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Risk and asset management models: theoretical and practical aspects

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Ai miei figli e mia moglie
che sono il bene più prezioso che possiedo

Contents

List of Tables

List of Figures

Chapter 1

Introduction

In the last 20 years the financial markets have been characterized by several crisis, that have produced crashes, corrections and bear markets[1]. In the past few decades, one may recall Black Monday (October 19, 1987), when the Dow Jones fell more than 20% and many quantitative portfolio insurances (option-based portfolio insurance (OBPI) and constant proportion portfolio insurance (CPPI)) collapsed, and Black Wednesday (September 16, 1992), when the British government was forced to withdraw the pound sterling from the European Exchange Rate Mechanism. Later, in 1997, the UK Treasury estimated the cost of Black Wednesday for £3.4 billion. At the end of the 1990s, the Russian government and the Russian Central Bank devalued the ruble and defaulted (1998 Russian financial crisis).

An other example of a flash crash occurred on May 6, 2010, when the Dow plummeted almost 1000 points in just a few minutes. This was caused by a technical malfunction of quantitative trading programs. We may recall an example of irrational exuberance that drove stock prices to unsustainable levels at the end of the dot com bubble in 1999, which caused the NASDAQ to drop in 2000 and became a bear market. From May 10, 1995 to March 10, 2000 the NASDAQ index earned 506.93%, whereas one year later, on March 9, 2001, this index lost 59% (as shown figure 1.1), back to price levels seen in the summer of 1998.

In 2004, B. Mandelbrot published a book with the emblematic title '*The (Mis)Behavior of Markets: A Fractal View of Risk, Ruin, and Reward*' in which he claimed that bubbles are unavoidable and endemic to the markets [73]. He also questioned other authors about their main assumption: normally

[1] A stock market crash is when a stock index drops severely in a day or two of trading. A correction is when the market falls 10% from its 52-week high over days, weeks, or even months. A bear market is defined when the market falls another 10%, for a total decline of 20% or more.

Figure 1.1: Time series of Nasdaq index cumulative returns from March 10, 2000 to March 9, 2001.

distributed asset returns. In his book he introduced new theories based on a fractal view of the financial world, highliting the fat tail behaviour of asset returns.

In 2007, the first signs of something going wrong with the American residential mortgage market could be observed. Bond markets had been flooded with securitizations of all kinds. From December 31, 1999 to March 31, 2007 the American real estate market, measured by the index FHFA US House Price, had grown almost like NASDAQ had done in the late 1990s (+68.46%). Suddenly house prices collapsed, from March 31, 2007 to September 30, 2009 that index lost 12.5% (see figure 1.2) and it come back to the March 2005 price (as shown figure 1.4). In that period borrowers did not repay loans that banks had entrusted them (in figure 1.3 we present the % of the loan delinquencies in the USA) and the delinquencies reached 10% of the total loans. This unexpected event led banks into a liquidity crisis. It was the beginning of a domino effect that led to:

- the introduction of liquidity by the FED;
- the expansive monetary policy with interest rates close to zero;
- the acquisition of Fannie Mae and Freddie Mac by the government[2];
- the rescue of Bear Stearns and AIG by the government;
- the acquisition of Merrill Lynch by Bank of America;

[2]On September 7, 2008 were being placed into conservator-ship of the Federal Housing Finance Agency (FHFA). This action was seen as one of the most relevant government intervention in the capital and bond markets.

Figure 1.2: Time series of FHFA US House price cumulative returns from March 30, 2007 to September 30, 2009.

Figure 1.3: Time series of US Delinquencies As % Of Total Loans SA from June 30, 2003 to December 31, 2017.

- the bankruptcy of Lehman Brothers.

In 2008, as a result of this big bubble, the market crumbled. The Standard and Poor's 500 index during the crisis fell roughly by 52%. In figure 1.5 we present the time series of Standard and Poor's 500 index where we can easily identify the deep drawdown periods.

Before the subprime crises, traditional strategic asset allocation theory was deeply rooted in the mean-variance portfolio optimization framework developed by Markowitz [74, 77] and, therefore, the risk-gain analysis has become a key issue for portfolio selection. However, the mean-variance optimization methodology can be very sensitive to the input parameters required by the model. Therefore, the problem of estimation errors becomes very relevant, particularly for the expected returns of the assets. In addition, subjective es-

Figure 1.4: Time series of FHFA US House price from December 31, 1999 to December 31, 2017.

Figure 1.5: Time series of S&P500 index from December 31, 1999 to May 7, 2018.

timates of expected future returns can frequently be influenced and modified by individual biases or by a priori investor view, such as the overestimation of expected returns due to the really strong momentum of a particular asset class or the economic cycle. Investors can also underestimate risk when they consider one particular type of distribution of returns which may result in ignoring fat tails when markets collapse. As such, parameter estimation based on past realized observations can contain a certain level of noise, especially if the risk premia and correlations are time-varying.

The reaction to any crises, once the imminent danger has subsided, should be to keep calm and to look back, evaluating what did not go as planned and developing methods and strategies to avoid or mitigate the impact of future crises with similar characteristics. The recent financial crisis is no exception. As the markets began to recover, rumors started circulating about 'next-generation solutions'. Indeed, there has been strong criticism of the process of portfolio selection based on mean-variance optimization and on the traditional 60/40 split between stocks and bonds. Both methods of constructing a portfolio were the core asset allocation process employed by many institutions and private investors.

The first and well-known rule in asset allocation is 'diversification'. This principle evidently has its roots in common sense, rather than in portfolio theory, and therefore dates back to the dawn of time. The Talmud, a collection of Jewish laws created over 2000 years ago, suggests investing the money as follows: *one third in real estate, one third in shares and one third in liquid reserves*. In the first act of 'The Merchant of Venice', written at the end of the sixteenth century by Shakespeare, the protagonist Antonio says that his wealth is *spread over different vessels, in different places and is destined to mature at different times*. And in 1738 the great Swiss mathematician Daniel Bernoulli stated precisely that '*...it is advisable to divide the assets exposed to risk, rather than putting them at risk together at the same time*'. In a nutshell, the idea of diversification comes from afar and is well rooted in all cultures. Some investors do not seem concerned by risk contribution and are thinking that a traditional 60/40 portfolio offers a real diversified portfolio.

This thesis will deal with the problem facing investors in seeking to determine a 'well' diversified asset allocation strategy. The thesis reviews the widely used models to build up a portfolio avoiding the explicit use of expected returns, so as minimum Risk, Risk Diversifications, and Capital Diversification strategies. Our contribution to the literature is to provide the Equal Risk Contribution (ERC) strategy based on Conditional Value at Risk (CVaR) and on Conditional Value at Risk-deviation (CVaR-deviation) as a risk measure. More precisely, we first tackle these problems by means of a least-squares approach and, then, under appropriate conditions, we provide an alternative approach to finding a CVaRERC portfolio as a solution to a convex optimization problem.

From a theoretical and practical viewpoint, the solution of the convex problem can be easy to find (at least for a discrete probability space), but another issue still remains: how do we can generate the discrete probability space? Is it sufficient to use past realized asset returns or can we use a more clever approach? In this thesis we consider both historical and simulated scenarios as the input to portfolio selection problem specification. This means that we use two different Risk Management approaches: Historical Simulation and Historical Filtered Bootstrap.

Summing up, the questions that we would like to answer are:

1. Could we formulate and solve an ERC problem based on a coherent risk measure as CVaR?

2. How intense is the effect of estimation errors on the ERCCVaR model?

3. Could we mitigate this effect?

4. Is there any difference between a historical scenario and a simulated scenario?

In order to address such questions, at the end of this chapter we discuss the motivations of our research. In chapter 2 we review the risk-based portfolio models and introduce the backtesting setup, also presenting the main results obtained using real data. In particular we explore seven datasets composed of equity, bonds and commodities as base for the backtests. In chapter 3 we extend the ERC to CVaR-deviation and we investigate the stability of the ERC CVaR-deviation solutions with respect to the input parameters. In chapter 4 we review the Historical Filtered Bootstrap procedure that allows us to generate future scenarios. We show how to use those scenarios for portfolio selection and we also validate the Historical Filtered Bootstrap model using both statistical accuracy and efficiency evaluation tests. In Chapter 5 we provide a new risk management model that is a nice tool to estimate risk as good as the sophisticated Historical Filtered Bootstrap strategy. In Chapter 6 we apply our Risk management approach to a balanced portfolio. That is a case study particular relevant for financial industry because any asset managers all over the world provide balanced investment solutions to their client. Indeed, according to the data provided by BlackRock Investment Management and State Street Global Advisor[3], the US institutional investors and pensions fund

[3]These two companies kindly provided their Asset Under Management on June, 30 2018.

are investing their wealth for about 5.5 Trillion US dollar on balanced portfolio in the bond and equity market. Finally, Chapter 7 concludes.

1.1 Motivations

The Equal Risk Contribution (or Risk Parity) approach selects a portfolio that is characterized by the requirement of having equal total risk contribution from each asset. The literature is typically focused on volatility as a risk measure, however, according to the axioms of a coherent risk measure provided by Artzner at al. (1999) (see [6]) volatility is not coherent risk measure. Our contribution to literature is to extend the formulation of the Equal Risk Contribution (ERC) strategy to CVaR that is a particular coherent risk measure. In addition, we study the ERC portfolios using the CVaR-deviation. that is a deviation risk measure (CVaR-deviation) similarly to volatility.

In this dissertation we investigate and compare the properties of these following three different types of portfolio selection approaches, that we named: *Minimum-Risk*, *Capital-Diversification* and *Risk-Diversification*. For the *Minimum-Risk* approach we consider two risk measures, variance and Conditional Value-at-Risk (CVaR). Then we extend the analysis to the Conditional Value-at-Risk deviation (CVaR-deviation). The *Capital-Diversification* strategy is represented by the Equally Weighted portfolio, which is also considered a benchmark in our experiments. For the *Risk-Diversification* models, we examine the Risk Parity (RP) portfolio as in Maillard et al (2010) (see [72]), and the Naive Risk Parity (NRP) portfolio, where the weight of an asset is proportional to the inverse of its volatility. Note that this is the solution to the RP model when the linear correlation among the assets is constant.

