n_\parallel}{n_1}\Big)^2}\, g(\zeta) + ...\Big] \\
&\qquad\qquad \times\Big[V_{\mathrm{t}}^0(n_\parallel) + V_{\mathrm{t}}^1(n_\parallel) + ...\Big]\,dn_\parallel
\end{aligned} \tag{1.3.47}$$

Balancing the equations (1.3.46) and (1.3.47) to the zero-*th* order gives

$$2\pi e^{in_0 n_\parallel^{\mathrm{i}}\zeta} + \int_{-\infty}^{\infty} e^{in_0 n_\parallel\zeta} V_{\mathrm{r}}^0(n_\parallel)\,dn_\parallel = \int_{-\infty}^{\infty} e^{in_1 n_\parallel\zeta} V_{\mathrm{t}}^0(n_\parallel)\,dn_\parallel \tag{1.3.48}$$

and

$$2\pi\frac{n_{\perp}^{\mathrm{i}}}{n_0}e^{in_{\parallel}^{\mathrm{i}}\zeta}-\int\limits_{-\infty}^{\infty}\left(\frac{n_{\perp}}{n_0}\right)e^{in_0n_{\parallel}\zeta}V_{\mathrm{r}}^0(n_{\parallel})dn_{\parallel}=\int\limits_{-\infty}^{\infty}\frac{1}{n_1}\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}e^{in_1n_{\parallel}\zeta}V_{\mathrm{t}}^0(n_{\parallel})dn_{\parallel} \tag{1.3.49}$$

Taking the Fourier transform of equation (1.3.48) yields

$$2\pi\delta(n_0n_{\parallel}-n_0n_{\parallel}^{\mathrm{i}})+V_{\mathrm{r}}^0(n_{\parallel})=V_{\mathrm{t}}^0(n_{\parallel}) \tag{1.3.50}$$

Putting equation (1.3.50) in equation (1.3.49), we get

$$\begin{aligned}&\left[\frac{n_{\perp}^{\mathrm{i}}}{n_0}-\frac{1}{n_1}\sqrt{1-\left(\frac{n_0n_{\parallel}^{\mathrm{i}}}{n_1}\right)^2}\right]e^{in_0n_{\parallel}^{\mathrm{i}}\zeta}\\ &=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{n_{\perp}}{n_0}+\frac{1}{n_1}\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}\right]e^{in_0n_{\parallel}\zeta}V_{\mathrm{r}}^0(n_{\parallel})\,dn_{\parallel}\end{aligned} \tag{1.3.51}$$

$$V_{\mathrm{r}}^0(n_{\parallel})=\left[\frac{n_1n_{\perp}^{\mathrm{i}}-n_0\sqrt{1-\left(\frac{n_0n_{\parallel}^{\mathrm{i}}}{n_1}\right)^2}}{n_1n_{\perp}^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_{\parallel}^{\mathrm{i}}}{n_1}\right)^2}}\right]2\pi\delta(n_0n_{\parallel}-n_0n_{\parallel}^{\mathrm{i}}) \tag{1.3.52}$$

From equation (1.3.50), we have

$$V_{\mathrm{t}}^0(n_{\parallel})=\left[\frac{2n_1n_{\perp}^{\mathrm{i}}}{n_1n_{\perp}^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_{\parallel}^{\mathrm{i}}}{n_1}\right)^2}}\right]2\pi\delta(n_0n_{\parallel}-n_0n_{\parallel}^{\mathrm{i}}) \tag{1.3.53}$$

Balancing the equations (1.3.46) and (1.3.47) to the first-order, we obtain

$$\begin{aligned}&i2\pi n_0n_{\perp}^{\mathrm{i}}g(\zeta)e^{in_0n_{\parallel}^{\mathrm{i}}\zeta}-i\int\limits_{-\infty}^{\infty}n_0n_{\perp}g(\zeta)e^{in_0n_{\parallel}\zeta}V_{\mathrm{r}}^0(n_{\parallel})\,dn_{\parallel}+\int\limits_{-\infty}^{\infty}e^{in_0n_{\parallel}\zeta}V_{\mathrm{r}}^1(n_{\parallel})\,dn_{\parallel}\\ &=i\int_{-\infty}^{\infty}\left[n_1\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}\right]g(\zeta)e^{in_1n_{\parallel}\zeta}V_{\mathrm{t}}^0(n_{\parallel})\,dn_{\parallel}+\int\limits_{-\infty}^{\infty}e^{in_1n_{\parallel}\zeta}V_{\mathrm{t}}^1(n_{\parallel})\,dn_{\parallel}\end{aligned} \tag{1.3.54}$$

and

$$i2\pi\left(n^{\mathrm{i}}_{\perp}\right)^{2}g(\zeta)e^{in_0n^{\mathrm{i}}_{\parallel}\zeta}-2\pi\frac{\partial g(\zeta)}{\partial\zeta}\left(\frac{n^{\mathrm{i}}_{\parallel}}{n_0}\right)e^{in_0n^{\mathrm{i}}_{\parallel}\zeta}-\int\limits_{-\infty}^{\infty}\left(\frac{n_{\perp}}{n_0}\right)e^{in_0n_{\parallel}\zeta}V^{1}_{\mathrm{r}}(n_{\parallel})\,dn_{\parallel}$$
$$+i\int\limits_{-\infty}^{\infty}\left(n_{\perp}\right)^{2}g(\zeta)e^{in_0n_{\parallel}\zeta}V^{0}_{\mathrm{r}}(n_{\parallel})\,dn_{\parallel}-\frac{\partial g(\zeta)}{\partial\zeta}\int\limits_{-\infty}^{\infty}\left(\frac{n_{\parallel}}{n_0}\right)e^{in_0n_{\parallel}\zeta}V^{0}_{\mathrm{r}}(n_{\parallel})\,dn_{\parallel}$$
$$=\int\limits_{-\infty}^{\infty}\left[\frac{1}{n_1}\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^{2}}\,\right]e^{in_1n_{\parallel}\zeta}V^{1}_{\mathrm{t}}(n_{\parallel})\,dn_{\parallel}-\frac{\partial g(\zeta)}{\partial\zeta}\int\limits_{-\infty}^{\infty}\left(\frac{n_{\parallel}}{n_1}\right)e^{in_1n_{\parallel}\zeta}V^{0}_{\mathrm{t}}(n_{\parallel})\,dn_{\parallel}$$
$$+i\int\limits_{-\infty}^{\infty}\left[\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^{2}}\,\right]^{2}g(\zeta)e^{in_1n_{\parallel}\zeta}V^{0}_{\mathrm{t}}(n_{\parallel})\,dn_{\parallel} \tag{1.3.55}$$

Putting the values of $V^{0}_{\mathrm{r}}(n_{\parallel})$ and $V^{0}_{\mathrm{t}}(n_{\parallel})$ in equation (1.3.54), we have

$$\int\limits_{-\infty}^{\infty}e^{in_1n_{\parallel}\zeta}V^{1}_{\mathrm{t}}(n_{\parallel})\,dn_{\parallel}-\int\limits_{-\infty}^{\infty}e^{in_0n_{\parallel}\zeta}V^{1}_{\mathrm{r}}(n_{\parallel})\,dn_{\parallel}$$
$$=\frac{i(n_0^2-n_1^2)2n^{\mathrm{i}}_{\perp}\sqrt{1-\left(\frac{n_0n^{\mathrm{i}}_{\parallel}}{n_1}\right)^{2}}}{\left[n_1n^{\mathrm{i}}_{\perp}+n_0\sqrt{1-\left(\frac{n_0n^{\mathrm{i}}_{\parallel}}{n_1}\right)^{2}}\,\right]}g(\zeta)e^{in_0n^{\mathrm{i}}_{\parallel}\zeta} \tag{1.3.56}$$

and taking Fourier transform, we get

$$V^{1}_{\mathrm{t}}(n_{\parallel})-V^{1}_{\mathrm{r}}(n_{\parallel})=\frac{i(n_0^2-n_1^2)2n^{\mathrm{i}}_{\perp}\sqrt{1-\left(\frac{n_0n^{\mathrm{i}}_{\parallel}}{n_1}\right)^{2}}}{\left[n_1n^{\mathrm{i}}_{\perp}+n_0\sqrt{1-\left(\frac{n_0n^{\mathrm{i}}_{\parallel}}{n_1}\right)^{2}}\,\right]}G(n_0n_{\parallel}-n_0n^{\mathrm{i}}_{\parallel}) \tag{1.3.57}$$

Putting the values of $V_{\mathrm{r}}^0(n_\parallel)$ and $V_{\mathrm{t}}^0(n_\parallel)$ in equation (1.3.55), we obtain

$$\int_{-\infty}^{\infty}\left[n_0\sqrt{1-\left(\frac{n_0n_\parallel}{n_1}\right)^2}\,\right]e^{in_1n_\parallel\zeta}V_t^1(n_\parallel)\,dn_\parallel+\int_{-\infty}^{\infty}\Big(n_1n_\perp\Big)e^{in_0n_\parallel\zeta}V_r^1(n_\parallel)\,dn_\parallel$$
$$=\frac{i(n_0^2-n_1^2)2n_0n_\perp^{\mathrm{i}}{n_\parallel^{\mathrm{i}}}^2g(\zeta)e^{in_0n_\parallel^{\mathrm{i}}\zeta}}{\left[n_1n_\perp^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\,\right]}+\frac{i(n_0^2-n_1^2)2n_\perp^{\mathrm{i}}n_\parallel^{\mathrm{i}}}{\left[n_1n_\perp^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\,\right]}\frac{\partial g(\zeta)}{\partial\zeta}e^{in_0n_\parallel^{\mathrm{i}}\zeta} \tag{1.3.58}$$

Taking Fourier transform, we get

$$\left[n_0\sqrt{1-\left(\frac{n_0n_\parallel}{n_1}\right)^2}\,\right]V_t^1(n_\parallel)+\Big(n_1n_\perp\Big)V_r^1(n_\parallel)$$
$$=\frac{i(n_0^2-n_1^2)2n_0n_\perp^{\mathrm{i}}n_\parallel^{\mathrm{i}}n_\parallel}{\left[n_1n_\perp^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_\parallel}{n_1}\right)^2}\,\right]}G(n_0n_\parallel-n_0n_\parallel^{\mathrm{i}}) \tag{1.3.59}$$

Solving Equations (1.3.57) and (1.3.59), we get

$$V_{\mathrm{r}}^1(n_\parallel)=\frac{i(n_0^2-n_1^2)2n_0n_\perp^{\mathrm{i}}}{\left[n_1n_\perp^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\,\right]}G(n_0n_\parallel-n_0n_\parallel^{\mathrm{i}})$$
$$\times\frac{\left[n_\parallel n_\parallel^{\mathrm{i}}-\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\sqrt{1-\left(\frac{n_0n_\parallel}{n_1^2}\right)^2}\,\right]}{\left[n_1\sqrt{1-(n_\parallel)^2}+n_0\sqrt{1-\left(\frac{n_0n_\parallel}{n_1}\right)^2}\,\right]} \tag{1.3.60}$$

$$V_{\mathrm{t}}^1(n_\parallel)=\frac{i(n_0^2-n_1^2)2n_\perp^{\mathrm{i}}}{\left[n_1n_\perp^{\mathrm{i}}+n_0\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\,\right]}G(n_0n_\parallel-n_0n_\parallel^{\mathrm{i}})$$
$$\times\frac{\left[n_0n_\parallel n_\parallel^{\mathrm{i}}+n_1\sqrt{1-\left(\frac{n_0n_\parallel^{\mathrm{i}}}{n_1}\right)^2}\sqrt{1-(n_\parallel)^2}\right]}{\left[n_1\sqrt{1-(n_\parallel)^2}+n_0\sqrt{1-\left(\frac{n_0n_\parallel}{n_1}\right)^2}\,\right]} \tag{1.3.61}$$

Putting equations (1.3.60) and (1.3.52) in equation (1.3.6), the reflected field can be written as a sum of unperturbed and perturbed terms

$$V_{\mathrm{r}}(\xi,\zeta)=\Gamma_{01}^{\mathrm{TE}}(n_{\|}^{\mathrm{i}})V_0 e^{in_0(-n_{\perp}^{\mathrm{i}}\xi+n_{\|}^{\mathrm{i}}\zeta)}+V_0\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\gamma_{01}^{\mathrm{TE}}(n_{\|})e^{in_0(-n_{\perp}\xi+n_{\|}\zeta)}dn_{\|} \quad (1.3.62)$$

where $\Gamma_{01}^{\mathrm{TE}}(n_{\|}^{\mathrm{i}})$ is the Fresnel reflection coefficient relevant to the zero-*th* order term, and it is given by

$$\Gamma_{01}^{\mathrm{T}E}(n_{\|}^{\mathrm{i}})=\frac{n_1 n_{\perp}^{\mathrm{i}}-n_0 n_{\perp}^{\mathrm{t}}}{n_1 n_{\perp}^{\mathrm{i}}+n_0 n_{\perp}^{\mathrm{t}}} \quad (1.3.63)$$

The reflection coefficient $\gamma_{01}^{\mathrm{TE}}(n_{\|})$ relevant to the first-order term is defined as

$$\begin{aligned}\gamma_{01}^{\mathrm{T}E}(n_{\|}) = & \frac{i(n_0^2-n_1^2)2n_0 n_{\perp}^{\mathrm{i}}}{n_1 n_{\perp}^{\mathrm{i}}+n_0 n_{\perp}^{\mathrm{t}}}G(n_0 n_{\|}-n_0 n_{\|}^{\mathrm{i}}) \\ & \times\frac{\left[n_{\|}n_{\|}^{\mathrm{i}}-n_{\perp}^{\mathrm{t}}\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1^2}\right)^2}\right]}{\left[n_1\sqrt{1-(n_{\|})^2}+n_0\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\right]}\end{aligned} \quad (1.3.64)$$

The expressions for the reflected field V_{r} can be written in a more compact way

$$V_{\mathrm{r}}(\xi,\zeta)=V_0\left[\Gamma_{01}^{\mathrm{TE}}(n_{\|}^{\mathrm{i}})e^{in_0(-n_{\perp}^{\mathrm{i}}\xi+n_{\|}^{\mathrm{i}}\zeta)}+\Delta A_{\mathrm{r}}^{\mathrm{TE}}\right] \quad (1.3.65)$$

where the term $\Delta A_{\mathrm{r}}^{\mathrm{TE}}$ in (1.3.65) denotes a small quantity due to a small surface perturbation and it is given by

$$\Delta A_{\mathrm{r}}^{\mathrm{TE}}=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\gamma_{01}^{\mathrm{TE}}(n_{\|})e^{in_0(-n_{\perp}\xi+n_{\|}\zeta)}dn_{\|} \quad (1.3.66)$$

Putting equations (1.3.61) and (1.3.53) in equation (1.3.7), the transmitted field V_{t} can be defined

$$\begin{aligned}V_{\mathrm{t}}(\xi,\zeta) &= T_{01}^{\mathrm{TE}}(n_{\|}^{\mathrm{i}})V_0 e^{in_1(n_{\perp}^{\mathrm{t}}\xi+n_{\|}^{\mathrm{t}}\zeta)} \\ &+ V_0\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\tau_{01}^{\mathrm{TE}}(n_{\|})e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\xi+n_{\|}\zeta\right]}dn_{\|}\end{aligned} \quad (1.3.67)$$

where $T^{\mathrm{TE}}_{01}(n^{\mathrm{i}}_{\parallel})$ and $\tau^{\mathrm{TE}}_{01}(n_{\parallel})$ are coefficients for unperturbed and perturbed transmitted fields, respectively. They are given by

$$T^{\mathrm{T}E}_{01}(n^{\mathrm{i}}_{\parallel}) = \frac{2n_1 n^{\mathrm{i}}_{\perp}}{n_1 n^{\mathrm{i}}_{\perp} + n_0 n^{\mathrm{t}}_{\perp}} \tag{1.3.68}$$

and

$$\begin{aligned}\tau^{\mathrm{T}E}_{01}(n_{\parallel}) = {} & \frac{i(n_0^2 - n_1^2)2n^{\mathrm{i}}_{\perp}}{n_1 n^{\mathrm{i}}_{\perp} + n_0 n^{\mathrm{t}}_{\perp}} G(n_0 n_{\parallel} - n_0 n^{\mathrm{i}}_{\parallel}) \\ & \times \frac{\left[n_0 n_{\parallel} n^{\mathrm{i}}_{\parallel} + n_1 n^{\mathrm{t}}_{\perp}\sqrt{1-(n_{\parallel})^2}\right]}{\left[n_1\sqrt{1-(n_{\parallel})^2} + n_0\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\right]}\end{aligned} \tag{1.3.69}$$

Introducing equation (1.1.1) in equation 1.3.67, the transmitted field V_{t} can be written in a point of coordinates (ξ, ζ) of (MRF) as a function of the coordinates $(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}})$ in (RF_{p}).

$$\begin{aligned} V_{\mathrm{t}}(\xi,\zeta) & = T^{\mathrm{TE}}_{01}(n^{\mathrm{i}}_{\parallel}) V_0 e^{in_1\left[n^{\mathrm{t}}_{\perp}(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}) + n^{\mathrm{t}}_{\parallel}(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}})\right]} \\ & + V_0 \frac{1}{2\pi}\int_{-\infty}^{\infty} \tau^{\mathrm{TE}}_{01}(n_{\parallel}) e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}(\xi_{\mathrm{p}}+\chi_{\mathrm{p}}) + n_{\parallel}(\zeta_{\mathrm{p}}+\eta_{\mathrm{p}})\right]} dn_{\parallel} \end{aligned} \tag{1.3.70}$$

Using the plane-wave expansion into Bessel function, the transmitted field is expressed in polar coordinates.

$$V_{\mathrm{t}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}}) = V_0 \sum_{\ell=-\infty}^{\infty} i^{\ell} J_{\ell}(n_1 \rho_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}}\left[A^{\mathrm{TE}}_{\mathrm{t}} + \Delta B^{\mathrm{TE}}_{\mathrm{t}}\right] \tag{1.3.71}$$

where

$$A^{\mathrm{TE}}_{\mathrm{t}} = T^{\mathrm{TE}}_{01}(n^{\mathrm{i}}_{\parallel}) e^{-i\ell\varphi_{\mathrm{t}}} e^{in_1(n^{\mathrm{t}}_{\perp}\chi_{\mathrm{p}} + n^{\mathrm{t}}_{\parallel}\eta_{\mathrm{p}})} \tag{1.3.72}$$

and

$$\Delta B^{\mathrm{TE}}_{\mathrm{t}} = \int_{-\infty}^{\infty} \tau^{\mathrm{TE}}_{01}(n_{\parallel}) e^{-i\ell\phi_{\mathrm{t}}} e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\chi_{\mathrm{p}} + n_{\parallel}\eta_{\mathrm{p}}\right]} dn_{\parallel} \tag{1.3.73}$$

being

$$\phi_{\mathrm{t}} = \tan^{-1}\left[\frac{n_{\parallel}}{\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}}\right] \tag{1.3.74}$$

1.4 Scattered fields

The scattered field $V_{\rm s}(\xi,\zeta)$ may be expressed as a superposition of cylindrical functions $CW_{\rm m}(n_1\xi_{\rm q},\zeta_{\rm q}) = H_{\rm m}^{(1)}(n_1\rho_{\rm q})e^{i{\rm m}\theta_{\rm q}}$ with unknown expansion coefficients $c_{\rm qm}$, where $H_{\rm m}^{(1)}$ is a first kind Hankel function of integer order m. Therefore, for $V_{\rm s}(\xi,\zeta)$ we have

$$V_{\rm s}(\xi,\zeta) = \sum_{\rm q=1}^{N}\sum_{\rm m=-\infty}^{\infty} V_0 i^{\rm m} e^{-i{\rm m}\varphi_{\rm t}} c_{\rm qm} CW_{\rm m}(n_1\xi_{\rm q}, n_1\zeta_{\rm q}) \tag{1.4.1}$$

The term $i^{\rm m}e^{-i{\rm m}\varphi_{\rm t}}$ has been added in order to simplify the imposition of boundary conditions in analogy with the definition of transmitted field $V_{\rm t}(\xi_{\rm p},\zeta_{\rm p})$. Moreover, it is easier to apply boundary condition on the p-*th* cylinder if the scattered field is expressed as a function of the coordinates of $RF_{\rm p}$. Isolating the term q=p, we get

$$\begin{aligned} V_{\rm s}(\xi,\zeta) = &\sum_{\rm m=-\infty}^{\infty} V_0 i^{\rm m} e^{-i{\rm m}\varphi_{\rm t}} c_{\rm pm} CW_{\rm m}(n_1\xi_{\rm p}, n_1\zeta_{\rm p}) \\ &+\sum_{\substack{\rm q=1\\ \rm q\neq p}}^{N}\sum_{\rm m=-\infty}^{\infty} V_0 i^{\rm m} e^{-i{\rm m}\varphi_{\rm t}} c_{\rm qm} CW_{\rm m}(n_1\xi_{\rm q}, n_1\zeta_{\rm q}) \end{aligned} \tag{1.4.2}$$

By using the addition theorem of Hankel functions [36], the cylindrical wave emitted by the q-*th* cylinder, with q $\neq$ p, can be expressed in the $RF_{\rm p}$ system as

$$H_{\rm m}^{(1)}(n_1\rho_{\rm q})e^{i{\rm m}\theta_{\rm q}} = e^{i{\rm m}\theta_{\rm qp}}\sum_{\ell=-\infty}^{\infty}(-1)^{\ell} H_{{\rm m}+\ell}^{(1)}(n_1\rho_{\rm qp})e^{i\ell\theta_{\rm qp}} J_{\ell}(n_1\rho_{\rm p})e^{-i\ell\theta_{\rm p}} \tag{1.4.3}$$

It follows

$$\begin{aligned} V_{\rm s}(\xi_{\rm p},\zeta_{\rm p}) = &\sum_{\rm m=-\infty}^{+\infty} V_0 i^{\rm m} e^{-i{\rm m}\varphi_{\rm t}} c_{\rm pm} CW_{\rm m}(n_1\xi_{\rm p}, n_1\zeta_{\rm p}) \\ +\sum_{\substack{\rm q=1\\ \rm q\neq p}}^{N} V_0 &\sum_{\rm m=-\infty}^{+\infty} i^{\rm m} e^{-i{\rm m}\varphi_{\rm t}} c_{\rm qm} \sum_{\ell=-\infty}^{+\infty} CW_{{\rm m}+\ell}(n_1\xi_{\rm qp}, n_1\zeta_{\rm qp})(-1)^{\ell} J_{\ell}(n_1\rho_{\rm p})e^{-i\ell\theta_{\rm p}} \end{aligned} \tag{1.4.4}$$

By using the property $(-1)^{\ell}J_{\ell}(.) = J_{-\ell}(.)$ in equation (1.4.4), and replacing ℓ with $-\ell$, we get

$$V_{\mathrm{s}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}}) = V_0 \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{pm}} CW_{\mathrm{m}}(n_1\xi_{\mathrm{p}}, n_1\zeta_{\mathrm{p}})$$
$$+\sum_{\substack{\mathrm{q}=1\\ \mathrm{q}\neq\mathrm{p}}}^{N} V_0 \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \sum_{\ell=-\infty}^{+\infty} CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}}) J_\ell(n_1\rho_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \tag{1.4.5}$$

The above equation can be written in more compact form representing the field associated to the point having coordinates (ξ,ζ) in MRF as a function of the coordinates $(\xi_{\mathrm{p}},\zeta_{\mathrm{p}})$ in RF_{p}

$$V_{\mathrm{s}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}}) = V_0 \sum_{\ell=-\infty}^{+\infty} J_\ell(n_1\rho_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}}$$
$$\times\left[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{H_\ell^{(1)}(n_1\rho_{\mathrm{p}})}{J_\ell(n_1\rho_{\mathrm{p}})}\delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}}\right] \tag{1.4.6}$$

1.4.1 Scattered-reflected field

The interaction of the scattered field V_{s} by the cylinders and the rough interface results in a scattered-reflected field. Two reference frames, a cylindrical coordinate system relevant to the cylinders and the Cartesian reference frame relevant to the planar discontinuity, are employed. The field scattered by each cylinder has been expressed in terms of cylindrical waves emitted by the cylinder itself, but properties of reflection and transmission through a planar surface are known only for plane waves, by means of reflection and transmission coefficients. This makes the description of such an interaction a difficult task.

There exists a solution of the problem that employs the plane-wave spectrum of a cylindrical wave [49]. It means that each cylindrical wave radiated by the p-*th* cylinder can be expressed as a superimposition of spectral plane-waves: reflection and transmission phenomena can be studied for each plane-wave of the spectrum. Finally, the reflected or transmitted plane waves are superimposed to express the scattered-reflected and scattered-transmitted fields, respectively. Once the scattered-reflected and scattered-transmitted fields of each cylinder have been evaluated, they are summed up in order to obtain the total scattered-reflected and scattered-transmitted fields.

Now we introduce Fourier spectrum F_{m} of the cylindrical wave function CW_{m} in medium 1

$$CW_{\mathrm{m}}(n_1\xi_{\mathrm{p}}, n_1\zeta_{\mathrm{p}}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_{\mathrm{m}}(n_1\xi_{\mathrm{p}}, n_{\parallel})e^{in_{\parallel}\zeta_{\mathrm{p}}}dn_{\parallel} \tag{1.4.7}$$

The spectrum F_{m} has been evaluated in [49] and has been further developed in [50] taking into account of a different behavior for $|n_{\parallel}| \geq 1$ and $|n_{\parallel}| \leq 1$.
For $\xi > 0$

$$F_{\mathrm{m}}(\xi, n_{\parallel}) = \frac{2e^{i\xi\sqrt{1-(n_{\parallel})^2}}}{\sqrt{1-(n_{\parallel})^2}} \times \begin{cases} (\sqrt{(n_{\parallel})^2-1}+n_{\parallel})^{\mathrm{m}} & \text{if } |n_{\parallel}| \geq 1 \\ e^{-i\mathrm{m}\arccos n_{\parallel}} & \text{if } |n_{\parallel}| \leq 1 \end{cases} \tag{1.4.8}$$

and for $\xi < 0$

$$F_{\mathrm{m}}(\xi, n_{\parallel}) = \frac{2e^{-i\xi\sqrt{1-(n_{\parallel})^2}}}{\sqrt{1-(n_{\parallel})^2}} \times \begin{cases} (\sqrt{(n_{\parallel})^2-1}+n_{\parallel})^{-\mathrm{m}} & \text{if } |n_{\parallel}| \geq 1 \\ e^{i\mathrm{m}\arccos n_{\parallel}} & \text{if } |n_{\parallel}| \leq 1 \end{cases} \tag{1.4.9}$$

Equations (1.4.8) and (1.4.9) can be written in a more compact way, making use of the properties of function arccosine in a complex domain [65], and considering $n_{\parallel} \in (-\infty, \infty)$

$$F_{\mathrm{m}}(\xi, n_{\parallel}) = \frac{2e^{i|\xi|\sqrt{1-(n_{\parallel})^2}}}{\sqrt{1-(n_{\parallel})^2}} \times \begin{cases} e^{-i\mathrm{m}\arccos n_{\parallel}} & \text{if } \xi \geq 0 \\ e^{i\mathrm{m}\arccos n_{\parallel}} & \text{if } \xi \leq 0 \end{cases} \tag{1.4.10}$$

For a cylinder buried below a flat interface, the evaluation of the scattered-reflected field has been done in [47] and [48]. In these works, the basis functions for the scattered-reflected field are *Reflected Cylindrical Functions*, where, in turn, each plane wave of the spectrum is obtained evaluating the reflection of the generic plane-wave defined in equation (1.4.10).

In this work for rough interface, we define *Reflected Cylindrical Functions* as a sum of *Reflected Cylindrical Functions* developed in [47], relevant to solution with a flat interface, and *Perturbed Reflected Cylindrical Functions* terms, taking into account the superimposed roughness. So, the scattered-reflected field is written by means of a modal expansion into *Reflected Cylindrical Functions* of order m RW_{m}

$$RW_{\mathrm{m}}(\xi,\zeta,\chi) = RW_{\mathrm{m}}^{un}(\xi,\zeta) + RW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) \tag{1.4.11}$$

where RW_{m}^{un} [48] and RW_{m}^{per} are the unperturbed and perturbed components, respectively, given by

$$RW_{\mathrm{m}}^{un}(\xi,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_{10}(n_\parallel)F_{\mathrm{m}}(\xi,n_\parallel)e^{in_\parallel\zeta}dn_\parallel \tag{1.4.12}$$

$$RW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{10}(n'_\parallel,n_\parallel)F_{\mathrm{m}}(-n_1\chi,n_\parallel)e^{i(-n'_\perp\xi+n'_\parallel\zeta)}dn_\parallel dn'_\parallel \tag{1.4.13}$$

The expression for the scattered-reflected field is

$$V_{\mathrm{sr}}(\xi,\zeta) = V_0\sum_{\mathrm{q}=1}^{N}\sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}}e^{-i\mathrm{m}\varphi_{\mathrm{t}}}c_{\mathrm{qm}}RW_{\mathrm{m}}(n_1\xi_{\mathrm{q}},n_1\zeta_{\mathrm{q}},\chi_{\mathrm{q}}) \tag{1.4.14}$$

Making use of the plane wave expansion into Bessel functions together with equation (1.4.10) in equations (1.4.12) and (1.4.13), the field V_{sr} can be written in the reference frame RF_{p} centered on the p-*th* cylinder axis

$$V_{\mathrm{sr}}(\xi,\zeta) = V_0 \sum_{\ell=-\infty}^{\infty} J_\ell(n_1\rho_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}}$$

$$\times\{RW^{un}_{\mathrm{m}+\ell}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})\} \tag{1.4.15}$$

where

$$RW^{un}_{\mathrm{m}}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma_{10}(n_\parallel) F_m[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_\parallel] e^{in_1 n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})} dn_\parallel \tag{1.4.16}$$

and

$$RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \gamma_{10}(n'_\parallel, n_\parallel) F_m(-n_1\chi_{\mathrm{q}}, n_\parallel) e^{-i(n_1 n_\parallel \eta_{\mathrm{q}} + \ell\phi_{\mathrm{sr}})} e^{in_1(n'_\parallel\eta_{\mathrm{p}} - n'_\perp\chi_{\mathrm{q}})}\, dn_\parallel\, dn'_\parallel \tag{1.4.17}$$

being

$$\phi_{\mathrm{sr}} = \tan^{-1}\left(\frac{n'_\parallel}{n'_\perp}\right) \tag{1.4.18}$$

1.4.2 Scattered-transmitted field

The scattered-transmitted field V_{st} is given by the sum of the fields scattered by each buried cylinder and transmitted to medium 0. It is defined as a sum of *Transmitted Cylindrical Functions* of order m $TW_{\mathrm{m}}(\xi,\zeta,\chi)$ weighted by unknown coefficients c_{qm}

$$V_{\mathrm{st}}(\xi,\zeta) = V_0 \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} TW_{\mathrm{m}}(\xi-\chi_{\mathrm{q}}, \zeta-\eta_{\mathrm{q}}, \chi_{\mathrm{q}}) \tag{1.4.19}$$

$TW_{\mathrm{m}}(\xi,\zeta,\chi)$ is sum of zero-*th* order unperturbed term and a first-order perturbed term:

$$TW_{\mathrm{m}}(\xi,\zeta,\chi) = TW^{un}_{\mathrm{m}}(\xi,\zeta,\chi) + TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi) \tag{1.4.20}$$

where the unperturbed $TW_{\mathrm{m}}^{un}(\xi,\zeta,\chi)$ in (1.4.20) relevant to flat interface has been developed in [48], and it is given by

$$TW_{\mathrm{m}}^{un}(\xi,\zeta,\chi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T_{10}(n_\parallel)F_{\mathrm{m}}(-n_1\chi, n_\parallel)e^{-i\sqrt{1-(n_1 n_\parallel)^2}(\xi+\chi)}e^{in_1 n_\parallel \zeta}\, dn_\parallel \tag{1.4.21}$$

The perturbed $TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi)$ in (1.4.20) corresponds to rough surface contribution given by

$$TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau_{10}(n'_\parallel, n_\parallel)F_{\mathrm{m}}(-n_1\chi, n_\parallel)e^{-i\sqrt{1-(n_1 n'_\parallel)^2}(\xi+\chi)}e^{in_1 n'_\parallel \zeta}\, dn_\parallel\, dn'_\parallel \tag{1.4.22}$$

1.4.3 Field inside the cylinder

As regards the case of dielectric scatterers, the field V_{cp} transmitted inside the p-*th* cylinder must be considered. The expression for $V_{\mathrm{cp}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}})$ is given by

$$V_{\mathrm{cp}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}}) = V_0 \sum_{\ell=-\infty}^{\infty} i^\ell e^{-i\ell\varphi_{\mathrm{t}}} d_{\mathrm{p}\ell} J_\ell(n_{\mathrm{cp}}\rho_{\mathrm{p}})e^{i\ell\theta_{\mathrm{p}}} \tag{1.4.23}$$

which is given by an expansion in terms of first-kind Bessel functions, with unknown coefficients $d_{\mathrm{p}\ell}$.

1.5 Imposition of boundary conditions

To determine the unknown coefficients c_{qm}, boundary conditions have to be imposed on the cylinder's surface, being the other boundary conditions already tackled using the CWA method. We are dealing with dielectric and perfectly-conducting cylinder both for TM and TE polarization.

1.5.1 Perfectly conducting cylinders

TM polarization

When a perfectly-conducting cylinder is considered, boundary conditions correspond to zero tangential electric field on cylinders surface. In TM polarization we have

$$V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = 0 \tag{1.5.1}$$

Substituting equations (1.3.38), (1.4.6), and (1.4.15) into equation (1.5.1)

$$\begin{aligned}
&\sum_{\ell=-\infty}^{\infty} i^{\ell} J_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \left[A_{\mathrm{t}}^{\mathrm{TM}} + \Delta B_{\mathrm{t}}^{\mathrm{TM}}\right] \\
&+ \sum_{\ell=-\infty}^{+\infty} J_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \\
&\times \left[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{H_{\ell}^{(1)}(n_1\alpha_{\mathrm{p}})}{J_{\ell}(n_1\alpha_{\mathrm{p}})} \delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}}\right] \\
&+ \sum_{\ell=-\infty}^{\infty} J_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \\
&\times \{RW_{\mathrm{m}+\ell}^{un}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW_{\mathrm{m},\ell}^{per}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})\} = 0
\end{aligned} \tag{1.5.2}$$

Multiplying equation (1.5.2) by $e^{-i\mathrm{h}\theta_{\mathrm{p}}}$, integrating from 0 to 2π in the variable θ_{p}, and employing the orthogonal property of exponential functions $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} e^{i\ell\theta_{\mathrm{p}}} e^{-i\mathrm{h}\theta_{\mathrm{p}}} d\theta_{\mathrm{p}} = \delta_{\ell\mathrm{h}}$, we obtain

$$\begin{aligned}
&i^{\mathrm{h}} J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}}) \left[A_{\mathrm{t}}^{\mathrm{TM}} + \Delta B_{\mathrm{t}}^{\mathrm{TM}}\right] + J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}}) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \\
&\times \left[CW_{\mathrm{m}-\mathrm{h}}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{H_{\mathrm{h}}^{(1)}(n_1\alpha_{\mathrm{p}})}{J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})} \delta_{\mathrm{qp}}\delta_{\mathrm{hm}}\right] \\
&+ J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}}) \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \\
&\times \left\{RW_{\mathrm{m}+\mathrm{h}}^{un}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW_{\mathrm{m},\mathrm{h}}^{per}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})\right\} = 0
\end{aligned} \tag{1.5.3}$$

Bringing the terms containing the unknown coefficients on left hand side, putting $G_{\mathrm{h}}(.) = J_{\mathrm{h}}(.)/H_{\mathrm{h}}^{(1)}(.)$, and multiplying by $G_{\mathrm{h}}(.)$, we get

$$\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}-\mathrm{h}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} G_{\mathrm{h}}(n_1\alpha_{\mathrm{p}}) \Big\{ \Big[CW_{\mathrm{m}-\mathrm{h}}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{\delta_{\mathrm{qp}}\delta_{\mathrm{hm}}}{G_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})} \Big]$$
$$+ \Big[RW^{un}_{\mathrm{m}+\mathrm{h}}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\mathrm{h}}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \Big] \Big\}$$
$$= -G_{\mathrm{h}}(n_1\alpha_{\mathrm{p}}) \big[A_{\mathrm{t}}^{\mathrm{TM}} + \Delta B_{\mathrm{t}}^{\mathrm{TM}} \big] \tag{1.5.4}$$

We can substitute h with ℓ, and the system can be written as

$$\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}} c_{\mathrm{qm}} = B_{\ell}^{\mathrm{p}} \tag{1.5.5}$$

being

$$A_{\ell\mathrm{m}}^{\mathrm{qp}} = i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} G_{\ell}(n_1\alpha_{\mathrm{p}}) \Big\{ \Big[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{\delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}}}{G_{\ell}(n_1\alpha_{\mathrm{p}})} \Big]$$
$$+ \Big[RW^{un}_{\mathrm{m}+\ell}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \Big] \Big\} \tag{1.5.6}$$

and

$$B_{\ell}^{\mathrm{p}} = -G_{\ell}(n_1\alpha_{\mathrm{p}}) \left[A_{\mathrm{t}}^{\mathrm{TM}} + \Delta B_{\mathrm{t}}^{\mathrm{TM}} \right] \tag{1.5.7}$$

TE polarization

In TE polarization, from Maxwell equations, the condition is the following

$$\frac{\partial}{\partial\rho_{\mathrm{p}}}(V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}})\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = 0 \tag{1.5.8}$$

Substituting equations (1.3.71), (1.4.6), and (1.4.15) into equation (1.5.8), we get

$$\sum_{\ell=-\infty}^{\infty} i^{\ell} J'_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \left[A_{\mathrm{t}}^{\mathrm{TE}} + \Delta B_{\mathrm{t}}^{\mathrm{TE}}\right]$$
$$+ \sum_{\ell=-\infty}^{+\infty} J'_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{+\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}}$$
$$\times \left[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{H_{\ell}'^{(1)}(n_1\alpha_{\mathrm{p}})}{J'_{\ell}(n_1\alpha_{\mathrm{p}})} \delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}} \right]$$
$$+ \sum_{\ell=-\infty}^{\infty} J'_{\ell}(n_1\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} i^{\mathrm{m}} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}}$$
$$\times \left\{ RW^{un}_{\mathrm{m}+\ell}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \right\} = 0 \tag{1.5.9}$$

where

$$J'_{\ell}(n_1\alpha_{\mathrm{p}}) = \frac{1}{n_1}\left[\frac{\partial J_{\ell}(n_1\rho_{\mathrm{p}})}{\partial\rho_{\mathrm{p}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}$$

and

$$H_{\ell}'^{(1)}(n_1\alpha_{\mathrm{p}}) = \frac{1}{n_1}\left[\frac{\partial H_{\ell}'^{(1)}(n_1\rho_{\mathrm{p}})}{\partial\rho_{\mathrm{p}}}\right]_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}}$$

Comparing equation (1.5.9) with equation (1.5.2), it turns out that system of equations for TM and TE polarization is equivalent with $G_{\ell}(.) = J'_{\ell}(.)/H'_{\ell}(.)$ in case of TE polarization.

