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Coherent sheaves on primitive multiple curves and their moduli

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Introduction

This work is devoted to the study of coherent sheaves on primitive multiple curves, which are a special kind of non-reduced curves, and of the moduli spaces of semistable ones. It concerns, in particular, the so-called generalized line bundles, extending, as far as possible, the results already known on ribbons (cf. [**CK**]), which are the easiest and most well-known type of primitive multiple curves, i.e. those of multiplicity 2. It is also a partial answer to the first questions posed in [**DEL**, §4], where it is suggested to study the moduli of sheaves on what is there called a ribbon of order n, which is precisely a primitive multiple curve of multiplicity n, with a special attention to a particular kind of sheaves which are exactly generalized line bundles.

Primitive multiple curves. This is only an extremely brief introduction to the subject, for more details we refer to Chapter 1.

A primitive multiple curve $C_n = X$ is a Cohen-Macauley non-reduced but irreducible scheme of dimension 1 over an algebraically closed field $\mathbb{K}$ such that its reduced subscheme $C = X_{red}$ is a smooth projective curve and locally its nilradical is a principal ideal (or, equivalently, X can be locally embedded in a smooth surface). Let $\mathscr{N} \subset \mathscr{O}_X$ be its nilradical, then X is said to be of multiplicity n if $\mathscr{N}^n = 0$ and $\mathscr{N}^{n-1} \neq 0$; in the case $n = 2$ it is just an irreducible ribbon (standard references about ribbons and generalized line bundles on them are [**BE**] and [**EG**]). C_n admits a filtration $C = C_1 \subset C_2 \subset \cdots \subset C_n$, where C_i is a primitive multiple curve of multiplicity i, for $2 \leq i \leq n-1$. Multiple curves were introduced by Bănică and Forster in [**BF**] and primitive ones have been studied by Drézet in various articles, among which there are [**D3**], parametrizing them and inspired by [**BE**], and [**D1**], [**D2**] and [**D4**], where coherent sheaves on them are studied. Note that in all Drézet articles $\mathbb{K}$ is assumed to be $\mathbb{C}$, but it seems that this hypothesis is not needed for the results we will use.

Any coherent sheaf $\mathscr{F}$ on X has two fundamental invariants, introduced in [**D1**]: the generalized rank $\mathrm{R}(\mathscr{F})$ and the generalized degree $\mathrm{Deg}(\mathscr{F})$. A generalized line bundle $\mathscr{F}$ on X is a pure coherent sheaf that is generically a line bundle (i.e. $\mathscr{F}_\eta \cong \mathscr{O}_{X,\eta}$, where η is the generic point of X); in particular, it has generalized rank n. It is relevant to observe that generalized line bundles coincide, in this context, with generalized divisors introduced by Hartshorne in [**H**] (at a level of generality sufficient to comprehend primitive multiple curves); the coincidence is due to [**H**, Proposition 2.8].

Primitive multiple curves and coherent sheaves on them are interesting objects of study mainly because they are a kind of non-reduced curves relatively easy to handle (particularly, in the case of multiplicity 2) and non-reduced schemes and sheaves on them are still quite unknown. There are also applications to the study of reduced curves, but these aspects are not directly connected to our study. They are also significant because of the fact that, when they have a retraction to the reduced subcurve, they are involved in the so-called spectral correspondence (for a brief introduction about twisted Higgs pairs and spectral correspondence cf. [**MRV2**, Appendix]): if C is a smooth projective curve, the spectral cover associated to nilpotent Higgs pairs of rank n over C is a primitive multiple curve of multiplicity n with reduced subcurve C and there is an isomorphism between the moduli space of (semistable) pure coherent sheaves of generalized rank n on it and (semistable) nilpotent Higgs pairs of rank n over C. This relation with the extremely active research area of Higgs bundles will be described with some more details in Appendix A.

Moduli space of sheaves. The study of spaces classifying coherent sheaves of a certain rank and degree on algebraic schemes has ancient roots. An attempt of writing an exhaustive history of the problem would exceed the limits of both my knowledge and an introduction. The firstly studied case, nowadays classical, is that of the Jacobian variety, parametrizing line bundles (or invertible sheaves, throughout this work they will be used as synonyms) of degree 0 over a smooth and projective curve; it has been introduced, in the case of curves over $\mathbb{C}$, in the nineteenth century; for an introduction to its classical theory cf. [**M**]. Now it is seen as the identity component of the group scheme parametrizing line bundles, without restrictions on the degree, on the smooth projective curve, called the Picard scheme (it exists for proper schemes of any dimension but only in the case of curves the identity component is called Jacobian variety; for its construction and main properties cf. [**K**]); for a brief account about the development of the theory (comprehending also that of the Jacobian variety) from the remote origins at the end of the seventeenth century until Grothendieck's contributions in the sixties of the twentieth one, cf. [**K**, pages 237-249].

A huge difficulty to parametrizing sheaves that are not invertible (present also for vector bundles of any rank strictly greater than 1 on a smooth projective curve) is that they form unbounded families. The key to overcoming this obstacle is the notion of stability, which depends on a polarization of the scheme in dimension greater than or equal to 2; for a general introduction to it cf. [**HL**, Chapter 1] and for an extremely concise history of its development see the comments at the end of the cited chapter. The most famous construction of the moduli space of semistable sheaves on a projective polarized scheme (in characteristic 0, but it has been later extended to any characteristic) is due to Simpson (cf. [**Si**]), although there are also older constructions, in particular in the case of integral curves. A presentation of Simpson moduli space, with special attention to the case of surfaces and also some historical references, can be found in the textbook [**HL**] (the construction is given in the fourth chapter). An account of the basic case of vector bundles over smooth projective curves can be found in [**LP**, Part I]. There are also results avoiding stability conditions, but in this case one gets a moduli stack and not a moduli scheme.

Clearly, the existence of the construction does not imply that all the properties of the moduli scheme are known. There are very few studies about sheaves on a base scheme which is non-reduced: according to my knowledge in arbitrary dimension (but with special attention to curves and degenerate quadric surfaces) there is Inaba's article [**I**], while in the case of curves there are Drézet's studies [**D1**], [**D2**] and [**D4**] for primitive multiple curves, that of Chen and Kass about the compactified Jacobian of a ribbon [**CK**] (see also [**Sa**], which completes their description of the irreducible components covering a case there left open), and Yang's one [**Y**] about coherent sheaves on fat curves (within which there are ribbons and, more generally, ropes, but not primitive multiple curves of higher multiplicity). Some of the results of [**CK**] had already been stated, without proofs and under more restrictive hypotheses, by Donagi, Ein and Lazarsfeld in [**DEL**].

Going back to the case of line bundles, it is convenient to point out that a connected component of the Picard scheme of a non-smooth scheme is not necessarily smooth and this fact leads to the problem of its compactification. It is a highly non-trivial problem also for singular curves; for the case of reduced curves, cf., e.g., [**MRV1**] and its references.

In the case of primitive multiple curves of multiplicity n the situation is easier because a standard compactification of the Picard scheme (when the line bundles are stable, so, as we will see later, when the degree of the conormal sheaf of the reduced subcurve in the primitive multiple curve is negative) is simply the moduli space of semistable sheaves of generalized rank n, which include generalized line bundles, and fixed generalized degree. This moduli scheme is projective and contains as an open the connected component of the Picard scheme of line bundles of the fixed generalized degree (which for some generalized degrees is empty, it depends on the congruency class with respect to n of the generalized degree). The main aim of this thesis is precisely the study of the moduli space of semistable sheaves of generalized rank n and fixed generalized degree on a primitive multiple curve of multiplicity n.

Structure of the work and main results. This thesis begins with an introductory chapter about the theory of coherent sheaves on a primitive multiple curve, collecting the results and tools which will be used in the next ones; it is almost entirely based on [**D1**] and [**D2**]. It is divided in six sections: the first one recalls the definition of a primitive multiple curve and its basic properties. The second one treats briefly line bundles and the Picard scheme of C_n, a primitive multiple curve of multiplicity n. The third one introduces the two canonical filtrations of a coherent sheaf on C_n and their main properties. The forth section is about two fundamental invariants of a coherent sheaf: the generalized rank and the generalized degree. There we explain also their relation with ordinary rank and degree. The fifth one recalls the equivalent (on a primitive multiple curve) notions of pure sheaf of dimension 1, torsion-free sheaf and reflexive sheaf. It treats also duality of sheaves and, in particular, the relations between the two canonical filtrations of dual sheaves. Finally, the sixth section is a brief overview about semistability of sheaves on a primitive multiple curve. We do not specify slope or Gieseker semistability because, as we will see in this section, these notions are equivalent on primitive multiple curves, as on smooth projective ones.

Chapter 2 is concerned with various properties of generalized line bundles on C_n, although the first section is more generally about pure sheaves of generalized rank n on C_n. The first two sections are inspired by the case of ribbons treated in [**CK**, §2], according to my knowledge all the results are new (in higher multiplicity, i.e. in multiplicity greater than or equal to 3). In particular, we introduce the indices $b_1(\mathscr{F}), \ldots, b_{n-1}(\mathscr{F})$ (which are non-negative integers) of a generalized line bundle $\mathscr{F}$ on C_n, which will play a significant role throughout the whole work, and the associated torsion sheaves $\mathscr{T}_i(\mathscr{F})$ for $1 \leq i \leq n-1$ (see Definition 2.16).

The third section of this chapter studies the structure of a generalized line bundle on a primitive multiple curve. While it is quite easy to describe it on a ribbon (cf. [**EG**, Theorem 1.1]), the situation is much more complicated in higher multiplicity. The main result is the following:

Theorem A. *Let $\mathscr{F}$ be a generalized line bundle on C_n. Then $\mathscr{F}$ is isomorphic to $\mathscr{I}_{Z/C_n} \otimes \mathscr{G}$, where $Z \subset C_{n-1}$ is a closed subscheme of finite support whose schematic intersection with C is* $\operatorname{Supp}(\mathscr{T}_{n-1}(\mathscr{F}))$, *called the subscheme associated to $\mathscr{F}$, and $\mathscr{G}$ is a line bundle on C_n.*
Moreover

(i) *Z is unique up to adding a Cartier divisor.*
(ii) *Locally isomorphic generalized line bundles have the same associated subscheme, up to adding a Cartier divisor. In particular, if $\mathscr{F}$ and $\mathscr{F}'$ are locally isomorphic generalized line bundles, then there exists a line bundle $\mathscr{E}$ such that $\mathscr{F} = \mathscr{F}' \otimes \mathscr{E}$. Equivalently, there is a transitive action of* $\operatorname{Pic}(X)$ *on the set of locally isomorphic generalized line bundles.*

In the text it appears as Corollary 2.32, because it is a consequence of the extremely involved local description, given in Theorem 2.27, and [**H**, Proposition 2.12]. The above cited action is studied with particular attention for some special types of generalized line bundles which will play a fundamental role in determining the irreducible components of the moduli space (see Corollaries 2.35 and 2.38).

The last section of the chapter, i.e. Section 2.4, studies semistability of generalized line bundles on C_n; the main result is Theorem 2.41:

Theorem B. *Let $\mathscr{F}$ be a generalized line bundle of generalized degree D on C_n with indices $b_1(\mathscr{F}), \ldots, b_{n-1}(\mathscr{F})$. Then $\mathscr{F}$ is semistable if and only if the following inequalities hold:*

$$i \sum_{j=i}^{n-1} b_j(\mathscr{F}) - (n-i) \sum_{j=1}^{i-1} b_j(\mathscr{F}) \leq -\frac{in(n-i)}{2} \deg(\mathscr{N}/\mathscr{N}^2), \ \forall\, 1 \leq i \leq n-1,$$

where $\mathscr{N}$ is the nilradical of $\mathscr{O}_{C_n}$.

It is stable if and only if all the inequalities are strict.

The case of ribbons had already been treated in [**CK**, §3]. Another significant result of this section is the computation of a surprisingly canonical

Jordan-Holder filtration of a strictly semistable generalized line bundle (see Proposition 2.44).

Chapter 3 studies the irreducible components of the moduli space of stable generalized rank n sheaves on C_n that contain stable generalized line bundles. It extends some results of [**CK**, §4] to higher multiplicity and it is divided into two sections, the first about multiplicity 3, about which we know something more, and the other about multiplicity greater than or equal to 4. These irreducible components of stable generalized line bundles, which are all of the same dimension, when they exist, are completely described. There are also some results about the local geometry of the moduli space; in particular, we compute the dimension of the tangent space to points representing some special (any in multiplicity 3) generalized line bundles (see Propositions 3.9 and 3.21), including, in particular, the generic elements of the irreducible components, which result to be generically smooth only when their generic element is a line bundle. The main results, which are Theorems 3.6 and 3.8 and Corollary 3.10 for multiplicity 3 and Theorems 3.16 and 3.19 and Corollary 3.22 for higher multiplicity, can be summarized in a simplified version as follows:

THEOREM C. *Let C_n be a primitive multiple curve of arithmetic genus g_n such that $\deg(\mathscr{N}/\mathscr{N}^2) < 0$, where $\mathscr{N}$ is the nilradical of $\mathscr{O}_{C_n}$.*

(i) *The closure of the locus of stable generalized line bundles of fixed indices $b_1, \dots, b_{n-1}$ on C_n, $\bar{Z}_{b_1,\dots,b_{n-1}}$, is a g_n-dimensional irreducible component of the moduli space of semistable sheaves of generalized rank n (when this locus is not empty).*

(ii) *The union of these loci is connected for $n = 3$ or for $n \geq 4$ and $\deg(\mathscr{N}/\mathscr{N}^2)$ sufficiently small.*

(iii) *The tangent space to the generic point of $\bar{Z}_{b_1,\dots,b_{n-1}}$ has dimension $g_n + \sum_{i=[\frac{n+1}{2}]}^{n-1} b_i - \sum_{i=1}^{[\frac{n-2}{2}]} b_i$.*

It was not possible to get a complete description of the irreducible components of the moduli space, because within semistable pure coherent sheaves of generalized rank n on C_n there are also direct images of semistable pure coherent sheaves of generalized rank n on C_i, for any $1 \leq i \leq n-1$ and they are quite hard to handle, in general. In the case of ribbons, treated in [**CK**], this is not a real problem because there is only C_1, i.e. the reduced subcurve, to be considered and pure sheaves of generalized rank 2 on it are just vector bundles of rank 2, whose moduli space is well-known.

We describe the state of the art about these other components in Chapter 4. There we give also the following conjecturally picture (which is Conjecture 4.49) of the irreducible components of $\mathrm{M}(C_n, P_D)$, the moduli space of semistable sheaves of generalized rank n and generalized degree D on C_n, in higher multiplicity:

CONJECTURE D. *Let C_n be a primitive multiple curve of multiplicity n such that $\delta = -\deg(\mathscr{N}/\mathscr{N}^2) > 0$ and $g_1 \geq 2$, where g_1 is the genus of its reduced subcurve.*

(i) *If $\delta \leq 2(g_1 - 1)$, then the irreducible components of $\mathrm{M}(C_n, P_D)$ are the closures of the loci of stable sheaves of fixed complete type, for* *each complete type for which stable sheaves exist. For each type, there is at least one irreducible component whose generic element is of that type.*

(ii) *If* $\delta > 2(g_1 - 1)$*, the only irreducible components of* $\mathrm{M}(C_n, P_D)$ *are those whose generic elements are generalized line bundles.*

See also its special, and more precise, case for $n = 3$, that is Conjecture 4.40. Until now, we were not able to prove it. In multiplicity 3 the first part is almost done, it remains only to prove that these components are really all the components, while for the second part we only know that the closure of the locus of rank 3 vector bundles on the reduced curve is not a component, for more details see the discussion after Conjecture 4.40. In higher multiplicity we have only few results and the conjecture is deeply inspired by the cases of low multiplicity and by the studies about the nilpotent cone of Higgs bundles.

This last chapter collects also some other complementary results about coherent sheaves on primitive multiple curves, with a special attention to those on ribbons. The latter are studied in detail in the long Section 4.2, whose main results are about the semistability conditions for quasi locally free sheaves on a ribbon (see Theorem 4.19) and about the deformation of vector bundles on C to sheaves defined on the ribbon (see Propositions 4.25 and 4.26). The following is a simplified version of Conjecture 4.39 about the irreducible components of the moduli space of stable sheaves on a ribbon.

Conjecture E. *Let* X *be a ribbon such that* $g_1 \geq 2$*, where* g_1 *is the genus of the reduced subcurve, let* $\delta = -\deg(\mathscr{N})$*, where* $\mathscr{N}$ *is the nilradical of* $\mathscr{O}_X$*, and let* $M = \mathrm{M}_s(X, R, D)$ *be the moduli space of stable sheaves of generalized rank* R *and generalized degree* D*.*

(i) *Assume* $0 < \delta \leq 2g_1 - 2$*. For any possible complete type* $((r_0, r_1), (d_0, d_1))$ *with* $r_0 > r_1 > 0$ *and verifying strictly inequalities* (4.6)*, the closure of the locus of quasi locally free stable sheaves of complete type* $((r_0, r_1), (d_0, d_1))$ *is a* $(1 + (r_0^2 + r_1^2)(g_1 - 1) + r_0 r_1 \delta)$*-dimensional irreducible component of* M*. Distinct complete types correspond to distinct irreducible components. Also the closure of the locus of stable rank* R *vector bundles of degree* D *on* C *is an irreducible component, which has dimension* $1 + R^2(g_1 - 1)$*. If* R *is odd, these are all the irreducible components of* M*. On the other hand, if* $R = 2r$ *is even, also the closure of the locus of stable generalized vector bundles of generalized rank* R *and degree* D *and fixed index* $b < r\delta$ *is an irreducible component of* M *of dimension* $1 + 2r^2(g_1 - 1) + r^2\delta$*. Distinct indices correspond to distinct components and there are no other irreducible components.*

(ii) *If* $\delta > 2g_1 - 2$*, then we have to distinguish two cases.*

   (a) *If* $R = 2r$ *is even, then the only irreducible components of* M *are the closures of the loci of stable generalized vector bundles of generalized rank* R *and degree* D *and fixed index* $b < r\delta$ *and they have dimension* $1 + 2r^2(g_1 - 1) + r^2\delta$*.*

   (b) *If* $R = 2a + 1$*, then the only irreducible components of* M *are the closures of the loci of stable quasi locally free sheaves of rigid type of generalized rank* R *and generalized degree* D

verifying strictly inequalities (4.6)*. They have dimension* $1 + (a^2 + a)\delta + (2a^2 + 2a + 1)(g_1 - 1)$.

The dimensional results are all known. Also the irreducibility of the loci of quasi locally free sheaves of fixed complete type is known. In the first part of the conjecture (i.e. $\delta \leq 2g_1 - 2$) the only conjectural parts are that the loci of generalized vector bundles of fixed index are irreducible, that the cited loci are irreducible components and that they are all the irreducible components. The second part, i.e. $\delta > 2g_1 - 2$ is a reformulation of Conjecture 4.28. It is also known that in this case the closure of the locus of stable vector bundles on the reduced curve is not a component. For more details we refer to the discussion after Conjecture 4.39.

The thesis ends with a brief appendix about the relation between coherent sheaves on primitive multiple curves and nilpotent Higgs bundles on primitive multiple curves. In particular, we recall briefly the spectral correspondence for the nilpotent cone and the results of [**Bo**] linked to our conjectures. We also give formulae relating the complete type of a coherent sheaf on a primitive multiple curve being a spectral cover with both the nilpotent type and the Jordan type of the corresponding nilpotent Higgs bundle.

CHAPTER 1

Generalities on sheaves on primitive multiple curves

As anticipated in the introduction, this chapter collects definitions and properties of primitive multiple curves (in the first section) and the basis of the theory of coherent sheaves on them developed in [**D1**], [**D2**] and [**D4**] (in the next ones). Here we fix also notations and conventions which will be used throughout the work.

1.1. Primitive multiple curves

This section is based on [**D1**, §2.1] and [**D2**, §2.1].

Definition 1.1. A *primitive multiple curve* of *multiplicity* n is an irreducible Cohen-Macaulay algebraic scheme $(X = C_n, \mathscr{O}_X = \mathscr{O}_{C_n})$ over an algebraically closed field $\mathbb{K}$ such that:

(i) its reduced subscheme $(X_{red}, \mathscr{O}_{X_{red}})$ is a smooth projective curve $(C, \mathscr{O}_C)$ over $\mathbb{K}$;

(ii) the multiplicity n is the least natural number such that $\mathscr{N}^n = 0$, where $\mathscr{N} = \ker(\mathscr{O}_X \twoheadrightarrow \mathscr{O}_C)$ is the *nilradical* ideal sheaf of $\mathscr{O}_X$;

(iii) it is locally embedded in a smooth surface, i.e. any closed point admits a neighbourhood that can be embedded in a smooth surface, or, equivalently, the nilradical is locally a principal ideal.

Remark 1.2. If hypothesis (iii) is omitted, then X is called a *multiple curve* or a *multiple structure* over C, but in this work only primitive ones are treated.

Observe that a primitive multiple curve of multiplicity 1 is just a smooth projective curve and a primitive multiple curve of multiplicity 2 is just a ribbon over a smooth projective curve (which from now on will be called simply a ribbon). The topological space underlying a primitive multiple curve is homeomorphic to that of its reduced subcurve, but the structure sheaves are quite different. Indeed, if $P \in X$ is a closed point, it holds that $\mathscr{O}_{X,P} = \mathscr{O}_{C,P} \otimes_{\mathbb{K}} (\mathbb{K}[y]/(y)^n)$.

The above definition is not the original one, used in [**D1**] and given in terms of an ambient three-fold, but this abstract one, used e.g. in [**D2**], is equivalent to the embedded one by [**D3**, Théorème 5.3.2].

The arithmetic genus of C_n, equal to $1 - \chi(C_n, \mathscr{O}_{C_n})$, will be denoted by $g(C_n) = g_n$ and will be called simply the *genus* of C_n (more generally for any curve Y that will appear throughout the work its genus will be $g(Y) = 1 - \chi(Y, \mathscr{O}_Y)$).

There is a *canonical filtration* of X by closed subschemes

$$C_1 = C \subset C_2 \subset \cdots \subset C_{n-1} \subset C_n = X,$$

where C_i is a primitive multiple curve of multiplicity i (whose genus will be denoted by g_i) with reduced subcurve C and such that its ideal sheaf in X is $\mathscr{I}_{C_i/X} = \mathscr{N}^i$, for $1 \leq i \leq n-1$. It holds that $\mathscr{N}$ is a line bundle over C_{n-1} and there exists a line bundle $\mathscr{C}$ over C_n extending it.

The *conormal* sheaf of C in X is $\mathscr{N}/\mathscr{N}^2$ and is denoted by $\mathcal{C}$. It is a line bundle over C and it plays a quite important role in the study of X: if its degree is negative, X has no non-constant global sections (cf. [**D2**, §2.6]) and so in this case $g_n = \mathrm{h}^1(X, \mathscr{O}_X)$. Moreover, $\mathscr{I}_{C_i/X}/\mathscr{I}_{C_{i+1}/X} = \mathscr{N}^i/\mathscr{N}^{i+1}$ is a line bundle on C and it is equal to $\mathcal{C}^{\otimes i}$, for $2 \leq i \leq n-1$. The nilradical ideal of $\mathscr{O}_{C_i}$ is $\mathscr{N}/\mathscr{N}^i$, for any $2 \leq i \leq n-1$. This implies that the conormal sheaf of C with respect to C_i is again $\mathcal{C}$ (indeed, it is evident that $(\mathscr{N}/\mathscr{N}^i)/(\mathscr{N}/\mathscr{N}^i)^2 = \mathscr{N}/\mathscr{N}^2$).

A primitive multiple curve is called *split* if it admits a retraction to the reduced subcurve. A primitive multiple curve of multiplicity n is *trivial*, if it is the n-th infinitesimal neighbourhood of a smooth projective curve C in the geometric vector bundle associated to $\mathcal{L}^*$, where $\mathcal{L}$ is a line bundle on C. These are the only primitive multiple curves that appear in the spectral correspondence. Any trivial primitive multiple curve is split. The converse holds in general only in multiplicity 2 (essentially by [**BE**, Proposition 1.1]), while it is false in higher multiplicity (cf. [**D3**, §1.1.6]).

1.2. Line bundles and the Picard scheme

In this short section we collect some useful facts about line bundles on C_n and its Picard group.

The first relevant properties are the following, which are, respectively, [**D1**, Théorème 3.1.1 and §3.1.5]:

FACT 1.3.

(i) *For $1 \leq i \leq n-1$, any line bundle (and more generally any vector bundle) on C_i extends to a line bundle (resp. vector bundle of the same rank) on C_n.*

(ii) *Let $\mathscr{L}$ be a line bundle on C_{n-1} and $\mathcal{L} = \mathscr{L}|_C$. Then there is a short exact sequence*

$$0 \to H^1(\mathcal{C}^{n-1}) \to \mathrm{Ext}^1_{\mathscr{O}_{C_n}}(\mathcal{L}, \mathscr{L} \otimes \mathscr{N}) \xrightarrow{\pi} \mathbb{K} \to 0.$$

Moreover, the set $P_{\mathscr{L}}$ of line bundles on C_n that extend $\mathscr{L}$ is identified with $\pi^{-1}(1)$, which is an affine space isomorphic to $H^1(\mathcal{C}^{n-1})$. In particular, the bijection between the latter and $P_{\mathscr{O}_{C_{n-1}}}$ is an isomorphism of abelian groups.

On a primitive multiple curve C_n there exists, by, e.g., [**BLR**, Theorem 8.2.3], the so-called Picard scheme $\mathrm{Pic}(C_n)$. It is a scheme locally of finite type parametrizing line bundles on C_n and endowed with a tautological line bundle, called the Poincaré line bundle, over $\mathrm{Pic}(C_n) \times C_n$. A general introduction to the rich theory of relative and absolute Picard schemes can be found in [**K**] or in [**BLR**, Chapter 8].

The following fact is an application to our case of the general theory; in particular, the first point follows from [**K**, Proposition 9.5.3] for separateness, from [**K**, Proposition 9.5.19] for the smoothness and from [**BLR**, Theorem 8.4.1] or, equivalently, [**K**, Corollary 9.5.13] for the dimension. The second and the third point are contained in [**D1**, §3.3], while the last assertion follows from the general theory and from the previous points and it is inspired by the case of ribbons treated in [**CK**, Fact 2.10].

FACT 1.4. *Let C_n be a primitive multiple curve. Then*

(i) *The Picard scheme* $\mathrm{Pic}(C_n)$ *for* C_n *is smooth, separated and of dimension* $\mathrm{h}^1(C_n, \mathscr{O}_{C_n})$.

(ii) *Its irreducible components are the varieties parametrizing the line bundles on* C_n *whose restrictions to* C *have fixed degree* j.

(iii) *There are two short exact sequences of abelian group schemes:*

$$0 \to P_{\mathscr{O}_{C_{n-1}}} \simeq H^1(\mathcal{C}^{n-1}) \to \mathrm{Pic}(C_n) \to \mathrm{Pic}(C_{n-1}) \to 0,$$

$$0 \to \mathbf{P}_n \to \mathrm{Pic}(C_n) \to \mathrm{Pic}(C) \to 0;$$

where $\mathbf{P}_n \subset \mathrm{Pic}(C_n)$ *is the affine subgroup scheme of line bundles with trivial restriction to* C*. Moreover, there exists a filtration of group schemes* $0 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} = \mathbf{P}_n$ *such that* $G_i/G_{i-1} \simeq H^1(\mathcal{C}^i)$.

(iv) *The component of the identity, which is called the* Jacobian variety*, is not proper if and only if* $\mathbf{P}_n \neq 0$*. The latter holds, in particular, if* $\mathrm{h}^1(C, \mathcal{C}) \neq 0$*, and, thus, if* $\deg(\mathcal{C}) \leq g_1 - 2$*, where* g_1 *is the genus of* C.

REMARK 1.5. The first assertion of Fact 1.4 is true for any projective curve over a field. In the following, we will consider the Picard scheme also of some blowing-ups of a primitive multiple curve which are not necessarily primitive multiple curves themselves.

1.3. Canonical filtrations

The aim of this section is to introduce two tools, which are fundamental to study a coherent sheaf on a primitive multiple curve: the so-called canonical filtrations. The first one has been introduced by Drézet in [**D1**, §4.1], while the second one has been studied for the first time by Inaba, although in a more general context (cf. [**I**, §1]), as Drézet himself points out. Our presentation will essentially follow Drézet's article.

Before starting with their definitions, which are given not only for sheaves but also for finitely generated $\mathscr{O}_{C_n,P}$-modules, where P is a closed point of C_n, it is useful to fix some more conventions.

Throughout the work, if $\mathscr{F}$ will be a coherent sheaf on C_i for some $1 \leq i \leq n-1$, its direct image on C_n will be denoted again by $\mathscr{F}$ and they will be treated as if they were the same object. All the sheaves studied throughout the thesis will be coherent, so this attribute will be omitted. Vector (resp. line) bundle will be used as a synonym of locally free sheaf of finite rank (resp. of rank 1).

It is also convenient to fix the notation for the local set-up: set $A_i := \mathscr{O}_{C_i,P}$, for $1 \leq i \leq n$, where P is a closed point. This implies that $A_n = A_1 \otimes_{\mathbb{K}} \mathbb{K}[y]/(y^n)$; moreover, $A_i = A_n/(y^i)$, for $1 \leq i \leq n-1$. The following definitions could more generally be made when A_1 is a DVR and A_n a local ring whose nilradical is principal, generated by an element y such that

$y^n = 0 \neq y^{n-1}$, and whose reduced ring is A_1 and, in this more general context, the A_i could be defined as $A_n/(y^i)$, but for this work we do not need that generality.

DEFINITION 1.6.

(i) The *first canonical filtration* of a finitely generated A_n-module M is

$$\{0\} = M_n \subseteq M_{n-1} = y^{n-1}M \subseteq \cdots \subseteq M_1 = yM \subseteq M_0 = M.$$

Equivalently, M_i is equal to $\ker(M_{i-1} \twoheadrightarrow M_{i-1} \otimes_{A_n} A_1)$, for $1 \leq i \leq n$.

The *first graded object* of M is $\mathrm{Gr}_1(M) := \bigoplus_{i=0}^{n-1} M_i/M_{i+1}$.

(ii) The *first canonical filtration* of a sheaf $\mathscr{F}$ on C_n is, analogously,

$$0 = \mathscr{F}_n \subseteq \mathscr{F}_{n-1} = \mathscr{N}^{n-1}\mathscr{F} \subseteq \cdots \subseteq \mathscr{F}_1 = \mathscr{N}\mathscr{F} \subseteq \mathscr{F}_0 = \mathscr{F}.$$

Equivalently, $\mathscr{F}_i$ is equal to $\ker(\mathscr{F}_{i-1} \twoheadrightarrow \mathscr{F}_{i-1}|_C)$, for $1 \leq i \leq n$.

The *first graded object* of $\mathscr{F}$ is

$$\mathrm{Gr}_1(\mathscr{F}) = \bigoplus_{i=0}^{n-1} G_i(\mathscr{F}) := \bigoplus_{i=0}^{n-1} \mathscr{F}_i/\mathscr{F}_{i+1}.$$

The *complete type* of $\mathscr{F}$ is

$$\Big(\big(\mathrm{rk}(G_0(\mathscr{F})), \ldots, \mathrm{rk}(G_{n-1}(\mathscr{F}))\big), \big(\deg(G_0(\mathscr{F})), \ldots, \deg(G_{n-1}(\mathscr{F}))\big)\Big).$$

The following remark collects some easy properties of the first canonical filtration.

REMARK 1.7. For any $1 \leq i \leq n-1$, $M_i/M_{i+1} = M_i \otimes_{A_n} A_1$, while $M/M_i \cong M \otimes_{A_n} A_i$ (analogously $\mathscr{F}_i/\mathscr{F}_{i+1} = \mathscr{F}_i|_C$ and $\mathscr{F}/\mathscr{F}_i \cong \mathscr{F}|_{C_i}$).

Again for any $1 \leq i \leq n-1$, $M_i = \{0\}$ (respectively $\mathscr{F}_i = 0$) if and only if M is an A_i-module (resp. $\mathscr{F}$ is a sheaf on C_i). M_i (resp. $\mathscr{F}_i$) is an A_{n-i}-module (resp. a sheaf on C_{n-i}) with first canonical filtration $\{0\} \subseteq M_{n-1} \subseteq \cdots \subseteq M_{i+1} \subseteq M_i$ (resp. $0 \subseteq \mathscr{F}_{n-1} \subseteq \cdots \subseteq \mathscr{F}_{i+1} \subseteq \mathscr{F}_i$).

Any morphism of A_n-modules (resp. of sheaves over C_n) maps the first canonical filtration of the first module (resp. sheaf) to that of the second one.

The first canonical filtration (and thus also the related invariants of generalized rank and degree, cf. Definition 1.11) could be defined exactly in the same way for a multiple curve not necessarily primitive.

DEFINITION 1.8.

(i) The *second canonical filtration* of a finitely generated A_n-module M is

$$\{0\} = M^{(0)} \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(n-1)} \subseteq M^{(n)} = M,$$

where $M^{(i)} := \{m \in M | y^i m = 0\}$, for $1 \leq i \leq n$.

The *second graded object* of M is $\mathrm{Gr}_2(M) := \bigoplus_{i=1}^{n} M^{(i)}/M^{(i-1)}$.

(ii) The *second canonical filtration* of a sheaf $\mathscr{F}$ on C_n is defined analogously and is denoted by

$$0 = \mathscr{F}^{(0)} \subseteq \mathscr{F}^{(1)} \subseteq \cdots \subseteq \mathscr{F}^{(n-1)} \subseteq \mathscr{F}^{(n)} = \mathscr{F}.$$

The *second graded object* of $\mathscr{F}$ is

$$\mathrm{Gr}_2(\mathscr{F}) = \bigoplus_{i=1}^{n} G^{(i)}(\mathscr{F}) := \bigoplus_{i=1}^{n} \mathscr{F}^{(i)}/\mathscr{F}^{(i-1)}.$$

For any $1 \leq i \leq n-1$, it holds that $M_{n-i} \subset M^{(i)}$ (resp. $\mathscr{F}_{n-i} \subset \mathscr{F}^{(i)}$) and that $M^{(i)}$ (resp. $\mathscr{F}^{(i)}$) is an A_i-module (resp. a sheaf on C_i) with second canonical filtration $\{0\} \subseteq M^{(1)} \subseteq \cdots \subseteq M^{(i-1)} \subseteq M^{(i)}$ (resp. $0 \subseteq \mathscr{F}^{(1)} \subseteq \cdots \subseteq \mathscr{F}^{(i-1)} \subseteq \mathscr{F}^{(i)}$). Any morphism of A_n-modules (resp. of sheaves on C_n) maps the second canonical filtration of the first module (resp. sheaf) to that of the second one.

The following fact collects some properties, proved by Drézet, about the two canonical filtrations.

FACT 1.9. *Let $\mathscr{F}$ be a sheaf on C_n.*

(i) *(*[**D2**, Proposition 3.1(i)]*) There is a canonical isomorphism between $\mathscr{F}_i$ and $(\mathscr{F}/\mathscr{F}^{(i)}) \otimes \mathscr{C}^{\otimes i}$, for any $0 \leq i \leq n$.*

(ii) *(*[**D2**, Proposition 3.7]*) For any $0 \leq i \leq n-1$, it holds that $\mathrm{rk}(G^{(i+1)}(\mathscr{F})) = \mathrm{rk}(G_i(\mathscr{F}))$ and $\deg(G^{(i+1)}(\mathscr{F})) = \deg(G_i(\mathscr{F})) + \big(\sum_{j=i+1}^{n-1} \mathrm{rk}(G_j(\mathscr{F})) - i\,\mathrm{rk}(G_i(\mathscr{F}))\big)\deg(\mathcal{C})$.*

(iii) *(*[**D2**, Proposition 3.3 and Corollaire 3.4]*) Consider the canonical morphism defined by multiplication $\nu : \mathscr{F} \otimes \mathscr{C} \to \mathscr{F}$. Then:*

   (a) *ν induces injective morphisms $\lambda_{i,k} = \lambda_{i,k}(\mathscr{F}) : G^{(i+1)}(\mathscr{F}) \otimes \mathcal{C}^k \hookrightarrow G^{(i+1-k)}(\mathscr{F})$, for any integers $0 < i < n$, $0 < k \leq i+1$;*

   (b) *ν induces surjective morphisms $\mu_{j,m} = \mu_{j,m}(\mathscr{F}) : G_j(\mathscr{F}) \otimes \mathcal{C}^m \twoheadrightarrow G_{j+m}(\mathscr{F})$, for any non-negative integer j and positive one m such that $j+m \leq n-1$;*

   (c) *there is a canonical isomorphism $\Gamma_i(\mathscr{F}) \simeq \Gamma^{(i)}(\mathscr{F}) \otimes \mathcal{C}^{i+1}$, where $\Gamma_i(\mathscr{F}) = \ker(\mu_{i,1})$ and $\Gamma^{(i)}(\mathscr{F}) = \mathrm{coker}(\lambda_{i+1,1})$, for $0 \leq i < n$.*

(iv) *(*[**D2**, Proposition 3.5]*) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on C_n. Then:*

   (a) *φ is surjective if and only if its restriction $\varphi|_C : G_0(\mathscr{F}) \to G_0(\mathscr{G})$ is surjective. In this case, all the induced morphisms $G_i(\mathscr{F}) \to G_i(\mathscr{G})$ are surjective for $1 \leq i < n$;*

   (b) *φ is injective if and only if the induced morphism $G^{(1)}(\mathscr{F}) \to G^{(1)}(\mathscr{G})$ is injective. If this is the case, all the induced morphisms $G^{(i)}(\mathscr{F}) \to G^{(i)}(\mathscr{G})$ are injective for $2 \leq i \leq n$.*

REMARK 1.10. By Fact 1.9(ii) the complete type of a sheaf on C_n can be characterized also in terms of the second canonical filtration, but we will not make it explicit.

1.4. Generalized rank and degree

This section is devoted to recall the definitions (cf. [**D1**, §§4.1.3-4.1.4] or [**D2**, §3.2]) and main properties of two fundamental invariants of a sheaf on C_n. The notation adopted is the same of the previous sections.

DEFINITION 1.11.

(i) Let M be a finitely-generated A_n-module, then its *generalized rank* is $\mathrm{R}(M) = \mathrm{rk}(\mathrm{Gr}_1(M)) = \mathrm{rk}(\mathrm{Gr}_2(M))$.

(ii) Let $\mathscr{F}$ be a sheaf on C_n. Its *generalized rank* $\mathrm{R}(\mathscr{F})$ is, by definition, $\mathrm{rk}(\mathrm{Gr}_1(\mathscr{F}))$, while its *generalized degree* is $\mathrm{Deg}(\mathscr{F}) = \deg(\mathrm{Gr}_1(\mathscr{F}))$. This is equivalent, by Fact 1.9(ii), to $\mathrm{R}(\mathscr{F}) = \mathrm{rk}(\mathrm{Gr}_2(\mathscr{F}))$ and $\mathrm{Deg}(\mathscr{F}) = \deg(\mathrm{Gr}_2(\mathscr{F}))$.

The following are some basic but fundamental properties of the generalized rank and degree.

FACT 1.12.

(i) ([**D1**, §§4.1.3-4.1.4]) *If $\mathcal{F}$ is a sheaf on C, then* $\mathrm{rk}(\mathcal{F}) = \mathrm{R}(\mathcal{F})$ *and* $\deg(\mathcal{F}) = \mathrm{Deg}(\mathcal{F})$.

More generally, if $\mathscr{F}$ is the direct image of a sheaf on C_i, for any $1 \leq i < n$, its generalized rank and degree as a sheaf on C_n and those as a sheaf on C_i coincide by definition of the first canonical filtration.

(ii) ([**D1**, §§4.1.3-4.1.4]). *Let $\mathscr{F}$ be a locally free sheaf of rank m on C_n, then* $\mathrm{R}(\mathscr{F}) = nm = n\,\mathrm{rk}(\mathscr{F}|_C)$ *and* $\mathrm{Deg}(\mathscr{F}) = n\deg(\mathscr{F}|_C) + (n(n-1)/2)m\deg(\mathcal{C})$. *In particular, any line bundle has generalized rank n and* $\mathrm{Deg}(\mathscr{O}_{C_n}) = (n(n-1)/2)\deg(\mathcal{C})$.

(iii) ([**D1**, Théorème 4.2.1]) *Let $\mathscr{F}$ be a sheaf on C_n. It verifies the so-called* generalized Riemann-Roch theorem, *i.e.* $\chi(\mathscr{F}) = \mathrm{Deg}(\mathscr{F}) + \mathrm{R}(\mathscr{F})\chi(\mathscr{O}_C)$.

(iv) ([**D1**, §4.2.2]) *Let $\mathscr{F}$ be a sheaf and $\mathscr{O}_{C_n}(1)$ be a very ample line bundle on C_n, let $\mathscr{O}_C(1)$ be its restriction to C and $d = \deg(\mathscr{O}_C(1))$. Then the Hilbert polynomial of $\mathscr{F}$ with respect to $\mathscr{O}_{C_n}(1)$ is*

$$P_{\mathscr{F}}(T) = \mathrm{Deg}(\mathscr{F}) + \mathrm{R}(\mathscr{F})\chi(\mathscr{O}_C) + \mathrm{R}(\mathscr{F})dT. \tag{1.1}$$

(v) ([**D1**, Corollaire 4.3.2]) *The generalized rank and degree are additive, i.e.:*
  (a) *if $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated A_n-modules, then* $\mathrm{R}(M) = \mathrm{R}(M') + \mathrm{R}(M'')$;
  (b) *if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is a short exact sequence of sheaves on C_n, then* $\mathrm{R}(\mathscr{F}) = \mathrm{R}(\mathscr{F}') + \mathrm{R}(\mathscr{F}'')$ *and* $\mathrm{Deg}(\mathscr{F}) = \mathrm{Deg}(\mathscr{F}') + \mathrm{Deg}(\mathscr{F}'')$.

(vi) ([**D1**, Proposition 4.3.3])
  (a) *The generalized rank of finitely generated A_n-modules is invariant by deformation.*
  (b) *The generalized rank and degree of sheaves on C_n are invariant by deformation.*

REMARK 1.13.

(i) Fact 1.12 (i) is one fundamental reason for which it is possible to do not distinguish between a sheaf on C_i and its direct image on C_n, for any $1 \leq i \leq n-1$. It is also a significant reason to use generalized rank and degree instead of the usual ones, which do not have this very useful property (cf. Lemma 1.14 and the brief discussion preceding it).

(ii) It is possible to give also another characterization of generalized rank and degree of a sheaf $\mathscr{F}$ on C_n without making use of the canonical filtrations: the generalized rank of $\mathscr{F}$ can be seen as its generic length, i.e. the length of the $\mathscr{O}_{C_n,\eta}$-module $\mathscr{F}_\eta$, where η is the generic point of C_n, while its generalized degree could be defined also as $\chi(\mathscr{F}) - \mathrm{R}(\mathscr{F})\chi(\mathscr{O}_C)$. The equivalence of these characterizations to the original definitions is almost immediate (for the generalized degree it has to be used the additivity of the Euler characteristic, which implies that $\chi(\mathscr{F}) = \chi(\mathrm{Gr}_1(\mathscr{F}))$.

Now we will describe the relation of generalized rank and degree with the usual rank and degree. Indeed, the latter can be defined also in this context and are often used for sheaves on ribbons (as in [**CK**] or, at least the degree, in [**EG**]). First of all, we need to recall the classical definitions: if $\mathscr{F}$ is a sheaf on C_n, then its rank, $\mathrm{rk}(\mathscr{F})$, and its degree, $\deg(\mathscr{F})$, are the rational numbers for which its Hilbert polynomial with respect to a fixed very ample line bundle has the form

$$P_{\mathscr{F}}(T) = \deg(\mathscr{F}) + \mathrm{rk}(\mathscr{F})\chi(\mathscr{O}_{C_n}) + nd\,\mathrm{rk}(\mathscr{F})T, \tag{1.2}$$

where d is as in Fact 1.12(iv), (for this definition, cf., e.g., [**HP**, Definition 3.7]). Observe that if $\mathscr{F}$ is a sheaf on C_i, then its rank and degree are not equal to those of its direct image on C_n.

The next lemma, which is implied by formulae (1.1) and (1.2), compares generalized rank and degree with the usual ones:

LEMMA 1.14. *Let $\mathscr{F}$ be a sheaf on C_n. Then* $\mathrm{R}(\mathscr{F}) = n\,\mathrm{rk}(\mathscr{F})$ *and* $\mathrm{Deg}(\mathscr{F}) = \deg(\mathscr{F}) + \mathrm{rk}(\mathscr{F})\,\mathrm{Deg}(\mathscr{O}_{C_n}) = \deg(\mathscr{F}) + \mathrm{rk}(\mathscr{F})\frac{n(n-1)}{2}\deg(\mathcal{C})$.