From a theoretical point of view, we suggest new developments of the ERC approach based on CVaR as risk measure, where the objective is to select a portfolio such that the total CVaR contributions of all assets are equal among them. Similar to the NRP portfolio we find a closed-form solution for the CVaR Equal Risk Contribution (CVaRERC) model in the case of the worst CVaR scenario. We call it Naive CVaR Equal Risk Contribution (NCVaR-ERC) portfolio. In this case, the weight of an asset is proportional to the inverse of its intrinsic CVaR. To achieve a uniform distribution of risk allocation in the general case, we investigate a least-squares model, where CVaR-ERC portfolios are included in the set of optimal solutions. However, since the existence of a CVaRERC solution is not always guaranteed (see [34]), our formulation can lead to a feasible portfolio which is as close as possible to a

CVaRERC portfolio. Furthermore, under appropriate conditions, we provide an alternative approach to finding a CVaRERC portfolio as a solution of a convex optimization problem, also showing that the CVaR of the CVaRERC portfolio is between those of the Minimum CVaR and of the EW portfolios. We also extend the formulation of the convex optimization to ERC using the CVaR-deviation as a risk measure, defined by Rockafellar et al. (2006) (see [97]).

We also investigate the effect of possible estimation errors to the input parameters of the portfolio selection models on the optimal portfolio weights, following a similar to that proposed by Kondor et al. (2007) ([61]).

For the real-world backtest of models we use both historical and simulated scenarios. The covariance matrix, which is needed in the *Minimum-Risk* and in the *Risk-Diversification* strategies, is estimated in different ways using daily returns on rolling windows of 500 days. More precisely, for historical scenarios we adopt: the Maximum Likelihood (ML) covariance; the Exponential Weighted Moving Average (EWMA) covariance; the Shrinkage estimator between ML and EWMA covariances; and the covariance matrix with Kendall correlation, obtained by combining the intrinsic variances of the assets returns and their Kendall correlations. We also use the covariance matrix obtained by a Monte Carlo Simulation technique known as Historical Filtered Bootstrap (see [8], [9] and [113]). Also for *Minimum-Risk* and *Risk-Diversification* strategies based on CVaR as a risk measure, we adopt both historical and simulated (Historical Filtered Bootstrap) scenarios. This means that, from a Risk Management viewpoint, we are analyzing two different methods to evaluate the future portfolio return distribution.

We propose a way to reduce the estimation error increasing the number of observations (T). We noted that in the real world there is the lack of data in financial time series. To let the estimate be accurate it could need more than ten years of history. Thus, we will show that using Historical Filtered Bootstrap we can generate millions of scenarios using only the last 500 daily returns (two years history).

The Historical Filtered Bootstrap as a risk management tool was introduced by Barone-Adesi et al. (1999) (see [8]). In order to validate this model that was already intensively backtested by some authors (see [24] and [1] and reference therein) we put in place a comparative backtest of the Historical Filtered Bootstrap versus the common risk models used in the financial industry. We contribute to the literature presenting a new risk model (called Shrunk Volatility VaR) that reaches the same performance of the HFB models that could be used by portfolio managers to have a first (good) estimate of the portfolio risk.

1.2 Literature Background

The popular Capital Asset Pricing Model (CAPM) shows that the *market* portfolio is an optimal choice for an investor and that the CAPM equation can be obtained by assuming that portfolio selection is done under the Markowitz's framework. In a nutshell, under the CAPM assumptions (see [104]), in condition of equilibrium, the optimal portfolio choice coincides with the *market* portfolio for all investors. Furthermore, it has been shown that asset expected excess returns must be proportional to the product between the market portfolio expected excess return and the beta coefficients of the assets, measuring not-diversifiable systematic risk (see [104]). During the 1990s, there was a rapid development of this passive management and, at the same time, a rapid growth of the number of institutional investors. Many of these players used passive management to invest in equity and bond markets. Regarding strategic asset allocation, they basically used the mean-variance analysis developed by Markowitz [74, 77] even though such an approach presents high sensitivity to input parameters, particularly for expected returns (see [20], [21] and [37]). One trivial reason is that there was no other alternative quantitative models for portfolio selection. Other motivations that favored the spread of the mean-variance approach are its simplicity, the easy concept of balancing gain (expected returns) and the corresponding risk (variance), and the subadditivity property due to diversification. One of the lack of this process is that investors believed that long-term historical expected returns, estimated from the past history would have been useful in explaining the future. An example of this way of working can be found during the dot-com crisis. Some institutional investors, in particular defined benefit pension plans lost a relevant part of the invested capital because of their high exposure to equities [101]. However, the performance of the equity market between 2003 and 2007 gave back the confidence that standard asset allocation models would continue to work as well as before the bear market and that the dot-com crisis was a non-recurring event. Nevertheless, the 2008 financial crisis highlighted the hidden risk in many strategic asset allocations. For institutional investors, this crisis was unprecedentedly severe. Indeed, in 2000, the dot-com crisis was limited to large capitalization stocks, technology and telecommunications sectors. In 2008, the financial crisis led to a violent plunge in credit strategies and other fixed-income related instruments as well. In addition, all equity sectors posted negative returns of approximately 50%, and the performance of hedge funds was not satisfying. Although the use of several asset classes and the exposure to different regions, the diversification methods *á* la Markowitz were not enough to protect investments. Accordingly, Markowitz's modern portfolio theory (MPT) was strongly criticized by professionals[4].

However, the issues were not totally due to the portfolio selection methods used by the financial institutions. Indeed, much of the failure was caused by the wrong estimation of the input parameters. Estimating expected returns by historical scenarios led the portfolio selection models to overweight the performance of equities. Investors also selected assets that were supposed to have a low correlation with equities. Conversely, during the crisis the correlations between different asset classes significantly increased and the diversification benefit did not work.

To reduce the influence of estimation errors on portfolio selection, several methods are suggested in the literature. These embrace approaches developed in the area of statistics, such as shrinkage methods (see [56], [57], [58], [65] and [66]) and bayesian approaches (see [10], [15], [71], [89] and [90]), robust statistics [18], robust optimization procedures (see [45], [49], [50], [100] and [107]), bayesian robust optimization (see [109]), robust estimation methods [46], and a portfolio resampling method [81].

Some other scholars suggest to only minimize risk to avoid the use of expected return. We call this class of models *Minimum-Risk* strategies. However, these approaches can lead to a portfolio poorly diversified in terms of risk (see [72], [93], [94] and [98]). Accordingly, the recent global financial crisis started in 2008 has given rise to a new research stream that aims at diversifying risk.

A straightforward method to diversify risk could be the choice of the Equally Weighted (EW) portfolio. The EW portfolio is the one where the capital is equally distributed among the assets. Clearly, its selection does not use any in-sample information nor involves any optimization approach. However, the EW portfolio is often used in practice (see [14] and [110]), and some authors claim that its practical out-of-sample performance is hard to beat on real-world datasets [46]. We call this method *Capital-Diversification* strategy. Nonetheless, if the market contains assets with very different intrinsic risks, then the EW approach can lead to a portfolio with limited total risk diversification among the assets.

Some very recent research effort are focused on risk allocation. Specifically, they aim at selecting a portfolio such that each asset equally contributes to the

[4]AsianInvestors, *Is Markowitz Dead?*, December 2012.

total portfolio risk. The formalization of the risk contribution of an individual asset is essentially based on Euler's decomposition of a homogeneous risk function. The risk measure commonly used in the Equal Risk Contribution (ERC), also called Risk Parity (RP), models is volatility (see [7], [72], [98]). It is known that the volatility of RP portfolios lies between that of Minimum Variance (MinV) and of EW portfolios (see [72]). We call this class of models *Risk-Diversification* strategies.

As regards the issue of the stability of portfolio selection models w.r.t. a pertubation of the inputs, we consider a reference the study of Kondor et al. (2007) (see [61]) that propose a methodology to analyze the impact of the estimation errors on the optimal portfolio weights obtained by the models. The authors examine the sensitivity to estimation errors of several minimum risk (variance, MAD, CVaR, Maximum Loss) portfolios by considering an artificial investment universe represented by multivariate standard normal returns with different numbers n of assets and lengths T of the time series. In particular, they show that Minimum CVaR portfolios already present high estimation errors for values of n/T around 0.4.

Chapter 2

Minimum Risk vs. Capital and Risk Diversification strategies for portfolio construction

In this chapter we test all models by using both historical and simulated scenarios. The covariance matrix, which is needed for the *Minimum-Risk* and in the *Risk-Diversification* strategies, is estimated in different ways using daily returns on rolling windows of 500 days. More precisely, for historical scenarios we adopt: the Maximum Likelihood (ML) covariance; the Exponential Weighted Moving Average (EWMA) covariance; the Shrinkage estimator between ML and EWMA covariances; and the covariance matrix with Kendall correlation, obtained by combining the intrinsic variances of the assets returns and their Kendall correlations. We also use the covariance matrix obtained by a Monte Carlo Simulation technique known as Historical Filtered Bootstrap. Also for *Minimum-Risk* and *Risk-Diversification* strategies based on CVaR as risk measure, we adopt both historical and simulated (Historical Filtered Bootstrap) scenarios.

For each portfolio strategy we evaluate daily out-of-sample portfolio returns with a rolling time windows approach. Then, for each out-of-sample portfolio return obtained, we compute the following performance measures: annualized expected return and volatility; Sharpe ratio; Maximum Drawdown (MDD); Ulcer index; turnover; and Herfindahl index as risk diversification index. Furthermore, to examine the statistical significance of the difference between the Sharpe ratios of the out-of-sample returns for two given portfolios, we use the bootstrapping methodology proposed by Ledoit and Wolf (2008) ([67]).