1.5.2 Dielectric cylinders

TM polarization

The boundary conditions for a dielectric cylinder in the case of TM polarization are given by

$$V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = V_{\mathrm{cp}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{1.5.10}$$

$$\frac{\partial}{\partial\rho_{\mathrm{p}}}(V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}})\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = \frac{\partial V_{\mathrm{cp}}}{\partial\rho_{\mathrm{p}}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{1.5.11}$$

Let us consider equation (1.5.10), and follow the similar steps that we have done on equation (1.5.2). On the left hand side, we get what we obtained at the left hand side of the equation (1.5.5), i.e., $A^{\mathrm{qp}}_{\ell\mathrm{m}}$ and B^{p}_{ℓ}. Here, we call these terms as $A^{\mathrm{qp}(1)}_{\ell\mathrm{m}}$ and $B^{\mathrm{p}(1)}_{\ell}$. Now, we perform the similar passages on the right hand side of the equation (1.5.10). Putting equation (1.4.23) with $\rho_{\mathrm{p}} = \alpha_{\mathrm{p}}$, we get

$$\sum_{\ell=-\infty}^{\infty} i^{\ell} e^{-i\ell\varphi_{\mathrm{t}}} d_{\mathrm{p}\ell} J_{\ell}(n_{\mathrm{cp}}\alpha_{\mathrm{p}}) e^{i\ell\theta_{\mathrm{p}}} \tag{1.5.12}$$

Multiplying equation (1.5.12) by $e^{-i\mathrm{h}\theta_{\mathrm{p}}}$, integrating from 0 to 2π in the variable θ_{p}, and employing the orthogonal property of exponential functions, we obtain

$$i^{\mathrm{h}} e^{-i\mathrm{h}\varphi_{\mathrm{t}}} d_{\mathrm{ph}} J_{\mathrm{h}}(n_{\mathrm{cp}}\alpha_{\mathrm{p}}) \tag{1.5.13}$$

Dividing by $i^{\mathrm{h}} J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})$, and multiplying by $\dfrac{J_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})}{H^{(1)}_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})}$

$$e^{-\mathrm{ih}\varphi_{\mathrm{t}}} d_{\mathrm{ph}} \frac{J_{\mathrm{h}}(n_{\mathrm{cp}}\alpha_{\mathrm{p}})}{H^{(1)}_{\mathrm{h}}(n_1\alpha_{\mathrm{p}})} \tag{1.5.14}$$

Replacing h with ℓ in equation (1.5.14), we have the following system

$$\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} A^{\mathrm{qp}(1)}_{\ell\mathrm{m}} c_{\mathrm{qm}} - B^{\mathrm{p}(1)}_{\ell} = d_{\mathrm{p}\ell} L^{\mathrm{p}(1)}_{\ell} \tag{1.5.15}$$

where $\ell = 0; \pm 1; \pm 2...$, $\mathrm{p} = 1, ..., N$.
The system coefficients are given by

$$A^{\mathrm{qp}(1)}_{\ell\mathrm{m}} = i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} G^{(1)}_{\ell}(n_1\alpha_{\mathrm{p}}) \Big\{ \Big[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{\delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}}}{G^{(1)}_{\ell}(n_1\alpha_{\mathrm{p}})} \Big]$$
$$+ \Big[RW^{un}_{\mathrm{m}+\ell}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \Big] \Big\} \tag{1.5.16}$$

where

$$B^{\mathrm{p}(1)}_{\ell} = -\left[A^{\mathrm{TM}}_{\mathrm{t}} + \Delta B^{\mathrm{TM}}_{\mathrm{t}} \right] G^{(1)}_{\ell}(n_1\alpha_{\mathrm{p}}) \tag{1.5.17}$$

and

$$L^{\mathrm{p}(1)}_{\ell} = e^{-i\ell\varphi_{\mathrm{t}}} \frac{J_{\ell}(n_{\mathrm{cp}}\alpha_{\mathrm{p}})}{H^{(1)}_{\ell}(n_1\alpha_{\mathrm{p}})} \tag{1.5.18}$$

and δ is Kronecker symbol and $G_\ell^{(1)}(\cdot) = J_\ell(\cdot)/H_\ell^{(1)}(\cdot)$.

Now, we consider equation (1.5.11), and follow the similar steps that we have done to obtain the equation (1.5.9). On the left hand side, we get what we obtained at the left hand side of the equation (1.5.9), i.e., $A_{\ell\mathrm{m}}^{\mathrm{qp}}$ and B_ℓ^{p}. Here, we call these terms as $A_{\ell\mathrm{m}}^{\mathrm{qp}(2)}$ and $B_\ell^{\mathrm{p}(2)}$. It is now $G_\ell(\cdot) = G_\ell^{(2)}(\cdot) = J'_\ell(\cdot)/H_\ell'^{(1)}(\cdot)$. Now, we perform the similar passages on the right hand side of the equation (1.5.11). Putting equation (1.4.23) with $\rho_\mathrm{p} = \alpha_\mathrm{p}$, we get

$$\sum_{\ell=-\infty}^{\infty} i^\ell e^{-i\ell\varphi_\mathrm{t}} d_{\mathrm{p}\ell} \frac{n_\mathrm{cp}}{n_1} J'_\ell(n_\mathrm{cp}\alpha_\mathrm{p}) e^{i\ell\theta_\mathrm{p}} \tag{1.5.19}$$

Comparing equation (1.5.19) and (1.5.12), it can be easily obtained

$$e^{-i\ell\varphi_\mathrm{t}} d_{\mathrm{p}\ell} \frac{n_\mathrm{cp}}{n_1} \frac{J'_\ell(n_\mathrm{cp}\alpha_\mathrm{p})}{H_\ell'^{(1)}(n_1\alpha_\mathrm{p})} \tag{1.5.20}$$

Finally, from equation (1.5.11), we have the following system

$$\sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}(2)} c_{\mathrm{q}m} - B_\ell^{\mathrm{p}(2)} = d_{\mathrm{p}\ell} L_\ell^{\mathrm{p}(2)} \tag{1.5.21}$$

The system coefficients are given by

$$\begin{aligned} A_{\ell\mathrm{m}}^{\mathrm{qp}(2)} &= i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_\mathrm{t}} G_\ell^{(2)}(n_1\alpha_\mathrm{p}) \Big\{ \Big[CW_{\mathrm{m}-\ell}(n_1\xi_\mathrm{qp}, n_1\zeta_\mathrm{qp})(1-\delta_\mathrm{qp}) + \frac{\delta_\mathrm{qp}\delta_{\ell\mathrm{m}}}{G_\ell^{(2)}(n_1\alpha_\mathrm{p})} \Big] \\ &+ \Big[RW_{\mathrm{m}+\ell}^{un}[-n_1(\chi_\mathrm{p}+\chi_\mathrm{q}), n_1(\eta_\mathrm{p}-\eta_\mathrm{q})] + RW_{\mathrm{m},\ell}^{per}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) \Big] \Big\} \end{aligned} \tag{1.5.22}$$

where

$$B_\ell^{\mathrm{p}(2)} = -\left[A_\mathrm{t}^\mathrm{TM} + \Delta B_\mathrm{t}^\mathrm{TM} \right] G_\ell^{(2)}(n_1\alpha_\mathrm{p}) \tag{1.5.23}$$

and

$$L_\ell^{\mathrm{p}(2)} = e^{-i\ell\varphi_\mathrm{t}} \frac{n_\mathrm{cp}}{n_1} \frac{J'_\ell(n_\mathrm{cp}\alpha_\ell)}{H_\ell'^{(1)}(n_1\alpha_\ell)} \tag{1.5.24}$$

A way to solve the system of equations (1.5.15) and (1.5.21) is to eliminate the coefficient $d_{\mathrm{p}\ell}$, thus obtaining a linear system for unknown coefficients c_{qm}. From equation (1.5.15), we can write

$$d_{\mathrm{p}\ell} = \frac{1}{L_\ell^{\mathrm{p}(1)}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}(1)} c_{\mathrm{qm}} - B_\ell^{\mathrm{p}(1)} \tag{1.5.25}$$

and from equation (1.5.21)

$$d_{\mathrm{p}\ell} = \frac{1}{L_\ell^{\mathrm{p}(2)}} \sum_{\mathrm{q}=1}^{N} \sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}(2)} c_{\mathrm{qm}} - B_\ell^{\mathrm{p}(2)} \tag{1.5.26}$$

Equating the right hand sides of the equations (1.5.25) and (1.5.26), we get the system of equations in unknown coefficients c_{qm}

$$\sum_{\mathrm{q}=1}^{N} \sum_{m=-\infty}^{\infty} D_{\mathrm{m}\ell}^{\mathrm{qp}} c_{\mathrm{qm}} = M_\ell^{\mathrm{p}} \tag{1.5.27}$$

where

$$D_{\mathrm{m}\ell}^{\mathrm{qp}} = L_\ell^{\mathrm{p}(2)} A_{\ell\mathrm{m}}^{\mathrm{qp}(1)} - L_\ell^{\mathrm{p}(1)} A_{\ell\mathrm{m}}^{\mathrm{qp}(2)} \tag{1.5.28}$$

$$M_\ell^{\mathrm{p}} = B_\ell^{\mathrm{p}(1)} L_\ell^{\mathrm{p}(2)} - B_\ell^{\mathrm{p}(2)} L_\ell^{\mathrm{p}(1)} \tag{1.5.29}$$

After the solution of system 1.5.27, the coefficients $d_{\mathrm{p}\ell}$ can be obtained from equations (1.5.25) and (1.5.26). After some algebra, we get

$$\begin{aligned} d_{\mathrm{p}\ell} = {} & n_1 \frac{J_\ell(n_1\alpha_{\mathrm{p}}) H_\ell^{\prime(1)}(n_1\alpha_{\mathrm{p}}) - J_\ell'(n_1\alpha_{\mathrm{p}}) H_\ell^{(1)}(n_1\alpha_{\mathrm{p}})}{n_1 J_\ell(n_{\mathrm{cp}}\alpha_{\mathrm{p}}) H_\ell^{\prime(1)}(n_1\alpha_{\mathrm{p}}) - n_{\mathrm{cp}} J_\ell'(n_{\mathrm{cp}}\alpha_{\mathrm{p}}) H_\ell^{(1)}(n_1\alpha_{\mathrm{p}})} \\ & \times \Big\{ \sum_{\mathrm{q}=1}^{N} \sum_{m=-\infty}^{\infty} i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} c_{\mathrm{qm}} \Big[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) \\ & + RW_{\mathrm{m}+\ell}^{un}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW_{\mathrm{m},\ell}^{per}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \Big] \\ & + [A_{\mathrm{t}}^{\mathrm{TM}} + \Delta B_{\mathrm{t}}^{\mathrm{TM}}] \Big\} \end{aligned} \tag{1.5.30}$$

TE polarization

The boundary conditions for a dielectric cylinder in the case of TE polarization are given by

$$V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = V_{\mathrm{cp}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{1.5.31}$$

$$\frac{\partial}{\partial\rho_{\mathrm{p}}}(V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}})\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = \left(\frac{n_1}{n_{\mathrm{cp}}}\right)^2 \frac{\partial V_{\mathrm{cp}}}{\partial\rho_{\mathrm{p}}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} \tag{1.5.32}$$

From the condition in equation (1.5.31), equal to equation (1.5.10) written for the TM polarization, we obtain an expression similar to equation (1.5.15). Moreover, from the condition in equation (1.5.32) that is similar to (1.5.11), an expression similar to equation (1.5.21) is obtained, where the only difference in this case is

$$L_\ell^{\mathrm{p}(2)} = e^{-i\ell\varphi_{\mathrm{t}}}\frac{n_1}{n_{\mathrm{cp}}}\frac{J'_{\mathrm{p}}(n_{\mathrm{cp}}\alpha_{\mathrm{p}})}{H'^{(1)}_\ell(n_1\alpha_{\mathrm{p}})} \tag{1.5.33}$$

$$\begin{aligned} d_{\mathrm{p}\ell} = {} & n_{\mathrm{cp}}\frac{J_\ell(n_1\alpha_{\mathrm{p}})H'^{(1)}_\ell(n_1\alpha_{\mathrm{p}}) - J'_\ell(n_1\alpha_{\mathrm{p}})H^{(1)}_\ell(n_1\alpha_{\mathrm{p}})}{n_{\mathrm{cp}}J_\ell(n_{\mathrm{cp}}\alpha_{\mathrm{p}})H'^{(1)}_\ell(n_1\alpha_{\mathrm{p}}) - n_1J'_\ell(n_{\mathrm{cp}}\alpha_{\mathrm{p}})H^{(1)}_\ell(n_1\alpha_{\mathrm{p}})} \\ & \times\Bigg\{\sum_{\mathrm{q}=1}^{N}\sum_{m=-\infty}^{\infty} i^{\mathrm{m}-\ell}e^{-i\mathrm{m}\varphi_{\mathrm{t}}}c_{\mathrm{qm}}\Big[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) \\ & + RW^{un}_{\mathrm{m}+\ell}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})\Big] \\ & + [A_{\mathrm{t}}^{\mathrm{TE}} + \Delta B_{\mathrm{t}}^{\mathrm{TE}}]\Bigg\} \end{aligned} \tag{1.5.34}$$

Chapter 2

Numerical implementation for sinusoidal surface

The physical insight into the scattering from objects buried below a rough surface can be gained by considering the objects buried below a rough surface with sinusoidal profile [43]. Also, the analytical theory developed in chapter 1 for the scattering problem of a plane wave by cylindrical objects buried beneath a rough surface, based on the *Cylindrical Wave Approach* is validated by making a comparison with the results available in the literature. The expressions for *Reflected* and *Transmitted Cylindrical Functions* developed for a general rough surface are simplified by putting the Fourier transform of the sinusoidal surface which contains two delta functions at a location depending on the period of the sinusoidal surface considered for numerical analysis.

To solve the theory presented in chapter 1, it is required to numerically evaluate reflected and transmitted wave functions. Moreover, it is necessary to truncate the series to a finite number of elements. The choice of a truncation index such as $M = 3n_1\alpha$ [62] is a good one, where α is the normalized radius of the cylinder: numerical tests in this respect have been performed in [48] and [47] for metallic and dielectric scatterers, respectively. In order to develop an efficient integration algorithm and to calculate the near field, one must consider the infinite extension of the integration domain as the evanescent spectrum's components cannot be neglected. The behavior of the integrands reveals to be highly oscillating as expansion order increases. A suitable algorithm for the integration of these functions has been developed in [48] and [52], with reference to the unperturbed terms. Evanescent spectrum is solved by means of Laguerre-Gauss formulas with small values of ζ, and with Gauss-Legendre formulas applied to small subintervals with higher values. In the homogeneous spectrum, partially and totally reflected plane-waves are distinguished. Integrals are solved through adaptive decompositions of the integration domain, where Gauss-Legendre quadrature formula is applied.

In Section 2.1, the integration of *Perturbed Reflected Cylindrical Functions* has been performed, where accurate evaluation of the integral has been performed. The evaluation of *Perturbed Transmitted Cylindrical Functions* is done in Section 2.2. An effort has also been done to get an asymptotic solution of the *Perturbed Transmitted Cylindrical Functions* in Section 2.3 up to the second order terms. The difference between the zero*th* and second order asymptotic solution has been calculated to check the convergence of the solution. The comparison of execution time for the accurate evaluation and asymptotic solution of the *Perturbed Transmitted Cylindrical Functions* is also performed.

2.1 Evaluation of perturbed reflected cylindrical functions

The numerical analysis of the *Perturbed Reflected Cylindrical Functions* $RW^{per}_{\mathrm{m},\ell}$ defined in equation (1.4.17) is here considered for TM polarization, in the case of a rough surface with sinusoidal profile defined by

$$g(\zeta) = \frac{2\pi}{\lambda_0} A \cos(n_{\mathrm{s}}\zeta) \tag{2.1.1}$$

where $n_{\mathrm{s}} = 2\pi/P_{\mathrm{s}}$ and P_{s} is the normalized period of the surface.
The Fourier transform of the function defined in equation (2.1.1) is given by

$$G(n_{\|}) = \frac{2\pi^2}{\lambda_0} A\Big[\delta(n_1 n'_{\|} - n_{\mathrm{s}}) + \delta(n_1 n'_{\|} + n_{\mathrm{s}})\Big] \tag{2.1.2}$$

The perturbed reflection coefficient $\gamma_{10}^{\mathrm{T}M}(n'_\parallel, n_\parallel)$ can therefore be expressed as a sum of two terms, relevant to the Dirac Delta function in $n'_\parallel = \left(n_\parallel + \frac{n_\mathrm{s}}{n_1}\right)$ and $n'_\parallel = \left(n_\parallel - \frac{n_\mathrm{s}}{n_1}\right)$, respectively:

$$\gamma_{10}^{\mathrm{T}M}(n'_\parallel, n_\parallel) = \gamma_{10}^{+,\mathrm{T}M}(n'_\parallel, n_\parallel) + \gamma_{10}^{-,\mathrm{T}M}(n'_\parallel, n_\parallel) \tag{2.1.3}$$

where

$$\gamma_{10}^{+,\mathrm{TM}}(n'_\parallel, n_\parallel) = i\frac{2\pi^2}{\lambda_0}A\frac{(n_1^2 - n_0^2)2n_1\sqrt{1-n_\parallel^2}}{\left[n_1\sqrt{1-n_\parallel^2} + n_0\sqrt{1-\left(\frac{n_1 n_\parallel}{n_0}\right)^2}\right]}\delta\left[n'_\parallel - \left(n_\parallel + \frac{n_\mathrm{s}}{n_1}\right)\right]$$
$$\times\frac{1}{\left[n_1\sqrt{1-(n'_\parallel)^2} + n_0\sqrt{1-\left(\frac{n_1 n'_\parallel}{n_0}\right)^2}\right]} \tag{2.1.4}$$

$$\gamma_{10}^{-,\mathrm{TM}}(n'_\parallel, n_\parallel) = i\frac{2\pi^2}{\lambda_0}A\frac{(n_1^2 - n_0^2)2n_1\sqrt{1-n_\parallel^2}}{\left[n_1\sqrt{1-n_\parallel^2} + n_0\sqrt{1-\left(\frac{n_1 n_\parallel}{n_0}\right)^2}\right]}\delta\left[n'_\parallel - \left(n_\parallel - \frac{n_\mathrm{s}}{n_1}\right)\right]$$
$$\times\frac{1}{\left[n_1\sqrt{1-(n'_\parallel)^2} + n_0\sqrt{1-\left(\frac{n_1 n'_\parallel}{n_0}\right)^2}\right]} \tag{2.1.5}$$

Taking into account of definitions (2.1.4) and (2.1.5) in *Perturbed Reflected Cylindrical Functions* (1.4.17), we obtain

$$RW_{\mathrm{m},\ell}^{per}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) =$$
$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\gamma_{10}^{+,\mathrm{TM}}(n_\parallel)F_\mathrm{m}(-n_1\chi_\mathrm{q}, n_\parallel)e^{-i(n_1 n_\parallel \eta_\mathrm{q} + \ell\phi_\mathrm{sr}^{+})}e^{in_1\left[\left(n_\parallel + \frac{n_\mathrm{s}}{n_1}\right)\eta_\mathrm{p} + \chi_\mathrm{p}\sqrt{1-\left(n_\parallel + \frac{n_\mathrm{s}}{n_1}\right)^2}\right]}dn_\parallel$$
$$+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\gamma_{10}^{-,\mathrm{TM}}(n_\parallel)F_\mathrm{m}(-n_1\chi_\mathrm{q}, n_\parallel)e^{-i(n_1 n_\parallel \eta_\mathrm{q} + \ell\phi_\mathrm{sr}^{-})}e^{in_1\left[\left(n_\parallel - \frac{n_\mathrm{s}}{n_1}\right)\eta_\mathrm{p} + \chi_\mathrm{p}\sqrt{1-\left(n_\parallel - \frac{n_\mathrm{s}}{n_1}\right)^2}\right]}dn_\parallel \tag{2.1.6}$$

due to a sampling of the integral in $n_{\|} \pm n_{\mathrm{s}}/n_1$. In (2.1.6), ϕ_{sr}^{+} and ϕ_{sr}^{-} are the reflection angles of the spectral plane waves relevant to the Dirac Delta functions in $n_{\|} + n_{\mathrm{s}}/n_1$ and $n_{\|} - n_{\mathrm{s}}/n_1$, respectively, and given by

$$
\begin{aligned}
\phi_{\mathrm{sr}}^{+} &= \tan^{-1}\left[\frac{n_{\|} + \frac{n_{\mathrm{s}}}{n_1}}{\sqrt{1-(n_{\|} + \frac{n_{\mathrm{s}}}{n_1})^2}}\right] \\
\phi_{\mathrm{sr}}^{-} &= \tan^{-1}\left[\frac{n_{\|} - \frac{n_{\mathrm{s}}}{n_1}}{\sqrt{1-(n_{\|} - \frac{n_{\mathrm{s}}}{n_1})^2}}\right]
\end{aligned}
\tag{2.1.7}
$$

The sampling of the reflection coefficient $\gamma_{10}(n'_{\|}, n_{\|})$ in (2.1.3) leads to the two following expressions

$$
\begin{aligned}
\gamma_{10}^{+,\mathrm{TM}}(n_{\|}) &= \frac{i4\pi^2 A(n_1^2 - n_0^2) n_1 \sqrt{1-n_{\|}^2}}{\lambda_0\left[n_1\sqrt{1-n_{\|}^2} + n_0\sqrt{1-\left(\frac{n_1 n_{\|}}{n_0}\right)^2}\right]} \\
&\times \frac{1}{\left[n_1\sqrt{1-(n_{\|}+\dfrac{n_{\mathrm{s}}}{n_1})^2} + n_0\sqrt{1-\left[\dfrac{n_1}{n_0}\left(n_{\|}+\dfrac{n_{\mathrm{s}}}{n_1}\right)\right]^2}\right]} \\
\gamma_{10}^{-,\mathrm{TM}}(n_{\|}) &= \frac{i4\pi^2 A(n_1^2 - n_0^2) n_1 \sqrt{1-n_{\|}^2}}{\lambda_0\left[n_1\sqrt{1-n_{\|}^2} + n_0\sqrt{1-\left(\frac{n_1 n_{\|}}{n_0}\right)^2}\right]} \\
&\times \frac{1}{\left[n_1\sqrt{1-(n_{\|}-\dfrac{n_{\mathrm{s}}}{n_1})^2} + n_0\sqrt{1-\left[\dfrac{n_1}{n_0}\left(n_{\|}-\dfrac{n_{\mathrm{s}}}{n_1}\right)\right]^2}\right]}
\end{aligned}
\tag{2.1.8}
$$

For the sake of simplicity, from now onwards we put $\nu = n_{\mathrm{s}}/n_1$. The integral in equation (2.1.6) can be written as

$$
\begin{aligned}
&RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&\frac{1}{(2\pi)^2}\int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\ell\phi^{+}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel+\nu)^2}}\,dn_\parallel \\
&+\frac{1}{(2\pi)^2}\int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\ell\phi^{-}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel-\nu)^2}}\,dn_\parallel \\
&+\frac{1}{(2\pi)^2}\int\limits_{|n_\parallel|<1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\ell\phi^{+}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel+\nu)^2}}\,dn_\parallel \\
&+\frac{1}{(2\pi)^2}\int\limits_{|n_\parallel|<1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\ell\phi^{-}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel-\nu)^2}}\,dn_\parallel
\end{aligned}
\tag{2.1.9}
$$

The expression for the $RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})$ can be written as homogeneous and evanescent spectral contributions to the *Perturbed Reflected Cylindrical Functions.*

$$
RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \frac{1}{\pi}\Big[I^{(per,int)} - iI^{(per,ext)}\Big] \tag{2.1.10}
$$

where

$$
\begin{aligned}
&I^{(per,int)}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&\int\limits_{|n_\parallel|<1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1}e^{i\mathrm{m}\cos^{-1}(n_\parallel)}e^{-i\ell\phi^{+}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}r_2}\,dn_\parallel \\
&+\int\limits_{|n_\parallel|<1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1}e^{i\mathrm{m}\cos^{-1}(n_\parallel)}e^{-i\ell\phi^{-}_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}r_3}\,dn_\parallel
\end{aligned}
\tag{2.1.11}
$$

and

$$
\begin{aligned}
&I_{\mathrm{m},\ell}^{(per,ext)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&\int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel + r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi_{\mathrm{q}}r_1} e^{-i\ell\phi_{\mathrm{sr}}^{+}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_2}\, dn_\parallel \\
&+ \int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel + r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi_{\mathrm{q}}r_1} e^{-i\ell\phi_{\mathrm{sr}}^{-}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_3}\, dn_\parallel
\end{aligned}
\tag{2.1.12}
$$

being $r_1 = \sqrt{|1 - n_\parallel^2|}$, $r_2 = \sqrt{|1 - (n_\parallel + \nu)^2|}$ and $r_3 = \sqrt{|1 - (n_\parallel - \nu)^2|}$.

2.1.1 Evanescent spectrum of perturbed reflected cylindrical functions

The evanescent spectrum can be decomposed in the following way

$$
\begin{aligned}
I_{\mathrm{m},\ell}^{per,ext}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = {} & I_{\mathrm{m},\ell}^{per,ext+}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\
& + I_{\mathrm{m},\ell}^{per,ext-}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})
\end{aligned}
\tag{2.1.13}
$$

where the contribution of the positive and negative spectral components $\pm\nu$ have been distinguished. Thus, integral $I_{\mathrm{m},\ell}^{per,ext+}$ is defined as follows

$$
\begin{aligned}
&I_{\mathrm{m},\ell}^{per,ext+}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&\int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel + r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi_{\mathrm{q}}r_1} e^{-i\ell\phi_{\mathrm{sr}}^{+}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_2}\, dn_\parallel
\end{aligned}
\tag{2.1.14}
$$

and the integral $I_{\mathrm{m},\ell}^{per,ext-}$ is defined as follows

$$
\begin{aligned}
&I_{\mathrm{m},\ell}^{per,ext-}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&\int\limits_{|n_\parallel|\geq 1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel + r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi_{\mathrm{q}}r_1} e^{-i\ell\phi_{\mathrm{sr}}^{-}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_3}\, dn_\parallel
\end{aligned}
\tag{2.1.15}
$$

According to the numerical value of ν, the square root r_2 can be real or imaginary in the intervals $n_\| \geq 1$ and $n_\| \leq 1$. As regards the decomposition of equation (2.1.14), for $n_\| \geq 1$,

$$r_2 = i\sqrt{(n_\| + \nu)^2 - 1} \tag{2.1.16}$$

when $n_\| \leq -1$.

$$r_2 = \begin{cases} i\sqrt{(n_\| + \nu)^2 - 1} & \text{if } -\infty \leq n_\| \leq (-1-\nu) \\ \sqrt{1 - (n_\| + \nu)^2} & \text{if } (-1-\nu) < n_\| < (1-\nu) \\ i\sqrt{(n_\| + \nu)^2 - 1} & \text{if } (1-\nu) \leq n_\| \leq -1 \end{cases} \tag{2.1.17}$$

On the basis of relations (2.1.16) and (2.1.17), equation (2.1.14) is decomposed into a sum of four terms:

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext+)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) &= I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \end{aligned} \tag{2.1.18}$$

where

$$I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$

$$\int_1^{+\infty} \gamma_{10}^{+,\mathrm{TM}}(n_\|) \frac{(n_\| + \sqrt{n_\|^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\|^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\|^2-1}} e^{-i\ell\phi_{\mathrm{sr}}^+} e^{in_1[n_\|(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\|+\nu)^2-1}} dn_\| \tag{2.1.19}$$

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int\limits_{-\infty}^{(-1-\nu)} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}}e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}}e^{-i\ell\phi_{\mathrm{sr}}^{+}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel+\nu)^2-1}}dn_\parallel \tag{2.1.20}$$

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int\limits_{(-1-\nu)}^{(1-\nu)} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}}e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}}e^{-i\ell\phi_{\mathrm{sr}}^{+}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{q}}\sqrt{1-(n_\parallel+\nu)^2}}dn_\parallel \tag{2.1.21}$$

$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int\limits_{(1-\nu)}^{-1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}}e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}}e^{-i\ell\phi_{\mathrm{sr}}^{+}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel+\nu)^2-1}}dn_\parallel \tag{2.1.22}$$

The integration in equation (2.1.19) is performed solving the singularity in $|n_\parallel| = 1$, by means of the substitution

$$t = \sqrt{n_\parallel^2-1} \Rightarrow \begin{cases} n_\parallel = \sqrt{t^2+1} \\ dn_\parallel = \dfrac{t}{\sqrt{t^2+1}}\,dt \\ 0 < t < \infty \end{cases} \tag{2.1.23}$$

$$I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{0}^{+\infty} \bar{\gamma}_{10}^{+,\mathrm{TM}}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}+\nu}{i\sqrt{(\sqrt{t^2+1}+\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(\sqrt{t^2+1}+\nu)^2-1}}\,dt \tag{2.1.24}$$

where

$$\bar{\gamma}_{10}^{+,\mathrm{TM}}(t) = \gamma_{10}^{+,\mathrm{TM}}(\sqrt{t^2+1})$$

To remove the singularity at $|n_\parallel| = 1$ in equation (2.1.20), we put

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\, dt \\ -\infty < t < -b_1 \end{cases} \tag{2.1.25}$$

being

$$b_1 = \sqrt{(-1-\nu)^2 - 1}$$

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{-\infty}^{-b_1} \bar{\gamma}_{10}^{+,\mathrm{TM}}(t) \frac{(-\sqrt{t^2+1}-t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{n_1\chi_{\mathrm{q}} t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]} e^{-i\ell \tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\, dt \tag{2.1.26}$$

Put $t = -t$ in the above equation and using the relation

$$(-\sqrt{t^2+1}+t)^{-\mathrm{m}} = (-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}$$

we get:

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{b_1}^{\infty} \bar{\gamma}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]} e^{-i\ell \tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\, dt \tag{2.1.27}$$

Following the similar steps, the integrals in equations (2.1.21) and (2.1.22) can be written as

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int_{a_1}^{b_1} \bar{\gamma}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^\mathrm{m}(\sqrt{t^2+1}+t)^\mathrm{m}}{\sqrt{t^2+1}} e^{-n_1\chi_\mathrm{q} t}$$

$$\times e^{in_1[-(\eta_\mathrm{p}-\eta_\mathrm{q})\sqrt{t^2+1}+\nu\eta_\mathrm{p}]} e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{\sqrt{1-(-\sqrt{t^2+1}+\nu)^2}}\right]} e^{in_1\chi_\mathrm{q}\sqrt{1-(-\sqrt{t^2+1}+\nu)^2}}\, dt \tag{2.1.28}$$

being

$$a_1 = \sqrt{(1-\nu)^2 - 1}$$

$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int_{0}^{a_1} \bar{\gamma}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^\mathrm{m}(\sqrt{t^2+1}+t)^\mathrm{m}}{\sqrt{t^2+1}} e^{-n_1\chi_\mathrm{q} t}$$

$$\times e^{in_1[-(\eta_\mathrm{p}-\eta_\mathrm{q})\sqrt{t^2+1}+\nu\eta_\mathrm{p}]} e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]} e^{-n_1\chi_\mathrm{q}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\, dt \tag{2.1.29}$$

Now, we consider the integral $I_{\mathrm{m},\ell}^{(per,ext-)}$ defined in equation (2.1.15), which can be solved decomposing its integration domain according to the behavior of r_3

For $n_\parallel \leq -1$,

$$r_3 = i\sqrt{(n_\parallel - \nu)^2 - 1} \tag{2.1.30}$$

when $n_\parallel \geq 1$.

$$r_3 = \begin{cases} i\sqrt{(n_\parallel - \nu)^2 - 1} & \text{if } (1+\nu) \leq n_\parallel \leq \infty \\ \sqrt{1-(n_\parallel - \nu)^2} & \text{if } (-1+\nu) < n_\parallel < (1+\nu) \\ i\sqrt{(n_\parallel - \nu)^2 - 1} & \text{if } 1 \leq n_\parallel \leq (-1+\nu) \end{cases} \tag{2.1.31}$$

On the basis of relations (2.1.30) and (2.1.31), equation (2.1.15) is decomposed into a sum of four terms:

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext-)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) &= I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \end{aligned} \tag{2.1.32}$$

where

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{-\infty}^{-1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}^-} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}}\, dn_\parallel \tag{2.1.33}$$

$$I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{1+\nu}^{\infty} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}^-} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}}\, dn_\parallel \tag{2.1.34}$$

$$I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{-1+\nu}^{1+\nu} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}^-} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{q}}\sqrt{1-(n_\parallel-\nu)^2}}\, dn_\parallel \tag{2.1.35}$$

$$I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{1}^{-1+\nu} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}^-} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}}\, dn_\parallel \tag{2.1.36}$$

With the position $n_\parallel = -\sqrt{t^2+1}$ in equation (2.1.33) and replacing t with $-t$, we get

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int\limits_0^\infty \bar{\gamma}_{10}^{-,\mathrm{TM}}(t)\frac{(-1)^\mathrm{m}(\sqrt{t^2+1}+t)^\mathrm{m}}{\sqrt{t^2+1}}e^{-n_1\chi_\mathrm{q}t}$$
$$\times e^{in_1[-(\eta_\mathrm{p}-\eta_\mathrm{q})\sqrt{t^2+1}-\nu\eta_\mathrm{p}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}-\nu}{i\sqrt{(-\sqrt{t^2+1}-\nu)^2-1}}\right]}e^{-n_1\chi_\mathrm{q}\sqrt{(-\sqrt{t^2+1}-\nu)^2-1}}dt \tag{2.1.37}$$

Comparing equations (2.1.37) and (2.1.24)

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = (-1)^\mathrm{m} I_{-\mathrm{m},-\ell}^{(per,ext+1)}(-n_1\chi_\mathrm{q}, -n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, -n_1\eta_\mathrm{p}) \tag{2.1.38}$$

To solve the integrals in equations (2.1.34), (2.1.35) and (2.1.36), we make the following substitution $n_\parallel = \sqrt{t^2+1}$

$$I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int\limits_{b_1}^\infty \bar{\gamma}_{10}^{-,\mathrm{TM}}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_\mathrm{q}t}$$
$$\times e^{in_1[(\eta_\mathrm{p}-\eta_\mathrm{q})\sqrt{t^2+1}-\nu\eta_\mathrm{p}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{i\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}\right]}e^{-n_1\chi_\mathrm{q}\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}dt \tag{2.1.39}$$

Comparing equations (2.1.39) and (2.1.27)

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = (-1)^\mathrm{m} I_{-\mathrm{m},-\ell}^{(per,ext-2)}(-n_1\chi_\mathrm{q}, -n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, -n_1\eta_\mathrm{p}) \tag{2.1.40}$$

$$I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int\limits_{a_1}^{b_1} \bar{\gamma}_{10}^{-,\mathrm{TM}}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_\mathrm{q}t}$$
$$\times e^{in_1[(\eta_\mathrm{p}-\eta_\mathrm{q})\sqrt{t^2+1}-\nu\eta_\mathrm{p}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{\sqrt{1-(\sqrt{t^2+1}-\nu)^2}}\right]}e^{in_1\chi_\mathrm{q}\sqrt{1-(\sqrt{t^2+1}-\nu)^2}}dt \tag{2.1.41}$$

Comparing equations (2.1.41) and (2.1.28)

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = (-1)^\mathrm{m} I_{-\mathrm{m},-\ell}^{(per,ext-3)}(-n_1\chi_\mathrm{q}, -n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, -n_1\eta_\mathrm{p}) \tag{2.1.42}$$

$$I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_0^{a_1} \bar{\gamma}_{10}^{-,\mathrm{TM}}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}-\nu\eta_{\mathrm{p}}]} e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{i\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}\right]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(\sqrt{t^2+1}-\nu)^2-1}} dt \tag{2.1.43}$$

Comparing equations (2.1.43) and (2.1.29)

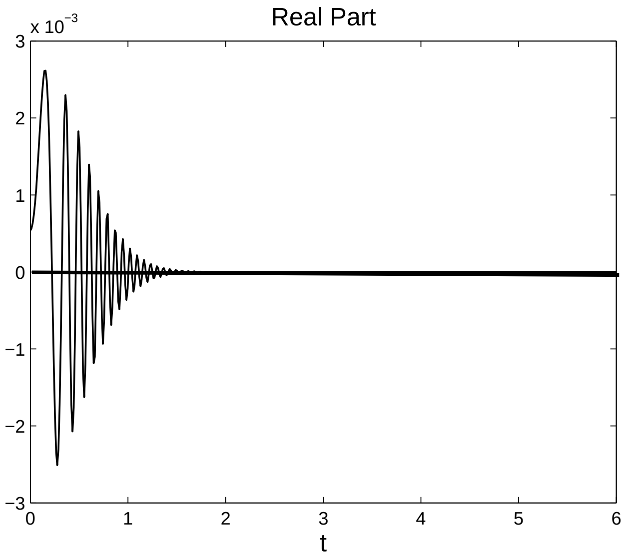
$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{2.1.44}$$

The evanescent spectrum can be decomposed as follows

$$\begin{aligned}
&I_{\mathrm{m},\ell}^{(per,ext)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \\
&I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\
&+I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\
&+I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\
&+I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}})
\end{aligned} \tag{2.1.45}$$

The integrals of the evanescent spectrum are solved adopting a generalized Gaussian integration method. For relatively small values of the parameter ζ, a simple Laguerre-Gauss integration formula has given accurate results. For large values of ζ, we adopted a generalized Gaussian integration method, consisting in a decomposition of the integration interval in subintervals of suitable length on which a fixed low-order Gaussian rule as Gauss-Legendre quadrature formula gives good accuracy.