COROLLARY 1.15. *Let $\mathscr{F}$ be a sheaf on C_n of generalized rank R and $\mathscr{E}$ a vector bundle of rank m (i.e. generalized rank nm) on C_n. Then*

$$\mathrm{Deg}(\mathscr{F} \otimes \mathscr{E}) = \frac{R}{n}\,\mathrm{Deg}(\mathscr{E}) + m\,\mathrm{Deg}(\mathscr{F}) - \frac{Rm(n-1)}{2}\deg(\mathcal{C}). \tag{1.3}$$

PROOF. It follows from the above Lemma and from [**SP**, Tag 0AYV], asserting that $\chi(\mathscr{F} \otimes \mathscr{E}) = \mathrm{rk}(\mathscr{F})\deg(\mathscr{E}) + \mathrm{rk}(\mathscr{E})\chi(\mathscr{F})$, in a wider context, i.e. if $\mathscr{F}$ is a sheaf and $\mathscr{E}$ a vector bundle on a proper irreducible curve over a field. *q.e.d.*

1.5. Purity and duality

The next step is to introduce the key (equivalent in our case) notions of pure, torsion-free and reflexive sheaves. The distinction between pure and torsion-free is taken from [**CK**, Definition 2.1]; Drézet speaks only of reflexive and torsion-free sheaves (*faisceaux sans torsion* in French), but he defines the latter as Chen and Kass define pure ones (cf. [**D2**, §3.3]).

Let us begin with pure and torsion-free sheaves:

DEFINITION 1.16. Let $\mathscr{F}$ be a sheaf on X. Its *dimension*, $\mathrm{d}(\mathscr{F})$, is the dimension of its support. A sheaf $\mathscr{F}$ on X is *pure* if it has dimension 1 and $\mathrm{d}(\mathscr{G}) = 1$ for any non-zero subsheaf $\mathscr{G} \subset \mathscr{F}$.

Let U be an open subscheme of X, a regular function $f \in H^0(U, \mathscr{O}_X)$ is a *nonzerodivisor on* $\mathscr{F}$ if the multiplication map $f \cdot \_ : \mathscr{F}|_U \to \mathscr{F}|_U$ is injective and the sheaf $\mathscr{F}$ is *torsion-free* if every nonzerodivisor on $\mathscr{O}_X$ is a nonzerodivisor also on $\mathscr{F}$.

REMARK 1.17. Our definition of pure is not completely equal to [**CK**, Definition 2.1]: there, as often in literature, it is only required that the dimension of any proper subsheaf equals that of the sheaf, so any sheaf of dimension 0 would be considered pure but we are not interested in them.

The following result is the extension of [**CK**, Lemma 2.2] from the case of ribbons to the case of primitive multiple curves of arbitrary multiplicity. Also the proof is almost identical to that of the cited place, which extends verbatim to our case (it holds also in wider generality, namely at least for sheaves on any irreducible algebraic scheme of dimension 1).

LEMMA 1.18. *Let $\mathscr{F}$ be a sheaf on a primitive multiple curve X. Then $\mathscr{F}$ is pure if and only if it is torsion-free.*

PROOF. By definition, $\mathscr{F}$ is not pure if and only if there exists a non-zero subsheaf of $\mathscr{F}$ with finite support. This is equivalent to the existence of an open affine subscheme $U \subset X$ and a non-zero $g \in H^0(U, \mathscr{F})$ with finite support. Equivalently, there exist an open affine subscheme $U \subset X$ and a non-zero $g \in H^0(U, \mathscr{F})$ such that $\mathrm{ann}(g) \not\subset \mathscr{N}|_U$. This is equivalent to the fact that there exist an open affine subscheme $U \subset X$, a nonzerodivisor $f \in H^0(U, \mathscr{O}_X)$ and a non-zero $g \in H^0(U, \mathscr{F})$ such that $fg = 0$. The last assertion means, by definition, that $\mathscr{F}$ is not torsion-free. *q.e.d.*

In order to introduce reflexiveness, we need first to recall the notion of dual of a sheaf on a primitive multiple curve.

DEFINITION 1.19. Let $\mathscr{F}$ be a sheaf on C_n. Its *dual* is $\mathscr{F}^{\vee_n} = \mathscr{F}^{\vee} = \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{O}_{C_n})$.

The sheaf $\mathscr{F}$ is *reflexive* if the canonical morphism $\mathscr{F} \to \mathscr{F}^{\vee\vee}$ is an isomorphism.

REMARK 1.20. If $\mathscr{F}$ is a sheaf on C_i, for $1 \leq i < n$, then $\mathscr{F}^{\vee_i} \neq \mathscr{F}^{\vee_n}$. But there is a canonical isomorphism $\mathscr{F}^{\vee_n} \simeq \mathscr{F}^{\vee_i} \otimes \mathscr{N}^{n-i}$ (this is [**D2**, Lemme 4.1]).

The following fact collects some properties of duality and of reflexive sheaves.

FACT 1.21. *Let $\mathscr{F}$ be a sheaf on C_n.*

(i) *(*[**D2**, Proposition 3.8 and Théorème 4.4]*) The following are equivalent:*
   (a) *$\mathscr{F}$ is pure;*

(b) $\mathscr{F}^{(1)} = G^{(1)}(\mathscr{F})$ *is a vector bundle on* C*;*
(c) $\mathscr{F}$ *is reflexive;*
(d) $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{C_n}}(\mathscr{F}, \mathscr{O}_{C_n}) = 0$.

Moreover, if the above conditions hold, $\underline{\mathrm{Ext}}^i_{\mathscr{O}_{C_n}}(\mathscr{F}, \mathscr{O}_{C_n}) = 0$ *for any* $i \geq 1$ *and* $G^{(j)}(\mathscr{F})$ *is a vector bundle on* C *for* $1 \leq j \leq n$*.*

(ii) *(*[**D2**, Corollaire 4.6]*) For any* $i \geq 2$, $\underline{\mathrm{Ext}}^i_{\mathscr{O}_{C_n}}(\mathscr{F}, \mathscr{O}_{C_n}) = 0$.

(iii) *(*[**D2**, Proposition 4.2]*) For any* $1 \leq i < n$, $(\mathscr{F}^\vee)^{(i)} = (\mathscr{F}|_{C_i})^\vee$.

(iv) *(*[**D4**, Proposition 4.4.1]*) It holds that* $\mathrm{R}(\mathscr{F}^\vee) = \mathrm{R}(\mathscr{F})$*, while* $\mathrm{Deg}(\mathscr{F}^\vee) = -\mathrm{Deg}(\mathscr{F}) + \mathrm{R}(\mathscr{F})(n-1)\deg(\mathcal{C}) + \mathrm{h}^0(\mathscr{T}(\mathscr{F}))$*, where* $\mathscr{T}(\mathscr{F})$ *is the torsion subsheaf of* $\mathscr{F}$*, i.e. its greatest subsheaf with finite support.*

(v) *Assume, moreover, that* $\mathscr{F}$ *is pure. Then, for any* $1 \leq i < n$*:*

(a) *(*[**D4**, Proposition 4.3.1(i)]*) There is a canonical isomorphism between* $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{C_n}}(\mathscr{T}(\mathscr{F}|_{C_i}), \mathscr{O}_{C_n}) \otimes \mathscr{C}^i$ *and* $\mathscr{T}(\mathscr{F}^\vee|_{C_i})$*, where* $\mathscr{T}(\mathscr{F}^\vee|_{C_i})$ *and* $\mathscr{T}(\mathscr{F}|_{C_i})$ *are the torsion subsheaves of, respectively,* $\mathscr{F}^\vee|_{C_i}$ *and* $\mathscr{F}|_{C_i}$*.*

(b) *(*[**D4**, Proposition 4.3.1(ii)]*) There is a canonical isomorphism between* $(\ker(\mathscr{F} \twoheadrightarrow (\mathscr{F}|_{C_i})^{\vee\vee}))^\vee$ *and* $(\mathscr{F}^\vee)_i \otimes \mathscr{C}^{-i}$*.*

(vi) *(*[**D2**, Corollaire 4.5]*) If* $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ *is a short exact sequence of sheaves on* C_n *with* $\mathscr{G}$ *pure, then also the dual sequence* $0 \to \mathscr{G}^\vee \to \mathscr{F}^\vee \to \mathscr{E}^\vee \to 0$ *is exact.*

Remark 1.22. The hypothesis of the last point of the Fact is a bit weaker than that of the cited place, where it is required that also $\mathscr{E}$ and $\mathscr{F}$ are pure, but the only significant point for the proof is that $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{C_n}}(\mathscr{G}, \mathscr{O}_{C_n})$ vanishes. So the assertion remains true also under our hypothesis.

There is a special type of pure sheaves which plays a major role in the theory of sheaves over a primitive multiple curve: the so-called *quasi locally free* sheaves.

Definition 1.23. (Cf. [**D1**, §5.1].) A finitely-generated A_n-module M is *quasi free* if there exist non-negative integers $m_1, \ldots, m_n$ such that $M \cong \bigoplus_{i=1}^n A_i^{\oplus m_i}$. The n-tuple $(m_1, \ldots, m_n)$ is called the *type* of M.

Let $\mathscr{F}$ be a sheaf on C_n. It is *quasi locally free in a closed point* P if there exists an open neighbourhood U of P and non-negative integers $m_1, \ldots, m_n$ such that $\mathscr{F}_Q$ is quasi free of type $(m_1, \ldots, m_n)$ for any $Q \in U$. It is *quasi locally free* if it is such in any closed point.

The following fact contains some significant results.

Fact 1.24. *Let* $\mathscr{F}$ *be a sheaf on* C_n *and let* P *be a closed point of* C*.*

(i) *(*[**D2**, Théorème 3.9 and Corollaire 3.10]*) The following are equivalent:*

(a) $\mathscr{F}$ *is quasi locally free (resp. quasi locally free in* P*);*
(b) *for* $0 \leq i < n$, $G_i(\mathscr{F})$ *is a vector bundle on* C *(resp. is free in* P*);*
(c) *for* $0 \leq i < n$, $\Gamma_i(\mathscr{F})$ *(or, equivalently,* $\Gamma^{(i)}(\mathscr{F})$*) is a vector bundle on* C *(resp. is free in* P*) (see Fact 1.9*(iii)(c)*).*

(ii) *(*[**D1**, Théorème 5.1.6]*) $\mathscr{F}$ is generically quasi locally free, i.e. there exists a non-empty open U of C_n such that $\mathscr{F}$ is quasi locally free in each point of U.*

The last point of the above fact allows to give the following definition:

Definition 1.25. The *type* of a sheaf $\mathscr{F}$ on C_n is the n-tuple of non-negative integers $(m_1, \ldots, m_n)$ such that $\mathscr{F}_\eta \cong \bigoplus_{i=1}^{n} \mathscr{O}_{C_i,\eta}^{\oplus m_i}$, where, as usual, η is the generic point of C_n.

Within quasi locally free sheaves there are those *of rigid type*, studied in [**D2**] and [**D4**]:

Definition 1.26. A sheaf $\mathscr{F}$ on C_n is said to be *quasi locally free of rigid type* if there exist two non-negative integers $a > 0$ and $j < n$ such that $\mathscr{F}$ is locally isomorphic to $\mathscr{O}_{C_n}^{\oplus a} \oplus \mathscr{O}_{C_j}$.

Observe that these comprehend vector bundles (they are the quasi locally free sheaves of rigid type with $j = 0$). They are relevant because being quasi locally free of rigid type is an open condition in flat families of sheaves on C_n, as the name suggests (see [**D2**, Proposition 6.9]). They are the only kind of pure sheaves on C_n such that there are some results in literature (precisely [**D2**, Proposition 6.12] and [**D4**, Théorème 5.3.3]) about loci containing them in the moduli space of semistable sheaves.

1.6. Semistability

The last argument that we quickly treat in this chapter is semistability. First of all, it is necessary to recall how it is defined on a primitive multiple curve (cf. [**D1**, §1.1]).

Definition 1.27. Let $\mathscr{F}$ be a pure sheaf on C_n. Its *slope* is $\mu(\mathscr{F}) = \mathrm{Deg}(\mathscr{F}) / \mathrm{R}(\mathscr{F})$. The definition of semistability of $\mathscr{F}$ is the usual definition of (slope-)semistability: $\mathscr{F}$ is (slope-)*semistable* if for any non-trivial subsheaf $\mathscr{E}$ it holds that $\mu(\mathscr{E}) \leq \mu(\mathscr{F})$ or, equivalently, if for any non-trivial pure quotient $\mathscr{G}$ it holds that $\mu(\mathscr{G}) \geq \mu(\mathscr{F})$. If the inequality is always strict, $\mathscr{F}$ is said to be *stable*.

Remark 1.28.

(i) Thanks to the description of the Hilbert polynomial given by formula (1.1), on a primitive multiple curve slope semistability coincides with Gieseker one, which considers the reduced Hilbert polynomial instead of the slope. So the latter results to be independent of the polarization, as in the case of vector bundles on a smooth projective curve.

(ii) The equivalence of the condition about subsheaves and pure quotients is almost trivial and is a well-known property of semistability (cf. e.g. [**HL**, Proposition 1.2.6]).

(iii) It is evident, by definition, that if $\mathscr{F}$ is a semistable sheaf on C_i, for some $1 \leq i \leq n-1$, then it is semistable also on C_n.

(iv) It is possible to verify (cf. e.g. [**D4**, §1.2]) that there are interesting (i.e. different from direct images of stable vector bundles on C) stable sheaves on C_n only if $\deg(\mathcal{C}) < 0$. It is quite easy to check the assertion for a vector bundle $\mathscr{F}$ on C_n: by Fact 1.12 (ii) it holds that $\mu(\mathscr{F}) = \mu(\mathscr{F}|_C) + ((n-1)/2)\deg(\mathcal{C})$; hence, it can be stable only if $\deg(\mathcal{C}) < 0$. Under this assumption all the line bundles on C_n are stable (it is almost trivial, but it is also a consequence of Theorem 2.41, which, moreover, confirms the necessity of $\deg(\mathcal{C}) < 0$ in order to have stable generalized line bundles).

The following easy lemma concludes this quick overview about semistability.

LEMMA 1.29. *Let $\mathscr{F}$ be a pure sheaf on C_n. It is (semi)stable if and only if $\mathscr{F}^\vee$ is (semi)stable.*

PROOF. The proof is done only in the case of semistability, because that for stabilty is essentially the same.

By the equivalence of purity and reflexiveness, it is sufficient to show that $\mathscr{F}$ semistable implies $\mathscr{F}^\vee$ semistable.

So, assume $\mathscr{F}$ semistable and let $\mathscr{G}$ be any pure quotient of $\mathscr{F}^\vee$. Hence, $\mathscr{G}^\vee$ is a subsheaf of $\mathscr{F}$ by Fact 1.21(vi). Thus, $\mu(\mathscr{G}) = -\mu(\mathscr{G}^\vee) + (n-1)\deg(\mathcal{C}) \geq -\mu(\mathscr{F}) + (n-1)\deg(\mathcal{C}) = \mu(\mathscr{F}^\vee) - (n-1)\deg(\mathcal{C}) + (n-1)\deg(\mathcal{C}) = \mu(\mathscr{F}^\vee)$, where the various equalities follow from Fact 1.21(iv) while the inequality is due to the semistabilty of $\mathscr{F}$. *q.e.d.*

CHAPTER 2

Generalized line bundles

This chapter is devoted to describe various properties of generalized line bundles on a primitive multiple curve. It is divided into four sections. The first two generalize, as far as possible, [**CK**, §2]; in particular, the first is more generally about pure sheaves of generalized rank n on a primitive multiple curve of multiplicity n while the second is more specific about generalized line bundles. The third one studies the local and global structure of generalized line bundles; its results are fundamental for the study of the components of stable generalized line bundles in the moduli space. The last section treats semistability conditions of generalized line bundles and is inspired by [**CK**, §3].

Throughout this chapter X will be a primitive multiple curve of multiplicity n.

2.1. Pure sheaves of generalized rank n

All the results of this section are completely trivial for $n = 1$, while the case $n = 2$ is that treated in [**CK**]. For $n \geq 3$ the properties described and also their proves are straightforward extensions of those by Chen and Kass.

First of all, we need to define properly one of the main objects of study of this work.

Definition 2.1. A *generalized line bundle* is a pure sheaf $\mathscr{F}$ on X such that $\mathscr{F}_\eta$ is isomorphic to $\mathscr{O}_{X,\eta}$, where η is the generic point of X.

Remark 2.2. Let $\mathscr{F}$ be a generalized line bundle on X. By definition $\mathrm{R}(\mathscr{F}) = n$ (or, equivalently, $\mathrm{rk}(\mathscr{F}) = 1$).

According to my knowledge this definition of generalized line bundle is new for $n \geq 3$, but it is an obvious extension of the notion for ribbons (i.e. $n = 2$). Furthermore, as in case $n = 2$ (cf. [**EG**, beggining of page 759]), generalized line bundles coincide with generalized divisors which have been introduced in a much more general context by Hartshorne (see, in particular, [**H**, Proposition 2.12]).

Following Chen and Kass, we prove some easy lemmata about pure sheaves on X. The first one extends to the general case [**CK**, Lemma 2.3].

Lemma 2.3. *If $\mathscr{F}$ is a pure sheaf on X, then $\underline{\mathrm{End}}(\mathscr{F})$ is pure, too.*

Proof. The proof of [**CK**] extends almost verbatim. We omit the trivial proof for sheaves of dimension 0, because, as pointed out in Remark 1.17, for us pure is a synonym of pure of dimension 1.

It suffices to prove that if φ is an element of $\underline{\mathrm{End}}(\mathscr{F})_x$ annihilated by $\mathfrak{m}_x \subset \mathscr{O}_{X,x}$, where $x \in X$ is a closed point and $\mathfrak{m}_x$ is the maximal ideal of

$\mathscr{O}_{X,x}$, then $\varphi = 0$. Indeed, given such a φ, it holds that for any $s \in \mathscr{F}_x$ its image $\varphi(s) \in \mathscr{F}_x$ is annihilated by $\mathfrak{m}_x$. Therefore, $\varphi(s) = 0$, because $\mathscr{F}$ is pure (or, equivalently, torsion-free, thanks to Lemma 1.18). *q.e.d.*

The next lemma is a generalization of [**CK**, Lemma 2.4] from the case of ribbons to any multiplicity.

LEMMA 2.4. *Let $\mathscr{F}$ be a pure sheaf on X. The kernel of the natural morphism $\varphi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{F})$ is equal to $\mathscr{N}^i$ for some $1 \le i \le n$. Hence, the schematic support of $\mathscr{F}$ is C_i.*

PROOF. Let $\mathscr{K} = \ker(\varphi)$. By definition, there is an injection $\mathscr{O}_X/\mathscr{K} \hookrightarrow \underline{\mathrm{End}}(\mathscr{F})$ and it is clear that $\mathscr{O}_X/\mathscr{K} \neq 0$ (indeed, e.g., any nonzerodivisor constant defines a non-zero endomorphism, being $\mathscr{F}$ pure). Because $\underline{\mathrm{End}}(\mathscr{F})$ is pure (by Lemma 2.3), $\mathscr{O}_X/\mathscr{K}$ is such too. By primary decomposition, $\mathscr{K}$ is contained in $\mathscr{N}$ (indeed, over every open affine U, the prime ideals associated to $\mathscr{K}(U)$ in $\mathscr{O}_X(U)$ must have height zero, but the only prime of $\mathscr{O}_X(U)$ with this property is $\mathscr{N}(U)$).

Let η be, as usual, the generic point of X. By definition of primitive multiple curve, $\mathscr{O}_{X,\eta}$ is isomorphic to $\mathscr{O}_{C,\eta'}[y]/(y^n)$, where η' is the generic point of C, and $\mathscr{O}_{C,\eta'}$ is a field. The only subideals of $\mathscr{N}_\eta$ are its powers, i.e. $\mathscr{N}_\eta^j$ with $1 \le j \le n$ (with $\mathscr{N}_\eta^n = 0$) and, thus, $\mathscr{K}_\eta = \mathscr{N}_\eta^i$ for an $i \in \{1, \dots, n\}$. At this point the conclusion follows from the purity of $\mathscr{N}^i/\mathscr{K}$ (it is a subsheaf of $\mathscr{O}_X/\mathscr{K}$, which is pure by the above argument): the fact that $\mathscr{N}_\eta^i/\mathscr{K}_\eta = 0$ implies that $\mathscr{N}^i/\mathscr{K}$ has finite support and so it must be zero, i.e. $\mathscr{K} = \mathscr{N}^i$. *q.e.d.*

REMARK 2.5. In the cited lemma of [**CK**] there is the adjunctive hypothesis of generic length 2 (equivalent to generalized rank 2) of the sheaves involved (and that would correspond to generalized rank n), but it is superfluous.

The previous lemma allows to get another characterization of generalized line bundles. The case of multiplicity 2 is [**CK**, Lemma 2.5].

COROLLARY 2.6. *Let $\mathscr{F}$ be a pure sheaf of generalized rank n on X. Then $\mathscr{F}$ is a generalized line bundle if and only if the morphism $\varphi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{F})$ is injective.*

PROOF. By the previous Lemma, it is sufficient to show that, if $\mathscr{F}$ is pure, $\mathscr{F}_\eta \cong \mathscr{O}_{X,\eta}$, where, as usual, η is the generic point of X, is equivalent to φ injective.

Assume that $\mathscr{F}$ is a generalized line bundle. The morphism φ gives rise to the following commutative diagram:

$$\begin{array}{ccc} \mathscr{O}_X & \xrightarrow{\varphi} & \underline{\mathrm{End}}(\mathscr{F}) \\ \downarrow & & \downarrow \\ \mathscr{O}_{X,\eta} & \xrightarrow{\varphi_\eta} & \underline{\mathrm{End}}(\mathscr{F})_\eta, \end{array}$$

where the vertical arrows are injective because both $\mathscr{O}_X$ and $\underline{\mathrm{End}}(\mathscr{F})$ are pure (the latter by Lemma 2.3). Moreover, being $\mathscr{F}$ a generalized line bundle, φ_η is an isomorphism; hence, φ is injective.

Conversely, assume that φ is injective and let y_η be a generator of the generic stalk $\mathscr{N}_\eta$. By hypothesis, multiplication by y_η^{n-1} on $\mathscr{F}_\eta$ is not the zero map. Hence, we can choose $s_0 \in \mathscr{F}_\eta$ such that $y_\eta^{n-1}s_0 \neq 0$; let us consider the morphism $\psi : \mathscr{O}_{X,\eta} \to \mathscr{F}_\eta$ defined by $\psi(f) = fs_0$: it is the desired isomorphism. Indeed, $\ker(\psi) = \mathrm{Ann}(s_0)$ is a submodule of $\mathscr{N}_\eta$, being $\mathscr{F}$ pure. Moreover, the fact that $y_\eta^{n-1}s_0 \neq 0$ implies that $\ker(\psi) \subsetneq \mathscr{N}_\eta^{n-1}$ and thus $\ker(\psi) = 0$. Surjectivity follows from the fact that $\mathscr{F}_\eta$ and the submodule generated by s_0 have both length n. *q.e.d.*

These lemmata and the last corollary imply the following classification of pure sheaves of generalized rank n on X, which extends [**D1**, §8.2] and [**CK**, Proposition 2.6].

PROPOSITION 2.7. *Let $\mathscr{F}$ be a pure sheaf of generalized rank n on X. Then $\mathscr{F}$ is either a generalized line bundle or the direct image of a pure sheaf of generalized rank n defined on C_{n-1} under the inclusion $C_{n-1} \subset X$ (in the following such sheaves will be called sheaves of generalized rank n defined on C_{n-1}).*

PROOF. Consider again the natural morphism $\varphi : \mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{F})$. If φ is injective, $\mathscr{F}$ is a generalized line bundle by Corollary 2.6. Otherwise, $\mathscr{N}^{n-1} \subseteq \ker(\varphi)$ by Lemma 2.4 and, thus, $\mathscr{F}$ can be seen as an $\mathscr{O}_X/\mathscr{N}^{n-1}$-module, i.e. an $\mathscr{O}_{C_{n-1}}$-module. *q.e.d.*

REMARK 2.8. When $n \geq 3$ this result is quite vague with respect to that for ribbons because it does not give a precise classification of pure sheaves of generalized rank n defined on C_{n-1}, which comprehend various different kinds of sheaves, from vector bundles of rank n on C to sheaves generically of the form $\mathscr{O}_{C_{n-1}} \oplus \mathscr{O}_C$. In general, they are pure sheaves generically of the form $\bigoplus_{i=1}^{n-1} \mathscr{O}_{C_i}^{\oplus a_i}$, with the a_i non-negative integers such that $\sum_{i=1}^{n-1} ia_i = n$ (cf. Fact 1.24(ii)). This will be a huge complication in the study of the moduli space of generalized rank n sheaves on X.

2.2. Generalities about generalized line bundles

In this section we will study generalized line bundles, so throughout it $\mathscr{F}$ will denote a generalized line bundle on X.

The following lemma extends to morphisms between generalized line bundles a well-known property of those between line bundles.

LEMMA 2.9. *If a morphism between generalized line bundles on X is surjective, then it is an isomorphism.*

PROOF. Let $\pi : \mathscr{F} \twoheadrightarrow \mathscr{G}$ be a surjective morphism between generalized line bundles. It is evident that $\pi_\eta : \mathscr{F}_\eta \to \mathscr{G}_\eta$ is an isomorphism. Hence, $\ker(\pi)$ is generically zero; in other words, it has finite support. Therefore, the purity of $\mathscr{F}$ implies that $\ker(\pi) = 0$. *q.e.d.*

The two canonical filtrations of $\mathscr{F}$ will play a crucial role in the following; in particular, we will use them to define other sheaves associated to $\mathscr{F}$ and some invariants. So we need to study them in some detail.

First of all, observe that, being $\mathscr{F}$ a generalized line bundle, all the containments in the two filtrations are strict, $\mathscr{F}_i$ is a generalized line bundle defined on C_{n-i} while $\mathscr{F}^{(i)}$ is a generalized line bundle defined on C_i. However $\mathscr{F}/\mathscr{F}_i = \mathscr{F}|_{C_i}$ is not necessarily a generalized line bundle on C_i, for any $1 \leq i \leq n-1$: in general, it has a non-zero torsion subsheaf; on the other side, $\mathscr{F}/\mathscr{F}^{(j)}$, being isomorphic, up to tensor product with a line bundle, to $\mathscr{F}_j$ (by Fact 1.9(i)), is a generalized line bundle on C_{n-j}, for any $1 \leq j \leq n-1$. This suggests to study the relation between these quotients and, in order to do that, it is useful to introduce some definitions and notations.

Definition 2.10. For any $1 \leq i \leq n$, the *i-th pure quotient* of $\mathscr{F}$ is $\overline{\mathscr{F}}_i := (\mathscr{F}|_{C_i})^{\vee\vee}$, while the kernel of the natural morphism $\mathscr{F}|_{C_i} \twoheadrightarrow \overline{\mathscr{F}}_i$ is denoted $\mathscr{T}_i(\mathscr{F})$, or simply $\mathscr{T}_i$ if it is clear which is the generalized line bundle involved. In order to avoid any risk of confusion between $\overline{\mathscr{F}}_i$ and $\mathscr{F}_i$ in the following the latter will be always denoted by $\mathscr{N}^i\mathscr{F}$.

It makes sense to call $\overline{\mathscr{F}}_i$ *the* i-th pure quotient of $\mathscr{F}$ by the next lemma, asserting that it is the only pure quotient of $\mathscr{F}$ supported exactly on C_i (in the sense that it is an $\mathscr{O}_{C_i}$-module but it does not have a structure of $\mathscr{O}_{C_{i-1}}$-module). It holds that $\overline{\mathscr{F}}_n = \mathscr{F}$ and $\mathscr{T}_n = 0$, for any generalized line bundle $\mathscr{F}$.

In the case of ribbons, the following lemma has not been explicitly enunciated in [**CK**] but it is contained in the proof of [**CK**, Lemma 3.2].

Lemma 2.11. *Let $\mathscr{G}$ be a pure sheaf on X and let $q : \mathscr{F} \twoheadrightarrow \mathscr{G}$ be a surjective morphism. Then $\mathscr{G} = \overline{\mathscr{F}}_i$ for some $1 \leq i \leq n$.*

Proof. There are two different cases to be discussed according to the generalized rank of $\mathscr{G}$, i.e. $\mathrm{R}(\mathscr{G}) = n$ and $\mathrm{R}(\mathscr{G}) < n$.

If $\mathrm{R}(\mathscr{G}) = n$, it is sufficient to show that $\mathscr{G}$ is a generalized line bundle on X because this implies that q is an isomorphism by Lemma 2.9 (and thus, $\mathscr{G} = \mathscr{F} = \overline{\mathscr{F}}_n$). Hence, by Lemma 2.7, it is sufficient to prove that $\mathscr{G}$ is not the direct image of a pure sheaf on C_{n-1} of generalized rank n. This is the case because $\mathscr{G}_\eta$ can be generated as $\mathscr{O}_{X,\eta}$-module by a single element (the image of a generator of $\mathscr{F}_\eta$) and the generic stalk of a $\mathscr{O}_{C_{n-1}}$-module of generalized rank n does not have this property (indeed, it is of the form $\bigoplus_{i=1}^{n-1} \mathscr{O}_{C_i}^{\oplus a_i}$ with the a_i non-negative integers such that $\sum_{i=1}^{n-1} i a_i = n$).

Now assume $\mathrm{R}(\mathscr{G}) = r < n$. The morphism $q : \mathscr{F} \twoheadrightarrow \mathscr{G}$ induces an epimorphism $q_\eta : \mathscr{F}_\eta \cong \mathscr{O}_{X,\eta} \twoheadrightarrow \mathscr{G}_\eta$. Thus, $\mathscr{G}_\eta$ can be generated by a single element, say s_0. Let $\mathscr{K} = \ker(\mathscr{O}_X \to \underline{\mathrm{End}}(\mathscr{G}))$. By Lemma 2.4, $\mathscr{K} \simeq \mathscr{N}^i$ for some $1 \leq i \leq n$. Moreover, $\mathscr{K}_\eta = \mathrm{Ann}(s_0)$ and $\mathscr{G}_\eta \cong \mathscr{O}_{X,\eta} s_0 \cong \mathscr{O}_{X,\eta}/\mathscr{K}_\eta$. The fact that the length of $\mathscr{G}_\eta$ is r implies that $\mathscr{K}_\eta$ is isomorphic to $\mathscr{N}_\eta^r$. Hence, $\mathscr{K} = \mathscr{N}^r$ and it follows that $\mathscr{G}$ is a pure $\mathscr{O}_X/\mathscr{N}^r$-module, i.e. a pure sheaf on C_r, or rather a generalized line bundle on C_r. Moreover, q can be factorized as $\mathscr{F} \to \mathscr{F}|_{C_r} \twoheadrightarrow \overline{\mathscr{F}}_r \overset{\bar{q}}{\twoheadrightarrow} \mathscr{G}$. So, $\bar{q}$ is a surjective morphism between generalized line bundles on C_r, hence an isomorphism, again by Lemma 2.9. *q.e.d.*

REMARK 2.12. The above lemma gives another useful characterization of the i-th pure quotient: it is isomorphic to $\mathscr{F}/\mathscr{F}^{(n-i)}$, because the latter is a pure quotient of $\mathscr{F}$ on C_i.

A priori, $\mathscr{T}_i$ has a structure of $\mathscr{O}_{C_i}$-module, but thanks to this remark it is possible to say something more, for $i > n/2$.

LEMMA 2.13. *For any* $1 \leq i \leq n-1$*, the torsion sheaf* $\mathscr{T}_i$ *is isomorphic to* $\mathscr{F}^{(n-i)}/\mathscr{N}^i\mathscr{F}$*; in particular, if* $n/2 < i < n$*, it is an* $\mathscr{O}_{C_{n-i}}$*-module.*

PROOF. The second assertion is an immediate consequence of the first one.

The following diagram is exact by definition of the various sheaves involved and by Remark 2.12:

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & & & \mathscr{T}_i & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \mathscr{N}^i\mathscr{F} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}|_{C_i} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}^{(n-i)} & \longrightarrow & \mathscr{F} & \longrightarrow & \overline{\mathscr{F}}_i & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & \mathscr{F}^{(n-i)}/\mathscr{N}^i\mathscr{F} & & & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

The first assertion follows from it by snake's lemma. *q.e.d.*

Now we investigate when a generalized line bundle is a line bundle.

PROPOSITION 2.14. *A generalized line bundle on* X *is a line bundle if and only if its restriction to* C *is a line bundle.*

PROOF. The necessity is obvious; hence, the only interesting part is sufficiency.

This proof proceeds by induction on n, the base is the completely trivial case $n = 1$, although also the case $n = 2$ is already known (cf., e.g, the proof of [**EG**, Theorem 1.1]). Assume that the statement is true for $n - 1 \geq 1$ and that $\mathscr{F}|_C$ is a line bundle. By Lemma 2.11, $\mathscr{F}|_C$ is a line bundle if and only if $\mathscr{F}|_C = \overline{\mathscr{F}}_1$ if and only if $\mathscr{T}_1 = 0$ (by definition) if and only if $\mathscr{N}\mathscr{F} = \mathscr{F}^{(n-1)}$ by Lemma 2.13. By Fact 1.24(i) and by the trivial observation that a generalized line bundle is quasi locally free if and only if it is a line bundle, it is sufficient to show that $\mathscr{F}^{(n-1)} = \mathscr{N}\mathscr{F}$ is a line bundle on C_{n-1}: indeed its second canonical filtration is the same of $\mathscr{F}$ and the fact it is a line bundle implies that $\mathscr{F}^{(j)}/\mathscr{F}^{(j-1)}$ is a line bundle on C for any $j \leq n-1$, while $\mathscr{F}/\mathscr{F}^{(n-1)}$ is $\overline{\mathscr{F}}_1 = \mathscr{F}|_C$, which is locally free by hypothesis. By Fact 1.9(i), $\mathscr{N}\mathscr{F} \simeq \overline{\mathscr{F}}_{n-1} \otimes \mathscr{N}$, thus it is sufficient to prove that $\overline{\mathscr{F}}_{n-1}$ is a line bundle on C_{n-1}. The fact that $\mathscr{F} \twoheadrightarrow \overline{\mathscr{F}}_{n-1}$ implies that $\mathscr{F}|_C \twoheadrightarrow \overline{\mathscr{F}}_{n-1}|_C$, too. But $\mathscr{F}|_C$ is a line bundle and $\overline{\mathscr{F}}_{n-1}|_C$ has rank 1 on C, so the epimorphism has to be an isomorphism. Hence, $\overline{\mathscr{F}}_{n-1}|_C$ is a line bundle on C and, by inductive hypothesis, $\overline{\mathscr{F}}_{n-1}$ is a line bundle on C_{n-1}, as required. *q.e.d.*

COROLLARY 2.15. *The following are equivalent:*

(i) *$\mathscr{F}$ is a line bundle on X;*
(ii) *$\mathscr{F}|_{C_i}$ is a line bundle on C_i for any $1 \leq i \leq n-1$;*
(iii) *there exists $i \in \{1, \ldots, n-1\}$ such that $\mathscr{F}|_{C_i}$ is a line bundle on C_i.*

PROOF. It is immediate that (i) implies (ii) and that the latter implies (iii).

If (iii) holds, then also $(\mathscr{F}|_{C_i})|_C = \mathscr{F}|_C$ is a line bundle and then the Proposition allows to conclude that (iii) implies (i). *q.e.d.*

The next step is to introduce the generalizations of the index and of the local index sequence of a generalized line bundle on a ribbon (cf. [**CK**, Definition 2.7]).

DEFINITION 2.16. The *i-th index* of $\mathscr{F}$ is $b_i = b_i(\mathscr{F}) := \mathrm{h}^0(C, \mathscr{T}_i(\overline{\mathscr{F}}_{i+1}))$ and the *indices-vector* of $\mathscr{F}$ is $b_{.} = b_{.}(\mathscr{F}) := (b_1, \ldots, b_{n-1})$.

Let $P \in X$ be a closed point, then the *local i-th index* of $\mathscr{F}$ at P is $b_{i,P} = b_{i,P}(\mathscr{F}) := \mathrm{lenght}((\mathscr{T}_i(\overline{\mathscr{F}}_{i+1}))_P)$ while its *local indices-vector* at P is $b_{.,P} = b_{.,P}(\mathscr{F}) := (b_{1,P}, \ldots, b_{n-1,P})$.

The *local indices sequence* of $\mathscr{F}$ is $b_{.,.}(\mathscr{F}) = b_{.,.} = (b_{.,P_1}, \ldots, b_{.,P_k})$, where $P_1, \ldots, P_k$ are the closed points supporting the torsion sheaves $\mathscr{T}_i(\overline{\mathscr{F}}_{i+1})$, for $1 \leq i \leq n-1$, i.e. the points in which $\mathscr{F}$ is not locally free.

The definition makes sense because $\overline{\mathscr{F}}_{i+1}$ is a generalized line bundle on C_{i+1} and thus, thanks to Lemma 2.13, $\mathscr{T}_i(\overline{\mathscr{F}}_{i+1})$ is an $\mathscr{O}_C$-module.

REMARK 2.17. For any $1 \leq i \leq n-1$, it holds that $b_i = \sum_{j=1}^k b_{i,P_j}$. By definition, $b_j(\overline{\mathscr{F}}_i) = b_j(\mathscr{N}^{n-i}\mathscr{F}) = b_j(\mathscr{F})$ for any $0 < j < i$.

LEMMA 2.18. *It holds that $\mathscr{T}_i(\overline{\mathscr{F}}_{i+1}) \subseteq \mathscr{T}_{i+1}(\overline{\mathscr{F}}_{i+2})$, for any $1 \leq i \leq n-2$. In particular, $b_i \leq b_{i+1}$ and $b_{i,P} \leq b_{i+1,P}$ for any $1 \leq i \leq n-2$ and for any closed point $P \in X$.*

PROOF. The second assertion is a straightforward consequence of the first one.

By the fact that $\overline{(\overline{\mathscr{F}}_i)}_j = \overline{\mathscr{F}}_j$ for any $1 \leq j \leq i \leq n-1$, which is an obvious consequence of Lemma 2.11, it is sufficient to show that $\mathscr{T}_{n-2}(\overline{\mathscr{F}}_{n-1}) \subseteq \mathscr{T}_{n-1}(\mathscr{F})$.

Indeed, $\overline{\mathscr{F}}_{n-1} \simeq \mathscr{N}\mathscr{F} \otimes \mathscr{N}^{-1}$ by Fact 1.9(i). Then, $\mathscr{T}_{n-2}(\overline{\mathscr{F}}_{n-1}) = \mathscr{T}_{n-2}(\mathscr{N}\mathscr{F})$, because $\mathscr{N}^{-1}$ is a line bundle on C_{n-1}. By the fact $\mathscr{N}\mathscr{F}$ is a generalized line bundle on C_{n-1} and by Lemma 2.13, $\mathscr{T}_{n-2}(\mathscr{N}\mathscr{F}) = (\mathscr{N}\mathscr{F})^{(1)}/\mathscr{N}^{n-1}\mathscr{F}$ (thanks to the fact that the non-trivial terms of the two canonical filtrations of $\mathscr{N}\mathscr{F}$ seen as a sheaf on C_{n-1} and seen as a sheaf on X coincide). Moreover, $\mathscr{N}\mathscr{F} \subset \mathscr{F}$ implies $(\mathscr{N}\mathscr{F})^{(1)} \subset \mathscr{F}^{(1)}$, hence $\mathscr{T}_{n-2}(\overline{\mathscr{F}}_{n-1}) = \mathscr{T}_{n-2}(\mathscr{N}\mathscr{F}) \subset \mathscr{F}^{(1)}/\mathscr{N}^{n-1}\mathscr{F} = \mathscr{T}_{n-1}(\mathscr{F})$, as wanted (the last equality holds again by Lemma 2.13). *q.e.d.*

COROLLARY 2.19. *The following are equivalent:*

(i) $\mathscr{F}$ *is a line bundle on* X*;*
(ii) $b_i = 0$ *for any* $1 \le i \le n-1$*;*
(iii) $b_{n-1} = 0$.

PROOF. It is evident that (i) implies (ii) which implies (iii); by the above Lemma (iii) implies (ii). The proof that (ii) implies (i) is by induction. The basis is the case of ribbons, i.e. $n = 2$, which is Proposition 2.14.

So let $n \ge 3$. By the fact $b_i = 0$ for $1 \le i \le n-2$, it holds that $\overline{\mathscr{F}}_{n-1}$ is a line bundle by inductive hypothesis; moreover, $b_{n-1} = 0$ means that $\mathscr{F}|_{C_{n-1}} = \overline{\mathscr{F}}_{n-1}$. Hence, $\mathscr{F}$ is a line bundle by Corollary 2.15. *q.e.d.*

An interesting problem, whose solution will be useful also in the study of stability conditions, is how to express the generalized degrees of the $\overline{\mathscr{F}}_i$'s in terms of that of $\mathscr{F}$. The solution is the following:

PROPOSITION 2.20. *Let* $\mathscr{F}$ *be a generalized line bundle on* X *of generalized degree* $\mathrm{Deg}(\mathscr{F}) = D$*. Then*

$$\mathrm{Deg}(\overline{\mathscr{F}}_i) = \frac{1}{n}\left[iD + (n-i)\sum_{j=1}^{i-1} b_j - i\sum_{j=i}^{n-1} b_j - \frac{in(n-i)}{2}\deg(\mathcal{C})\right], \qquad (2.1)$$

for any $1 \le i \le n$.

PROOF. The proof is by induction on n. The basis is given by the trivial case $n = 1$, where there is only the equality $D = D$.

In order to simplify the notation, let $D' = \mathrm{Deg}(\overline{\mathscr{F}}_{n-1})$ and $\delta = -\deg(\mathcal{C})$.

By inductive hypothesis, it holds that $\mathrm{Deg}(\overline{\mathscr{F}}_i) = \frac{1}{n-1}\Big[iD' + (n-1-i)\sum_{j=1}^{i-1} b_j - i\sum_{j=i}^{n-2} b_j + \frac{i(n-1)(n-1-i)}{2}\delta\Big]$ for $1 \le i \le n-1$.

Now let us calculate D' in terms of D: $D' = \chi(\overline{\mathscr{F}}_{n-1}) - (n-1)\chi(\mathscr{O}_C) = \chi(\mathscr{F}) - \chi(\mathscr{F}^{(1)}) - (n-1)\chi(\mathscr{O}_C) = D - \chi(\mathscr{N}^{n-1}\mathscr{F}) - b_{n-1} + \chi(\mathscr{O}_C) = D - \chi(\overline{\mathscr{F}}_1 \otimes \mathcal{C}^{\otimes n-1}) + \chi(\mathscr{O}_C) - b_{n-1} = D - \deg(\overline{\mathscr{F}}_1 \otimes \mathcal{C}^{\otimes n-1}) - b_{n-1} = D - \deg(\overline{\mathscr{F}}_1) + (n-1)\delta - b_{n-1} = D - \frac{1}{n-1}\Big[D' - \sum_{j=1}^{n-2} b_j + \frac{(n-1)(n-2)}{2}\delta\Big] + (n-1)\delta - b_{n-1}$; where the first equality holds by definition, the second by Remark 2.12 (and additivity of the Euler characteristic), the third by the definitions of D and b_{n-1}, the forth by Fact 1.9(i), the fifth by definition of degree of a line bundle on C, the sixth by its additivity and the last by inductive hypothesis and by the fact that for a line bundle on C degree and generalized degree coincide.

Thus we have that $D' = \frac{1}{n}\Big[(n-1)D + \sum_{j=1}^{n-2} b_j - (n-1)b_{n-1} + \frac{n(n-1)}{2}\delta\Big]$, as desired. In order to obtain the claim for $\mathrm{Deg}(\overline{\mathscr{F}}_i)$ with $1 \le i \le n-2$, it is sufficient to substitute this value of D' in the formulae obtained by inductive hypothesis. The case $i = n$ is a trivial identity. *q.e.d.*

COROLLARY 2.21. *For any* $1 \le i \le n-1$*, it holds that*

$$\mathrm{Deg}(\mathscr{F}^{(i)}) = \frac{1}{n}\left[iD - i\sum_{j=1}^{n-i-1} b_j + (n-i)\sum_{j=n-i}^{n-1} b_j + \frac{in(n-i)}{2}\deg(\mathcal{C})\right], \quad (2.2)$$

where $D = \mathrm{Deg}(\mathscr{F})$, *as in the proposition.*

PROOF. The assertion follows from the Proposition, because $\mathscr{F}/\mathscr{F}^{(i)}$ is isomorphic to $\overline{\mathscr{F}}_{n-i}$ (cf. Remark 2.12) and the generalized degree is additive (cf. Fact 1.12(v)). *q.e.d.*

Proposition 2.20 can be used also to give another, apparently surprising, characterization of the indices of a generalized line bundle in terms of the torsion parts of the quotients of the first canonical filtration.