2.1 Models and Methodologies

In this section we present three different categories of portfolio construction models, each focused on different objectives: the minimization of risk, called *Minimum-Risk* strategy; the requirement that the capital is equally distributed among the assets, called *Capital-Diversification* strategy; and the requirement that each asset contributes equally to the total portfolio risk, called *Risk-Diversification* strategy. We describe here the models analyzed and we provide a formulation for all of them. The complete list of portfolio strategies that we examine is reported in Table 2.1.

Let us introduce some notations. In our analysis we use linear returns, so if p_{it} is the price of asset i at time t, then $r_{it} = \frac{p_{it} - p_{i(t-1)}}{p_{i(t-1)}}$ represents its return at time t. We denote by x the vector of decision variables whose components x_i represent the assets weights in a portfolio. Thus assuming that n assets are available in an investment universe, the portfolio return $R_P(x)$ at time t is $R_t(x) = \sum_{i=1}^{n} x_i r_{it}$. Let μ denote the vector whose components μ_i are the expected returns of n assets, and let Σ denote their covariance matrix, where the generic element σ_{ij} is the covariance of the returns of asset i and asset j with $i, j = 1, \ldots, n$.

For all models the feasible portfolios are determined by the budget constraint ($\sum_{i=1}^{n} x_i = 1$) and by the no short-selling condition ($x_i \geq 0$ for all $i = 1, \ldots, n$).

2.1.1 *Minimum-Risk* strategy

2.1.1.1 Minimum Variance portfolio

In the classical Markowitz approach the Minimum Variance (MinV) portfolio is the one obtained by minimizing the variance of the return of the whole portfolio. Then, the MinV portfolio is achieved by solving the following convex quadratic programming problem:

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j \\ s.t. \quad & \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad i = 1, \ldots, n \end{aligned} \tag{2.1}$$

Model	Abbreviation
Minimum-Risk strategy	
MinV portfolio with ML covariance	MinV-ML
MinV portfolio with EWMA covariance	MinV-EWMA
MinV portfolio with shrinkage estimator	MinV-Shr
MinV portfolio with covariance matrix obtained by Kendall correlation	MinV-Kend
MinV portfolio with covariance matrix obtained by Historical Filtered Bootstrap	MinV-HFB
MinCVaR portfolio with historical scenarios ($\varepsilon = 0.05$)	MinCVaR-Hist
MinCVaR portfolio with Historical Filtered Bootstrap scenarios ($\varepsilon = 0.05$)	MinCVaR-HFB
Capital-Diversification strategy	
Equally Weighted portfolio	EW
Risk-Diversification strategy	
Naive RP portfolio with ML covariance	NRP-ML
Naive RP portfolio with EWMA covariance	NRP-EWMA
Naive RP portfolio with shrinkage estimator	NRP-Shr
Naive RP portfolio with covariance matrix obtained by Historical Filtered Bootstrap	NRP-HFB
RP portfolio with ML covariance	RP-ML
RP portfolio with EWMA covariance	RP-EWMA
RP portfolio with shrinkage estimator	RP-Shr
RP portfolio with covariance matrix obtained by Kendall correlation	RP-Kend
RP portfolio with covariance matrix obtained by Historical Filtered Bootstrap	RP-HFB
Naive CVaR ERC portfolio with historical scenarios ($\varepsilon = 0.01, 0.05, 0.10$)	NCVaRERC-Hist-ε
Naive CVaR ERC portfolio with Historical Filtered Bootstrap scenarios ($\varepsilon = 0.01, 0.05, 0.10$)	NCVaRERC-HFB-ε
CVaR ERC portfolio with historical scenarios ($\varepsilon = 0.01, 0.05, 0.10$)	CVaRERC-Hist-ε
CVaR ERC portfolio with Historical Filtered Bootstrap scenarios ($\varepsilon = 0.01, 0.05, 0.10$)	CVaRERC-HFB-ε

Table 2.1: List of portfolio strategies.

To implement this model only the estimate of the covariance matrix of asset returns is needed, thus completely avoiding the problems due to the estimates of the expected returns. In our experiments we compute the covariance matrix using daily returns on a rolling window of 500 days. More specifically, we use:

1. the Maximum Likelihood (ML) covariance

$$\sigma_{ij}^{ML} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - \mu_i)(r_{jt} - \mu_j)$$

   where $\mu_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}$;

2. the Exponential Weighted Moving Average (EWMA) covariance

$$\sigma_{ij}^{EWMA} = \frac{1-\lambda}{1-\lambda^T}\sum_{t=1}^{T}\lambda^{T-t}\left(r_{it} - \mu_i^{EWMA}\right)\left(r_{jt} - \mu_j^{EWMA}\right)$$

   where $\mu_i^{EWMA} = \frac{1-\lambda}{1-\lambda^T}\sum\limits_{t=1}^{T}\lambda^{T-t}r_{it}$ and the decay factor λ is fixed to 0.94 as in JP Morgan Risk Metrics technical document ([82]);

3. the Shrinkage estimator between ML and EWMA covariance

$$\sigma_{ij}^{Shr} = \gamma\sigma_{ij}^{ML} + (1-\gamma)\sigma_{ij}^{EWMA}$$

with the shrinkage factor $\gamma = 0.5$;

4. the covariance matrix with Kendall correlation, where the generic element σ_{ij}^{Kend} is obtained by combining the shrunk volatilities of assets returns and their Kendall correlations ρ_{ij}^{Kend}:

$$\sigma_{ij}^{Kend} = \rho_{ij}^{Kend}\sigma_i^{Shr}\sigma_j^{Shr}$$

   We recall that the Kendall correlation is defined as follows. For given assets i and j we say that a pair of their realizations $(r_{it'}, r_{jt'})$, $(r_{it''}, r_{jt''})$ is concordant (discordant), if $(r_{it'} - r_{jt'})(r_{it''} - r_{jt''}) > 0$ $((r_{it'} - r_{jt'})(r_{it''} - r_{jt''}) < 0)$. The Kendall rank correlation (often called Kendall's tau) ρ_{ij}^{Kend} measures the intensity of dependence between the returns of assets i and j (see [60]) in the following way

$$\rho_{ij}^{Kend} = \frac{n_{ij}^c - n_{ij}^d}{\frac{1}{2}T(T-1)}$$

   where $\frac{1}{2}T(T-1)$ is the total number of pair combinations, n_{ij}^c is the number of concordant pairs, and n_{ij}^d is the number of discordant pairs. We support the intuition in [47], where the authors state that rank correlation is much more robust than linear Pearson's correlation coefficient. They also provide an efficient algorithm to quickly calculate the Kendall τ.

5. the covariance matrix is obtained with a Monte Carlo simulation technique known as Historical Filtered Bootstrap (see [8], [9], [24] and [113]).

2.1.1.2 Minimum CVaR portfolio

The Minimum CVaR (MinCVaR) model is a minimum risk model like the previous one, but instead of variance it minimizes the Conditional Value-at-Risk at a specified confidence level ε ($CVaR_\varepsilon$), namely the average of losses in the worst $100\varepsilon\%$ of the cases (see [5]), where losses are defined as negative outcomes. The use of CVaR[1] as a risk measure for asset allocation and risk management is increasingly popular (see [3], [102] and [95]). This is due to theoretical and computational reasons. From the theoretical viewpoint, CVaR satisfies the axioms of a coherent risk measure (see [6]) and, furthermore, the mean-CVaR model is consistent with second-order stochastic dominance

[1] CVaR is often also called *average value-at-risk* or *expected shortfall*

(see [88]). From the computational viewpoint, the mean-CVaR model can be efficiently solved by means of linear programming (see [96]).

The formal definition of CVaR is:

$$CVaR_\varepsilon(x) = -\frac{1}{\varepsilon}\int_0^\varepsilon Q_{R_P(x)}(\alpha)d\alpha\,,$$

where $Q_{R_P(x)}(\alpha)$ is the quantile function of the portfolio return $R_P(x)$. In our analysis we set ε equal to 0.01, 0.05 and 0.10. However, since for these values the portfolio performances are very similar, in Section 2.2 we report only the results for $\varepsilon = 0.05$. Complete results are available in the web page ([30]).

The MinCVaR model can be written as follows:

$$\begin{array}{ll} \min & CVaR_\varepsilon(x) \\ s.t. & \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \qquad i = 1,\ldots,n \end{array} \tag{2.2}$$

As described in Rockafellar and Uryasev (2000) (see [96]), in the discrete case we have

$$\min_x CVaR_\varepsilon(x) = \min_{(x,\zeta)} \zeta + \frac{1}{\varepsilon}\sum_{t=1}^{T} p_t[\sum_{i=1}^{n} -r_{it}x_i - \zeta]^+, \tag{2.3}$$

where $\zeta \in \mathbb{R}$, $[b]^+ = \max\{0, b\}$, and p_t is the probability of the scenario of the portfolio losses $l_t = \sum_{i=1}^{n} -r_{it}x_i$. We assume that the T scenarios considered are equally likely, i.e., $p_t = 1/T$. Furthermore, in order to linearize the objective function, we introduce T auxiliary variables d_t that are defined as the deviations of the portfolio losses l_t from ζ when $l_t > \zeta$, and 0 otherwise. Note that, under some assumptions, the optimal value of ζ in (2.3) coincides with VaR_ε of the optimal portfolio x that minimizes $CVaR_\varepsilon$ (see [96]). Thus, Problem (2.2) can be reformulated as the following Linear Programming (LP) problem

$$
\begin{aligned}
\min \quad & \zeta + \frac{1}{\varepsilon}\frac{1}{T}\sum_{t=1}^{T} d_t \\
s.t. \quad & \\
& d_t \geq \sum_{i=1}^{n} -r_{it}x_i - \zeta, \quad t = 1, \ldots, T \\
& d_t \geq 0, \qquad\qquad\qquad\quad t = 1, \ldots, T \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0 \qquad\qquad\qquad\quad\; i = 1, \ldots, n \\
& \zeta \in \mathbb{R}
\end{aligned}
\tag{2.4}
$$