The real and imaginary parts of the integrand in equation (2.1.24) have been shown in figures (2.1) for $\bar{\gamma}_{10}^{+,\mathrm{TM}}(t) = 1$, $\chi_{\mathrm{p}} = 2.57$, $\chi_{\mathrm{q}} = 2.57$, $\eta_{\mathrm{p}} = 30$, $\eta_{\mathrm{q}} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.8\pi$, while the real and imaginary parts of the integrand in equation (2.1.41) are presented in figure 2.2 considering $\bar{\gamma}_{10}^{-,\mathrm{TM}}(t) = 1$, $\chi_{\mathrm{p}} = 2.57$, $\chi_{\mathrm{q}} = 2.57$, $\eta_{\mathrm{p}} = 30$, $\eta_{\mathrm{q}} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.4\pi$.



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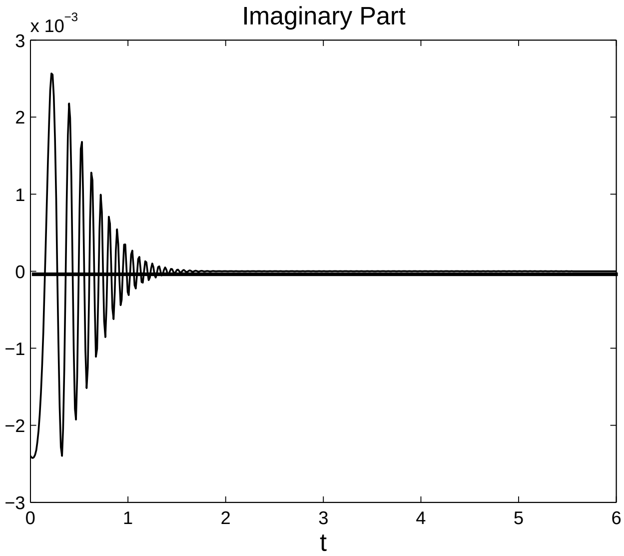
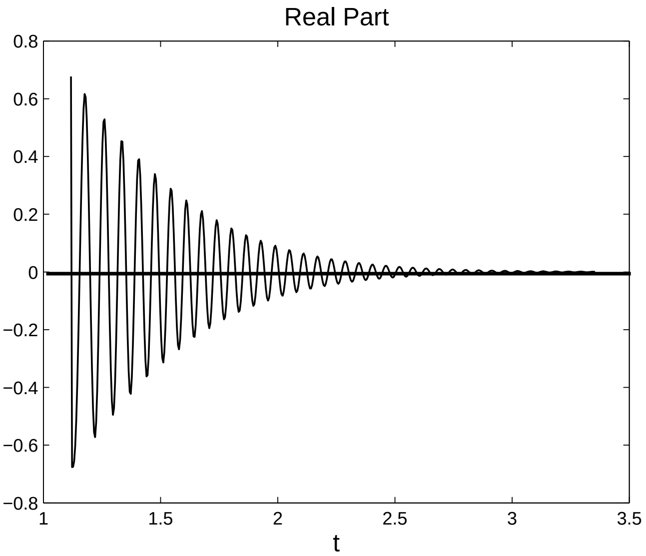
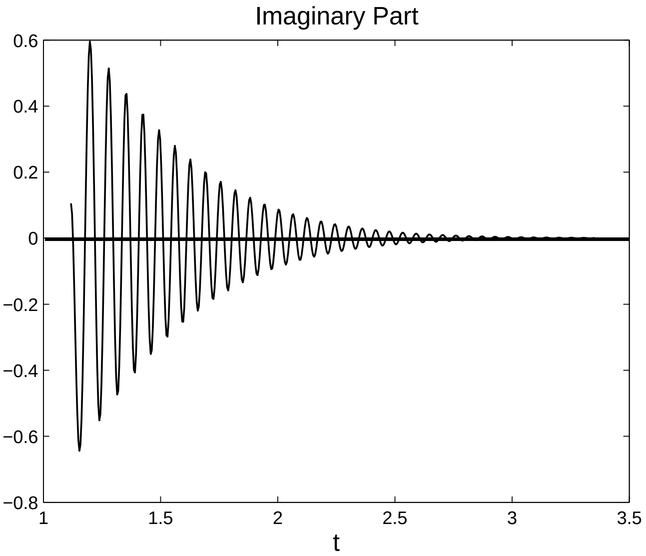


(b)

Figure 2.1: a) Real part of the integrand in equation (2.1.24) for $\bar{\gamma}_{10}^{+,\mathrm{TM}}(t) = 1$, $\chi_{\mathrm{p}} = 2.57$, $\chi_{\mathrm{q}} = 2.57$, $\eta_{\mathrm{p}} = 30$, $\eta_{\mathrm{q}} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.8\pi$; b) Imaginary part.



(a)



(b)

Figure 2.2: a) Real part of the integrand in equation (2.1.41) for $\bar{\gamma}_{10}^{-,\mathrm{TM}}(t) = 1$, $\chi_{\mathrm{p}} = 2.57$, $\chi_{\mathrm{q}} = 2.57$, $\eta_{\mathrm{p}} = 30$, $\eta_{\mathrm{q}} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.4\pi$; b) Imaginary part.

2.1.2 Homogeneous spectrum of perturbed reflected cylindrical functions

Let us consider homogeneous spectrum defined in equation (2.1.11) that can be decomposed into two terms:

$$\begin{aligned} I_{\mathrm{m},\ell}^{per,int}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) &= I_{\mathrm{m},\ell}^{per,int+}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+ I_{\mathrm{m},\ell}^{per,int-}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \end{aligned} \tag{2.1.46}$$

where

$$I_{\mathrm{m},\ell}^{(per,int+)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{|n_\parallel|<1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}^{+}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_2}\, dn_\parallel \tag{2.1.47}$$

$$I_{\mathrm{m},\ell}^{(per,int-)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{|n_\parallel|<1} \gamma_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}^{-}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_3}\, dn_\parallel \tag{2.1.48}$$

Consider the integral $I_{\mathrm{m},\ell}^{per,int+}$, the square root r_1 is a real number, while the sign of the argument in r_2 depends on the value of ν:

$$r_2 = \begin{cases} \sqrt{1-(n_\parallel+\nu)^2} & \text{if } -1 < n_\parallel < (1-\nu) \\ i\sqrt{(n_\parallel+\nu)^2-1} & \text{if } (1-\nu) < n_\parallel < 1 \end{cases} \tag{2.1.49}$$

Equation (2.1.47) can be written as a sum of two terms:

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{-1}^{(1-\nu)} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}^{+}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel+\nu)^2}}\, dn_\parallel \tag{2.1.50}$$

$$I_{\mathrm{m},\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{(1-\nu)}^{1} \gamma_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}}e^{i\mathrm{m}\cos^{-1}(n_\parallel)}e^{-i\ell\phi_{\mathrm{sr}}^{+}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{-n_1\chi_{\mathrm{p}}\sqrt{(n_\parallel+\nu)^2-1}}\,dn_\parallel \tag{2.1.51}$$

Substituting $n_\parallel = \cos t$, equation (2.1.50) can be written as

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{\cos^{-1}(1-\nu)}^{\pi} \gamma_{10}^{+,\mathrm{TM}}(\cos t)e^{i(n_1\chi_{\mathrm{q}}\sin t+\mathrm{m}t)} \times e^{-i\ell\tan^{-1}\left[\frac{\cos t+\nu}{\sqrt{1-(\cos t+\nu)^2}}\right]}e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(\cos t+\nu)^2}}\,dt \tag{2.1.52}$$

In equation (2.1.52) using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(-1+\nu) < t' < 0 \end{cases} \tag{2.1.53}$$

we get

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = e^{i\mathrm{m}\pi}\int\limits_{0}^{\cos^{-1}(-1+\nu)} \gamma_{10}^{+,\mathrm{TM}}(\cos t)e^{i(n_1\chi_{\mathrm{q}}\sin t-\mathrm{m}t)} \times e^{-i\ell\tan^{-1}\left[\frac{-\cos t+\nu}{\sqrt{1-(-\cos t+\nu)^2}}\right]}e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(-\cos t+\nu)^2}}\,dt \tag{2.1.54}$$

where the identity $\cos^{-1}(-1+\nu) = \pi - \cos^{-1}(1-\nu)$ has been used.

With the position $n_\parallel = \cos t$, equation (2.1.51) can be written as

$$I^{(per,int+2)}_{\mathrm{m},\ell}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) = \int_0^{\cos^{-1}(1-\nu)} \gamma^{+,\mathrm{TM}}_{10}(\cos t) e^{i(n_1\chi_\mathrm{q}\sin t + \mathrm{m}t)}$$

$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t+\nu}{i\sqrt{(\cos t+\nu)^2-1}}\right]} e^{in_1[(\eta_\mathrm{p}-\eta_\mathrm{q})\cos t + \nu\eta_\mathrm{p}]} e^{-n_1\chi_\mathrm{p}\sqrt{(\cos t+\nu)^2-1}}\, dt \tag{2.1.55}$$

Now, let us consider the integral $I^{per,int-}_{\mathrm{m},\ell}$ in equation (2.1.48). The expression of integral $I^{per,int-}_{\mathrm{m},\ell}$ is analogous to the one $I^{per,int+}_{\mathrm{m},\ell}$ in equation (2.1.47), and a similar decomposition into two terms on the integration domain is performed on the basis of r_3:

$$r_3 = \begin{cases} \sqrt{1-(n_\parallel-\nu)^2} & \text{if } (-1+\nu) < n_\parallel < 1 \\ i\sqrt{(n_\parallel-\nu)^2-1} & \text{if } -1 < n_\parallel < (-1+\nu) \end{cases} \tag{2.1.56}$$

Equation (2.1.48) can be written as a sum of two terms:

$$I^{(per,int-1)}_{\mathrm{m},\ell}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) =$$

$$\int_{(-1+\nu)}^{1} \gamma^{-,\mathrm{TM}}_{10}(n_\parallel)\frac{e^{in_1\chi_\mathrm{q}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{i m}\cos^{-1}(n_\parallel)} e^{-i\ell\phi^-_\mathrm{sr}} e^{in_1[n_\parallel(\eta_\mathrm{p}-\eta_\mathrm{q})-\nu\eta_\mathrm{p}]} e^{in_1\chi_\mathrm{p}\sqrt{1-(n_\parallel-\nu)^2}}\, dn_\parallel \tag{2.1.57}$$

$$I^{(per,int-2)}_{\mathrm{m},\ell}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) =$$

$$\int_{-1}^{(-1+\nu)} \gamma^{-,\mathrm{TM}}_{10}(n_\parallel)\frac{e^{in_1\chi_\mathrm{q}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{i m}\cos^{-1}(n_\parallel)} e^{-i\ell\phi^-_\mathrm{sr}} e^{in_1[n_\parallel(\eta_\mathrm{p}-\eta_\mathrm{q})-\nu\eta_\mathrm{p}]} e^{-n_1\chi_\mathrm{p}\sqrt{(n_\parallel-\nu)^2-1}}\, dn_\parallel \tag{2.1.58}$$

Substituting $n_{\parallel} = \cos t$, equations (2.1.57) and (2.1.58) can be written as

$$I_{\mathrm{m},\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{0}^{\cos^{-1}(-1+\nu)} \gamma_{10}^{-,\mathrm{TM}}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t-\nu}{\sqrt{1-(\cos t-\nu)^2}}\right]} e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t - \nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}\sqrt{1-(\cos t-\nu)^2}}\, dt \tag{2.1.59}$$

Comparing equations (2.1.54) and (2.1.59), we obtain

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{2.1.60}$$

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{\cos^{-1}(-1+\nu)}^{\pi} \gamma_{10}^{-,\mathrm{TM}}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t-\nu}{i\sqrt{(\cos t-\nu)^2-1}}\right]} e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t - \nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{p}}\sqrt{(\cos t-\nu)^2-1}}\, dt \tag{2.1.61}$$

In equation (2.1.61) using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(1-\nu) < t' < 0 \end{cases} \tag{2.1.62}$$

we obtain

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}) = e^{i\mathrm{m}\pi}\int\limits_{0}^{\cos^{-1}(1-\nu)}\gamma_{10}^{+,\mathrm{TM}}(\cos t)e^{i(n_1\chi_{\mathrm{q}}\sin t-\mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{-\cos t-\nu}{i\sqrt{(-\cos t-\nu)^2-1}}\right]}e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t-\nu\eta_{\mathrm{p}}]}e^{-n_1\chi_{\mathrm{p}}\sqrt{(-\cos t-\nu)^2-1}}\,dt \tag{2.1.63}$$

Comparing equations (2.1.63) and (2.1.55)

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}},-n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},-n_1\eta_{\mathrm{p}}) \tag{2.1.64}$$

The final decomposition of the homogeneous spectrum is as follows

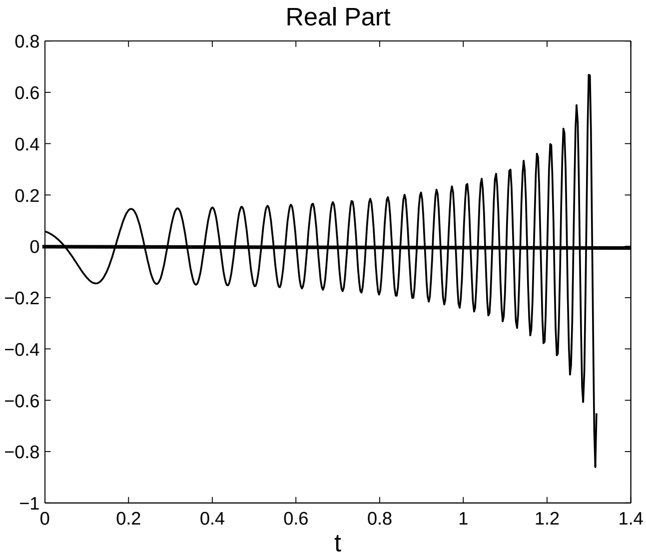
$$\begin{aligned}&I_{\mathrm{m},\ell}^{(per,int)}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}) =\\ &I_{\mathrm{m},\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}},-n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},-n_1\eta_{\mathrm{p}})\\ &+I_{\mathrm{m},\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}) + (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}},-n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},-n_1\eta_{\mathrm{p}})\end{aligned} \tag{2.1.65}$$

When the square roots r_2 and r_3 in equations (2.1.49) and (2.1.56) are real, the integrands are the product of a purely-oscillating function times the coefficient $\gamma_{10}^{\pm,\mathrm{TM}}(\cos t)$, and they represent totally-reflected plane waves. The integration technique is essentially determined by the exponential term, which is highly oscillating as the parameters χ, m, and ℓ are increased. The adaptive generalized Gaussian quadrature rule proposed in [52] is applied, which is based on a decomposition of the integration interval in a suitable number of sub-domains, where the local frequency oscillation rate behaves monotonically. In each subinterval, Gauss-Legendre quadrature formulas are employed. The adaptive integration algorithm sets the number of subdomains and their amplitudes depending on the oscillatory behavior of the integrand, evaluating the local oscillation rate on the function:

$$f_{\mathrm{i}}(t) = \frac{1}{2\pi}\left|\frac{d}{dt}\left\{n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t - \ell\tan^{-1}\left[\frac{\cos t \pm \nu}{\sqrt{1-(\cos t\pm\nu)^2}}\right] + n_1\chi_{\mathrm{p}}\sqrt{1-(\cos t\pm\nu)^2}\right\}\right| \tag{2.1.66}$$

The algorithm assumes a monotonic oscillating rate, but the function $f_{\mathrm{i}}(t)$ in equation (2.1.66) is not monotonic on the integration interval. Thus, a decomposition is performed on (2.1.66) in order to identify sub-interval in which $f_{\mathrm{i}}(t)$ behaves monotonically. The real and imaginary part of the integrand in equation (2.1.59) has been shown in figure 2.3 for $\bar{\gamma}_{10}^{-,\mathrm{TM}}(\cos t) = 1$, $\chi_{\mathrm{p}} = 1$, $\chi_{\mathrm{q}} = 1$, $\eta_{\mathrm{p}} = 50$, $\eta_{\mathrm{q}} = -50$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.8\pi$.

Otherwise, in the intervals where r_2 and r_3 are imaginary numbers, the integrands are the product of an oscillating function times an exponentially decaying one, representing partially-reflected plane waves. The oscillation rate increases as the variables χ, m, and ℓ are increased. The variable χ is included also in the decaying term, which is dominant for high values of χ. An adaptive integration algorithm applied to the oscillating part is employed, similar to the one used for the totally-reflected plane waves. In figure 2.4, the real and imaginary part of the integrand in equation (2.1.55) is shown for $\bar{\gamma}_{10}^{+,\mathrm{TM}}(\cos t) = 1$, $\chi_{\mathrm{p}} = 2$, $\chi_{\mathrm{q}} = 2$, $\eta_{\mathrm{p}} = 30$, $\eta_{\mathrm{q}} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.8\pi$.



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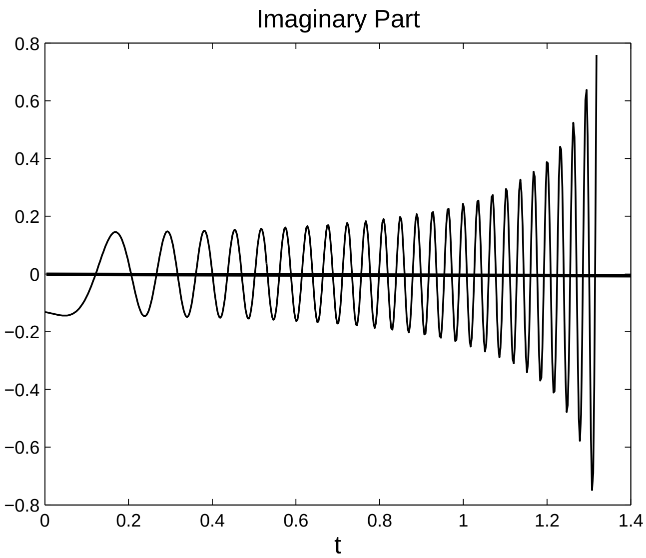
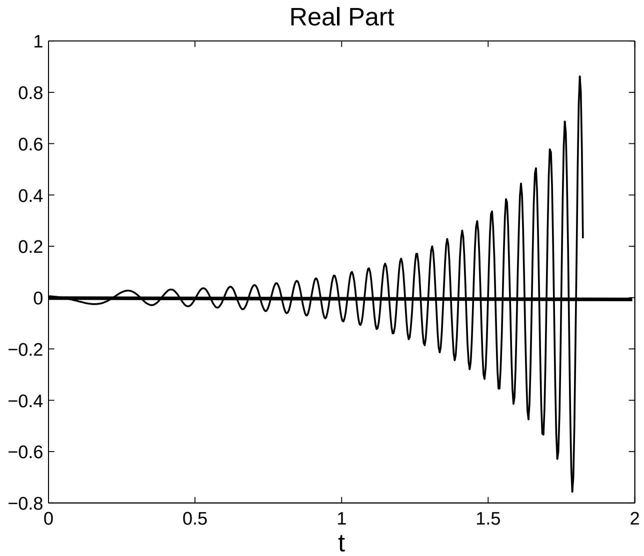


(b)

Figure 2.3: a) Real part of the integrand in equation (2.1.59) for $\bar{\gamma}_{10}^{-,\mathrm{TM}}(\cos t) = 1$, $\chi_{\mathrm{p}} = 1$, $\chi_{\mathrm{q}} = 1$, $\eta_{\mathrm{p}} = 50$, $\eta_{\mathrm{q}} = -50$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_{\mathrm{s}} = 0.8\pi$; b) Imaginary part.



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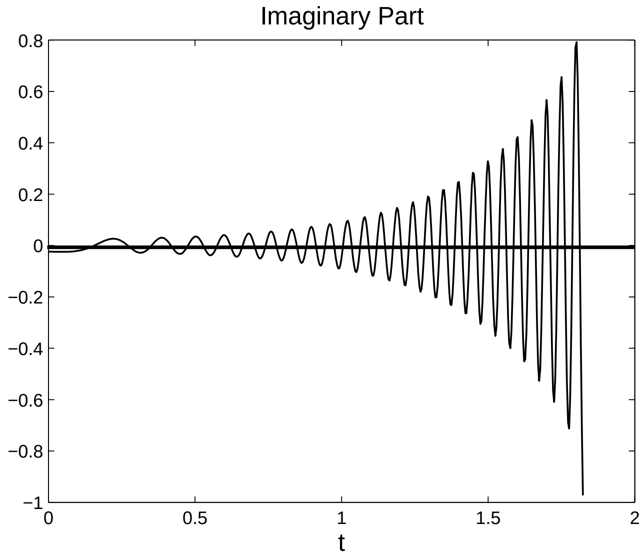


(b)

Figure 2.4: a) Real part of the integrand in equation (2.1.55) for $\bar{\gamma}_{10}^{+,\mathrm{TM}}(\cos t) = 1$, $\chi_\mathrm{p} = 2$, $\chi_\mathrm{q} = 2$, $\eta_\mathrm{p} = 30$, $\eta_\mathrm{q} = -30$, $\mathrm{m} = -6$, $\ell = -3$, $n_1 = 2$ and $P_\mathrm{s} = 0.8\pi$; b) Imaginary part.

2.2 Evaluation of perturbed transmitted cylindrical functions

Let us consider the *Transmitted Cylindrical Functions* defined in equation (1.4.22)

$$TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\tau_{10}^{\mathrm{TM}}(n'_{\parallel},n_{\parallel})F_{\mathrm{m}}(-n_1\chi,n_{\parallel}) \times e^{-i\sqrt{1-(n_1 n'_{\parallel})^2}(\xi+\chi)}e^{in_1 n'_{\parallel}\zeta}\,dn_{\parallel}\,dn'_{\parallel} \tag{2.2.1}$$

From equation (1.3.36), the expression for the transmission coefficients $\tau_{10}^{\mathrm{TM}}(n'_{\parallel},n_{\parallel})$ is equivalent to the the expression for the reflection coefficients $\gamma_{10}^{\mathrm{TM}}(n'_{\parallel},n_{\parallel})$, therefore, in a similar way to *Reflected Cylindrical Functions*, equation (2.2.1) can be decomposed into sum of two terms:

$$\begin{aligned} TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = \\ \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\tau_{10}^{+,\mathrm{TM}}(n_{\parallel})F_{\mathrm{m}}(-n_1\chi,n_{\parallel})e^{in_1(n_{\parallel}+\frac{ns}{n_1})\zeta}e^{-i\sqrt{1-[n_1(n_{\parallel}+\frac{ns}{n_1})]^2}(\xi+\chi)}\,dn_{\parallel} \\ +\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\tau_{10}^{-,\mathrm{TM}}(n_{\parallel})F_{\mathrm{m}}(-n_1\chi,n_{\parallel})e^{in_1(n_{\parallel}-\frac{ns}{n_1})\zeta}e^{-i\sqrt{1-[n_1(n_{\parallel}-\frac{ns}{n_1})]^2}(\xi+\chi)}\,dn_{\parallel} \end{aligned} \tag{2.2.2}$$

For the sake of simplicity, from now onwards we put $\nu = n_{\mathrm{s}}/n_1$. The integral in equation (2.2.2) can be written as

$$\begin{aligned} TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = \\ \frac{1}{(2\pi)^2}\int\limits_{|n_{\parallel}|\geq 1}\tau_{10}^{+,\mathrm{TM}}(n_{\parallel})F_{\mathrm{m}}(-n_1\chi,n_{\parallel})e^{in_1(n_{\parallel}+\nu)\zeta}e^{-i\sqrt{1-[n_1(n_{\parallel}+\nu)]^2}(\xi+\chi)}\,dn_{\parallel} \\ +\frac{1}{(2\pi)^2}\int\limits_{|n_{\parallel}|\geq 1}\tau_{10}^{-,\mathrm{TM}}(n_{\parallel})F_{\mathrm{m}}(-n_1\chi,n_{\parallel})e^{in_1(n_{\parallel}-\nu)\zeta}e^{-i\sqrt{1-[n_1(n_{\parallel}-\nu)]^2}(\xi+\chi)}\,dn_{\parallel} \end{aligned} \tag{2.2.3}$$

$$+\frac{1}{(2\pi)^2}\int\limits_{|n_\|\|<1} \tau_{10}^{+,\mathrm{TM}}(n_\|)F_{\mathrm{m}}(-n_1\chi,n_\|)e^{in_1(n_\|+\nu)\zeta}e^{-i\sqrt{1-[n_1(n_\|+\nu)]^2}(\xi+\chi)}\,dn_\|$$
$$+\frac{1}{(2\pi)^2}\int\limits_{|n_\|\|<1} \tau_{10}^{-,\mathrm{TM}}(n_\|)F_{\mathrm{m}}(-n_1\chi,n_\|)e^{in_1(n_\|-\nu)\zeta}e^{-i\sqrt{1-[n_1(n_\|-\nu)]^2}(\xi+\chi)}\,dn_\| \tag{2.2.4}$$

The expression for the $TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi)$ can be written as homogeneous and evanescent spectrum:

$$TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi)=\frac{1}{\pi}\Big[I^{(per,int)}-iI^{(per,ext)}\Big] \tag{2.2.5}$$

where

$$I_{\mathrm{m}}^{(per,int)}(\xi,\zeta,\chi)=$$
$$\int\limits_{|n_\|\|<1}\tau_{10}^{+,\mathrm{TM}}(n_\|)\frac{e^{in_1\chi r_1}}{r_1}e^{\mathrm{im}\cos^{-1}(n_\|)}e^{in_1(n_\|+\nu)\zeta}e^{-ir_4(\xi+\chi)}\,dn_\|$$
$$+\int\limits_{|n_\|\|<1}\tau_{10}^{-,\mathrm{TM}}(n_\|)\frac{e^{in_1\chi r_1}}{r_1}e^{\mathrm{im}\cos^{-1}(n_\|)}e^{in_1(n_\|-\nu)\zeta}e^{-ir_5(\xi+\chi)}\,dn_\| \tag{2.2.6}$$

and

$$I_{\mathrm{m}}^{(per,ext)}(\xi,\zeta,\chi)=$$
$$\int\limits_{|n_\|\|\geq 1}\tau_{10}^{+,\mathrm{TM}}(n_\|)\frac{(n_\|+r_1)^{-\mathrm{m}}}{r_1}e^{-n_1\chi r_1}e^{-ir_4(\xi+\chi)}e^{in_1(n_\|+\nu)\zeta}\,dn_\|$$
$$+\int\limits_{|n_\|\|\geq 1}\tau_{10}^{-,\mathrm{TM}}(n_\|)\frac{(n_\|+r_1)^{-\mathrm{m}}}{r_1}e^{-n_1\chi r_1}e^{-ir_5(\xi+\chi)}e^{in_1(n_\|-\nu)\zeta}\,dn_\| \tag{2.2.7}$$

being $r_1=\sqrt{|1-n_\|^2|}$, $r_4=\sqrt{|1-[n_1(n_\|+\nu)]^2|}$ and $r_5=\sqrt{|1-[n_1(n_\|-\nu)]^2|}$.

2.2.1 Evanescent spectrum of perturbed transmitted cylindrical functions

The evanescent spectrum can be decomposed in the following way

$$I_{\mathrm{m}}^{(per,ext)}(\xi,\zeta,\chi)=I_{\mathrm{m}}^{(per,ext+)}(\xi,\zeta,\chi)+I_{\mathrm{m}}^{(per,ext-)}(\xi,\zeta,\chi) \tag{2.2.8}$$

where

$$I_{\mathrm{m}}^{(per,ext+)}(\xi,\zeta,\chi) = \int\limits_{|n_{\|}|\geq 1} \tau_{10}^{+,\mathrm{TM}}(n_{\|}) \frac{(n_{\|}+r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi r_1} e^{-ir_4(\xi+\chi)} e^{in_1(n_{\|}+\nu)\zeta}\, dn_{\|} \tag{2.2.9}$$

and

$$I_{\mathrm{m}}^{(per,ext-)}(\xi,\zeta,\chi) = \int\limits_{|n_{\|}|\geq 1} \tau_{10}^{-,\mathrm{TM}}(n_{\|}) \frac{(n_{\|}+r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi r_1} e^{-ir_5(\xi+\chi)} e^{in_1(n_{\|}-\nu)\zeta}\, dn_{\|} \tag{2.2.10}$$

According to the numerical value of ν, the square root r_4 can be real or imaginary in the intervals $n_{\|} \geq 1$ and $n_{\|} \leq 1$. As regards the decomposition of equation (2.2.9), for $n_{\|} \geq 1$, we have

$$r_4 = i\sqrt{[n_1(n_{\|}+\nu)]^2 - 1} \tag{2.2.11}$$

while with $n_{\|} \leq -1$.

$$r_4 = \begin{cases} i\sqrt{[n_1(n_{\|}+\nu)]^2-1} & \text{if } -\infty \leq n_{\|} \leq (-\frac{1}{n_1}-\nu) \\ \sqrt{1-[n_1(n_{\|}+\nu)]^2} & \text{if } (-\frac{1}{n_1}-\nu) < n_{\|} < (\frac{1}{n_1}-\nu) \\ i\sqrt{[n_1(n_{\|}+\nu)]^2-1} & \text{if } (\frac{1}{n_1}-\nu) \leq n_{\|} \leq -1 \end{cases} \tag{2.2.12}$$

On the basis of relations (2.2.11) and (2.2.12), equation (2.2.9) is decomposed into a sum of four terms.

$$I_{\mathrm{m}}^{(per,ext+1)}(\xi,\zeta,\chi) =$$

$$\int_1^{+\infty} \tau_{10}^{+,\mathrm{TM}}(n_{\|}) \frac{(n_{\|}+\sqrt{1-n_{\|}^2})^{-\mathrm{m}}}{\sqrt{1-n_{\|}^2}} e^{-n_1\chi\sqrt{1-n_{\|}^2}} e^{\sqrt{[n_1(n_{\|}+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_{\|}+\nu)\zeta}\, dn_{\|} \tag{2.2.13}$$

$$I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) =$$
$$\int_{-\infty}^{(-\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{\sqrt{[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{2.2.14}$$

$$I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) =$$
$$\int_{(-\frac{1}{n_1}-\nu)}^{(\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{i\sqrt{1-[n_1(n_\parallel+\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{2.2.15}$$

$$I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) =$$
$$\int_{(\frac{1}{n_1}-\nu)}^{-1} \tau_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{\sqrt{[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{2.2.16}$$

The singularity in $|n_\parallel| = 1$, in equation (2.2.13) can be removed by the substitution

$$t = \sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = \sqrt{t^2+1} \\ dn_\parallel = \dfrac{t}{\sqrt{t^2+1}}\, dt \\ 0 < t < \infty \end{cases} \tag{2.2.17}$$

leading to

$$I_{\mathrm{m}}^{(per,ext+1)}(\xi,\zeta,\chi) = \int_0^{+\infty} \bar{\tau}_{10}^{+,\mathrm{TM}}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}+\nu)\zeta}\, dt \tag{2.2.18}$$

where

$$\bar{\tau}_{10}^{+,\mathrm{TM}}(t) = \tau_{10}^{+,\mathrm{TM}}(\sqrt{t^2+1})$$

The integration in equation (2.2.14) is performed solving the singularity in $|n_\parallel| = 1$, by means of the substitution

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\, dt \\ -\infty < t < -d_1 \end{cases} \tag{2.2.19}$$

being

$$d_1 = \sqrt{(-\frac{1}{n_1} - \nu)^2 - 1}$$

Putting these values in equation (2.2.14) and changing $t = -t$, we get

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) = &\int_{d_1}^{+\infty} \bar{\tau}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ &\times e^{(\xi+\chi)\sqrt{[n_1(-\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{2.2.20}$$

In a similar way, the integrals in equations (2.2.15) and (2.2.16) can be written as

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) = &\int_{c_1}^{d_1} \bar{\tau}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ &\times e^{-i(\xi+\chi)\sqrt{1-[n_1(-\sqrt{t^2+1}+\nu)]^2}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{2.2.21}$$

being

$$c_1 = \sqrt{(\frac{1}{n_1} - \nu)^2 - 1}$$

and

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) = &\int_{0}^{c_1} \bar{\tau}_{10}^{+,\mathrm{TM}}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ &\times e^{(\xi+\chi)\sqrt{[n_1(-\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{2.2.22}$$

Now, we consider the integral $I_{\mathrm{m}}^{(per,ext-)}(\xi,\zeta,\chi)$ defined in equation (2.2.10), which can be solved decomposing its integration domain according to the behavior of r_5. For $n_\parallel \leq -1$,

$$r_5 = i\sqrt{[n_1(n_\parallel - \nu)]^2 - 1} \tag{2.2.23}$$

when $n_\parallel \geq 1$,

$$r_4 = \begin{cases} i\sqrt{[n_1(n_\parallel - \nu)]^2 - 1} & \text{if } (\frac{1}{n_1} + \nu) \leq n_\parallel \leq \infty \\ \sqrt{1 - [n_1(n_\parallel - \nu)]^2} & \text{if } (-\frac{1}{n_1} + \nu) < n_\parallel < (\frac{1}{n_1} + \nu) \\ i\sqrt{[n_1(n_\parallel - \nu)]^2 - 1} & \text{if } 1 \leq n_\parallel \leq (-\frac{1}{n_1} + \nu) \end{cases} \tag{2.2.24}$$

On the basis of relations (2.2.23) and (2.2.24), equation (2.2.10) is decomposed into a sum of four terms.

$$I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) = \int_{-\infty}^{-1} \tau_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{\sqrt{[n_1(n_\parallel-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{2.2.25}$$

$$I_{\mathrm{m}}^{(per,ext-2)}(\xi,\zeta,\chi) = \int_{(\frac{1}{n_1}+\nu)}^{\infty} \tau_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{(n_\parallel + \sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{\sqrt{[n_1(n_\parallel-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{2.2.26}$$

$$I_{\mathrm{m}}^{(per,ext-3)}(\xi,\zeta,\chi) =$$

$$\int_{(-\frac{1}{n_1}+\nu)}^{(\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{-i\sqrt{1-[n_1(n_\parallel-\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\,dn_\parallel \tag{2.2.27}$$

$$I_{\mathrm{m}}^{(per,ext-4)}(\xi,\zeta,\chi) =$$

$$\int_{1}^{(-\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{(n_\parallel+\sqrt{1-n_\parallel^2})^{-\mathrm{m}}}{\sqrt{1-n_\parallel^2}} e^{-n_1\chi\sqrt{1-n_\parallel^2}} e^{\sqrt{[n_1(n_\parallel-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\,dn_\parallel \tag{2.2.28}$$

The integration in equation (2.2.25) is performed solving the singularity in $|n_\parallel| = 1$, by means of the substitution

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\,dt \\ -\infty < t < 0 \end{cases} \tag{2.2.29}$$

Putting these values in equation (2.2.25) and changing $t = -t$, we get

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) = \int_{0}^{+\infty} \bar{\tau}_{10}^{-,\mathrm{TM}}(t)\frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}+\nu)]^2-1}} e^{-in_1(\sqrt{t^2+1}+\nu)\zeta}\,dt \end{aligned} \tag{2.2.30}$$

where

$$\bar{\tau}_{10}^{-,\mathrm{TM}}(t) = \tau_{10}^{-,\mathrm{TM}}(\sqrt{t^2+1})$$

Comparing equations (2.2.30) and (2.2.18)

$$I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext+1)}(\xi,-\zeta,\chi) \tag{2.2.31}$$

The singularity in $|n_{\|}| = 1$, in equation (2.2.26) can be removed by the substitution

$$t = \sqrt{n_{\|}^2 - 1} \Rightarrow \begin{cases} n_{\|} = \sqrt{t^2+1} \\ dn_{\|} = \dfrac{t}{\sqrt{t^2+1}}\, dt \\ d_1 < t < \infty \end{cases} \tag{2.2.32}$$

thus obtaining

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-2)}(\xi,\zeta,\chi) = & \int_{d_1}^{+\infty} \bar{\tau}_{10}^{-,\mathrm{TM}}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ & \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}-\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{2.2.33}$$

Comparing equations (2.2.33) and (2.2.20)

$$I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-2)}(\xi,-\zeta,\chi) \tag{2.2.34}$$

In a similar way, the integrals in equations (2.2.27) and (2.2.28) can be written as

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-3)}(\xi,\zeta,\chi) = & \int_{c_1}^{d_1} \bar{\tau}_{10}^{-,\mathrm{TM}}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ & \times e^{-i(\xi+\chi)\sqrt{1-[n_1(\sqrt{t^2+1}-\nu)]^2}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{2.2.35}$$

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-4)}(\xi,\zeta,\chi) = & \int_{0}^{c_1} \bar{\tau}_{10}^{-,\mathrm{TM}}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi t} \\ & \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}-\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{2.2.36}$$

Note that

$$I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-3)}(\xi,-\zeta,\chi) \tag{2.2.37}$$

$$I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-4)}(\xi,-\zeta,\chi) \tag{2.2.38}$$

The final decomposition of the evanescent spectrum is

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext)}(\xi,\zeta,\chi) &= I_{\mathrm{m}}^{(per,ext+1)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext+1)}(\xi,-\zeta,\chi) \\ &+ I_{\mathrm{m}}^{(per,ext-2)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-2)}(\xi,-\zeta,\chi) \\ &+ I_{\mathrm{m}}^{(per,ext-3)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-3)}(\xi,-\zeta,\chi) \\ &+ I_{\mathrm{m}}^{(per,ext-4)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-4)}(\xi,-\zeta,\chi) \end{aligned} \tag{2.2.39}$$

The technique applied for the evaluation of the evanescent spectrum of the *Perturbed Reflected Cylindrical Functions* in Section 2.1 has been used for the evaluation of the evanescent spectrum of the *Perturbed Transmitted Cylindrical Functions*.