PROPOSITION 2.22. *Let* $\mathscr{F}$ *be a generalized line bundle on* X. *For any* $1 \leq i \leq n-1$, $b_i(\mathscr{F}) = \mathrm{h}^0(\mathcal{T}_{n-1-i}(\mathscr{F}))$, *where* $\mathcal{T}_{n-1-i}(\mathscr{F})$ *is the torsion part of* $G_{n-1-i}(\mathscr{F})$.

PROOF. In order to simplify notations throughout the proof, we will set $b_i = b_i(\mathscr{F})$ and $\beta_i = \beta_i(\mathscr{F}) = \mathrm{h}^0(\mathcal{T}_{n-1-i}(\mathscr{F}))$. We proceed by induction on n, the multiplicity of X. The basis is constituted by $n = 2$. In this case, it has to be considered only b_1 and the desired equality is verified by definition.

So let $n \geq 3$ and assume that the statement holds for $n-1$. Let $\mathscr{F}$ be a generalized line bundle on X of generalized degree D and let $d_i = \deg(G_i(\mathscr{F}))$. By definition and by additivity of the generalized degree, $D = \sum_{i=0}^{n-1} d_i$. It holds also that $d_i = d_{n-1} - (n-1-i)\deg(\mathcal{C}) + \beta_{n-1-i}$, for any $0 \leq i \leq n-2$. Indeed, for any $0 \leq i \leq n-2$, by Fact 1.9(iii)(b) there is a surjective morphism $\mu_{i,n-1-i} : G_i(\mathscr{F}) \otimes \mathcal{C}^{n-1-i} \twoheadrightarrow G_{n-1}(\mathscr{F}) = \mathscr{N}^{n-1}\mathscr{F}$. Moreover, by the fact $\mathscr{F}$ is a generalized line bundle, $G_{n-1}(\mathscr{F})$ is a line bundle over C, while $G_i(\mathscr{F}) \otimes \mathcal{C}^{n-1-i}$ has rank 1 over C. Hence, its locally free part is isomorphic to $G_{n-1}(\mathscr{F})$ and the kernel of $\mu_{i,n-1-i}$ is isomorphic to $\mathcal{T}_i(\mathscr{F})$. Therefore, $D = nd_{n-1} - n(n-1)/2\deg(\mathcal{C}) + \sum_{i=1}^{n-1}\beta_i$.

Recall that by Fact 1.9(ii), $d_{n-1} = \deg(\overline{\mathscr{F}}_1) + (n-1)\deg(\mathcal{C})$.

It follows that $D = n\deg(\overline{\mathscr{F}}_1) + n(n-1)/2\deg(\mathcal{C}) + \sum_{i=1}^{n-1}\beta_i$. Substituting in this equality the value of $\deg(\overline{\mathscr{F}}_1)$ given by formula (2.1) we get that $\sum_{i=1}^{n-1} b_i = \sum_{i=1}^{n-1}\beta_i$.

By Fact 1.9(i), $\mathscr{N}\mathscr{F}$ (which is a generalized line bundle over C_{n-1}) is isomorphic to $\overline{\mathscr{F}}_{n-1} \otimes \mathscr{N}$; hence, $b_i(\mathscr{N}\mathscr{F}) = b_i(\overline{\mathscr{F}}_{n-1}) = b_i$, for $1 \leq i \leq n-2$, where the last equality holds by definition. Again by definition, $\beta_i = \beta_i(\mathscr{N}\mathscr{F})$ for $1 \leq i \leq n-2$. Thus we can use inductive hypothesis to assert that $b_i = \beta_i$, for any $1 \leq i \leq n-2$.

Therefore, the previous equality $\sum_{i=1}^{n-1} b_i = \sum_{i=1}^{n-1}\beta_i$ implies that also $b_{n-1} = \beta_{n-1}$. *q.e.d.*

The next lemma and corollary describe some relations between a generalized line bundle and its dual.

LEMMA 2.23. *Let* $\mathscr{F}$ *be, as usual, a generalized line bundle on* X *and let* $\mathscr{F}^\vee$ *be its dual (which is a generalized line bundle, too). Then there are the following canonical isomorphisms*

(i) $\mathscr{N}^i(\mathscr{F}^\vee) \simeq \left(\mathscr{F}^{(n-i)}\right)^\vee \otimes \mathscr{C}^{\otimes i}$, *i.e.* $\overline{(\mathscr{F}^\vee)}_{n-i} \simeq \left(\mathscr{F}^{(n-i)}\right)^\vee$;

(ii) $\mathscr{T}_i(\mathscr{F}^\vee) \simeq \underline{\mathrm{Ext}}^1_{\mathscr{O}_X}(\mathscr{T}_i(\mathscr{F}), \mathscr{O}_X) \otimes \mathscr{C}^{\otimes i}$, *and then there is a non-canonical isomorphism between* $\mathscr{T}_i(\mathscr{F}^\vee)$ *and* $\mathscr{T}_i(\mathscr{F})$.

PROOF. The first assertion is Fact 1.21(v)(b), thanks to the fact that for a generalized line bundle $(\mathscr{F}|_{C_i})^{\vee\vee} = \overline{\mathscr{F}}_i$ by Lemma 2.11 and, thus, $\ker(\mathscr{F} \twoheadrightarrow (\mathscr{F}|_{C_i})^{\vee\vee})$ coincides with $\mathscr{F}^{(n-i)}$. The second one is Fact 1.21(v)(a). *q.e.d.*

COROLLARY 2.24. *For $1 \leq i \leq n-1$, the following formula holds:*

$$b_i(\mathscr{F}^\vee) = b_{n-1}(\mathscr{F}) - b_{n-1-i}(\mathscr{F}), \tag{2.3}$$

where $b_0(\mathscr{F})$ is posed equal to 0.

PROOF. The case $i = n-1$ is implied by the second statement of the Lemma.

Thus, let $i \leq n-2$. By the first point of the Lemma, $b_i(\mathscr{F}^\vee) = b_i(\mathscr{F}^{(i+1)})$; hence, it is sufficient to show that $b_i(\mathscr{F}^{(i+1)}) = b_{n-1}(\mathscr{F}) - b_{n-1-i}(\mathscr{F})$.

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{N}^{n-1}\mathscr{F} & \longrightarrow & \mathscr{F}^{(1)} & \longrightarrow & \mathscr{T}_{n-1}(\mathscr{F}) & \longrightarrow & 0 \\ & & \downarrow f & & \| & & \downarrow g & & \\ 0 & \longrightarrow & \mathscr{N}^{i}\mathscr{F}^{(i+1)} & \longrightarrow & \mathscr{F}^{(1)} & \longrightarrow & \mathscr{T}_{i}(\mathscr{F}^{(i+1)}) & \longrightarrow & 0 \end{array}$$

By snake's lemma, f is injective (as obvious), g is surjective, as expected, and $\ker(g) \simeq \operatorname{coker}(f)$; thus, it suffices to prove that $\operatorname{coker}(f) \simeq \mathscr{T}_{n-1-i}(\overline{\mathscr{F}}_{n-i})$. The fact that, by their definitions, $\mathscr{N}^i\mathscr{F}^{(i+1)} \simeq (\mathscr{N}^i\mathscr{F})^{(1)}$ implies that $\operatorname{coker}(f) \simeq \mathscr{T}_{n-1-i}(\mathscr{N}^i\mathscr{F}) \simeq \mathscr{T}_{n-1-i}(\overline{\mathscr{F}}_{n-i})$, where the latter isomorphism is due to Fact 1.9(i). *q.e.d.*

The next corollary will be useful in order to determine a surprisingly canonical Jordan-Holder filtration of a semistable generalized line bundle (see Proposition 2.44).

COROLLARY 2.25. *For any $2 \leq i \leq n-1$ and for any $1 \leq j \leq i-1$, it holds that*

$$b_j(\mathscr{F}^{(i)}) = b_{n-i+j}(\mathscr{F}) - b_{n-i}(\mathscr{F}). \tag{2.4}$$

PROOF. Note that $\mathscr{F}^{(i)} \simeq (\overline{(\mathscr{F}^\vee)}_i)^\vee$ by Lemma 2.23(i). Thus, by a double application of the previous Corollary, it holds that $b_j(\mathscr{F}^{(i)}) = b_{i-1}(\mathscr{F}^\vee) - b_{i-j-1}(\mathscr{F}^\vee) = b_{n-1}(\mathscr{F}) - b_{n-i}(\mathscr{F}) - b_{n-1}(\mathscr{F}) + b_{n-i+j}(\mathscr{F}) = b_{n-i+j}(\mathscr{F}) - b_{n-i}(\mathscr{F})$, as desired. *q.e.d.*

2.3. Structure theorem

This section is devoted firstly to study the local and global structure of a generalized line bundle on a primitive multiple curve X and then to describe the action of $\operatorname{Pic}(X)$ on the set of locally isomorphic generalized line bundles. It is not possible to extend straightforwardly to higher multiplicity [**CK**, Lemma 2.9], which asserts that two generalized line bundles with the same local index sequence on a ribbon differ by the tensor product by a line bundle and makes also explicit the stabilizer of this action of the Picard group. Indeed, we will show that having the same local indices sequence does not mean being locally isomorphic and that, moreover, in general, there is not a natural blow up on which a generalized line bundle becomes a line bundle (which in the case of ribbons is [**EG**, Theorem 1.1] and is the fundamental argument beyond the cited lemma by Chen and Kass). However there is an action of the Picard group on the set of locally isomorphic generalized line bundles (see Corollary 2.32) whose stabilizer is completely known in multiplicity 3 (see Corollary 2.35) and in some special cases in higher multiplicity (see Corollary 2.38). These special cases comprehend those of the generic elements of irreducible components of stable generalized line bundles in the moduli space (see Theorems 3.6 and 3.16).

The next lines recall the local set-up introduced before Definition 1.6, adding also some more notation. Let $P \in C$ be a closed point; then, in local arguments, $A_1 = A_{\text{red}}$ denotes $\mathscr{O}_{C,P}$ (which is a DVR) and $\mathfrak{m}_{A_1} = \mathfrak{m}_1$ is its maximal ideal, while $A = A_n$ denotes $\mathscr{O}_{X,P}$ with maximal ideal $\mathfrak{m}_A = \mathfrak{m}_n$ and A_i denotes $\mathscr{O}_{C_i,P}$ and $\mathfrak{m}_{A_i} = \mathfrak{m}_i$ is its maximal ideal, for $2 \leq i \leq n-1$; moreover, π_i denotes the projection $A \twoheadrightarrow A_i$, for $1 \leq i \leq n-1$. Let y denote a generator of the nilradical of A and let $\bar{y}_i = \pi_i(y)$, for any $1 \leq i \leq n-1$; fix a nonzero divisor x such that $(x, y) = \mathfrak{m}_A$ and let $\bar{x}_i = \pi_i(x)$, for any $1 \leq i \leq n-1$.

Definition 2.26. An A-module M is said to be *generalized invertible* if there exists a generalized line bundle $\mathscr{F}$ on X such that the stalk $\mathscr{F}_P$ is isomorphic to M. In algebraic terms, this means that M is a torsion-free A-module (in the sense that $\operatorname{ann}(m) \subseteq (y) = \operatorname{Nil}(A)$ for any $0 \neq m \in M$) and $M^{(i)}/M^{(i-1)}$ is an invertible A_1-module, i.e., being in a local context, it is isomorphic to A_1, for $1 \leq i \leq n$ (in particular, $\mathrm{R}(M) = n$). In the trivial case $n = 1$ generalized invertible is just invertible.

By the theory of generalized line bundles developed in the previous section, M admits only one torsion-free quotient $\overline{M}_i = M/(M^{(n-i)})$, which is an A_i generalized invertible module, for any $1 \leq i \leq n-1$.

The indices-vector of M is $b_{.} = b_{.}(M) := b_{.,P}(\mathscr{F})$ and its i-th index is $b_i = b_i(M) := b_{i,P}(\mathscr{F})$, for $1 \leq i \leq n-1$.

The following theorem describes the structure of generalized invertible A-modules. It is called Local Structure Theorem because it describes all the stalks at closed points of a generalized line bundle on X.

Theorem 2.27 (Local Structure Theorem). *Let M be a generalized invertible A-module with indices-vector $b_{.}$. Then there exist elements (possibly equal to zero) $\alpha_{i,j} \in A$, with $3 \leq i \leq n$ and $1 \leq j \leq i-2$, well-defined modulo $(x^{b_{n-j}-b_{n-j-1}}, y)$, and $m_i \in M$, with $1 \leq i \leq n$ such that*

$$M \cong \frac{\bigoplus_{i=1}^{n} m_i A}{\left(ym_1, ym_i - x^{b_{n-i+1}-b_{n-i}} m_{i-1} - \sum_{j=1}^{i-2} \alpha_{i,j} m_j \,\middle|\, 2 \leq i \leq n \right)}$$

$$\cong \left(y^{n-i} x^{b_{n-1}-b_{n-i}} + \sum_{j=2}^{i-1} \left(\sum_{h=0}^{j-2} (-1)^h \alpha_{i-h,j-1-h} x^{b_{n-2(j+h)+5}-b_{n-2(j+h)+4}} \right) \cdot \right.$$

$$\left. \cdot\, y^{n-j} \,\middle|\, 1 \leq i \leq n \right),$$

where $b_0 = b_n = 0$ *and the last module is an ideal of* A.

PROOF. First of all, observe that the second isomorphism is trivial: indeed, the relations between the generators of the ideal are those required for the m_i's. So the only point is to show the first isomorphism.

The easiest way to prove such a statement is induction. The basis is the trivial case $n = 1$, where generalized invertible modules are exactly invertible A-modules and there is no y: the statement reduces to the obvious observation that the only invertible modules on a local domain are free modules of rank 1. In the case $n = 2$ the statement is quite simpler than the general one: it reduces to the assertion that M is isomorphic to (x^b, y). This is already known, although I do not know any explicit reference for it: it is a consequence of [**EG**, Theorem 1.1] and it is used various times in [**CK**].

So, let the statement hold for $n - 1 \geq 1$ and let us prove it for n. Consider $M^{(n-1)} \subset M$: it is a generalized invertible module on A_{n-1} with $b_i(M^{(n-1)}) = b_{i+1} - b_1$, for $1 \leq i \leq n-2$, by a local application of Corollary 2.25. Thus, $b_{n-i}(M^{(n-1)}) - b_{n-i-1}(M^{(n-1)}) = b_{n-i+1} - b_{n-i}$ and, by inductive hypothesis, it holds that $M^{(n-1)} \cong \bigoplus_{i=1}^{n-1} \tilde{m}_i A_{n-1}/(\bar{y}_{n-1}\tilde{m}_1, \bar{y}_{n-1}\tilde{m}_i - \bar{x}_{n-1}^{b_{n-i}-b_{n-i-1}}\tilde{m}_{i-1} - \sum_{j=1}^{i-2} \tilde{\alpha}_{i,j}\tilde{m}_j \big| 1 \leq i \leq n-2)$. The $\tilde{m}_i$'s belong to M; rename $\tilde{m}_i = m_i$ and choose $\alpha_{i,j} \in A$ over $\tilde{\alpha}_{i,j}$, for each pair (i,j); so, we get that $M^{(n-1)} \cong \bigoplus_{i=2}^{n} m_i A/(ym_1, ym_i - x^{b_{n-i+1}-b_{n-i}} m_{i-1} - \sum_{j=1}^{i-2} \alpha_{i,j} m_j \big| 2 \leq i \leq n-1)$. Moreover, $M/M^{(n-1)} = \overline{M}_1 \cong A_1$ by hypothesis. Therefore, choosing $m_n \in M$ over a generator of $\overline{M}_1$ we obtain a set of generators of M, i.e. $m_1, \ldots, m_n$. In order to complete the proof, we need to find the relations between m_n and the other generators. Indeed, the submodule generated by $y^{n-1}m_n$ is isomorphic to $y^{n-1}M$; hence, substituting, if necessary, m_n with another element with the same image in $\overline{M}_1$, we have that $y^{n-1}m_n = x^{b_{n-1}}m_1$, by a local application of Lemma 2.13. Now using the other relations we get the desired one $ym_n - x^{b_1}m_{n-1} - \sum_{j=1}^{n-2} \alpha_{n,j} m_j$. By the fact we can substitute again m_n with m_n plus a linear combination of the other m_i's we obtain that the $\alpha_{n,j}$'s are defined only modulo $(x^{b_{n-j}-b_{n-j-1}}, y)$. *q.e.d.*

REMARK 2.28. In order to apply [**H**, Proposition 2.12] to derive the global structure of generalized line bundles from the local one described in the previous theorem (see the proof of Corollary 2.32), generalized invertible A-modules should be classified up to *linear equivalence* (i.e. up to product by an element of the total ring of fractions of A) and not up to isomorphism. It could also be sufficient to work with ideals (cf. [**H**, Lemma 2.13]) of the completion of A (cf. [**H**, Proposition 2.14]). A priori linear equivalence is stronger than being isomorphic, but the inductive step could be adapted to show that the description given in the theorem holds up to linear equivalence and not only up to isomorphism. The classification of the ideals of A_2 up to linear equivalence had already been worked out in [**H**, Example 3.9].

Observe that sometimes a generalized invertible module can be generated by a smaller set of generators; a quite important case, which will be fundamental to describe the irreducible components of generalized line bundles in the moduli space of semistable pure sheaves of generalized rank n in Chapter 3, is that treated in the following corollary.

COROLLARY 2.29. *Let M be a generalized invertible module over A with indices-vector $b_.$ such that there exists an integer $1 \leq j \leq n-1$ such that $0 = b_{j-1} < b_j = b_{n-1} = b$. Then there exists $\alpha \in A$ such that $M \cong (x^b + \alpha y,\, y^j)$. Moreover, there exist unique $z_{h,i} \in \mathbb{K}$, for $1 \leq h \leq \bar{\jmath}$, where $\bar{\jmath} = \min\{j, n-j\} - 1$, and $0 \leq i \leq b-1$, such that $M \cong \big(x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z_{h,i} x^i\big) y^h,\, y^j\big)$.*

PROOF. The first assertion is a trivial consequence of the Theorem.

In order to simplify the notation in the proof of the second assertion, set $M(\beta) = (x^b + \beta y,\, y^j)$, for any $\beta \in A$. The existence of the $z_{h,i} \in \mathbb{K}$ is equivalent to the fact that there exists $\alpha' \in A$, defined modulo $(x^b,\, y^{\bar{\jmath}+1})$, such that $M(\alpha) \cong M(\alpha')$. First of all, observe that $M(\gamma + \beta y^{\bar{\jmath}}) \cong M(\gamma)$, for any $\gamma, \beta \in A$, which implies that α is defined modulo $y^{\bar{\jmath}+1}$. Indeed, if $\bar{\jmath} = j-1$, the two modules are equal, while, if $\bar{\jmath} = n-j-1$, there is a trivial isomorphism, say φ defined on the generators as $\varphi(x^b + (\gamma + \beta y^{\bar{\jmath}})y) = x^b + \gamma y$ and $\varphi(y^j) = y^j$: in order to check that φ is not only a bijection of sets but also a morphism of A-modules it is sufficient to verify that $y^j \varphi(x^b + (\gamma + \beta y^{\bar{\jmath}})y) = (x^b + (\gamma + \beta y^{\bar{\jmath}})y)\varphi(y^j)$, which is a trivial equality: in this case $\beta y^{\bar{\jmath}} y \varphi(y^j) = \beta y^n = 0$.

The next step is to show that $M(\gamma + \beta x^b)$ is isomorphic to the module $M\big(\sum_{l=1}^{\bar{\jmath}} (-\beta y)^{l-1}\gamma\big)$, for any $\gamma, \beta \in A$. Setting $\alpha = \bar{\alpha} + \beta x^b$, where $\bar{\alpha} \in A$ is an element without terms in x^k, with $k \geq b$ (looking for a while at A as a $\mathbb{K}$-vector space of infinite dimension), and iterating, if necessary, the proceeding, such an isomorphism is sufficient to conclude the existence of the desired α' (which is not necessarily congruent to α modulo $(x^b, y^{\bar{\jmath}+1})$). The point is to check that $M(\gamma + \beta x^b)$ is equal to $M\big(\sum_{l=1}^{2^c-1} (-\beta y)^{l-1}\gamma\big)$, where $c = [\log_2(\bar{\jmath})] + 1$, because the latter is isomorphic to $M\big(\sum_{l=1}^{\bar{\jmath}} (-\beta y)^{l-1}\gamma\big)$, by the first step. The equality holds by the fact that $x^b + \sum_{l=1}^{2^c-1} (-\beta)^{l-1} y^l \gamma = (x^b + (\gamma + \beta x^b)y) \prod_{r=0}^{c} (1 + (-1)^r (\beta y)^{2^r})$ and the latter is an invertible element of A.

It remains to prove the uniqueness of the $z_{h,i}$. So, we need to show that if $M\big(\sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z_{h,i} x^i\big) y^{h-1}\big)$ and $M\big(\sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z'_{h,i} x^i\big) y^{h-1}\big)$ are isomorphic, then $z_{h,i} = z'_{h,i}$, for any h and i. In order to simplify notations, set $s = x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z_{h,i} x^i\big) y^h$ and $s' = x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z'_{h,i} x^i\big) y^h$. Let ψ be such an isomorphism and ψ^{-1} its inverse. It holds that $\psi(y^j) = a_1 y^j + a_2 s'$ and $\psi(s) = a_3 y^j + a_4 s'$ and, analogously, $\psi^{-1}(y^j) = a'_1 y^j + a'_2 s$ and $\psi^{-1}(s') = a'_3 y^j + a'_4 s$. By the fact $y^{n-j}\psi y^j = 0$ and $y^{n-1}\psi^{-1}(y^j) = 0$, it follows that a_1 and a'_1 can be chosen so that $a_2 = a'_2 = 0$; moreover, $y^j = \psi^{-1}(\psi(y^j)) = a'_1 a_1 y^j$ implies that a_1 and a'_1 are invertible and $a'_1 = a_1^{-1}$ (set $a_1 = u_1 + m_1$ with $u_1 \in \mathbb{K}$ and $m_1 \in \mathfrak{m}_A$). Moreover, $s = \psi^{-1}\psi(s) = a'_4 a_4 s + (a'_3 a_4 + a_1 a_3) y^j$ implies that $a'_4 = a_4^{-1}$ and $(a'_3 a_4 + a_1 a_3)$ is a multiple of y^{n-j}. As usual, ψ is really a morphism if and only if $y^j \psi(s) = s\psi(y^j)$. But $y^j \psi(s) = a_3 y^{2j} + a_4 y^j s' = a_3 y^{2j} + a_4 y^j x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} a_4 z'_{h,i} x^i\big) y^{j+h}$ and $s\psi(y^j) = a_1 y^j s = a_1 y^j x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} a_1 (z_{h,i} - z'_{h,i} + z'_{h,i}) x^i\big) y^{j+h}$. Hence, they are equal if and only if $(a_1 - a_4)x^b y^j = \sum_{h=1}^{\bar{\jmath}} \sum_{i=0}^{b-1} \big((a_4 - a_1)z'_{h,i} - (u_1 + m_1)(z_{h,i} - z'_{h,i})\big)x^i y^{j+h} + a_3 y^{2j}$. So it has to be $a_1 - a_4 = \epsilon y$, for some $\epsilon \in A$, and the equality becomes $\epsilon x^b y^{j+1} = \sum_{h=1}^{\bar{\jmath}} \sum_{i=0}^{b-1} \big(- \epsilon z'_{h,i} y - u_1(z_{h,i} - z'_{h,i}) - m_1(z_{h,i} - z'_{h,i})\big)x^i y^{j+h} + c y^{2j}$. Observing the powers of x and y in the right term (and remembering that m_1 belongs to $\mathfrak{m}_A = (x, y)$, while u_1, $z_{h,i}$ and $z'_{h,i}$ belong to $\mathbb{K}$), it has to hold that $z_{1,i} = z'_{1,i}$ for any i. But then all the terms on the right are divided by y^{j+2} so $\epsilon = \zeta y$, for some $\zeta \in A$, and by the same considerations $z_{2,i} = z'_{2,i}$ for any i, and so on. The same argument continues to hold at any step and it follows that $z_{h,i} = z'_{h,i}$, for any h and i, as wanted. *q.e.d.*

REMARK 2.30.

(i) The Corollary classifies these kind of modules also up to linear equivalence (cf. Remark 2.28): indeed, if two modules are not isomorphic they are also not linearly equivalent, and the only isomorphism used throughout the proof, i.e. $\varphi : M(\gamma+\beta y^{\bar{\jmath}}) \xrightarrow{\sim} M(\gamma)$, in the case $\bar{\jmath} = n - j - 1$, could be substituted with multiplication by $x^{-(2j-n)b} \prod_{l=0}^{2j-n-1}(x^b + (-\beta)^{l+1}\gamma^l y^{n-j+l})$.

(ii) By the Theorem, in multiplicity greater than or equal to 3 the local indices sequence is not always sufficient to characterize up to isomorphism stalks of generalized line bundles. Indeed, e.g. in the case $n = 3$, using local notation, $(x^2 + y, xy, y^2)$ has the same indices-vector of (x^2, xy, y^2) but it is easy to show that they are not isomorphic.

Only in some special cases two generalized line bundles having the same local indices sequence are necessarily locally isomorphic; by the above Corollary it happens in particular for those having either $b_{1,P} = b_{n-1,P}$ or $b_{1,P} = b_{n-2,P} = 0$, for any closed point $P \in C$.

It is easy to pass from the local description to the following affine picture.

COROLLARY 2.31. *Let $\mathscr{F}$ be a generalized line bundle on X and let P be a closed point where $\mathscr{F}$ has non-trivial local indices sequence $b_{.,P} = b_{.}$. There exists an affine neighbourhood $P \in U \subset X$, where $\mathscr{F}(U)$ is isomorphic to the ideal* $\Big(y^{n-i}x^{b_{n-1}-b_{n-i}} + \sum_{j=2}^{i-1}\big(\sum_{h=0}^{j-2}(-1)^h \alpha_{i-h,j-1-h} x^{b_{n-2(j+h)+5}-b_{n-2(j+h)+4}}\big)y^{n-j} \,\Big|\, 1 \leq i \leq n\Big)$, *where y is a generator of the nilradical of $\mathscr{O}_X(U)$ and x is a nonzerodivisor in $\mathscr{O}_X(U)$ such that (x, y) is the ideal of P in U.*

In the special case in which there exists an integer $1 \leq j \leq n-1$ such that $0 = b_{j-1} < b_j = b_{n-1} = b$, then there exist and are unique $z_{h,i} \in \mathbb{K}$, for $1 \leq h \leq \bar{\jmath}$, where $\bar{\jmath} = \min\{j, n-j\} - 1$, and $0 \leq i \leq b-1$, such that $\mathscr{F}(U) \cong \Big(x^b + \sum_{h=1}^{\bar{\jmath}} \big(\sum_{i=0}^{b-1} z_{h,i} x^i\big) y^h,\ y^j\Big)$.

PROOF. It is a trivial application of the Theorem and of the above Corollary, considering that there are only finitely many points on which the stalks of a generalized line bundle are not free. *q.e.d.*

In general it is possible to obtain only the following global description, which remains quite vague.

Corollary 2.32 (Global structure). *Let $\mathscr{F}$ be a generalized line bundle on X. Then $\mathscr{F}$ is isomorphic to $\mathscr{I}_{Z/X} \otimes \mathscr{G}$, where $Z \subset C_{n-1}$ is a closed subscheme of finite support whose schematic intersection with C is* $\mathrm{Supp}(\mathscr{T}_{n-1}(\mathscr{F}))$*, called the* subscheme associated *to $\mathscr{F}$, and $\mathscr{G}$ is a line bundle on X.*
Moreover, it holds that

(i) *Z is unique up to adding a Cartier divisor.*
(ii) *Locally isomorphic generalized line bundles have the same associated subscheme, up to adding a Cartier divisor. In particular, if $\mathscr{F}$ and $\mathscr{F}'$ are locally isomorphic generalized line bundles, then there exists a line bundle $\mathscr{E}$ such that $\mathscr{F} = \mathscr{F}' \otimes \mathscr{E}$. Equivalently, there is a transitive action of* $\mathrm{Pic}(X)$ *on the set of locally isomorphic generalized line bundles.*

Proof. Let I_P denote the ideal isomorphic to $\mathscr{F}_P$ described in the Theorem. Observe that the sum of I_P with $\mathscr{N}_P$ defines locally the support of $\mathscr{T}_{n-1}(\mathscr{F})$. Moreover, it is evident that $\mathscr{O}_{X,P}/I_P$ is an $\mathscr{O}_{C_{n-1},P}$-module.

Let $\mathscr{I} \subset \mathscr{O}_X$ be the ideal sheaf defined locally as $\mathscr{I}_P = I_P$ for any closed point P (hence, it is isomorphic to $\mathscr{F} \otimes \mathscr{G}$ for some line bundle $\mathscr{G}$ by [**H**, Proposition 2.12]): by the local observations, it defines a closed subscheme of finite support $Z \subset C_{n-1}$ such that $Z \cap C = \mathrm{Supp}(\mathscr{T}_{n-1}(\mathscr{F}))$.

The two last assertions are trivial. *q.e.d.*

As anticipated at the beginning of this section, it seems impossible to extend [**EG**, Theorem 1.1] to higher multiplicity for any generalized line bundle. However, it is possible to get something similar for some special choices of the local indices sequence. We will begin the study of this problem examining the case of multiplicity 3.

Lemma 2.33. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_3$, let Z be the subscheme associated to it (cf. Corollary 2.32) and let $q : X' \to X$ be the blow up of X along Z. Then*

(i) *$\mathscr{F}$ is the direct image of a line bundle $\mathscr{F}'$ on X' if and only if in any closed point P such that $\mathscr{F}_P$ is not a free $\mathscr{O}_{X,P}$-module it holds that $2b_{1,P} \leq b_{2,P}$.*
(ii) *X' is a primitive multiple curve of multiplicity 3 with reduced subcurve C if and only if in any closed point P such that $\mathscr{F}_P$ is not a free $\mathscr{O}_{X,P}$-module it holds that $2b_{1,P} \geq b_{2,P}$.*

Proof. We can restrict our attention to the local setting, because there exists an affine cover where the situation is essentially equal to the local one (cf. Corollary 2.31). This is due to the fact that a generalized line bundle is not free only in a finite set of closed points.

Let us study the local setting, using the same notation of the beginning of the section. If M is an invertible generalized A-module for $A = A_3$, then, by the Local Structure Theorem, i.e. Theorem 2.27, $M \cong (x^{b_2}+\alpha y, x^{b_2-b_1}y, y^2)$. If it were possible to extend the cited theorem by Eisenbud and Green, there would exist a natural blow up A' of A, having the same reduced ring and possibly being again the local ring of a primitive multiple curve of multiplicity 3, such that M admits a structure of free A'-module of rank 1.

Being M isomorphic to an ideal, there is only one natural blow up A' of A to consider: the one with respect to this ideal. By computations similar to those of the proof of [**BE**, Theorem 1.9] based on the fact that y is nilpotent, it holds that A' is isomorphic to $A[yx^{b_2-b_1}/(x^{b_2}+\alpha y), y^2/(x^{b_2}+\alpha y)]$, which reduces to $A[y/x^{b_1}, y^2/x^{b_2}]$, in the simplest case of $\alpha = 0$. By the fact A' is contained in the total ring of fractions of A, M admits a structure of A'-module if and only if it is closed under multiplication by $yx^{b_2-b_1}/(x^{b_2} + \alpha y)$ and $y^2/(x^{b_2} + \alpha y)$; this happens only if $2b_1 \leq b_2$. In this case, M is isomorphic to xA' and, thus, is a free A'-module of rank 1.

On the other hand, by definition such an A' is the local ring of a primitive multiple curve when its nilradical is a principal ideal, and this happens only for $2b_1 \geq b_2$.

Hence, we can summarize these results saying that M admits a structure of free A'-module of rank 1 if and only if $2b_1 \leq b_2$ while A' is the local ring of a primitive multiple curve (of multiplicity 3) if and only if $2b_1 \geq b_2$. Clearly, both conditions hold if and only if $2b_1 = b_2$. *q.e.d.*

REMARK 2.34. In particular, any generalized line bundle with $b_1 = 0$ verifies the hypotheses of the first point of the Lemma.

The following corollary is somehow an extension to multiplicity 3 of [**CK**, Lemma 2.9], although it requires more restrictive hypotheses.

COROLLARY 2.35. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_3$, let $\mathscr{E}$ be a line bundle on X and let $q : X' \to X$ be the blow up of X with respect to the ideal sheaf $\mathscr{I}$ such that $\mathscr{I}_P \cong \mathscr{F}_P$ if $2b_{1,P} \leq b_{2,P}$ and $\mathscr{I}_P \cong \mathscr{F}_P^\vee$ otherwise, for any closed point $P \in C$. Then $\mathscr{F} = \mathscr{F} \otimes \mathscr{E}$ if and only if $\mathscr{E}$ belongs to* $\ker(q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X'))$. *In other words, the stabilizer of the action of* $\mathrm{Pic}(X)$ *on the set of locally isomorphic generalized line bundles (see Corollary 2.32) is* $\ker(q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X'))$.

PROOF. First of all, observe that $\mathscr{I}$ is the direct image of a line bundle on X', by Lemma 2.33(i) (recall that $b_{1,P}(\mathscr{F}^\vee) = b_{2,P} - b_{1,P}$ by a local application of Corollary 2.24). So, the assertion follows for $\mathscr{I}$ and any $\mathscr{F}$ locally isomorphic to it from an easy application of the projection formula.

There are other two possibilities to consider. The first one is that $\mathscr{F}^\vee$ is locally isomorphic to $\mathscr{I}$. In this case, we can conclude by the previous case and by the trivial observation that $(\mathscr{G} \otimes \mathscr{E})^\vee \simeq \mathscr{G}^\vee \otimes \mathscr{E}^\vee$ if $\mathscr{E}$ is a line bundle and $\mathscr{G}$ any sheaf.

The last case is the mixed one, in which nor $\mathscr{F}$ neither $\mathscr{F}^\vee$ are locally isomorphic to $\mathscr{I}$. It follows easily from the previous ones. Indeed, in this case $\mathscr{F}$ is locally isomorphic to $\mathscr{I}_1 \otimes \mathscr{I}_2$, where $\mathscr{I}_1$ is the ideal sheaf everywhere trivial except in the points for which $2b_{1,P} \leq b_{2,P}$ where $\mathscr{I}_{1,P} \cong \mathscr{F}_P$ and $\mathscr{I}_2$ is the ideal sheaf everywhere trivial except in the points for which $2b_{1,P} \geq b_{2,P}$ where $\mathscr{I}_{2,P} \cong \mathscr{F}_P$. The line bundles that fix $\mathscr{F}$ are those fixing both $\mathscr{I}_1$ and $\mathscr{I}_2$. Hence, the assertion follows from the previous cases (essentially, because $\mathscr{I}_1$ and $\mathscr{I}_2$ are non-trivial in distinct points). *q.e.d.*

REMARK 2.36. It is not difficult to show that, for any generalized line bundle $\mathscr{F}$ on X, it holds that $\underline{\mathrm{End}}(\mathscr{F}) \simeq q_*(\mathscr{O}_{X'})$, where $q : X' \to X$ is the blow up of the previous Corollary. Indeed, if $\mathscr{F} \simeq \mathscr{I} \otimes \mathscr{E}$ or $\mathscr{F} \simeq \mathscr{I}^\vee \otimes \mathscr{E}$

(where $\mathscr{I}$ is as in the statement of the Corollary and $\mathscr{E}$ is a line bundle), it is immediate. Otherwise, $\underline{\mathrm{End}}(\mathscr{F})$ is locally isomorphic to $q_*(\mathscr{O}_{X'})$ (because they are both locally isomorphic to $\mathscr{I}$), so it is sufficient to show that there exists a morphism between the two sheaves. The latter is guaranteed by the universal property of the blow up, because, thanks to the local isomorphisms, the inverse image ideal sheaf of $\mathscr{I}$ on the relative spectrum $\underline{\mathrm{Spec}}\big(\underline{\mathrm{End}}(\mathscr{F})\big)$ is invertible.

The above Remark will be useful in Chapter 3 in order to study the dimension of the tangent space to a point corresponding to a stable generalized line bundle in the moduli space.

The next step is to get similar results for any multiplicity $n > 3$. The basic ideas are essentially the same of multiplicity 3. Indeed, using again local notation, if M is a generalized invertible A-module, with $A = A_n$, by the Local Structure Theorem, a representative of its isomorphism class is the ideal $\Big(y^{n-i}x^{b_{n-1}-b_{n-i}}+\sum_{j=2}^{i-1}\big(\sum_{h=0}^{j-2}(-1)^h\alpha_{i-h,j-1-h}x^{b_{n-2(j+h)+5}-b_{n-2(j+h)+4}}\big)y^{n-j}\Big|\ 1 \leq i \leq n\Big)$ and for our purposes we can identify M with it. It is quite complicate to write down explicitly the blow up of A with respect to M in full generality; hence, we restrict our attention to the case with all the α's zero. In this case, by the nilpotency of y, it holds that the blow up of A with respect to M is $A' = A[y/x^{b_1}, y^2/x^{b_2}, \ldots, y^{n-2}/x^{b_{n-2}}, y^{n-1}/x^{b_{n-1}}]$. By similar considerations to the case of $n = 3$, we have that M (under the hypothesis that all the α's are zero) admits a structure of A'-module (and, moreover, $M = xA'$) if and only if $b_j + b_i \leq b_{j+i}$ for $1 \leq j \leq n-2$ and $j \leq i \leq n-j-1$, while the nilradical of A' is a principal ideal (and, thus, A' can be seen as the local ring of a primitive multiple curve) if and only if $ib_1 \geq b_i$ for $1 \leq i \leq n-1$. The situation is more intricate when there are non-zero α's.

However the description is quite easy in the special case in which there exists a positive integer $h \leq n-1$ such that $0 = b_{h-1} < b_h = b_{n-1} = b$. In this case, as pointed out in Corollary 2.29, there exists $\alpha \in A$ such that $M \cong (x^b+\alpha y, y^h)$ and the blow up results to be simply $A' = A[y^h/(x^b+\alpha y)]$: hence, M admits a structure of A'-module (and, moreover, it is a free A'-module of rank one) if and only if $h \geq n/2$, while the nilradical of A' is a principal ideal if and only if $h = 1$. By these observations and by essentially the same arguments of multiplicity 3, the following lemma, corollaries and remarks hold:

LEMMA 2.37. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_n$ of local indices $b_{\cdot,\cdot}$, let Z be the subscheme associated to it (cf. Corollary 2.32) and let $q : X' \to X$ be the blow up of X along Z. Then*

(i) *If for any closed point P such that $\mathscr{F}_P$ is not a free $\mathscr{O}_{X,P}$-module $b_{j,P}+b_{i,P} \leq b_{j+i,P}$ for $1 \leq j \leq n-2$ and $j \leq i \leq n-j-1$ and all the α's are zero or there exists a positive integer $n/2 \leq h(P) \leq n-1$ such that $0 = b_{h(P)-1,P} < b_{h(P),P} = b_{n-1,P}$, then $\mathscr{F}$ is the direct image of a line bundle $\mathscr{F}'$ on X'.*

(ii) *If for any closed point P such that $\mathscr{F}_P$ is not a free $\mathscr{O}_{X,P}$-module $ib_{1,P} \geq b_{i,P}$ for $2 \leq i \leq n-1$ and all the α's are zero or $b_{1,P} =$*

$b_{n-1,P}$, then X' is a primitive multiple curve of multiplicity n with reduced subcurve C.

Corollary 2.38. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_n$ of local indices $b_{\cdot,\cdot}$ and let $\mathscr{E}$ be a line bundle on it.*

(i) *If $\mathscr{F}$ verifies the hypotheses of the first point of the previous lemma, then $\mathscr{F} \otimes \mathscr{E} \simeq \mathscr{F}$ if and only if $\mathscr{E}$ belongs to $\ker(q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X'))$, where $q : X' \to X$ is the blow up of X with respect to the ideal sheaf locally isomorphic to $\mathscr{F}$. Equivalently, this kernel is the stabilizer of the transitive action of $\mathrm{Pic}(X)$ on the set of generalized line bundles locally isomorphic to $\mathscr{F}$.*

(ii) *If $\mathscr{F}^\vee$ verifies the hypotheses of the first point of the previous lemma, then $\mathscr{F} \otimes \mathscr{E} \simeq \mathscr{F}$ if and only if $\mathscr{E}$ belongs to $\ker(q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X'))$, where $q : X' \to X$ is the blow up of X with respect to the ideal sheaf locally isomorphic to $\mathscr{F}^\vee$. In other words, this kernel is the stabilizer of the transitive action of $\mathrm{Pic}(X)$ on the set of generalized line bundles locally isomorphic to $\mathscr{F}$.*

(iii) *If for any closed point P there exists a positive integer $1 \leq h(P) \leq n-1$ such that $0 = b_{h(P)-1,P} < b_{h(P),P} = b_{n-1,P}$, then $\mathscr{F} \otimes \mathscr{E} \simeq \mathscr{F}$ if and only if $\mathscr{E}$ belongs to $\ker(q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X'))$, where $q : X' \to X$ is the blow up of X with respect to the ideal sheaf $\mathscr{I}$ such that $\mathscr{I}_P \cong \mathscr{F}_P$ when $h(P) \geq n/2$ and $\mathscr{I}_P \cong \mathscr{F}_P^\vee$ otherwise, for any closed point P. Equivalently, this kernel is the stabilizer of the transitive action of $\mathrm{Pic}(X)$ on the set of locally isomorphic generalized line bundles whose local indices verify the hypothesis.*

Proof. The proof is essentially the same of Corollary 2.35, with Lemma 2.33 replaced by Lemma 2.37. *q.e.d.*

Remark 2.39. It is not difficult to show that, for any generalized line bundle $\mathscr{F}$ on X whit the same hypotheses of the last point of the corollary, it holds that $\underline{\mathrm{End}}(\mathscr{F}) \simeq q_*(\mathscr{O}_{X'})$, where $q : X' \to X$ is the blow up of X with respect to the same ideal sheaf $\mathscr{I}$ of the last point of the corollary. Indeed, if $\mathscr{F} \simeq \mathscr{I} \otimes \mathscr{E}$ or $\mathscr{F} \simeq \mathscr{I}^\vee \otimes \mathscr{E}$ (where $\mathscr{E}$ is a line bundle on X), it is clear. Otherwise it is again evident that $\underline{\mathrm{End}}(\mathscr{F})$ is locally isomorphic to $q_*(\mathscr{O}_{X'})$ (because they are both locally isomorphic to $\mathscr{I}$), so it is sufficient to show that there exists a morphism between the two sheaves of $\mathscr{O}_X$-algebras $\underline{\mathrm{End}}(\mathscr{F})$ and $\mathscr{O}_{X'}$. The latter is guaranteed by the universal property of the blow up, because thanks to the local isomorphisms the inverse image ideal sheaf of $\mathscr{I}$ on the relative spectrum $\underline{\mathrm{Spec}}\big(\underline{\mathrm{End}}(\mathscr{F})\big)$ is invertible.

The Corollary and the Remark will be useful to replace as far as possible [**CK**, Lemma 2.9] (which is an essential tool in the proof of [**CK**, Lemma 4.4]) in the study of the moduli space in higher multiplicity.

2.4. Semistable generalized line bundles

This section, as the title suggests, studies semistability of generalized line bundles; it extends to higher multiplicity the results of [**CK**, §3]. We will assume throughout this section that $\deg(\mathcal{C}) < 0$, for otherwise there will not be stable generalized line bundles, as pointed out in Section 1.6.

We start with a quick remark about the slope and the Hilbert polynomial of a generalized line bundle and about an apparent discrepancy with [**CK**].

REMARK 2.40.

(i) Let $\mathscr{F}$ be a generalized line bundle on $X = C_n$. Its slope is $\mu(\mathscr{F}) = \mathrm{Deg}(\mathscr{F})/n$ and its Hilbert polynomial is $P_{\mathscr{F}}(T) = \mathrm{Deg}(\mathscr{F}) + n(1-g_1) + ndT$, while its reduced Hilbert polynomial $p_{\mathscr{F}}(T)$ is equal to $T + (\mu(\mathscr{F}) + 1 - g_1)/d$, where d is the degree of a polarization on C, cf. Fact 1.12(iv).

(ii) In [**CK**, §3] there is a different definition of the slope of a generalized line bundle on a ribbon and, thus, an apparently different notion of its (semi)stability. However, it is equivalent to that used in this work, being both equivalent to Gieseker's semistability.