Similar to (2.1), also this model is based only on the minimization of risk, thus avoiding problems in estimating expected returns. However, when varying the confidence level ε, $CVaR_\varepsilon$ can depend on wider ranges of the portfolio return distribution. For the extreme case where $\varepsilon = 1$, $-CVaR_\varepsilon(x)$ is equal to the portfolio expected return. However, as shown by Kondor et al. (2007) (see [61]), for appropriate values of the confidence level ε and of the ratio n/T, the effect of the estimation error on $CVaR_\varepsilon$ tends to be minimal. More precisely, the simulation results reported in Figs. 16, 17 and 18 of Kondor et al. (2007) (see [61]) show that the estimation errors of $CVaR_\varepsilon(x)$ tend to be small for small values of ε and n/T, and comparable to those of variance. Since in our analysis we set $\varepsilon = 0.01, 0.05, 0.10$, and n/T is between 0.0028 and 0.0560, we expect to have small estimation errors for the minimum risk portfolio.
For Problem (2.4) we use as inputs:

1. historical scenarios on a rolling window of 500 days;

2. Historical Filtered Bootstrap (HFB) scenarios.

Note that with the HFB approach we generate 10000 scenarios 20 days forward that we use as inputs to Problem (2.4). The HFB approach is described in the Chapter 4.

2.1.2 *Capital-Diversification* strategy

2.1.2.1 Equally Weighted portfolio

The Equally Weighted (EW) portfolio is the one where the capital is equally distributed among the assets. In terms of relative weights we have $x_i = 1/n$. Clearly the choice of the EW portfolio does not use any in-sample information nor involve any optimization approach. We use this portfolio as a benchmark to compare the performance of the portfolios constructed by the other models.

2.1.3 *Risk Diversification* strategy

2.1.3.1 Risk Parity portfolio

The Risk Parity (RP) portfolio, introduced by Maillard at al. (2010) [72], is characterized by the requirement of having equal total risk contribution from each asset:

$$TRC_i^\sigma(x) = TRC_j^\sigma(x) \Leftrightarrow x_i\,(\Sigma x)_i = x_j\,(\Sigma x)_j \quad \forall i, j,$$

where $TRC_i^\sigma(x) = x_i \frac{\partial \sigma(x)}{\partial x_i} = \frac{x_i (\Sigma x)_i}{\sigma(x)} = \frac{1}{\sigma(x)} \sum_{k=1}^{n} \sigma_{ik} x_i x_k$ is the total risk contribution of asset i. It is easy to show that $\sigma(x) = \sum_{i=1}^{n} TRC_i^\sigma(x)$. Thus, the RP portfolio can be found by solving the following system of equations and inequalities:

$$\begin{cases} x_i\,(\Sigma x)_i = \lambda \qquad i = 1, ..., n \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \quad i = 1, \ldots, n \end{cases} \tag{2.5}$$

This system of linear and quadratic equations and inequalities has a unique solution (x^{RP}, λ^{RP}), at least when the covariance matrix Σ is positive definite (for a complete proof see [105]). As inputs to Problem (2.5) we consider the same covariance matrices used for the MinV portfolio (see Section 2.1.1.1).

2.1.3.2 Naive RP portfolio

When using volatility as risk measure, a naive approach to reach approximately equal risk contributions of all assets is to consider the Naive Risk Parity (NRP) portfolio with weights proportional to the inverse of the intrinsic volatility of the assets, i.e.,

$$x_i^{NRP} = \frac{\sigma_i^{-1}}{\sum_{k=1}^{n} \sigma_k^{-1}} \qquad \forall i = 1, ..., n.$$

It is proved by Maillard et al. (2010) [72] that this coincides with the portfolio obtained by solving Problem (2.5) when the correlations among the assets are constant.

2.1.3.3 CVaRERC portfolio

The CVaR Equal Risk Contribution (CVaRERC) portfolio is characterized by the same requirement of the RP portfolio (see Section 2.1.3.1), i.e., to obtain a portfolio composition that achieves equal total risk contribution among all assets, where the risk is measured by CVaR. However, unlike volatility, CVaR may be positive or negative, and in the latter case CVaR indicates a gain. Since the Equal Risk Contribution approach makes sense only in the case of positive risk, we decided to apply it only when the minimum CVaR is positive. In the very few cases[2] where the optimal value of Problem (2.2) is negative, instead of the ERC approach we consider the minimum risk one.

Similar to volatility, $CVaR_\varepsilon(x)$ is a homogeneous function of degree 1, therefore we can write

$$CVaR_\varepsilon(x) = \sum_{i=1}^{n} x_i \frac{\partial CVaR_\varepsilon(x)}{\partial x_i} = \sum_{i=1}^{n} TRC_i^{CVaR}(x),$$

where $TRC_i^{CVaR}(x) = x_i \frac{\partial CVaR_\varepsilon(x)}{\partial x_i}$ is the total risk contribution of asset i. Then, the CVaRERC portfolio is obtained by imposing the following conditions:

$$\begin{cases} TRC_i^{CVaR}(x) = \lambda \qquad \forall i = 1, ..., n \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \qquad\qquad i = 1, \dots, n \end{cases} \tag{2.6}$$

In the case of continuous portfolio returns we have

$$TRC_i^{CVaR}(x) = x_i \frac{\partial CVaR_\varepsilon(x)}{\partial x_i} = -x_i E\left[r_i | R_P \leq -VaR_\varepsilon(x)\right] \tag{2.7}$$

(see [19] and [103]), while in the discrete case

$$CVaR_\varepsilon(x) = -\frac{1}{\lfloor \varepsilon T \rfloor} \sum_{k=1}^{\lfloor \varepsilon T \rfloor} R_P^{(k)}(x),$$

where $R_P^{(1)}(x) \leq R_P^{(2)}(x) \leq \ldots \leq R_P^{(T)}(x)$ indicate the sorted outcomes of

[2]The analysis involves a total of 2830 optimization problems that arise by combining 7 datasets, around 200 different sampling periods, and 2 methods to represent the future portfolio return distribution (historical and simulated). The minimum CVaR is negative only 18 times out of 2830, i.e., approximately 0.6%.

portfolio returns. Thus, Expression (2.7) becomes

$$TRC_i^{CVaR}(x) = -x_i \frac{1}{\lfloor \varepsilon T \rfloor} \sum_{k=1}^{\lfloor \varepsilon T \rfloor} {r_i}^{(k)} .$$

However, since the existence of a CVaRERC solution is not always guaranteed (see [34]), we formulate Problem (2.6) so as to lead to a feasible portfolio which is as close as possible to a CVaRERC portfolio. More precisely, we minimize the deviations of relative risk contributions $\frac{TRC_i^{CVaR}(x)}{CVaR_\varepsilon(x)}$ from $\frac{1}{n}$, where $\frac{1}{n}$ is the relative risk contribution when $TRC_i^{CVaR}(x) = \lambda \quad \forall i$. Thus, we obtain the following least-squares formulation:

$$\begin{cases} \min_x \quad F(x) = \sum_{i=1}^n \left(\frac{TRC_i^{CVaR}(x)}{CVaR_\varepsilon(x)} - \frac{1}{n} \right)^2 \\ \quad s.t. \\ \qquad \sum_{i=1}^n x_i = 1 \\ \qquad x_i \geq 0 \qquad\qquad i = 1, \ldots, n \end{cases} \tag{2.8}$$

We briefly explain, with an example, (see [79] for more details) that there may be some cases in which the ERCCVaR does not exist while conversely the sigma parity exists. Let us consider a correlation matrix $C = \begin{bmatrix} 1 & -0.95 & 0.1 \\ -0.95 & 1 & 0 \\ 0.1 & 0 & 1 \end{bmatrix}$ and the single asset standard deviation as $s = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.01 \end{bmatrix}$. With this specification of C and s the problem 2.5 has a solution $x^* = \begin{bmatrix} 1.46\% \\ 3.99\% \\ 94.55\% \end{bmatrix}$. The $TRC_i = 0.558\%$. This asset universe could not admit an ERCCVaR solutions provided the non negativity of portfolio weights. The first two asset are highly negatively correlated and the third is quite not correlated with the others. Once the CVaR is the conditional expectation in the bad tail, may not exists scenarios that contributes equally to CVaR. While asset 1 is plunging the asset 2 is gaining and asset 3 is going for its own way. The mean also plays a relevant role, assuming that the expected returns are defined as: $\mu = \begin{bmatrix} -10\% \\ 10\% \\ 1\% \end{bmatrix}$, than the portfolio expected return will be 1.2% and (under normality assumption) the $CVaR_{5\%} = 2.25\%$. Try to figure out in the previous situation it could be impossible to find that each asset contribute to the portfolio CVaR while one asset collapse and the other will gain.