2.2.2 Homogeneous Spectrum of perturbed transmitted cylindrical functions

Let us consider homogeneous spectrum defined in equation (2.2.6), that can be decomposed into two terms:

$$I_{\mathrm{m}}^{(per,int)}(\xi,\zeta,\chi) = I_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) + I_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) \tag{2.2.40}$$

where

$$\begin{aligned} I_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) = \int\limits_{|n_\parallel|<1} & \tau_{10}^{+,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi r_1}}{r_1} e^{\mathrm{im}\cos^{-1}(n_\parallel)} \\ & \times e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \end{aligned} \tag{2.2.41}$$

and

$$\begin{aligned} I_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = \int\limits_{|n_\parallel|<1} & \tau_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi r_1}}{r_1} e^{\mathrm{im}\cos^{-1}(n_\parallel)} \\ & \times e^{-ir_5(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \end{aligned} \tag{2.2.42}$$

Consider the integral $I_{\mathrm{m}}^{per,int+}$, the square root r_1 is a real number, while the sign of the argument in r_4 depends on the value of ν:

$$r_4 = \begin{cases} i\sqrt{[n_1(n_\| + \nu)]^2 - 1} & \text{if } -1 \le n_\| \le (-\frac{1}{n_1} - \nu) \\ \sqrt{1 - [n_1(n_\| + \nu)]^2} & \text{if } (-\frac{1}{n_1} - \nu) < n_\| < (\frac{1}{n_1} - \nu) \\ i\sqrt{[n_1(n_\| + \nu)]^2 - 1} & \text{if } (\frac{1}{n_1} - \nu) \le n_\| \le 1 \end{cases} \tag{2.2.43}$$

Equation (2.2.41) can be written as a sum of three terms:

$$\begin{aligned} I_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) &= O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int+1)}(\xi,\zeta,\chi) \\ &\quad + S_{\mathrm{m}}^{(per,int+2)}(\xi,\zeta,\chi) \end{aligned} \tag{2.2.44}$$

where

$$O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) = \int\limits_{(-\frac{1}{n_1}-\nu)}^{(\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(n_\|) \frac{e^{in_1\chi\sqrt{1-n_\|^2}}}{\sqrt{1-n_\|^2}} e^{\mathrm{im}\cos^{-1}(n_\|)} e^{-i\sqrt{1-[n_1(n_\|+\nu)]^2}(\xi+\chi)} e^{in_1(n_\|+\nu)\zeta}\, dn_\| \tag{2.2.45}$$

$$S_{\mathrm{m}}^{(per,int+1)}(\xi,\zeta,\chi) = \int\limits_{(\frac{1}{n_1}-\nu)}^{1} \tau_{10}^{+,\mathrm{TM}}(n_\|) \frac{e^{in_1\chi\sqrt{1-n_\|^2}}}{\sqrt{1-n_\|^2}} e^{\mathrm{im}\cos^{-1}(n_\|)} e^{\sqrt{[n_1(n_\|+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\|+\nu)\zeta}\, dn_\| \tag{2.2.46}$$

$$S_{\mathrm{m},\ell}^{(per,int+2)}(\xi,\zeta,\chi) = \int\limits_{-1}^{(-\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(n_\|) \frac{e^{in_1\chi\sqrt{1-n_\|^2}}}{\sqrt{1-n_\|^2}} e^{\mathrm{im}\cos^{-1}(n_\|)} e^{\sqrt{[n_1(n_\|+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\|+\nu)\zeta}\, dn_\| \tag{2.2.47}$$

Let us consider the integral $O_{\mathrm{m}}^{(per,int+)}$ in equation (2.2.45), using the substitution

$$t = \cos^{-1}(n_\parallel) \Rightarrow \begin{cases} n_\parallel = \cos t \\ dn_\parallel = -\sin t \, dn_\parallel \\ \cos^{-1}(-\frac{1}{n_1} - \nu) < t < \cos^{-1}(\frac{1}{n_1} - \nu) \end{cases} \tag{2.2.48}$$

we get

$$O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) = \int\limits_{\cos^{-1}(\frac{1}{n_1}-\nu)}^{\cos^{-1}(-\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t + \mathrm{m}t)} \times e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t+\nu)]^2}} e^{in_1(\cos t+\nu)\zeta} \, dt \tag{2.2.49}$$

Substituting $n_\parallel = \cos t$, equations (2.2.46) and (2.2.47) can be written as

$$S_{\mathrm{m},\ell}^{(per,int+1)}(\xi,\zeta,\chi) = \int\limits_{0}^{\cos(\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t + \mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}} e^{in_1(\cos t+\nu)\zeta} \, dt \tag{2.2.50}$$

$$S_{\mathrm{m}}^{(per,int+2)}(\xi,\zeta,\chi) = \int\limits_{\cos^{-1}(-\frac{1}{n_1}-\nu)}^{\pi} \tau_{10}^{+,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t + \mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}} e^{in_1(\cos t+\nu)\zeta} \, dt \tag{2.2.51}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\, dt' \\ \cos^{-1}(\frac{1}{n_1} + \nu) < t' < 0 \end{cases} \tag{2.2.52}$$

Equation (2.2.51) can be written as follows

$$S_{\mathrm{m}}^{(per,int+2)}(\xi,\zeta,\chi) = e^{\mathrm{im}\pi} \int_{0}^{\cos^{-1}(\frac{1}{n_1}+\nu)} \tau_{10}^{+,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t - \mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}} e^{-in_1(\cos t-\nu)\zeta}\, dt \tag{2.2.53}$$

where the identity $\cos^{-1}(\frac{1}{n_1}+\nu) = \pi - \cos^{-1}(-\frac{1}{n_1}-\nu)$ is used.
Now, let us consider the integral in equation (2.2.42). The expression of integral $I_{\mathrm{m}}^{per,int-}$ in equation (2.2.42) is analogous to the one in equation (2.2.41), and a similar decomposition into three terms on the integration domain is performed on the basis of r_5:

$$r_5 = \begin{cases} i\sqrt{[n_1(n_\parallel-\nu)]^2-1} & \text{if } -1 \le n_\parallel \le (-\frac{1}{n_1}+\nu), \\ \sqrt{1-[n_1(n_\parallel-\nu)]^2} & \text{if } (-\frac{1}{n_1}+\nu) < n_\parallel < (\frac{1}{n_1}+\nu) \\ i\sqrt{[n_1(n_\parallel-\nu)]^2-1} & \text{if } (\frac{1}{n_1}+\nu) \le n_\parallel \le 1 \end{cases} \tag{2.2.54}$$

Equation (2.2.42) can be written as a sum of three terms:

$$I_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int-1)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) \tag{2.2.55}$$

where

$$O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = \int_{(-\frac{1}{n_1}+\nu)}^{(\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{e^{in_1\chi\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\sqrt{1-[n_1(n_\parallel-\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{2.2.56}$$

$$S_{\mathrm{m}}^{(per,int-1)}(\xi,\zeta,\chi) = \int_{(\frac{1}{n_1}+\nu)}^{1} \tau_{10}^{-,\mathrm{TM}}(n_\parallel) \frac{e^{in_1\chi\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{\sqrt{[n_1(n_\parallel-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{2.2.57}$$

$$S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) = \int\limits_{-1}^{(-\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(n_\parallel)\frac{e^{in_1\chi\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{\sqrt{[n_1(n_\parallel-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{2.2.58}$$

With the position $n_\parallel = \cos t$, the integral $O_{\mathrm{m}}^{(per,int-)}$ in equations (2.2.56) can be written as:

$$O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = \int\limits_{\cos^{-1}(\frac{1}{n_1}+\nu)}^{\cos^{-1}(-\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t+\mathrm{m}t)} \times e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t-\nu)]^2}} e^{in_1(\cos t-\nu)\zeta}\, dt \tag{2.2.59}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\, dt' \\ \cos^{-1}(-\dfrac{1}{n_1} - \nu) < t' < \cos^{-1}(\dfrac{1}{n_1} - \nu) \end{cases} \tag{2.2.60}$$

The equation (2.2.59) can be written as

$$O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = e^{\mathrm{im}\pi} \int\limits_{\cos^{-1}(\frac{1}{n_1}-\nu)}^{\cos^{-1}(-\frac{1}{n_1}-\nu)} \tau_{10}^{-,\mathrm{TM}}(\cos t) e^{i(n_1\chi\sin t-\mathrm{m}t)} \times e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t+\nu)]^2}} e^{-in_1(\cos t+\nu)\zeta}\, dt \tag{2.2.61}$$

Note that

$$O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} O_{-\mathrm{m}}^{(per,int+)}(\xi,-\zeta,\chi) \tag{2.2.62}$$

Substituting $n_{\parallel} = \cos t$, equation (2.2.57) can be written as

$$S_{\mathrm{m}}^{(per,int-1)}(\xi,\zeta,\chi) = \int_{0}^{\cos(\frac{1}{n_1}+\nu)} \tau_{10}^{-,\mathrm{TM}}(\cos t)e^{i(n_1\chi\sin t+\mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}}e^{in_1(\cos t-\nu)\zeta}\,dt \tag{2.2.63}$$

Comparing equation (2.2.53) and equation (2.2.63)

$$S_{\mathrm{m}}^{(per,int+2)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}}S_{-\mathrm{m}}^{(per,int-1)}(\xi,-\zeta,\chi) \tag{2.2.64}$$

With the position $n_{\parallel} = \cos t$, equation (2.2.58) can be written as

$$S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) = \int_{\cos^{-1}(-\frac{1}{n_1}+\nu)}^{\pi} \tau_{10}^{-,\mathrm{TM}}(\cos t)e^{i(n_1\chi\sin t+\mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}}e^{in_1(\cos t-\nu)\zeta}\,dt \tag{2.2.65}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(\frac{1}{n_1} - \nu) < t' < 0 \end{cases} \tag{2.2.66}$$

The equation (2.2.65) can be written as

$$S_{\mathrm{m},\ell}^{(per,int-2)}(\xi,\zeta,\chi) = e^{i\mathrm{m}\pi} \int_{0}^{\cos^{-1}(\frac{1}{n_1}-\nu)} \tau_{10}^{+,\mathrm{TM}}(\cos t)e^{i(n_1\chi\sin t-\mathrm{m}t)} \times e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}}e^{-in_1(\cos t+\nu)\zeta}\,dt \tag{2.2.67}$$

Comparison of equation (2.2.67) and equation (2.2.50) gives

$$S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}}S_{-\mathrm{m}}^{(per,int+1)}(\xi,-\zeta,\chi) \tag{2.2.68}$$

The final decomposition of the homogeneous spectrum is

$$\begin{aligned} I_{\mathrm{m}}^{(per,int)}(\xi,\zeta,\chi) &= O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} O_{-\mathrm{m}}^{(per,int+)}(\xi,-\zeta,\chi) \\ &+ S_{\mathrm{m}}^{(per,int+1)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} S_{-\mathrm{m}}^{(per,int+1)}(\xi,-\zeta,\chi) \\ &+ S_{\mathrm{m}}^{(per,int-1)}(\xi,-\zeta,\chi) + (-1)^{\mathrm{m}} S_{-\mathrm{m}}^{(per,int-1)}(\xi,-\zeta,\chi) \end{aligned} \tag{2.2.69}$$

For the evaluation of homogeneous spectrum of *Perturbed Transmitted Cylindrical Functions*, we apply the same technique that has been used in the evaluation of the homogeneous spectrum of the *Perturbed Reflected Cylindrical Functions* in Section 2.1.

2.3 Asymptotic solution of perturbed transmitted cylindrical functions

The Stationary phase Method [64], is an approximate technique for the evaluation of integrals for large values of ρ. In this method, the path of integration can be deformed without affecting the value of the integral, provided that the path does not pass through the singularities of the integrand. The new path of the integration can also be deformed in such a way that the only small segment of the new path including the saddle point will make the most of the contribution to the integral.

Here we are going to apply Stationary phase Method for the evaluation of *Perturbed Transmitted Cylindrical Functions* as an alternative solution to one described in Section 2.2 with large value of ρ. The purpose is to compare the results obtained by accurate evaluation of *Transmitted Cylindrical Functions* with those of obtained by applying asymptotic technique to *Transmitted Cylindrical Functions* which will give us accuracy of the results. Also we wish to perform a comparison of estimated computer time to obtain the curve by two techniques.

The expression for the *Transmitted Cylindrical Functions* in equation (1.4.22) can be written as

$$TW_{\mathrm{m}}^{per}(\xi,\zeta,\chi) = TW_{\mathrm{m}}^{per+}(\xi,\zeta,\chi) + TW_{\mathrm{m}}^{per-}(\xi,\zeta,\chi) \tag{2.3.1}$$

where

$$TW_{\mathrm{m}}^{per+}(\xi,\zeta,\chi) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\tau_{10}^{+,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi,n_\parallel)e^{in_1(n_\parallel+\frac{n_{\mathrm{s}}}{n_1})\zeta} \times e^{-i\sqrt{1-[n_1(n_\parallel+\frac{n_{\mathrm{s}}}{n_1})]^2}(\xi+\chi)}\,dn_\parallel \tag{2.3.2}$$

and

$$TW_{\mathrm{m}}^{per-}(\xi,\zeta,\chi) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\tau_{10}^{-,\mathrm{TM}}(n_\parallel)F_{\mathrm{m}}(-n_1\chi,n_\parallel)e^{in_1(n_\parallel-\frac{n_{\mathrm{s}}}{n_1})\zeta} \times e^{-i\sqrt{1-[n_1(n_\parallel-\frac{n_{\mathrm{s}}}{n_1})]^2}(\xi+\chi)}\,dn_\parallel \tag{2.3.3}$$

being

$$\tau_{10}^{\pm,\mathrm{TM}}(n_1n'_\parallel) = \frac{i4\pi^2A(n_1^2-n_0^2)n_1\sqrt{1-n_\parallel^2}}{\lambda_0\left\{n_1\sqrt{1-n_\parallel^2}+n_0\sqrt{1-\left(\frac{n_1n_\parallel}{n_0}\right)^2}\right\}} \times\frac{1}{\left\{n_1\sqrt{1-(n_\parallel\pm\frac{n_{\mathrm{s}}}{n_1})^2}+n_0\sqrt{1-[\frac{n_1}{n_0}(n_\parallel\pm\frac{n_{\mathrm{s}}}{n_1})]^2}\right\}} \tag{2.3.4}$$

Consider the first integral in equation (2.3.2), and put

$$n_1\left(n_\parallel+\frac{n_{\mathrm{s}}}{n_1}\right)=\sin t,\quad \zeta=\rho\sin\varphi,\quad \xi=\rho\cos\varphi$$

we get

$$TW_{\mathrm{m}}^{per+}(\rho) = \int_C Q(t)e^{\rho q(t)}dt \tag{2.3.5}$$

where

$$Q(t) = \tau_{10}^{+,\mathrm{TM}}(t)h(t) \tag{2.3.6}$$

$$\tau_{10}^{+,\mathrm{TM}}(t) = \frac{iA(n_1^2-n_0^2)\cos t}{\lambda_0\left\{\sqrt{n_0^2-\sin^2 t}+\sqrt{n_1^2-\sin^2 t}\right\}} \times\frac{1}{\left\{\sqrt{n_0^2-(\sin t-n_{\mathrm{s}})^2}+\sqrt{n_1^2-(\sin t-n_{\mathrm{s}})^2}\right\}} \tag{2.3.7}$$

$$\begin{aligned} h(t) &= e^{i\left\{\chi_q\sqrt{n_1^2-(\sin t-n_{\mathrm{s}})^2}-\chi_q\sqrt{1-(\sin t)^2}+\mathrm{m}\cos^{-1}\left[\frac{(\sin t-n_{\mathrm{s}})}{n_1}\right]\right\}} \\ q(t) &= i(\sin t\sin\varphi-\cos t\cos\varphi) \end{aligned} \tag{2.3.8}$$

As $Q(t)$ is a well behaved function, the Method of Stationary Phase [64] can be used to obtain approximate asymptotic expressions, for large values of ρ. To find the saddle point, we put the first derivative of the $q(t)$ equal to zero. The saddle point occurs at $t_{\mathrm{s}}=\pi-\varphi$. Putting $q(t)=q(t_{\mathrm{s}})-\mathrm{s}^2$ in equation (2.3.5), we get

$$TW_{\mathrm{m}}^{per+}(\rho)=e^{\rho q(t_{\mathrm{s}})}\int\limits_C Q(\mathrm{s})e^{-\rho\mathrm{s}^2}\mathrm{ds} \tag{2.3.9}$$

The function $Q(s)$ can be expanded into a power series

$$Q(\mathrm{s})=Q(0)+\mathrm{s}Q'(0)+\frac{\mathrm{s}^2}{2!}Q''(0)+... \tag{2.3.10}$$

A complete asymptotic expansion is given by [64]. Putting the expansion (2.3.10) in equation (2.3.9), we obtain

$$TW_{\mathrm{m}}^{per+}(\rho)=e^{\rho q(t_{\mathrm{s}})}\sum_{\mathrm{n}=0}^{\infty}\frac{Q^{\mathrm{n}}(0)}{\mathrm{n}!}I_{\mathrm{n}}(\rho) \tag{2.3.11}$$

where

$$I_{\mathrm{n}}(\rho)=\begin{cases}\displaystyle\int\limits_{-\infty}^{\infty}\mathrm{s}^{\mathrm{n}}e^{-\rho\mathrm{s}^2}\mathrm{ds}=\frac{\Gamma[(1+\mathrm{n})/2]}{\rho^{(1+\mathrm{n})/2}} & \text{if n} = 0,2,4,...... \\ 0 & \text{if n} = 1,3,5........\end{cases} \tag{2.3.12}$$

and

$$Q(0) = Q(t_{\mathrm{s}})\beta(0) \tag{2.3.13}$$

$$Q'(0) = Q'(t_{\mathrm{s}})[\beta(0)]^2+Q(t_{\mathrm{s}})\beta'(0) \tag{2.3.14}$$

$$Q''(0) = Q''(t_{\mathrm{s}})[\beta(0)]^3+3Q'(t_{\mathrm{s}})\beta(0)\beta'(0)+Q(t_{\mathrm{s}})\beta''(0) \tag{2.3.15}$$

with

$$\beta(0) = \sqrt{\frac{-2}{q^{(2)}(t_s)}} = \sqrt{\frac{2}{i}} \tag{2.3.16}$$

$$\beta'(0) = \frac{2}{3}\frac{q'''(t_\mathrm{s})}{[q''(t_\mathrm{s})]^2} = 0 \tag{2.3.17}$$

$$\begin{aligned}\beta''(0) &= \frac{-3}{\beta(0)}\Big\{\Big(\frac{\beta'(0)}{2}\Big)^2 + \frac{2}{q''(t_\mathrm{s})}\Big[\frac{q''''(t_s)}{4!}\Big(\beta(0)\Big)^4 + \frac{q'''(t_s)}{4}\Big(\beta(0)\Big)^2\beta'(0)\Big]\Big\}\\ &= \frac{1}{4}\Big(\sqrt{\frac{2}{i}}\Big)^3\end{aligned} \tag{2.3.18}$$

The zero-*th* order solution can be obtained by putting n = 0 in equation (2.3.11)

$$TW^{per+}_{\mathrm{m}}(\rho) = \sqrt{\frac{2\pi}{i\rho}}e^{i\rho}Q(t_\mathrm{s}) \tag{2.3.19}$$

From equation (2.3.12), the first order solution corresponding to n=1 is equal to zero. For the second order solution, we need to evaluate the first and second derivatives of $Q(\mathrm{s})$ at s = 0, i.e $t = t_\mathrm{s}$.

The derivatives of $Q(\mathrm{s})$ are given by

$$Q'(t) = \tau^{+,\mathrm{TM}}_{10}(t)h'(t) + \tau'^{+,\mathrm{TM}}_{10}(t)h(t) \tag{2.3.20}$$

$$Q''(t) = \tau^{+,\mathrm{TM}}_{10}(t)h''(t) + 2\tau'^{+,\mathrm{TM}}_{10}(t)h'(t) + \tau''^{+,\mathrm{TM}}_{10}(t)h(t) \tag{2.3.21}$$

The expression for the first and second derivatives of the transmission coefficient $\tau^{+,\mathrm{TM}}_{10}(t)$ is given by

$$\begin{aligned}\tau'^{+,\mathrm{TM}}_{10}(t) =& \tau^{+,\mathrm{TM}}_{10}(t)\left\{\frac{\cos t\sin t}{\sqrt{n_0^2-(\sin t)^2}\sqrt{n_1^2-(\sin t)^2}} - \tan t\right\}\\ &+\tau^{+,\mathrm{TM}}_{10}(t)\left\{\frac{\cos t(\sin t - n_\mathrm{s})}{\sqrt{n_0^2-(\sin t-n_\mathrm{s})^2}\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}\right\}\end{aligned} \tag{2.3.22}$$

$$
\begin{aligned}
\tau_{10}''^{+,\mathrm{TM}}(t) =& \tau_{10}'^{+,\mathrm{TM}}(t)\left\{\frac{\cos t\sin t}{\sqrt{n_0^2-(\sin t)^2}\sqrt{n_1^2-(\sin t)^2}}-\tan t\right\}\\
&+\tau_{10}'^{+,\mathrm{TM}}(t)\left\{\frac{\cos t(\sin t-n_\mathrm{s})}{\sqrt{n_0^2-(\sin t-n_\mathrm{s})^2}\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}\right\}\\
&+\tau_{10}^{+,\mathrm{TM}}(t)\left\{\frac{[\cos^2 t-\sin^2 t]}{(\sqrt{n_0^2-(\sin t)^2}\sqrt{n_1^2-(\sin t)^2})}-\sec^2 t\right\}\\
&+\tau_{10}^{+,\mathrm{TM}}(t)\left\{\frac{[\cos^2 t-\sin t(\sin t-n_\mathrm{s})]}{(\sqrt{n_0^2-(\sin t-n_\mathrm{s})^2}\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2})}\right\}\\
&+\tau_{10}^{+,\mathrm{TM}}(t)\left\{\frac{\cos^2 t\sin^2 t\left[n_0^2+n_1^2-2\sin^2 t\right]}{\left[\sqrt{n_0^2-(\sin t)^2}\sqrt{n_1^2-(\sin t)^2}\right]^3}\right\}\\
&+\tau_{10}^{+,\mathrm{TM}}(t)\left\{\frac{\cos^2 t(\sin t-n_\mathrm{s})^2\left[n_0^2+n_1^2-2(\sin t-n_\mathrm{s})^2\right]}{\left[\sqrt{n_0^2-(\sin t-n_\mathrm{s})^2}\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}\right]^3}\right\}
\end{aligned}
\tag{2.3.23}
$$

The first and second derivatives of $h(t)$ can be written as

$$
h'(t) = ih(t)\left\{\chi\sin t-\frac{\chi\cos t(\sin t-n_\mathrm{s})}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}-\frac{\mathrm{m}\cos t}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}\right\}
\tag{2.3.24}
$$

$$
\begin{aligned}
h''(t) = ih'(t)&\left\{\chi\sin t-\frac{\chi\cos t(\sin t-n_\mathrm{s})}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}-\frac{\mathrm{m}\cos t}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}\right\}\\
+ih(t)&\left\{\frac{\chi[\sin t(\sin t-n_\mathrm{s})-\cos^2 t]}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}-\frac{\chi\cos^2 t(\sin t-n_\mathrm{s})^2}{\left[\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}\right]^3}\right\}\\
+ih(t)&\left\{\chi\cos t+\frac{\mathrm{m}\sin t}{\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}}-\frac{\mathrm{m}\cos^2 t(\sin t-n_\mathrm{s})}{\left[\sqrt{n_1^2-(\sin t-n_\mathrm{s})^2}\right]^3}\right\}
\end{aligned}
\tag{2.3.25}
$$

We can solve the second integral given in equation (2.3.3) in a similar way.

Chapter 3

Numerical results for sinusoidal surface

In this chapter, numerical results for the sinusoidal surface are presented using the theory developed in chapter 1 and numerical evaluation of the spectral integrals done in chapter 2.

In Section 3.1, validation of the proposed technique is performed comparing the results with the literature, where a single dielectric cylinder buried beneath a rough surface with sinusoidal profile is considered. The effect of the geometrical and physical parameters on the scattering pattern of a cylinder is presented in Section 3.2. It is noted that the period of the rough surface is an important parameter in the study of scattering pattern. Arrangement of two interacting air filled cylinders placed at a symmetric distance from the vertical axis has been considered in Section 3.3. In Section 3.4, simulation of a cylinder by means of N cylinders is considered. Both perfectly and dielectric cylinders are simulated using *Same Area Rule* and *Same Volume Rule*, respectively. Results obtained by asymptotic solution of the *Perturbed Transmitted Cylindrical Functions* are reported in Section 3.5. An effort has also been done to make a comparison of execution time to obtain a curve using accurate analysis of the *Perturbed Transmitted Cylindrical Functions* with that of obtained by asymptotic solution.

3.1 Comparison with literature

To validate the proposed technique, let us consider a single cylinder buried below a rough surface with sinusoidal profile as shown in figure 3.1. The results are obtained implementing a rough surface with sinusoidal profile, given by

$$g(\zeta) = \frac{2\pi}{\lambda_0} A \cos(n_s \zeta) \tag{3.1.1}$$

Figure 3.2 shows the very good comparison of scattering cross-section σ with the

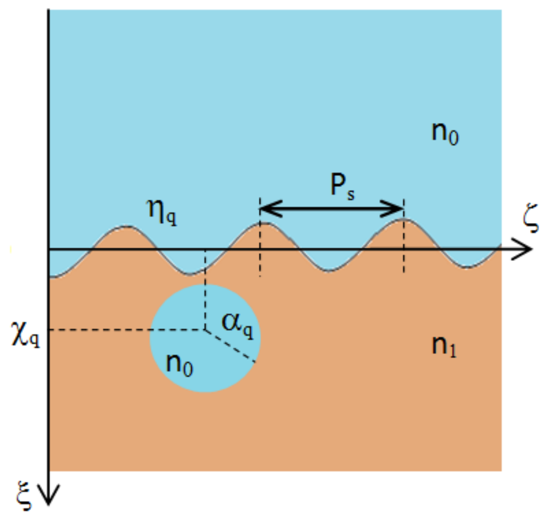


Figure 3.1: A cylinder buried below a rough surface with sinusoidal profile .

results given in the literature [43], [45], where $\alpha = 0.32\pi$, $\chi = 2.6\pi$, $\eta = 0$, $\varphi_i = 30°$, $A = 0.0064\lambda_0$, $n_1 = 2$, $n_c = 1.5$, $P_s = 0.8\pi$ and TM polarization has been considered. The estimated computer time to obtain the coefficient set is 1.71 seconds, on an Intel Pentium Dual CPU T3200 @2 GHz, RAM 2 GB.

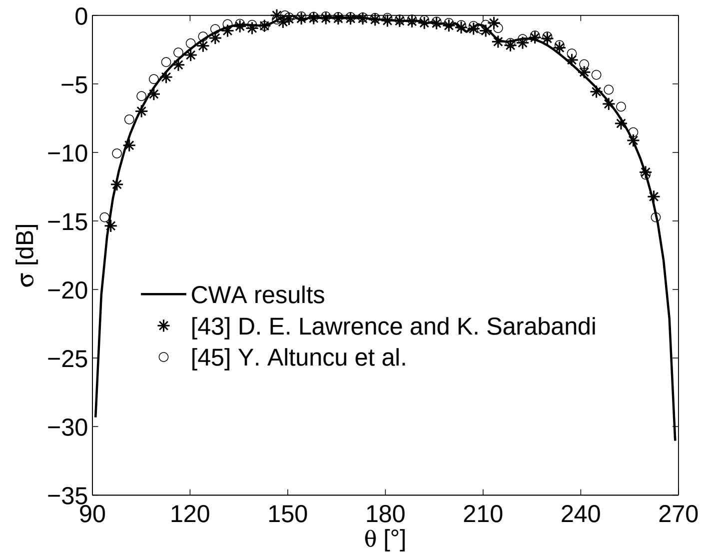


Figure 3.2: Comparison of normalized scattering pattern (normalized to its maximum value) of a dielectric cylinder buried beneath a sinusoidal profile.

3.2 Numerical results for a single cylinder

The change of some physical and geometrical parameters gives us a chance to simulate different scenarios for the scattering problem. These scenarios are very helpful to demonstrate the effect of the statistics of the rough surface, radius and permittivity of the dielectric cylinder on the scattering pattern. Also, the permittivity of the host medium is an important parameter affecting the scattering pattern. Figure 3.3 shows the normalized scattering pattern (normalized to its maximum value) for different values of the cylinder's radius α. The calculations are performed with $\chi = 2.6\pi$, $\eta = 0$, $\varphi_{\mathrm{i}} = 30°$, $A = 0.0064\lambda_0$, $n_1 = 2$, $n_{\mathrm{c}} = 1.5$, $P_{\mathrm{s}} = 0.8\pi$. As the radius of the cylinder is increased, the value of the scattered field is also increased. Note that the specular component of the initial reflected wave from the rough surface is not included.

Figure 3.4 shows the normalized scattering pattern for different values of n_{c}, the

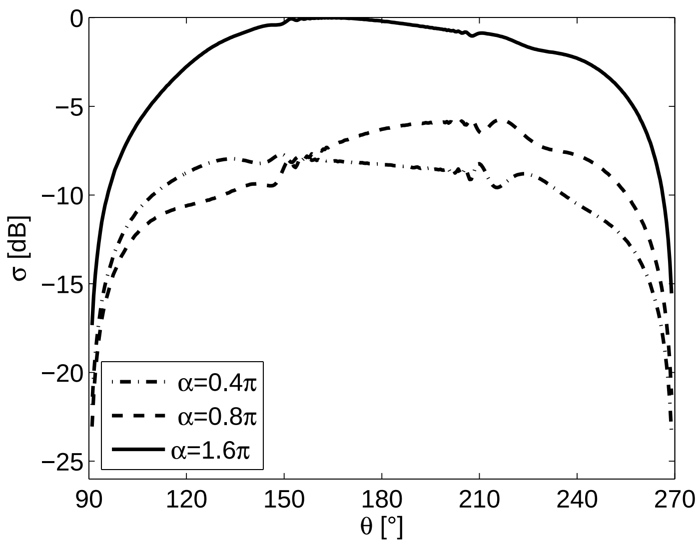


Figure 3.3: Normalized scattering pattern (normalized to its maximum value) of a dielectric cylinder for different values of the normalized radius α.

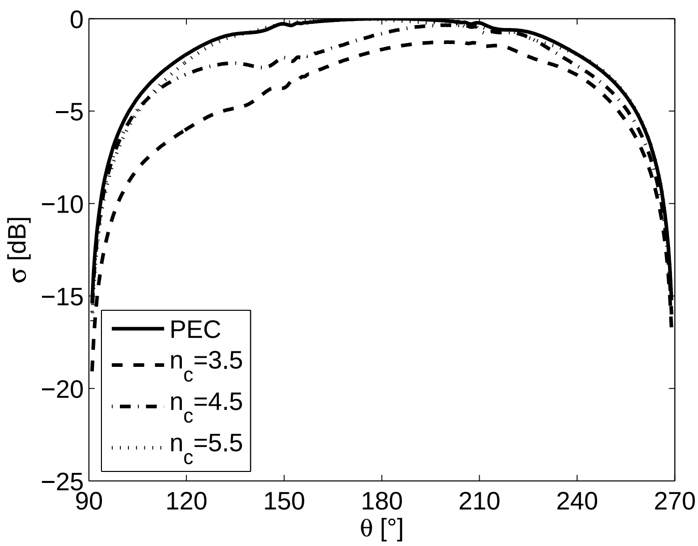


Figure 3.4: Normalized scattering pattern (normalized to its maximum value) of a dielectric cylinder for different values of n_{c}.

refractive index of the dielectric cylinder. A large value of n_c may correspond to a highly-reflecting cylinder. A curve for a perfectly conducting cylinder has also been included. As the values of the n_c is increased, the pattern approaches to that of a perfectly conducting cylinder, verifying the coherence of the results shown.

The period of the rough surface is an important parameter in the analysis of the

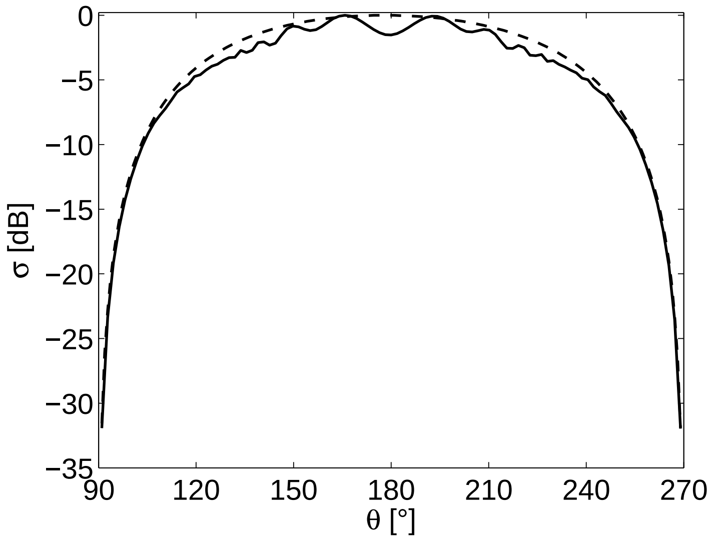


Figure 3.5: Normalized scattering pattern (normalized to its maximum value) for $P_s = 0.8\pi$.

scattering pattern of a buried cylinder below a periodic sinusoidal surface. A buried air cavity has been simulated for different values of the period of the rough surface.

Figure 3.5 shows the normalized scattering pattern with normalized period $P_s = 0.8\pi$. The scattering pattern for a flat interface has also been shown in the same figure. The parameters selected for this simulation are $n_1 = 3$, $n_c = 1$, $\alpha = 0.32\pi$, $\chi = 2.6\pi$. In figure 3.6, the period of the surface is increased to $P_s = 1.2\pi$. The result is shown for period $P_s = 4\pi$ in figure 3.7. The pattern approaches to that of a flat surface.