The following theorem characterizes (semi)stability of a generalized line bundle on a primitive multiple curve in terms of a system of inequalities relating its indices and $\deg(\mathcal{C})$; it is the extension to higher multiplicity of [**CK**, Lemma 3.2]:

THEOREM 2.41. *Let $\mathscr{F}$ be a generalized line bundle of generalized degree D on X and indices-vector $b_{.}$. Then $\mathscr{F}$ is semistable if and only if the following inequalities hold:*

$$i\sum_{j=i}^{n-1} b_j - (n-i)\sum_{j=1}^{i-1} b_j \leq -\frac{in(n-i)}{2}\deg(\mathcal{C}),\ \forall\, 1 \leq i \leq n-1. \tag{2.5}$$

It is stable if and only if all the inequalities are strict.

PROOF. It is a straightforward application of previous results. Indeed, by Lemma 2.11 it is sufficient to verify that $\mu(\mathscr{F}) \leq \mu(\overline{\mathscr{F}}_i)$, i.e. that $D \leq n\,\mathrm{Deg}(\overline{\mathscr{F}}_i)/i$, for any $1 \leq i \leq n-1$. The assertion is easily obtained by substituting in these inequalities the formulae (2.1): $\mathrm{Deg}(\overline{\mathscr{F}}_i) = \frac{1}{n}\Big[iD + (n-i)\sum_{j=1}^{i-1} b_j - i\sum_{j=i}^{n-1} b_j - \frac{in(n-i)}{2}\deg(\mathcal{C})\Big]$. *q.e.d.*

COROLLARY 2.42. *Let $\mathscr{F}$ be a generalized line bundle on X, then it is semistable (resp. stable) if and only if $\mathscr{F}^\vee$ is semistable (resp. stable).*

PROOF. This is a special case of Lemma 1.29, but it follows also from the Theorem. Indeed, using formulae (2.3), the i-th inequality for $\mathscr{F}$ is equivalent to the $(n-i)$-th for $\mathscr{F}^\vee$. *q.e.d.*

REMARK 2.43. As anticipated in Remark 1.28(iv), this Theorem implies that there can exist stable generalized line bundles if and only if $\deg(\mathcal{C})$ is negative because the left hand side of inequalities (2.5) is always non-negative (thanks to Lemma 2.18 and to the obvious observation that $b_1 \geq 0$). Line bundles are always stable, if $\deg(\mathcal{C}) < 0$, because their indices are all 0, while they are the only type of strictly semistable generalized line bundles in the case $\deg(\mathcal{C}) = 0$.

The next step is to describe a Jordan-Holder filtration (which results to be in a certain sense canonical, being related to the second canonical filtration) and the Jordan-Holder graduate object of a strictly semistable generalized line bundle; it is an extension of [**CK**, Lemma 3.3] to higher multiplicity.

Proposition 2.44. *Let $\mathscr{F}$ be a generalized line bundle of generalized degree D on X strictly semistable, i.e. such that in $k \geq 1$ of the inequalities* (2.5) *the equality holds. Let $0 < i_1 < \cdots < i_k < n$ be the indices such that in the i_h-th inequality equality holds, for $1 \leq h \leq k$. Then a Jordan-Holder filtration of $\mathscr{F}$ is*

$$0 \subsetneq \mathscr{F}^{(n-i_k)} \subsetneq \cdots \subsetneq \mathscr{F}^{(n-i_1)} \subsetneq \mathscr{F};$$

and its Jordan-Holder graduate is

$$\mathrm{Gr}_{\mathrm{JH}}(\mathscr{F}) = \bigoplus_{h=0}^{k} \mathscr{F}^{(n-i_h)}/\mathscr{F}^{(n-i_{h+1})} = \bigoplus_{h=0}^{k-1} \overline{(\mathscr{F}^{(n-i_h)})}_{i_{h+1}} \oplus \mathscr{F}^{(n-i_k)},$$

where $i_0 = 0$ and $i_{k+1} = n$.

Proof. Set $\delta = -\deg(\mathcal{C})$ in order to simplify notation.

The proof is by strong induction. The basis is the trivial case $n = 1$, i.e. the case of line bundles on a reduced smooth projective curve, which, as well-known, are all stable.

So assume that the statement holds for generalized line bundles defined on C_l with $1 \leq l \leq n-1$. First of all, observe that the greatest term in the Jordan-Holder filtration has to be a semistable pure subsheaf of $\mathscr{F}$, having its same reduced Hilbert polynomial $p_{\mathscr{F}}(T)$, i.e. having its same slope, and such that the quotient is a pure stable sheaf. Observe that $\mathscr{F}^{(n-r)}$, being a generalized line bundle on C_{n-r} has the same slope of $\mathscr{F}$ if and only if $\mathrm{Deg}(\mathscr{F}^{(n-r)}) = \frac{n-r}{n} D$, which is equivalent, by formula (2.2), to having the equality in the r-th inequality of $\mathscr{F}$. Moreover, by similar considerations and by formula (2.1), in this case also the pure quotient $\overline{\mathscr{F}}_r$ has the same slope of $\mathscr{F}$.

Hence, if i_1 is the greatest index i such that the i-th inequality is an equality, $\mathscr{F}^{(n-i_1)}$ is a plausible candidate as greatest term of the Jordan-Holder filtration of $\mathscr{F}$. In order to check that it is really so, we need to verify that $\overline{\mathscr{F}}_{i_1}$ is stable and that $\mathscr{F}^{(n-i_1)}$ is semistable.

The i_1-th pure quotient is stable if and only if $l \sum_{j=l}^{i_1-1} b_j - (i_1-l) \sum_{j=1}^{l-1} b_j \leq \frac{l i_1 (i_1-l)}{2}\delta$ for any $1 \leq l \leq i_1 - 1$, by Theorem 2.41. The choice of i_1 implies that $i_1 \sum_{j=i_1}^{n-1} b_j - (n-i_1) \sum_{j=1}^{i_1-1} b_j = \frac{i_1 n (n-i_1)}{2}\delta$ and $i \sum_{j=i}^{n-1} b_j - (n-i) \sum_{j=1}^{i-1} b_j < \frac{in(n-i)}{2}\delta$ for any $1 \leq i \leq i_1 - 1$; substituting in the latter inequalities the value obtained for $\sum_{j=i_1}^{n-1} b_j$ one gets exactly those proving the stability of $\mathscr{F}^{(i_1)}$.

On the other side, $\mathscr{F}^{(n-i_1)}$ is semistable if and only if the inequalities $i \sum_{j=i}^{n-i_1-1} b_j(\mathscr{F}^{(n-i_1)}) - (n-i_1-i) \sum_{j=1}^{i-1} b_j(\mathscr{F}^{(n-i_1)}) \leq \frac{i(n-i_1)(n-i_1-i)}{2}\delta$ hold for any $1 \leq i \leq n-i_1-1$. The equality $i_1 \sum_{j=i_1}^{n-1} b_j - (n-i_1) \sum_{j=1}^{i_1-1} b_j = \frac{i_1 n (n-i_1)}{2}\delta$ and the fact that, by formulae (2.4), $b_j(\mathscr{F}^{(n-i_1)}) = b_{i_1+j} - b_{n-i}$, for any $i \leq j \leq n-i_1-1$, imply that the i-th of these inequalities is equivalent to the (i_1+i)-th of those giving the semistability of $\mathscr{F}$; thus $\mathscr{F}^{(n-i_1)}$ is semistable as wanted.

Now there are two distinct cases to consider. If $i_1 = i_k$, i.e. all the other inequalities are strict, then $\mathscr{F}^{(n-i_1)}$ is stable; therefore a Jordan Holder filtration of $\mathscr{F}$ is simply $0 \subset \mathscr{F}^{(n-i_1)} \subset \mathscr{F}$ and the graduate is $\mathrm{Gr}_{\mathrm{JH}}(\mathscr{F}) \cong \mathscr{F}^{(n-i_1)} \oplus \overline{\mathscr{F}}_{i_1}$.

Otherwise, $\mathscr{F}^{(n-i_1)}$ is strictly semistable and one can conclude by strong induction, getting the desired Jordan-Holder filtration and Jordan-Holder graduate of $\mathscr{F}^{(n-i_1)}$ (and, hence, those of $\mathscr{F}$, for which one has to pay attention to the shift of indices: $i_h(\mathscr{F}^{(n-i_1)}) = i_{h+1}(\mathscr{F}) - i_1(\mathscr{F})$). *q.e.d.*

CHAPTER 3

The moduli space: components of generalized line bundles

The aim of this chapter is to study the moduli space of semistable generalized line bundles on a primitive multiple curve. Throughout this chapter, any primitive multiple curve X will be such that $\deg(\mathcal{C}) < 0$, because, as observed in the previous one, only in this case there exist stable generalized line bundles on it.

After a brief common introduction, the chapter is divided into two sections: in the first we will treat the case of multiplicity 3, which is easier to handle and describe, while the second is devoted to higher multiplicity (where some results, especially about local geometry, are less complete). Most of the results about the irreducible components containing generalized line bundles could be stated and proved simultaneously for both the cases, but the proofs are clearer and more readable in multiplicity 3, so we preferred to treat this case separately.

It is well-known (cf. e.g. [**HL**]) that there exists a good moduli space parametrizing semistable pure sheaves of fixed Hilbert polynomial P on any projective scheme, and thus, in particular, on X. We will denote by $\mathrm{M}^\sharp(X,P)$ the moduli functor, by $\mathrm{M}(X,P)$ the projective scheme whose $\mathbb{K}$-valued points parametrize the S-equivalence classes of semistable pure sheaves of Hilbert polynomial P and by $\mathrm{M}_s(X,P)$ its subscheme whose $\mathbb{K}$-valued points parametrize stable sheaves with the same Hilbert polynomial. The general theory works for polarized projective schemes, but, as observed in Fact 1.12(iv), semistability on a primitive multiple curve is independent of the choice of a polarization. In the following, we will restrict our attention to Hilbert polynomials of the form $P_D(T) = D + n(1-g_1) + ndT$ (where d is the degree of a polarization on C), i.e. to the Hilbert polynomials of generalized line bundles on C_n of generalized degree D (cf. Remark 2.40(i)). As a consequence of Proposition 2.7, the only other pure sheaves having the same Hilbert polynomial are direct images of sheaves on C_{n-1} of generalized rank n and generalized degree D.

The following methods are inspired by the case of ribbons treated in [**CK**, §4.1] (where ordinary degree and rank are used instead of the generalized ones, but it is elementary to translate their results in terms of the latter). It seems very difficult to extend the main result of the cited section, i.e. [**CK**, Theorem 4.7], to higher multiplicity: in the case of ribbons the involved sheaves which are not generalized line bundles are direct images of vector bundles of rank 2 on a smooth projective curve, whose moduli spaces are well-known, while in the general case also the direct images of pure sheaves of generalized rank n on C_i, for any $1 \leq i \leq n-1$, are involved and their moduli spaces have never been studied in general (except, obviously, vector bundles of rank n on $C = C_1$). This is still an open problem, about which we will give some partial results and formulate some conjectures in Chapter 4.

After this brief introduction to the problem and its difficulties, it is time to begin the study. First of all, we generalize [**CK**, Lemma 4.2] from ribbons to the general case, noting that a sheaf of rank n on C_{n-1} cannot specialize to a generalized line bundle on X.

LEMMA 3.1. *Let T be a $\mathbb{K}$-scheme, let $\mathscr{F}$ be a sheaf representing a T-valued point of $\mathrm{M}^\sharp(X,P)$ and let $T_0 \subset T$ the locus of points $t \in T$ such that the restriction of $\mathscr{F}$ to the fibre $X \times_{\mathbb{K}} T \times_T \operatorname{Spec}(\mathbb{K}(t))$ is a generalized line bundle. Then $T_0 \subset T$ is open.*

PROOF. It is possible to prove this assertion in at least two different ways.

The first one is almost verbatim the proof of the cited Lemma by Chen-Kass: by general results (cf., e.g., [**HL**, Theorem 4.3.4]) the moduli space of semistable pure sheaves on C_{n-1} is projective; hence, it is universally closed and this implies that $T \setminus T_0$ is closed.

The second one is maybe easier: it is well-known that the number of generators of a module can only decrease under specialization and generalized line bundles are the only sheaves of generalized rank n whose generic stalk has only one generator. *q.e.d.*

Now it is time to distinguish the two cases of multiplicity 3 and higher one.

3.1. Multiplicity 3

In this section, as already anticipated, we restrict our attention to multiplicity 3, which is easier to be treated. Hence, throughout it, $X = C_3$. It is divided into two subsections: one about the global geometry of the moduli space and the other about the local one.

3.1.1. Global geometry: irreducible components. This subsection is about the irreducible components of the moduli space of semistable sheaves of generalized rank 3 on a primitive multiple curve of multiplicity 3 whose generic elements are generalized line bundles. In particular, we describe them and we show that they are connected.

The first step is to introduce some interesting loci of generalized line bundles in $\mathrm{M}_s(X, P_D)$. As we will see, within their closures there are the irreducible components of the moduli space containing generalized line bundles.

DEFINITION 3.2. Let X be a primitive multiple curve of multiplicity 3 such that $\delta = -\deg(\mathcal{C}) > 0$ and let $(b_{1,1}, \dots, b_{1,i})$ and $(b_{2,1}, \dots, b_{2,j})$ be two (one of which possibly empty) sequences of positive integers such that $2\sum_{h=1}^{i} b_{1,h} + \sum_{l=1}^{j} b_{2,l} < 3\delta$ and $\sum_{h=1}^{i} b_{1,h} + 2\sum_{l=1}^{j} b_{2,l} < 3\delta$. Set $\underline{b} := ((b_{1,1}, b_{1,1}), \dots, (b_{1,i}, b_{1,i}), (0, b_{2,1}), \dots, (0, b_{2,j}))$. Define $Z_{\underline{b}} \subset \mathrm{M}_s(X, P_D)$ as the subset of stable generalized line bundles of generalized degree D with local indices sequence $\underline{b}$.

The inequalities in the previous definition, which is inspired by [**CK**, Definition 4.3], are the conditions for the existence of stable generalized line bundles, cf. formulae 2.5. The following lemma is similar to [**CK**, Lemma 4.4].

LEMMA 3.3. *If* $3 \not| D - 2b_{1,1} - \cdots - 2b_{1,i} - b_{2,1} - \cdots - b_{2,j}$*, then* $Z_{\underline{b}}$ *is empty. Otherwise, it is a constructible, irreducible subset of dimension* $g_3 - (b_{1,1} - 1) - \cdots - (b_{1,i} - 1) - (b_{2,1} - 1) - \cdots - (b_{2,j} - 1)$*, where* g_3 *is the genus of* X*.*

PROOF. In order to simplify notations we denote $(b_{1,1}, \ldots, b_{1,i})$ by $\underline{\mathrm{b}}_1$ and $(b_{2,1}, \ldots, b_{2,j})$ by $\underline{\mathrm{b}}_2$. We set also $\beta_1 = b_1 = \sum_{h=1}^{i} b_{1,h}$ and $\beta_2 = \sum_{l=1}^{j} b_{2,l}$. Finally we pose $b_2 = b_1 + \beta_2$.

The first assertion follows from the first formula (2.1), which implies that $3|D - b_1(\mathscr{F}) - b_2(\mathscr{F})$ for any generalized line bundle $\mathscr{F}$.

So, assume $3|D - b_1 - b_2$. As in the proof of [**CK**, Lemma 4.4], the assertion is proved by parametrizing $Z_{\underline{b}}$ with an irreducible variety of the required dimension.

Consider $C^{(\beta_s)}$, i.e. the β_s-th symmetric product of the reduced subcurve C, and the diagonal $\Delta_{\underline{\mathrm{b}}_s}$ associated to the partition $\underline{\mathrm{b}}_s$ of β_s (for $s = 1, 2$), i.e. the image of the i-th (resp. j-th) direct product of C with itself in $C^{(\beta_1)}$ (resp. $C^{(\beta_2)}$) under the morphism sending $(P_1, \ldots, P_i)$ to $\sum_{h=1}^{i} b_{1,h} P_h$ (resp. $(Q_1, \ldots, Q_j)$ to $\sum_{l=1}^{j} b_{2,l} Q_l$). Let $U \subset \Delta_{\underline{\mathrm{b}}_1} \times \Delta_{\underline{\mathrm{b}}_2}$ be the locus such that the points P_h's and Q_l's are all distinct. It is clear that U is locally closed in $C^{(\beta_1)} \times C^{(\beta_2)}$ and irreducible of dimension $i + j$.

Let $\Sigma \in U$ and associate to it the ideal sheaf $\mathscr{I}(\Sigma)$ defined as $(x^{b_{1,h}}, y)$ at the points P_h (for $1 \le h \le i$) and as $(x^{b_{2,l}}, y^2)$ at the points Q_l (for $1 \le l \le j$): $\mathscr{I}(\Sigma)$ is a stable generalized line bundle of generalized degree $-2b_2 + b_1 - 3\delta$ and local indices sequence $\underline{b}$. So it is possible to define a map $U \times \mathrm{Pic}^{D+2b_2-b_1}(X) \to \mathrm{M}_s(X, P_D)$ by the rule $\Sigma \times \mathscr{E} \mapsto \mathscr{I}(\Sigma) \otimes \mathscr{E}$, where $\mathrm{Pic}^{D+2b_2-b_1}(X)$ is the Picard variety of line bundles on X of generalized degree $D + 2b_2 - b_1$ (it is the right generalized degree that has to be used by Corollary 1.15). By the definition of U, for any set of $i + j$ points $P_1, \ldots, P_i, Q_1, \ldots, Q_j$ there is a unique closed subscheme $\Sigma \subset C$, corresponding to a point of U, such that $b_{1,P_h}(\mathscr{I}(\Sigma)) = b_{2,P_h}(\mathscr{I}(\Sigma)) = b_{1,h}$, for $1 \le h \le i$, and $b_{1,Q_l}(\mathscr{I}(\Sigma)) = 0$ and $b_{2,Q_l}(\mathscr{I}(\Sigma)) = b_{2,l}$, for $1 \le l \le j$. Hence, by Corollary 2.35, the image of the just defined map is $Z_{\underline{b}}$ and, moreover, the fibre over a point is an irreducible variety of dimension $\mathrm{h}^1(X, \mathscr{O}_X) - \mathrm{h}^1(X', \mathscr{O}_{X'}) = g_3 - g(X') = b_2$ (X' is the blow up considered in the cited Corollary), where the second equality is trivial and the first one holds by the fact that both X and X' have no non-trivial global sections (for X this is easily implied by the fact $\deg(\mathcal{C}) < 0$, while for X' this is Lemma 3.4). Therefore, $Z_{\underline{b}}$ is irreducible and constructible of dimension $i+j+g_3-b_2$, i.e. $g_3-(b_{1,1}-1)-\cdots-(b_{1,i}-1)-(b_{2,1}-1)-\cdots-(b_{2,j}-1)$. *q.e.d.*

LEMMA 3.4. *Let* $\mathscr{F}$ *be a stable generalized line bundle on* X *with local indices sequence* $b_{.,.}$*, let* $\mathscr{I}$ *be the ideal sheaf locally isomorphic to* $\mathscr{F}$ *in the points* P *where* $2b_{1,p} \le b_{2,P}$ *and to* $\mathscr{F}^\vee$ *in the other points and let*

$q : X' \to X$ be the blow up of X with respect to $\mathscr{I}$. Then X' has only trivial global sections, equivalently $g(X') = \mathrm{h}^1(X', \mathscr{O}_{X'})$.

PROOF. Using local notation, by the observations done in the proof of Lemma 2.33, it holds that $\mathscr{O}_{X',P} = \mathscr{O}_{X,P}[x^{\max\{b_{1,p},\, b_{2,P}-b_{1,P}\}}y/(x^{b_{2,p}} + \alpha_P y),\, y^2/(x^{b_{2,P}} + \alpha_P y)]$ in each point P. Let $\mathscr{K} \subset \mathscr{O}_{X'}$ be the ideal sheaf defined locally as $\mathscr{K}_P = (y^2/x^{b_{2,P}})$. We have that $q_*(\mathscr{O}_{X'}/\mathscr{K}) = \mathscr{O}_{C_2'}$, where C_2' is the blow up of C_2 with respect to $\overline{\mathscr{I}}_2$, and that there is an exact sequence $0 \to \mathscr{N}^2 \to q_*\mathscr{K} \to \mathscr{O}_\Sigma \to 0$, where Σ is the effective divisor of C supported at the points P where $\mathscr{I}$ is not free and having length $b_{2,P}$ at each of them.

The assertion follows from proving that $q_*(\mathscr{K})$ does not have global sections and $\mathscr{O}_{C_2}$ does not have non-trivial global sections. Let us begin with the former. By the fact its square is 0, it is a line bundle on C and it is sufficient to show that its degree is negative. By the above exact sequence, $\deg(q_*(\mathscr{K})) = \deg(\mathscr{N}^2) + b_2 = 2\deg(\mathcal{C}) + b_2$; hence, it is enough to prove that $b_2 < 2\delta$. Recall the stability inequalities (2.5) of $\mathscr{F}$:

$$\begin{cases} b_2 < 2\delta + \delta - b_1 \\ 2b_2 < 2\delta + \delta + b_1. \end{cases}$$

There is an elementary dichotomy: or $b_1 \geq \delta$ either $b_1 < \delta$. If the former holds, the first inequality implies the desired one; while if the latter is verified, $b_2 < 2\delta$ follows from the second inequality (indeed in this case the right hand term is less than 4δ).

It remains to prove that $\mathscr{O}_{C_2'}$ does not have non-trivial global sections. It is a ribbon with reduced subcurve C; hence, it suffices to show that its nilpotent sheaf, seen as a line bundle on C, has negative degree or, equivalently, that $g(C_2') > 2g_1$. It holds that $g(C_2') = g_2 - \sum \min\{b_{1,p},\, b_{2,P} - b_{1,P}\}$, where P varies within the closed points where $\mathscr{F}$ is not locally free, and that $g_1 = g_2 - \delta$. Thus, it is sufficient to prove that $\sum \min\{b_{1,p},\, b_{2,P} - b_{1,P}\} < \delta$. But $\sum \min\{b_{1,p},\, b_{2,P} - b_{1,P}\} \leq b_2/2$ by definition and $b_2 < 2\delta$, as it has been shown above. *q.e.d.*

The next step is to show that the Zariski closures of some of the loci studied in Lemma 3.3 are irreducible components of the moduli space of semistable generalized line bundles. In order to achieve this result we will use the next lemma, which is inspired by [**CK**, Lemma 4.9], about deformations of generalized line bundles.

LEMMA 3.5. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_3$ of local indices sequence $b_{.,.}$ and let P be a closed point of C such that $b_{2,P} \geq 2$.*

(i) *If $0 < b_{1,P} < b_{2,P}$, then $\mathscr{F}$ is specialization of generalized line bundles $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $b_{1,P}(\mathscr{F}') = 0$ and $b_{2,P}(\mathscr{F}') = b_{2,P} - b_{1,P}$, and in another closed point Q, where $b_{1,Q} = b_{2,Q} = 0$ and $b_{1,Q}(\mathscr{F}') = b_{2,Q}(\mathscr{F}') = b_{1,P}$.*

(ii) *If $b_{1,P} = b_{2,P}$, then $\mathscr{F}$ is specialization of generalized line bundles $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $b_{1,P}(\mathscr{F}') = b_{2,P}(\mathscr{F}') = b_{2,P} - 1$, and in another closed point Q, where $b_{1,Q} = b_{2,Q} = 0$ and $b_{1,Q}(\mathscr{F}') = b_{2,Q}(\mathscr{F}') = 1$.*

(iii) *If* $b_{1,P} = 0$*, then* $\mathscr{F}$ *is specialization of generalized line bundles* $\mathscr{F}'$ *with the same local indices sequence of* $\mathscr{F}$ *except in* P*, where* $b_{1,P}(\mathscr{F}') = 0$ *and* $b_{2,P}(\mathscr{F}') = b_{2,P} - 1$*, and in another closed point* Q*, where* $b_{1,Q} = b_{2,Q} = 0$*, while* $b_{1,Q}(\mathscr{F}') = 0$ *and* $b_{2,Q}(\mathscr{F}') = 1$*.*

PROOF. For all the three points we will exhibit explicit deformations over $\mathbb{K}[t]$. In order to simplify notation, throughout the proof $b_i = b_i(P)$, for $i = 1, 2$.

First of all, recall that by Corollary 2.32, $\mathscr{F}$ is isomorphic to $\mathscr{I}_{Z/X} \otimes \mathscr{E}$, where $Z \subset C_2$ is a closed subscheme of finite support and $\mathscr{E}$ is a line bundle on X. Thus, it is sufficient to find an appropriate deformation of $\mathscr{I} = \mathscr{I}_{Z/X}$, say $\mathscr{I}'$, because then the desired deformation of $\mathscr{F}$ would be $\mathscr{I}' \otimes \mathscr{E}$, where, by a slight abuse of notation, $\mathscr{E}$ here denotes the constant family with fibre $\mathscr{E}$. Deforming $\mathscr{I}$ is equivalent to deforming Z. In order to do that we will use Corollary 2.31, which is an affine application of the Local Structure Theorem. So let $U = \operatorname{Spec}(A)$ be an affine neighbourhood of P, in which the thesis of the cited corollary holds (in particular, P is the only point in U such that $\mathscr{F}_P$ is not free). Now it is necessary to distinguish the three cases.

Let us begin with (i). In this case, using the notation of the cited corollary, $\mathscr{I}(U) \cong (x^{b_2} + \alpha y, x^{b_1} y, y^2)$ and the desired deformation is given by the extension of the ideal $(x^{b_1}(x-t)^{b_2-b_1} + \alpha y, x^{b_1} y, y^2)$ to a proper flat family over $\operatorname{Spec}(\mathbb{K}[t])$ (it is possible to have such an extension by, e.g., the properness of the Hilbert scheme). This generic fibre is the expression over U of a generalized line bundle having the desired local indices sequence, while the special fibre is $\mathscr{I}(U)$.

The proof of (ii) is analogous, $\mathscr{I}(U) \cong (x^{b_2}, y)$ and, similarly, the desired deformation is the extension of the ideal $(x^{b_2-1}(x-t), y)$ to a proper flat family over $\operatorname{Spec}(\mathbb{K}[t])$.

Finally, also that of (iii) is quite similar: $\mathscr{I}(U) \cong (x^{b_2}, y^2)$ and the deformation is the flat family having generic fibre $(x^{b_2-1}(x-t), y^2)$.

q.e.d.

The following theorem describes the irreducible components of the moduli space containing stable generalized line bundles. It is similar to the case of ribbons treated in [**CK**, Theorem 4.6].

THEOREM 3.6. *Let* X *be a primitive multiple curve of multiplicity* 3 *and genus* g_3 *and let* $b_1 \leq b_2$ *be two non-negative integers satisfying* $3 | D - b_1 - b_2$*,* $0 \leq b_2 + b_1 < 3\delta$ *and* $0 \leq 2b_2 - b_1 < 3\delta$ *(where, as in other circumstances,* $\delta = \deg(\mathcal{C})$*).*

Let $\bar{Z}_{b_1,b_2} \subset \mathrm{M}(X, P_D)$ *be the Zariski closure of the locus of stable generalized line bundles of generalized rank* D *and indices-vector* (b_1, b_2)*.*

First of all, $\bar{Z}_{b_1,b_2}$ *is equal to the Zariski closure of* $Z_{\underline{b}}$*, where* $\underline{b}$ *is the sequence* $(\underbrace{(1,1),\dots,(1,1)}_{b_1 \text{ times}}, \underbrace{(0,1),\dots,(0,1)}_{b_2 - b_1 \text{ times}})$*, for* $(b_1, b_2) \neq (0, 0)$*.*

Then $\bar{Z}_{b_1,b_2}$ *is a* g_3*-dimensional irreducible component of* $\mathrm{M}(X, P_D)$*. Moreover, any irreducible component containing a stable generalized line bundle is equal to* $\bar{Z}_{b_1,b_2}$*, for a unique pair* $(b_1,\, b_2)$ *satisfying the above conditions.*

Proof. The theorem is a straightforward application of the above lemmata. First of all, it is clear from Theorem 2.41 that $\cup\bar{Z}_{b_1,b_2}$ contains the locus of stable generalized line bundles. The first assertion is implied by a repeated application of Lemma 3.5. Moreover, by this identification and by Lemma 3.3, each $\bar{Z}_{b_1,b_2}$ is irreducible and of dimension g_3.

Now let $\bar{Z}$ be an irreducible component containing a stable generalized line bundle. Observe that its subset consisting of stable generalized line bundles is open (by Lemma 3.1) and non-empty; hence, it is dense. This implies that $\bar{Z}$ is contained in the union $\cup\bar{Z}_{b_1,b_2}$, so $\bar{Z} = \bar{Z}_{b_1,b_2}$, for some (b_1, b_2), by the fact these loci are irreducible.

This proves that some of the $\bar{Z}_{b_1,b_2}$ are irreducible components. But each of them is a component: indeed, fix (b_1, b_2) and consider $\bar{Z}_{b_1,b_2}$. It is certainly contained in an irreducible component containing stable generalized line bundles, say $\bar{Z}_{b'_1,b'_2}$. But they are both irreducible and of the same dimension, thus it must hold that they are equal, hence, $\bar{Z}_{b_1,b_2}$ is an irreducible component.

Furthermore, if $(b_1, b_2) \neq (b'_1, b'_2)$ the generic elements of $\bar{Z}_{b_1,b_2}$ and $\bar{Z}_{b'_1,b'_2}$ are different, so they are distinct irreducible components. *q.e.d.*

The next step is to show that the locus of stable generalized line bundles of generalized degree D over X is connected. In order to do that, we need other deformations which are introduced in the following lemma (which is analogous to [**CK**, Lemma 4.5]).

Lemma 3.7. *Let X be a primitive multiple curve of multiplicity 3; let $\mathscr{F}$ be a generalized line bundle of local indices sequence $b_{.,.}$ on X and let P be a closed point.*

(i) *Assume that $b_{1,P} = 0$ and $b_{2,P} \geq 3$. Then $\mathscr{F}$ is specialization of another generalized line bundle $\mathscr{F}'$ having the same local indices sequence except in P, where $b_{1,P}(\mathscr{F}') = 0$ and $b_{2,P}(\mathscr{F}') = b_{2,P} - 3$.*

(ii) *If $b_{1,P} = b_{2,P} \geq 3$, then $\mathscr{F}$ is the specialization of another generalized line bundle $\mathscr{F}'$ having the same local indices sequence except in P, where $b_{1,P}(\mathscr{F}') = b_{2,P}(\mathscr{F}') = b_{2,P} - 3$.*

(iii) *Assume that $0 \neq b_{1,P} < b_{2,P}$ and $\alpha_{3,1,P} = 0$, where, here and in the following statements, $\alpha_{3,1,P}$ is the $\alpha_{3,1}$ attached to $\mathscr{F}_P$ by Theorem 2.27. Then $\mathscr{F}$ is the specialization of another generalized line bundle $\mathscr{F}'$ having the same local indices sequence except in P, where $b_{1,P}(\mathscr{F}') = b_{1,P} - 1$ and $b_{2,P}(\mathscr{F}') = b_{2,P} - 2$.*

(iv) *If $2 \leq b_{1,P} \leq b_{2,P}$ and $\alpha_{3,1,P} = 0$, then $\mathscr{F}$ is the specialization of a generalized line bundle $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $b_{1,P}(\mathscr{F}') = b_{1,P} - 2$ and $b_{2,P}(\mathscr{F}') = b_{2,P} - 2$, and in another closed point Q, where $b_{1,Q} = b_{2,Q} = 0$ and $b_{1,Q}(\mathscr{F}') = 0$ while $b_{2,Q}(\mathscr{F}') = 1$.*

(v) *Assume $b_{2,P} \geq b_{2,P} - b_{1,P} \geq 2$ and $\alpha_{3,1,P} = 0$. Then $\mathscr{F}$ is the specialization of a generalized line bundle $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $b_{1,P}(\mathscr{F}') = b_{1,P}$ and $b_{2,P}(\mathscr{F}') = b_{2,P} - 2$, and in another closed point Q, where $b_{1,Q} = b_{2,Q} = 0$ and $b_{1,Q}(\mathscr{F}') = b_{2,Q}(\mathscr{F}') = 1$.*

PROOF. As in the proof of Lemma 3.5, it is possible, without loss of generality, to work within an affine neighbourhood U of P in which P is the only point where $\mathscr{F}_P$ is not free and we can also assume that $\mathscr{F}$ is the ideal of its associated subscheme Z. Again as in the cited proof, it is sufficient to give an appropriate deformation of $\mathscr{F}(U) = I$ over $\mathbb{K}[t]$ exhibiting its generic fibre. Throughout the proof $b_i = b_{i,P}$ for $i = 1, 2$.

For (i), using affine notation (cf. Corollary 2.31), $I = (x^{b_2}, y)$ and the generic fibre of the deformation is the ideal $I'_t = (x^{b_2-3}, y) \cap ((x-t)^3, y - t^{b_2-2}(x-t)^2)$. Indeed, from the fact $x^{b_2-3}(x-t)^3$ and $y - x^{b_2-3}(t^3 - 2xt^2 + x^2t) = y - t^{b_2-2}(x-t)^2 - (x-t)^3 t \sum_{i=0}^{b_2-4} x^i t^{b_2-4-i}$ (if $b_2 \geq 4$; if $b_2 = 3$, instead of the latter consider $y - t(x-t)^2$) belong to I'_t, for any $t \neq 0$, it follows that I is contained in the special fibre. The fact they coincide is due to degree considerations: A/I (where $A = \mathscr{O}_X(U)$) has length b_2 while $A/(x^{b_2-3}, y)$ has length $b_2 - 3$ and $A/((x-t)^3, y - t^{b_2-2}(x-t)^2)$ has length 3, for any non-zero value of the parameter t. Moreover, the ideal $((x-t)^3, y - t^{b_2-2}(x-t)^2)$ defines a Cartier divisor of X, and so the part contributing to the local indices sequence of I'_t is only (x^{b_2-3}, y).

The second assertion is dual (by Corollary 2.24) to the first one so it would not be necessary to exhibit an explicit deformation. Anyway, in this case $I = (x^{b_2}, y^2)$ and the generic fibre of the desired deformation is the ideal $(x^{b_2-3}, y^2) \cap ((x-t)^3, y^2 - t^{b_2-2}(x-t)^2)$ and the proof is almost identical to the previous one.

Also the proof of (iii) is similar: this time $I = (x^{b_2}, x^{b_2-b_1}y, y^2)$ and the generic fibre of the deformation is the ideal $I'_t = (x^{b_2-2}, x^{b_2-b_1-1}y, y^2) \cap ((x-t)^2, (x-t)y, y^2 - t^{b_2-1}(x-t))$. The special fibre contains I because $x^{b_2-2}(x-t)^2$, $x^{b_2-b_1-1}y(x-t)$ and $y^2 - tx^{b_2-1} + t^2x^{b_2-2} = y^2 - t^{b_2-1}(x-t) - (x-t)^2 t \sum_{i=0}^{b_2-3} x^i t^{b_2-3-i}$ (if $b_2 > 2$, in the case $b_2 = 2$ the last element has to be substituted by $y^2 - t(x-t)$) belong to I'_t for any $t \neq 0$. The fact I is the special fibre is due to degree considerations similar to those of the first assertion: A/I has length $2b_2 - b_1$, while $A/(x^{b_2-2}, x^{b_2-b_1-1}y, y^2)$ has length $2b_2 - b_1 - 3$ and $A/((x-t)^2, (x-t)y, y^2 - t^{b_2-1}(x-t))$ has length 3, for any fixed $t \neq 0$. Also in this case the ideal $((x-t)^2, (x-t)y, y^2 - t^{b_2-1}(x-t))$ defines a Cartier divisor of X, for $t \neq 0$.

It is time to prove (iv). Also in this case $I = (x^{b_2}, x^{b_2-b_1}y, y^2)$ but the generic fibre of the deformation is $I'_t = H'_t \cap J'_t = (x^{b_2-2}, x^{b_2-b_1}y, y^2) \cap (y - t^{b_2-1}(x-t), t^{2(b_2-1)}y^2 + (x-t)^2)$.

I is really contained in the special fibre because $x^{b_2-2}(t^{2(b_2-1)}y^2 + (x-t)^2)$, $x^{b_2-b_1}y + x^{b_2-2}x^{b_2-b_1}t(-x+t) - y^2t^{2(b_2-2)} \sum_{i=0}^{b_2-3} t^i x^{b_2-3-i} = \{y - t^{b_2-1}(x-t) - [t^{2(b_2-1)}y^2 + (x-t)^2] t \sum_{i=0}^{b_2-3} t^i x^{b_2-3-i}\} x^{b_2-b_1}$ and $y^2 + x^{b_2-2}ty(t-x) = \{y - t^{b_2-1}(x-t) - [t^{2(b_2-1)}y^2 + (x-t)^2] t \sum_{i=0}^{b_2-3} t^i x^{b_2-3-i}\} y$ belong to I'_t for any $t \neq 0$ (this makes sense only if $b_2 \geq 3$, but the remaining case $b_2 = b_1 = 2$ is easy because then $H'_t = A$). It remains to prove that I coincides with the special fibre and that I'_t defines a generalized line bundle with the desired local index sequence. The first fact is due, as usual, to easy degree considerations. Also the second one is almost trivial.

It remains (v). It is the dual of (iv) so it does not need an explicit proof. For completeness we point out that in this case I is again $(x^{b_2}, x^{b_2-b_1}y, y^2)$, while the generic fibre of the deformation is $I'_t = (x^{b_2-2}, x^{b_2-b_1-2}y, y^2) \cap (y^2 - t^{b_2-1}(x-t), tx^{b_2-b_1-2)}y + (x-t)^2)$.

q.e.d.

THEOREM 3.8. *Let X be a primitive multiple curve of multiplicity 3 and let D be an integer. The locus of stable generalized line bundles in* $\mathrm{M}(X, P_D)$ *is connected.*

PROOF. First of all, notice that the case $\delta = -\deg(\mathcal{C}) = 1$ is trivial, because according to Theorem 3.6, there is only one irreducible component of $\mathrm{M}(X, P_D)$ containing stable generalized line bundles (that is $\bar{Z}_{0,0}$ if $D \equiv 0 \pmod 3$, $\bar{Z}_{0,1}$ if $D \equiv 1 \pmod 3$ and $\bar{Z}_{1,1}$ if $D \equiv 2 \pmod 3$). So we can assume $\delta \geq 2$.

Let $0 \leq b_1 \leq b_2$ be two non-negative integers such that $b_1 + b_2 \equiv D \pmod 3$. Then there exists a generalized line bundle $\mathscr{F}$ on X being locally free except in at most two closed point P and Q, where $b_{i,P}(\mathscr{F}) = b_1$ for $i = 1, 2$ (so it is locally free also in P if $b_1 = 0$) and $b_{1,Q}(\mathscr{F}) = 0$ and $b_{2,Q}(\mathscr{F}) = b_2 - b_1$ (so, also its stalk in Q is free if $b_1 = b_2$). By an iterated application of Lemma 3.7(i) and (ii) and, if necessary, an application of Lemma 3.5(ii) or (iii), it results that $\mathscr{F}$ is the specialization of the generic element of $\bar{Z}_{\overline{b_1},\overline{b_1}+\overline{b_2-b_1}}$, where $0 \leq \overline{b_1} \leq 2$ is congruent to b_1 modulo 3 and $0 \leq \overline{b_2 - b_1} \leq 2$ is congruent to $b_2 - b_1$ modulo 3. Thus, if $\mathscr{F}$ is stable, it belongs to both $\bar{Z}_{\overline{b_1},\overline{b_1}\overline{b_2-b_1}}$ and $\bar{Z}_{b_1,b_2}$.

Hence, recalling that, by formula (2.1), $b_1(\mathscr{F}) + b_2(\mathscr{F}) \equiv D \pmod 3$ for any generalized line bundle $\mathscr{F}$ of generalized degree D, the locus of stable generalized line bundles of generalized degree D has at most three connected components: the one containing $\bar{Z}_{0,0}$, that containing $\bar{Z}_{1,2}$ and that containing $\bar{Z}_{2,4}$ (which there isn't if $\delta = 2$) if $D \equiv 0 \pmod 3$; that containing $\bar{Z}_{0,1}$, that containing $\bar{Z}_{1,3}$ and that containing $\bar{Z}_{2,2}$ if $D \equiv 1 \pmod 3$ and finally the connected component of $\bar{Z}_{1,1}$, that of $\bar{Z}_{0,2}$ and that of $\bar{Z}_{2,3}$ if $D \equiv 2 \pmod 3$.

To conclude it is necessary to use Lemma 3.7(iii). If $D \equiv 0 \pmod 3$, there exists a stable generalized line bundle $\mathscr{G}$ whose stalks are free in all closed points except one, say P, where $(b_{1,P}(\mathscr{G}), b_{2,P}(\mathscr{G})) = (1, 2)$ and $\alpha_{3,1,P} = 0$. Applying Lemma 3.7(iii), it follows that $\mathscr{G}$ connects $\bar{Z}_{0,0}$ and $\bar{Z}_{1,2}$. If $\delta > 2$, there exists also a stable generalized line bundle $\mathscr{G}'$ with the same properties of $\mathscr{G}$ except that $(b_{1,P}(\mathscr{G}'), b_{2,P}(\mathscr{G}')) = (2, 4)$, which, again by Lemma 3.7(iii), is a specialization of $\mathscr{G}$, but it belongs also to $\bar{Z}_{2,4}$ (by definition) and thus there is only one connected component, if $\delta \geq 1$.

If $D \equiv 1 \pmod 3$, there exists a stable generalized line bundle $\mathscr{G}$ whose stalks are free in all closed points except one, say P, where it holds that $(b_{1,P}(\mathscr{G}), b_{2,P}(\mathscr{G})) = (1, 3)$ and $\alpha_{3,1,P} = 0$. By Lemma 3.7(iii) $\mathscr{G}$ belongs to $\bar{Z}_{0,1}$; hence, $\mathscr{G}$ connects $\bar{Z}_{0,1}$ with $\bar{Z}_{1,3}$, to which it belongs by definition. Moreover, by Lemma 3.7(iv), $\mathscr{G}$ belongs also to $\bar{Z}_{2,2}$ and so the locus of stable generalized line bundles with $D \equiv 1 \pmod 3$ is connected.

If $D \equiv 2 \pmod 3$, the situation is similar: there exists a stable generalized line bundle $\mathscr{G}$ whose stalks are free in all closed points except one, say P, where $(b_{1,P}(\mathscr{G}), b_{2,P}(\mathscr{G})) = (2, 3)$ and $\alpha_{3,1,P} = 0$. By definition, $\mathscr{G}$

belongs to $\bar{Z}_{2,3}$. By, respectively, Lemmata 3.7(iii) and 3.7(v), $\mathscr{G}$ belongs also to $\bar{Z}_{1,1}$ and $\bar{Z}_{0,2}$. *q.e.d.*

In order to have an almost complete description of $\mathrm{M}(X, P_D)$, similar to [**CK**, Theorem 4.7], we need to study also the irreducible components which do not contain stable generalized line bundles. For their study we refer to Chapter 4 and, in particular, to Conjecture 4.40 and the discussion about it.

3.1.2. Local geometry: Zariski tangent space. This subsection about the local geometry of $\mathrm{M}(X, P_D)$ is mainly devoted to the computation of the dimension of the tangent space to points corresponding to generalized line bundles. The case of points corresponding to rank 3 vector bundles on C will be treated in Section 4.1, while we are not able to handle the case of generalized rank 3 sheaves on C_2. The results are quite similar to the first part of [**CK**, Proposition 4.11] and also the lemma used to get them is similar, both in the enunciation and in the proof, to [**CK**, Lemma 4.12]. This is a good point to observe that there is a little mistake in the second assertion of [**CK**, Lemma 4.12]: the right hypothesis to simplify the formula about the dimension of the Ext^1 of a generalized line bundle on a ribbon is that the associated blow up does not have non-trivial global sections; hence, its genus (and not that of the ribbon) has to be greater than or equal to two times the genus of the reduced curve, i.e. $2g_1 + b_1 \leq g_2$ (using their notation $2\bar{g} + b \leq g$, and not $2\bar{g} \leq g$, as asserted in the cited Lemma); in any case this error does not affect [**CK**, Proposition 4.11], because it is about stable generalized line bundles, for which it holds also the right hypothesis.

Proposition 3.9. *Let X be a primitive multiple curve of multiplicity* 3 *and let x be a point of $\mathrm{M}_s(X, P_D)$. If x corresponds to a stable generalized line bundle $\mathscr{F}$ of indices sequence $b_{\cdot,\cdot}$, then*

$$\dim T_x \mathrm{M}(X, P_D) = g_3 + b_2 + \sum_{j=1}^{r} \min\{b_{1,P_j},\, b_{2,P_j} - b_{1,P_j}\}, \tag{3.1}$$

where $P_1, \ldots, P_r$ are the points of C where $\mathscr{F}$ is not locally free.