Note that, assuming positive minimum CVaR, $F(x)$ is well-defined and that if $F(x^*) = 0$, then x^* is guaranteed to be an optimal solution to (2.8). Thus we can use a local optimization method to solve the Problem (2.8) with a certificate of global optimality when it is attained. Here we solve the Problem (2.8) with Matlab 8.1 by using the built-in function *fmincon*. All experiments are executed on a workstation with Intel Core Duo CPU (E7400, 2.8 GHz, 4 Gb RAM) under MS Windows XP. We set the starting point equal to the Naive CVaR ERC portfolio described in the following section, namely

$$x_0 = \left\{ \frac{CVaR_\varepsilon^{-1}(r_i)}{\sum_{j=1}^n CVaR_\varepsilon^{-1}(r_j)} \right\}_{i=1,\ldots,n}.$$

As for Problem (2.4), we use as inputs to (2.8) both historical and Historical Filtered Bootstrap (simulated) scenarios, and the running time to solve the CVaRERC model is always within 10 secs. When a global solution is not found by the local optimizer we can reformulate and solve the CVaRERC model by using the following convex optimization problem

$$\begin{aligned} \min \quad & CVaR_\varepsilon(y) \\ s.t. \quad & \\ & \sum_{i=1}^n \ln y_i \geq c \end{aligned} \tag{2.9}$$

where c is an arbitrary constant. Indeed, the Lagrangian of Problem (2.9) is

$$L(y, \lambda) = CVaR_\varepsilon(y) + \lambda(c - \sum_{i=1}^n \ln y_i)$$

and every solution y^* of (2.9) must satisfy the KKT conditions

$$\begin{aligned} \frac{\partial L}{\partial y_k^*} &= \frac{\partial CVaR_\varepsilon(y^*)}{\partial y_k^*} - \lambda\frac{1}{y_k^*} = 0 \qquad k = 1,\dots,n \\ & c - \sum_{i=1}^{n} \ln y_i^* \leq 0 \\ & \lambda \geq 0 \\ & \lambda\left(c - \sum_{i=1}^{n} \ln y_i^*\right) = 0 \end{aligned} \tag{2.10}$$

Obviously $y_i > 0$ for all i and from (2.10) we have

$$y_k^* \frac{\partial CVaR_\varepsilon(y^*)}{\partial y_k^*} = \lambda \qquad \forall k = 1,\dots,n$$

meaning that all asset risk contributions are all equal as required in the system (2.6). As mentioned above, since we seek ERC portfolios only when the portfolios are risky, we assume $CVaR_\varepsilon(y) > 0$ (as is almost always the case in our empirical analysis). Since $CVaR_\varepsilon(y^*) = n\lambda$, this clearly implies $\lambda > 0$ and, in turn, $\sum_{i=1}^{n} \ln y_i^* = c$. Note that Problem (2.9) consists in minimizing the convex function $CVaR_\varepsilon(y)$ on a convex set and thus any KKT point is also a solution to (2.9). Furthermore, using the results in Rockafellar and Uryasev (2000) ([96]), Problem (2.9) can be reformulated as follows:

$$\begin{cases} \min\limits_{(y,\zeta,d)} & \zeta + \frac{1}{\epsilon}\frac{1}{T}\sum\limits_{t=1}^{T} d_t \\ \text{s.t.} & \\ & d_t \geq -\sum\limits_{i=1}^{n} r_{it}y_i - \zeta \quad t = 1,\dots,T \\ & d_t \geq 0 \qquad\qquad\qquad\quad t = 1,\dots,T \\ & \sum\limits_{i=1}^{n} \ln y_i \geq c \\ & \zeta \in \mathbb{R} \end{cases}$$

From a solution y^* of (2.9) we easily obtain a CVaRERC portfolio that satisfies the budget constraint by setting $x_i^{CVaRERC} = y_i^* / \sum_{j=1}^{n} y_j^*$. Note that $x^{CVaRERC}$ is a solution to (2.9) with the constant $\tilde{c} = \sum_{i=1}^{n} \ln x_i^{CVaRERC} = c - n \ln \sum_{j=1}^{n} y_j^*$. With this constant, if we add to (2.9) the additional constraints $y \in \Delta = \{y \in \mathbb{R}^n : \sum_{i=1}^{n} y_i = 1, y_i \geq 0, \quad i = 1,\dots,n\}$, the optimal solution remains unchanged.

Remark 2.1 *Let $y^*(c)$ denote the optimal solution to Problem (2.9) with the additional constraint $y \in \Delta$. Obviously, if $c_1 \leq c_2$, then $CVaR_\varepsilon(y^*(c_1)) \leq CVaR_\varepsilon(y^*(c_2))$. Recall that $x^{CVaRERC} = y^*(\tilde{c})$. Furthermore, $x^{MinCVaR}$ coincides with $y^*(-\infty)$, i.e., with the solution to Problem (2.9) with the additional constraint $y \in \Delta$, but without the logarithmic constraint. Thus clearly $CVaR_\varepsilon(x^{MinCVaR}) \leq CVaR_\varepsilon(x^{CVaRERC})$. Note that by Jensen's inequality we have $\sum_{i=1}^{n} \ln(x_i) \leq -n\ln(n)$. Hence Problem (2.9) has feasible solutions only when $c \leq -n\ln(n)$. Furthermore, for $c = -n\ln(n)$ the constraint of (2.9) implies $\sum_{i=1}^{n} \ln(x_i) = -n\ln(n)$, which has the only solution $x_i^{EW} = \frac{1}{n}$ for all i, namely the Equally-Weighted (EW) portfolio which also belongs to Δ. Thus we have $x^{EW} = y^*(-n\ln(n))$, so that*

$$CVaR_\varepsilon(x^{CVaRERC}) \leq CVaR_\varepsilon(x^{EW}).$$

Therefore, the CVaR of the ERC portfolio is bounded between the CVaR of the Minimum Risk portfolio and that of the EW portfolio.

2.1.3.4 Naive CVaRERC portfolio

We present here a new *naive* approach to reach approximately equal risk contributions of all assets when the portfolio risk measure is CVaR. Specifically, we are interested in the CVaRERC portfolio based on the worst case scenario in terms of CVaR. The CVaR properties of sub-additivity and of positive homogeneity imply that CVaR is a convex function so that:

$$CVaR_\varepsilon(x) = CVaR_\varepsilon(R_P(x)) = CVaR_\varepsilon(\sum_{i=1}^{n} x_i r_i) \leq \sum_{i=1}^{n} x_i CVaR_\varepsilon(r_i)\,. \tag{2.11}$$

For a fixed portfolio x the RHS of (2.11) represents the scenario with the maximum risk (in terms of CVaR). We call it Worst-Conditional Value-at-Risk, $CVaR_\varepsilon^W(x) = \sum_{i=1}^{n} x_i CVaR_\varepsilon(r_i)$. This scenario represents the case where there is no diversification benefit in the portfolio construction, namely the portfolio risk is equal to the weighted sum of intrinsic assets risk. This property is well known as comonotonicity (see [92] and reference therein). It mainly refers to the perfect positive dependence between the components of a random vector, essentially saying that they can be represented as increasing functions of a single random variable. In two dimensions it is also possible to consider perfect negative dependence, which is called countermonotonicity.

In particular, the sum of the components $X_1 + X_2 + \ldots + X_n$ is the riskiest if the joint probability distribution of the random vector $X_1 + X_2 + \ldots + X_n$ is comonotonic. Furthermore, the α-quantile of the sum equals of the sum of the α-quantiles of its components, hence comonotonic random variables are quantile-additive.

Let $TRC_i^W(x)$ denote the total risk contribution of asset i to $CVaR_\varepsilon^W(x)$, then it is straightforward to see that $TRC_i^W(x) = x_i CVaR_\varepsilon(r_i)$. Obviously, $CVaR_\varepsilon^W(x) = \sum_{i=1}^n TRC_i^W(x)$, where in general $TRC_i^W(x) \neq TRC_j^W(x)$ $\forall i \neq j$. Let us consider now a portfolio $\bar{x}$ for which $TRC_i^W(\bar{x}) = TRC_j^W(\bar{x})$ $\forall i, j$. This means that $TRC_i^W(\bar{x}) = CVaR_\varepsilon^W(\bar{x})/n$ $\forall i = 1, \ldots, n$. We denote this common value of total risk contribution of each assets by $TRC^E = CVaR_\varepsilon^W(\bar{x})/n$. As a consequence, the Equal Risk Contribution portfolio $\bar{x}$ is such that $TRC^E = \bar{x}_i CVaR_\varepsilon(r_i)$ for $i = 1, \ldots, n$, where in general $\sum_{i=1}^n \bar{x}_i \neq 1$. Thus, we can write that

$$\begin{aligned} \bar{x}_i &= \frac{TRC^E}{CVaR_\varepsilon(r_i)} \\ &= \frac{CVaR_\varepsilon^W(\bar{x})}{nCVaR_\varepsilon(r_i)} \,. \end{aligned}$$

Requiring the normalization of such a portfolio we obtain

$$\begin{aligned} x_k^* &= \frac{\bar{x}_k}{\sum_{j=1}^n \bar{x}_j} \\ &= \frac{\frac{CVaR_\varepsilon^W(\bar{x})}{nCVaR_\varepsilon(r_k)}}{\sum_{j=1}^n \frac{CVaR_\varepsilon^W(\bar{x})}{nCVaR_\varepsilon(r_j)}} \\ &= \frac{CVaR_\varepsilon^{-1}(r_k)}{\sum_{j=1}^n CVaR_\varepsilon^{-1}(r_j)} \,. \end{aligned} \tag{2.12}$$

We call x^* the Naive Conditional Value-at-Risk Equal Risk Contribution (NCVaRERC) portfolio, where the weights of the assets are proportional to the inverse of their intrinsic risk. The higher (lower) the CVaR of an asset return, the lower (higher) its weight in the ERC portfolio. The resulting total risk contribution of asset k is

$$\begin{aligned} TRC_k(x^*) &= x_k^* CVaR_\varepsilon(r_k) \qquad \forall k \\ &= \frac{1}{\sum_{j=1}^n CVaR_\varepsilon^{-1}(r_j)} \,. \end{aligned}$$

We point out that if $CVaR_\varepsilon(r_k) > 0$ $\forall k$, then x^* represents the unique CVaR-ERC portfolio with $x_i^* > 0$, $\sum_{i=1}^n x_i^* = 1$, corresponding to the Worst CVaR scenario $CVaR_{\varepsilon}^{W}$, where

$$CVaR_{\varepsilon}^{W}(x^{*}) = \frac{n}{\sum_{j=1}^{n} CVaR_{\varepsilon}^{-1}(r_j)} .$$

It is the harmonic mean of the assets CVaRs. Note that the NCVaRERC portfolio is similar to the Naive Risk Parity (NRP) portfolio, where the weights of the assets are the inverse of their volatilities divided by the sum of the assets volatility reciprocals.