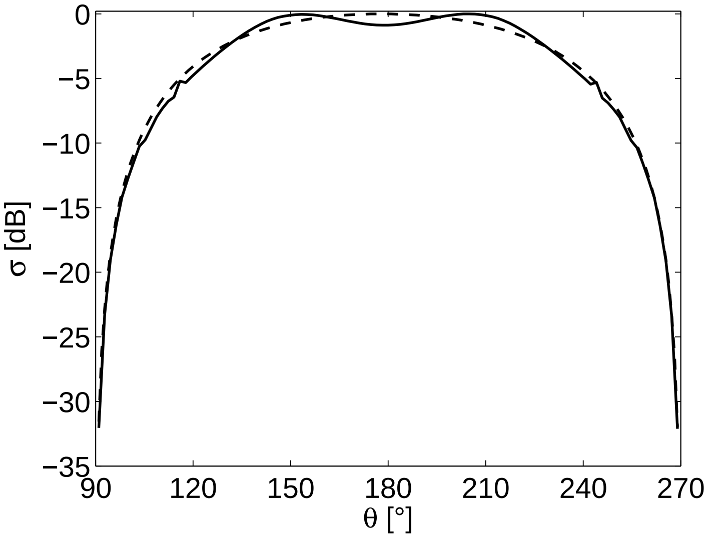


Figure 3.6: Normalized scattering pattern (normalized to its maximum value) for $P_{\mathrm{s}} = 1.2\pi$.

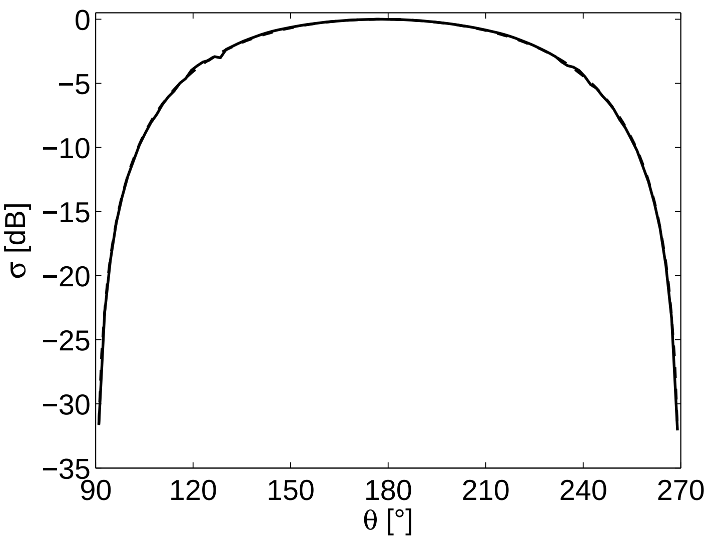


Figure 3.7: Normalized scattering pattern (normalized to its maximum value) for $P_{\mathrm{s}} = 4\pi$.

It can be observed from the results shown that the scattering patterns are significantly different from the flat surface case just for a limited range of spatial frequencies contained in the surface roughness. Surface frequencies outside this range do not cause a significant change in the scattering pattern. This corresponds to the interference due to the interaction of the cylinder with the periodic surface considered.

3.3 Two interacting cylinders

We are going to consider two interacting cylinders placed as shown in figure 3.8. A comparison between field response with flat and rough surface is shown in figure 3.9. The amplitude of V_{st} is plotted as a function of θ in $\rho = 50$, for two interacting air-filled cylinders ($n_c = 1$), buried at a depth $\chi = 2.57$ at a symmetric distance from the ξ-axis, and excited by a normally incident TM-polarized plane-wave. The hosting medium is a ground of permittivity $\epsilon_1 = 5$. The period of the surface is $P_s = 1.2\pi$ with $A = 0.0064\lambda_0$. As expected, the scattered field is symmetric showing a good self-consistency check.

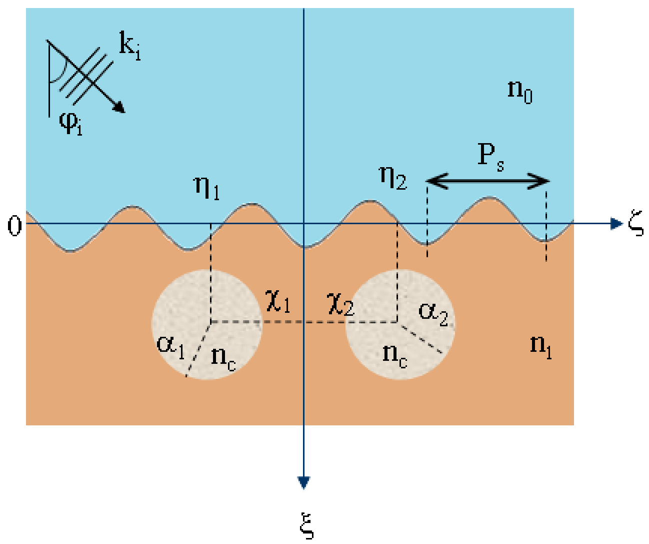
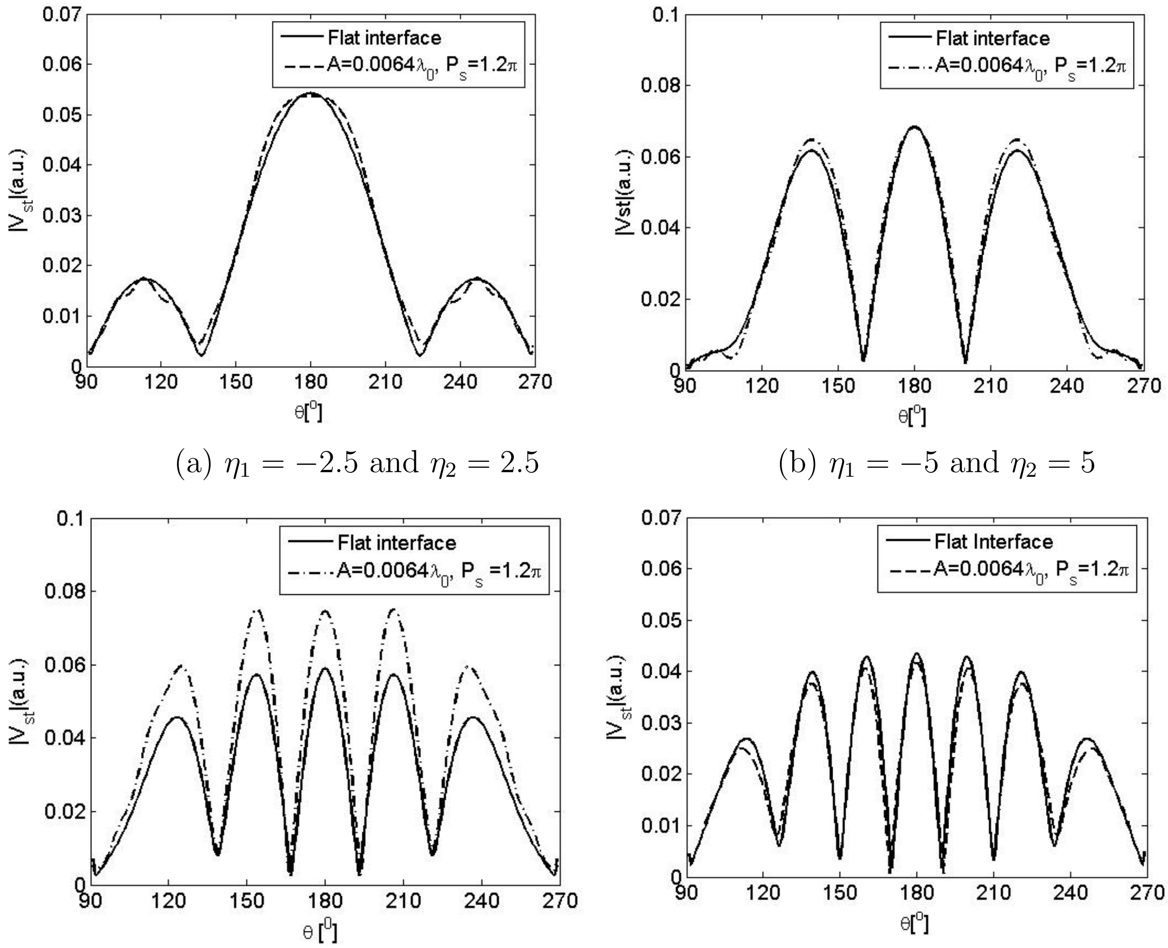


Figure 3.8: Two identical cylinders placed at equal distance from vertical axis.



(a) $\eta_1 = -2.5$ and $\eta_2 = 2.5$

(b) $\eta_1 = -5$ and $\eta_2 = 5$

(c) $\eta_1 = -7.5$ and $\eta_2 = 7.5$

(d) $\eta_1 = -10$ and $\eta_2 = 10$

Figure 3.9: Polar plot of the field scattered by two air cylinders with $\chi_1 = \chi_2 = 2.57$, buried in ground with $\epsilon_1 = 5$ $\varphi_{\rm i} = 0$, and TM polarization.

3.4 Simulation of a cylinder by means of N cylinders

Multiple scatterers can simulate a cylindrical object of arbitrary cross-section, if suitably placed along its border. Moreover, such an arrangement can offer a self-consistency check in the simulation of an isolated circular cross-section cylinder by means of a suitable set of N cylinders.

3.4.1 Simulation of a perfectly conducting cylinder

The simulation of the perfectly conducting cylinder by means of N cylinders is here considered. The geometry of the problem is shown in Figure 3.10.

The alignment proposed in [59] is employed, which satisfies *Same Area Rule*: the

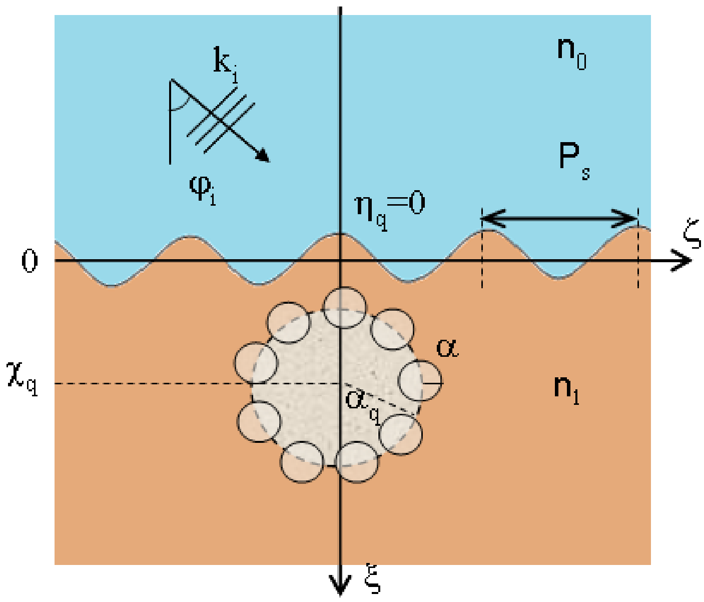


Figure 3.10: Geometry for the simulation of a perfectly-conducting cylinder by means of N cylinders.

total surface area, for unit length along y, of the N cylinders must be equal to the one of the isolated cylinder. Being R the radius of the reference cylinder, and α the

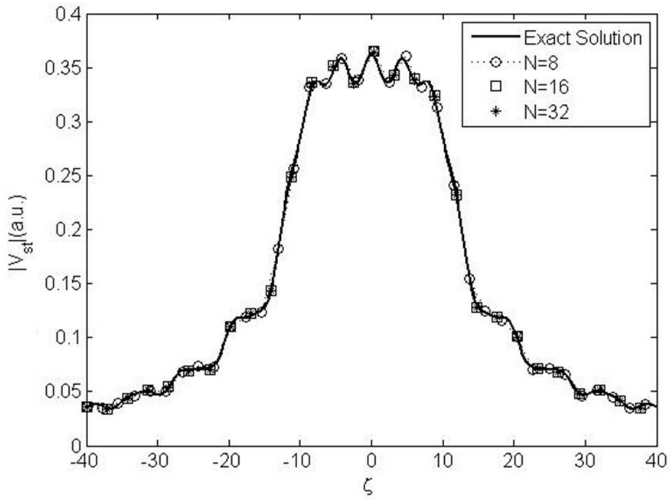


Figure 3.11: Simulation of a perfectly-conducting cylinder (radius $\alpha = 0.32\pi$, depth $\chi = 2.6\pi$, $n_1 = 2$, TM polarization, normal incidence) by means on N cylinders; interface with sinusoidal profile (normalized period $P_s = 1.2\pi$, amplitude $A = 0.0064\lambda_0$).

one of the N modeling cylinders, then

$$2\pi R = N 2\pi\alpha \Rightarrow \alpha = \frac{R}{N}. \tag{3.4.1}$$

In figure 3.11, a perfectly-conducting cylinder of normalized radius $\alpha = 0.32\pi$, buried at a depth $\chi = 2.6\pi$ ($\eta = 0$) in a half-space with $n_1 = 2$ (TM polarization, normal incidence), is simulated by means on N cylinders satisfying the *Same Area Rule*. The interface follows the sinusoidal profile defined in equation (3.1.1), with normalized period $P_s = 1.2\pi$ and amplitude $A = 0.0064\lambda_0$. The near-field response is evaluated, in terms of the amplitude of the scattered-transmitted field V_{st}, along a line parallel to the interface, in $\xi = -0.1$.

3.4.2 Simulation of an air filled cylinder

A geometry shown in figure 3.12 has been applied to the simulation of a dielectric cylinder by means of N cylinders. As far as dielectric scatterers are considered, the *Same Volume Rule* may be implemented, according to which the total volume of the N modeling cylinders, for unit length along y, must be equal to the volume of the reference cylinder. The rule has been applied in the particular case of $N' = N - 1$ external cylinders with same radius α_i, i $= 1, ..., N-1$, tangent to an internal cylinder of radius α_{int}. In Figure 3.13, an air-filled cylinder ($n_\mathrm{c} = 1$) of radius $\alpha = 0.32\pi$, placed in $\chi = 2.57$ ($\eta = 0$) in a half-space with $n_1 = 2$ (TM polarization, normal incidence), is simulated by means of $N' = 7, 15, 31, 54$ external cylinders, and an internal cylinder with $\alpha_{\mathrm{int}} = 0.8$. Results of both figures 3.11 and 3.13 show a very good agreement between the reference solution relevant to the isolated scatterer and its simulation with a set of N cylinders.

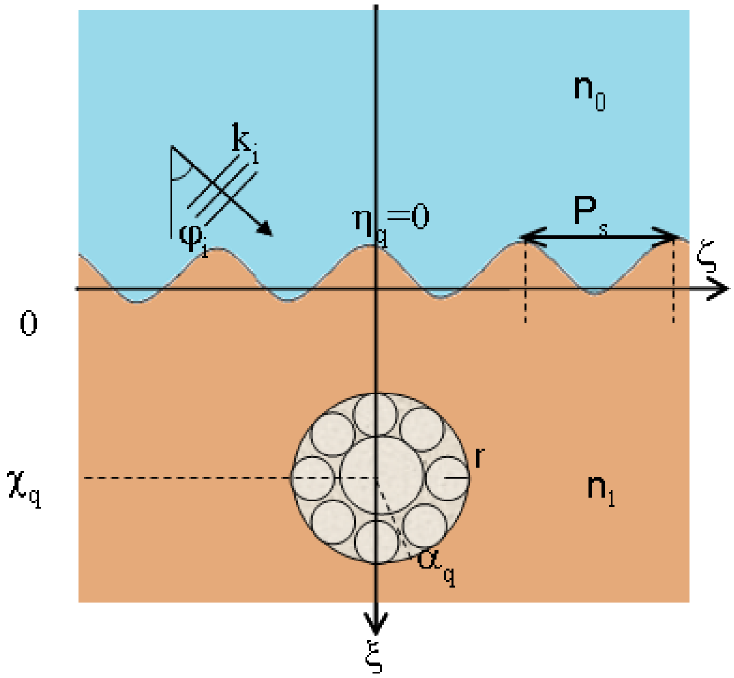


Figure 3.12: Geometry for the simulation of a dielectric cylinder by means of N cylinders.

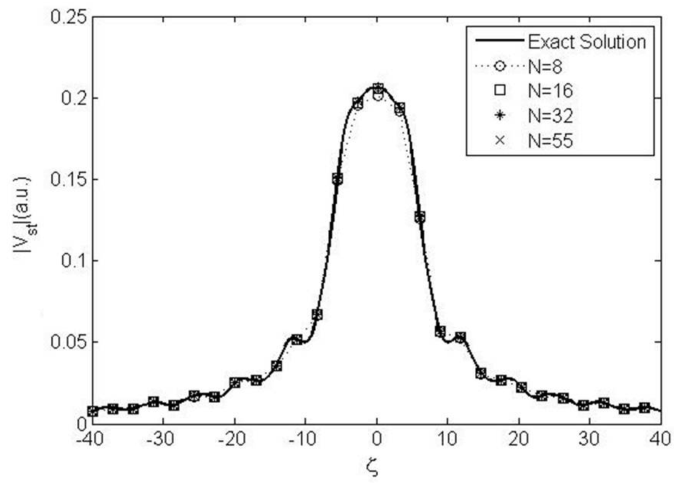


Figure 3.13: Simulation of a air cylinder (radius $\alpha = 0.32\pi$, depth $\chi = 2.57$, $n_c = 1$, $n_1 = 2$, TM polarization, normal incidence) by means on N cylinders; interface with sinusoidal profile (normalized period $P_s = 1.2\pi$, amplitude $A = 0.0064\lambda_0$).

3.5 Asymptotic results

The numerical implementation of the theory developed in the Section 1.4 has been done to evaluate the scattered field in region 0 for TM polarization using asymptotic solution of the Section 2.3.

In figure 3.14, results of figure 3.2 obtained with accurate solution of the spectral integrals, are compared to the numerical results with the asymptotic evaluation developed in Section 2.3. The normalized scattering cross section σ is plotted as a function of the scattered angle θ where TM polarization has been considered. The calculations are performed with $P_s = 0.8\pi, A = 0.0064\lambda_0, n_1 = 2, n_c = 1.5, \varphi_i = 30^o, \alpha = 0.32\pi, \chi = 2.6\pi$ and $\eta = 0$. There is a fair agreement between the two results confirming the accuracy of the technique. These results are also in agreement with the results reported in [43] and [45]. The estimated computer time to obtain the curve for asymptotic technique is 1061 seconds on an Intel Pentium Dual CPU T3200@ 2GHz, RAM 2 GB, while the time for the curve obtained by accurate solution is 3137 seconds, on the same computer.

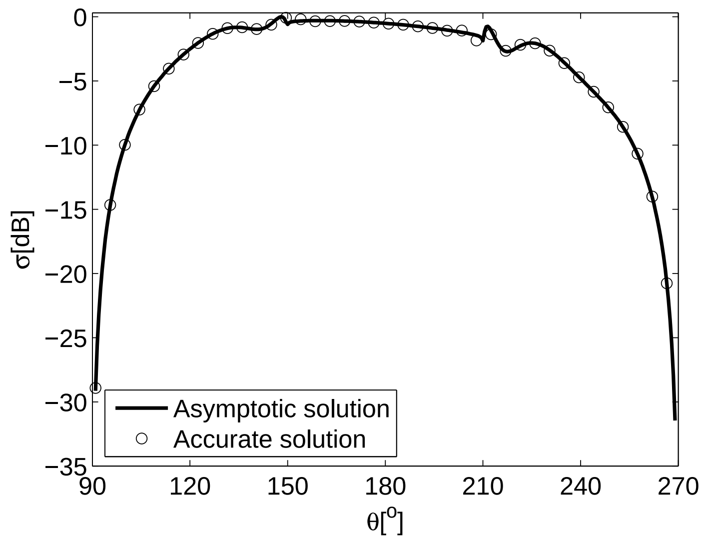


Figure 3.14: Comparison between asymptotic and accurate solution for normalized period $P_s = 0.8\pi$.

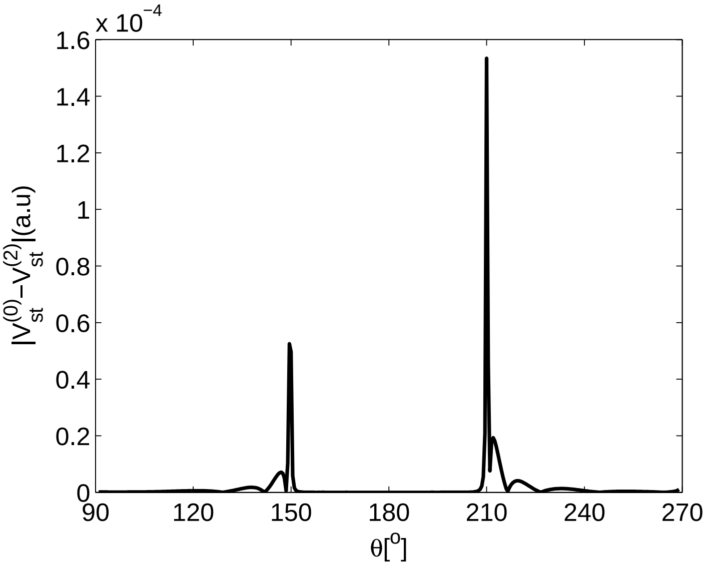


Figure 3.15: Absolute value of the difference between zeroth and second order solution $P_s = 0.8\pi$.

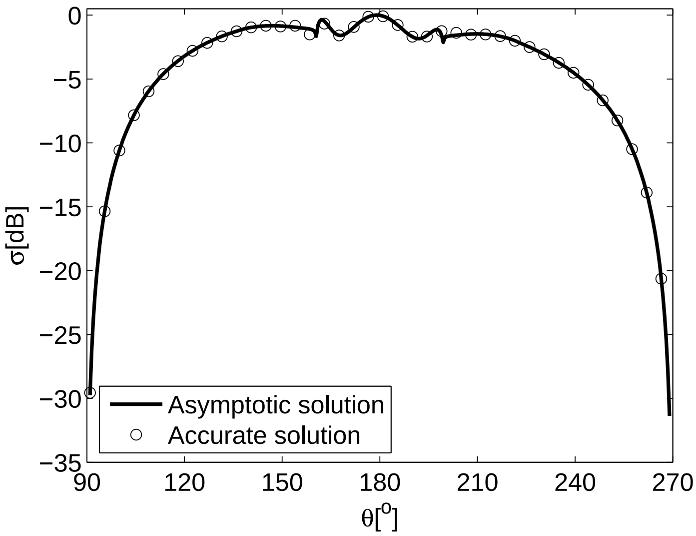


Figure 3.16: Comparison between asymptotic and accurate solution for normalized period $P_{\mathrm{s}} = 0.8\pi$.

The absolute value of the difference between zero-*th* and second order asymptotic solution are shown in figure 3.15. This figure gives an idea of the fast convergence of the asymptotic solution. Figure 3.16 shows another comparison of exact and asymptotic technique for $P_{\mathrm{s}} = 1.2\pi$ with the same values of figure 3.14.

Chapter 4

Scattering from cylinders buried below a Gaussian surface

Scattering from buried objects below a one-dimensional periodic rough surface with Gaussian roughness spectrum is addressed in this chapter. In the evaluation of scattering from rough surfaces, we need to find out spectral density, which is the Fourier transform of the correlation function [60].

There are several models of spectral density to represent practical situations of rough surface scattering [61]. Among the mostly used non-Gaussian spectrum models, one is with exponential correlation function, the other one is with power law spectral density. It has been found that the surfaces with exponential correlation function are without rms slope, which is required for numerical simulations of wave scattering from random rough surfaces. Power law spectra are multi-scale, which complicates the rough surface scattering [9].

The Gaussian choice is a useful simplification because it restricts the roughness to a single spatial scale. It provides a less complicated environment for investigating rough surface scattering problems as compared to other composite roughness models for the multi-scale case [55]-[56] available in the literature. Also, in the theory of rough surface scattering, the model of Gaussian, statistically homogeneous rough surface is usually used, which gives a good chance to compare the results with the literature.

Since the surface is periodic, it is necessary to to limit the area illuminated by the incident field in order to limit the amount of reflected energy from the rough surface. It is done by either tapering the incident wave or limiting the extent of the rough surface. A tapering of the incident field has been discussed by many authors [9]-[20]. In this chapter, a Gaussian beam has been used as incident field. The Gaussian beam is widely used as incident field in the rough surface literature [57] -[58]. The spot size at the waist can be changed to select the area illuminated by the incident field.

In Section 4.1, some details on the geometry of the problem is discussed. The decomposition of the total field in each medium is presented in Section 4.2, while the description of each component has been described separately in Section 4.3. Boundary conditions are imposed for both TM and TE polarization in Section 4.4.

4.1 Geometry of the problem

Let us consider N infinite dielectric cylinders of radius $\alpha_{\rm p} = k_0 a_{\rm p}$, and permittivity $\epsilon_{\rm cp} = \epsilon_0 n_{\rm cp}^2$ buried beneath a one dimensional periodic rough surface with Gaussian roughness spectrum as shown in figure 4.1. Both media are linear , isotropic, homo-

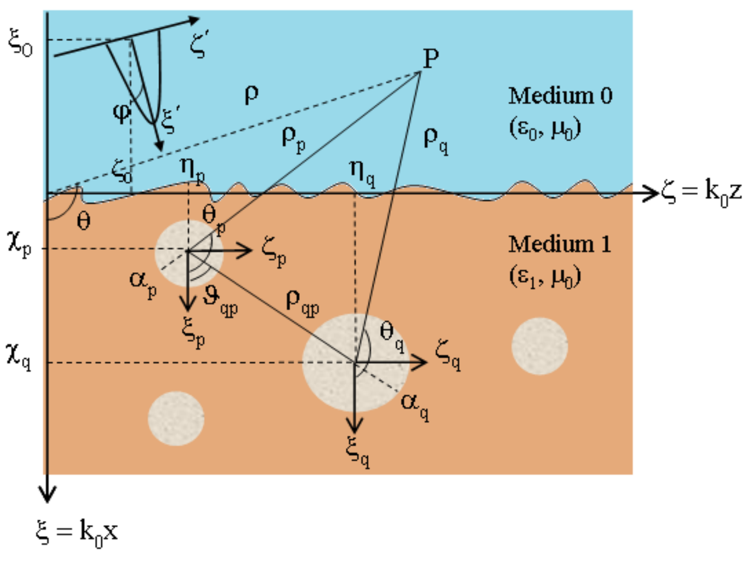


Figure 4.1: Geometry of the problem.

geneous, dielectric and lossless with permittivities ϵ_0 and $\epsilon_1 = \epsilon_0 n_1^2$. We introduce a main reference frame MRF(O,ξ,ζ) with normalized coordinates $\xi = k_0 x$ and $\zeta = k_0 z$, where $k_0 = 2\pi/\lambda_0$ is the the propagation constant in medium 0. Moreover, a reference frame RF_{p} is centered on the p-*th* cylinder axis. Both rectangular (ξ_{p},ζ_{p}) and polar (ρ_{p},θ_{p}) coordinates are considered, with $\xi_{\mathrm{p}} = k_0 x_{\mathrm{p}} = \xi - \chi_{\mathrm{p}}$, $\zeta_{\mathrm{p}} = k_0 z_{\mathrm{p}} = \zeta - \eta_{\mathrm{p}}$, $\rho_{\mathrm{p}} = k_0 r_{\mathrm{p}}$.

4.2 Decomposition of the total field

At any point $P(\xi, \zeta)$, the scalar function $V(\xi, \zeta)$ is a representation for the components of the field parallel to the axis of the cylinder, the sum of all the contributions following the interaction of the incident field with the rough surface and the circular scatterers. In medium 0,

$$V^{(0)} = V_{\mathrm{i}}(\xi, \zeta) + V_{\mathrm{r}}(\xi, \zeta) + V_{\mathrm{st}}(\xi, \zeta)$$

In medium 1,

$$V^{(1)} = V_{\mathrm{t}}(\xi, \zeta) + V_{\mathrm{s}}(\xi, \zeta) + V_{\mathrm{sr}}(\xi, \zeta) + V_{\mathrm{cp}}(\xi, \zeta)$$

For perfectly conducting cylinder, The terms $V_{\mathrm{cp}}(\xi, \zeta)$ becomes zero. In the next Section, every field contribution will be described separately.

4.3 Evaluation of reflected and transmitted fields by rough surface

Using Small Perturbation Method, the reflected and transmitted fields by rough surface without buried objects are calculated in this Section. Since the first order SPM is applied, the reflected and transmitted fields are the sum of zero-*th* and first-order fields representing the contribution relevant to flat interface and rough surface, respectively. Both TM and TE polarization states have been considered.

4.3.1 TM polarization

Consider a one-dimensional periodic rough surface as shown in figure 4.1. Since we are considering TM polarization, the scalar function $V(\xi, \zeta)$ corresponds to the field E_y.

The incident field is a two dimensional Gaussian beam [53], propagating in medium 0, and impinging on the interface, which in MRF with normalized coordinates may be expressed as:

$$V_{\mathrm{i}}(\xi,\zeta)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\hat{V}_0^{\mathrm{i}}(n_\parallel)e^{in_0[n_\perp(\xi-\xi_0)+n_\parallel(\zeta-\zeta_0)]}\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)\,dn_\parallel \tag{4.3.1}$$

where φ is the angle between ζ and ζ' as shown in figure 4.1, and $\hat{V}_0^i(n_\parallel)$ is the angular spectrum of the incident field given by [53]:

$$\hat{V}_0^{\mathrm{i}}(n_\parallel)=V_0\sqrt{\pi}\mathrm{w}_0e^{-[\mathrm{w}_0(n_\parallel\cos\varphi-n_\perp\sin\varphi)]^2}$$

V_0 is the complex amplitude of the incident field and w_0 is the spot size at the waist. By making use of the Rayleigh hypothesis [2], the reflected field in medium 0 can be expressed as:

$$V_{\mathrm{r}}(\xi,\zeta)=\frac{1}{2\pi}V_0\int\limits_{-\infty}^{\infty}V_{\mathbf{r}}(n'_\parallel)e^{in_0(-n'_\perp\xi+n'_\parallel\zeta)}\,dn'_\parallel \tag{4.3.2}$$

where $n'_\perp=\sqrt{1-(n'_\parallel)^2}$

The transmitted field in medium 1 can be written as:

$$V_{\mathrm{t}}(\xi,\zeta)=\frac{1}{2\pi}V_0\int\limits_{-\infty}^{\infty}V_{\mathrm{t}}(n'_\parallel)e^{in_1\left[\sqrt{1-\left(\frac{n_0n'_\parallel}{n_1}\right)^2}\xi+n'_\parallel\zeta\right]}\,dn'_\parallel \tag{4.3.3}$$

The boundary conditions for TM polarization are defined in Section 1.3. Putting equation (4.3.1) together with equations (4.3.2) and (4.3.3) in boundary condition defined in equation (1.3.10) at $\xi=g(\zeta)$, we get

$$\begin{aligned}&\int\limits_{-\infty}^{\infty}\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\Big[1+in_0n_\perp g(\zeta)+...\Big]\,dn_\parallel\\&+\int\limits_{-\infty}^{\infty}e^{in_0n'_\parallel\zeta}\Big[1-in_0n'_\perp g(\zeta)+....\Big]\Big[V_{\mathrm{r}}^0(n'_\parallel)+V_{\mathrm{r}}^1(n'_\parallel)+...\Big]\,dn'_\parallel\\&=\int\limits_{-\infty}^{\infty}e^{in_1n'_\parallel\zeta}\Big[1+in_1\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}g(\zeta)+...\Big]\Big[V_{\mathrm{t}}^0(n'_\parallel)+V_{\mathrm{t}}^1(n'_\parallel)+...\Big]\,dn'_\parallel\end{aligned} \tag{4.3.4}$$

For magnetic field, we impose the boundary condition in equation (1.3.11) at $\xi = g(\zeta)$ on the rough interface

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0 n_\perp \xi_0}e^{in_0 n_\parallel(\zeta-\zeta_0)}(n_0 n_\perp)\Big[1 + in_0 n_\perp g(\zeta) + ...\Big]\,dn_\parallel \\
&- \int_{-\infty}^{\infty} n_0 n'_\perp e^{in_0 n'_\parallel \zeta}\Big[1 - in_0 n'_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n'_\parallel) + V_{\mathrm{r}}^1(n'_\parallel) + ...\Big]\,dn'_\parallel \\
&= \int_{-\infty}^{\infty} \left[n_1\sqrt{1 - \Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}\right] e^{in_1 n'_\parallel \zeta}\Big[1 + in_1\sqrt{1 - \Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}g(\zeta) + ...\Big] \\
&\qquad \times\Big[V_{\mathrm{t}}^0(n'_\parallel) + V_{\mathrm{t}}^1(n'_\parallel) + ...\Big]\,dn'_\parallel
\end{aligned}
\tag{4.3.5}
$$

Balancing the equations (4.3.4) and (4.3.5) to the zero-*th* order gives

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0 n_\perp \xi_0}e^{in_0 n_\parallel(\zeta-\zeta_0)}\,dn_\parallel + \int_{-\infty}^{\infty} e^{in_0 n'_\parallel \zeta}V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel \\
&= \int_{-\infty}^{\infty} e^{in_1 n'_\parallel \zeta}V_{\mathrm{t}}^0(n'_\parallel)\,dn'_\parallel
\end{aligned}
\tag{4.3.6}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty} (n_0 n_\perp)\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0 n_\perp \xi_0}e^{in_0 n_\parallel(\zeta-\zeta_0)}\,dn_\parallel \\
&- \int_{-\infty}^{\infty} (n_0 n'_\perp)e^{in_0 n'_\parallel \zeta}V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel = \int_{-\infty}^{\infty} \left[n_1\sqrt{1 - \Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}\right] e^{in_1 n'_\parallel \zeta}V_{\mathrm{t}}^0(n'_\parallel)\,dn'_\parallel
\end{aligned}
\tag{4.3.7}
$$

Taking the Fourier transform of equation (4.3.6), we obtain

$$
\hat{V}_0^{\mathrm{i}}(n'_\parallel)\Big(\cos\varphi + \frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)e^{-in_0 n'_\perp \xi_0}e^{-in_0 n'_\parallel \zeta_0} + V_{\mathrm{r}}^0(n'_\parallel) = V_{\mathrm{t}}^0(n'_\parallel) \tag{4.3.8}
$$

Putting in equation (4.3.7), we get

$$\int_{-\infty}^{\infty}\left[n_0 n'_\perp - n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}\,\right] e^{-in_0 n'_\perp \xi_0} e^{in_0 n'_\parallel(\zeta-\zeta_0)} \hat{V}_0^{\mathrm{i}}(n'_\parallel)\Big(\cos\varphi + \frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)\,dn'_\parallel$$
$$= \int_{-\infty}^{\infty}\left[n_0 n'_\perp + n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}\,\right] e^{in_0 n'_\parallel \zeta} V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel \tag{4.3.9}$$

$$V_{\mathrm{r}}^0(n'_\parallel) = \left[\frac{n_0 n'_\perp - n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}}{n_0 n'_\perp + n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}}\right] \hat{V}_0^{\mathrm{i}}(n'_\parallel)\Big(\cos\varphi + \frac{n'_\parallel}{n'_\perp}\sin\varphi\Big) e^{-in_0(n'_\perp \xi_0 + n'_\parallel \zeta_0)} \tag{4.3.10}$$

From equation (4.3.8), we have

$$V_{\mathrm{t}}^0(n'_\parallel) = \left[\frac{2n_0 n'_\perp}{n_0 n'_\perp + n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}}\right] \hat{V}_0^{\mathrm{i}}(n'_\parallel)\Big(\cos\varphi + \frac{n'_\parallel}{n'_\perp}\sin\varphi\Big) e^{-in_0(n'_\perp \xi_0 + n'_\parallel \zeta_0)} \tag{4.3.11}$$

Balancing the equations (4.3.4) and (4.3.5) to the first-order leads to

$$i\int_{-\infty}^{\infty} n_0 n_\perp g(\zeta) e^{-in_0 n_\perp \xi_0} e^{in_0 n_\parallel(\zeta-\zeta_0)} \hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)\,dn_\parallel$$
$$-i\int_{-\infty}^{\infty} n_0 n'_\perp g(\zeta) e^{in_0 n'_\parallel \zeta} V_{\mathrm{r}}^0(n'_\parallel) dn'_\parallel + \int_{-\infty}^{\infty} e^{in_0 n'_\parallel \zeta} V_{\mathrm{r}}^1(n'_\parallel) dn'_\parallel \tag{4.3.12}$$
$$= i\int_{-\infty}^{\infty}\left[n_1\sqrt{1-\Big(\frac{n_0 n'_\parallel}{n_1}\Big)^2}\,\right] g(\zeta) e^{in_1 n'_\parallel \zeta} V_{\mathrm{t}}^0(n'_\parallel) dn'_\parallel + \int_{-\infty}^{\infty} e^{in_1 n'_\parallel \zeta} V_{\mathrm{t}}^1(n'_\parallel) dn'_\parallel$$

and

$$i\int_{-\infty}^{\infty}(n_0n_\perp)^2 g(\zeta)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)\,dn_\parallel$$
$$+i\int_{-\infty}^{\infty}(n_0n'_\perp)^2 g(\zeta)e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel-\int_{-\infty}^{\infty}(n_0n'_\perp)e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^1(n'_\parallel)\,dn'_\parallel$$
$$=i\int_{-\infty}^{\infty}\left[n_1^2-(n_0n'_\parallel)^2\right]g(\zeta)e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^0(n'_\parallel)dn'_\parallel+\int_{-\infty}^{\infty}\left[n_1\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^1(n'_\parallel)dn'_\parallel \tag{4.3.13}$$