Corollary 3.10. *The tangent space to a generic point of the irreducible component $\bar{Z}_{b_1,b_2}$ has dimension $g_3 + b_2$. In particular, only the component of stable line bundles, i.e. $\bar{Z}_{0,0}$, is generically reduced.*

Proof. The first assertion is a consequence of formula (3.1) and Theorem 3.6, which describes the generic elements of $\bar{Z}_{b_1,b_2}$. The second assertion is implied by the first one, by the fact $\bar{Z}_{b_1,b_2}$ has dimension g_3 (again by Theorem 3.6) and by Corollary 2.15. *q.e.d.*

The Proposition follows from the well-known fact that the Zariski tangent space to a point corresponding to a stable sheaf $\mathscr{G}$ in the moduli space is canonically isomorphic to $\mathrm{Ext}^1(\mathscr{G}, \mathscr{G})$ (see, e.g., [**HL**, Corollary 4.5.2]) and from the next lemma which calculates the dimension of this Ext^1 for a generalized line bundle and is analogue to [**CK**, Lemma 4.12].

LEMMA 3.11. *If $\mathscr{F}$ is a generalized line bundle on $X = C_3$ of local indices sequence $b_{\cdot,\cdot}$, then*

$$\dim(\mathrm{Ext}^1(\mathscr{F},\mathscr{F})) = g_3 + b_2 + \tilde{b}_1 + \mathrm{h}^0(X', \mathscr{O}_{X'}) - 1, \tag{3.2}$$

where $\tilde{b}_1 = \sum_{j=1}^{r} \min\{b_{1,P_j}, b_{2,P_j} - b_{1,P_j}\}$, being $P_1, \dots, P_r$ the points of C where $\mathscr{F}$ is not locally free and X' is the blow up associated to $\mathscr{F}$ as in Corollary 2.35.

If, moreover, $\mathscr{F}$ is stable, then this formula simplifies to

$$\dim(\mathrm{Ext}^1(\mathscr{F},\mathscr{F})) = g_3 + b_2 + \tilde{b}_1. \tag{3.3}$$

PROOF. The fundamental ideas of the proof are the same of that of the cited place in [**CK**].

The Ext-spectral sequence $H^p(X, \underline{\mathrm{Ext}}^q(\mathscr{F},\mathscr{F})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\mathscr{F})$ implies the existence of the following short exact sequence

$$0 \to H^1(X, \underline{\mathrm{End}}(\mathscr{F})) \to \mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \to H^0(X, \underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F})) \to 0.$$

Hence, it is sufficient to compute the dimensions of the two external terms in order to get the result. By Remark 2.36, it holds that $\underline{\mathrm{End}}(\mathscr{F}) \simeq q_*(\mathscr{O}_{X'})$, where $q : X' \to X$ is the blow up there studied. It follows that $H^1(X, \underline{\mathrm{End}}(\mathscr{F})) = H^1(X', \mathscr{O}_{X'})$ and the latter has dimension $g(X') - \mathrm{h}^0(X', \mathscr{O}_{X'}) + 1 = g_3 - b_2 - \tilde{b}_1 - \mathrm{h}^0(X', \mathscr{O}_{X'}) + 1$ (this formula is implied by the definition of the blow up X'); by Lemma 3.4, if $\mathscr{F}$ is stable, then $\mathrm{h}^0(X', \mathscr{O}_{X'}) = 1$, justifying the difference between formulae (3.2) and (3.3).

It remains to calculate $\mathrm{h}^0(X, \underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F}))$. As in the case of ribbons, it is clear that $\underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F})$ is supported on $P_1, \dots, P_r$ and that it can be decomposed as $\bigoplus_{j=1}^{r} \mathrm{Ext}^1(\mathscr{F}_{P_j}, \mathscr{F}_{P_j})$.

In the following lines we will show that $\dim(\mathrm{Ext}^1(\mathscr{F}_{P_j}, \mathscr{F}_{P_j})) = 2b_{2,P_j} + 2\min\{b_{1,P_j}, b_{2,P_j} - b_{1,P_j}\}$; therefore, formulae (3.2) and (3.3) hold, as desired.

In order to do the explicit computations, it is useful to distinguish three different cases, according to the indices of $\mathscr{F}$ in the point P_j:

(i) $0 = b_{1,P_j} < b_{2,P_j}$;
(ii) $0 < b_{1,P_j} = b_{2,P_j}$;
(iii) $0 < b_{1,P_j} < b_{2,P_j}$.

In all the three cases we will us local notation with $A = \mathscr{O}_{X,P_j}$ and $\mathscr{F}_{P_j}$ will be denoted by I, while $b_{i,P_j} = b_i$, for $i = 1$ or 2.

Let us begin with case (i). By Local Structure Theorem (i.e. Theorem 2.27), I is isomorphic to the ideal (x^{b_2}, y^2).

It has the following periodic free resolution:

$$\cdots \longrightarrow A^2 \xrightarrow{M_2} A^2 \xrightarrow{M_1} A^2 \xrightarrow{f} I \longrightarrow 0,$$

where

$$M_1 = \begin{pmatrix} y^2 & x^{b_2} \\ 0 & -y \end{pmatrix}, \; M_2 = \begin{pmatrix} y & x^{b_2} \\ 0 & -y^2 \end{pmatrix} \text{ and } \begin{cases} f((1,0)) = y^2 \\ f((0,1)) = x^{b_2}. \end{cases}$$

From the resolution one gets the complex

$$\cdots \longleftarrow \mathrm{Hom}(A^2, I) \xleftarrow{a_2} \mathrm{Hom}(A^2, I) \xleftarrow{a_1} \mathrm{Hom}(A^2, I),$$

where a_i is the homomorphism induced by multiplication by M_i, for $i = 1, 2$. By definition, $\mathrm{Ext}^1(I, I) = \ker(a_2)/\operatorname{im}(a_1)$. It holds that

$$\varphi \in \operatorname{im}(a_1) \iff \begin{cases} \varphi((1,0)) = \beta_1 y x^{b_2} \\ \varphi((0,1)) = \beta_1 x^{2b_2} + \beta_2 x^{b_2} y^2, \end{cases}$$

with β_1, $\beta_2 \in A$; while

$$\psi \in \ker(a_2) \iff \begin{cases} \psi((1,0)) = \gamma_1 x^{b_2} y + \gamma_2 y^2 \\ \psi((0,1)) = \gamma_1 x^{2b_2} + \gamma_2 x^{b_2} y + \gamma_3 y^2, \end{cases}$$

with $\gamma_i \in A$, for $1 \le i \le 3$. Therefore, $\mathrm{Ext}^1(I, I)$ has length $2b_2$, as asserted.

Now consider case (ii), which is quite similar to the previous one. This time $I \cong (x^{b_2}, y)$, again by Local Structure Theorem. Its free resolution can be written similarly to that of the previous case:

$$\cdots \longrightarrow A^2 \xrightarrow{M_2'} A^2 \xrightarrow{M_1'} A^2 \xrightarrow{f'} I \longrightarrow 0,$$

with $M_1' = M_2$, $M_2' = M_1$, $f'((1,0)) = y$ and $f'((0,1)) = x^{b_2}$. From the resolution one gets the complex

$$\cdots \longleftarrow \mathrm{Hom}(A^2, I) \xleftarrow{a_2'} \mathrm{Hom}(A^2, I) \xleftarrow{a_1'} \mathrm{Hom}(A^2, I),$$

where a_i' is the homomorphism induced by multiplication by M_i', for $i = 1, 2$. By definition, $\mathrm{Ext}^1(I, I) = \ker(a_2')/\operatorname{im}(a_1')$. It holds that

$$\varphi \in \operatorname{im}(a_1') \iff \begin{cases} \varphi((1,0)) = \beta_1 y^2 x^{b_2} \\ \varphi((0,1)) = \beta_1 x^{2b_2} + \beta_2 x^{b_2} y + \beta_3 y^2, \end{cases}$$

with $\beta_i \in A$, for $1 \le i \le 3$; while

$$\psi \in \ker(a_2') \iff \begin{cases} \psi((1,0)) = \gamma_1 y^2 \\ \psi((0,1)) = \gamma_1 x^{b_2} + \gamma_2 y, \end{cases}$$

with γ_1, $\gamma_2 \in A$. Thus, $\mathrm{Ext}^1(I, I)$ has length $2b_2$, as asserted.

It remains (iii). In this case, again by Local Structure Theorem, $I \cong (x^{b_2} + \alpha y,\, x^{b_2 - b_1} y,\, y^2)$ (observe that it is possible to assume that not only $x^{b_2 - b_1}$ and y do not divide α but also that x^{b_1} does not divide it: indeed, it holds that $(x^{b_2} + x^{b_1}\epsilon y,\, x^{b_2 - b_1} y,\, y^2) \cong (x^{b_2},\, x^{b_2 - b_1} y,\, y^2)$, for any $\epsilon \in A$). The method of calculation is the same of the previous cases, but the computations are harder, having one more generator. The following is a periodic free resolution of I:

$$\cdots \longrightarrow A^2 \xrightarrow{M_2} A^2 \xrightarrow{M_1} A^2 \xrightarrow{f} I \longrightarrow 0,$$

where

$$M_1 = \begin{pmatrix} y & -x^{b_2 - b_1} y & -\alpha \\ 0 & y & -x^{b_1} \\ 0 & 0 & y \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} y^2 & x^{b_2 - b_1} y & x^{b_2} + \alpha y \\ 0 & y^2 & -x^{b_1} y \\ 0 & 0 & y^2 \end{pmatrix}$$

while $f((1,0,0)) = y^2$, $f((0,1,0)) = x^{b_2 - b_1} y$ and $f((0,0,1)) = x^{b_2} + \alpha y$.

From the resolution one gets the complex

$$\cdots \longleftarrow \mathrm{Hom}(A^3, I) \xleftarrow{a_2} \mathrm{Hom}(A^3, I) \xleftarrow{a_1} \mathrm{Hom}(A^3, I),$$

where a_i is the homomorphism induced by multiplication by M_i, for $i = 1, 2$. By definition, $\mathrm{Ext}^1(I, I) = \ker(a_2)/\operatorname{im}(a_1)$. It holds that $\varphi \in \operatorname{im}(a_1)$ if and only if

$$\begin{cases} \varphi((1,0,0)) = \beta_1 x^{b_2-b_1}y^2 + \beta_2 x^{b_2} y \\ \varphi((0,1,0)) = (\beta_3 x^{b_2-b_1} + \beta_4\alpha)y^2 + (-\beta_1 x^{b_2-b_1} + \beta_4 x^{b_1})x^{b_2-b_1}y \\ \qquad\qquad - \beta_2 x^{b_2-b_1}(x^{b_2} + \alpha y) \\ \varphi((0,0,1)) = (\beta_5\alpha + \beta_6 x^{b_1} + \beta_7 x^{b_2-b_1})y^2 + (-\beta_1\alpha + (\beta_5 - \beta_3)x^{b_1})x^{b_2-b_1}y \\ \qquad\qquad - (\beta_2\alpha + \beta_4 x^{b_1})(x^{b_2} + \alpha y), \end{cases}$$

with $\beta_i \in A$, for any $1 \le i \le 7$; on the other side $\psi \in \ker(a_2)$ if and only if

$$\begin{cases} \psi((1,0,0)) = \gamma_1 y^2 + \gamma_2 x^{\max\{0,\, 2b_1-b_2\}} x^{b_2-b_1} y \\ \psi((0,1,0)) = -\gamma_2 x^{\max\{b_2-2b_1,\, 0\}}(x^{b_2} + \alpha y) + \gamma_3 y^2 + \gamma_4 x^{b_2-b_1} y \\ \psi((0,0,1)) = (-\gamma_1 - \gamma_4)(x^{b_2} + \alpha y) + \gamma_5 y^2 + \gamma_6 x^{b_2-b_1} y, \end{cases}$$

with $\gamma_i \in A$, for $1 \le i \le 6$. Hence, the desired result follows from these direct computations (observing that each β_i can be used to *limit* almost one γ_j). *q.e.d.*

3.2. Higher multiplicity

Now it is time to turn our attention to higher multiplicity, so throughout this section X will be a primitive multiple curve of multiplicity $n \ge 4$ such that $\delta = -\deg(\mathcal{C}) > 0$. The results about the moduli space $\mathrm{M}(X, P)$ are analogous to those of multiplicity 3, but the statements and the proofs are often more involved and some of them are also more vague (e.g. Theorem 3.19 does not guarantee the connection of the locus of stable generalized line bundles for any value of δ neither furnishes a precise estimate of the value of δ from which this connection holds). Moreover, the problem of the existence of components whose generic elements are defined on subcurves grows up with n; we will say few words and formulate a conjecture about this question in Section 4.3.

3.2.1. Global geometry: irreducible components. As in multiplicity 3 (see §3.1.1), first of all, we introduce some loci of generalized line bundles in $\mathrm{M}_s(X, P)$, among whose closures there are the irreducible components containing stable generalized line bundles, as we will show later.

Definition 3.12. (Cf. Definition 3.2). Let X be a primitive multiple curve of multiplicity n and let $(b_{1,1}, \ldots, b_{1,r_1}), \ldots, (b_{n-1,1}, \ldots, b_{n-1,r_{n-1}})$ be $n-1$ (possibly empty except one of them) sequences of positive integers such that the inequalities (2.5) are strictly verified by $b_j = \sum_{h=1}^{j}\sum_{l=1}^{r_h} b_{h,l}$, for $1 \le j \le n-1$. Set $\underline{b} := ((\underbrace{0, \ldots, 0}_{j-1 \text{ times}}, \underbrace{b_{j,h}, \ldots, b_{j,h}}_{n-j \text{ times}}))_{1 \le j \le n-1,\, 1 \le h \le r_j}$. Define $Z_{\underline{b}} \subset \mathrm{M}_s(X, P_D)$ as the subset of stable generalized line bundles of generalized degree D and local indices sequence $\underline{b}$.

As for Definition 3.2 the inequalities are the stability conditions of Theorem 2.41.

LEMMA 3.13. *(Cf. Lemma 3.3). If* $n \not| D + (n(n-1)/2)\delta - b_1 - \cdots - b_{n-1}$*, then* $Z_{\underline{b}}$ *is empty. Otherwise, it is a constructible, irreducible subset of dimension* $g_n - b_{n-1} + \sum_{h=1}^{n-1} r_h$*, where* g_n *is the genus of* X*.*

PROOF. In order to simplify notations set $\underline{\beta}_j = (b_{j,1}, \ldots, b_{j,r_j})$ and $\beta_j = \sum_{l=1}^{r_j} b_{j,l}$, for $1 \le j \le n-1$.

The first assertion follows from the fact that the first of formulae (2.1) implies that $n | D + (n(n-1)/2)\delta - b_1(\mathscr{F}) - \cdots - b_{n-1}(\mathscr{F})$ for any generalized line bundle $\mathscr{F}$.

So, assume $n | D + (n(n-1)/2)\delta - b_1 - \cdots - b_{n-1}$. As in the proof of Lemma 3.3, the key point is to parametrize $Z_{\underline{b}}$ with an irreducible variety of the required dimension.

Consider $C^{(\beta_j)}$, i.e. the β_j-th symmetric product of the reduced subcurve C, and within it the diagonal $\Delta_{\underline{\beta}_j}$ associated to the partition $\underline{\beta}_j$ of β_j (for $j = 1, \ldots, n-1$), i.e. the image of the r_j-th direct product of C with itself in $C^{(\beta_j)}$ under the morphism sending $(P_{j,1}, \ldots, P_{j,r_j})$ to $\sum_{l=1}^{r_j} b_{j,l} P_{j,l}$. Let $U \subset \Delta_{\underline{\beta}_1} \times \cdots \times \Delta_{\underline{\beta}_{n-1}}$ be the locus such that the points $P_{j,l}$'s are all distinct (for $1 \le j \le n-1$ and $1 \le l \le r_j$). It is clear that U is locally closed in $C^{(\beta_1)} \times \cdots \times C^{(\beta_{n-1})}$ and irreducible of dimension $r_1 + \cdots + r_{n-1}$.

Set $m = \sum_{j=2}^{n-2} \sum_{h=1}^{\bar{\jmath}} \sum_{l=1}^{r_j} \sum_{i=0}^{b_{j,l}-1} 1 = \sum_{j=2}^{n-2} \bar{\jmath}\beta_j$, where $\bar{\jmath} = \min\{j, n-j\} - 1$, and consider the affine space $\mathbb{A}_k^m$; any of its closed points will be denoted in a completely non-standard way as $a = (z_{h,i}^{(j,l)})$, with h, i, j and l varying as in the definition of m.

For any $\Sigma \in U$ and $a \in \mathbb{A}_k^m$, consider the ideal sheaf $\mathscr{I}(\Sigma, a)$ defined as, using local notation, $\left(x^{b_{j,l}} + \sum_{h=1}^{\bar{\jmath}} \sum_{i=0}^{b_{j,l}-1} z_{h,i}^{(j,l)} x^i y^h, y^l\right)$ at the point $P_{j,l}$ for any $1 \le j \le n-1$ and $1 \le l \le r_j$. It holds that $\mathscr{I}\left(\Sigma, \left(z_{h,i}^{(j,l)}\right)\right)$ is a stable generalized line bundle of generalized degree $-\sum_{j=1}^{n-1} j\beta_j - (n(n-1)/2)\delta$ and local indices sequence $\underline{b}$. So it is possible to define a map $\mathbb{A}_k^m \times U \times \mathrm{Pic}^{D + \sum_{j=1}^{n-1} j\beta_j}(X) \to \mathrm{M}_s(X, P_D)$ by the rule $a \times \Sigma \times \mathscr{E} \mapsto \mathscr{I}(\Sigma, a) \otimes \mathscr{E}$, where $\mathrm{Pic}^{D + \sum_{j=1}^{n-1} j\beta_j}(X)$ is the variety of line bundles on X of generalized degree $D + \sum_{j=1}^{n-1} j\beta_j$ (it is the right generalized degree to be used by Corollary 1.15). Let $a \in \mathbb{A}_j^m$; by the definition of U, for any set of $r_1 + \cdots + r_{n-1}$ points $P_{j,l}$, with $1 \le j \le n-1$ and $1 \le l \le r_j$, there is a unique closed subscheme $\Sigma \subset C$, corresponding to a point of U, such that $b_{1,P_{j,l}}(\mathscr{I}(\Sigma, a)) = b_{j-1,P_{j,l}}(\mathscr{I}(\Sigma, a)) = 0$ and $b_{j,P_{j,l}}(\mathscr{I}(\Sigma, a)) = b_{n-1,P_{j,l}}(\mathscr{I}(\Sigma, a)) = b_{j,l}$, for $1 \le j \le n-1$ and $1 \le j \le r_j$. Hence, by Corollary 2.29 and by Corollary 2.38(iii), the image of the just defined map is $Z_{\underline{b}}$ and, moreover, if X' is the blow up described in the second of the cited Corollaries, the fibre over a point is an irreducible variety of dimension $\mathrm{h}^1(X, \mathscr{O}_X) - \mathrm{h}^1(X', \mathscr{O}_{X'}) = g_n - g(X') = \sum_{j=1}^{n-1} (\bar{\jmath} + 1) b_{j,l}$ (with $\bar{\jmath}$ as above), where the second equality is trivial and the first one holds because both X and X' do not have non-trivial global sections (for X it is easily implied by $\deg(\mathcal{C}) < 0$, while for X' it is Lemma 3.14). Hence, $Z_{\underline{b}}$ is irreducible and constructible of dimension $m + r_1 + \cdots + r_{n-1} + g_n - \sum_{j=1}^{n-1} (\bar{\jmath} + 1) b_{j,l} = g_n - b_{n-1} + \sum_{h=1}^{n-1} r_h$. *q.e.d.*

In order to complete the above proof we need the following:

LEMMA 3.14. *(Cf. Lemma 3.4.) Let $q : X' \to X$ be the blow up considered in the proof of the previous Lemma. Then it has only trivial global sections, equivalently $g(X') = \mathrm{h}^1(X', \mathscr{O}_{X'})$.*

PROOF. The notation is as in the proof of the previous Lemma and, moreover, we set $\tilde{\jmath} = n - (\bar{\jmath} + 1)$. The idea of the proof is similar to that of Lemma 3.4, but instead of going immediately from X' to a primitive multiple curve of multiplicity i, with $1 \leq i \leq n-1$, we need, in general, various steps through non-primitive multiple curves before arriving there.

It follows from the definition that in any point P different from the $P_{j,l}$'s $\mathscr{O}_{X',P} \cong \mathscr{O}_{X,P}$, while $\mathscr{O}_{X',P_{j,l}} \cong \mathscr{O}_{X,P_{j,l}}[y^{\tilde{\jmath}}/(x^{b_{j,l}} + \alpha y)]$ for an appropriate $\alpha \in \mathscr{O}_{X,P_{j,l}}$ which is not relevant to make explicit for the following counts. So, we can consider the ideal sheaf $\mathscr{K}_{n-1} \subset \mathscr{O}_{X'}$ defined as $\mathscr{N}_P^{n-1}$ for $P \notin \{P_{j,l}\}$ and as the ideal generated by $y^{n-1}/(x^{b_{j,l}} + \alpha y)$ in any $P_{j,l}$. The scheme X'_{n-1} defined as $(C, \mathscr{O}_{X'}/\mathscr{K}_{n-1})$ is a multiple curve such that $\mathscr{O}_{X'_{n-1},P} \cong \mathscr{O}_{C_{n-1},P}$ for $P \notin \{P_{j,l}\}$ (for $2 \leq j \leq n-2$ only) while $\mathscr{O}_{X'_{n-1},P_{j,l}} \cong \mathscr{O}_{C_{n-1},P_{j,l}}[\bar{y}_{n-1}^{\tilde{\jmath}}/(\bar{x}_{n-1}^{b_{j,l}} + \bar{\alpha}_{n-1}\bar{y}_{n-1})]$ (excluded the $P_{j,l}$ with $j = 1$ or $n-1$, where $\bar{y}_{n-1}^{\tilde{\jmath}} = 0$). Observe that if $r_j = 0$ for $2 \leq j \leq n-2$, then X'_{n-1} is just C_{n-1}. If it is not the case, one can define $\mathscr{K}_{n-2} \subset \mathscr{O}_{X'_{n-1}}$ as the ideal isomorphic to $((\mathscr{N}/\mathscr{N}^{n-1})^{n-2})_P$ for $P \notin \{P_{j,l}\}$ (for $2 \leq j \leq n-2$) and to the ideal generated by $\bar{y}_{n-1}^{n-2}/(\bar{x}_{n-1}^{b_{j,l}} + \bar{\alpha}_{n-1}\bar{y}_{n-1})$ in any $P_{j,l}$, with $2 \leq j \leq n-2$. So it is possible to consider the scheme X'_{n-2} defined as $(C, \mathscr{O}_{X'_{n-1}}/\mathscr{K}_{n-1})$. If it is not isomorphic to C_{n-2}, i.e. if there is at least one $r_j \neq 0$, with $3 \leq j \leq n-3$, define similarly $\mathscr{K}_{n-3}$ and X'_{n-3} and continue in the same way defining $\mathscr{K}_{n-i}$ and X'_{n-i} for increasing i until you get $X'_{n-\bar{\imath}} = C_{n-\bar{\imath}}$ ($\bar{\imath}$ is at most the integral part of $n/2$).

The point of the proof is to show that all the $\mathscr{K}_{n-i}$ do not have global sections, for $1 \leq i \leq n/2$, so that each X'_{n-i} (and thus also X') has only trivial global sections (because $C_{n-\bar{\imath}}$ has this property).

There are two distinct cases to be treated: $i < n/2$ and $i = n/2$ (the latter is possible only if n is even).

For any $1 \leq i < n/2$, the sheaf $\mathscr{K}_{n-i}$ is a line bundle on C and there is an exact sequence $0 \to (\mathscr{N}/\mathscr{N}^{n-i+1})^{n-i} \to \mathscr{K}_{n-i} \to \mathscr{O}_{D_i} \to 0$, where $D_i \subset C$ is an effective divisor of length $b_{n-i} - b_{i-1}$. Hence, it is sufficient to show that $\deg(\mathscr{K}_{n-i}) = -(n-i)\delta + b_{n-i} - b_{i-1} < 0$, i.e. that $b_{n-i} - b_{i-1} < (n-i)\delta$.

Consider the i-th and the $(n-i)$-th stability inequalities (2.5), which hold strictly by hypothesis. They can be written as:

$$\begin{cases} i\left(\displaystyle\sum_{j=n-i}^{n-1} b_j - \sum_{j=1}^{i-1} b_j\right) + i \displaystyle\sum_{j=i}^{n-i-1} b_j - (n-2i)\sum_{j=1}^{i-1} b_j < \dfrac{in(n-i)}{2}\delta \\ (n-i)\left(\displaystyle\sum_{j=n-i}^{n-1} b_j - \sum_{j=1}^{i-1} b_j\right) - i \displaystyle\sum_{j=i}^{n-i-1} b_j + (n-2i)\sum_{j=1}^{i-1} b_j < \dfrac{in(n-i)}{2}\delta. \end{cases}$$

By Lemma 2.18, it holds that $b_{n-i} \leq b_j$ for any $n-i \leq j \leq n-1$, that $b_{i-1} \geq b_h$ for any $1 \leq h \leq i-1$ and that $b_{n-1} \geq b_{n-i} - b_{i-1}$; thus, each of the above inequalities implies the corresponding one within the following

$$\begin{cases} i\displaystyle\sum_{j=i}^{n-i-1} b_j-(n-2i)(i-1)b_{i-1}<\frac{in(n-i)}{2}\delta-i^2(b_{n-i}-b_{i-1}) \\ (n-i)i(b_{n-i}-b_{i-1})<\dfrac{in(n-i)}{2}\delta+i\displaystyle\sum_{j=i}^{n-i-1} b_j-(n-2i)(i-1)b_{i-1}. \end{cases}$$

Hence, substituting the first one in the right hand term of the second one it follows that

$$(n-i)i(b_{n-i}-b_{i-1})<in(n-i)\delta-i^2(b_{n-i}-b_{i-1}),$$

which is equivalent to the desired inequality.

Now assume n even and consider the case of $i=n/2$. As in the previous case, it holds that $\mathscr{K}_{n/2}$ is a line bundle on C and that there is an exact sequence $0\to(\mathscr{N}/\mathscr{N}^{n/2+1})^{n/2}\to\mathscr{K}_{n/2}\to\mathscr{O}_{D_{n/2}}\to 0$, where $D_{n/2}\subset C$ is an effective divisor of length $b_{n/2}-b_{(n-2)/2}$. Hence, it is sufficient to show that $\deg(\mathscr{K}_{n/2})=-(n/2)\delta+b_{n/2}-b_{(n-2)/2}<0$, i.e. that $b_{n/2}-b_{(n-2)/2}<(n/2)\delta$. In this case, we need only the $(n/2)$-th stability inequality (2.5), which can be written as

$$\frac{n}{2}\left(\sum_{j=n/2}^{n-1} b_j-\sum_{j=1}^{(n-2)/2} b_j\right)<\left(\frac{n}{2}\right)^3\delta.$$

Again by the basic inequalities between indices due to Lemma 2.18, the left hand term is greater than or equal to $(n/2)^2(b_{n/2}-b_{(n-2)/2})$. Therefore, it holds that $b_{n/2}-b_{(n-2)/2}<(n/2)\delta$, as wanted. *q.e.d.*

As in the case of multiplicity 3, among the Zariski closures of the loci introduced in Definition 3.12 there are the irreducible components of the moduli space containing stable generalized line bundles. In order to prove this fact, it is convenient to study some deformations of generalized line bundles. The following is the extension of Lemma 3.5 to higher multiplicity.

LEMMA 3.15. *Let $\mathscr{F}$ be a generalized line bundle on $X=C_n$ of local indices sequence $b_{\cdot,\cdot}$. Let P be a closed points of C, such that $b_{n-1,P}\geq 2$.*

(i) *If it does not exist an integer h such that $0=b_{h-1,P}<b_{h,P}=b_{n-1,P}$, then $\mathscr{F}$ is the specialization of a generalized line bundle $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $b_{n-2,P}(\mathscr{F}')=0$ and $b_{n-1,P}(\mathscr{F}')=b_{n-1,P}-b_{n-2,P}$, and in at most other $n-2$ closed points $Q_1,\dots,Q_{n-2}$, where $b_{1,Q_j}=b_{n-1,Q_j}=0$, while $b_{j-1,Q_j}(\mathscr{F}')=0$ and $b_{j,Q_j}(\mathscr{F}')=b_{n-1,Q_j}(\mathscr{F}')=b_{j,P}-b_{j-1,P}$, for any $1\leq j\leq n-2$.*
(ii) *If there exists an integer h such that $0=b_{h-1,P}<b_{h,P}=b_{n-1,P}$, then $\mathscr{F}$ is the specialization of a generalized line bundle $\mathscr{F}'$ with the same local indices sequence of $\mathscr{F}$ except in P, where $0=b_{h-1,P}(\mathscr{F}')<b_{h,P}(\mathscr{F}')=b_{n-1,P}(\mathscr{F}')=b_{n-1,P}-1$, and in another closed point Q, where $b_{1,Q}=b_{n-1,Q}=0$ and $0=b_{h-1,Q}(\mathscr{F}')<b_{h,Q}(\mathscr{F}')=b_{n-1,Q}(\mathscr{F}')=1$.*

PROOF. For both the two points we will exhibit explicit deformations, respectively over $\mathbb{K}[t_1, \dots, t_{n-2}]$ and over $\mathbb{K}[t]$. In order to simplify notation, throughout the proof $b_i = b_{i,P}$, for $i = 1, \dots, n-1$.

First of all, recall that by Corollary 2.32, $\mathscr{F}$ is isomorphic to $\mathscr{I}_{Z/X} \otimes \mathscr{L}$, where $Z \subset C_{n-1}$ is a closed subscheme of finite support and $\mathscr{L}$ is a line bundle on X. Thus, it is sufficient to find an appropriate deformation of $\mathscr{I} = \mathscr{I}_{Z/X}$, say $\mathscr{I}'$, because, then, the desired deformation of $\mathscr{F}$ would be $\mathscr{I}' \otimes \mathscr{L}$, where by a slight abuse of notation $\mathscr{L}$ here denotes the constant family with fibre $\mathscr{L}$. Deforming $\mathscr{I}$ is equivalent to deforming Z. In order to do that we will use the local affine description of generalized line bundles given in Corollary 2.31. So let $U = \mathrm{Spec}(A)$ be an affine neighbourhood of P, in which the thesis of the cited Corollary holds (in particular, P is the only closed point in U where $\mathscr{I}$ is not locally free). Now it is necessary to distinguish the two cases.

We start with (i): using the notation of the cited Corollary, it holds that $\mathscr{I}(U) \cong (x^{b_{n-1}-b_{i-1}} y^{i-1} + \alpha_i y^i)_{i=1,\dots,n}$ (where $\alpha_i \in A$, for $1 \leq i \leq n-1$, are not completely arbitrary, they could be written as in the cited Corollary) and the desired deformation is given by the extension of the ideal $\left(x^{b_{n-1}-b_{n-2}} y^i \prod_{j=i}^{n-2} (x - t_j)^{b_j - b_{j-1}} + \alpha_i y^{i-1}, y^{n-1}\right)_{i=1,\dots,n-1}$ to a proper flat family over $\mathrm{Spec}(\mathbb{K}[t_1, \dots, t_{n-1}])$ (it is possible to have such an extension by, e.g., the properness of the Hilbert scheme). This generic fibre is the expression over U of a generalized line bundle having the desired local indices sequence, while the special fibre is $\mathscr{I}(U)$.

The proof of (ii) is similar: $\mathscr{I}(U) \cong (x^{b_{n-1}} + \alpha y, y^{n-h})$ (also this $\alpha \in A$ is not completely arbitrary, it can be expressed as in the last statement of Corollary 2.31) and, analogously, the desired deformation is the extension of the ideal $(x^{b_{n-1}-1}(x - t) + \alpha y, y^{n-h})$ to a proper flat family over $\mathrm{Spec}(\mathbb{K}[t])$. Also in this case the needed verifies are almost trivial. *q.e.d.*

The next statement is the extension of Theorem 3.6 to higher multiplicity.

THEOREM 3.16. *Let X be a primitive multiple curve of multiplicity n and let $b_1 \leq \cdots \leq b_{n-1}$ be non-negative integers satisfying $n | D + (n(n-1)/2)\delta - \sum_{i=1}^{n-1} b_i$ and the strict inequalities (2.5). Let $\bar{Z}_{b_1,\dots,b_{n-1}} \subset \mathrm{M}(X, P_D)$ be the Zariski closure of the locus of stable generalized line bundles of indices-vector $(b_1, \dots, b_{n-1})$. If $(b_1, \dots, b_{n-1}) \neq (0, \dots, 0)$, then $\bar{Z}_{b_1,\dots,b_{n-1}}$ coincides with the Zariski closure of $Z_{\underline{b}}$, where $\underline{b}$ is the sequence*

$$\Big(\underbrace{(\underbrace{0,\dots,0}_{i-1},\underbrace{1,\dots,1}_{n-i})}_{b_i - b_{i-1}\ \text{times}}\Big)_{1 \leq i \leq n-1},$$

where $b_0 = 0$.

Then any $\bar{Z}_{b_1,\dots,b_{n-1}}$ is an irreducible component of $\mathrm{M}(X, P_D)$ of dimension $g_n = g(X)$. Moreover, any irreducible component containing a stable generalized line bundle is equal to $\bar{Z}_{b_1,\dots,b_{n-1}}$, for a unique choice of $b_1 \leq \ldots \leq b_{n-1}$ satisfying the above conditions.

PROOF. The proof is almost identical to that of Theorem 3.6. There are only few adaptations to the more general setting.

By definition, $\cup \bar{Z}_{b_1,\dots,b_{n-1}}$ contains the locus of stable generalized line bundles.

The first assertion is implied by a repeated application of Lemma 3.15. Combined with Lemma 3.13 it implies that each $\bar{Z}_{b_1,\dots,b_{n-1}}$ is irreducible of dimension g_n.

Now let $\bar{Z}$ be an irreducible component containing a stable generalized line bundle. Its subset consisting of stable generalized line bundles is open (by Lemma 3.1) and non-empty; hence, it is dense. Thus, $\bar{Z}$ is contained in the union $\cup \bar{Z}_{b_1,\dots,b_{n-1}}$; hence, $\bar{Z} = \bar{Z}_{b_1,\dots,b_{n-1}}$, for some $(b_1,\dots,b_{n-1})$, because these loci are irreducible.

So, some of the $\bar{Z}_{b_1,\dots,b_{n-1}}$ are irreducible components. Moreover, by their irreducibility and equidimensionality, each of them is a component. Furthermore they are all distinct, because, if $(b_1,\dots,b_{n-1}) \neq (b'_1,\dots,b'_{n-1})$, the generic elements of $\bar{Z}_{b_1,\dots,b_{n-1}}$ and $\bar{Z}_{b'_1,\dots,b'_{n-1}}$ are different. *q.e.d.*

Again as in the case of multiplicity 3, it is useful to introduce some other deformations, which allow to study the connectedness of the locus of stable generalized line bundles.

LEMMA 3.17. *Let $\mathscr{F}$ be a generalized line bundle on $X = C_n$ of local indices sequence $b_{\cdot,\cdot}$ and let P be a closed point of C such that $b_{n-1,P} \neq 0$.*

(i) *Assume that $\mathscr{F}_P \cong (x^{b_{n-1,P}}, y^{n-h})$, for an integer $1 \leq h \leq n-1$, and that $b_{n-1,P} \geq k$, where $k = n/\gcd(n,h)$. Then $\mathscr{F}$ is the specialization of generalized line bundles $\mathscr{F}'$, whose local indices sequence is equal to $b_{\cdot,\cdot}$ except in P, where $b_{n-h-1,P}(\mathscr{F}') = b_{n-h-1,P} = 0$ and $b_{n-h,P}(\mathscr{F}') = b_{n-1,P}(\mathscr{F}') = b_{n-1,P} - l$.*

(ii) *Assume that $\mathscr{F}_P \cong (x^{b_{n-1,P}-b_{i,P}} y^i)_{i=0,\dots,n-1}$. Let $0 = j_0 \leq j_1 \leq \dots \leq j_{n-1}$ be integers whose sum is divided by n and such that $b_i - j_i \geq b_{i-1} - j_{i-1} \geq 0$, for any $2 \leq i \leq n-1$. Then $\mathscr{F}$ is the specialization of generalized line bundles $\mathscr{F}'$, whose local indices sequence is equal to $b_{\cdot,\cdot}$ except in P, where $b_{i,P}(\mathscr{F}') = b_{i,P}(\mathscr{F}) - j_i$, for any $1 \leq i \leq n-1$.*

(iii) *If $\mathscr{F}_P \cong (x^{b_{n-1,P}-b_{i,P}} y^i)_{i=0,\dots,n-1}$ and $b_{1,P} \geq 2$, then $\mathscr{F}$ is the specialization of generalized line bundles $\mathscr{F}'$, whose local indices sequence is equal to $b_{\cdot,\cdot}$ except in P, where $b_{i,P}(\mathscr{F}') = b_{i,P}(\mathscr{F}) - 2$, for any $1 \leq i \leq n-1$, and in another point Q where $b_{1,Q} = b_{n-1,Q} = b_{1,Q}(\mathscr{F}') = 0$ and $b_{2,Q}(\mathscr{F}') = b_{n-1,Q}(\mathscr{F}') = 1$.*

(iv) *If $\mathscr{F}$ is the dual of a generalized line bundle verifying the hypotheses of the previous point, then $\mathscr{F}$ is the specialization of generalized line bundles $\mathscr{F}'$, whose local indices sequence is equal to $b_{\cdot,\cdot}$ except in P, where $b_{i,P}(\mathscr{F}') = b_{i,P}(\mathscr{F})$, for any $1 \leq i \leq n-2$ and $b_{n-1,P}(\mathscr{F}') = b_{n-1,P}(\mathscr{F}) - 2$, and in another point Q where $b_{1,Q} = b_{n-1,Q} = b_{n-3,Q}(\mathscr{F}') = 0$ and $b_{n-2,Q}(\mathscr{F}') = b_{n-1,Q}(\mathscr{F}') = 1$.*

PROOF. The idea of the proof is essentially the same of that of Lemma 3.7, which is a particular case of the present one. As in that case, we will exhibit explicit deformations over $\mathbb{K}[t]$ and it is sufficient to work with $\mathscr{I}$, the ideal sheaf in the orbit of $\mathscr{F}$ under the action of $\mathrm{Pic}(X)$; as usual deforming $\mathscr{I}$ is the same thing of deforming the associated subscheme $Z \subset C_{n-1}$. It is also sufficient to work in an open neighbourhood, $U = \mathrm{Spec}(A)$, of P

in which $\mathscr{F}$ (or, equivalently, $\mathscr{I}$) is locally free in all points except P. In order to simplify the notation, throughout the proof we set $b_i = b_{i,P}$, for $1 \leq i \leq n-1$, and $I = \mathscr{I}(U)$. It is time to distinguish the various cases.

We start with (i): it holds by Corollary 2.31 that $I = (x^{b_{n-1}}, y^{n-h}) \subset A$ (by a slight abuse of notation, the affine one is identical to the local one used in the statement, but there is no risk of confusion because throughout the proof it will be used only the affine one; to be more precise in the local one x and y should be substituted by x_P and y_P). As usual, it is sufficient to give the generic fibre of the deformation, which, in this case, is the ideal $I'_t = (x^{b_{n-1}-k}, y^{n-h}) \cap ((x-t)^k, y^{n-k} - t^{b_{n-1}-k+1}(x-t)^{k-1})$. Indeed by the fact $x^{b_{n-1}-k}(x-t)^k$ and $y^{n-h} - x^{b_{n-1}-k}t(x-t)^{k-1} = y^{n-h} - t^{b_{n-1}-k+1}(x-t)^{k-1} - (x-t)^k t \sum_{i=k+1}^{b_{n-1}} x^{i-k-1} t^{b_{n-1}-i}$ (if $b_{n-1} \geq k+1$; if $b_{n-1} = k$, instead of the latter consider $y^{n-j} - t(x-t)^{k-1}$) belong to I'_t, for any $t \neq 0$, it follows that I is contained in the special fibre. The fact they coincide is due to degree considerations: A/I has length $(n-h)b_{n-1}$ while $A/(x^{b_{n-1}-k}, y^{n-h})$ has length $(n-h)(b_{n-1}-k)$ and $A/((x-t)^k, y^{n-k} - t^{b_{n-1}-k+1}(x-t)^{k-1})$ has length $(n-h)k$, for any non-zero value of the parameter t. Moreover, the ideal $((x-t)^k, y^{n-k} - t^{b_{n-1}-k+1}(x-t)^{k-1})$ defines a Cartier divisor of X (for any fixed non-zero t), and so the part contributing to the local indices sequence of I'_t is only $(x^{b_{n-1}-k}, y^{n-k})$.

The proof of (ii) is similar. In this case $I = (x^{b_{n-1}-b_i} y^i)_{i=0,\dots,n-1}$ and the deformation has generic fibre $I'_t = J_t \cap H_t = (x^{b_{n-1}-b_i-j_{n-1}+j_i} y^i)_{i=0,\dots,n-1} \cap \big((x-t)^{j_{n-1}-j_i} y^i)_{i=0,\dots,\tilde{\imath}}, y^{\tilde{\imath}+1} - t^{b_{n-1}-j_{n-1}+j_{\tilde{\imath}}+1}(x-t)^{j_{n-1}-j_{\tilde{\imath}}-1}\big)$, where $\tilde{\imath}$ is the greatest integer within 0 and $n-2$ such that $j_{\tilde{\imath}} < j_{n-1}$. The special fibre is really I. Indeed, it is clear if $j_{n-1} = b_{n-1}$; otherwise, $(x-t)^{j_{n-1}-j_i} y^i x^{b_{n-1}-b_i-j_{n-1}+j_i}$ belongs to I'_t, for any $0 \leq i \leq \tilde{\imath}$ and for any $t \neq 0$; hence, $x^{b_{n-1}-b_i} y^i$ belongs to the special fibre for any $0 \leq i \leq \tilde{\imath}$. Moreover, $x^{b_{n-1}-b_i} y^i$ belongs to the special fibre, also for $\tilde{\imath} < i \leq n-1$, being the limit of the following element (which is in I'_t for any $t \neq 0$): $x^{b_{n-1}-b_i} y^{i-\tilde{\imath}-1}(y^{\tilde{\imath}+1} - t^{b_{n-1}-j_{n-1}+j_{\tilde{\imath}}+1}(x-t)^{j_{n-1}-j_{\tilde{\imath}}-1} - (x-t)^{j_{n-1}} w) = x^{b_{n-1}-b_i} y^i - y^{i-\tilde{\imath}-1}(x-t)^{j_{n-1}-j_{n-2}-1} t x^{2b_{n-1}-b_i-j_{n-1}} z$, where $w = \sum_{r=0}^{b_{n-1}-j_{n-1}-1} w_r t^{b_{n-1}-j_{n-1}-1-r} x^r$, in which $w_0 = (-1)^{j_{n-2}}$, and recursively $w_r = \sum_{l=0}^{r-1} (-1)^{r-l} \binom{j_{n-2}+1}{r-l} w_l$ for $1 \leq r \leq j_{n-2}+1$ and $w_r = \sum_{l=r-j_{n-2}-1}^{r-1} (-1)^{r-l} \binom{j_{n-2}+1}{r-l} w_l$ for $j_{n-2}+1 \leq r \leq b_{n-1} - j_{n-1} - 1$, while $z = \sum_{r=1}^{j_{n-2}+1} x^{j_{n-2}+1-r} t^{r-1} \big(\sum_{l=1}^{r} w_{b_{n-1}-j_{n-1}-l} \binom{j_{n-2}+1}{r-l}\big)$.

Hence, I is contained in the special fibre; furthermore, they are equal by degree reasons: indeed, A/I has length $(n-1)b_{n-1} - \sum_{i=1}^{n-2} b_i$, while A/J_t has length $(n-1)b_{n-1} - \sum_{i=1}^{n-2} b_i - (\tilde{\imath}+1)j_{n-1} + \sum_{i=1}^{\tilde{\imath}} j_i$ and A/H_t has length $(\tilde{\imath}+1)j_{n-1} - \sum_{i=1}^{\tilde{\imath}} j_i$, for any $t \neq 0$. The generalized line bundle defined by this deformation has the desired local indices sequence, because H_t defines a Cartier divisor of X and, hence, only J_t contributes to the local indices.