We observe that in our empirical analysis we almost always found that $CVaR_{\varepsilon}(r_k) > 0$ for all $k = 1, \ldots, n$. However, as stated above, CVaR could be non-positive, thus constituting a gain. Clearly, in this case the NCVaRERC portfolio (2.12) does not exist. However, the very unlikely case of $CVaR_{\varepsilon}(r_k) \leq 0$ (for some k) corresponds to an asset k which can be seen as risk free asset. In this case, we obviously force the NCVaRERC portfolio to have positive (equal) weights for all assets with $CVaR_{\varepsilon}(r_k) \leq 0$ and zero otherwise.

2.1.4 Description of Performance Measures

Our goal is to study the performance of all models summarized in Table 2.1 across a variety of data sets. Our analysis relies on a rolling time windows approach. More specifically, given a daily frequency data set of asset returns with T outcomes, we consider an in-sample time window of $M = 500$ days. Then, we evaluate the portfolio performance in the following 20 days (out-of-sample), during which no rebalances are allowed. After this, we shift the mentioned in-sample window by 20 days in order to cover the out-of-sample period, we recompute the optimal portfolio w.r.t. the new in-sample window and repeat. Then, for each portfolio strategy the rolling time windows approach generates $T - M$ daily out-of-sample portfolio returns on which we compute some performance measures, described in the following sections.

2.1.4.1 Sharpe ratio

For each portfolio strategy the Sharpe ratio is defined as the annualized average of out-of-sample portfolio returns $\hat{\mu}^{out}$ (called Annualized Expected Return, AER) divided by their annualized sample standard deviation $\hat{\sigma}^{out}$ (called Annualized Volatility, AV):

$$SR = \frac{\hat{\mu}^{out}}{\hat{\sigma}^{out}}.$$

Furthermore, we examine the statistical significance of the difference between the Sharpe ratios of the out-of-sample returns for two given portfolios, using the robust test proposed by Ledoit and Wolf (2008) [67]. Specifically, we test the statistical significant differences of each portfolio strategy w.r.t. the EW portfolio. For this purpose, we use the Matlab code provided by the authors ([111]). We conduct this analysis using out-of-sample portfolio returns with two different frequencies: daily and monthly. Furthermore, for each frequency we consider two block sizes in the bootstrap procedure of the Ledoit and Wolf test to investigate its sensitivity: 1 and 10 days for daily returns; 1 and 3 months for monthly returns. In addition, we consider 5000 bootstrap resamples for monthly data and 3000 bootstrap resamples for daily data, and a confidence level of 95%. However, looking at the resulting p-values we find that for daily data the robust test rejects the null hypothesis (the Sharpe ratio of a portfolio is equal to that of another portfolio) more frequently than for monthly data.

In Section 2.2.2.1 we report only a summary of the empirical results, but full results are available in the web page (see [30]).

2.1.4.2 Maximum Drawdown and Ulcer Index

Let us denote by R_τ^{out} the out-of-sample portfolio returns for each portfolio strategy and let us consider the cumulative out-of-sample portfolio returns, which correspond to the values of wealth after τ periods

$$W_\tau = W_{\tau-1}(1 + R_\tau^{out}) \qquad \tau = M+1, \ldots, T \tag{2.13}$$

with initial wealth $W_M = 1$. We define the drawdowns as

$$dd_\tau = \frac{W_\tau - \max_{M+1 \le s \le \tau}(W_s)}{\max_{M+1 \le s \le \tau}(W_s)}.$$

Note that the drawdowns dd_τ are obviously negative. The Maximum Drawdown Mdd corresponds to the worst drawdown or, equivalently, to the maximum potential loss achieved over the entire out-of-sample period:

$$Mdd = \min_{M+1 \le \tau \le T}(dd_\tau). \tag{2.14}$$

For a detailed discussion of this performance measure, see Chekhlov et al. (2005) ([36] and references therein). In addition to the Maximum Drawdown, we also examine the Ulcer Index (UI), that evaluates the depth and the duration of drawdowns in prices over the out-of-sample period (see MacCann (1989) [70]). Technically, UI is the square root of the mean of the squared percentage drawdowns dd_τ with $\tau = M+1, \ldots, T$:

$$UI = \sqrt{\frac{\sum_{\tau=M+1}^{T} dd_\tau^2}{T-M}}. \tag{2.15}$$

The greater a drawdown in absolute value and the longer it takes to go back to earlier highs, the higher the UI. Furthermore, the effect of the squaring operation amplifies the contribution of high drawdowns with respect to the small ones. In other words, UI highlights the impact of long and deep drawdowns.

2.1.4.3 Turnover

To evaluate the amount of trading required to perform in practice each portfolio strategy, we use a measure of portfolio turnover. This is defined as the mean on all rebalances of the sum of the absolute values of the trades across the n available assets in the market:

$$Turn = \frac{1}{Q} \sum_{j=1}^{Q} \sum_{i=1}^{n} \left| w_{j,i} - w_{j-1,i} \right|, \tag{2.16}$$

where Q is the number of rebalances realized (see [46]). We point out that with this definition of portfolio turnover we examine only the amount of trading generated by the models at each rebalance, without considering the trades due to changes in asset prices between one rebalance and the next. Thus, by definition, the turnover of the EW portfolio is zero.

2.1.4.4 Risk Diversification Index

The distribution of risk diversification provides a description of the portfolio risk concentration structure. A syntectic index widely used to measure risk concentration is the Herfindahl index (see [7] and [55])

$$HI = \sum_{i=1}^{n} (RCR_i)^2$$

where RCR_i represents the relative contribution of asset i to total risk. We consider here

$$RCR_i = \frac{TRC_i^{CVaR}(x)}{CVaR_\varepsilon(x)},$$

	Data set	♯ of assets	time interval	abbreviation
1	Global diversified portfolio	6	03/01/1995-16/10/2014	GDP-Mix1
2	Italian Bond and Global Equity Portfolio	7	03/01/1995-16/10/2014	IBGEP-Mix2
3	Worldwide Asset	28	01/01/1999-16/10/2014	WWA-Mix3
4	Stock Picking on Eurostoxx50	9	04/01/2000-16/10/2014	Euro-Eq1
5	World Equity Sectors Portfolio	9	03/01/1995-16/10/2014	WES-Eq2
6	Equity Emerging Countries	11	03/01/1999-16/10/2014	EEC-Eq3
7	Euro Government Bond Portfolio	10	03/01/2000-16/10/2014	Euro-Bond

Table 2.2: List of data sets analyzed.

and we define a normalized version of the Herfindahl index (NHI) as follows

$$NHI = \frac{1 - HI}{1 - \frac{1}{n}},$$

so that $NHI = 0$ when all the risk is completely concentrated in one single asset, while $NHI = 1$ when the risk is uniformly distributed among all the n assets. Furthermore, since we have a value of NHI for each portfolio rebalance, in this analysis we consider its average on all the rebalances as follows:

$$ANHI = \frac{1}{Q}\sum_{j=1}^{Q} NHI_j, \tag{2.17}$$

where Q is the number of rebalances realized.

2.2 Computational Results

In this section we present computational results for three different categories of portfolio models, *Minimum-Risk*, *Capital-Diversification and Risk-Diversification* on seven investment universes. In order to better appreciate the behavior of different portfolio strategies, we consider data sets characterized by different sources of risk. Thus, we select investment universes which consist of equities, bonds and mixed assets as described below.

2.2.1 Data sets

We provide here some details about the seven real-world data sets, that are summarized in Table 2.2. The data sets consist of daily prices obtained from Bloomberg. We report below only a qualitative description of the seven data sets. However, for a complete list of indices (including the Bloomberg tickers) refer to the web site (see [29]).

1. **GDP-Mix1**: it consists of global stocks, government bonds, high-yield

corporate bonds and commodities. These data generally are Total Return indices that include dividends; when they are not available we use Price Return indices, where dividends are not included. This data set is characterized by low correlations among the assets. Furthermore, in terms of risk factors GDP-Mix1 can be featured by duration risk, credit risk, equity risk, inflation risk.

2. **IBGEP-Mix2**: it consists of a mixture of equities, bonds and commodities. The equities come from the global market, while the bonds are Italian short and long term government bonds. Furthermore, we include a gold commodity that represents a safe asset for investors when there are stress conditions in the market. The main risk factors that characterize this investment universe are duration risk, credit risk, equity risk, country risk.

3. **WWA-Mix3**: it consists of a mixture of equities (big and small capitalization equities both of developed and of emerging countries), of bonds (corporate and government bonds with different maturity, rating, and currency), of commodities (agriculture, precious and industrial metals, energy) and real estate. We recognize many risk factors that characterize this investment universe as duration risk, credit risk, inflation risk, equity risk, country risk, currency risk, real estate risk.

4. **Euro-Eq1**: it is based on a stock picking in Eurostoxx50 market, where for each sector we choose (at the beginning of 2002) the companies with greater weight on the Eurostoxx50 index. These data consist of Total Return indices, where dividends are included, namely they are reinvested in the company. In term of risk factors this data set is characterized only by equity risk.

5. **WES-Eq2**: it comes from the world equity sectors provided by MSCI. All indices are in USD and in term of risk factors this investment universe is characterized only by equity risk.

6. **EEC-Eq3**: it consists of equities of emerging countries provided by MSCI. All indices are in USD. The risk factors that characterize this investment universe are equity risk and country risk.

7. **Euro-Bond**: it consists of Euro government bonds with constant maturity. The maturity bucket is 7-10y for all government bonds and currency is Euro. In term of risk factors we could recognize duration risk and credit risk.

We stress that the choice of data sets with a small number of assets (from 6 to 28) is done by focusing on the viewpoint of small investors, for whom it would be hard to manage a large number of assets in a portfolio. Indeed, any asset class in the investment universes listed in Table 2.2 is replicable by means of mutual funds and/or ETFs, that are easily available to small investors.