Putting the values of $V_{\mathrm{r}}^0(n'_\parallel)$ and $V_{\mathrm{t}}^0(n'_\parallel)$ in equation (4.3.12), we have

$$\int_{-\infty}^{\infty}e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^1(n'_\parallel)\,dn'_\parallel=\int_{-\infty}^{\infty}e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^1(n'_\parallel)dn'_\parallel \tag{4.3.14}$$

It follows,

$$V_{\mathrm{r}}^1(n'_\parallel)=V_{\mathrm{t}}^1(n'_\parallel) \tag{4.3.15}$$

Using equation (4.3.15) along with the values of $V_{\mathrm{r}}^0(n'_\parallel)$ and $V_{\mathrm{t}}^0(n'_\parallel)$ in equation (4.3.13), we get

$$i\int_{-\infty}^{\infty}\frac{(n_0^2-n_1^2)2n_0n'_\perp}{\left[n_0n'_\perp+n_1\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]}g(\zeta)e^{-in_0n'_\perp\xi_0}e^{in_0n'_\parallel(\zeta-\zeta_0)}\hat{V}_0^{\mathrm{i}}(n'_\parallel)\Big(\cos\varphi+\frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)\,dn'_\parallel$$
$$=\int_{-\infty}^{\infty}V_{\mathrm{r}}^1(n'_\parallel)e^{in_0n'_\parallel\zeta}\left[n_0n'_\perp+n_1\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]\,dn'_\parallel \tag{4.3.16}$$

$$V_{\mathrm{r}}^{1}(n'_{\|}, n''_{\|}) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{i(n_0^2 - n_1^2) 2 n_0 n''_{\perp} G(n_0 n'_{\|} - n_0 n''_{\|})}{\left[n_0 n''_{\perp} + n_1 \sqrt{1 - \left(\frac{n_0 n''_{\|}}{n_1} \right)^2} \right]} e^{-i n_0 (n''_{\perp} \xi_0 + n''_{\|} \zeta_0)} \times \frac{\hat{V}_0^{\mathrm{i}}(n''_{\|}) \Big(\cos\varphi + \frac{n''_{\|}}{n''_{\perp}} \sin\varphi \Big)}{\left[n_0 \sqrt{1 - (n'_{\|})^2} + n_1 \sqrt{1 - \left(\frac{n_0 n'_{\|}}{n_1} \right)^2} \right]} \, dn''_{\|} \tag{4.3.17}$$

From equation (4.3.15), we get

$$V_{\mathrm{t}}^{1}(n'_{\|}, n''_{\|}) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{i(n_0^2 - n_1^2) 2 n_0 n''_{\perp} G(n_0 n'_{\|} - n_0 n''_{\|})}{\left[n_0 n''_{\perp} + n_1 \sqrt{1 - \left(\frac{n_0 n''_{\|}}{n_1} \right)^2} \right]} e^{-i n_0 (n''_{\perp} \xi_0 + n''_{\|} \zeta_0)} \times \frac{\hat{V}_0^{\mathrm{i}}(n''_{\|}) \Big(\cos\varphi + \frac{n''_{\|}}{n''_{\perp}} \sin\varphi \Big)}{\left[n_0 \sqrt{1 - (n'_{\|})^2} + n_1 \sqrt{1 - \left(\frac{n_0 n'_{\|}}{n_1} \right)^2} \right]} \, dn''_{\|} \tag{4.3.18}$$

Putting equations (4.3.17) and (4.3.10) in equation (4.3.2) and after a change of variables, the reflected field can be written as

$$V_{\mathrm{r}}(\xi, \zeta) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \Gamma_{01}^{\mathrm{TM}}(n_{\|}) \hat{V}_0^{\mathrm{i}}(n_{\|}) e^{i n_0 (-n_{\perp}(\xi + \xi_0) + n_{\|}(\zeta - \zeta_0))} \Big(\cos\varphi + \frac{n_{\|}}{n_{\perp}} \sin\varphi \Big) \, dn_{\|} + \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \gamma_{01}^{\mathrm{TM}}(n'_{\|}, n_{\|}) \hat{V}_0^{i}(n_{\|}) e^{-i n_0 (n_{\perp} \xi_0 + n_{\|} \zeta_0)} e^{i n_0 (-n'_{\perp} \xi + n'_{\|} \zeta)} \Big(\cos\varphi + \frac{n_{\|}}{n_{\perp}} \sin\varphi \Big) \, dn_{\|} \, dn'_{\|} \tag{4.3.19}$$

where

$$\Gamma_{01}^{\mathrm{TM}}(n_{\|}) = \left[\frac{n_0 n_{\perp} - n_1 \sqrt{1 - \left(\frac{n_0 n_{\|}}{n_1} \right)^2}}{n_0 n_{\perp} + n_1 \sqrt{1 - \left(\frac{n_0 n_{\|}}{n_1} \right)^2}} \right] \tag{4.3.20}$$

and

$$\gamma_{01}^{\mathrm{TM}}(n'_{\|}, n_{\|}) = \frac{i(n_0^2 - n_1^2)2n_0 n_\perp G(n_0 n'_{\|} - n_0 n_{\|})}{\left[n_0 n_\perp + n_1\sqrt{1 - \left(\frac{n_0 n_{\|}}{n_1}\right)^2}\,\right]} \times \frac{1}{\left[n_0\sqrt{1-(n'_{\|})^2} + n_1\sqrt{1-\left(\frac{n_0 n'_{\|}}{n_1}\right)^2}\right]} \tag{4.3.21}$$

Putting equations (4.3.18) and (4.3.11) in equation (4.3.3), the transmitted field V_t can be defined

$$\begin{aligned} V_\mathrm{t}(\xi,\zeta) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} & T_{01}^{\mathrm{TM}}(n_{\|})\hat{V}_0^{\mathrm{i}}(n_{\|})e^{-in_0(n_\perp\xi_0 + n_{\|}\zeta_0)}e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\xi + n_{\|}\zeta\right]} \\ & \times\left(\cos\varphi + \frac{n_{\|}}{n_\perp}\sin\varphi\right)dn_{\|} \\ + \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} & \tau_{01}^{\mathrm{TM}}(n'_{\|}, n_{\|})\hat{V}_0^{\mathrm{i}}(n_{\|})e^{-in_0(n_\perp\xi_0 + n_{\|}\zeta_0)}e^{in_1\left[\sqrt{1-\left(\frac{n_0 n'_{\|}}{n_1}\right)^2}\xi + n'_{\|}\zeta\right]} \\ & \times\left(\cos\varphi + \frac{n_{\|}}{n_\perp}\sin\varphi\right)\, dn_{\|}\, dn'_{\|} \end{aligned} \tag{4.3.22}$$

where

$$T_{01}^{\mathrm{T}M}(n_{\|}) = \left[\frac{2n_0 n_\perp}{n_0 n_\perp + n_1\sqrt{1 - \left(\frac{n_0 n_{\|}}{n_1}\right)^2}}\right] \tag{4.3.23}$$

and

$$\tau_{01}^{\mathrm{TM}}(n'_{\|}, n_{\|}) \equiv \gamma_{01}^{\mathrm{TM}}(n'_{\|}, n_{\|}) \tag{4.3.24}$$

The transmitted field V_t can be written in polar coordinates as follows

$$V_\mathrm{t}(\xi_\mathrm{p}, \zeta_\mathrm{p}) = \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell(n_1\rho_\mathrm{p})e^{i\ell\theta_\mathrm{p}}\left[A_\mathrm{t}^{\mathrm{TM}} + B_\mathrm{t}^{\mathrm{TM}}\right] \tag{4.3.25}$$

where

$$A_{\mathrm{t}}^{\mathrm{TM}} = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} T_{01}^{\mathrm{TM}}(n_\parallel) \hat{V}_0^i(n_\parallel) e^{-in_0(n_\perp \xi_0 + n_\parallel \zeta_0)} e^{-i\ell\phi_{\mathrm{t}}^A} e^{in_1 \left[\sqrt{1-\left(\frac{n_0 n_\parallel}{n_1}\right)^2} \chi_{\mathrm{p}} + n_\parallel \eta_{\mathrm{p}}\right]} \times \left(\cos\varphi + \frac{n_\parallel}{n_\perp} \sin\varphi \right) dn_\parallel \tag{4.3.26}$$

with

$$\phi_{\mathrm{t}}^A = \tan^{-1} \left[\frac{n_\parallel}{\sqrt{1 - \left(\frac{n_0 n_\parallel}{n_1}\right)^2}} \right] \tag{4.3.27}$$

and

$$B_{\mathrm{t}}^{\mathrm{TM}} = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \int\limits_{-\infty}^{\infty} \tau_{01}^{\mathrm{TM}}(n'_\parallel, n_\parallel) \hat{V}_0^i(n_\parallel) e^{-in_0(n_\perp \xi_0 + n_\parallel \zeta_0)} e^{-i\ell\phi_{\mathrm{t}}^B} e^{in_1 \left[\sqrt{1-\left(\frac{n_0 n'_\parallel}{n_1}\right)^2} \chi_{\mathrm{p}} + n'_\parallel \eta_{\mathrm{p}}\right]} \times \left(\cos\varphi + \frac{n_\parallel}{n_\perp} \sin\varphi \right) dn_\parallel \, dn'_\parallel \tag{4.3.28}$$

being

$$\phi_{\mathrm{t}}^B = \tan^{-1} \left[\frac{n'_\parallel}{\sqrt{1 - \left(\frac{n_0 n'_\parallel}{n_1}\right)^2}} \right] \tag{4.3.29}$$

where $T_{01}^{\mathrm{TM}}(n_\parallel)$ and $\tau_{01}^{\mathrm{TM}}(n'_\parallel, n_\parallel)$ are coefficients for unperturbed and perturbed transmitted fields, respectively.

4.3.2 TE Polarization

In this case, the scalar function $V_{\mathrm{i}}(\xi, \zeta)$ corresponds to the magnetic field H_y. Putting equations (4.3.1), (4.3.2) and (4.3.3) in boundary condition (1.3.44) at $\xi = g(\zeta)$

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\Big[1 + in_0n_\perp g(\zeta) + ...\Big]\,dn_\parallel \\
&+\int_{-\infty}^{\infty} e^{in_0n'_\parallel\zeta}\Big[1 - in_0n'_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n'_\parallel) + V_{\mathrm{r}}^1(n'_\parallel) + ...\Big]\,dn'_\parallel \\
&= \int_{-\infty}^{\infty} e^{in_1n'_\parallel\zeta}\Big[1 + in_1\sqrt{1 - \Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}g(\zeta) + ...\Big]\Big[V_{\mathrm{t}}^0(n'_\parallel) + V_{\mathrm{t}}^1(n'_\parallel) + ...\Big]dn'_\parallel
\end{aligned}
\tag{4.3.30}
$$

From equation(1.3.45), we have the following boundary condition for E-field on the rough interface

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \Big(\frac{n_\perp}{n_0}\Big)\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\Big[1 + in_0n_\perp g(\zeta) + ...\Big]\,dn_\parallel \\
&-\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty} \Big(\frac{n_\parallel}{n_0}\Big)\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi + \frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\Big[1 + in_0n_\perp g(\zeta) + ...\Big]\,dn_\parallel \\
&-\int_{-\infty}^{\infty} \Big(\frac{n'_\perp}{n_0}\Big)e^{in_0n'_\parallel\zeta}\Big[1 - in_0n'_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n'_\parallel) + V_{\mathrm{r}}^1(n'_\parallel) + ...\Big]\,dn'_\parallel \\
&-\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty} \Big(\frac{n'_\parallel}{n_0}\Big)e^{in_0n'_\parallel\zeta}\Big[1 - in_0n'_\perp g(\zeta) + ...\Big]\Big[V_{\mathrm{r}}^0(n'_\parallel) + V_{\mathrm{r}}^1(n'_\parallel) + ...\Big]\,dn'_\parallel
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left[\frac{1}{n_1} \sqrt{1 - \left(\frac{n_0 n'_\parallel}{n_1} \right)^2} \right] e^{i n_1 n'_\parallel \zeta} \left[1 + i n_1 \sqrt{1 - \left(\frac{n_0 n'_\parallel}{n_1} \right)^2} g(\zeta) + ... \right] \\
&\qquad \times \left[V_{\mathrm{t}}^0(n'_\parallel) + V_{\mathrm{t}}^1(n'_\parallel) + ... \right] dn'_\parallel \\
&- \frac{\partial g(\zeta)}{\partial \zeta} \int_{-\infty}^{\infty} \left(\frac{n'_\parallel}{n_1} \right) e^{i n'_1 n_\parallel \zeta} \left[1 + i n_1 \sqrt{1 - \left(\frac{n_0 n'_\parallel}{n_1} \right)^2} g(\zeta) + ... \right] \\
&\qquad\qquad \times \left[V_{\mathrm{t}}^0(n'_\parallel) + V_{\mathrm{t}}^1(n'_\parallel) + ... \right] dn'_\parallel
\end{aligned}
\tag{4.3.31}
$$

Balancing equations (4.3.30) and (4.3.31) to the zero-*th* order, we get

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \hat{V}_0^{\mathrm{i}}(n_\parallel) \left(\cos\varphi + \frac{n_\parallel}{n_\perp} \sin\varphi \right) e^{-i n_0 n_\perp \xi_0} e^{i n_0 n_\parallel (\zeta - \zeta_0)} \, dn_\parallel + \int_{-\infty}^{\infty} e^{i n_0 n'_\parallel \zeta} V_{\mathrm{r}}^0(n'_\parallel) \, dn'_\parallel \\
&= \int_{-\infty}^{\infty} e^{i n_1 n'_\parallel \zeta} V_{\mathrm{t}}^0(n'_\parallel) \, dn'_\parallel
\end{aligned}
\tag{4.3.32}
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{n_\perp}{n_0} \right) \hat{V}_0^{\mathrm{i}}(n_\parallel) \left(\cos\varphi + \frac{n_\parallel}{n_\perp} \sin\varphi \right) e^{-i n_0 n_\perp \xi_0} e^{i n_0 n_\parallel (\zeta - \zeta_0)} dn_\parallel - \int_{-\infty}^{\infty} \left(\frac{n'_\perp}{n_0} \right) e^{i n_0 n'_\parallel \zeta} V_{\mathrm{r}}^0(n'_\parallel) dn'_\parallel \\
&= \int_{-\infty}^{\infty} \left[\frac{1}{n_1} \sqrt{1 - \left(\frac{n_0 n'_\parallel}{n_1} \right)^2} \right] e^{i n_1 n'_\parallel \zeta} V_{\mathrm{t}}^0(n'_\parallel) dn'_\parallel
\end{aligned}
\tag{4.3.33}
$$

Taking the Fourier transform of equation (4.3.32)

$$
\hat{V}_0^{\mathrm{i}}(n'_\parallel) \left(\cos\varphi + \frac{n'_\parallel}{n'_\perp} \sin\varphi \right) e^{-i n_0 n'_\perp \xi_0} e^{-i n_0 n'_\parallel \zeta_0} + V_{\mathrm{r}}^0(n'_\parallel) = V_{\mathrm{t}}^0(n'_\parallel) \tag{4.3.34}
$$

Putting in equation (4.3.33), we get

$$\left[\frac{n'_\perp}{n_0}-\frac{1}{n_1}\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]\hat{V}_0^i(n'_\parallel)\Big(\cos\varphi+\frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)e^{-in_0n'_\perp\xi_0}e^{-in_0n'_\parallel\zeta_0}$$
$$=\int\limits_{-\infty}^{\infty}\left[\frac{n'_\perp}{n_0}+\frac{1}{n_1}\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]e^{in_0n'_\parallel\zeta}V_\mathrm{r}^0(n'_\parallel)\,dn'_\parallel \tag{4.3.35}$$

$$V_\mathrm{r}^0(n'_\parallel)=\left[\frac{n_1n'_\perp-n_0\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}}{n_1n'_\perp+n_0\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}}\right]\hat{V}_0^\mathrm{i}(n'_\parallel)\Big(\cos\varphi+\frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)e^{-in_0(n'_\perp\xi_0+n'_\parallel\zeta_0)} \tag{4.3.36}$$

From equation (4.3.34), we have

$$V_\mathrm{t}^0(n'_\parallel)=\left[\frac{2n_1n'_\perp}{n_1n'_\perp+n_0\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}}\right]\hat{V}_0^\mathrm{i}(n'_\parallel)\Big(\cos\varphi+\frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)e^{-in_0(n'_\perp\xi_0+n'_\parallel\zeta_0)} \tag{4.3.37}$$

Balancing equations (4.3.30) and (4.3.31) to the first-order gives

$$i\int\limits_{-\infty}^{\infty}(n_0n_\perp)\hat{V}_0^\mathrm{i}(n_\parallel)\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\,dn_\parallel$$
$$-i\int\limits_{-\infty}^{\infty}(n_0n'_\perp)g(\zeta)e^{in_0n'_\parallel\zeta}V_\mathrm{r}^0(n'_\parallel)\,dn'_\parallel+\int\limits_{-\infty}^{\infty}e^{in_0n'_\parallel\zeta}V_\mathrm{r}^1(n'_\parallel)\,dn'_\parallel \tag{4.3.38}$$
$$=i\int\limits_{-\infty}^{\infty}\left[n_1\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]g(\zeta)e^{in_1n'_\parallel\zeta}V_\mathrm{t}^0(n'_\parallel)\,dn'_\parallel+\int\limits_{-\infty}^{\infty}e^{in_1n'_\parallel\zeta}V_\mathrm{t}^1(n'_\parallel)\,dn'_\parallel$$

$$
\begin{aligned}
& i\int_{-\infty}^{\infty}\Big(n_\perp\Big)^2\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\,dn_\parallel \\
& -\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty}\Big(\frac{n_\parallel}{n_0}\Big)\hat{V}_0^{\mathrm{i}}(n_\parallel)\Big(\cos\varphi+\frac{n_\parallel}{n_\perp}\sin\varphi\Big)e^{-in_0n_\perp\xi_0}e^{in_0n_\parallel(\zeta-\zeta_0)}\,dn_\parallel \\
& +i\int_{-\infty}^{\infty}\Big(n'_\perp\Big)^2 g(\zeta)e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel-\int_{-\infty}^{\infty}\Big(\frac{n'_\perp}{n_0}\Big)e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^1(n'_\parallel)\,dn'_\parallel \\
& -\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty}\Big(\frac{n'_\parallel}{n_0}\Big)e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^0(n'_\parallel)\,dn'_\parallel=\int_{-\infty}^{\infty}\left[\frac{1}{n_1}\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^1(n'_\parallel)\,dn'_\parallel \\
& -\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty}\Big(\frac{n'_\parallel}{n_1}\Big)e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^0(n'_\parallel)\,dn'_\parallel+i\int_{-\infty}^{\infty}\left[\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]^2 g(\zeta)e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^0(n'_\parallel)\,dn'_\parallel
\end{aligned}
\tag{4.3.39}
$$

Putting the values of $V_{\mathrm{r}}^0(n'_\parallel)$ and $V_{\mathrm{t}}^0(n'_\parallel)$ in equation (4.3.38), we have

$$
\begin{aligned}
& \int_{-\infty}^{\infty}e^{in_1n'_\parallel\zeta}V_{\mathrm{t}}^1(n'_\parallel)\,dn'_\parallel-\int_{-\infty}^{\infty}e^{in_0n'_\parallel\zeta}V_{\mathrm{r}}^1(n'_\parallel)\,dn'_\parallel \\
& =\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n'_\perp\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}}{\left[n_1n'_\perp+n_0\sqrt{1-\Big(\frac{n_0n'_\parallel}{n_1}\Big)^2}\,\right]}g(\zeta)\hat{V}_0^{i}(n'_\parallel)\Big(\cos\varphi+\frac{n'_\parallel}{n'_\perp}\sin\varphi\Big)e^{-in_0n'_\perp\xi_0}e^{in_0n'_\parallel(\zeta-\zeta_0)}dn'_\parallel
\end{aligned}
\tag{4.3.40}
$$

Taking Fourier transform, we get

$$
\begin{aligned}
V_{\mathrm{t}}^1(n'_\parallel,n''_\parallel)-V_{\mathrm{r}}^1(n'_\parallel)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n''_\perp\sqrt{1-\Big(\frac{n_0n''_\parallel}{n_1}\Big)^2}}{\left[n_1n''_\perp+n_0\sqrt{1-\Big(\frac{n_0n''_\parallel}{n_1}\Big)^2}\,\right]}G(n_0n'_\parallel-n_0n''_\parallel) \\
\times\hat{V}_0^{\mathrm{i}}(n''_\parallel)\Big(\cos\varphi+\frac{n''_\parallel}{n''_\perp}\sin\varphi\Big)e^{-in_0n''_\perp\xi_0}e^{-in_0n''_\parallel\zeta_0}\,dn''_\parallel
\end{aligned}
\tag{4.3.41}
$$

Putting the values of $V_{\mathrm{r}}^{0}(n'_{\parallel})$ and $V_{\mathrm{t}}^{0}(n'_{\parallel})$ in equation (4.3.39), we get

$$\int_{-\infty}^{\infty}\left[n_0\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1}\Big)^2}\right]e^{in_1 n'_{\parallel}\zeta}V_{\mathrm{t}}^{1}(n'_{\parallel})\,dn'_{\parallel}+\int_{-\infty}^{\infty}\Big(n_1 n'_{\perp}\Big)e^{in_0 n'_{\parallel}\zeta}V_{\mathrm{r}}^{1}(n'_{\parallel})\,dn'_{\parallel}$$

$$=\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n_0 n'_{\perp}{n'_{\parallel}}^2 g(\zeta)}{\left[n_1 n'_{\perp}+n_0\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1}\Big)^2}\right]}\hat{V}_0^{\mathrm{i}}(n'_{\parallel})\Big(\cos\varphi+\frac{n'_{\parallel}}{n'_{\perp}}\sin\varphi\Big)e^{-in_0 n'_{\perp}\xi_0}e^{in_0 n'_{\parallel}(\zeta-\zeta_0)}\,dn'_{\parallel}$$

$$+\frac{\partial g(\zeta)}{\partial\zeta}\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n_{\perp}n'_{\parallel}}{\left[n_1 n'_{\perp}+n_0\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1}\Big)^2}\right]}\hat{V}_0^{\mathrm{i}}(n'_{\parallel})\Big(\cos\varphi+\frac{n'_{\parallel}}{n'_{\perp}}\sin\varphi\Big)e^{-in_0 n'_{\perp}\xi_0}e^{in_0 n'_{\parallel}(\zeta-\zeta_0)}\,dn'_{\parallel} \tag{4.3.42}$$

Taking Fourier transform, we get

$$\left[n_0\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1}\Big)^2}\right]V_{\mathrm{t}}^{1}(n'_{\parallel})+\Big(n_1 n'_{\perp}\Big)V_{\mathrm{r}}^{1}(n'_{\parallel})$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n_0 n''_{\perp}n''_{\parallel}n''_{\parallel}}{\left[n_1 n''_{\perp}+n_0\sqrt{1-\Big(\frac{n_0 n''_{\parallel}}{n_1}\Big)^2}\right]}G(n_0 n'_{\parallel}-n_0 n''_{\parallel}) \tag{4.3.43}$$

$$\times\hat{V}_0^{i}(n''_{\parallel})\Big(\cos\varphi+\tfrac{n''_{\parallel}}{n''_{\perp}}\sin\varphi\Big)e^{-in_0 n''_{\perp}\xi_0}e^{-in_0 n''_{\parallel}\zeta_0}\,dn''_{\parallel}$$

Solving equations (4.3.41) and (4.3.43), we obtain

$$V_{\mathrm{r}}^{1}(n'_{\parallel},n''_{\parallel})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i(n_0^2-n_1^2)2n_0 n''_{\perp}}{\left[n_1 n''_{\perp}+n_0\sqrt{1-\Big(\frac{n_0 n''_{\parallel}}{n_1}\Big)^2}\right]}G(n_0 n'_{\parallel}-n_0 n''_{\parallel})$$

$$\times\frac{\left[n''_{\parallel}n'_{\parallel}-\sqrt{1-\Big(\frac{n_0 n''_{\parallel}}{n_1}\Big)^2}\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1^2}\Big)^2}\right]}{\left[n_1\sqrt{1-(n'_{\parallel})^2}+n_0\sqrt{1-\Big(\frac{n_0 n'_{\parallel}}{n_1}\Big)^2}\right]}\hat{V}_0^{\mathrm{i}}(n''_{\parallel})\Big(\cos\varphi+\frac{n''_{\parallel}}{n''_{\perp}}\sin\varphi\Big)e^{-in_0(n''_{\perp}\xi_0+n''_{\parallel}\zeta_0)}\,dn''_{\parallel} \tag{4.3.44}$$

$$V_{\rm t}^{1}(n'_{\parallel}) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{i(n_0^2-n_1^2)2n''_{\perp}}{\left[n_1 n''_{\perp} + n_0\sqrt{1-\left(\frac{n_0 n''_{\parallel}}{n_1}\right)^2}\right]} G(n_0 n'_{\parallel} - n_0 n''_{\parallel})$$

$$\times\frac{\left[n_0 n''_{\parallel} n'_{\parallel} + n_1\sqrt{1-\left(\frac{n_0 n''_{\parallel}}{n_1}\right)^2}\sqrt{1-(n'_{\parallel})^2}\right]}{\left[n_1\sqrt{1-(n'_{\parallel})^2} + n_0\sqrt{1-\left(\frac{n_0 n'_{\parallel}}{n_1}\right)^2}\right]} \hat{V}_0^{i}(n''_{\parallel})\Big(\cos\varphi + \frac{n''_{\parallel}}{n''_{\perp}}\sin\varphi\Big)e^{-in_0(n''_{\perp}\xi_0 + n''_{\parallel}\zeta_0)}\,dn''_{\parallel} \tag{4.3.45}$$

Putting equations (4.3.44) and (4.3.36) in equation (4.3.2), the reflected field can be written as

$$V_{\rm r}(\xi,\zeta) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \Gamma_{01}^{\rm TE}(n_{\parallel})\hat{V}_0^{\rm i}(n_{\parallel})e^{in_0[-n_{\perp}(\xi+\xi_0)+n_{\parallel}(\zeta-\zeta_0)]}\Big(\cos\varphi + \frac{n_{\parallel}}{n_{\perp}}\sin\varphi\Big)\,dn_{\parallel}$$

$$+\frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \gamma_{01}^{\rm TE}(n'_{\parallel}, n_{\parallel})\hat{V}_0^{i}(n_{\parallel})e^{-in_0(n_{\perp}\xi_0 + n_{\parallel}\zeta_0)}e^{i(-n'_{\perp}\xi + n'_{\parallel}\zeta)}\Big(\cos\varphi + \frac{n_{\parallel}}{n_{\perp}}\sin\varphi\Big)\,dn_{\parallel}\,dn'_{\parallel} \tag{4.3.46}$$

$$\Gamma_{01}^{\rm TE}(n_{\parallel}) = \left[\frac{n_1 n_{\perp} - n_1\sqrt{1-\left(\frac{n_1 n_{\parallel}}{n_1}\right)^2}}{n_1 n_{\perp} + n_0\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}}\right] \tag{4.3.47}$$

and

$$\gamma_{01}^{\rm TE}(n_{\parallel}) = \frac{i(n_0^2-n_1^2)2n_0 n_{\perp}}{\left[n_1 n_{\perp} + n_0\sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\right]} G(n_0 n'_{\parallel} - n_0 n_{\parallel})$$

$$\times\frac{\left[n_{\parallel} n'_{\parallel} - \sqrt{1-\left(\frac{n_0 n_{\parallel}}{n_1}\right)^2}\sqrt{1-\left(\frac{n_0 n'_{\parallel}}{n_1^2}\right)^2}\right]}{\left[n_1\sqrt{1-(n'_{\parallel})^2} + n_0\sqrt{1-\left(\frac{n_0 n'_{\parallel}}{n_1}\right)^2}\right]} \tag{4.3.48}$$

Putting equations (4.3.45) and (4.3.37) in equation (4.3.3), the transmitted field V_{t} can be defined

$$
\begin{aligned}
V_{\mathrm{t}}(\xi,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} & T_{01}^{\mathrm{TE}}(n_{\parallel})\hat{V}_0^{\mathrm{i}}(n_{\parallel})e^{-in_0(n_{\perp}\xi_0+n_{\parallel}\zeta_0)}e^{in_1\left[\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}\xi+n_{\parallel}\zeta\right]} \\
& \times\left(\cos\varphi+\frac{n_{\parallel}}{n_{\perp}}\sin\varphi\right)dn_{\parallel} \\
+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & \tau_{01}^{\mathrm{TE}}(n_{\parallel}',n_{\parallel})\hat{V}_0^{\mathrm{i}}(n_{\parallel})e^{-in_0(n_{\perp}\xi_0+n_{\parallel}\zeta_0)}e^{in_1\left[\sqrt{1-\left(\frac{n_0n_{\parallel}'}{n_1}\right)^2}\xi+n_{\parallel}'\zeta\right]} \\
& \times\left(\cos\varphi+\frac{n_{\parallel}}{n_{\perp}}\sin\varphi\right)dn_{\parallel}\,dn_{\parallel}'
\end{aligned}
\tag{4.3.49}
$$

where

$$
T_{01}^{TE}(n_{\parallel}) = \left[\frac{2n_0n_{\perp}}{n_0n_{\perp}+n_1\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}}\right] \tag{4.3.50}
$$

and

$$
\begin{aligned}
\tau_{01}^{\mathrm{TE}}(n_{\parallel}',n_{\parallel}) = & \frac{i(n_0^2-n_1^2)2n_{\perp}}{\left[n_1n_{\perp}+n_0\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}\right]}G(n_0n_{\parallel}'-n_0n_{\parallel}) \\
& \times\frac{\left[n_0n_{\parallel}n_{\parallel}'+n_1\sqrt{1-\left(\frac{n_0n_{\parallel}}{n_1}\right)^2}\sqrt{1-(n_{\parallel}')^2}\right]}{\left[n_1\sqrt{1-(n_{\parallel}')^2}+n_0\sqrt{1-\left(\frac{n_0n_{\parallel}'}{n_1}\right)^2}\right]}
\end{aligned}
\tag{4.3.51}
$$

The transmitted field V_{t} can be written in polar coordinates as follows

$$
V_{\mathrm{t}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}}) = \sum_{\ell=-\infty}^{\infty} i^{\ell}J_{\ell}(n_1\rho_{\mathrm{p}})e^{i\ell\theta_{\mathrm{p}}}\left[A_{\mathrm{t}}^{\mathrm{TE}}+B_{\mathrm{t}}^{\mathrm{TE}}\right] \tag{4.3.52}
$$

where

$$A_{\mathrm{t}}^{\mathrm{TE}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} T_{01}^{\mathrm{TE}}(n_{\|}) \hat{V}_0^i(n_{\|}) e^{-in_0(n_\perp \xi_0 + n_{\|}\zeta_0)} e^{-i\ell\phi_{\mathrm{t}}^A} e^{in_1\left[\sqrt{1-\left(\frac{n_0 n_{\|}}{n_1}\right)^2}\chi_{\mathrm{p}} + n_{\|}\eta_{\mathrm{p}}\right]} \times \left(\cos\varphi + \frac{n_{\|}}{n_\perp}\sin\varphi\right) dn_{\|} \tag{4.3.53}$$

with

$$\phi_{\mathrm{t}}^A = \tan^{-1}\left[\frac{n_{\|}}{\sqrt{1-\left(\dfrac{n_0 n_{\|}}{n_1}\right)^2}}\right] \tag{4.3.54}$$

and

$$B_{\mathrm{t}}^{\mathrm{TE}} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau_{01}^{\mathrm{TE}}(n'_{\|}, n_{\|}) \hat{V}_0^i(n_{\|}) e^{-in_0(n_\perp \xi_0 + n_{\|}\zeta_0)} e^{-i\ell\phi_{\mathrm{t}}^B} e^{in_1\left[\sqrt{1-\left(\frac{n_0 n'_{\|}}{n_1}\right)^2}\chi_{\mathrm{p}} + n'_{\|}\eta_{\mathrm{p}}\right]} \times \left(\cos\varphi + \frac{n_{\|}}{n_\perp}\sin\varphi\right) dn_{\|} dn'_{\|} \tag{4.3.55}$$

being

$$\phi_{\mathrm{t}}^B = \tan^{-1}\left[\frac{n'_{\|}}{\sqrt{1-\left(\dfrac{n_0 n'_{\|}}{n_1}\right)^2}}\right] \tag{4.3.56}$$

where $T_{01}^{\mathrm{TE}}(n_{\|})$ and $\tau_{01}^{\mathrm{TE}}(n'_{\|}, n_{\|})$ are coefficients for unperturbed and perturbed transmitted fields, respectively.

4.3.3 Scattered fields

The explicit expressions of the scattered field $V_{\mathrm{s}}(\xi,\zeta)$, scattered-reflected field $V_{\mathrm{sr}}(\xi,\zeta)$ and scattered-transmitted field $V_{\mathrm{st}}(\xi,\zeta)$ have been given in the Section 1.4 of chapter 1. As regards the case of dielectric scatterers, the field V_{cp} transmitted inside the p-*th* cylinder must be considered. The expression for $V_{\mathrm{cp}}(\xi_{\mathrm{p}},\zeta_{\mathrm{p}})$ is also given there.

4.4 Boundary conditions

Once the expressions of all the fields have been given, the boundary conditions on the cylinders surfaces have to be imposed to determine the unknown coefficients c_{qm}. In this section, a linear system of equations is obtained for dielectric and perfectly conducting cylinders.

4.4.1 Perfectly conducting cylinders

The following equation has to be written for TM polarization:

$$V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}}\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = 0 \tag{4.4.1}$$

In the TE case, using Maxwells equations we have

$$\frac{\partial}{\partial\rho_{\mathrm{p}}}(V_{\mathrm{t}} + V_{\mathrm{s}} + V_{\mathrm{sr}})\Big|_{\rho_{\mathrm{p}}=\alpha_{\mathrm{p}}} = 0 \tag{4.4.2}$$

The boundary conditions are the same that defined in the Section 1.5 of chapter 1. Here, the only change is the expression of transmitted field $V_{\mathrm{t}}(\xi_{\mathrm{p}}, \zeta_{\mathrm{p}})$, i.e A_{t} and ΔB_{t}. Following the similar steps, we have

$$\sum_{\mathrm{q}=1}^{N}\sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp(TM,TE)}} c_{\mathrm{qm}} = B_{\ell}^{\mathrm{p(TM,TE)}} \tag{4.4.3}$$

with

$$\begin{aligned}
A_{\ell\mathrm{m}}^{\mathrm{qp(TM,TE)}} &= i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_{\mathrm{t}}} G_{\ell}^{\mathrm{(TM,TE)}}(n_1\alpha_{\mathrm{p}}) \\
&\times\Big\{\Big[CW_{\mathrm{m}-\ell}(n_1\xi_{\mathrm{qp}}, n_1\zeta_{\mathrm{qp}})(1-\delta_{\mathrm{qp}}) + \frac{\delta_{\mathrm{qp}}\delta_{\ell\mathrm{m}}}{G_{\ell}^{\mathrm{(TM,TE)}}(n_1\alpha_{\mathrm{p}})}\Big] \\
&+\Big[RW_{\mathrm{m}+\ell}^{un}[-n_1(\chi_{\mathrm{p}}+\chi_{\mathrm{q}}), n_1(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})] + RW_{\mathrm{m},\ell}^{per}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})\Big]\Big\}
\end{aligned} \tag{4.4.4}$$

and

$$B_{\ell}^{\mathrm{(TM,TE)}} = -G_{\ell}^{\mathrm{(TM,TE)}}(n_1\alpha_{\mathrm{p}})\left[A_{\mathrm{t}}^{\mathrm{(TM,TE)}} + B_{\mathrm{t}}^{\mathrm{(TM,TE)}}\right] \tag{4.4.5}$$

being $G_\ell^{(\mathrm{TM})}(\cdot) = J_\ell(.)/H_\ell(.)$, and $G_\ell^{(\mathrm{TE})}(\cdot) = J'_\ell(.)/H'_\ell(.)$.