Now let us prove (iii). This time $I = (x^{b_{n-1}-b_i} y^i)_{i=0,\dots,n-1}$ and the deformation has generic fibre $I'_t = J_t \cap H_t = (x^{b_{n-1}-2}, x^{b_{n-1}-b_i} y^i)_{i=1,\dots,n-1} \cap (y - t^{b_{n-1}-1}(x-t), t^{2(b_{n-1}-1)} y^2 + (x-t)^2)$. The ideal I is contained in the special fibre because both $x^{b_{n-1}-2}\big[(x-t)^2 + t^{2(b_{n-1}-1)} y^2\big]$ and $x^{b_{n-1}-b_i} y^{i-1}\big\{y - t^{b_{n-1}-1}(x-t) - \big[t^{2(b_{n-1}-1)} y^2 + (x-t)^2\big] t \sum_{j=0}^{b_{n-1}-3} t^j x^{b_{n-1}-3-j}\big\} = x^{b_{n-1}-b_i} y^i (1+

$t^{2(b_{n-1}-1)}y\sum_{j=0}^{b_{n-1}-3}t^jx^{b_{n-1}-3-j}) + x^{b_{n-1}-2}tx^{b_{n-1}-b_i}y^i(t-tx)$, for $1 \leq i \leq n-2$ belong to both H_t and J_t. The conclusion holds, as usual in these kind of proofs, by easy degree considerations. In the previous verifications it was implicitly assumed that $b_{n-1} \geq 3$; if $b_{n-1} = 2$, the ideal I is simply (x^2, y) while the generic fibre I' reduces to $((x-t)^2 + t^2y^2, y - t(x-t))$.

Finally, (iv) is simply the dual of (iii). *q.e.d.*

REMARK 3.18. The first point of the above Lemma could be seen as a special case of the second one; it is separated by its relevance, which will be perspicuous in the proof of the next theorem.

Now it is possible to state the following theorem which is a partial generalization of Theorem 3.8.

THEOREM 3.19. *Let X be a primitive multiple curve of multiplicity n and let $\delta = -\deg(\mathcal{C})$. The locus of stable generalized line bundles in* $\mathrm{M}(X, P_D)$ *is connected for δ sufficiently large. In the case $n|D - (n(n-1)/2)\delta$, then this locus is connected for any value of δ.*

PROOF. Let $\mathscr{F}$ be a generalized line bundle of indices sequence $b_.$, not free in only one point P, where $\mathscr{F}_P = (x^{b_{n-1}-b_i}y^i)_{i=0,\ldots,n-1}$. By definition, $\mathscr{F}$ belongs to $\bar{Z}_{b_1,\ldots,b_{n-1}}$. It belongs also to $\bar{Z}_{\overline{b_1},\overline{b_1}+\overline{b_2-b_1},\ldots,\overline{b_1}+\overline{b_2-b_1}+\cdots+\overline{b_{n-1}-b_{n-2}}}$, where $\overline{b_i - b_{i-1}}$ is the representative of the congruency class modulo n of $b_i - b_{i-1}$ contained in $\{0, \ldots, n-1\}$, for $1 \leq i \leq n-1$ (as usual $b_0 = 0$). This follows applying, if needed, Lemma 3.15(i), then various times Lemma 3.17(i) and, finally, also Lemma 3.15(ii).

When δ is sufficiently large, the locus is connected: a repeated application of 3.17(ii) shows that $\mathscr{F}$ is the generization of a generalized line bundle $\mathscr{G}$ being not free only in P and such that $D|b_i(\mathscr{G})$ for any $1 \leq i \leq n-2$ and $b_{n-1}(\mathscr{G}) \equiv b_1 + \cdots + b_{n-1} \pmod{n}$. If δ is sufficiently large, $\mathscr{G}$ connects the above cited irreducible components where lies $\mathscr{F}$, and so $\mathscr{G}$, with $\bar{Z}_{0,\ldots,0,\overline{b_1+\cdots+b_{n-1}}}$ (to which $\mathscr{G}$ belongs by a repeated application of 3.15(ii)) where $\overline{b_1 + \cdots + b_{n-1}}$ is the representative between 0 and $n-1$ of the congruency class of $b_1 + \cdots + b_{n-1}$, i.e. of $D + (n(n-1)/2)\delta$.

The last assertion of the statement follows form the fact that, under this hypothesis, $b_1 + \cdots + b_{n-1}$ is a multiple of n and, hence, $\mathscr{F}$ is the specialization of a line bundle by Lemma 3.17(ii). *q.e.d.*

REMARK 3.20. The first part of the proof implies also that there could be at most $n^{(n-2)}$ connected components for fixed D, i.e. those containing the irreducible components $\bar{Z}_{b_1,\ldots,b_{n-1}}$, with $0 \leq b_i - b_{i-1} \leq n-1$, for $1 \leq i \leq n-1$, and such that $n|D - (n(n-1)/2)\delta - \sum_{i=1}^{n-1} b_i$ (when δ is small, some of this components do not exist, because their indices are too big to satisfy the stability inequalities). But many (maybe all) of them coincide, as in the case of $n|D-(n(n-1)/2)\delta$ or in that of $n = 3$ (cf. Theorem 3.8). The case $n = 4$, where the number of candidates is not excessive, is relatively easy to be treated by hand and the result is the following: for $\delta = 1$ or 2 the locus of generalized line bundle is certainly connected for D even (one has to use also Lemma 3.17(iv)) and it has at most 2 connected components for D odd, while for $\delta \geq 3$ it is always connected. In general the situation is not easy to handle directly and I do not know explicitly from which value of δ

the known deformations are sufficient to conclude the connection. However I think that there are other deformations implying that this locus is always connected, although I had not been able to find them until now.

We delay the whole conjectural picture about the irreducible components of the moduli space $\mathrm{M}(X, P_D)$ to the next chapter, see Conjecture 4.49.

3.2.2. Local geometry: Zariski tangent space. Here we will compute the dimension of the tangent space to the moduli space $\mathrm{M}(X, P_D)$, where X is a primitive multiple curve of multiplicity ≥ 4, in some of its closed points. The main result is the next proposition, which extends to higher multiplicity some of the results of Proposition 3.9. The method of proof is essentially the same.

PROPOSITION 3.21. *Let X be a primitive multiple curve of multiplicity $n \geq 4$ and let x be a point of $\mathrm{M}_s(X, P_D)$. If x corresponds to a stable generalized line bundle $\mathscr{F}$ of local indices sequence $b_{.,.}$ such that in each point P where $\mathscr{F}$ is not free there exists an integer $1 \leq h(P) \leq n-1$ such that $0 = b_{h(P)-1,P} < b_{h(P),P} = b_{n-1,P}$, then*

$$\dim T_x \mathrm{M}(X, P_D) = g_n + \sum_{j=1}^{r} \min\{h(P_j),\, n - h(P_j)\} b_{n-1,P_j}, \tag{3.4}$$

where $P_1, \ldots, P_r$ are the points of C where $\mathscr{F}$ is not locally free.

The following corollary generalizes to higher multiplicity Corollary 3.10.

COROLLARY 3.22. *The tangent space to a generic point of the irreducible component $\bar{Z}_{b_1,\ldots,b_{n-1}}$ has dimension*

$$g_n + \sum_{i=[\frac{n+1}{2}]}^{n-1} b_i - \sum_{i=1}^{[\frac{n-2}{2}]} b_i = g_n + b_{n-1} + \sum_{i=[\frac{n+1}{2}]}^{n-2} (b_i - b_{n-1-i}).$$

In particular, only the component of line bundles, i.e. $\bar{Z}_{0,\ldots,0}$, is generically reduced.

PROOF. The first assertion follows from formula (3.4) and Theorem 3.16, which describes the generic points of $\bar{Z}_{b_1,\ldots,b_{n-1}}$. The first assertion implies, in particular, that this generic dimension is always greater than or equal to $g_n + b_{n-1}$, by Lemma 2.18. Hence, recalling that, again by Theorem 3.16, each $\bar{Z}_{b_1,\ldots,b_{n-1}}$ has dimension g_n, the second assertion is a consequence of the first one and of Corollary 2.15. *q.e.d.*

As in the case of multiplicity 3, the Proposition is an immediate consequence of the well-known characterization of the Zariski tangent space to the moduli space in terms of the first Ext-group of a sheaf by itself and of a lemma computing explicitly the latter for a special type of generalized line bundles on C_n:

LEMMA 3.23. *(Cf. Lemma 3.11). If $\mathscr{F}$ is a generalized line bundle on $X = C_n$ with local indices sequence $b_{.,.}$ such that in each point P where*

$\mathscr{F}$ is not locally free there exists an integer $1 \leq h(P) \leq n-1$ such that $0 = b_{h(P)-1,P} < b_{h(P),P} = b_{n-1,P}$, then

$$\dim(\mathrm{Ext}^1(\mathscr{F},\mathscr{F})) = g_n + \tilde{b}_{n-1} + \mathrm{h}^0(X', \mathscr{O}_{X'}) - 1, \tag{3.5}$$

where $\tilde{b}_{n-1} = \sum_{j=1}^{r} \min\{h(P_j),\, n-h(P_j)\} b_{n-1,P_j}$, where $P_1, \dots, P_r$ are the points of C in which $\mathscr{F}$ is not free and X' is the blow up associated to $\mathscr{F}$ as in Corollary 2.38(iii)*.*

If, moreover, $\mathscr{F}$ is stable, then this formula simplifies to

$$\dim(\mathrm{Ext}^1(\mathscr{F},\mathscr{F})) = g_n + \tilde{b}_{n-1}. \tag{3.6}$$

PROOF. The proof is quite similar to that of Lemma 3.11.

The Ext-spectral sequence $H^p(X, \underline{\mathrm{Ext}}^q(\mathscr{F},\mathscr{F})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_X}(\mathscr{F},\mathscr{F})$ implies the existence of the following short exact sequence

$$0 \to H^1(X, \underline{\mathrm{End}}(\mathscr{F})) \to \mathrm{Ext}^1(\mathscr{F},\mathscr{F}) \to H^0(X, \underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F})) \to 0.$$

Hence, in order to get the result, it suffices to compute the dimensions of the two external terms. By Remark 2.39, we have $\underline{\mathrm{End}}(\mathscr{F}) \simeq q_*(\mathscr{O}_{X'})$, where $q : X' \to X$ is the blow up there studied. Therefore, $H^1(X, \underline{\mathrm{End}}(\mathscr{F})) = H^1(X', \mathscr{O}_{X'})$ and the latter has dimension $g(X') - \mathrm{h}^0(X', \mathscr{O}_{X'}) + 1 = g_n - \tilde{b}_{n-1} - \mathrm{h}^0(X', \mathscr{O}_{X'}) + 1$ (this formula is due to the definition of the blow up X'). By Lemma 3.14, if $\mathscr{F}$ is stable, then $\mathrm{h}^0(X', \mathscr{O}_{X'}) = 1$, justifying the difference between formulae (3.5) and (3.6).

It remains to calculate $\mathrm{h}^0(X, \underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F}))$. As in multiplicity 3, it is obvious that $\underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F})$ is supported on $P_1, \dots, P_r$ and that it can be decomposed as $\bigoplus_{j=1}^{r} \mathrm{Ext}^1(\mathscr{F}_{P_j}, \mathscr{F}_{P_j})$.

In the next lines we will show that $\dim(\mathrm{Ext}^1(\mathscr{F}_{P_j}, \mathscr{F}_{P_j})) = 2\tilde{b}_{n-1}$ and, thus, formulae (3.5) and (3.6) hold, as desired.

In order to do this computation, we will use local notation with $A = \mathscr{O}_{X,P_j}$ and $\mathscr{F}_{P_j}$ will be denoted by I, while $b_{n-1,I} = b$ and $h = h(P_j)$.

By Corollary 2.29, I is isomorphic to the ideal (x^b, y^h).

It has the following periodic free resolution:

$$\cdots \longrightarrow A^2 \xrightarrow{M_2} A^2 \xrightarrow{M_1} A^2 \xrightarrow{f} I \longrightarrow 0,$$

where

$$M_1 = \begin{pmatrix} y^{n-h} & -x^b - \alpha y \\ 0 & y^h \end{pmatrix}, M_2 = \begin{pmatrix} y^h & x^b + \alpha y \\ 0 & y^{n-h} \end{pmatrix} \text{ and } \begin{cases} f((1,0)) = y^h \\ f((0,1)) = x^b + \alpha y. \end{cases}$$

From the resolution one gets the complex

$$\cdots \longleftarrow \mathrm{Hom}(A^2, I) \xleftarrow{a_2} \mathrm{Hom}(A^2, I) \xleftarrow{a_1} \mathrm{Hom}(A^2, I),$$

where a_i is the homomorphism induced by multiplication by M_i, for $i = 1, 2$. By definition, $\mathrm{Ext}^1(I, I) = \ker(a_2)/\operatorname{im}(a_1)$. It holds that

$$\varphi \in \operatorname{im}(a_1) \iff \begin{cases} \varphi((1,0)) = \beta_1 y^{n-h}(x^b + \alpha y) \\ \varphi((0,1)) = -\beta_1 (x^b + \alpha y)^2 + \beta_2 y^h (x^b + \alpha y) + \beta_3 y^{2h}, \end{cases}$$

with $\beta_i \in A$, for $1 \leq i \leq 3$. In order to study $\ker(a_2)$ it is convenient to distinguish two cases: $n - h \leq h$ and $h < n - h$. In the first one, we have that

$$\psi \in \ker(a_2) \iff \begin{cases} \psi((1,0)) = \gamma_1(x^b + \alpha y)y^{n-h} + \gamma_2 y^h \\ \psi((0,1)) = \gamma_2(x^b + \alpha y)y^{2h-n} + \gamma_3 y^h, \end{cases}$$

with $\gamma_i \in A$, for $1 \leq i \leq 3$.
Otherwise, it holds that

$$\psi \in \ker(a_2) \iff \begin{cases} \psi((1,0)) = \gamma_1 y^{n-h} \\ \psi((0,1)) = (\gamma_2 - \gamma_1)(x^b + \alpha y) + \gamma_3 y^{2h}, \end{cases}$$

with $\gamma_i \in A$, for $1 \leq i \leq 3$. In both cases the length of $\mathrm{Ext}^1(I, I)$ is the desired one. *q.e.d.*

REMARK 3.24. The beginning of the proof, i.e. the existence of the short exact sequence $H^1(X, \underline{\mathrm{End}}(\mathscr{F})) \hookrightarrow \mathrm{Ext}^1(\mathscr{F}, \mathscr{F}) \twoheadrightarrow H^0(X, \underline{\mathrm{Ext}}^1(\mathscr{F}, \mathscr{F}))$ and also the identification of the right hand term with $\bigoplus_{j=1}^r \mathrm{Ext}^1(\mathscr{F}_{P_j}, \mathscr{F}_{P_j})$, is true for any generalized line bundle $\mathscr{F}$ on X. However, the interpretation of $\underline{\mathrm{End}}(\mathscr{F})$ in terms of an appropriate blow up is known only for those verifying the hypotheses of Lemma 2.37(i) and their duals, within which there are those studied in the above Lemma. These are particularly significant because within them there are the generic elements of the irreducible components of the moduli space containing stable generalized line bundles (cf. Theorem 3.16). Moreover, in this case the explicit calculation of the extensions of the stalks is not too hard, because they have only two local generators.

CHAPTER 4

Other components: partial results and open problems

This chapter is concerned with the study of sheaves on C_n which are not generalized line bundles, their semistability and their moduli space. There are various partial results and explanations of the open problems; in particular, we formulate and justify the two Conjectures 4.40 and 4.49. It is divided into three sections: the first one is about vector bundles on C, the second, which is the main part of this chapter studies sheaves on the ribbon C_2, the last one is quite short and treats higher multiplicity.

4.1. Vector bundles on the reduced subcurve

The moduli space of semistable vector bundles of rank n and degree D on a smooth projective curve C is well-known. We have to distinguish three cases. If C is rational, by a famous result which is usually attributed to Grothendieck (see, e.g., [**LP**, Lemma 4.4.1]), although it had already been proved (over $\mathbb{C}$) in the language of classical algebraic geometry by C. Segre at the end of the nineteenth century, any vector bundle of rank n decomposes in a direct sum of n line bundles, so there exist only strictly semistable vector bundles of rank n and their moduli space is a point, if D is a multiple of n, and, otherwise, it is empty. If C is elliptic, the situation changes depending on whether n and D are coprime or not: in the first case there are not stable vector bundles of rank n and degree D and the moduli space is isomorphic to the h-th symmetric product of C, where $h = \gcd(n, D)$; in the second one the moduli space of stable vector bundles of rank n and degree D is non-empty; hence, it is open and dense in that of semistable ones which is isomorphic to C (see, e.g., [**LP**, Thoerem 8.6.2 and §8.7]). Finally, if C has genus $g_1 \geq 2$, the moduli space of stable vector bundles of rank n and arbitrary degree d is non-empty; therefore, it is dense in that of semistable ones, which is irreducible and smooth of dimension $n^2(g_1 - 1) + 1$ (see, e.g., [**LP**, Theorems 8.3.2, 8.5.2 and 8.6.1]).

In each of these three cases, the moduli of semistable vector bundles of rank n and degree D over C is irreducible, if non-empty; thus, they can be the generic elements of at most one irreducible component of $\mathrm{M}(C_n, P_D)$, which is the whole space if $\delta = -\deg(\mathcal{C}) \leq 0$ (here we are using the notation introduced at the beginning of Chapter 3).

We can also compute the dimension of the tangent space to a stable vector bundle on C in $\mathrm{M}(C_n, P_D)$; the case $n = 2$ is the second part of [**CK**, Proposition 4.11].

Proposition 4.1. *Let C_n be a primitive multiple curve of multiplicity $n \geq 2$ and let x be a point of $\mathrm{M}_s(X, P_D)$ corresponding to a stable vector* *bundle $\mathcal{E}$ of rank n over C, then*

$$\dim T_x \mathrm{M}(X, P_D) = n^2(g_1 - 1) + 1 + \mathrm{h}^0(C, \underline{\mathrm{End}}(\mathcal{E}) \otimes \mathcal{C}^{-1}) \quad (4.1)$$

$$= n^2\delta + 1 \text{ if } \delta > \deg(\omega_C) \ (\iff \ g_2 > 4g_1 - 3). \quad (4.2)$$

As in the computation of the tangent space to generalized line bundles done in §§3.1.2 and 3.2.2, the Proposition follows from the well-known fact that the tangent space to a stable point in the moduli space equals the dimension of the group of extensions of the sheaf by itself and from the explicit computation of the latter. This computation is done in the following lemma, which generalizes to higher multiplicity [**CK**, Lemma 4.13]. The first assertion is essentially [**I**, Remark 2.7(iii)], from which the first part of the proof is taken, too.

Lemma 4.2. *Let $\mathcal{E}$ be a stable vector bundle of rank $n \geq 2$ over C. It holds that*

$$\dim(\mathrm{Ext}^1_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})) = n^2(g_1 - 1) + 1 + \mathrm{h}^0(C, \mathcal{C}^{-1} \otimes \underline{\mathrm{End}}(\mathcal{E})). \quad (4.3)$$

If, furthermore, $\delta = -\deg(\mathcal{C}) > 2g_1 - 2$, then this formula simplifies to

$$\dim(\mathrm{Ext}^1_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})) = n^2\delta + 1. \quad (4.4)$$

Proof. The Ext-spectral sequence $H^p(\underline{\mathrm{Ext}}^q_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})) \Rightarrow \mathrm{Ext}^{p+q}_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})$ implies that the following sequence is exact:

$$0 \to H^1(\underline{\mathrm{End}}(\mathcal{E})) \to \mathrm{Ext}^1_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E}) \to H^0(\underline{\mathrm{Ext}}^1_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})) \to 0.$$

It is well-known that $H^1(\underline{\mathrm{End}}(\mathcal{E})) = \mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{E}, \mathcal{E})$, being $\mathcal{E}$ a vector bundle on C. It holds also that $H^0(\underline{\mathrm{Ext}}^1_{\mathscr{O}_{C_n}}(\mathcal{E}, \mathcal{E})) \cong \mathrm{Hom}(\mathcal{C} \otimes \mathcal{E}, \mathcal{E})$ (it can be checked using, e.g., the locally free periodical resolution $\cdots \to \mathscr{C}^n \otimes \mathscr{E} \to \mathscr{C} \otimes \mathscr{E} \to \mathscr{E} \to \mathcal{E} \to 0$, where $\mathscr{E}$ is a vector bundle on C_n extending $\mathcal{E}$, cf. also the proof of [**D2**, Proposition 3.14]). Hence, formula (4.3) is implied by well-known properties of stable vector bundles over smooth projective curves and by the trivial identity $\mathrm{Hom}(\mathcal{C} \otimes \mathcal{E}, \mathcal{E}) = H^0(C, \mathcal{C}^{-1} \otimes \underline{\mathrm{End}}(\mathcal{E}))$.

Assume now $\delta = \deg(\mathcal{C}^{-1}) > 2g_1 - 2$. Consider $\mathcal{C}^{-1} \otimes \underline{\mathrm{End}}(\mathcal{E})$: it is a semistable vector bundle of rank n^2 and degree $n^2\delta$ on C, because $\mathcal{C}^{-1}$ is a line bundle and $\underline{\mathrm{End}}(\mathcal{E})$ is a semistable vector bundle of rank n^2 and degree 0. Hence, $\chi(C, \mathcal{C}^{-1} \otimes \underline{\mathrm{End}}(\mathcal{E})) = n^2(1 - g_1) + n^2\delta$. Furthermore, by Serre duality, $\mathrm{h}^1(C, \mathcal{C}^{-1} \otimes \underline{\mathrm{End}}(\mathcal{E})) = \mathrm{h}^0(C, \omega_C \otimes \mathcal{C} \otimes \underline{\mathrm{End}}(\mathcal{E}))$ and the latter vanishes, because $\omega_C \otimes \mathcal{C} \otimes \underline{\mathrm{End}}(\mathcal{E})$ is semistable of degree $n^2(2g_1 - 2) - n^2\delta < 0$, by hypothesis. Therefore, $\mathrm{h}^0(\omega_C \otimes \mathcal{C} \otimes \underline{\mathrm{End}}(\mathcal{E})) = \chi(\omega_C \otimes \mathcal{C} \otimes \underline{\mathrm{End}}(\mathcal{E}))$ and formula (4.4) holds. *q.e.d.*

4.2. Sheaves on a ribbon

This section studies various aspects of sheaves over a ribbon $X = C_2$. It contains mainly results about conditions for semistability, families and loci in the moduli space. Sometimes, also the tools used to investigate these questions have their own interest, as an extension of [**EG**, Theorem 1.1] from generalized line bundles to pure sheaves of type $(k, 1)$, for any non-negative integer k (see Proposition 4.32), which is useful to reduce the study of the loci of non quasi locally free sheaves of this type to that of quasi locally free ones of the same type on an appropriate blow up of X. There are also description of some open problems and conjectures.

Recall that, being X a ribbon, $\mathscr{N}^2 = 0$ so that $\mathscr{N}$ and $\mathcal{C}$ are equal. We will usually use $\mathscr{N}$ to denote it, also when interpreted as a line bundle on C.

First of all, it is useful to collect some facts and definitions which are specific of sheaves on ribbons.

Let us begin listing the following properties:

FACT 4.3. *Let X be a ribbon and let $\mathscr{F}$ be a sheaf on X.*

(i) *There is a canonical exact sequence:*

$$0 \to \mathscr{N}\mathscr{F} \to \mathscr{F}^{(1)} \to \mathscr{F}|_C \to \mathscr{N}\mathscr{F} \otimes \mathscr{N}^{-1} \to 0. \tag{4.5}$$

This exact sequence encodes the complete type $((r_0, r_1), (d_0, d_1))$ of $\mathscr{F}$, indeed $r_0 = \mathrm{rk}(\mathscr{F}|_C) = \mathrm{rk}(\mathscr{F}^{(1)})$ and $r_1 = \mathrm{rk}(\mathscr{N}\mathscr{F}) = \mathrm{rk}(\mathscr{N}\mathscr{F} \otimes \mathscr{N}^{-1})$, while $d_0 = \deg(\mathscr{F}|_C) = \deg(\mathscr{F}^{(1)}) + r_1 \deg(\mathscr{N})$ and $d_1 = \deg(\mathscr{N}\mathscr{F}) = \deg(\mathscr{N}\mathscr{F} \otimes \mathscr{N}^{-1}) + r_1 \deg(\mathscr{N})$ (see Definition 1.6 and Remark 1.10, which we make explicit in the easy case of multiplicity 2).

(ii) *(See [**D5**, §3.2.4]) $\mathscr{F}^{(1)}$ is the greatest subsheaf of $\mathscr{F}$ defined on C, meaning that a subsheaf $\mathcal{F} \subset \mathscr{F}$ is defined on C if and only if $\mathcal{F} \subseteq \mathscr{F}^{(1)}$.*

On the other hand, $\mathscr{F}/\mathcal{F}$ is a sheaf on C if and only if $\mathscr{N}\mathscr{F} \subseteq \mathcal{F}$.

Moreover, there is a canonical morphism $\mathscr{F}/\mathcal{F} \otimes \mathscr{N} \to \mathcal{F}$, which is surjective if and only if $\mathcal{F} = \mathscr{N}\mathscr{F}$, while it is injective if and only if $\mathcal{F} = \mathscr{F}^{(1)}$.

(iii) *(See [**D5**, §3.4]) Let $\mathcal{F}$ be a sheaf on C and $\mathcal{E}$ a vector bundle on C; there exists the following canonical exact sequence:*

$$0 \to \mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{E}, \mathcal{F}) \to \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F}) \xrightarrow{\pi} \mathrm{Hom}(\mathcal{E} \otimes \mathscr{N}, \mathcal{F}) \to 0.$$

By the previous point, if $\mathscr{F}$ is a sheaf on X which sits in a short exact sequence $0 \to \mathcal{F} \to \mathscr{F} \to \mathcal{E} \to 0$ represented by $\sigma \in \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{E}, \mathcal{F})$, then $\mathcal{F} = \mathscr{N}\mathscr{F}$ if and only if $\pi(\sigma)$ is surjective, while $\mathcal{F} = \mathscr{F}^{(1)}$ if and only if $\pi(\sigma)$ is injective.

The following definition is that of the *index* for any pure sheaf on a ribbon.

DEFINITION 4.4. Let $\mathscr{F}$ be a pure sheaf on X. The *index* of $\mathscr{F}$ is $b(\mathscr{F}) = \mathrm{h}^0(\mathcal{T}(\mathscr{F}))$, where $\mathcal{T}(\mathscr{F}) = \mathcal{T}$ is the torsion sheaf of $\mathscr{F}|_C$ (i.e. $\mathcal{T}$ is the kernel of $\mathscr{F}_C \twoheadrightarrow (\mathscr{F}|_C)^{\vee\vee}$). For any closed point P, the *local index* of $\mathscr{F}$ at P, denoted by $b_P(\mathscr{F})$, is the length of $\mathcal{T}_P$ as an $\mathscr{O}_{C,P}$-module. The *local index sequence* of $\mathscr{F}$, denoted by $b_.(\mathscr{F})$, is the collection $\{b_P(\mathscr{F}) : P \in \mathrm{Supp}(\mathcal{T})\}$.

REMARK 4.5. Let $\mathscr{F}$ be a pure sheaf on X. By definition, $b(\mathscr{F})$ is a non-negative integer which vanishes if and only if $\mathscr{F}$ is quasi locally free, by Fact 1.24(i).

The definition of index is due to Drézet (see cite[§6.3.7]DR1), while those of local index and of local index sequence are inspired by [**CK**, Definition 2.7])

A relevant fact about non quasi locally free sheaves on a ribbon is the following:

FACT 4.6. *(See* [**D1**, Lemme 6.3.4 and Corollaire 6.4.2]*) Let $\mathscr{F}$ be a pure sheaf on X and let $\mathcal{T}$ be the torsion part of $\mathscr{F}|_C$. There exist two quasi locally free sheaves $\mathscr{E}$ and $\mathscr{G}$ on X (generically isomorphic to $\mathscr{F}$ and not necessarily unique) such that the following exact sequences are exact:*

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathcal{T} \longrightarrow 0,$$
$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathcal{T} \longrightarrow 0.$$

Moreover, for any such $\mathscr{E}$ and $\mathscr{G}$, it holds that $\mathscr{N}\mathscr{E} = \mathscr{N}\mathscr{F}$, while $\mathscr{G}^{(1)} = \mathscr{F}^{(1)}$.

A kind of sheaves that seems particularly significant for the study of the moduli space is the following generalization of generalized line bundle:

DEFINITION 4.7. A *generalized vector bundle $\mathscr{F}$* is a pure sheaf on X such that $\mathscr{F}_\eta$ is a free $\mathscr{O}_{X,\eta}$-module of finite rank, where η is the generic point of X.
This is equivalent to requiring that $\mathrm{rk}(\mathscr{N}\mathscr{F}) = \mathrm{rk}(\mathscr{F}|_C)$, or, in other words, that the complete type of $\mathscr{F}$ is $((r, r), (d_0, d_1))$, with r a positive integer.

A generalized vector bundle being quasi locally free (or, equivalently, with index 0) is a vector bundle.

This definition is new, although deeply inspired by that of generalized line bundle, which is the case of generalized rank 2.

This kind of sheaves seems significant for the following conjecture:

CONJECTURE 4.8. *Let $\mathscr{F}$ be a pure sheaf on X which is not a generalized vector bundle. Then $\mathscr{F}$ generizes to a quasi locally free sheaf.*

If the conjecture holds, then it will follow that the only kind of sheaves on a ribbon that can be generic elements of an irreducible component of the moduli space are quasi locally free sheaves and generalized vector bundles.

The first observation justifying the conjecture is that there exist quasi locally free sheaves for any complete type $((r_0, r_1), (d_0, d_1))$ such that $r_0 > r_1$ (with $r_1 \geq 0$; on the other hand, this is not the case if $r_0 = r_1$, i.e. if $\mathscr{F}$ is a generalized vector bundle, because then quasi locally free is equivalent to locally free and then d_1 has to be equal to $d_0 + r_0 \deg(\mathscr{N})$). This fact (which is Corollary 4.10) is relevant for the conjecture because the irreducible components of the moduli stack of Higgs bundles are the closures of the loci of fixed complete type (cf. [**Bo**], although we need to translate his language into ours, as we will do in Appendix A); hence, the irreducible components of the moduli space of sheaves on ribbons involved in the spectral correspondence (as subschemes of the spectral cover) have to be the closures of the loci of fixed complete type; it seems reasonable that the same holds also for other ribbons (although not for any ribbon, see Conjecture 4.39). If it were the case, the above conjecture would have to hold by Corollary 4.10 and by the fact that being quasi locally free is an open condition for sheaves of fixed type on a ribbon (see Proposition 4.13 below).

Before saying more about the previous conjecture, we state and prove Proposition 4.9, which implies the already cited Corollary 4.10. It extends [**D4**, Proposition 3.4.1] from quasi locally free sheaves of rigid type to all quasi locally free sheaves, in the case of ribbons (the cited result is about primitive multiple curves of any multiplicity). The method of proof is inspired by [**D4**, §3.2].

PROPOSITION 4.9. *Let* (r_0, r_1) *be a pair of positive integers with* $r_0 > r_1$ *and let*

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{e} \mathcal{G} \xrightarrow{g} \mathcal{F} \otimes \mathscr{N}^{-1} \to 0 \qquad (*)$$

be an exact sequence of vector bundles on C*, with* $\mathrm{rk}(\mathcal{F}) = r_1$ *and* $\mathrm{rk}(\mathcal{G}) = r_0$*. Then there exists a quasi locally free sheaf* $\mathscr{F}$ *on* X *such that its associated canonical exact sequence* (4.5) *is isomorphic to* $(*)$*.*

PROOF. In this proof we use the same notation of Fact 4.3(iii). Let $\mathscr{F}$ be a sheaf on X corresponding to an element $\sigma_{\mathscr{F}} \in \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{G}, \mathcal{F})$ such that $\pi(\sigma_{\mathscr{F}}) = g \otimes \mathrm{id}_{\mathscr{N}}$. Hence, $\mathscr{N}\mathscr{F} = \mathcal{F}$ and $\mathscr{F}|_C = \mathcal{G}$, by the surjectivity of g and by Fact 4.3(iii). Moreover, by Fact 1.9(i), it holds that $\mathscr{F}/\mathscr{F}^{(1)} = \mathcal{F} \otimes \mathscr{N}^{-1}$ and by Fact 1.24(i) such an $\mathscr{F}$ is quasi locally free.

For all these sheaves it is also fixed $\mathcal{K} = \ker(\mathscr{F}|_C \to \mathscr{F}/\mathscr{F}^{(1)}) = \ker(g) = \mathrm{im}(e)$, which is also equal to $\mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}$ (see Fact 4.3(i)). Therefore, $\mathscr{F}^{(1)}$ is represented by an element $\sigma'_{\mathscr{F}} \in \mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{K}, \mathcal{F})$. Thus, we need $\sigma'_{\mathscr{F}} = \sigma_{\mathcal{E}}$, where $\sigma_{\mathcal{E}}$ is the element in $\mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{K}, \mathcal{F})$ associated to the short exact sequence $0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{e} \mathcal{K} \to 0$.

The following diagram is commutative with exact rows:

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{G}, \mathcal{F}) & \hookrightarrow & \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{G}, \mathcal{F}) & \xrightarrow{\pi} & \mathrm{Hom}(\mathcal{G} \otimes \mathscr{N}, \mathcal{F}) \\ \downarrow & & \downarrow p & & \downarrow \\ \mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{K}, \mathcal{F}) & \hookrightarrow & \mathrm{Ext}^1_{\mathscr{O}_X}(\mathcal{K}, \mathcal{F}) & \twoheadrightarrow & \mathrm{Hom}(\mathcal{K} \otimes \mathscr{N}, \mathcal{F}) \end{array}$$

By definition of $\mathcal{K}$, $p(\sigma_{\mathscr{F}})$ belongs to $\mathrm{Ext}^1_{\mathscr{O}_C}(\mathcal{K}, \mathcal{F})$ for any $\mathscr{F}$ as above (because $\pi(\sigma_{\mathscr{F}}) = g \otimes \mathrm{id}_{\mathscr{N}}$). Moreover, by Fact 4.3(ii), it holds that $p(\sigma_{\mathscr{F}}) = \sigma'_{\mathscr{F}}$.

Hence, there exists an $\mathscr{F}$ such that $\sigma'_{\mathscr{F}} = \sigma_{\mathcal{E}}$, by the surjectivity of the first vertical arrow of the commutative diagram (this surjectivity can be easily checked looking at the long exact sequence of Ext's on C associated to the short exact sequence $0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{F} \otimes \mathscr{N}^{-1} \to 0$ and remembering that both $\mathcal{F}$ and $\mathcal{F} \otimes \mathscr{N}^{-1}$ are locally free on C so that $\mathrm{Ext}^2_{\mathscr{O}_C}(\mathcal{F} \otimes \mathscr{N}^{-1}, \mathcal{F}) = 0$). *q.e.d.*

The following corollary is an immediate consequence of the above Proposition.

COROLLARY 4.10. *For any pair of positive integers* $r_0 > r_1$ *and any pair of integers* (d_0, d_1)*, there exists a quasi locally free sheaf on* X *of complete type* $((r_0, r_1), (d_0, d_1))$*.*

Coming back to Conjecture 4.8, a possible strategy of demonstration is the following: let $\mathscr{F}$ be a pure sheaf on X of complete type $((r_0, r_1), (d_0, d_1))$ with $r_0 > r_1$. If $\mathscr{F}$ is quasi locally free, there is nothing to prove; so, in particular, we can assume $r_1 > 0$ and $\mathscr{F}$ of index $b > 0$. Look at $\mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}$, which is a sheaf defined on C with a locally free part of rank $r_0 - r_1$ and a torsion part of length b. By Proposition 4.11 below, there is a flat family of sheaves on C with a fibre isomorphic to $\mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}$ and the generic fibre locally free. An idea for the proof of the Conjecture is the following: first of all, one could try to obtain from this family a flat family of short exact sequences of sheaves on C with a fibre isomorphic to the exact sequence $\mathscr{N}\mathscr{F} \hookrightarrow \mathscr{F}^{(1)} \twoheadrightarrow \mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}$ and with generic fibre such that all the terms were vector bundles on C. Then we would have to get from it another flat family of short exact sequences, this time of sheaves on X, with a fibre isomorphic to $\mathscr{F}^{(1)} \hookrightarrow \mathscr{F} \twoheadrightarrow \mathscr{F}/\mathscr{F}^{(1)}$ and the generic fibre with central term quasi locally free.

A possible method to do that could be using the relative ext sheaf of the first family by the constant family over the same base with fibres isomorphic to $\mathscr{N}\mathscr{F}$; if there were an universal family of extensions attached to this relative ext sheaf (it is true, e.g., if this relative ext sheaf commutes with base change, see [**Lan**]), then the universal family would be an appropriate family of short exact sequences on C. After that one should repeat a similar argument with the relative ext sheaf of the constant family with fibre isomorphic to $\mathscr{F}/\mathscr{F}^{(1)}$ by the previous family and then using the universal family of extensions (which exists if this second relative ext commutes with base change, see again [**Lan**]). Until now I was able neither to control these relative ext sheaves nor to check if they commute with base change.

It is time to state and prove the previously cited result about sheaves with torsion on C:

PROPOSITION 4.11. *Let $\mathcal{G}$ be a sheaf of degree d and rank $r > 0$ on a smooth projective curve C over an algebraically closed field. Then there exists a flat family of sheaves on C with a fibre isomorphic to $\mathcal{G}$ and generic fibre a vector bundle of the same rank.*

PROOF. Let $\mathcal{G} = \mathcal{E} \oplus \mathcal{T}$ with $\mathcal{E}$ a vector bundle of rank r and $\mathcal{T}$ a torsion sheaf. Assume $\mathcal{T}$ is generated by s global sections. It holds that $\mathcal{E}(m)$ is generated by global sections for any $m >> 0$. Hence, $\mathscr{O}_C^{\oplus r+1} \oplus \mathscr{O}_C^{\oplus s} \twoheadrightarrow \mathcal{E}(m) \oplus \mathcal{T}$. For m sufficiently large, $\mathrm{Quot}_{r,d}(\mathscr{O}_C^{\oplus r+s+1})$, i.e. the Quot scheme parametrizing quotients of $\mathscr{O}_C^{\oplus r+s+1}$ of rank r and degree d, is irreducible and the generic quotient is a vector bundle, by [**PR**, Theorems 6.2 and 6.4]. Hence, the universal quotient family of $\mathrm{Quot}_{r,d}(\mathscr{O}_C^{\oplus r+s+1})$ twisted by $-m$ is a family with the desired properties. *q.e.d.*

REMARK 4.12. I have not found a reference for the above Proposition but it is so elementary, both as statement and as proof, that I would be surprised if it were really new.

There is also another possible strategy to deal with Conjecture 4.8, as suggested before Proposition 4.9. Indeed, one could try to show that the locus of pure sheaves of fixed complete type is irreducible (i.e. to prove

Conjecture 4.42 at least in the case of ribbons). If this holds, then the conjecture follows from the fact that for any complete type with $r_0 > r_1$ there exists a quasi locally free sheaf by Corollary 4.10) and from the following proposition:

PROPOSITION 4.13. *Let Z be a $\mathbb{K}$-scheme and let $\mathfrak{F}$ be a family of sheaves on X of fixed type (m_1, m_2) parametrized by Z. Then the set of closed points $z \in Z$ where $\mathfrak{F}_z$ is quasi locally free is open.*

PROOF. Let $z_0 \in Z$ be a point where $\mathfrak{F}_z$ is quasi locally free on X. If such a z_0 exists, then the set of points $(z, P) \in Z \times_{\mathbb{K}} X$ such that there exists a surjective morphism $\mathscr{O}_{X,P}^{\oplus r_0} \twoheadrightarrow \mathfrak{F}_{z,P}$, with $r_0 := m_1 + m_2$, is non-empty. For any such (z, P) there exists a neighbourhood $U \subset Z \times_{\mathbb{K}} X$ such that $\mathscr{O}_U^{\oplus r_0} \twoheadrightarrow \mathfrak{F}|_U$. Hence, there exists an open $W \subset Z \times_{\mathbb{K}} X$ such that for any $(z, P) \in W$ there is an epimorphism $\mathscr{O}_{C,P}^{\oplus r_0} \twoheadrightarrow M/(y_p M)$, where $M = \mathfrak{F}_{z,P}$ and $0 \subset y_p M \subset M$ is its first canonical filtration (the surjective morphism is induced restricting to C the previous one). By the fact the family is of sheaves of fixed type, $M/(y_p M)$ has to be of the form $\mathscr{O}_{C,P}^{\oplus r_0} \oplus N$, where N is a torsion module. Therefore, it follows that the epimorphism is an isomorphism, i.e. that $\mathscr{O}_{C,P}^{\oplus r_0} \cong M/(y_p M)$. This implies that $\mathfrak{F}_{z,P}$ is quasi free of type (m_1, m_2).

If T denotes the projection of $(Z \times_{\mathbb{K}} X) \setminus W$ in Z, then the desired open is $Z \setminus T$. *q.e.d.*

REMARK 4.14. The hypothesis that the sheaves in the family are of fixed type cannot be removed, at least in general (it is not necessary only in some special cases as that of quasi locally free sheaves of rigid type, cf. [**D2**, Proposition 6.9]). An example in which without this hypothesis the Proposition would fail is given by those families that deform rank 2 vector bundles over C to generalized line bundles over C_2 (see [**D1**, Théorème 7.2.3] and [**Sa**, Theorem 1]).

Assuming Conjecture 4.8, the only pure sheaves that are really relevant for determining the irreducible components of the moduli space of semistable sheaves of fixed generalized rank and degree over a ribbon are quasi locally free sheaves and generalized vector bundles. Before stating and proving our results, it can be useful to recall what has already been proved by Drézet about loci of quasi locally free sheaves of rigid type on primitive multiple curves of any multiplicity. The first point of the following fact is an adaptation of [**D2**, Proposition 6.12] and [**D4**, Théorème 5.3.3] to the case of ribbons, while the second one is essentially [**D4**, §5.2.2].

FACT 4.15.

(i) *Let a be a positive integer and let d_0 and d_1 be two integers and let $\mathrm{N}(a, d_0, d_1) \subset \mathrm{M}_s(X, 2a+1, d_0+d_1)$ be the locus of stable quasi locally free sheaves of rigid type $\mathscr{F}$ of complete type $((a+1, a), (d_0, d_1))$, where $\mathrm{M}_s(X, 2a+1, d_0+d_1)$ is the moduli space of stable sheaves on X of generalized rank $2a+1$ and generalized degree d_0+d_1. The locus $\mathrm{N}(a, d_0,\, d_1)$ is open and irreducible. If it is non-empty, it has dimension $1+(a^2+a)\delta+(2a^2+2a+1)(g_1-1)$,* *where as usual $\delta = -\deg(\mathscr{N})$ and g_1 is the genus of the reduced subcurve C.*

If $g_1 \geq 2$, then it is non-empty if $d_0/(a+1) - \delta < d_1/a < (d_0 - a\delta)/(a+1)$.

(ii) *The locus of stable vector bundle of rank r (i.e. generalized rank $2r$) and generalized degree D is non-empty if and only if $D = 2d - r\delta$ for some integer d. In this case, it is a smooth irreducible open of $\mathrm{M}_s(X, 2r, D)$ of dimension $1+r^2\delta+(2r^2)(g_1-1) = 1+r^2(g_2-1)$.*

Remark 4.16. In the cited articles the condition about the genus is not explicitly stated, but it is needed because non-emptiness is proved applying the so-called Lange's conjecture on C, that is about the existence of exact sequences of (semi)stable vector bundles on smooth projective curves of genus greater than or equal to 2 (see [**Ba**] or [**RT**] for details).

Our inequality could seem different, at a first glance, from the original one because the latter is in terms of $\deg(\mathscr{F}|_C)$ and $\deg(\mathscr{F}/\mathscr{F}^{(1)})$, while ours is in terms of $d_0 = \deg(\mathscr{F}|_C)$ and $d_1 = \deg(\mathscr{N}\mathscr{F}) = \deg(\mathscr{F}/\mathscr{F}^{(1)}) - a\delta$ (and in Drézet's notation our δ is $-\deg(L)$).

Our next aim is to improve the inequality in the first assertion of the fact and to extend it to quasi locally free sheaves of any type giving necessary and sufficient conditions for the existence of semistable quasi locally free sheaves of a fixed complete type over a ribbon.

Before doing that, it is useful to state the following easy lemma, which is essentially [**D4**, Lemme 5.1.1] with a small improvement.