2.2.2 Empirical analysis

In this section we discuss the main results of the empirical analysis on the behavior of the models listed in Table 2.1. Specifically, for each portfolio we compute the following performance measures:

- annualized expected return, volatility, and Sharpe ratio. Furthermore, we report a synthesis of some results obtained by tests that try to evaluate the statistical significance of the difference between the Sharpe ratios of the out-of-sample returns for two given portfolios (Section 2.2.2.1).
- Maximum Drawdown and Ulcer index (Section 2.2.2.2).
- Turnover (Section 2.2.2.3).
- Risk Diversification index (Section 2.2.2.4).

In the tables of the following sections, for each asset universe we show with different colors the rank of the performance results of the proposed models. More in details, for each row (asset universe) the colors span from deep-green to deep-red, where deep-green represents the best performance while deep-red the worst one. Furthermore, this style of visualization allows for easier detection of possible persistent behavioral pattern of a portfolio strategy (corresponding to a column) on all data sets.

As mentioned above, we report here only a synthesis of the empirical results. For a more detailed analysis, that includes all models and significance tests discussed in the previous sections, we refer to the web page (see [30]).

2.2.2.1 Sharpe ratio

Table 2.3 reports the out-of-sample annualized expected return (Panel A), variance (Panel B), and the Sharpe Ratio (Panel C) defined in Section 2.1.4.1. From the results on annualized expected return (AER) we can observe that *Minimum-Risk* strategies show generally poor performance for each data set, while the EW portfolio often presents the best AER (4 out of 7 data sets). This is because *Capital-Diversification* strategy is more exposed to assets with high returns (high volatilities) than *Minimum-Risk* strategies. Regarding the Euro-Bond data set, MinCVaR-HFB has the highest AER, followed by MinV-Shr and MinCVaR-HS. In this case it seems that minimizing tail risk measures allows to avoid the Eurozone governments bond crisis, thus reducing the investment in the so called PIGS countries. *Risk-Diversification* strategies seem to be a compromise between *Minimum-Risk* and EW strategies. In particular, *Risk-Diversification* portfolios show a slightly superior AER than *Minimum-Risk* ones. Actually, *Risk-Diversification* models impose to invest in all asset, while *Minimum-Risk* strategies allow to have some portfolio weights equal to 0. Therefore, *Risk-Diversification* models, like the EW strategy, are more exposed to assets with high returns (high volatilities) than *Minimum-Risk* models.

From the results on annualized volatility (AV) we can notice that *Minimum-Risk* portfolios, as expected, are less volatile than the other strategies, except for the MinV-EWMA portfolio. Nevertheless, among *Risk Diversification* strategies the RP-ML portfolio presents good performance in terms of volatility. Actually, Maximum Likelihood (ML) estimators determine low risk portfolios both when minimizing variance and when diversifying it among the assets by means of the Risk Parity strategy. It seems that the in-sample ML estimation tends to better represent the out-of-sample projection of AV than the in-sample EWMA estimation. On the other hand, the EW portfolio shows the worst AV for each data set, again because this strategy by construction selects all assets including those with high volatilities (high returns).

Regarding the Sharpe ratio (SR), *Minimum-Risk* portfolios seem to exhibit high values in mixed investment universes (GDP-Mix1, IBGEP-Mix2, WWA-Mix3) and in the Euro-Bond data set. The EW portfolio, due to high volatility, shows the worst Sharpe ratios in these latter investment universes. On the other hand, on mixed asset universes and on Euro-Bond, *Risk-Diversification* portfolios present intermediate performance, except for the RP-ML portfolio with the best SR on WWA-Mix3, and for the NRP-EWMA portfolio with the best SR on GDP-Mix1. Regarding the equities universes (Euro-Eq1, WES-Eq2, EEC-Eq3) there is no a clear dominance pattern. However, in 2 data sets out of 3 (Euro-Eq1, EEC-Eq3) *Risk-Diversification* portfolios present the best Sharpe ratios, while only in WES-Eq2 *Minimum-Risk* portfolios have the highest values of SR. Panel D of Table 2.3 shows, in a synthetic way, the results of several tests to evaluate the statistical significance of the difference between Sharpe ratios of the out-of-sample returns for a given portfolio and the EW portfolio. More precisely, we report the percentages of rejections of the null hypothesis, i.e, the percentages where the Sharpe ratios for two given portfolios are statistically different. Then, the highest value (100% in green)

Panel A — Annualized ER

	Minimum-Risk							Risk Diversification													CD
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW
GDP-Mix1	5,92%	6,68%	6,33%	6,23%	6,52%	5,75%	6,60%	6,83%	6,84%	6,80%	6,81%	7,06%	6,42%	7,32%	6,76%	7,18%	6,70%	7,40%	6,99%	6,43%	7,47%
IBGEP-Mix2	4,16%	5,04%	4,13%	4,15%	4,10%	4,25%	4,70%	4,18%	5,31%	5,14%	5,17%	5,18%	4,56%	4,34%	4,45%	4,49%	5,26%	5,62%	5,33%	4,98%	6,59%
WWA-Mix3	3,71%	5,49%	3,80%	3,87%	3,58%	3,75%	4,42%	4,36%	5,77%	5,59%	5,37%	5,28%	4,39%	4,71%	4,41%	4,28%	5,40%	5,40%	5,61%	5,44%	6,48%
Euro-Eq1	4,85%	5,38%	3,61%	4,01%	0,48%	3,03%	2,24%	4,41%	4,88%	5,07%	5,25%	4,71%	5,65%	5,11%	5,28%	4,32%	5,55%	5,77%	5,23%	5,17%	5,76%
WES-Eq2	6,23%	6,42%	5,18%	5,08%	5,31%	6,40%	5,68%	4,55%	6,07%	6,22%	6,35%	6,48%	6,32%	5,85%	6,06%	6,44%	6,33%	6,50%	6,59%	6,37%	6,25%
EEC-Eq3	12,02%	14,63%	12,08%	12,21%	10,64%	9,63%	11,19%	13,43%	15,16%	14,79%	14,64%	14,10%	13,50%	13,74%	13,50%	12,58%	14,23%	13,63%	13,82%	14,30%	15,43%
Euro-Bond	5,87%	5,69%	6,13%	6,09%	5,61%	6,12%	6,38%	6,02%	5,76%	5,73%	5,72%	5,73%	5,79%	5,93%	5,85%	5,83%	5,64%	5,83%	5,83%	5,54%	4,88%

Panel B — Annualized Volatility

	Minimum-Risk							Risk Diversification													CD
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW
GDP-Mix1	4,82%	6,10%	4,57%	4,62%	4,63%	4,82%	5,09%	4,69%	5,70%	5,83%	5,95%	6,07%	5,23%	4,90%	4,92%	4,99%	6,31%	6,09%	5,96%	6,03%	9,54%
IBGEP-Mix2	1,97%	3,45%	1,94%	1,94%	2,01%	2,03%	2,54%	2,00%	3,41%	3,40%	3,47%	3,55%	2,38%	2,40%	2,35%	2,48%	3,51%	3,50%	3,39%	3,40%	9,35%
WWA-Mix3	1,08%	2,32%	0,99%	1,02%	1,05%	1,13%	1,18%	1,04%	2,25%	2,24%	2,37%	2,49%	1,38%	1,33%	1,33%	1,38%	2,63%	2,36%	2,67%	2,64%	8,41%
Euro-Eq1	22,21%	24,73%	21,72%	21,86%	23,23%	22,07%	23,82%	22,31%	24,27%	24,43%	24,47%	24,45%	24,07%	23,60%	23,72%	23,79%	24,85%	24,60%	24,48%	24,68%	25,73%
WES-Eq2	11,92%	14,69%	12,00%	12,26%	12,32%	12,07%	12,46%	12,26%	14,58%	14,62%	14,63%	14,56%	14,06%	13,95%	13,99%	13,88%	14,72%	14,61%	14,59%	14,70%	15,50%
EEC-Eq3	14,90%	18,51%	14,42%	14,75%	15,41%	15,26%	15,62%	14,76%	18,07%	18,19%	18,22%	18,47%	17,43%	16,68%	16,86%	17,44%	18,56%	18,56%	18,51%	18,58%	19,84%
Euro-Bond	4,43%	4,73%	4,33%	4,36%	4,43%	4,45%	4,42%	4,37%	4,56%	4,62%	4,59%	4,49%	4,53%	4,42%	4,45%	4,38%	4,59%	4,40%	4,42%	4,71%	5,28%

Panel C — Sharpe Ratio

	Minimum-Risk							Risk Diversification													CD
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW
GDP-Mix1	1,23	1,09	1,39	1,35	1,41	1,19	1,30	1,46	1,20	1,17	1,14	1,16	1,23	1,49	1,37	1,44	1,06	1,21	1,17	1,07	0,78
IBGEP-Mix2	2,11	1,46	2,13	2,14	2,03	2,10	1,85	2,09	1,56	1,51	1,49	1,46	1,92	1,81	1,89	1,81	1,50	1,61	1,57	1,46	0,71
WWA-Mix3	3,44	2,37	3,85	3,78	3,42	3,32	3,74	4,20	2,56	2,50	2,26	2,12	3,19	3,53	3,30	3,09	2,05	2,29	2,10	2,06	0,77
Euro-Eq1	0,22	0,22	0,17	0,18	0,02	0,14	0,09	0,20	0,20	0,21	0,21	0,19	0,23	0,22	0,22	0,18	0,22	0,23	0,21	0,21	0,22
WES-Eq2	0,52	0,44	0,43	0,41	0,43	0,53	0,46	0,37	0,42	0,43	0,43	0,44	0,45	0,42	0,43	0,46	0,43	0,44	0,45	0,43	0,40
EEC-Eq3	0,81	0,79	0,84	0,83	0,69	0,63	0,72	0,91	0,84	0,81	0,80	0,76	0,77	0,82	0,80	0,72	0,77	0,73	0,75	0,77	0,78
Euro-Bond	1,33	1,20	1,42	1,40	1,27	1,38	1,44	1,38	1,26	1,24	1,25	1,28	1,28	1,34	1,32	1,33	1,23	1,32	1,32	1,18	0,92