4.4.2 Dielectric cylinders

The boundary conditions for a dielectric cylinder have also been defined in the Section 1.5 of chapter 1. In particular, we have

$$\begin{aligned} V_\mathrm{t} + V_\mathrm{s} + V_\mathrm{sr}|_{\rho_\mathrm{p}=\alpha_\mathrm{p}} &= V_\mathrm{cp}|_{\rho_\mathrm{p}=\alpha_\mathrm{p}} \\ \frac{\partial}{\partial\rho_\mathrm{p}}(V_\mathrm{t} + V_\mathrm{s} + V_\mathrm{sr})\Big|_{\rho_\mathrm{p}=\alpha_\mathrm{p}} &= t_\mathrm{p}\frac{\partial V_\mathrm{cp}}{\partial\rho_\mathrm{p}}\Big|_{\rho_\mathrm{p}=\alpha_\mathrm{p}} \end{aligned} \tag{4.4.6}$$

In equation (4.4.6), $t_\mathrm{p} = 1$ or $(n_1/n_\mathrm{cp})^2$ for TM or TE polarization, respectively. Introducing the values of V_t, V_s, V_sr and V_cp, a linear system in the unknown coefficients c_qm and $d_{\mathrm{p}\ell}$ can be derived

$$\begin{gathered} \sum_{\mathrm{q}=1}^{N}\sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}(1)} c_\mathrm{qm} - B_\ell^{\mathrm{p}(1)} = d_{\mathrm{p}\ell} L_\ell^{\mathrm{p}(1)} \\ \sum_{\mathrm{q}=1}^{N}\sum_{\mathrm{m}=-\infty}^{\infty} A_{\ell\mathrm{m}}^{\mathrm{qp}(2)} c_\mathrm{qm} - B_\ell^{\mathrm{p}(2)} = d_{\mathrm{p}\ell} L_\ell^{\mathrm{p}(2)} \\ \ell = 0; \pm1; \pm2... \\ \mathrm{p} = 1, ..., N. \end{gathered} \tag{4.4.7}$$

where

$$\begin{aligned} A_{\ell\mathrm{m}}^{\mathrm{qp}(1,2)} = i^{\mathrm{m}-\ell} e^{-i\mathrm{m}\varphi_\mathrm{t}} G_\ell^{(1,2)}(n_1\alpha_\mathrm{p}) \Big\{ \Big[CW_{\mathrm{m}-\ell}(n_1\xi_\mathrm{qp}, n_1\zeta_\mathrm{qp})(1-\delta_\mathrm{qp}) + \frac{\delta_\mathrm{qp}\delta_{\mathrm{m}\ell}}{G_\ell^{(1,2)}(n_1\alpha_\mathrm{p})} \Big] \\ + \Big[RW_{\mathrm{m}+\ell}^{un}[-n_1(\chi_\mathrm{p}+\chi_\mathrm{q}), n_1(\eta_\mathrm{p}-\eta_\mathrm{q})] + RW_{\mathrm{m},\ell}^{per}(-n_1\chi_\mathrm{q}, n_1\eta_\mathrm{q}, -n_1\chi_\mathrm{p}, n_1\eta_\mathrm{p}) \Big] \Big\} \end{aligned} \tag{4.4.8}$$

$$B_\ell^{\mathrm{p}(1,2)} = -G_\ell^{(1,2)}(n_1\alpha_\mathrm{p}) \left[A_\mathrm{t}^{(\mathrm{TM,TE})} + B_\mathrm{t}^{(\mathrm{TM,TE})} \right] \tag{4.4.9}$$

$$
\begin{aligned}
L_\ell^{\mathrm{p}(1)} &= e^{-i\ell\varphi_{\mathrm{t}}}\frac{J_\ell(n_{\mathrm{cp}}\alpha_{\mathrm{p}})}{H_\ell^{(1)}(n_1\alpha_{\mathrm{p}})} \\
L_\ell^{\mathrm{p}(2)} &= \ell_{\mathrm{p}}e^{-i\ell\varphi_{\mathrm{t}}}\frac{J'_\ell(n_{\mathrm{cp}}\alpha_{\mathrm{p}})}{H_\ell'^{(1)}(n_1\alpha_{\mathrm{p}})}
\end{aligned}
\tag{4.4.10}
$$

being $\ell_{\mathrm{p}} = n_{\mathrm{cp}}/n_1$ or n_1/n_{cp} for TM or TE polarization, respectively. Moreover, it is $G_\ell^{(1)}(\cdot) = J_\ell(\cdot)/H_\ell^{(1)}(\cdot)$, and $G_\ell^{(2)}(\cdot) = J'_\ell(\cdot)/H_\ell'^{(1)}(\cdot)$.

Chapter 5

Numerical implementation for Gaussian surface

In this chapter, the integration of the *Perturbed Reflected Cylindrical Functions* and *Perturbed Transmitted Cylindrical Functions* defined in Sections 1.4.1 and 1.4.2 , respectively, of chapter 1 has been done considering Gaussian rough surface. The double integral is written in the form of summation of a single integral weighted by Fourier series coefficients of the rough surface. The evaluation of the single integral is performed by differentiating the evanescent and homogeneous part of the spectrum, separately for *Perturbed Reflected Cylindrical Functions*, as well as for *Perturbed Transmitted Cylindrical Functions*.

In Section 5.1, a discussion on the statistics of a Gaussian rough surface is presented, while the generation of the rough surface is described in Section 5.2. Evaluation of the *Perturbed Reflected Cylindrical Functions* is done in Section 5.3 while *Perturbed Transmitted Cylindrical Functions* are evaluated in Section 5.4 of this chapter.

5.1 Statistics of rough surface

A process $g(\zeta)$ is Gaussian if the random variables $g(\zeta_1)$, $g(\zeta_1)$,..., $g(\zeta_n)$ are jointly Gaussian for any n, $\zeta_1, \zeta_2, ..., \zeta_n$ [60]. The Gaussian process is completely characterized by the correlation function $\langle g(\zeta_1)g(\zeta 2)\rangle = h_{\text{nor}}^2 C(\zeta_1, \zeta_2)$. If the rough surface $g(\zeta)$ is statistically translational invariant, then $C(\zeta_1, \zeta_2) = C(\zeta_1 - \zeta_2)$. The Fourier transform of $h_{\text{nor}}^2 C(\zeta_1, \zeta_2)$ is the spectral density $W(n_\parallel)$. A periodic rough surface of finite length L_{nor} is to be generated, where L_{nor} is the normalized length of realization. We make $g(\zeta)$ periodic outside L_{nor}, so

$$g(\zeta) = g(\zeta + L_{\text{nor}}) \tag{5.1.1}$$

A Fourier series is used to represent $g(\zeta)$

$$g(\zeta) = \frac{1}{L_{\text{nor}}} \sum_{\text{u}=-\infty}^{\infty} G\Big(\frac{2\pi \text{u}}{L_{\text{nor}}}\Big) e^{i\frac{2\pi \text{u}}{L_{\text{nor}}}\zeta} \tag{5.1.2}$$

$G\Big(\frac{2\pi \text{u}}{L_{\text{nor}}}\Big)$ is a Gaussian random variable, and $\Big\langle g(\zeta)\Big\rangle = 0$ by Papoulis [60], where angular brackets denote mean value.

Let us consider the Correlation Function

$$\Big\langle g(\zeta_1)g(\zeta_2)\Big\rangle = \frac{1}{L_{\text{nor}}^2} \sum_{\text{u}=-\infty}^{\infty} \sum_{\text{v}=-\infty}^{\infty} \Big\langle G\Big(\frac{2\pi \text{u}}{L_{\text{nor}}}\Big) G^*\Big(\frac{2\pi \text{v}}{L_{\text{nor}}}\Big)\Big\rangle e^{i\frac{2\pi \text{u}}{L_{\text{nor}}}\zeta_1} e^{-i\frac{2\pi \text{v}}{L_{\text{nor}}}\zeta_2} \tag{5.1.3}$$

by using Wiener-Khinchine theorem defined in [60], we can also write

$$\Big\langle g(\zeta_1)g(\zeta_2)\Big\rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} W(n_\parallel) e^{in_\parallel(\zeta_1-\zeta_2)} dn_\parallel \tag{5.1.4}$$

In equation (5.1.4), $W(n_\parallel) = \sqrt{\pi} h_{\text{nor}}^2 \ell_{\text{nor}} e^{-\{n_\parallel^2 \frac{\ell_{\text{nor}}^2}{4}\}}$ is the Gaussian spectrum amplitude density function, ℓ_{nor} is the normalized correlation length, and h_{nor} is the normalized root mean square (rms) height of the rough surface. Comparing equation (5.1.3) and equation (5.1.4)

$$\Big\langle G\Big(\frac{2\pi \text{u}}{L_{\text{nor}}}\Big) G^*\Big(\frac{2\pi \text{v}}{L_{\text{nor}}}\Big)\Big\rangle = \delta_{\text{uv}} D_{\text{u}} \tag{5.1.5}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} W(n_\parallel) e^{in_\parallel(\zeta_1-\zeta_2)} dn_\parallel = \frac{1}{L_{\text{nor}}^2} \sum_{\text{u}=-\infty}^{\infty} D_{\text{u}} e^{i\frac{2\pi \text{u}}{L_{\text{nor}}}(\zeta_1-\zeta_2)} \tag{5.1.6}$$

In equation (5.1.6), the sampling of $n_{\|}$ at $\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}$

$$\sum_{\mathrm{u}=-\infty}^{\infty} W\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) e^{i\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}(\zeta_1-\zeta_2)} = \frac{1}{L_{\mathrm{nor}}} \sum_{\mathrm{u}=-\infty}^{\infty} D_{\mathrm{u}} e^{i\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}(\zeta_1-\zeta_2)} \tag{5.1.7}$$

leads to

$$D_{\mathrm{u}} = L_{\mathrm{nor}} W\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) \tag{5.1.8}$$

So, using the above equation in equation (5.1.5),

$$\Big\langle \Big| G(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}) \Big|^2 \Big\rangle = L_{\mathrm{nor}} W\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) \tag{5.1.9}$$

These coefficients also obey the following condition

$$G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) = G^*\Big(\frac{-2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) \tag{5.1.10}$$

If we let $\mathrm{u} = -\mathrm{v}$ in equation (5.1.5),

$$\Big\langle G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}) G^*(\frac{-2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) \Big\rangle = 0 \tag{5.1.11}$$

and

$$\Big\langle G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) \Big\rangle = 0 \tag{5.1.12}$$

it leads to the following expression for the Fourier series coefficients

$$G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) = \sqrt{L_{\mathrm{nor}} W\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big)} \tag{5.1.13}$$

5.2 Gaussian rough surface generation

The Gaussian rough surface can be easily generated by the spectral method given by E. Thorsos [9], which is widely used in the calculation of wave scattering [46]-[44]. Next step is to generate independent random samples taken from a zero mean, unit variance Gaussian distribution, and then to multiply by coefficients given by equation (5.1.13). Some of the properties of the Fourier series coefficients are useful in generating the rough surface. $G(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}})$ is a periodic sequence with period $N = \Big(L_{\mathrm{nor}}/\Delta\zeta\Big)$, that is

$$G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) = G\Big[\frac{2\pi}{L_{\mathrm{nor}}}(\mathrm{u}+N)\Big] \tag{5.2.1}$$

For $\mathrm{u} = -N/2$, it follows from equation (5.2.1)

$$G\Bigg[\frac{2\pi}{L_{\mathrm{nor}}}\Big(-\frac{N}{2}\Big)\Bigg] = G\Bigg[\frac{2\pi}{L_{\mathrm{nor}}}\Big(\frac{N}{2}\Big)\Bigg] \tag{5.2.2}$$

Also from equation (5.1.10)

$$G\Bigg[\frac{2\pi}{L_{\mathrm{nor}}}\Big(-\frac{N}{2}\Big)\Bigg] = G\Bigg[\frac{2\pi}{L_{\mathrm{nor}}}\Big(\frac{N}{2}\Big)\Bigg] = G^{*}\Bigg[\frac{2\pi}{L_{\mathrm{nor}}}\Big(-\frac{N}{2}\Big)\Bigg] \tag{5.2.3}$$

Then $G\Big[\frac{2\pi}{L_{\mathrm{nor}}}\Big(\frac{N}{2}\Big)\Big]$ is real, and from equation (5.1.10), G(0) is also real. To summarize, we have two real gaussian random numbers and the remaining random numbers are complex. The coefficients for $\mathrm{u} = 1, 2.....(N/2-1)$ can be computed by using the condition in equation (5.1.10), and other values obey the periodic relation of equation (5.2.1). All the random numbers to be generated are independent and they need not be arranged in any order. So, after multiplying these random numbers by equation (5.1.13), we get the final expression for Fourier series coefficients [20]:

$$G\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big) = \sqrt{L_{\mathrm{nor}} W\Big(\frac{2\pi \mathrm{u}}{L_{\mathrm{nor}}}\Big)} \times \begin{cases} \frac{\Big(N(0,1) + iN(0,1)\Big)}{\sqrt{2}} & \text{if } \mathrm{u} \neq 0, \frac{N}{2} \\ N(0,1) & \text{if } \mathrm{u} = \frac{N}{2} \end{cases} \tag{5.2.4}$$

where N(0,1) is a random variant with a Gaussian distribution of zero mean and unit variance.

5.3 Evaluation of perturbed reflected cylindrical functions

The numerical analysis of the *Perturbed Reflected Cylindrical Functions* $RW^{per}_{\mathrm{m},\ell}$ defined in equation (1.4.17) is here considered for TM polarization, in the case of a rough surface with Gaussian roughness spectrum defined in equation (5.1.2). Let us consider the *Perturbed Reflected Cylindrical Wave Functions* defined in equation (1.4.13)

$$RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\gamma_{10}(n_\parallel, n'_\parallel)$$
$$\times F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\left(n_1 n_\parallel \eta_{\mathrm{q}}+\ell\phi_{\mathrm{sr}}\right)}e^{in_1\left[n'_\parallel\eta_{\mathrm{p}}+\chi_{\mathrm{p}}\sqrt{1-(n'_\parallel)^2}\right]}\,dn_\parallel\,dn'_\parallel \tag{5.3.1}$$

where $\gamma^G_{10}(n'_\parallel, n_\parallel)$ is the reflection coefficient from first order Small Perturbation Method:

$$\gamma_{10}(n'_\parallel, n_\parallel) = \frac{i(n_1^2-n_0^2)2n_1\sqrt{1-(n_\parallel)^2}G(n'_\parallel-n_\parallel)}{\left[n_0\sqrt{1-(n_1n_\parallel)^2}+n_1\sqrt{1-(n_\parallel)^2}\right]} \times\frac{1}{\left[n_0\sqrt{1-(n_1n'_\parallel)^2}+n_1\sqrt{1-(n'_\parallel)^2}\right]} \tag{5.3.2}$$

$$\phi_{\mathrm{sr}} = \tan^{-1}\left[\frac{n'_\parallel}{\sqrt{1-(n'_\parallel)^2}}\right] \tag{5.3.3}$$

where $G(n'_\parallel, n_\parallel)$ is the Fourier series coefficients of the periodic rough surface, which are given by equation (5.2.4). For a periodic rough surface, equation (5.3.1) can be written as a summation of a single integral weighted by Fourier series coefficients of the periodic rough surface

$$RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{-\infty}^{+\infty}\gamma_{10}(n_\parallel, \mathrm{u})$$
$$\times F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_\parallel)e^{-i\left(n_1n_\parallel\eta_{\mathrm{q}}+\ell\phi_{\mathrm{sr}}\right)}e^{in_1\left[\left(n_\parallel+\frac{2\pi\mathrm{u}}{n_1L_{\mathrm{nor}}}\right)\eta_{\mathrm{p}}+\chi_{\mathrm{p}}\sqrt{1-\left(n_\parallel+\frac{2\pi\mathrm{u}}{n_1L_{\mathrm{nor}}}\right)^2}\right]}\,dn_\parallel \tag{5.3.4}$$

where

$$\gamma_{10}(n_{\parallel},\mathrm{u}) = \frac{i(n_1^2-n_0^2)2n_1\sqrt{1-(n_{\parallel})^2}G\Big(\frac{2\pi\mathrm{u}}{n_1L_{\mathrm{nor}}}\Big)}{\Big[n_0\sqrt{[1-(n_1n_{\parallel})^2}+n_1\sqrt{1-(n_{\parallel})^2}\Big]} \times\frac{1}{\Big[\sqrt{1-n_1^2\Big(n_{\parallel}+\frac{2\pi\mathrm{u}}{n_1L_{\mathrm{nor}}}\Big)^2}+n_1\sqrt{1-\Big(n_{\parallel}+\frac{2\pi\mathrm{u}}{n_1L_{\mathrm{nor}}}\Big)^2}\Big]} \tag{5.3.5}$$

For the sake of simplicity, from now onwards we put $\nu=\Big[2\pi\mathrm{u}/(n_1L_{\mathrm{nor}})\Big]$. The integral in equation (5.3.4) can be written as

$$RW^{per}_{\mathrm{m},\ell}[-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}}]=$$

$$\frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{|n_{\parallel}|\geq 1}\gamma_{10}(n_{\parallel},\mathrm{u})F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}},n_{\parallel})e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_{\parallel}(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_{\parallel}+\nu)^2}}\,dn_{\parallel}$$

$$+\frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{|n_{\parallel}|<1}\gamma_{10}(n_{\parallel},\mathrm{u})F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}},n_{\parallel})e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_{\parallel}(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_{\parallel}+\nu)^2}}\,dn_{\parallel} \tag{5.3.6}$$

The expression for the $RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}})$ can be written as homogeneous and evanescent spectrum:

$$RW^{per}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}})=\frac{1}{\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\Big[I^{(per,int)}-iI^{(per,ext)}\Big] \tag{5.3.7}$$

where

$$I^{(per,int)}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}})=$$

$$\int\limits_{|n_{\parallel}|<1}\gamma_{10}(n_{\parallel},\mathrm{u})\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1}e^{i\mathrm{m}\cos^{-1}(n_{\parallel})}e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_{\parallel}(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}r_2}\,dn_{\parallel} \tag{5.3.8}$$

and

$$I^{(per,ext)}_{\mathrm{m},\ell}(-n_1\chi_{\mathrm{q}},n_1\eta_{\mathrm{q}},-n_1\chi_{\mathrm{p}},n_1\eta_{\mathrm{p}})=$$

$$\int\limits_{|n_{\parallel}|\geq 1}\gamma_{10}(n_{\parallel},\mathrm{u})\frac{(n_{\parallel}+r_1)^{-\mathrm{m}}}{r_1}e^{in_1\chi_{\mathrm{q}}r_1}e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_{\parallel}(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}r_2}\,dn_{\parallel} \tag{5.3.9}$$

being $r_1 = \sqrt{|1 - n_\parallel^2|}$ and $r_2 = \sqrt{|1 - (n_\parallel + \nu)^2|}$.

5.3.1 Evanescent spectrum of perturbed reflected cylindrical functions

For the evanescent spectrum , the value of square root r_1 is $i\sqrt{n_\parallel^2 - 1}$. According to the numerical value of ν, the square root r_2 can be real or imaginary in the intervals $n_\parallel \geq 1$ and $n_\parallel \leq 1$. As regards the decomposition of equation (5.3.9), for $\mathrm{u} > 0$ and $n_\parallel \geq 1$,

$$r_2 = i\sqrt{(n_\parallel + \nu)^2 - 1} \tag{5.3.10}$$

when $n_\parallel \leq -1$.

$$r_2 = \begin{cases} i\sqrt{(n_\parallel + \nu)^2 - 1} & \text{if } -\infty \leq n_\parallel \leq (-1 - \nu) \\ \sqrt{1 - (n_\parallel + \nu)^2} & \text{if } (-1 - \nu) < n_\parallel < (1 - \nu) \\ i\sqrt{(n_\parallel + \nu)^2 - 1} & \text{if } (1 - \nu) \leq n_\parallel \leq -1 \end{cases} \tag{5.3.11}$$

On the basis of relations (5.3.10) and (5.3.11), equation (5.3.9) is decomposed into a sum of four terms:

$$I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$

$$\int_1^{+\infty} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2 - 1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}} - \eta_{\mathrm{q}}) + \nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel + \nu)^2 - 1}} dn_\parallel \tag{5.3.12}$$

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{-\infty}^{(-1-\nu)} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel+\nu)^2-1}} dn_\parallel \tag{5.3.13}$$

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{(-1-\nu)}^{(1-\nu)} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{q}}\sqrt{1-(n_\parallel+\nu)^2}} dn_\parallel \tag{5.3.14}$$

$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{(1-\nu)}^{-1} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel+\nu)^2-1}} dn_\parallel \tag{5.3.15}$$

The integration in equation (5.3.12) is performed solving the singularity in $|n_\parallel| = 1$, by means of the substitution

$$t = \sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = \sqrt{t^2+1} \\ dn_\parallel = \dfrac{t}{\sqrt{t^2+1}}\, dt \\ 0 < t < \infty \end{cases} \tag{5.3.16}$$

$$I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_0^{+\infty} \bar{\gamma}_{10}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}+\nu}{i\sqrt{(\sqrt{t^2+1}+\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(\sqrt{t^2+1}+\nu)^2-1}}dt \tag{5.3.17}$$

where

$$\bar{\gamma}_{10}(t) = \gamma_{10}(\sqrt{t^2+1})$$

To remove the singularity at $|n_\parallel| = 1$ in equation (5.3.13), we put

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\,dt \\ -\infty < t < -b_1 \end{cases} \tag{5.3.18}$$

being

$$b_1 = \sqrt{\nu^2 + 2\nu}$$

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{-\infty}^{-b_1} \bar{\gamma}_{10}(t)\frac{(-\sqrt{t^2+1}-t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}dt \tag{5.3.19}$$

Putting $t = -t$ in the above equation, and using the relation

$$(-\sqrt{t^2+1}+t)^{-\mathrm{m}} = (-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}$$

we get:

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{b_1}^{\infty} \bar{\gamma}_{10}(t)\frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}dt \tag{5.3.20}$$

Following similar steps, the integrals in equations (5.3.14) and (5.3.15) can be written as

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{a_1}^{b_1} \bar{\gamma}_{10}(t)\frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{\sqrt{1-(-\sqrt{t^2+1}+\nu)^2}}\right]}e^{in_1\chi_{\mathrm{q}}\sqrt{1-(-\sqrt{t^2+1}+\nu)^2}}dt \tag{5.3.21}$$

being

$$a_1 = \sqrt{\nu^2 - 2\nu}$$

$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{0}^{a_1} \bar{\gamma}_{10}(t)\frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}+\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}+\nu}{i\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}+\nu)^2-1}}dt \tag{5.3.22}$$

For u < 0,

$$r_2 = \sqrt{1-(n_\parallel - \nu)^2}$$

as

$$\nu = \frac{2\pi|\mathrm{u}|}{n_1 L_{\mathrm{nor}}}$$

For $n_\parallel \leq -1$,

$$r_2 = i\sqrt{(n_\parallel - \nu)^2 - 1} \tag{5.3.23}$$

when $n_\parallel \geq 1$,

$$r_2 = \begin{cases} i\sqrt{(n_\parallel-\nu)^2-1} & \text{if } (1+\nu) \leq n_\parallel \leq \infty \\ \sqrt{1-(n_\parallel-\nu)^2} & \text{if } (-1+\nu) < n_\parallel < (1+\nu) \\ i\sqrt{(n_\parallel-\nu)^2-1} & \text{if } 1 \leq n_\parallel \leq (-1+\nu) \end{cases} \tag{5.3.24}$$

On the basis of relations (5.3.23) and (5.3.24), equation (5.3.9) is decomposed into a sum of four terms.

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{-\infty}^{-1} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}} dn_\parallel \tag{5.3.25}$$

$$I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{1+\nu}^{\infty} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}} dn_\parallel \tag{5.3.26}$$

$$I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{-1+\nu}^{1+\nu} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{q}}\sqrt{1-(n_\parallel-\nu)^2}} dn_\parallel \tag{5.3.27}$$

$$I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int_{1}^{-1+\nu} \gamma_{10}(n_\parallel, \mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2 - 1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2 - 1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(n_\parallel-\nu)^2-1}} dn_\parallel \tag{5.3.28}$$

With the position $n_\parallel = -\sqrt{t^2+1}$ in equation (5.3.25), and replacing t with $-t$, we get

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_0^\infty \bar{\gamma}_{10}(t)\frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}-\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{-\sqrt{t^2+1}-\nu}{i\sqrt{(-\sqrt{t^2+1}-\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(-\sqrt{t^2+1}-\nu)^2-1}}dt \tag{5.3.29}$$

Comparing equations (5.3.17) and (5.3.29)

$$I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.30}$$

To solve the integrals in equations (5.3.26), (5.3.27) and (5.3.28), we make the substitution $n_\parallel = \sqrt{t^2+1}$

$$I_{\mathrm{m},\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{b_1}^\infty \bar{\gamma}_{10}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}-\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{i\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}\right]}e^{-n_1\chi_{\mathrm{q}}\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}dt \tag{5.3.31}$$

Comparing equations (5.3.20) and (5.3.31)

$$I_{\mathrm{m},\ell}^{(per,ext+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.32}$$

$$I_{\mathrm{m},\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{a_1}^{b_1} \bar{\gamma}_{10}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}}e^{-n_1\chi_{\mathrm{q}}t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}-\nu\eta_{\mathrm{p}}]}e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{\sqrt{1-(\sqrt{t^2+1}-\nu)^2}}\right]}e^{in_1\chi_{\mathrm{q}}\sqrt{1-(\sqrt{t^2+1}-\nu)^2}}dt \tag{5.3.33}$$

Comparing equations (5.3.21) and (5.3.33)

$$I_{\mathrm{m},\ell}^{(per,ext+3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.34}$$

$$I_{\mathrm{m},\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_0^{a_1} \bar{\gamma}_{10}(t)\frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t}$$
$$\times e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\sqrt{t^2+1}-\nu\eta_{\mathrm{p}}]} e^{-i\ell\tan^{-1}\left[\frac{\sqrt{t^2+1}-\nu}{i\sqrt{(\sqrt{t^2+1}-\nu)^2-1}}\right]} e^{-n_1\chi_{\mathrm{q}}\sqrt{(\sqrt{t^2+1}-\nu)^2-1}} dt \tag{5.3.35}$$

Comparing equations (5.3.22) and (5.3.35)

$$I_{\mathrm{m},\ell}^{(per,ext+4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.36}$$

The evanescent spectrum can be decomposed as follows

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) &= I_{\mathrm{m},\ell}^{(per,ext+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+(-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\ &+(-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\ &+(-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \end{aligned} \tag{5.3.37}$$

for u > 0 while with u < 0,

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) &= (-1)^{\mathrm{m}} I_{\mathrm{m},\ell}^{(per,ext-1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\ &+I_{-\mathrm{m},-\ell}^{(per,ext-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+I_{-\mathrm{m},-\ell}^{(per,ext-3)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \\ &+I_{-\mathrm{m},-\ell}^{(per,ext-4)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \end{aligned} \tag{5.3.38}$$

5.3.2 Homogeneous spectrum of perturbed reflected cylindrical functions

Let us consider equation (5.3.8)

$$I_{\mathrm{m},\ell}^{(per,int)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{|n_\parallel|<1} \gamma_{10}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}r_2}\, dn_\parallel \tag{5.3.39}$$

In case of homogeneous spectrum, $r_1 = \sqrt{1-n_\parallel^2}$ since $|n_\parallel| < 1$.
For u > 0,

$$r_2 = \begin{cases} \sqrt{1-(n_\parallel+\nu)^2} & \text{if } -1 < n_\parallel < (1-\nu) \\ i\sqrt{(n_\parallel+\nu)^2-1} & \text{if } (1-\nu) < n_\parallel < 1 \end{cases} \tag{5.3.40}$$

Equation (5.3.39) can be written as a sum of two terms:

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{-1}^{(1-\nu)} \gamma_{10}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel+\nu)^2}}\, dn_\parallel \tag{5.3.41}$$

and

$$I_{\mathrm{m},\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{(1-\nu)}^{1} \gamma_{10}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{-i\ell\phi_{\mathrm{sr}}} e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})+\nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{p}}\sqrt{(n_\parallel+\nu)^2-1}}\, dn_\parallel \tag{5.3.42}$$

Using the substitution $n_{\parallel} = \cos t$, equation (5.3.41) can be written as

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{\cos^{-1}(1-\nu)}^{\pi} \gamma_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t+\nu}{\sqrt{1-(\cos t+\nu)^2}}\right]} e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t + \nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}\sqrt{1-(\cos t+\nu)^2}}\, dt \tag{5.3.43}$$

In equation (5.3.43) using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(-1+\nu) < t' < 0 \end{cases} \tag{5.3.44}$$

we get

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = e^{\mathrm{i}m\pi} \int_{0}^{\cos^{-1}(-1+\nu)} \gamma_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t - \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{-\cos t+\nu}{\sqrt{1-(-\cos t+\nu)^2}}\right]} e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t + \nu\eta_{\mathrm{p}}]} e^{in_1\chi_{\mathrm{p}}\sqrt{1-(-\cos t+\nu)^2}}\, dt \tag{5.3.45}$$

With the position $n_{\parallel} = \cos t$, equation (5.3.42) can be written as

$$I_{\mathrm{m},\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int_{0}^{\cos^{-1}(1-\nu)} \gamma_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t+\nu}{i\sqrt{(\cos t+\nu)^2-1}}\right]} e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t + \nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{p}}\sqrt{(\cos t+\nu)^2-1}}\, dt \tag{5.3.46}$$

For u < 0, we put

$$\nu = \frac{2\pi|\mathrm{u}|}{n_1 L_{\mathrm{nor}}}$$

and

$$r_2 = \begin{cases} \sqrt{1-(n_\parallel-\nu)^2} & \text{if } (-1+\nu) < n_\parallel < 1 \\ i\sqrt{(n_\parallel-\nu)^2-1} & \text{if } -1 < n_\parallel < (-1+\nu) \end{cases} \tag{5.3.47}$$

Equation (5.3.39) can be written as a sum of two terms:

$$I_{\mathrm{m},\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int\limits_{(-1+\nu)}^{1} \gamma_{10}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}}e^{i\mathrm{m}\cos^{-1}(n_\parallel)}e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(n_\parallel-\nu)^2}}\,dn_\parallel \tag{5.3.48}$$

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) =$$
$$\int\limits_{-1}^{(-1+\nu)} \gamma_{10}(n_\parallel)\frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}}e^{i\mathrm{m}\cos^{-1}(n_\parallel)}e^{-i\ell\phi_{\mathrm{sr}}}e^{in_1[n_\parallel(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})-\nu\eta_{\mathrm{p}}]}e^{-n_1\chi_{\mathrm{p}}\sqrt{(n_\parallel-\nu)^2-1}}\,dn_\parallel \tag{5.3.49}$$

Substituting $n_\parallel = \cos t$, equations (5.3.48) and (5.3.49) can be written as

$$I_{\mathrm{m},\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{0}^{\cos^{-1}(-1+\nu)} \gamma_{10}(\cos t)e^{i(n_1\chi_{\mathrm{q}}\sin t+\mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t-\nu}{\sqrt{1-(\cos t-\nu)^2}}\right]}e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t-\nu\eta_{\mathrm{p}}]}e^{in_1\chi_{\mathrm{p}}\sqrt{1-(\cos t-\nu)^2}}\,dt \tag{5.3.50}$$

Comparing equations (5.3.50) and (5.3.45)

$$I_{\mathrm{m},\ell}^{(per,int+1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}}I_{-\mathrm{m},-\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.51}$$

and

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = \int\limits_{\cos^{-1}(-1+\nu)}^{\pi} \gamma_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{\cos t-\nu}{i\sqrt{(\cos t-\nu)^2-1}}\right]} e^{in_1[(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t - \nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{p}}\sqrt{(\cos t-\nu)^2-1}}\, dt \tag{5.3.52}$$

In equation (5.3.52), using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\, dt' \\ \cos^{-1}(1-\nu) < t' < 0 \end{cases} \tag{5.3.53}$$

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = e^{i\mathrm{m}\pi} \int\limits_{0}^{\cos^{-1}(1-\nu)} \gamma_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t - \mathrm{m}t)}$$
$$\times e^{-i\ell\tan^{-1}\left[\frac{-\cos t-\nu}{i\sqrt{(-\cos t-\nu)^2-1}}\right]} e^{in_1[-(\eta_{\mathrm{p}}-\eta_{\mathrm{q}})\cos t - \nu\eta_{\mathrm{p}}]} e^{-n_1\chi_{\mathrm{p}}\sqrt{(-\cos t-\nu)^2-1}}\, dt \tag{5.3.54}$$

Comparing equations (5.3.54) and (5.3.46)

$$I_{\mathrm{m},\ell}^{(per,int-2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) = (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.55}$$

The final decomposition of the homogeneous spectrum is the following one

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,int)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \quad &= \quad (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \\ &\quad\; + I_{\mathrm{m},\ell}^{(per,int+2)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \end{aligned} \tag{5.3.56}$$

for u > 0, while for u < 0,

$$I_{\mathrm{m},\ell}^{(per,int)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}}) \quad = \quad I_{\mathrm{m},\ell}^{(per,int-1)}(-n_1\chi_{\mathrm{q}}, n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, n_1\eta_{\mathrm{p}})$$

$$+(-1)^{\mathrm{m}} I^{(per,int+2)}_{-\mathrm{m},-\ell}(-n_1\chi_{\mathrm{q}}, -n_1\eta_{\mathrm{q}}, -n_1\chi_{\mathrm{p}}, -n_1\eta_{\mathrm{p}}) \tag{5.3.57}$$

5.4 Evaluation of perturbed transmitted cylindrical functions

Let us consider the *Perturbed Transmitted Cylindrical Functions* defined in equation (1.4.22)

$$\begin{aligned} TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi) =& \frac{1}{(2\pi)^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\tau_{10}(n'_{\|}, n_{\|})F_{\mathrm{m}}(-n_1\chi, n_{\|}) \\ &\times e^{-i\sqrt{1-(n_1 n'_{\|})^2}(\xi+\chi)}e^{in_1 n'_{\|}\zeta}\,dn_{\|}\,dn'_{\|} \end{aligned} \tag{5.4.1}$$

where $\tau_{10}(n'_{\|}, n_{\|})$ is the transmission coefficient from Small Perturbation Method. In a similar way as to *Perturbed Reflected Cylindrical Functions*, equation (5.4.1) can be written as a summation of a single integral weighted by Fourier series coefficients of periodic rough surface.

$$\begin{aligned} TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi) = & \frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{-\infty}^{+\infty}\tau_{10}(n_{\|},\mathrm{u})F_{\mathrm{m}}(-n_1\chi, n_{\|}) \\ &\times e^{-i\sqrt{1-[n_1(n_{\|}+\frac{2\pi\mathrm{u}}{n_1 L_{\mathrm{nor}}})]^2}(\xi+\chi)}e^{in_1(n_{\|}+\frac{2\pi\mathrm{u}}{n_1 L_{\mathrm{nor}}})\zeta}\,dn_{\|} \end{aligned} \tag{5.4.2}$$

For the sake of simplicity, from now onwards we put $\nu = \Big(2\pi\mathrm{u}/n_1 L_{\mathrm{nor}}\Big)$. The integral in equation (5.4.2) can be written as

$$\begin{aligned} &TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi) = \\ &\frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{|n_{\|}|\geq 1}\tau_{10}(n_{\|},\mathrm{u})F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_{\|})e^{-i\sqrt{1-[n_1(n_{\|}+\nu)]^2}(\xi+\chi)}e^{in_1(n_{\|}+\nu)\zeta}\,dn_{\|} \\ &+\frac{1}{2\pi L_{\mathrm{nor}}}\sum_{\mathrm{u}=-\infty}^{\infty}\int\limits_{|n_{\|}|<1}\tau_{10}(n_{\|},\mathrm{u})F_{\mathrm{m}}(-n_1\chi_{\mathrm{q}}, n_{\|})e^{-i\sqrt{1-[n_1(n_{\|}+\nu)]^2}(\xi+\chi)}e^{in_1(n_{\|}+\nu)\zeta}\,dn_{\|} \end{aligned} \tag{5.4.3}$$

The expression for the $TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi)$ can be written as a sum of homogeneous and evanescent spectrum.

$$TW^{per}_{\mathrm{m}}(\xi,\zeta,\chi) = \frac{1}{\pi L_{\mathrm{nor}}} \sum_{\mathrm{u}=-\infty}^{\infty} \left[I^{(per,int)} - i I^{(per,ext)} \right] \tag{5.4.4}$$

where

$$I^{(per,int)}_{\mathrm{m}}(\xi,\zeta,\chi) = \int\limits_{|n_\parallel|<1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}} r_1}}{r_1} e^{\mathrm{i}\mathrm{m}\cos^{-1}(n_\parallel)} e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.5}$$

and

$$I^{(per,ext)}_{\mathrm{m}}(\xi,\zeta,\chi) = \int\limits_{|n_\parallel|\geq 1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{(n_\parallel + r_1)^{-\mathrm{m}}}{r_1} e^{-n_1\chi_{\mathrm{q}} r_1} e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.6}$$

being $r_1 = \sqrt{|1-n_\parallel^2|}$ and $r_4 = \sqrt{|1-[n_1(n_\parallel+\nu)]^2|}$.