Lemma 4.17. *Let $\mathscr{F}$, $\mathscr{F}'$, $\mathscr{G}$, $\mathscr{G}'$, $\mathscr{H}$ and $\mathscr{H}'$ be sheaves of positive generalized rank on a primitive multiple curve C_n such that*

$$\mathrm{R}(\mathscr{F}) = \mathrm{R}(\mathscr{H}) + \mathrm{R}(\mathscr{G}),\ \mathrm{R}(\mathscr{F}') = \mathrm{R}(\mathscr{H}') + \mathrm{R}(\mathscr{G}'),$$

$$\mathrm{Deg}(\mathscr{F}) = \mathrm{Deg}(\mathscr{H}) + \mathrm{Deg}(\mathscr{G}),\ \mathrm{Deg}(\mathscr{F}') = \mathrm{Deg}(\mathscr{H}') + \mathrm{Deg}(\mathscr{G}').$$

Assume that $\mu(\mathscr{G}) \geq \mu(\mathscr{H})$ or $\mu(\mathscr{G}') \geq \mu(\mathscr{H}')$ and that $\mu(\mathscr{H}') \geq \mu(\mathscr{H})$, $\mu(\mathscr{G}') \geq \mu(\mathscr{G})$ and $\mathrm{R}(\mathscr{F}')/\mathrm{R}(\mathscr{F}) \geq \mathrm{R}(\mathscr{H}')/\mathrm{R}(\mathscr{H})$. Then it holds that $\mu(\mathscr{F}') \geq \mu(\mathscr{F})$.
If, moreover, one of the above inequalities on the slopes is strict, then $\mu(\mathscr{F}') > \mu(\mathscr{F})$.

Proof. The case $\mu(\mathscr{G}) \geq \mu(\mathscr{H})$ is [**D4**, Lemme 5.1.1]. The proof of the case $\mu(\mathscr{G}') \geq \mu(\mathscr{H}')$ is almost identical to that of the cited result, so we give only a sketch of it.

Under our hypotheses, $\mu(\mathscr{F}')-\mu(\mathscr{F}) \geq (\mathrm{R}(\mathscr{F})\,\mathrm{R}(\mathscr{F}'))^{-1}(\mathrm{R}(\mathscr{G}')\,\mathrm{R}(\mathscr{H}) - \mathrm{R}(\mathscr{G})\,\mathrm{R}(\mathscr{H}'))(\mu(\mathscr{G}')-\mu(\mathscr{H}')) \geq 0$. The last inequality is due to the fact that, in our case, $\mathrm{R}(\mathscr{F}')/\mathrm{R}(\mathscr{F}) \geq \mathrm{R}(\mathscr{H}')/\mathrm{R}(\mathscr{H})$ is equivalent to $\mathrm{R}(\mathscr{G}')/\mathrm{R}(\mathscr{G}) \geq \mathrm{R}(\mathscr{H}')/\mathrm{R}(\mathscr{H})$.

The last assertion of the statement holds because the first inequality is strict if $\mu(\mathscr{H}') > \mu(\mathscr{H})$ or $\mu(\mathscr{G}') > \mu(\mathscr{G})$ while the second is strict if $\mu(\mathscr{G}') > \mu(\mathscr{H}')$. *q.e.d.*

Remark 4.18. The above Lemma holds in a much more wider context than that of primitive multiple curves. Indeed, in the proof we use only the relations within the various numbers involved and the fact that the generalized ranks are positive. So, it could be stated as a result about real numbers verifying the hypotheses. It is quite elementary, but it is useful to check semistability, in the original form looking at quotients and in the modified one looking at subsheaves.

Now, we can turn our attention to semistability conditions.

THEOREM 4.19. *Let X be a ribbon such that $g_1 \geq 2$. There exists a semistable quasi locally free sheaf $\mathscr{F}$ on X of complete type $((r_0, r_1), (d_0, d_1))$, with $r_0 > r_1$, if and only if*

$$\frac{d_0 - (r_0 + r_1)\delta}{r_0} \leq \frac{d_1}{r_1} \leq \frac{d_0}{r_0}, \tag{4.6}$$

where, as usual, $\delta = -\deg(\mathscr{N})$.

There exists a stable sheaf as above if and only if the inequalities are strict.

PROOF. The necessity is quite trivial, for both semistability and stability. Indeed, if $\mathscr{F}$ is semistable, then $\mu(\mathscr{F}^{(1)}) \leq \mu(\mathscr{F}) \leq \mu(\mathscr{F}|_C)$ and this inequalities are equivalent to (4.6), because $\mu(\mathscr{F}^{(1)}) = (d_0 - a\delta)/r_0$, while $\mu(\mathscr{F}|_C) = d_0/r_0$ and $\mu(\mathscr{F}) = (d_0 + d_1)/(r_0 + r_1)$, by definition. In the stable case both the inequalities are strict.

In order to prove the sufficiency part we want to make use of Proposition 4.9. So, we need to find an appropriate exact sequence $0 \to \mathcal{F} \xrightarrow{f} \mathcal{E} \xrightarrow{e} \mathcal{G} \xrightarrow{g} \mathcal{F} \otimes \mathscr{N}^{-1} \to 0$ of vector bundles on C, with $\mathcal{F}$ of rank r_1 and degree d_1 and $\mathcal{G}$ of rank r_0 and degree d_0, such that an associated quasi locally free sheaf $\mathscr{F}$ on C_2 is (semi)stable (recall that $\mathcal{F} = \mathscr{N}\mathscr{F}$, $\mathcal{E} = \mathscr{F}^{(1)}$ and $\mathcal{G} = \mathscr{F}|_C$).

We can always work with a stable vector bundle $\mathcal{F}$ and we can also assume that $\mathcal{K} = \ker(g) = \operatorname{coker}(f)$ is a stable vector bundle of rank $r_0 - r_1$ and degree $d_0 - d_1 - r_1\delta$.

It is useful to distinguish the three following cases:

(i) $d_0/r_0 - \delta < d_1/r_1 < (d_0 - r_1\delta)/r_0$; in this case both $\mathcal{E}$ and $\mathcal{G}$ can be stable, because the right inequality is $\mu(\mathcal{F}) < \mu(\mathcal{E})$ and the left is equivalent to $\mu(\mathcal{G}) < \mu(\mathcal{F} \otimes \mathscr{N}^{-1})$;

(ii) $(d_0 - r_1\delta)/r_0 \leq d_1/r_1 \leq d_0/r_0$; this time only $\mathcal{G}$ can be stable while $\mathcal{E}$ is surely unstable and it can be strictly semistable only if the left inequality is an equality.

(iii) $[d_0 - (r_0 + r_1)\delta]/r_0 \leq d_1/r_1 \leq d_0/r_0 - \delta$; in this case only $\mathcal{E}$ can be stable while $\mathcal{G}$ is surely unstable and it can be strictly semistable only if the right inequality holds as an equality.

If $\mathscr{F}$ verifies the hypotheses of (ii), then its dual $\mathscr{F}^\vee$ verifies that of (iii); hence, by Lemma 1.29, it is sufficient to handle only one of the two cases.

Let us start with case (i). In this case, the numerical data allow to assume that both $\mathcal{G} = \mathscr{F}|_C$ and $\mathcal{E} = \mathscr{F}^{(1)}$ are stable and this is really possible thanks to Lange's conjecture.

Let $\mathscr{G} \subset \mathscr{F}$ be a saturated subsheaf. If $\mathscr{G} \subset \mathscr{F}^{(1)}$ or $\mathscr{F}|_C \twoheadrightarrow (\mathscr{F}/\mathscr{G})$, we have done by hypothesis. So, assume that nor $\mathscr{G}$ neither $\mathscr{F}/\mathscr{G}$ are defined on C.

In this case, we have, by Fact 1.9(iv)(b), that $0 \subsetneq \mathscr{G}^{(1)} \subseteq \mathscr{F}^{(1)}$ and $0 \subsetneq \mathscr{G}/\mathscr{G}^{(1)} \subseteq \mathscr{F}/\mathscr{F}^{(1)}$; hence, $\mu(\mathscr{G}^{(1)}) \leq \mu(\mathscr{F}^{(1)})$ and $\mu(\mathscr{G}/\mathscr{G}^{(1)}) \leq \mu(\mathscr{F}/\mathscr{F}^{(1)})$. We can conclude that $\mu(\mathscr{G}) < \mu(\mathscr{F})$ by Lemma 4.17 if we have that $(r_0 + r_1)/(r_0(\mathscr{G}) + r_1(\mathscr{G})) \geq r_0/r_0(\mathscr{G})$ (where $r_0(\mathscr{G})$ is the rank of $\mathscr{G}^{(1)}$ and $r_1(\mathscr{G})$ is the rank of $\mathscr{G}/\mathscr{G}^{(1)}$); this condition is equivalent to $r_0(\mathscr{G})r_1 \geq r_0 r_1(\mathscr{G})$. We can cover the remaining cases looking at $\mathscr{F}/\mathscr{G}$: indeed, it holds that, by Fact 1.9(iv)(a), $\mathscr{N}\mathscr{F} \twoheadrightarrow \mathscr{N}(\mathscr{F}/\mathscr{G})$ and $\mathscr{F}|_C \twoheadrightarrow (\mathscr{F}/\mathscr{G})|_C$. Moreover, under our hypothesis about $\mathscr{F}/\mathscr{G}$, $\mathrm{rk}(\mathscr{N}(\mathscr{F}/\mathscr{G})) = r_1(\mathscr{F}/\mathscr{G}) > 0$ and $\mathrm{rk}((\mathscr{F}/\mathscr{G})|_C) = r_0(\mathscr{F}/\mathscr{G}) > 0$; thus, $\mu(\mathscr{N}\mathscr{F}) \leq \mu(\mathscr{N}(\mathscr{F}/\mathscr{G}))$ and $\mu(\mathscr{F}|_C) \leq \mu((\mathscr{F}/\mathscr{G})|_C)$. Therefore, we can conclude that $\mu(\mathscr{F}) < \mu(\mathscr{F}/\mathscr{G})$, again by Lemma 4.17, if it holds that $(r_0(\mathscr{F}/\mathscr{G}) + r_1(\mathscr{F}/\mathscr{G}))/(r_0 + r_1) \geq r_1(\mathscr{F}/\mathscr{G})/r_1$, equivalently if $r_1 r_0(\mathscr{F}/\mathscr{G}) \geq r_0 r_1(\mathscr{F}/\mathscr{G})$. The last inequality is implied by $r_1(r_0 - r_0(\mathscr{G})) \geq r_0(r_1 - r_1(\mathscr{G}))$, which is equivalent to $r_0(\mathscr{G})r_1 \leq r_0 r_1(\mathscr{G})$.

We can turn our attention to case (iii). In this case we assume $\mathcal{E}$ stable (it is possible by Lange's conjecture), while we choose $\mathcal{G} = \mathcal{K} \oplus (\mathcal{F} \otimes \mathscr{N}^{-1})$. Let $\mathscr{G}$ be a saturated subsheaf of $\mathscr{F}$. If $\mathscr{G} \subset \mathscr{F}^{(1)} = \mathcal{E}$, we have done by hypothesis (and it is possible that $\mu(\mathscr{G}) = \mu(\mathscr{F})$ only if $\mathscr{G} = \mathcal{E}$ and $\mu(\mathcal{E}) = \mu(\mathscr{F})$, i.e. if $[d_0 - (r_0 + r_1)\delta]/r_0 = d_1/r_1$). If $\mathscr{F}/\mathscr{G}$ is defined on C, we have also done, because in this case $\mu(\mathscr{F}/\mathscr{G}) \geq \mu(\mathscr{F}/\mathscr{F}^{(1)}) = \mu(\mathcal{F} \otimes \mathscr{N}^{-1}) \geq \mu(\mathscr{F})$ (this time, the equalities are equivalent to $\mathscr{F}/\mathscr{G} = \mathscr{F}/\mathscr{F}^{(1)}$ and $[d_0 - (r_0 + r_1)\delta]/r_0 = d_1/r_1$; the first inequality is due to the fact $\mathcal{G} = \mathscr{F}/\mathscr{F}^{(1)} \oplus \mathcal{K}$, with both the addends stable and $\mu(\mathscr{F}/\mathscr{F}^{(1)}) \leq \mu(\mathcal{K})$ (with the equality if and only if $d_1/r_1 = d_0/r_0 - \delta$).

Therefore, the only case that remains to handle is that of $\mathscr{G} \subset \mathscr{F}$ such that both $\mathscr{G}$ and $\mathscr{F}/\mathscr{G}$ are not defined on C. In this case, by Fact 1.9(iv)(b), $0 \neq \mathscr{G}^{(1)} \subset \mathscr{F}^{(1)}$ and $0 \neq \mathscr{G}/\mathscr{G}^{(1)} \subset \mathscr{F}/\mathscr{F}^{(1)}$; so, by the stability of $\mathscr{F}^{(1)}$ and of $\mathscr{F}/\mathscr{F}^{(1)}$, it holds that $\mu(\mathscr{G}^{(1)}) \leq \mu(\mathscr{F}^{(1)})$ and $\mu(\mathscr{G}/\mathscr{G}^{(1)}) \leq \mu(\mathscr{F}/\mathscr{F}^{(1)})$ (with the equalities if and only if the sheaves are equal, and this cannot happen for both the sheaves at the same time). As above, we can conclude that $\mu(\mathscr{G}) < \mu(\mathscr{F})$ by Lemma 4.17 if we have $(r_0 + r_1)/(r_0(\mathscr{G}) + r_1(\mathscr{G})) \geq r_0/r_0(\mathscr{G})$ or, equivalently, $r_0(\mathscr{G})r_1 \geq r_0 r_1(\mathscr{G})$. Thus, only the case in which $0 < r_0(\mathscr{G})r_1 < r_0 r_1(\mathscr{G})$ remains open.

The following diagram is commutative:

$$\begin{array}{ccccc}
\mathscr{N}\mathscr{G} & \hookrightarrow & \mathscr{G}^{(1)} & \twoheadrightarrow & \mathscr{G}^{(1)}/\mathscr{N}\mathscr{G} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \varphi \\
\mathscr{N}\mathscr{F} & \hookrightarrow & \mathscr{F}^{(1)} & \twoheadrightarrow & \mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}
\end{array}$$

It implies, by snake's lemma, that $\mathcal{I} := \ker(\varphi) \subset (\mathscr{N}\mathscr{F})/(\mathscr{N}\mathscr{G})$ and $\mathcal{H} := \mathrm{im}(\varphi) \subset (\mathscr{F}^{(1)}/\mathscr{N}\mathscr{F})$. Let $\mathcal{J} := \ker(\mathscr{N}\mathscr{F} \twoheadrightarrow (\mathscr{N}\mathscr{F}/\mathscr{N}\mathscr{G})/\mathcal{I})$. It holds that $\mathrm{rk}(\mathcal{J}) = \mathrm{rk}(\ker(\mathscr{G}^{(1)} \twoheadrightarrow \mathcal{H}))$ and $\deg(\mathcal{J}) = \deg(\ker(\mathscr{G}^{(1)} \twoheadrightarrow \mathcal{H}))$. Hence, we have that

$$\mu(\mathscr{G}) = \frac{r_0(\mathscr{G})}{\mathrm{R}(\mathscr{G})}\left(\frac{\mathrm{rk}(\mathcal{J})}{r_0(\mathscr{G})}\mu(\mathcal{J}) + \frac{\mathrm{rk}(\mathcal{H})}{r_0(\mathscr{G})}\mu(\mathcal{H})\right) + \frac{r_1(\mathscr{G})}{\mathrm{R}(\mathscr{G})}\mu(\mathscr{G}/\mathscr{G}^{(1)}).$$

By the stability of $\mathcal{F} = \mathscr{N}\mathscr{F}$ and of $\mathcal{K}$, it holds that $\mu(\mathcal{J}) \leq \mu(\mathcal{F})$, $\mu(\mathcal{H}) \leq \mu(\mathcal{K})$ and $\mu(\mathscr{G}/\mathscr{G}^{(1)}) \leq \mu(\mathcal{F} \otimes \mathscr{N}^{-1})$. We have also that $\mu(\mathcal{K}) \geq \mu(\mathcal{F} \otimes \mathscr{N}^{-1})$ (by hypothesis of case (iii)), and, so, $\mu(\mathcal{K}) \geq \mu(\mathcal{F})$, too. Moreover, we are under the condition $0 < r_0(\mathscr{G})r_1 < r_0 r_1(\mathscr{G})$, which implies that $\mathrm{R}(\mathscr{G})r_1 - \mathrm{R}(\mathscr{F})\,\mathrm{rk}(\mathcal{J}) < 0$ (because $\mathrm{rk}(\mathcal{J}) \geq r_1(\mathscr{G})$) and $\mathrm{R}(\mathscr{F})r_0(\mathscr{G}) - \mathrm{R}(\mathscr{G})r_0 < 0$. Therefore, we can conclude that $\mu(\mathscr{G}) < \mu(\mathscr{F})$ by Lemma 4.22. *q.e.d.*

The following corollary is a straightforward consequence of the Theorem:

COROLLARY 4.20. *Assume* $g_1 \geq 2$. *The locus* $\mathrm{N}(a, d_0, d_1)$ *is non-empty if and only if* $(d_0 - (2a+1)\delta)/(a+1) < d_1/a < d_0/(a+1)$.

REMARK 4.21.

(i) In order to avoid any possibility of misunderstanding, it is better to point out explicitly that the Theorem does not mean that *any* quasi locally free sheaf of a complete type verifying the inequalities (4.6) is (semi)stable. It is extremely easy to find counterexamples, e.g. using split sheaves. The statement is just that there exist *a* quasi locally free sheaf of that complete type which is (semi)stable. This implies that a *generic* (in some suitable sense) quasi locally free sheaf of that complete type is (semi)stable.

(ii) The hypothesis $g_1 \geq 2$ is due to the use of Lange's conjecture (see [**RT**] and [**Ba**]), which holds for these genera. For the elliptic case (i.e. when the reduced subcurve is elliptic), it can be replaced looking at short exact sequences of indecomposable vector bundles (recall that on smooth elliptic curves indecomposable is equivalent to semistable and that the indecomposable vector bundles are completely classified, see, e.g., [**T**]), at least in the external cases (i.e. cases (ii) and (iii) in the proof of the Theorem), in which we need only one short exact sequence of semistable vector bundles, whose existence is guaranteed by [**BR**, Theorem 0.1]. If also in the elliptic case, the existence of one such exact sequence were sufficient to conclude that for generic semistable bundles the generic extension is semistable, then it could be used also for the central case (i.e. case (i) in the proof of the Theorem).

For the rational case, i.e. when C is a rational curve, it is well-known that there are not stable bundle of rank greater than or equal to 2 and that the only semistable bundles are polystable ones. These sheaves probably could be used to do alternative computations. I did by hand some explicit computations only in the case of generalized rank 3. I omit them, because they are quite tedious, but the result is the following: there exists a stable quasi locally free sheaf of generalized rank 3 if and only if

$$\begin{cases} \delta \geq 3 \text{ and } \frac{d_0-3\delta+3}{2} < d_1 < \frac{d_0-3}{2} \text{ or } d_1 = \frac{d_0-3\delta}{2}+1,\ \frac{d_0-\delta}{2}-1; \\ \delta = 2 \text{ and } d_1 = \frac{d_0}{2}-\delta,\ \frac{d_0}{2}+1-\delta. \end{cases}$$

The cases of the equalities hold only if the numerators are even.

On the other hand there exists a strictly semistable such sheaf if and only if

$$\begin{cases} \delta \geq 3 \text{ and } 2d_1 = d_0-3\delta,\ d_0-3\delta+3,\ d_0-3,\ d_0; \\ \delta = 2 \text{ and } 2d_1 = d_1-2\delta-2,\ d_0-2\delta+1,\ d_0-2\delta+4; \\ \delta = 1 \text{ and } 2d_1 = d_0-2\delta-1,\ d_0-2\delta+2; \\ \delta = 0 \text{ and } 2d_1 = d_0-2\delta. \end{cases}$$

The following lemma has already been used at the end of the proof of the previous Theorem.

LEMMA 4.22. *Let $m_1 > m_2 > m_3$ and $m'_1 > m'_2 > m'_3$ be non-negative integers and let q_1, q_2, q_3 and q'_1, q'_2, q'_3 be real numbers. Assume $q_1 \le q'_1$, $q_2 \le q'_2$, $q_3 \le q'_3$, $q'_1 \le q'_2$ and $q'_3 \le q'_2$, $m_1 m'_3 - m'_1 m_3 \le 0$ and $m_2 m'_1 - m'_2 m_1 \le 0$. Then $w \le w'$, where $w = [m_3 q_1 + (m_2 - m_3) q_2 + (m_1 - m_2) q_3]/m_1$ and $w' = [m'_3 q'_1 + (m'_2 - m'_3) q'_2 + (m'_1 - m'_2) q'_3]/m'_1$. If one of the inequalities in the hypotheses is strict, then $w < w'$.*

PROOF. It is an easy calculation: $w' - w = \frac{m'_3}{m'_1} q'_1 - \frac{m_3}{m_1} q_1 + \frac{m'_2 - m'_3}{m'_1} q'_2 - \frac{m_2 - m_3}{m_1} q_2 + \frac{m'_1 - m'_2}{m'_1} q'_3 - \frac{m_1 - m_2}{m_1} q_3 \ge \frac{1}{m'_1 m_1} [q'_1 (m_1 m'_3 - m'_1 m_3) + q'_2 (m'_2 m_1 - m'_3 m_1 - m_2 m'_1 + m_3 m'_1) + q'_3 (m'_1 m_1 - m'_2 m_1 - m_1 m'_1 + m_2 m'_1)] = \frac{1}{m'_1 m_1} [(m_1 m'_3 - m'_1 m_3)(q'_1 - q'_2) + (m_2 m'_1 - m'_2 m_1)(q'_3 - q'_2)] \ge 0$. If one of the inequalities in the hypotheses is strict, then $w' - w > 0$ because, then, one of the two above inequalities has to be strict, too. *q.e.d.*

The next result is a computation of the dimension of the locus of quasi locally free sheaves of fixed complete type (for $g_1 \ge 2$).

PROPOSITION 4.23. *Let X be a ribbon such that $\delta = -\deg(\mathscr{N}) > 0$ and $g_1 \ge 2$ and let $((r_0, r_1), (d_0, d_1))$ be integers verifying the hypotheses of Theorem 4.19, with strict inequalities. The locus of semistable quasi locally free sheaves on X of complete type $((r_0, r_1), (d_0, d_1))$ has dimension $1 + (r_0^2 + r_1^2)(g_1 - 1) + r_0 r_1 \delta$.*

PROOF. First of all, observe that we can restrict our attention to the range in which $\mathscr{F}^{(1)}$ can be stable, because the other cases are covered by duality. So we can assume $[d_0 - (r_0 + r_1)\delta]/r_0 < d_1/r_1 < (d_0 - r_1\delta)/r_0$.

Observe also that there are not conditions about $\mathcal{F} = \mathscr{N}\mathscr{F}$ and about $\mathcal{K} = \mathscr{F}^{(1)}/\mathscr{N}\mathscr{F}$; so, we can start with these two vector bundles on C generic. They give rise to $r_1^2(g_1 - 1) + 1$ and $(r_0 - r_1)^2(g_1 - 1) + 1$ moduli, respectively. Then, we have to compute how many vector bundles on C are extensions of $\mathcal{K}$ by $\mathcal{F}$ and then look at the extensions on C_2 of these vector bundles by $\mathcal{F} \otimes \mathscr{N}^{-1}$.

The possible $\mathscr{F}^{(1)}$'s have $\mathrm{ext}^1_C(\mathcal{K}, \mathcal{F}) - 1$ moduli; so, we have to compute: $\mathrm{ext}^1_C(\mathcal{K}, \mathcal{F}) = \mathrm{h}^1(\mathcal{K}^* \otimes \mathcal{F}) = -\deg(\mathcal{K}^* \otimes \mathcal{F}) + \mathrm{h}^0(\mathcal{K}^* \otimes \mathcal{F}) + r_1(r_0 - r_1)(g_1 - 1) = -(r_0 - r_1)d_1 + r_1(d_0 - d_1 - r_1\delta) + r_1(r_0 - r_1)(g_1 - 1) = -r_0 d_1 + r_1 d_0 - r_1^2\delta + r_1(r_0 - r_1)(g_1 - 1)$; observe that the h^0 vanishes because $d_1/r_1 < (d_0 - r_1\delta)/r_0$.

Now, it seems that it remains to compute $\mathrm{ext}^1_X(\mathcal{F} \otimes \mathscr{N}^{-1}, \mathscr{F}^{(1)})$, which is equal to $\mathrm{ext}^1_C(\mathcal{F} \otimes \mathscr{N}^{-1}, \mathscr{F}^{(1)}) + \hom(\mathcal{F}, \mathscr{F}^{(1)})$ by Fact 4.3(iii). The extensions corresponding to sheaves of the desired complete type are those whose associated morphism from $\mathcal{F}$ to $\mathscr{F}^{(1)}$ is injective, again by Fact 4.3(iii); but the endomorphism has been fixed when constructing $\mathscr{F}^{(1)}$ as an extension of $\mathcal{K}$ by $\mathcal{F}$, apart from automorphisms of $\mathcal{F}$ in itself. So, the only remaining moduli are given by $\mathrm{ext}^1_C(\mathcal{F} \otimes \mathscr{N}^{-1}, \mathscr{F}^{(1)}) = \mathrm{h}^1(\mathcal{F}^* \otimes \mathscr{N} \otimes \mathscr{F}^{(1)}) = -\deg(\mathcal{F}^* \otimes \mathscr{N} \otimes \mathscr{F}^{(1)}) + \mathrm{h}^0(\mathcal{F}^* \otimes \mathscr{N} \otimes \mathscr{F}^{(1)}) + r_0 r_1(g_1 - 1) = r_0 d_1 + r_0 r_1 \delta - r_1(d_0 - r_1\delta) + r_0 r_1(g_1 - 1)$; indeed, the h^0 vanishes because under our hypotheses $[d_0 - (r_0 + r_1)\delta]/r_0 < d_1/r_1$, which is equivalent to $\deg(\mathcal{F}^* \otimes \mathscr{N} \otimes \mathscr{F}^{(1)}) < 0$.

It remains to sum up these moduli: $r_1^2(g_1-1)+1+(r_0-r_1)^2(g_1-1)+1-r_0d_1+r_1d_0-r_1^2\delta+r_1(r_0-r_1)(g_1-1)-1+r_0d_1+r_0r_1\delta-r_1(d_0-r_1\delta)+r_0r_1(g_1-1)=(r_0^2+r_1^2)(g_1-1)+r_0r_1\delta+1$, as wanted. *q.e.d.*

REMARK 4.24.

(i) The loci studied in the previous Proposition are irreducible by [**D2**, Théorème 6.8]. If Conjecture 4.8 holds, then their closures are irreducible components of the moduli space of semistable sheaves on X when $0<\delta\le 2g_1-2$. This would be due to dimensional reasons and to the upper (resp. lower) semicontinuity of r_0 (resp. r_1), for which see [**D1**, Proposition 7.3.1], paying attention to the fact that what is there denoted r_0 is our r_1. Indeed, if (r_0,r_1) and (s_0,s_1) are two pairs of non-negative integers such that $r_0+r_1=s_0+s_1$ and $s_1<s_0<r_0$, then $1+(r_0^2+r_1^2)(g_1-1)+r_0r_1\delta>1+(s_0^2+s_1^2)(g_1-1)+s_0s_1\delta$; hence, a locus of sheaves with (r_0,r_1) as rank-part of the complete type cannot be contained in a locus of those with (s_0,s_1) as rank-part of the complete type. On the other hand, by the above cited semicontinuity, any sheaf with (s_0,s_1) as rank-part of the complete type is not contained in the closure of a locus of sheaves with (r_0,r_1) as rank-part of the complete type.

(ii) It follows from [**D2**, Proposition 3.12] that, as in the case of quasi locally free sheaves of rigid type, the dimension obtained in the Proposition equals $\mathrm{h}^1(\underline{\mathrm{End}}(\mathscr{F}))$, for any stable quasi locally free $\mathscr{F}$ of that complete type. This implies that these loci are not smooth, out of the locally free case, because the tangent space has dimension $\mathrm{ext}^1(\mathscr{F},\mathscr{F})$, i.e., by the Ext-spectral sequence, $\mathrm{h}^1(\underline{\mathrm{End}}(\mathscr{F}))+\mathrm{h}^0(\underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F}))$, and, if $\mathscr{F}$ is not locally free, $\mathrm{h}^0(\underline{\mathrm{Ext}}^1(\mathscr{F},\mathscr{F}))\neq 0$ (although I have not computed it explicitly).

Before passing to the study of sheaves of positive index, we end this tour about quasi locally free sheaves with some results about deformations of vector bundles on C to sheaves on C_2; they are inspired by [**D1**, Théorème 7.2.3] and [**Sa**, Theorem 1], which are about rank 2 vector bundles on C, but they are less precise.

PROPOSITION 4.25. *Let $\mathcal{E}$ be a vector bundle of rank $r\ge 3$ on C. If there exists a non-trivial subsheaf $\mathcal{F}\subset\mathcal{E}$ of rank $r'<r$ such that* $\mathrm{Hom}((\mathcal{E}/\mathcal{F})\otimes\mathscr{N},\mathcal{F})\neq 0$*, then $\mathcal{E}$ deforms to pure sheaves defined on C_2 (and not on C).*

If, moreover, the generic element of this homomorphism group has maximal rank, i.e. $\min\{r',r-r'\}$*, then $\mathcal{E}$ deforms to pure sheaves of type $(|r-2r'|,\min\{r',r-r'\})$.*

PROOF. We can restrict our attention to the case in which $\mathcal{F}$ is a saturated subsheaf of $\mathcal{E}$, because, if $\mathcal{F}$ is not saturated and $\mathcal{F}^{\mathrm{sat}}$ is its saturation, then $\mathrm{Hom}((\mathcal{E}/\mathcal{F})\otimes\mathscr{N},\mathcal{F})\neq 0$ implies that $\mathrm{Hom}((\mathcal{E}/\mathcal{F}^{\mathrm{sat}})\otimes\mathscr{N},\mathcal{F}^{\mathrm{sat}})\neq 0$.

In the saturated case, both the assertions are trivial consequences of Fact 4.3(iii); indeed, the latter implies that the generic element of the universal family of extensions of $\mathcal{E}/\mathcal{F}$ by $\mathcal{F}$ is defined on C_2 and that it is of the asserted type if the generic element in $\mathrm{Hom}((\mathcal{E}/\mathcal{F})\otimes\mathscr{N},\mathcal{F})$ has maximal rank. *q.e.d.*

PROPOSITION 4.26. *Let X be a ribbon such that $\delta = -\deg(\mathscr{N}) > 2g_1 - 2$ and $g_1 \geq 2$. Any vector bundle of rank $r \geq 2$ and degree d on C deforms to pure sheaves on X of type $(r-2, 1)$ (hence, of generalized rank r) and generalized degree d, with the possible exception of the case in which $\delta = 2g_1 - 1$, $r = 3$ and 3 divides both d and g_1.*

PROOF. The case $r = 2$ is [**Sa**, Theorem 1].

We have to show that any vector bundle $\mathcal{E}$ of rank $r \geq 2$ and degree d as in the statement verifies the hypothesis of Proposition 4.25 for $r' = n - 1$ or for $r' = 1$ (these r''s are due to the hypothesis about the type).

Throughout the proof, we will denote by $s_{r'}$ the r'-Segre invariant of $\mathcal{E}$, i.e. the number $r'd - r \max\{\deg(\mathcal{E}') \,|\, \mathrm{rk}(\mathcal{E}') = r', \mathcal{E}' \subset \mathcal{E}\}$, for any $0 < r' < r$.

For any $\mathcal{F}$ saturated subbundle of $\mathcal{E}$, it holds that $\mathrm{Hom}((\mathcal{E}/\mathcal{F}) \otimes \mathscr{N}, \mathcal{F}) \neq 0$ if and only if $\mathrm{h}^0((\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}) > 0$. Let $\mathcal{F} \subset \mathcal{E}$ be a subbundle of rank r' of maximal degree. In this case, $\deg((\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}) = r'(r-r')\delta - s_{r'}$. Therefore, $\mathrm{h}^0((\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}) = r'(r-r')\delta - s_{r'} + \mathrm{h}^1((\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}) - r'(r-r')(g_1 - 1)$. By the basic properties of the Segre invariants (for which see, e.g., the introduction of [**RT**] and its references), the right hand term is always positive for any r' if $\delta \geq 2g_1 + r - 1$. But we need that the right term is positive only in the case $r' = r - 1$. This is the case if $\delta \geq 2g_1$ or if $\delta = 2g_1 - 1$ and d is not congruent to g_1 modulo r. Also the case $\delta = 2g_1 - 1$, $r' = 2$ and $r \geq 4$ (and also $r' = r - i$ and $r \geq i + 2$ for any $2 \leq i \leq r - 2$) follows from an almost trivial calculation.

Only the case in which $\delta = 2g_1 - 1$, $r = 3$ and 3 divides both d and g_1 remains open. In this case, for $\mathcal{E}$ generic, one obtains, for both $r' = 1$ and 2, that $(\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}$ is a stable rank 2 vector bundle of degree $2(g_1 - 1)$ and that there is a 2-dimensional family of $\mathcal{F}$ of maximal degree (by [**RT**, Theorem 0.2]). If one were able to show that the bundles $(\mathcal{E}/\mathcal{F})^* \otimes \mathscr{N}^{-1} \otimes \mathcal{F}$ are distinct, one could conclude by [**Su**, Thoerem III.2.4], which asserts, in particular, that the Brill-Noether locus of stable rank 2 vector bundles of degree $2(g_1 - 1)$ with at least one global section is a divisor in the moduli space of stable vector bundles of rank 2 and degree $2(g_1 - 1)$. *q.e.d.*

REMARK 4.27. The requirement about the type $(r-2, 1)$ in the statement of the Proposition does not mean that a vector bundle of rank r on C deforms only to sheaves of this type, but only that it surely deforms to sheaves of this type. This is due to the fact that we proved the existence of a subsheaf of rank $r - 1$ verifying the hypothesis of Proposition 4.25 for which any non-zero morphism has automatically maximal rank, i.e. rank 1.

The above Proposition, together with Conjecture 4.8, suggests the following conjecture:

CONJECTURE 4.28. *If $\delta > 2g_1 - 1$, the only irreducible components of the moduli space of coherent sheaves on C_2 are those whose generic elements are either quasi locally free sheaves of rigid type (for generalized rank odd) or generalized vector bundles (for generalized rank even).*

Indeed, if one were able to show that deformations of subsheaves or quotients of the canonical filtrations of a quasi locally sheaf $\mathscr{F}$ on C_2 induce deformations of $\mathscr{F}$ itself (maybe using the relative ext sheaves), the Proposition could be used to prove the conjecture. Indeed, one could proceed by induction on the generalized rank, starting from the first interesting case, i.e. generalized rank 3 (for generalized rank 2 the conjecture holds: it is [**Sa**, Corollary 1]). In generalized rank 3 there are only sheaves of type $(3,0)$ and $(1,1)$: the first are rank 3 vector bundles on C, all of which deform, under our hypotheses, to sheaves on C_2 by the Proposition (with that possible exception cited in its statement, but we think that it is not a real exception); so, the only possible generic elements of an irreducible component are sheaves of type $(1,1)$. The quasi locally free sheaves of this type are of rigid type; hence, Conjecture 4.28 reduces to Conjecture 4.8. In generalized rank 4, by the Proposition, one has to consider only sheaves of type $(2,1)$ and $(0,2)$, the latter being generalized vector bundles. Within sheaves of type $(2,1)$ we have to consider only quasi locally free ones (assuming, as usual, Conjecture 4.8). If $\mathscr{F}$ is a sheaf of type $(2,1)$, $\mathscr{F}|_C$ is a rank 3 vector bundle on C. So, by the Proposition, it deforms to a sheaf of type $(1,1)$ and the generic extension of a sheaf of this type by the line bundle (on C) $\mathscr{N}\mathscr{F}$ should be a generalized vector bundle. If the last assertion were not correct, one could look to a rank two quotient of $\mathscr{F}|_C$ and to the kernel of the composed morphism from $\mathscr{F}$ to it, which is either a rank two vector bundle on C or a generalized line bundle on X. In the first case the rank two vector bundles deform, by the Proposition, to two generalized line bundles on X, whose extensions are generalized vector bundle on X; in the second one, only one of them has to be deformed to a generalized line bundle and the conclusion is the same. This idea could be formalized by induction for any n, if one were able to prove Conjecture 4.8 and to control when and how deformations of subsheaves and quotients related to the canonical filtrations induce deformations of the sheaf itself.

It is time to left the world of quasi locally free sheaves in order to explore that of pure sheaves of positive index.

We start with some properties of generalized vector bundles.

PROPOSITION 4.29. *Let X be a ribbon, let $\delta = -\deg(\mathscr{N})$, let b be a non-negative integer and let r be a positive integer. There exists a semistable (resp.stable) generalized vector bundle $\mathscr{E}$ of generalized rank $2r$, generalized degree D and of index b if and only if $b+\delta \equiv D \pmod 2$ and $b \le r\delta$ (resp. $b < r\delta$).*

PROOF. The necessity is trivial: indeed, the inequality is equivalent to $\mu(\mathscr{E}^{(1)}) \le \mu(\mathscr{E}/\mathscr{E}^{(1)})$ (resp. $<$). On the other hand, the required parity is due to the fact that $\deg(()\mathscr{N}\mathscr{E}) = (D-b-r\delta)/2$ (it is an easy computation).

Also the sufficiency is not difficult to be checked directly, but it is an immediate consequence of Fact 4.6 and of [**D4**, Théorème 5.4.2]. *q.e.d.*

PROPOSITION 4.30. *Let X be a ribbon and let, as usual, $\delta = -\deg(\mathscr{N})$. Assume also $\delta > 0$. The locus of semistable generalized vector bundles of generalized rank $2r$, of generalized degree D and of fixed index b such that $b+\delta \equiv D \pmod 2$ and $b < r\delta$ has dimension $1 + 2r^2(g_1-1) + r^2\delta$.*

PROOF. The assertion is a trivial consequence of [**I**, Proposition 2.1 and Remark 2.7(i)]. *q.e.d.*

The case of generalized line bundles and the correspondence with Higgs bundles suggest the following conjecture:

CONJECTURE 4.31. *The locus of semistable generalized vector bundles of generalized rank* $2r$*, of fixed generalized degree and index (less than* $r\delta$*) on a ribbon* X *is irreducible.*

Now, we turn our attention to a particular class of pure sheaves which are nor quasi locally free neither generalized vector bundles: those of type $(k, 1)$, for a positive integer k. They are interesting because they are the push-forward of quasi locally free sheaves on a blow up of X (this assertion is made precise in Proposition 4.32 below) and this fact allows to derive easily many of their properties from those of quasi locally free sheaves.

First of all, we state the promised generalization of [**EG**, Theorem 1.1] (which is the case $k = 0$).

PROPOSITION 4.32. *Let* k *be a non-negative integer, let* $\mathscr{F}$ *be a pure sheaf on* X *of type* $(k, 1)$*, i.e. generically isomorphic to* $\mathscr{O}_X \oplus \mathscr{O}_C^{\oplus k}$*, and let* $\mathcal{T}$ *be the torsion part of* $\mathscr{F}|_C$*. There is a unique divisor* $D \subset C$ *such that* $\mathcal{T}$ *is isomorphic to* $\mathscr{O}_D$ *and a unique quasi locally free sheaf* $\mathscr{F}'$ *of type* $(k, 1)$*, i.e. locally isomorphic to* $\mathscr{O}_{X'} \oplus \mathscr{O}_C^{\oplus k}$*, on the blow up* $q : X' \to X$ *of* X *at* D *such that* $q_*\mathscr{F}' \simeq \mathscr{F}$*.*

PROOF. The proof is essentially the same of the cited place. Indeed, the key point of that proof is that $\mathscr{N}\mathscr{F}$ and $\mathscr{K} = \ker(\mathscr{F} \to (\mathscr{F}|_C)^{\vee\vee})$ are line bundles on C (such that $\mathscr{N}\mathscr{F} \subset \mathscr{K}$), which implies that $\mathscr{K}/\mathscr{N}\mathscr{F}$, isomorphic to $\mathcal{T}$ by snake's lemma, can be written as $\mathscr{O}_D$ for a unique effective divisor D of C.

The fact that $\mathscr{N}\mathscr{F}$ and $\mathscr{K}$ are line bundles on C is trivial: they are surely pure because are subsheaves of $\mathscr{F}$ and they have generalized rank 1 by additivity of the generalized rank; hence, they are pure sheaves of rank 1 on C, i.e. line bundles on it.

At this point the proof is *verbatim* the same of [**EG**, Theorem 1.1]: it is possible to give to $\mathscr{F}$ a structure of $\mathscr{O}_{X'}$-module (which is unique because it is derived only from the $\mathscr{O}_X$-module structure of $\mathscr{F}$) and, writing $\mathscr{F}'$ for $\mathscr{F}$ with this structure, it is clear that $q_*\mathscr{F}' \simeq \mathscr{F}$. Also the uniqueness of the divisor follows as there. Let us recall how to define such a structure.

Let $f \in \mathrm{H}^0(\mathscr{O}_C(D))$ be a section vanishing on D, let σ' be a section of $\mathscr{O}_{X'}$ defined on an open set U of X (recall that X and X' are homeomorphic) and m a section of $\mathscr{F}(U)$. Shrinking U, if necessary, it is possible to find a section σ of $\mathscr{O}_X(U)$ with the same image of σ' in $\mathscr{O}_C(U)$. Hence, $\sigma' = \sigma + f^{-1}\tau$, where τ is an appropriate section of $\mathscr{N}(U)$. The sheaf $\mathscr{F}$ admits a structure of $\mathscr{O}_{X'}$-module if we can define $\sigma' m$ as $\sigma m + f^{-1}(\tau m)$; the latter is well defined because $\tau m \in \mathscr{N}\mathscr{F}$ and $\mathscr{K} = \mathscr{O}_C(\mathscr{F}) \otimes \mathscr{N}\mathscr{F}$. It is possible to verify that this definition is independent of the choice of σ. *q.e.d.*

A similar result cannot hold for any pure sheaf on X; e.g., the blow up $q : X' \to X$ associated to $\mathscr{I} \oplus \mathscr{O}_X$, where $\mathscr{I}$ is a generalized line bundle with positive index, is the same associated to $\mathscr{I}$ and it is impossible to find a quasi locally free sheaf on X' such that $\mathscr{I} \oplus \mathscr{O}_X$ is its direct image via q. It is easy to see it looking at the local descriptions: if P is a closed point where $\mathscr{I}$ is not free, $\mathscr{I}_P \cong (x^b, y)$, where b is the index of $\mathscr{I}$ in P, y is a generator of the nilradical of $A = \mathscr{O}_{X,P}$ and x is a nonzerodivisor whose image in $\mathscr{O}_{C,P}$ is a generator of the maximal ideal, while $\mathscr{O}_{X',P} = A[y/x^b] = A'$. The module $\mathscr{I}_P \oplus A$ is the direct image of a module on A' if it is closed under multiplication by y/x^b (indeed A' and A have the same ring of fractions) but this is impossible, e.g., for the element $(y, 1)$ that is mapped to $(0, y/x^b)$ which does not belong to $\mathscr{I}_P \oplus A$.

The sheaves involved in the above Proposition have a quite nice behaviour with respect to semistability. Indeed, it is characterized by the following generalization of [**D1**, Lemme 9.1.2], which is about quasi locally free sheaves of generalized rank 3.

PROPOSITION 4.33. *Let k be a positive integer and let $\mathscr{F}$ be a pure sheaf on X of type $(k, 1)$. Then $\mathscr{F}$ is semistable if and only if the two following conditions are verified:*

(i) *for any subbundle $\mathcal{E} \subseteq \mathscr{F}^{(1)}$ of rank $\leq k$ we have $\mu(\mathcal{E}) \leq \mu(\mathscr{F})$;*
(ii) *for any pure quotient $(\mathscr{F}|_C)^{\vee\vee} \twoheadrightarrow \mathcal{G}$ of rank $\leq k$ it holds that $\mu(\mathcal{G}) \geq \mu(\mathscr{F})$.*

Furthermore, $\mathscr{F}$ is stable if and only if the inequalities in (i) *and* (ii) *are strict.*

PROOF. Necessity is obvious, we have to prove only sufficiency.

Throughout the proof we will denote $(\mathscr{F}|_C)^{\vee\vee}$ by $\mathcal{F}$.

We will prove only the semistable case, because the stable one is essentially identical.

Let $\mathscr{E} \subset \mathscr{F}$ be a saturated subsheaf. If $\mathscr{E}$ is defined on C, then $\mathscr{E} \subseteq \mathscr{F}^{(1)}$. If it has rank $\leq k$, then $\mu(\mathscr{E}) \leq \mu(\mathscr{F})$ by (i). On the other hand, if it has rank $k+1$, it has the same rank of $\mathscr{F}^{(1)}$ and it is contained in it. Hence, $\mu(\mathscr{E}) \leq \mu(\mathscr{F}^{(1)})$ and it suffices to check that $\mu(\mathscr{F}^{(1)}) \leq \mu(\mathscr{F})$. This follows from condition (ii), because $\mathscr{F}/\mathscr{F}^{(1)}$ is a pure quotient of $\mathcal{F}$ of rank 1 and thus $\mu(\mathscr{F}/\mathscr{F}^{(1)}) \geq \mu(\mathscr{F})$.