5.4.1 Evanescent spectrum of perturbed transmitted cylindrical functions

According to the numerical value of ν, the square root r_4 can be real or imaginary in the intervals $n_\parallel \geq 1$ and $n_\parallel \leq 1$ while $r_1 = i\sqrt{n_\parallel^2 - 1}$. As regards the decomposition of equation (5.4.6), for $\mathrm{u} > 0$ and $n_\parallel \geq 1$,

$$r_4 = i\sqrt{[n_1(n_\parallel+\nu)]^2 - 1} \tag{5.4.7}$$

while when $n_\parallel \leq -1$.

$$r_4 = \begin{cases} i\sqrt{[n_1(n_\parallel+\nu)]^2-1} & \text{if } -\infty \leq n_\parallel \leq (-\frac{1}{n_1}-\nu) \\ \sqrt{1-[n_1(n_\parallel+\nu)]^2} & \text{if } (-\frac{1}{n_1}-\nu) < n_\parallel < (\frac{1}{n_1}-\nu) \\ i\sqrt{[n_1(n_\parallel+\nu)]^2-1} & \text{if } (\frac{1}{n_1}-\nu) \leq n_\parallel \leq -1 \end{cases} \tag{5.4.8}$$

On the basis of relations (5.4.7) and (5.4.8), equation (5.4.6) is decomposed into a sum of four terms:

$$I_{\mathrm{m}}^{(per,ext+1)}(\xi,\zeta,\chi) = \int_{1}^{+\infty} \tau_{10}(n_\parallel,\mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{\sqrt{[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.9}$$

$$I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) = \int_{-\infty}^{(-\frac{1}{n_1}-\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{\sqrt{[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.10}$$

$$I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) = \int_{(-\frac{1}{n_1}-\nu)}^{(\frac{1}{n_1}-\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{-i\sqrt{1-[n_1(n_\parallel+\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.11}$$

$$I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) = \int_{(\frac{1}{n_1}-\nu)}^{-1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{(n_\parallel + \sqrt{n_\parallel^2-1})^{-\mathrm{m}}}{\sqrt{n_\parallel^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_\parallel^2-1}} e^{\sqrt{[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.12}$$

The singularity in $|n_\parallel| = 1$ in equation (5.4.9) can be removed by the substitution

$$t = \sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = \sqrt{t^2+1} \\ dn_\parallel = \dfrac{t}{\sqrt{t^2+1}}\, dt \\ 0 < t < \infty \end{cases} \tag{5.4.13}$$

leading to

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext+1)}(\xi,\zeta,\chi) = \int\limits_0^{+\infty} \bar{\tau}_{10}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t} \\ \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{5.4.14}$$

where

$$\bar{\tau}_{10}(t) = \tau_{10}(\sqrt{t^2+1})$$

The integration in equation (5.4.10) is performed solving the singularity in $|n_\parallel| = 1$, by means of the substitution

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\, dt \\ -\infty < t < -d_1 \end{cases} \tag{5.4.15}$$

being

$$d_1 = \sqrt{(-\frac{1}{n_1} - \nu)^2 - 1}$$

Putting these values in equation (5.4.10) and changing $t = -t$, we get

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) = \int\limits_{d_1}^{+\infty} \bar{\tau}_{10}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t} \\ \times e^{(\xi+\chi)\sqrt{[n_1(-\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{5.4.16}$$

In a similar way, the integrals in equations (5.4.11) and (5.4.12) can be written as

$$I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) = \int_{c_1}^{d_1} \bar{\tau}_{10}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1 \chi_{\mathrm{q}} t} \times e^{-i(\xi+\chi)\sqrt{1-[n_1(-\sqrt{t^2+1}+\nu)]^2}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \tag{5.4.17}$$

being

$$c_1 = \sqrt{(\frac{1}{n_1}-\nu)^2 - 1}$$

$$I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) = \int_{0}^{c_1} \bar{\tau}_{10}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1 \chi_{\mathrm{q}} t} \times e^{(\xi+\chi)\sqrt{[n_1(-\sqrt{t^2+1}+\nu)]^2-1}} e^{in_1(-\sqrt{t^2+1}+\nu)\zeta}\, dt \tag{5.4.18}$$

For u < 0,

$$r_4 = \sqrt{1 - [n_1(n_{\parallel} - \nu)]^2}$$

being

$$\nu = \frac{2\pi|\mathrm{u}|}{n_1 L_{\mathrm{nor}}}$$

For $n_{\parallel} \leq -1$,

$$r_4 = i\sqrt{[n_1(n_{\parallel} - \nu)]^2 - 1} \tag{5.4.19}$$

when $n_{\parallel} \geq 1$,

$$r_4 = \begin{cases} i\sqrt{[n_1(n_{\parallel}-\nu)]^2 - 1} & \text{if } (\frac{1}{n_1}+\nu) \leq n_{\parallel} \leq \infty \\ \sqrt{1-[n_1(n_{\parallel}-\nu)]^2} & \text{if } (-\frac{1}{n_1}+\nu) < n_{\parallel} < (\frac{1}{n_1}+\nu) \\ i\sqrt{[n_1(n_{\parallel}-\nu)]^2 - 1} & \text{if } 1 \leq n_{\parallel} \leq (\frac{1}{n_1}-\nu) \end{cases} \tag{5.4.20}$$

On the basis of relations (5.4.19) and (5.4.20), equation (5.4.6) is decomposed into a sum of four terms:

$$I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) =$$
$$\int_{-\infty}^{-1} \tau_{10}(n_{\parallel},\mathrm{u}) \frac{(n_{\parallel}+\sqrt{n_{\parallel}^2-1})^{-\mathrm{m}}}{\sqrt{n_{\parallel}^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_{\parallel}^2-1}} e^{\sqrt{[n_1(n_{\parallel}-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_{\parallel}-\nu)\zeta}\, dn_{\parallel} \tag{5.4.21}$$

$$I_{\mathrm{m}}^{(per,ext-2)}(\xi,\zeta,\chi) =$$
$$\int_{(\frac{1}{n_1}+\nu)}^{\infty} \tau_{10}(n_{\parallel},\mathrm{u}) \frac{(n_{\parallel}+\sqrt{n_{\parallel}^2-1})^{-\mathrm{m}}}{\sqrt{n_{\parallel}^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_{\parallel}^2-1}} e^{\sqrt{[n_1(n_{\parallel}-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_{\parallel}-\nu)\zeta}\, dn_{\parallel} \tag{5.4.22}$$

$$I_{\mathrm{m}}^{(per,ext-3)}(\xi,\zeta,\chi) =$$
$$\int_{(-\frac{1}{n_1}+\nu)}^{(\frac{1}{n_1}+\nu)} \tau_{10}(n_{\parallel},\mathrm{u}) \frac{(n_{\parallel}+\sqrt{n_{\parallel}^2-1})^{-\mathrm{m}}}{\sqrt{n_{\parallel}^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_{\parallel}^2-1}} e^{-i\sqrt{1-[n_1(n_{\parallel}-\nu)]^2}(\xi+\chi)} e^{in_1(n_{\parallel}-\nu)\zeta}\, dn_{\parallel} \tag{5.4.23}$$

$$I_{\mathrm{m}}^{(per,ext-4)}(\xi,\zeta,\chi) =$$
$$\int_{1}^{(-\frac{1}{n_1}+\nu)} \tau_{10}(n_{\parallel},\mathrm{u}) \frac{(n_{\parallel}+\sqrt{n_{\parallel}^2-1})^{-\mathrm{m}}}{\sqrt{n_{\parallel}^2-1}} e^{-n_1\chi_{\mathrm{q}}\sqrt{n_{\parallel}^2-1}} e^{\sqrt{[n_1(n_{\parallel}-\nu)]^2-1}(\xi+\chi)} e^{in_1(n_{\parallel}-\nu)\zeta}\, dn_{\parallel} \tag{5.4.24}$$

The integration in equation (5.4.21) is performed solving the singularity in $|n_{\parallel}| = 1$, by means of the substitution

$$t = -\sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = -\sqrt{t^2+1} \\ dn_\parallel = \dfrac{-t}{\sqrt{t^2+1}}\, dt \\ -\infty < t < 0 \end{cases} \tag{5.4.25}$$

Putting these values in equation (5.4.21) and changing $t = -t$, we get

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) = & \int_0^{+\infty} \bar{\tau}_{10}(t) \frac{(-1)^{\mathrm{m}}(\sqrt{t^2+1}+t)^{\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t} \\ & \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}+\nu)]^2-1}} e^{-in_1(\sqrt{t^2+1}+\nu)\zeta}\, dt \end{aligned} \tag{5.4.26}$$

where

$$\bar{\tau}_{10}(t) = \tau_{10}(\sqrt{t^2+1})$$

Comparing equations (5.4.26) and (5.4.14)

$$I_{\mathrm{m}}^{(per,ext-1)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext+1)}(\xi,-\zeta,\chi) \tag{5.4.27}$$

The singularity in $|n_\parallel| = 1$ in equation (5.4.21) can be removed by the substitution

$$t = \sqrt{n_\parallel^2 - 1} \Rightarrow \begin{cases} n_\parallel = \sqrt{t^2+1} \\ dn_\parallel = \dfrac{t}{\sqrt{t^2+1}}\, dt \\ d_1 < t < \infty \end{cases} \tag{5.4.28}$$

thus obtaining

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-2)}(\xi,\zeta,\chi) = & \int_{d_1}^{+\infty} \bar{\tau}_{10}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}} t} \\ & \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}-\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{5.4.29}$$

Comparing equations (5.4.29) and (5.4.16)

$$I_{\mathrm{m}}^{(per,ext+2)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-2)}(\xi,-\zeta,\chi) \tag{5.4.30}$$

In a similar way, the integrals in equations (5.4.23) and (5.4.24) can be written as

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-3)}(\xi,\zeta,\chi) = & \int_{c_1}^{d_1} \bar{\tau}_{10}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}}t} \\ & \times e^{-i(\xi+\chi)\sqrt{1-[n_1(\sqrt{t^2+1}-\nu)]^2}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{5.4.31}$$

$$\begin{aligned} I_{\mathrm{m}}^{(per,ext-4)}(\xi,\zeta,\chi) = & \int_{0}^{c_1} \bar{\tau}_{10}(t) \frac{(\sqrt{t^2+1}+t)^{-\mathrm{m}}}{\sqrt{t^2+1}} e^{-n_1\chi_{\mathrm{q}}t} \\ & \times e^{(\xi+\chi)\sqrt{[n_1(\sqrt{t^2+1}-\nu)]^2-1}} e^{in_1(\sqrt{t^2+1}-\nu)\zeta}\, dt \end{aligned} \tag{5.4.32}$$

Note that

$$I_{\mathrm{m}}^{(per,ext+3)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-3)}(\xi,-\zeta,\chi) \tag{5.4.33}$$

$$I_{\mathrm{m}}^{(per,ext+4)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} I_{-\mathrm{m}}^{(per,ext-4)}(\xi,-\zeta,\chi) \tag{5.4.34}$$

The evanescent spectrum can be decomposed as follows

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext)}(\xi,\zeta,\chi) &= I_{\mathrm{m},\ell}^{(per,ext+1)}(\xi,\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-2)}(\xi,-\zeta,\chi) \\ &\quad +(-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-3)}(\xi,-\zeta,\chi) + (-1)^{\mathrm{m}} I_{-\mathrm{m},-\ell}^{(per,ext-4)}(\xi,-\zeta,\chi) \end{aligned} \tag{5.4.35}$$

for u > 0 and while for u < 0,

$$\begin{aligned} I_{\mathrm{m},\ell}^{(per,ext)}(\xi,\zeta,\chi) &= (-1)^{\mathrm{m}} I_{\mathrm{m},\ell}^{(per,ext-1)}(\xi,-\zeta,\chi) + I_{-\mathrm{m},-\ell}^{(per,ext-2)}(\xi,\zeta,\chi) \\ &\quad + I_{-\mathrm{m},-\ell}^{(per,ext-3)}(\xi,\zeta,\chi) + I_{-\mathrm{m},-\ell}^{(per,ext-4)}(\xi,\zeta,\chi) \end{aligned} \tag{5.4.36}$$

The technique applied to the evaluation of the evanescent spectrum of the *Perturbed Reflected Cylindrical Functions* in Section 2.1 has been used for the evaluation of evanescent spectrum of the *Perturbed Transmitted Cylindrical Functions*.

5.4.2 Homogeneous Spectrum of perturbed transmitted cylindrical functions

Let us consider homogeneous spectrum defined in equation (5.4.5)

$$I_{\mathrm{m},\ell}^{(per,int)}(\xi,\zeta,\chi) = \int\limits_{|n_\parallel|<1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}}r_1}}{r_1} e^{\mathrm{i m}\cos^{-1}(n_\parallel)} e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.37}$$

In this case $r_1 = \sqrt{1-n_\parallel^2}$ and the value of r_4 depends on the value of u.
For $\mathrm{u} > 0$,

$$r_4 = \begin{cases} i\sqrt{[n_1(n_\parallel+\nu)]^2-1} & \text{if } -1 \le n_\parallel \le (-\frac{1}{n_1}-\nu) \\ \sqrt{1-[n_1(n_\parallel+\nu)]^2} & \text{if } (-\frac{1}{n_1}-\nu) < n_\parallel < (\frac{1}{n_1}-\nu) \\ i\sqrt{[n_1(n_\parallel+\nu)]^2-1} & \text{if } (\frac{1}{n_1}-\nu) \le n_\parallel \le 1 \end{cases} \tag{5.4.38}$$

On the basis of equation (5.4.38), equation (5.4.37) can be written as a sum of three terms:

$$\begin{aligned} I_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) &= O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int+1)}(\xi,\zeta,\chi) \\ &\quad + S_{\mathrm{m}}^{(per,int+2)}(\xi,\zeta,\chi) \end{aligned} \tag{5.4.39}$$

where

$$O_{\mathrm{m},\ell}^{(per,int+)}(\xi,\zeta,\chi) = \int\limits_{(-\frac{1}{n_1}-\nu)}^{(\frac{1}{n_1}-\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{i m}\cos^{-1}(n_\parallel)} e^{-i\sqrt{1-[n_1(n_\parallel+\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.40}$$

$$S_{\mathrm{m},\ell}^{(per,int+1)}(\xi,\zeta,\chi) =$$
$$\int_{(\frac{1}{n_1}-\nu)}^{1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_\mathrm{q}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{\sqrt{1-[n_1(n_\parallel+\nu)]^2-1}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.41}$$

$$S_{\mathrm{m},\ell}^{(per,int+2)}(\xi,\zeta,\chi) =$$
$$\int_{-1}^{(-\frac{1}{n_1}-\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_\mathrm{q}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{\mathrm{im}\cos^{-1}(n_\parallel)} e^{\sqrt{1-[n_1(n_\parallel+\nu)]^2}(\xi+\chi)} e^{in_1(n_\parallel+\nu)\zeta}\, dn_\parallel \tag{5.4.42}$$

In equation (5.4.40), using the substitution

$$t = \cos^{-1}(n_\parallel) \Rightarrow \begin{cases} n_\parallel = \cos t \\ dn_\parallel = -\sin t\, dn_\parallel \\ \cos^{-1}(-\frac{1}{n_1}-\nu) < t < \cos^{-1}(\frac{1}{n_1}-\nu) \end{cases} \tag{5.4.43}$$

we get

$$O_{\mathrm{m},\ell}^{(per,int+)}(\xi,\zeta,\chi) =$$
$$\int_{\cos^{-1}(\frac{1}{n_1}-\nu)}^{\cos^{-1}(-\frac{1}{n_1}-\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_\mathrm{q}\sin t+\mathrm{m}t)} e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t+\nu)]^2}} e^{in_1(\cos t+\nu)\zeta}\, dt \tag{5.4.44}$$

Substituting $n_\parallel = \cos t$, equations (5.4.41) and (5.4.42) can be written as

$$S_{\mathrm{m},\ell}^{(per,int+1)}(\xi,\zeta,\chi) =$$
$$\int_{0}^{\cos^{-1}(\frac{1}{n_1}-\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_\mathrm{q}\sin t+\mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}} e^{in_1(\cos t+\nu)\zeta}\, dt \tag{5.4.45}$$

and

$$S_{\mathrm{m},\ell}^{(per,int+2)}(\xi,\zeta,\chi) = \int\limits_{\cos^{-1}(-\frac{1}{n_1}-\nu)}^{\pi} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t+\mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}} e^{in_1(\cos t+\nu)\zeta}\, dt \tag{5.4.46}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\, dt' \\ \cos^{-1}(\frac{1}{n_1}+\nu) < t' < 0 \end{cases} \tag{5.4.47}$$

equation (5.4.46) can be written as

$$S_{\mathrm{m},\ell}^{(per,int+2)}(\xi,\zeta,\chi) = e^{\mathrm{i}\mathrm{m}\pi} \int\limits_{0}^{\cos^{-1}(\frac{1}{n_1}+\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t-\mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}} e^{-in_1(\cos t-\nu)\zeta}\, dt \tag{5.4.48}$$

where the identity $\cos^{-1}(\frac{1}{n_1}+\nu) = \pi - \cos^{-1}(-\frac{1}{n_1}-\nu)$ has been used.

For u < 0

$$r_4 = \sqrt{1-[n_1(n_\parallel-\nu)]^2}$$

as

$$\nu = \frac{2\pi|\mathrm{u}|}{n_1 L_{\mathrm{nor}}}$$

and

$$r_4 = \begin{cases} i\sqrt{[n_1(n_\parallel-\nu)]^2-1} & \text{if } -1 \le n_\parallel \le (-\frac{1}{n_1}+\nu), \\ \sqrt{1-[n_1(n_\parallel-\nu)]^2} & \text{if } (-\frac{1}{n_1}+\nu) < n_\parallel < (\frac{1}{n_1}+\nu) \\ i\sqrt{[n_1(n_\parallel-\nu)]^2-1} & \text{if } (\frac{1}{n_1}+\nu) \le n_\parallel \le 1 \end{cases} \tag{5.4.49}$$

Equation (5.4.37) can be written as a sum of three terms:

$$I_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) = O_{\mathrm{m}}^{(per,int-)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int-1)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) \tag{5.4.50}$$

where

$$O_{\mathrm{m},\ell}^{(per,int-)}(\xi,\zeta,\chi) = \int\limits_{(-\frac{1}{n_1}+\nu)}^{(\frac{1}{n_1}+\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{i\mathrm{m}\cos^{-1}(n_\parallel)} \times e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{5.4.51}$$

$$S_{\mathrm{m},\ell}^{(per,int-1)}(\xi,\zeta,\chi) = \int\limits_{(\frac{1}{n_1}+\nu)}^{1} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{i\mathrm{m}\cos^{-1}(n_\parallel)} \times e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{5.4.52}$$

$$S_{\mathrm{m},\ell}^{(per,int-2)}(\xi,\zeta,\chi) = \int\limits_{-1}^{(-\frac{1}{n_1}+\nu)} \tau_{10}(n_\parallel,\mathrm{u}) \frac{e^{in_1\chi_{\mathrm{q}}\sqrt{1-n_\parallel^2}}}{\sqrt{1-n_\parallel^2}} e^{i\mathrm{m}\cos^{-1}(n_\parallel)} \times e^{-ir_4(\xi+\chi)} e^{in_1(n_\parallel-\nu)\zeta}\, dn_\parallel \tag{5.4.53}$$

With the position $n_\parallel = \cos t$, the integral in equation (5.4.51) can be written as:

$$O_{\mathrm{m},\ell}^{(per,int-)}(\xi,\zeta,\chi) = \int\limits_{\cos^{-1}(\frac{1}{n_1}+\nu)}^{\cos^{-1}(-\frac{1}{n_1}+\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t+\mathrm{m}t)} e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t-\nu)]^2}} e^{in_1(\cos t-\nu)\zeta}\, dt \tag{5.4.54}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(-\frac{1}{n_1} - \nu) < t' < \cos^{-1}(\frac{1}{n_1} - \nu) \end{cases} \tag{5.4.55}$$

the equation (5.4.54) can be written as

$$O^{(per,int-)}_{\mathrm{m},\ell}(\xi,\zeta,\chi) = \\ e^{\mathrm{im}\pi} \int\limits_{\cos^{-1}(\frac{1}{n_1}-\nu)}^{\cos^{-1}(-\frac{1}{n_1}-\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t - \mathrm{m}t)} e^{-i(\xi+\chi)\sqrt{1-[n_1(\cos t+\nu)]^2}} e^{-in_1(\cos t+\nu)\zeta}\, dt \tag{5.4.56}$$

Comparing equation (5.4.56) and equation (5.4.44), we get

$$O^{(per,int-)}_{\mathrm{m}}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} O^{(per,int+)}_{-\mathrm{m}}(\xi,-\zeta,\chi) \tag{5.4.57}$$

Substituting $n_\parallel = \cos t$ in equations (5.4.52) and (5.4.53), we obtain

$$S^{(per,int-1)}_{\mathrm{m},\ell}(\xi,\zeta,\chi) = \\ \int\limits_{0}^{\cos(\frac{1}{n_1}+\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}} e^{in_1(\cos t-\nu)\zeta}\, dt \tag{5.4.58}$$

Comparing equation (5.4.58) and equation (5.4.48)

$$S^{(per,int+2)}_{\mathrm{m}}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} S^{(per,int-1)}_{-\mathrm{m}}(\xi,-\zeta,\chi) \tag{5.4.59}$$

$$S^{(per,int-2)}_{\mathrm{m},\ell}(\xi,\zeta,\chi) = \\ \int\limits_{\cos^{-1}(-\frac{1}{n_1}+\nu)}^{\pi} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t + \mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t-\nu)]^2-1}} e^{in_1(\cos t-\nu)\zeta}\, dt \tag{5.4.60}$$

Using the substitution

$$t' = \pi - t \Rightarrow \begin{cases} t = \pi - t' \\ dt = -\,dt' \\ \cos^{-1}(\frac{1}{n_1} - \nu) < t' < 0 \end{cases} \tag{5.4.61}$$

the equation (5.4.60) can be written as

$$S_{\mathrm{m},\ell}^{(per,int-2)}(\xi,\zeta,\chi) = \\ e^{\mathrm{im}\pi} \int_0^{\cos^{-1}(\frac{1}{n_1}-\nu)} \tau_{10}(\cos t) e^{i(n_1\chi_{\mathrm{q}}\sin t - \mathrm{m}t)} e^{(\xi+\chi)\sqrt{[n_1(\cos t+\nu)]^2-1}} e^{-in_1(\cos t+\nu)\zeta}\, dt \tag{5.4.62}$$

Comparison between equation (5.4.62) and equation (5.4.45) gives

$$S_{\mathrm{m}}^{(per,int-2)}(\xi,\zeta,\chi) = (-1)^{\mathrm{m}} S_{-\mathrm{m}}^{(per,int+1)}(\xi,-\zeta,\chi) \tag{5.4.63}$$

The final decomposition of the homogeneous spectrum is

$$\begin{aligned} I_{\mathrm{m}}^{(per,int)}(\xi,\zeta,\chi) &= O_{\mathrm{m}}^{(per,int+)}(\xi,\zeta,\chi) + S_{\mathrm{m}}^{(per,int+1)}(\xi,\zeta,\chi) \\ &\quad +(-1)^{\mathrm{m}} S_{-\mathrm{m}}^{(per,int-1)}(\xi,-\zeta,\chi) \end{aligned} \tag{5.4.64}$$

for u > 0 while with u < 0

$$\begin{aligned} I_{\mathrm{m}}^{(per,int)}(\xi,\zeta,\chi) &= (-1)^{\mathrm{m}} O_{-\mathrm{m}}^{(per,int+)}(\xi,-\zeta,\chi) + (-1)^{\mathrm{m}} S_{-\mathrm{m}}^{(per,int+1)}(\xi,-\zeta,\chi) \\ &\quad +S_{-\mathrm{m}}^{(per,int-1)}(\xi,\zeta,\chi) \end{aligned} \tag{5.4.65}$$

For the evaluation of homogeneous spectrum of *Perturbed Transmitted Cylindrical Functions*, we apply the same technique that has been used in the evaluation of the homogeneous spectrum of the *Perturbed Reflected Cylindrical Functions* in Section 2.1.

Chapter 6

Numerical results for Gaussian surface

In this chapter, numerical results for buried cylinders below a Gaussian rough surface using the theory presented in chapter 1 and numerical evaluation of the spectral integrals done in chapter 5 are presented. Different scattering scenarios are investigated for Gaussian beam incidence.

In Section 6.1, scattering from rough surface without any buried object is addressed. The reflected field from a dielectric and perfectly-conducting rough surface has been examined, in order to check the consistency of the technique used to take into account the rough surface scattering. The scattering pattern of a buried object below a rough surface is presented in Section 6.2. Comparison of the results is performed with the available literature. For all simulations, TM polarization has been considered. Results have also been simulated for different values of the spot size of the Gaussian beam at the waist, and radius of the cylinder.

6.1 Scattering from rough surface

In this Section, scattering from a rough surface without any buried scatterer is considered. Figure 6.1 shows the scattering cross-section σ as a function of scattered angle θ. We use a Gaussian beam as incident field on a perfectly-conducting rough surface. The parameters used for this simulation are $\varphi_i = 30^0$, $L_{\text{nor}} = 51.2\pi$, $\ell_{\text{nor}} = 0.7\pi$, and $h_{\text{nor}} = 0.1\pi$. The value of the spot size at the waist is $\mathrm{w}_0 = 10$, with waist center in (-20,0). The results reported in [20] are also shown for comparison. They are obtained in the two cases of MOM solution, developed for a tapered incident field, and analytical Small Perturbation Method solution for plane-wave incidence. In the MOM curve, a distinct wave peak in the specular direction can be noted, which is absent in the other two curves as the results shown in these plots are of incoherent wave only. The result obtained by numerical solution considering Gaussian beam incidence are in good agreement with those obtained by MOM and analytical SPM.

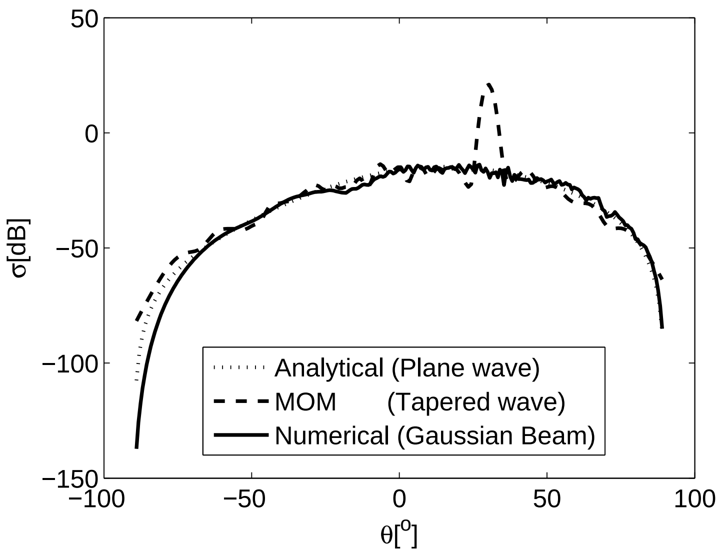


Figure 6.1: Comparison of scattering pattern of a PEC surface for Gaussian beam incidence (solid curve) with MOM and analytical Small Perturbation Method solutions reported in [20], where $\varphi_i = 30^o$, $L_{\text{nor}} = 51.2\pi$, $\ell_{\text{nor}} = 0.7\pi$ and $h_{\text{nor}} = 0.1\pi$.

In figure 6.2, the comparison between scattering pattern of a dielectric rough surface with different values of refractive index of the lower medium n_1 and a perfectly-conducting rough surface shown in figure 6.1 has been performed for a Gaussian beam incidence with $\varphi_i = 30^0$, $L_{\rm nor} = 51.2\pi$, $\ell_{\rm nor} = 0.7\pi$ and $h_{\rm nor} = 0.1\pi$. The value of spot size at the waist is $\mathrm{w}_0 = 10$, with waist center in (-20,0). As the permittivity of the lower medium is increased, the scattering pattern of the dielectric rough surface approaches to that of perfectly-conducting rough surface.

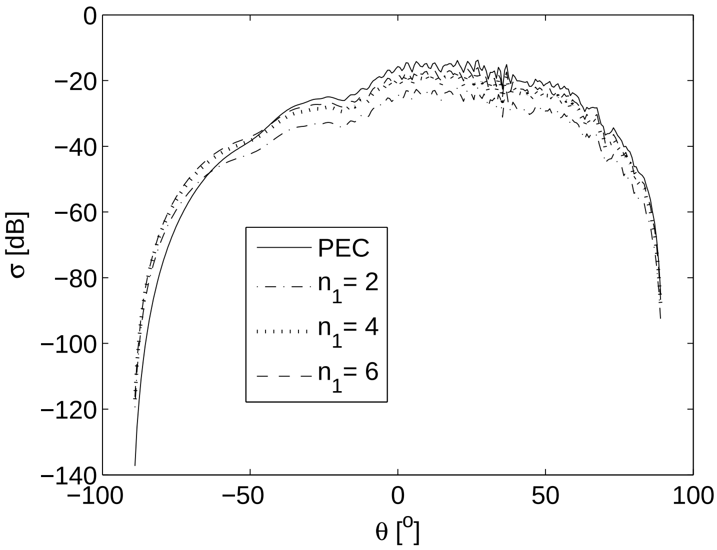


Figure 6.2: Comparison of scattering patterns of a PEC surface and a dielectric rough surface for $\varphi_i = 30^o$, $L_{\rm nor} = 51.2\pi$, $\ell_{\rm nor} = 0.7\pi$ and $h_{\rm nor} = 0.1\pi$.

6.2 Scattering from buried objects

In this Section, the scattering from a buried object below a Gaussian rough surface is addressed. A comparison of the scattering cross-section σ as a function of the scattered angle θ with the one reported in figure 10 in [46] is shown in figure 6.3, where $\varphi_i = -60^o, n_1 = 2, n_{\rm c} = 1.5, \alpha = 0.32\pi, \chi = 2.6\pi$, and $\eta = 0$. The incident field with TM polarization has been considered. The value of the normalized correlation length $\ell_{\rm nor} = 0.6\pi$ has been selected with normalized correlation height $h_{\rm nor} = 0.02\pi$.

The period of the rough surface is $L_{\mathrm{nor}} = 80\pi$. The spot size of the Gaussian beam at the waist is $\mathrm{w}_0 = 40$ with center at (-20,0), while the result in [46] are for plane-wave incidence.

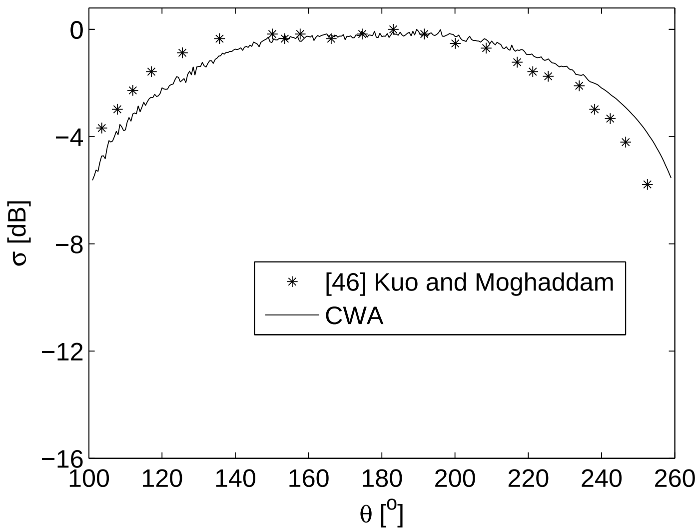


Figure 6.3: Comparison of normalized scattering pattern (normalized to its maximum value) for a Gaussian beam incidence with center in (-20,0) for $n_1 = 2, n_{\mathrm{c}} = 1.5, \alpha = 0.32\pi, \chi = 2.6\pi, \eta = 0, \ell_{\mathrm{nor}} = 0.6\pi, h_{\mathrm{nor}} = 0.02\pi, L_{\mathrm{nor}} = 80\pi$. The spot size at the waist is $\mathrm{w}_0 = 40$.

In figure 6.4, the normalized scattering pattern (normalized to its maximum value) has been shown for different values of the radius of the cylinder for $n_1 = 2, n_{\mathrm{c}} = 1.5, \alpha = 0.32\pi, \chi = 2.6\pi, \eta = 0, \ell_{\mathrm{nor}} = 0.6\pi, h_{\mathrm{nor}} = 0.02\pi, L_{\mathrm{nor}} = 20\pi$, and $\mathrm{w}_0 = 10\pi$. The Gaussian beam with center in (0,0) is normally incident on the rough surface, and TM polarization has been considered. Figure 6.5 shows the normalized scattering pattern (normalized to its maximum value) for different values of w_0, the spot size at the waist of the Gaussian beam, with center in (0,0), for $\varphi_{\mathrm{i}} = 0, n_1 = 2, n_{\mathrm{c}} = 1.5, \alpha = 1, \chi = 2.57, \eta = 0, \ell_{\mathrm{nor}} = 0.6\pi, h_{\mathrm{nor}} = 0.02\pi, L_{\mathrm{nor}} = 40\pi$.

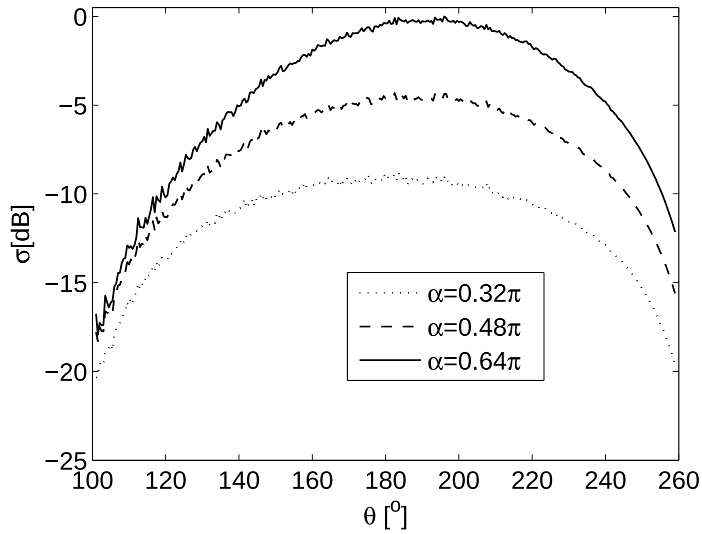


Figure 6.4: Normalized scattering pattern (normalized to its maximum value) for different values of the radius of the cylinder.

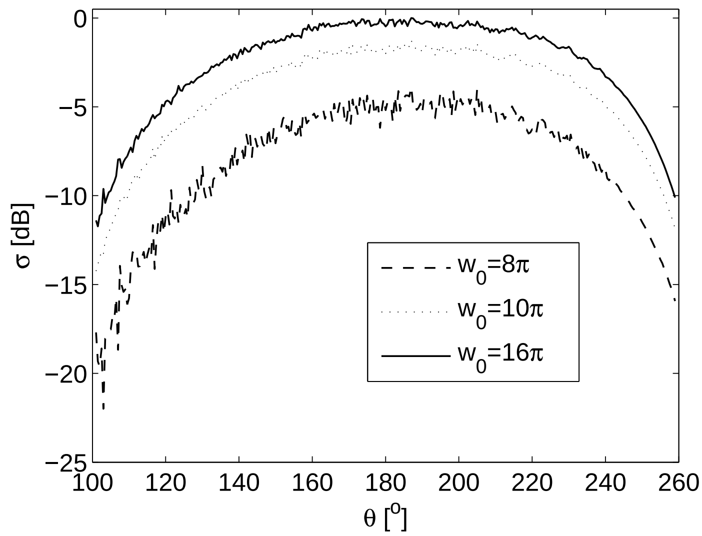


Figure 6.5: Normalized scattering pattern (normalized to its maximum value) for different values of w_0, the spot size at the waist.

Conclusions

A Cylindrical Wave Approach solution to the scattering of a plane wave by cylindrical objects under a rough interface has been presented. Small Perturbation Method has been applied, and suitable reflected and transmitted cylindrical functions dealing with surface perturbations have been introduced. The case of perfectly-conducting or dielectric scatterers can be considered, with a radius at a distance satisfying the limits of validity of Small Perturbation Method.

The numerical implementation has been performed in a Fortran code which can deal both with TM and TE polarization states. It has been obtained through an accurate evaluation of the spectral integrals relevant to the cylindrical functions, thus giving results for both near- and far-field regions.

First, scattering from a buried object beneath a one dimensional rough surface with sinusoidal profile has been considered. Comparisons with results available in the literature for a rough surface with sinusoidal profile have been performed, showing a very good agreement. The case of a single cylinder has been addressed to observe the effect of different values of the radius and permittivity of the cylinder. Period of the sinusoidal surface has been changed to simulate different scenarios and the scattering pattern is compared with the one obtained for flat interface. It is inferred from the results that the scattering pattern of a buried object below a rough surface is only different from the flat surface for a limited values of the period of the sinusoidal surface. Asymptotic solution of the scattered-transmitted field has also been obtained.

The work has been extended to deal with scattering of a plane wave by N cylinders under a rough interface through the *Cylindrical Wave Approach* combined with the

Small Perturbation Method. A set of perfectly-conducting or dielectric scatterers has been considered. In the numerical results, arrangements of cylinders providing self-consistency checks have been presented, and the possibility to simulate scatterers with non-circular cross-section has also been described.

Finally, scattering from a buried object beneath a periodic rough surface with Gaussian roughness spectrum has been considered. For a Gaussian rough surface, a two dimensional Gaussian beam has been used as an incident field to limit the amount of reflected energy, which is required for the detection of the object. To validate the theory, the comparison has been performed with results available in the literature for a Gaussian rough surface. Results has been presented for different values of the spot size at the waist of the Gaussian beam and correlation length of the Gaussian rough surface. The size of the object has also been increased to see the change in the scattering cross-section when the Gaussian beam is incident normally on the rough surface.

The presence of surface irregularities may be an obstacle in both the operational use of *Ground Penetrating Radar* equipment and scattering detection. This work is a first contribution to the possibility of getting a solution to direct scattering in many *Ground Penetrating Radar* applications.

Future works may regard the case of electromagnetic scattering by N cylinders in a layered medium with rough interfaces using *Cylindrical Wave Approach*. The rough perturbation can be dealt with by means of Extended Boundary Conditions to overcome the problem of small roughness limit imposed by Small Perturbation Method. Analysis can also be done for a buried object in a lossy media. The development of the inversion algorithms for buried objects may be done in an efficient manner using *Cylindrical Wave Approach* for rough surface.

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