So, assume that $\mathscr{E}$ is not defined on C; this means that $\mathscr{E}$ is generically isomorphic to $\mathscr{O}_X \oplus \mathscr{O}_C^{\oplus h}$ with $0 \leq h < k$. Hence, $\mathscr{F}/\mathscr{E}$ is generically isomorphic to $\mathscr{O}_C^{\oplus(k-h)}$. Furthermore, being pure ($\mathscr{E}$ is saturated), this quotient is a rank $k-h$ vector bundle on C. Thus, $\mathscr{F}/\mathscr{E}$ is a pure quotient of $\mathcal{F}$ of rank $\leq k$ and so, by (ii), $\mu(\mathscr{F}/\mathscr{E}) \geq \mu(\mathscr{F})$, which is equivalent to $\mu(\mathscr{E}) \leq \mu(\mathscr{F})$. *q.e.d.*

REMARK 4.34.

(i) The case $k = 0$, i.e. that of generalized line bundles, is not covered by the Proposition. In order to cover also their case, one should drop the hypothesis *of rank* $\leq k$ in the two conditions. Indeed if $\mathscr{I}$ is a generalized line bundle, it holds that $(\mathscr{I}|_C)^{\vee\vee} = \mathscr{I}/\mathscr{I}^{(1)}$ and the two conditions (without the cited hypothesis) are both equivalent to $b(\mathscr{I}) \leq -\deg(\mathscr{N})$, which is equivalent to the semistablity of $\mathscr{I}$ (see [**CK**, Lemma 3.2]).
(ii) The hypothesis $\deg(\mathscr{N}) < 0$, which, as pointed out in Remark 1.28(iv), is necessary for the existence of stable sheaves not defined on C, does not appear in the statement of the Proposition because it would be redundant. Indeed, it follows from the two conditions and from the observation that they cover the two sheaves used in the cited remark: $\ker(\mathscr{F} \twoheadrightarrow (\mathscr{F}|_C)^{\vee\vee})$ is a line subbundle of $\mathscr{F}^{(1)}$

while $\mathscr{F}/\mathscr{F}^{(1)} = \mathscr{N}\mathscr{F} \otimes \mathscr{N}^{-1}$ is a pure quotient of $(\mathscr{F}|_C)^{\vee\vee}$ of rank 1.

Corollary 4.35. *Let $\mathscr{F}$ be as in the Proposition, let $q : X' \to X$ be the blow up of X with respect to the divisor associated to the torsion part of $\mathscr{F}|_C$ and let $\mathscr{F}'$ be the quasi locally free sheaf on X' such that $q_*(\mathscr{F}') = \mathscr{F}$ (see Proposition 4.32). Then $\mathscr{F}$ is (semi)stable if and only if $\mathscr{F}'$ is (semi)stable.*

Proof. It holds by definition that $\mathrm{Deg}(\mathscr{F}) = \mathrm{Deg}(\mathscr{F}')$ and $\mathrm{R}(\mathscr{F}) = \mathrm{R}(\mathscr{F}')$; hence, $\mu(\mathscr{F}) = \mu(\mathscr{F}')$. The construction of $\mathscr{F}'$ implies that $\mathscr{F}^{(1)} = \mathscr{F}'^{(1)}$ and $(\mathscr{F}|_C)^{\vee\vee} = \mathscr{F}'|_C$. Therefore, the assertion follows from the Proposition. *q.e.d.*

Remark 4.36. The Corollary holds also for generalized line bundles. The proof is the same except that Remark 4.34(i) has to be used instead of the Proposition.

Let k be a positive integer, b a non-negative one and d_0 and d_1 two integers and let $\mathrm{L}(k, b, d_0, d_1) \subset \mathrm{M}_s(X, k+2, d_0+d_1)$ be the locus of stable sheaves $\mathscr{F}$ of complete type $((k+1, 1), (d_0, d_1))$ with index b.

The above Corollary allows to describe $\mathrm{L}(k, b, d_0, d_1)$ in terms of loci of quasi locally free sheaves on appropriate blow ups.

More precisely, set $S_b := \mathrm{Sym}^b C$ (which is, as well-known, isomorphic to the Hilbert scheme of zero dimensional subschemes of C of length b) and let $\mathfrak{D}$ be the tautological divisor of $C \times S_b$. By the fact that $\mathfrak{D}$ is also a subscheme of $X \times S_b$, we can consider $\rho : \mathfrak{X} \twoheadrightarrow X \times S_b$, the blow up of $X \times S_b$ along $\mathfrak{D}$.

It is clear that $\mathfrak{X}$ can be seen as an S_b-scheme. Furthermore, for any closed point $s \in S_b$ corresponding to an effective divisor D of C of length b, the fibre $\mathfrak{X}_s$ is isomorphic to the blow up X'_D of X along D.

We can consider the relative moduli space of semistable sheaves of fixed Hilbert polynomial of $\mathfrak{X}/S_b$. By the properties of this moduli space (see, e.g., [**HL**, Theorem 4.3.7]), its fibre at any closed point $s \in S_b$ is isomorphic to the moduli space of semistable sheaves of the same Hilbert polynomial on X'_D (where, as above, D is the divisor corresponding to s).

It is also clear, combining Proposition 4.32 and Corollary 4.35, that $\mathrm{L}(k, b, d_0, d_1) \subset \mathrm{M}_s(X, k+2, d_0+d_1)$ is the direct image of the sublocus of the relative moduli space whose fibre in s is isomorphic to $\mathrm{L}(X'_D, k, 0, d_0, d_1)$. This method leads also to a decomposition of $\mathrm{L}(k, b, d_0, d_1)$ as the disjoint union $\amalg\, \mathrm{L}(k, (b_1, \ldots, b_j), d_0, d_1)$, where the union is taken over all the (unordered and integral) partitions $(b_1, \ldots, b_j)$ of b and $\mathrm{L}(k, (b_1, \ldots, b_j), d_0, d_1)$ parametrizes the sheaves of $\mathrm{L}(k, b, d_0, d_1)$ having local index sequence $b_{.} = \{b_1, \ldots, b_j\}$. Indeed, in order to describe $\mathrm{L}(k, (b_1, \ldots, b_j), d_0, d_1)$ it is sufficient to look at the appropriate diagonal in S_b.

This gives a quite precise description of $\mathrm{L}(k, b, d_0, d_1)$, thanks to the knowledge of $\mathrm{L}(X'_D, k, 0, d_0, d_1)$ (which are irreducible of dimension $1+(k^2+2k+2)(g_1-1)+(k+1)(-\deg(\mathscr{N}(X')))$, see Proposition 4.23 and Remark 4.24(i)).

Theorem 4.37. *Let X be a ribbon such that $\delta = -\deg(\mathscr{N}) > 0$ and $g_1 \geq 2$. Let k and b be positive integers and d_0 and d_1 integers.*

The locus $\mathrm{L}(k,b,d_0,d_1)$ *is non-empty if and only if* $b<\delta$ *and* $(d_0-b-(k+2)\delta)/(k+1)<d_1<(d_0-b)/(k+1)$.

In this case, it is irreducible in $\mathrm{M}_s(k+2,d_0+d_1)$ *and has dimension* $1+(k^2+2k+2)(g_1-1)+(k+1)(\delta-b)$.

Under the same hypotheses, for any (unordered) partition $b_1,\dots,b_j$ *of* b *(with all the* b_i *positive integers)* $\mathrm{L}(k,(b_1,\dots,b_j),d_0,d_1)$ *is non-empty and irreducible of dimension* $1+(k^2+2k+2)(g_1-1)+(k+1)(\delta-b)+j$.

PROOF. The assertion follows from the above discussion together with Proposition 4.23 and Remark 4.24 and from the easy observation that if $\mathcal{N}'$ is the nilradical of $\mathcal{O}_{X'}$, where $q: X'\to X$ is the blow up of X along an effective divisor D of C of length b, then $\deg(\mathcal{N}')=\deg(\mathcal{N})+b$. *q.e.d.*

REMARK 4.38.

(i) The dimension of $\mathrm{L}(k,b,d_0,d_1)$, for $b>0$, is strictly smaller than that of the locus of quasi locally free sheaves of the same complete type. Also this fact suggests that Conjecture 4.8 is true.

(ii) This method can be applied to generalized line bundles, using the relative Picard scheme instead of the relative moduli space, in order to give an alternative demonstration of [**CK**, Lemma 4.4 and Theorem 4.6] (the only difference is that in the case of generalized rank greater than or equal to 3 there are not conditions about the parity of the index).

Before concluding this section we collect Conjectures 4.31, 4.28 and 4.8 in a unique statement about the irreducible components of the moduli space of stable sheaves on X, when $g_1\ge 2$.

CONJECTURE 4.39. *Let* X *be a ribbon such that* $g_1\ge 2$, *let* $\delta=-\deg(\mathcal{N})$ *and let* $M=\mathrm{M}_s(X,R,D)$ *be the moduli space of stable sheaves of generalized rank* R *and generalized degree* D *on* X.

(i) *Assume* $0<\delta\le 2g_1-2$, *equivalently* $g_2\le 4g_1-3$. *For any sequence of integers* $((r_0,r_1),(d_0,d_1))$ *such that* $r_0>r_1>0$ *and* $r_0+r_1=R$ *and* $d_0+d_1=D$ *and verifying strictly inequalities (4.6), i.e. such that* $(d_0-(r_0+r_1)\delta)/r_0<d_1/r_1<d_0/r_0$, *then the closure of the locus of quasi locally free stable sheaves of complete type* $((r_0,r_1),(d_0,d_1))$ *is a* $(1+(r_0^2+r_1^2)(g_1-1)+r_0r_1\delta)$*-dimensional irreducible component of* M. *Distinct complete types correspond to distinct irreducible components. Also the closure of the locus of stable rank* R *vector bundles of degree* D *on* C *is an irreducible component, which has dimension* $1+R^2(g_1-1)$. *If* R *is odd, these are all the irreducible components of* M. *On the other hand, if* $R=2r$ *is even, also the closure of the locus of stable generalized vector bundles of generalized rank* R *and degree* D *and fixed index* $b<r\delta$ *(with* b *of the same parity of* $D-r\delta$*) is an irreducible component of* M *of dimension* $1+2r^2(g_1-1)+r^2\delta$. *Distinct indices correspond to distinct components and there are not other irreducible components.*

(ii) *If* $\delta>2g_1-2$, *equivalently* $g_2>4g_1-3$, *then we have to distinguish two cases.*

(a) *If* $R = 2r$ *is even, then the only irreducible components of* M *are the closures of the loci of stable generalized vector bundles of generalized rank* R *and degree* D *and fixed index* $b < r\delta$ *(and of the same parity of* $D - r\delta$*) and they have dimension* $1 + 2r^2(g_1 - 1) + r^2\delta$.

(b) *If* $R = 2a + 1$ *is odd, then the only irreducible components of* M *are the closures of the loci* $\mathrm{N}(a, d_0, d_1)$ *(which are the loci of stable quasi locally free sheaves of rigid type of generalized rank* R *and generalized degree* D*, see Fact 4.15*(i)*) with* $(d_0 - (2a+1)\delta)/(a+1) < d_1/a < d_0/(a+1)$. *They have dimension* $1 + (a^2 + a)\delta + (2a^2 + 2a + 1)(g_1 - 1)$.

The dimensional results are all known (see Fact 4.15(i), Propositions 4.23 and 4.30 and the review about vector bundles on C in Section 4.1). Also the irreducibility of the loci of quasi locally free sheaves is known (see Remark 4.24(i)). In the first part of the conjecture (i.e. $\delta \leq 2g_1 - 2$) the only conjectural parts are that the loci of generalized vector bundles of fixed index are irreducible (which is Conjecture 4.31), that the cited loci are irreducible components and that there are no other irreducible components. The fact they are irreducible components is implied by Conjecture 4.8, as explained in Remark 4.24(i), whose argument extends immediately to say that the closure of the irreducible loci of stable generalized vector bundles are, indeed, irreducible components. The inequalities on the complete type and the index are the stability conditions given by Theorem 4.19 and Proposition 4.29. The second part, i.e. $\delta > 2g_1 - 2$, is a reformulation of Conjecture 4.28 with the addition of the stability conditions.

The sheaves of generalized rank 3 on a primitive multiple curve C_3 of multiplicity 3 are rank 3 vector bundles on C and sheaves of type $(1, 1)$ on C_2, apart from generalized line bundles on C_3 itself. So, this is a good point to formulate a conjecture about the irreducible components of the moduli space $\mathrm{M}(C_3, P_D)$ (which, as already observed, is the compactified Jacobian of C_3 when 3 divides D).

Conjecture 4.40. *Let* C_3 *be a primitive multiple curve of multiplicity* 3 *such that* $\delta = -\deg(\mathcal{C}) > 0$ *and such that* $g_1 \geq 2$*, where* g_1 *is the genus of its reduced subcurve.*

(i) *If* $\delta \leq 2(g_1 - 1)$*, then the irreducible components of* $\mathrm{M}(C_3, P_D)$ *are the following:*
  (a) $\mathrm{M}(C, P_D)$*, i.e. the moduli scheme of semistable rank* 3 *vector bundles of degree* D *on* C*;*
  (b) $\mathrm{N}(1, d_0, d_1)$ *for any pair of integers* d_0 *and* d_1 *such that* $d_0 + d_1 = D$ *and* $(d_0 - 3\delta)/2 < d_1 < d_0/2$*;*
  (c) $\bar{Z}_{b_1,b_2}$ *for any pair of non-negative integers* $b_1 \leq b_2$ *satisfying* $3 | D - b_1 - b_2$, $0 \leq b_2 + b_1 < 3\delta$ *and* $0 \leq 2b_2 - b_1 < 3\delta$.

(ii) *If* $\delta > 2(g_1 - 1)$*, the only irreducible components of* $\mathrm{M}(C_3, P_D)$ *are the* $\bar{Z}_{b_1,b_2}$*, with* b_1 *and* b_2 *as above.*

The first part of this conjecture is implied by Conjecture 4.39(i), by Lemma 3.1 and dimensional reasons (indeed, $1 + 2\delta + 5(g_1 - 1) \geq g_3 = 1 + 3\delta + 3(g_1 - 1)$ if $\delta \leq 2g_1 - 2$). The second part would follow from Conjecture

4.39(ii), if one were able to show that, if $\mathscr{F}$ is a sheaf of type $(1,1)$ on C_2, then a deformation of $\mathscr{F}^{(1)}$, which is a rank 2 vector bundle on C, to a generalized line bundle on C_2 (this deformations exists by Proposition 4.26, or rather by its special case [**Sa**, Theorem 1], being under the hypothesis $\delta > 2g_1 - 2$), induces a deformation of $\mathscr{F}$ to a generalized line bundle on C_3. It is inspired by the case of generalized rank 2 sheaves on ribbons (see [**CK**, Theorem 4.7] and [**Sa**, Corollary 1]).

In the special case in which C_3 is the spectral cover associated to nilpotent Higgs bundles of rank 3 on C, the conjecture has to hold by the spectral correspondence: all the candidate irreducible components (i.e. the irreducible components of stable generalized line bundles and the closures of the loci of semistable vector bundles of rank 3 on C and of stable quasi locally free sheaves of rigid type of generalized rank 3 on C_2), as we have already observed above, are really irreducible components by the fact they are all of the same dimension g_3 (in this case $\delta = 2g_1 - 2$ and $g_3 = 9g_1 - 8$) and they have different generic elements. In order to understand if further components should exist, it is possible to compare their number (that can be computed using the stability conditions) with that of irreducible components of the moduli space of semistable Higgs bundles of rank 3 on C, which is $2g_1(g_1 - 1) + g_1$ (when D is coprime to 3, see [**Sc**, Examples at page 306]): it follows that the above cited components should be all the components if the generalized degree is coprime to 3.

Moreover, also without doing this computation, translating [**Bo**, Corollary 2.4] about irreducible components of the nilpotent cone of Higgs bundles (at level of stacks) in our language (for this translation see Appendix A), it follows that in the cases involved in the spectral correspondence any irreducible component is the closure of the locus of sheaves with fixed complete type and quasi locally free sheaves of rigid type of generalized rank 3 on C_2 cover all the possible complete types for sheaves of generalized rank 3 on the ribbon (excluding rank 3 vector bundles on C, so, at least in this special case, Conjecture 4.8 has to hold).

4.3. Higher multiplicity

About higher multiplicity there are only extremely partial results. So, we will give only a general conjecture and some properties of generalized vector bundles, where a generalized vector bundle is defined precisely as on ribbons (see Definition 4.7): a generalized vector bundle on C_n is a pure sheaf $\mathscr{F}$ generically isomorphic to $\mathscr{O}_{C_n}^{\oplus r}$ for some $r \geq 1$ (if $r = 1$, then $\mathscr{F}$ is a generalized line bundle). But before stating the conjecture, we need to define the indices of a generalized vector bundle (defined as on a ribbon) of any generalized rank. We introduce them only in this case because both the possible definitions (looking at the pure quotients or looking at graduate pieces of the first canonical filtration) coincide, as for generalized line bundles (see Proposition 2.22) and it is not clear which of them is the right one for the general case (maybe looking at graduate pieces of the first canonical filtration, but we have not worked out this question sufficiently).

DEFINITION 4.41. Let $\mathscr{F}$ be a generalized vector bundle on C_n, then its indices are $b_i(\mathscr{F}) = b_i = \mathrm{h}^0(\mathcal{T}_i)$ for $1 \leq i \leq n-1$, where $\mathcal{T}_i$ is the torsion subsheaf of $G_{n-1-i}(\mathscr{F})$.

We could define also the local indices, as for generalized line bundles. Now we can state the full conjecture.

CONJECTURE 4.42. *Assume $g_1 \geq 2$. The locus of stable pure sheaves of complete type $((r_0, \dots, r_{n-1}), (d_0, \dots, d_{n-1}))$ on C_n is, if non-empty, irreducible of dimension $1 + \sum_{i=0}^{n-1} (r_i)^2(g_1 - 1) - \sum_{0 \leq i < j \leq n-1} (r_i r_j) \deg(\mathcal{C})$.*

It is not empty if and only if all the inequalities coming from comparing the slope of the sheaf with that of the sheaves associated to the canonical filtrations are verified.

In particular, there exists a stable generalized vector bundle of generalized rank nr with indices $b_1, \dots, b_{n-1}$ if and only if $i \sum_{j=i}^{n-1} b_j - (n-i) \sum_{j=1}^{i-1} b_j < -\frac{in(n-i)}{2} r \deg(\mathcal{C})$, for any $1 \leq i \leq n-1$.

If there exists a quasi locally free sheaf of a given complete type, then the generic element of the locus of sheaves with that associated complete type is quasi locally free.

If $0 < -\deg(\mathcal{C}) \leq 2g_1 - 2$, then the closure of the locus associated to any complete type is an irreducible component of the moduli space. If $-\deg(\mathcal{C}) > 2g_1 - 2$, then the irreducible components are the loci corresponding to either sheaves generically of rigid type (i.e. sheaves whose generic stalk is isomorphic to $\mathscr{O}_{C_n,\eta}^{\oplus l} \oplus \mathscr{O}_{C_h,\eta}$ for some positive numbers l and h with $1 \leq h \leq n-1$) or generalized vector bundles, according to the congruency class of the generalized rank.

Probably, the irreducibility and the dimension are true also at level of stacks, so without assumptions of stability.

The semistability conditions of generalized vector bundles on C_n assume that special form because, as we will see later (see Lemma 4.44), the complete type of a generalized vector bundle is completely determined by its indices, as it happens for generalized line bundles. We will prove them for small n (i.e. $n = 3, 4, 5$) in the following, but first we need some easy lemmata about generalized vector bundles on C_n, for any n. They extend various results about generalized line bundles. We omit the proofs because they are almost immediate, using the above definition of the indices. They are almost identical to some of the properties of generalized line bundles seen in Section 2.2.

LEMMA 4.43. *Let $\mathscr{F}$ be a generalized vector bundle on C_n with indices $b_1, \dots, b_{n-1}$. It holds that $0 \leq b_i \leq b_{i+1}$, for any $1 \leq i \leq n-2$.*

LEMMA 4.44. *Let $\mathscr{F}$ be a generalized vector bundle on C_n of generalized rank nr, generalized degree D and indices $b_1, \dots, b_{n-1}$. For any $1 \leq i \leq n-1$, it holds that*

$$\mathrm{Deg}(\mathscr{F}^{(i)}) = \frac{1}{n}\left[iD - i \sum_{j=1}^{n-i-1} b_j + (n-i) \sum_{j=n-i}^{n-1} b_j + \frac{in(n-i)r}{2} \deg(\mathcal{C}) \right];$$

$$\mathrm{Deg}(\mathscr{N}^i\mathscr{F}) = \frac{1}{n}\left[(n-i)D + i\sum_{j=1}^{n-i-1} b_j - (n-i)\sum_{j=n-i}^{n-1} b_j + \frac{in(n-i)r}{2}\deg(\mathcal{C})\right].$$

For any $0 \leq i \leq n-1$, *it holds that*

$$\deg(G_i(\mathscr{F})) = \frac{1}{n}\left[D - \sum_{j=1}^{n-1} b_j + nb_{n-i-1} + \frac{2i+1-n}{2} r\deg(\mathcal{C})\right];$$

$$\deg(G^{(i+1)}(\mathscr{F})) = \frac{1}{n}\left[D - \sum_{j=1}^{n-1} b_j + nb_{n-i-1} + \frac{n-2i-1}{2} r\deg(\mathcal{C})\right],$$

where $b_0 = 0$.

COROLLARY 4.45. *Let* $\mathscr{F}$ *be a generalized vector bundle on* C_n *of generalized rank* nr, *generalized degree* D *and indices* $b_1, \ldots, b_{n-1}$. *Then* $D - \sum_{j=1}^{n-1} b_j$ *is congruent to* $(n-2i-1)r\deg(\mathcal{C})/2$ *modulo* n.

We will need also the next lemma relating the indices of the subsheaves of the canonical filtrations of a generalized vector bundle $\mathscr{F}$ with those of $\mathscr{F}$ itself.

LEMMA 4.46. *Let* $\mathscr{F}$ *be a generalized vector bundle on* C_n. *For any* $2 \leq i \leq n-1$ *and for any* $1 \leq j \leq i-1$, *it holds that*

$$b_j(\mathscr{F}^{(i)}) = b_{n-i+j}(\mathscr{F}) - b_{n-i}(\mathscr{F}),$$

while $1 \leq i \leq n-2$ *and for any* $1 \leq j \leq n-i-1$, *it holds that*

$$b_j(\mathscr{N}^i\mathscr{F}) = b_j(\mathscr{F}/\mathscr{F}^{(i)}) = b_j(\mathscr{F}).$$

Now we can state the proposition about the semistability conditions for generalized vector bundles in small multiplicities.

PROPOSITION 4.47. *Let* $n = 3, 4, 5$.
There exists a semistable generalized vector bundle $\mathscr{F}$ *of generalized rank* nr, *of generalized degree* D *and of indices* $b_1, \ldots, b_{n-1}$ *on* C_n, *with* $D - \sum_{j=1}^{n-1} b_j$ *congruent to* $(n-2i-1)r\deg(\mathcal{C})/2$ *modulo* n, *if and only if the following inequalities hold:*

$$i\sum_{j=i}^{n-1} b_j - (n-i)\sum_{j=1}^{i-1} b_j \leq -\frac{irn(n-i)}{2}\deg(\mathcal{C}),\ \forall\, 1 \leq i \leq n-1.$$

There exists such a stable $\mathscr{F}$ *if and only if all the above inequalities are strict.*

PROOF. Throughout the proof, we will set $\delta = -\deg(\mathcal{C})$. We prove only the semistable case, because the stable one is almost identical.

The necessity is obvious: the inequalities are equivalent to $\mu(\mathscr{F}^{(i)}) \leq \mu(\mathscr{F})$, for any $1 \leq i \leq n-1$ (this holds for any positive integer n, not only for those in the statement).

For the sufficiency, we start with the case $n = 3$. Let us consider three stable vector bundles of rank r on C of the right degrees (see the end of Lemma 4.44) so that they can play the role of $G^{(1)}(\mathscr{F})$, $G^{(2)}(\mathscr{F})$

and $G^{(3)}(\mathscr{F})$. We need to find a semistable generalized vector of generalized rank $2r$ on C_2 playing the role either of $\mathscr{F}^{(2)}$ or of $\mathscr{F}/\mathscr{F}^{(1)}$. Indeed, if we build $\mathscr{F}$ as an extension of the appropriate sheaves, we have that $\mu(\mathscr{G}) \leq \mu(\mathscr{F})$, for any $\mathscr{G} \subset \mathscr{F}$. More precisely, if $\mathscr{F}^{(2)}$ is semistable, $\mu(\mathscr{G}^{(2)}) \leq \mu(\mathscr{F}^{(2)})$ and $\mu(G^{(3)}(\mathscr{G})) \leq \mu(G^{(3)}(\mathscr{F}))$ and we can conclude (if $G^{(3)}(\mathscr{G}) \neq 0$, otherwise $\mathscr{G} \subset \mathscr{F}^{(2)}$ and we have already done) by Lemma 4.17, because $3r\,\mathrm{R}((\mathscr{G}^{(2)})) \geq 2r\,\mathrm{R}(\mathscr{G})$ (this inequality holds being equivalent to $\mathrm{R}((\mathscr{G}^{(2)})) \geq 2\,\mathrm{rk}(G^{(3)}(\mathscr{G}))$, which is trivial). On the other side, if $\mathscr{F}/\mathscr{F}^{(1)}$ is semistable, we can conclude by the same Lemma, looking at $\mathscr{G}^{(1)} \subset \mathscr{F}^{(1)}$ and $\mathscr{G}/\mathscr{G}^{(1)} \subset \mathscr{F}/\mathscr{F}^{(1)}$ (the proof is analogous).

By Proposition 4.29, we can find such an appropriate semistable generalized vector bundle on C_2 if either $b(\mathscr{F}^{(2)}) = b_2 - b_1$ (see Lemma 4.46) is less than or equal to $r\delta$ or $b(\mathscr{F}/\mathscr{F}^{(1)}) = b_1$ is less than or equal to $r\delta$. One of the two options has to hold. Indeed, if they were both false, we would obtain that $b_2 + b_1 > 3\delta$, contradicting the hypothesis.

The proceeding is similar for $n = 4$, so we omit the details. We need to find either a semistable generalized vector bundle of generalized rank $3r$ on C_3, in order to play the role of either $\mathscr{F}/\mathscr{F}^{(1)}$ or $\mathscr{F}^{(3)}$, or a pair of semistable generalized vector bundles of generalized rank $2r$ which could play the role of $\mathscr{F}/\mathscr{F}^{(2)}$ and $\mathscr{F}^{(2)}$. Thus, by the previous case of multiplicity 3, by Proposition 4.29 and by Lemma 4.46, we need that one of the three following systems of inequalities holds:

$$\begin{cases} b_2 + b_1 \leq 3r\delta \\ 2b_2 - b_1 \leq 3r\delta \end{cases}, \qquad \begin{cases} b_3 + b_2 - 2b_1 \leq 3r\delta \\ 2b_3 - b_2 - b_1 \leq 3r\delta \end{cases}, \qquad \begin{cases} b_1 \leq r\delta \\ b_3 - b_2 \leq r\delta \end{cases}.$$

It is easy, although tedious, to check that they cannot be simultaneously false, under our hypotheses.

Indeed, if $b_1 + b_2 > 3r\delta$ and $2b_3 - b_2 - b_1 > 3r\delta$, then it would follow that $b_1 + b_2 + b_3 > 6r\delta$, contradicting the first inequality assumed.

If $b_3 - b_2 > r\delta$, $b_1 + b_2 > 3r\delta$ and $b_3 + b_2 - 2b_1 > 3r\delta$, then we would have again $b_1 + b_2 + b_3 > 6r\delta$.

Also if $b_1 > r\delta$ and $b_3 + b_2 - 2b_1 > 3r\delta$, then it would hold that $b_1 + b_2 + b_3 > 6r\delta$.

So, if $b_1 + b_2 > 3r\delta$, one of the two last systems has to hold.

On the other hand, if $2b_2 - b_1 > 3r\delta$ and $b_3 - b_2 > r\delta$, then we would have $3b_3 - b_2 - b_1 > 6\delta$, contradicting the last inequality assumed.

If $2b_2 - b_1 > 3r\delta$, $2b_3 - b_2 - b_1 > 3r\delta$ and $b_1 > r\delta$, then it would hold that $b_3 + b_2 - b_1 > 4r\delta$. But this contradicts the second inequality of the hypothesis.

As already observed, it cannot happen that both $b_1 > r\delta$ and $b_3 + b_2 - 2b_1 > 3r\delta$. Hence, also if $2b_2 - b_1 > 3r\delta$, one of the last two systems has to hold.

Now we can assume $n = 5$. Also this case is similar to the previous ones and we omit most of the details. We want to find appropriate semistable generalized vector bundles of generalized rank ir on C_i, with $i = 2, 3, 4$, playing the role of $\mathscr{F}/\mathscr{F}^{(1)}$ or $\mathscr{F}/\mathscr{F}^2$ and $\mathscr{F}^2$ or $\mathscr{F}/\mathscr{F}^3$ and $\mathscr{F}^3$ or $\mathscr{F}^{(4)}$.

This reduces, by the same arguments of above, to show that the following systems cannot be simultaneously false:

$$\begin{cases} b_3 + b_2 + b_1 \leq 6r\delta \\ b_3 + b_2 - b_1 \leq 4r\delta \\ 3b_3 - b_2 - b_1 \leq 6r\delta \end{cases}, \qquad \begin{cases} b_2 + b_1 \leq 3r\delta \\ 2b_2 - b_1 \leq 3r\delta \\ b_4 - b_3 \leq r\delta \end{cases},$$

$$\begin{cases} b_1 \leq r\delta \\ b_4 + b_3 - 2b_2 \leq 3r\delta \\ 2b_4 - b_3 - b_2 \leq 3r\delta \end{cases}, \qquad \begin{cases} b_4 + b_3 + b_2 - 3b_1 \leq 6r\delta \\ b_4 + b_3 - b_2 - b_1 \leq 4r\delta \\ 3b_4 - b_3 - b_2 - b_1 \leq 6r\delta \end{cases}.$$

It can be checked by hand that, under our hypotheses, one of these systems has to hold; we omit the verification, because it is excessively long and the ideas are essentially the same of the case $n = 4$. *q.e.d.*

REMARK 4.48. Probably, the above proof could be adapted for any n. The problem is that it is not possible to check by hand that the appropriate systems cannot be simultaneously false (for arbitrary n) and I have not found a way to prove it in general.

We end this brief section stating separately the special case of Conjecture 4.42 for generalized rank equal to n. It is similar to Conjecture 4.40, although more vague:

CONJECTURE 4.49. *Let C_n be a primitive multiple curve of multiplicity n such that $\delta = -\deg(\mathcal{C}) > 0$ and $g_1 \geq 2$, where g_1 is the genus of its reduced subcurve.*

(i) *If $\delta \leq 2(g_1 - 1)$, then the irreducible components of $\mathrm{M}(C_n, P_D)$ are the closures of the loci of stable sheaves of fixed complete type, for each complete type for which stable sheaves exist. For each type, there is at least one irreducible component whose generic element is of that type.*

(ii) *If $\delta > 2(g_1 - 1)$, the only irreducible components of $\mathrm{M}(C_n, P_D)$ are those whose generic elements are generalized line bundles (described in Theorem 3.16).*

APPENDIX A

Relation with Higgs bundles

One of the motivations for the study of pure sheaves on a primitive multiple curve is, as anticipated in the introduction, their relation with nilpotent Higgs bundles (or, in other words, with the nilpotent cone of the Hitchin system), due to the spectral correspondence (cf., e.g., for a brief introduction, [**MRV2**, Appendix] or, for a wider treatment, [**HP**]; another introduction to the Hitchin system and its nilpotent cone, with also relations with the so-called Mukai system, can be found in [**DEL**]). Although only a special class of primitive multiple curves, i.e. trivial ones, is involved in it, this correspondence can be used as a confirmation of some of our results, thanks to the fact that Higgs bundles are much more studied in literature than sheaves on a primitive multiple curve. The aim of this appendix is to explore briefly this relation.

Let C be a connected smooth and projective curve of genus g over an algebraically closed field $\mathbb{K}$. Recall that a Higgs bundle of rank n on C is a pair $(\mathcal{E},\phi)$, where $\mathcal{E}$ is a vector bundle of rank n on C and ϕ is a morphism from $\mathcal{E}$ to $\mathcal{E}\otimes\omega_C$. A Higgs bundle $(\mathcal{E},\phi)$ is said to be nilpotent if there exists a positive integer $m\leq n$ such that $\phi^m=0$, where $\phi^i:\mathcal{E}\otimes\omega_C^{\otimes i-1}\to\mathcal{E}\otimes\omega_C^{\otimes i}$ is defined recursively as $\phi^{i-1}\otimes\mathrm{id}_{\omega_C}$. Recall also that a Higgs bundle $(\mathcal{E},\phi)$ is said to be semistable (resp. stable) if for all proper subsheaves $\mathcal{F}$ such that $\phi(\mathcal{F})\subset\mathcal{F}\otimes\omega_C$ it holds that $\mu(\mathcal{F})\leq\mu(\mathcal{E})$ (resp. $\mu(\mathcal{F})<\mu(\mathcal{E})$), where μ denotes the slope. Consider the $\mathbb{P}^1$-fibration $p:\mathbb{P}(\mathscr{O}_C\oplus\omega_C^{-1})\to C$ and let $\mathscr{O}(1)$ be the relatively ample sheaf on $\mathbb{P}(\mathscr{O}_C\oplus\omega_C^{-1})$. The spectral cover of C with respect to the zero vector $\underline{0}=(0,\dots,0)$ in $\bigoplus_{i=1}^n H^0(C,\omega_C^{\otimes i})$, i.e. that inducing the spectral correspondence for nilpotent Higgs bundle of rank n, is a degree-n finite morphism $\pi_{\underline{0}}:X\to C$, with $X=C_n$ the primitive multiple curve of multiplicity n in $\mathbb{P}(\mathscr{O}_C\oplus\omega_C^{-1})$ defined as the zero locus of y^n, where y is the section of $\mathscr{O}(1)\otimes p^*(\omega_C)$ whose pushforward via p corresponds to the constant section $(0,1)$ of the vector bundle $p_*(\mathscr{O}(1)\otimes p^*(\omega_C))=(\mathscr{O}_C\oplus\omega_C^{-1})\otimes\omega_C=\omega_C\oplus\mathscr{O}_C$. In this case the conormal bundle $\mathcal{C}$ of C in X is exactly ω_C^{-1}. The restriction $y|_X$ can be seen also as a section of $[\mathscr{O}(1)\otimes p^*(\omega_C)]|_X=p^*(\omega_C)|_X=\pi_{\underline{0}}^*(\omega_C)$.

In this context, the spectral correspondence is an isomorphism Π between the moduli space of pure semistable sheaves of generalized rank n on X and nilpotent semistable Higgs bundles of rank n on C (it holds also at level of stacks, hence without assuming semistability). If $\mathscr{F}$ is a stable pure sheaf of generalized rank n on X, then $\Pi(\mathscr{F})=((\pi_{\underline{0}})_*(\mathscr{F}),\phi)$, with $\phi:(\pi_{\underline{0}})_*(\mathscr{F})\to(\pi_{\underline{0}})_*(\mathscr{F})\otimes\omega_C$ given by multiplication with $y|_X\in\mathrm{H}^0(X,\pi_{\underline{0}}^*(\omega_C))$. The above lines remain true for $\mathcal{L}$-twisted Higgs pairs, where ω_C is substituted by an arbitrary line bundle $\mathcal{L}$ on C.

A nilpotent Higgs bundle $(\mathcal{E},\phi)$ admits the following filtration introduced by Laumon in [**Lau**]:

$$0=\mathcal{K}_0(\mathcal{E},\phi)=\mathcal{K}_0\subset\mathcal{K}_1(\mathcal{E},\phi)=\mathcal{K}_1\subset\cdots\subset\mathcal{E},$$

where $\mathcal{K}_i(\mathcal{E},\phi)=\mathcal{K}_i=\ker(\phi^i)$. Denote by $\bar{\phi}_i$ the monomorphism from $\mathcal{K}_i/\mathcal{K}_{i-1}$ to $\mathcal{K}_{i-1}/\mathcal{K}_{i-2}\otimes\omega_C$ induced by ϕ (for basic properties of the filtration and of its quotients, see [**Lau**, Lemme 1.6]). Let $m+1$ be the length of the filtration, i.e. $\mathcal{K}_m=\mathcal{E}$; we can define the *nilpotent type* of $(\mathcal{E},\phi)$ as the pair $(\nu_\cdot,\lambda_\cdot)=((\nu_1,\lambda_1),\ldots,(\nu_m,\lambda_m))$, where $\nu_i=\operatorname{rk}(\mathcal{K}_i/\mathcal{K}_{i-1})$ and $\lambda_i=\deg(\mathcal{K}_i/\mathcal{K}_{i-1})$ (for more details, see [**Lau**, Définition 1.7]).

It follows from the various definitions that, if $\mathscr{F}$ is a pure sheaf of generalized rank n on X and $\Pi(\mathscr{F})=(\mathcal{E},\phi)$ is the corresponding nilpotent Higgs bundle of rank n on C under spectral correspondence, then $(\pi_0)_*\mathscr{F}^{(i)}=\mathcal{K}_i(\mathcal{E})$ for any $1\leq i\leq m$, where $0=\mathscr{F}^{(0)}\subset\mathscr{F}^{(1)}\subset\cdots\subset\mathscr{F}^{(m-1)}\subset\mathscr{F}^{(m)}=\mathscr{F}$ is, as usual, the second canonical filtration of $\mathscr{F}$. This implies that

$$\lambda_i(\mathcal{E})+\nu_i(\mathcal{E})\chi(\mathscr{O}_C)=\chi(\mathscr{F}^{(i)})-\chi(\mathscr{F}^{(i-1)}),\ \text{for } 1\leq i\leq m. \tag{A.1}$$

Moreover, $\mathscr{F}$ is a generalized line bundle if and only if $m=n$ and $\nu_1(\mathcal{E})=\cdots=\nu_n(\mathcal{E})=1$. More generally, the complete type of a sheaf $\mathscr{F}$ on X and the nilpotent type of $\Pi(\mathscr{F})$ are related by the following formulae: $\nu_i=\operatorname{rk}(G^{(i)}(\mathscr{F}))=\operatorname{rk}(G_{i-1}(\mathscr{F}))$ and $\lambda_i=\deg(G^{(i)}(\mathscr{F}))=\deg(G_{i-1}(\mathscr{F}))+\big(\sum_{j=i}^{n-1}\operatorname{rk}(G_j(\mathscr{F}))+(i-1)\operatorname{rk}(G_{i-1}(\mathscr{F}))\big)\deg(\omega_C)$.

In order to justify our conjectures, we cited a couple of times [**Bo**] and we said we needed to translate Bozec's language into ours. In particular, our conjectures are related to [**Bo**, Corollary 2.4], asserting that the set of the irreducible components of the global nilpotent cone (which is a stack) is given by the set of the closures of the loci of fixed Jordan type (which, as we will see in the following lines, is equivalent to the complete type), and to [**Bo**, Theorem 3.1 and Proposition 4.1], giving necessary and sufficient conditions for a Jordan type to be semistable (i.e. for the existence of semistable nilpotent Higgs bundles of that type) in terms of the so-called canonical regions of the associated polytope (we do not recall these definitions, but these conditions are equivalent to checking the semistability inequality only for subsheaves related to the canonical filtrations). The *Jordan type* of a nilpotent Higgs bundle $(\mathcal{E},\phi)$ with nilpotency order n is a vector (r_k,d_k) of pairs of integers and, as anticipated few lines above, it contains the same information of our complete type. Indeed, r_k is defined as the rank of $\ker((\mathcal{E}_{k-1}/\mathcal{E}_k)\otimes\omega_C^{\otimes k-1}\twoheadrightarrow(\mathcal{E}_k/\mathcal{E}_{k+1})\otimes\omega_C^{\otimes k})$ while d_k is defined as the degree of the same vector bundle, where $\mathcal{E}_k=\operatorname{im}(\phi^k)\otimes\omega_C^{\otimes -k}=(\mathcal{E}/\mathcal{K}_k)\otimes\omega_C^{\otimes -k}$. Hence, the filtration

$$0=\mathcal{E}_n\subset\mathcal{E}_{n-1}\subset\cdots\subset\mathcal{E}_1\subset\mathcal{E}_0=\mathcal{E},$$

corresponds, under spectral correspondence, to the first canonical filtration of $\mathscr{F}$. Therefore, if $\mathscr{F}$ is the coherent sheaf on X corresponding to $(\mathcal{E},\phi)$, we have that $r_k=\operatorname{rk}(G_{k-1}(\mathscr{F}))-\operatorname{rk}(G_k(\mathscr{F}))$ and $d_k=\deg(G_{k-1}(\mathscr{F}))-\deg(G_k(\mathscr{F}))-((k-1)\operatorname{rk}(G_{k-1}(\mathscr{F}))-k\operatorname{rk}(G_k(\mathscr{F})))\deg(\mathcal{C})$.

Related to Higgs bundles there are also chains (for a short introduction to them, cf. e.g. [**GPH**, §2.2]). A split nilpotent Higgs bundle $(\mathcal{E},\phi)$ (also called a Hodge Higgs bundle) corresponding to a generalized line bundle is essentially the same thing as the chain $((0,\mathcal{K}_i/\mathcal{K}_{i-1}\otimes\omega_C^{\otimes -i})_{i=1,\dots,n},(\bar{\phi}_i\otimes 1_{\omega_C^{-i+1}})_{i=1,\dots,n})$. Thus the n_i's of [**GPH**] are equal to the ν_i's of Laumon, while the d_i's of [**GPH**] are equal to $\lambda_i - i(2g-2)$. It is possible to introduce a slope of chains and a notion of slope-(semi)stabilty of chains with respect to parameters $\alpha_. = (\alpha_i)$ (for precise definitions cf. again [**GPH**]). The (semi)stability of $(\mathcal{E},\phi)$ is equivalent to that of the associated chain with respect to the parameter $\alpha_i = i(2g-2)$.

Thanks to these observations, in the special case where $(\mathcal{E},\phi)$ corresponds under spectral correspondence to a generalized line bundle $\mathscr{F}$ on X, the inequalities in [**GPH**, Proposition 4(1)], i.e. $\sum_{j=0}^{i}\lambda_j/i \le \sum_{j=0}^{n}\lambda_j/n$, for $1\le j\le n-1$, are equivalent to those in Theorem 2.41 and that those in [**GPH**, Proposition 4(2)], i.e. $\lambda_j\le\lambda_{j-1}+2g-2$, are equivalent to the fact $b_i(\mathscr{F})\le b_{i+1}(\mathscr{F})$ (cf. Lemma 2.18). This means that our results generalize [**GPH**, Proposition 4] to any primitive multiple curve, in the special case of generalized line bundles.

Indeed, it holds that $\sum_{j=0}^{i}\nu_j = \mathrm{R}(\mathscr{F}^{(i)})$ and, by equations (A.1), it holds also that $\sum_{j=0}^{i}\lambda_j = \chi(\mathscr{F}^{(i)}) - \mathrm{R}(\mathscr{F}^{(i)})\chi(\mathscr{O}_C) = \mathrm{Deg}(\mathscr{F}^{(i)})$, for $1\le i\le m$; thus the m inequalities in [**GPH**, Proposition 4(1)] are $\mu(\mathscr{F}^{(i)})\le\mu(\mathscr{F})$, for $1\le i\le m$, which are always necessary conditions for the (semi)stability of $\mathscr{F}$. Moreover, when $\mathscr{F}$ is a generalized line bundle, it holds that $m=n$ and, by Remark 2.12, these inequalities are $\mu(\overline{\mathscr{F}}_{n-i})\ge\mu(\mathscr{F})$, for $1\le i\le n$, which are exactly the inequalities (2.5) (see the proof of Theorem 2.41).

About the second inequalities, it follows from equations (A.1) and (2.2) that

$$\lambda_j = \frac{1}{n}\left[D - \sum_{h=1}^{n-j} b_h + (n-1)b_{n-j} + \sum_{h=n-j+1}^{n-1} b_h - n(n-2j+1)(g-1)\right],$$

for $1\le j\le n-1$. Therefore, $\lambda_j\le\lambda_{j-1}+2g-2$ is equivalent to $b_{n-1}\le b_{n-j+1}$.

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