Panel D — Robust Sharpe Test

	Minimum-Risk							Risk Diversification													
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW
GDP-Mix1	0,0%	50,0%	50,0%	50,0%	50,0%	0,0%	0,0%	50,0%	50,0%	50,0%	50,0%	50,0%	50,0%	50,0%	50,0%	50,0%	50,0%	100,0%	50,0%	50,0%	
IBGEP-Mix2	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	
WWA-Mix3	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	
Euro-Eq1	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	
WES-Eq2	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	25,0%	0,0%	
EEC-Eq3	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	50,0%	0,0%	0,0%	
Euro-Bond	50,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	100,0%	50,0%	100,0%	50,0%	75,0%	75,0%	100,0%	100,0%	100,0%	

Table 2.3: *Panel A:* out-of-sample Annualized Expected Return. *Panel B:* out-of-sample Annualized Volatility. *Panel C:* out-of-sample daily Sharpe Ratio. *Panel D:* percentages of rejections of the null hypothesis, i.e, the percentages of the cases where the Sharpe ratios for two given portfolios are statistically different.

means that in the four tests performed (i.e., two block sizes for daily returns and two block sizes for monthly returns, see Section 2.1.4.1) we can always reject the hypothesis that the Sharpe Ratios of two portfolios are equal. On the other hand, the lowest percentage (0% in red) shows that in any test the Sharpe ratios are not statistically different. Note that on equity asset universes (Euro-Eq1, WES-Eq2 and EEC-Eq3) *Minimum-Risk* and *Risk Diversification* portfolios are almost always not different from the EW portfolio in terms of Sharpe ratio. Indeed, we record only one rejection (when the block size is equal to 1 month) for CVaRERC-HFB05 portfolio on the WES-Eq2 data set, and two rejections (when the block sizes are equal to 1 and 3 months) for the NCVaRERC-HFB05 portfolio on EEC-Eq3. On the other hand, in mixed assets and bond assets universes the results are the opposite. *Minimum-Risk* and *Risk-Diversification* portfolios yield Sharpe ratios statistically different from those of the EW portfolio.

It seems that when investigating a data set with a single risk source and when no active risk management procedure is allowed (see [42]), such as to sell part of risky assets of a portfolio and to hold cash[3]), then no statistically significant differences of SR occur. These results are quite similar to the findings in Demiguel et al. (2009) [46] and more recently in Chow et al. (2011) (see [38]), where the authors investigate on an investment universe of equities and on the Fama-French portfolios. On the other hand, considering an assets universe with multiple risk factors (such as GDP-Mix1, IBGEP-Mix2, WWA-Mix3 and Euro-Bond) *Minimum-Risk* and *Risk-Diversification* portfolios seem to achieve statistically significant superior Sharpe ratios with respect to the EW portfolio.

2.2.2.2 Maximum Drawdown and Ulcer index

Table 2.4 reports the Maximum Drawdown (MDD) in Panel A, and the Ulcer index (UI) in Panel B, computed by Expressions (2.14) and (2.15), respectively. From the results on MDD and UI we observe that both performance measures present similar patterns of dominance. Actually, for almost all data sets, *Minimum-Risk* models lie in the "green zone" (i.e., low MDD and low UI), except for the MinV-EWMA portfolio that presents medium-high values ("yellow-orange zone"). The EW portfolio, as for the case of volatility, exhibits in almost all analyzed cases high MDD and high UI ("red zone"). *Risk-Diversification* strategies present medium performance ("yellow zone").

[3] A risk management procedure consists of a set of rules that change the risky assets composition in a portfolio so that one could have $\sum_{i=1}^{n} x_i < 1$, where x_i with $i = 1, \ldots, n$ are the weights of risky assets.

However, NRP-ML and RP-ML portfolios show low MDD and low UI ("green zone"). From preliminary results it seems that the EW portfolio tends to have good performance when the market grows. However, during crisis periods (like in 2008) the EW portfolio tends to show a severe Maximum Drawdown for all the data sets but the Euro-Bond. In the latter data set, the EW portfolio presents heavy Maximum Drawdowns after the Eurozone crisis in 2011 (see Figure 2.1).

2.2.2.3 Turnover

In Panel C of Table 2.4 we report the turnover, defined as in (2.16). Again, we stress that we are interested in the turnover generated by the models, and not in the price adjustment turnover. Therefore, by construction, in this analysis the EW portfolio has no turnover. It is interesting to observe that, among *Risk-Diversification* strategies, RP-ML, NRP-ML, NCVaRERC-Hist05 and CVaRERC-Hist05 portfolios have very low turnovers for each data set (with the maximum monthly mean value of 2%). Among *Minimum-Risk* models the MinV-ML portfolio presents good performance in terms of turnover for each data set but the Euro-bond. The highest turnovers can be observed for MinV-EWMA, MinV-HFB and MinCVaR-HFB portfolios. This can be due to the fact that in these strategies the covariance estimations are less stable during the portfolio rebalances. Actually, it seems that models characterized from more stable inputs, such as ML covariance matrix or Historical CVaR, allow to achieve low turnovers.

We stress that high turnovers can reduce the performance of models, due to transaction costs.

2.2.2.4 Risk Diversification Index

In Panel D of Table 2.4 is reported the Risk Diversification Index (average of the normalized Herfindahl index) defined as in (2.17), where $\varepsilon = 0.05$ and CVaR is computed by HFB approach.

As expected, it is straightforward to verify that CVaRERC-HFB05 portfolio has the best risk diversification. However, RP, NCVaRERC and CVaR-ERC portfolios also present a good diversification in terms of risk. On the other hand, *Minimum-Risk* strategies tend to concentrate risk in few assets, thus showing the lowest risk diversification. The EW portfolio shows medium performance ("yellow zone") similar to Naive Risk Parity portfolios.

MaxDD	Minimum-Risk							Risk Diversification													CD	
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW	Panel A
GDP-Mix1	-21,06%	-23,26%	-15,50%	-15,41%	-14,18%	-18,52%	-11,08%	-15,18%	-18,89%	-19,67%	-21,70%	-21,52%	-23,59%	-16,91%	-18,10%	-17,36%	-24,97%	-20,67%	-19,05%	-22,23%	-35,47%	
IBGEP-Mix2	-5,38%	-7,70%	-6,35%	-7,16%	-7,16%	-5,74%	-6,17%	-6,92%	-10,18%	-9,71%	-10,16%	-9,29%	-6,85%	-11,74%	-9,64%	-10,40%	-8,63%	-8,11%	-7,10%	-7,90%	-29,09%	
WWA-Mix3	-4,88%	-8,34%	-3,26%	-4,24%	-2,63%	-4,55%	-2,32%	-3,61%	-7,80%	-7,38%	-9,60%	-10,36%	-5,33%	-4,86%	-4,47%	-4,25%	-11,05%	-9,96%	-7,82%	-11,95%	-34,16%	
Euro-Eq1	-53,82%	-54,56%	-50,59%	-49,45%	-55,59%	-49,78%	-55,59%	-49,66%	-52,85%	-52,92%	-53,19%	-54,17%	-54,40%	-52,98%	-52,26%	-54,84%	-54,31%	-54,19%	-53,65%	-53,52%	-54,80%	
WES-Eq2	-40,90%	-52,13%	-46,90%	-45,41%	-47,71%	-41,52%	-48,67%	-50,23%	-52,17%	-52,11%	-51,80%	-51,47%	-50,23%	-49,95%	-49,98%	-49,56%	-52,25%	-51,67%	-51,49%	-52,11%	-54,05%	
EEC-Eq3	-53,36%	-61,60%	-49,80%	-52,63%	-55,01%	-57,41%	-54,98%	-48,01%	-59,34%	-60,18%	-60,64%	-62,24%	-60,74%	-57,22%	-58,23%	-61,43%	-61,94%	-62,44%	-62,20%	-62,21%	-62,81%	
Euro-Bond	-9,33%	-9,55%	-8,70%	-8,61%	-9,43%	-7,03%	-7,32%	-10,18%	-11,74%	-10,27%	-10,85%	-10,80%	-7,47%	-8,58%	-7,90%	-7,89%	-9,53%	-8,11%	-8,84%	-10,83%	-16,98%	

Ulcer Index	Minimum-Risk							Risk Diversification													CD	
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW	Panel B
GDP-Mix1	3,49%	4,21%	2,93%	2,90%	2,68%	3,78%	2,94%	3,19%	3,63%	3,61%	4,02%	3,99%	3,78%	2,95%	3,08%	2,84%	4,51%	3,88%	4,08%	4,22%	7,49%	
IBGEP-Mix2	0,55%	1,19%	0,59%	0,65%	0,62%	0,75%	0,71%	0,60%	1,28%	1,27%	1,34%	1,27%	0,75%	1,18%	0,95%	0,98%	1,25%	1,08%	0,98%	1,20%	9,16%	
WWA-Mix3	0,72%	1,43%	0,52%	0,67%	0,43%	0,77%	0,44%	0,60%	1,44%	1,32%	1,74%	1,90%	0,77%	0,81%	0,70%	0,59%	1,99%	1,75%	1,54%	2,14%	8,24%	
Euro-Eq1	19,69%	20,34%	20,51%	20,75%	25,87%	19,66%	24,03%	19,51%	21,10%	20,82%	20,57%	21,60%	19,66%	20,40%	20,25%	21,74%	20,41%	20,07%	20,54%	20,62%	20,72%	
WES-Eq2	13,93%	14,71%	17,71%	15,62%	14,21%	13,45%	14,30%	18,27%	15,31%	15,09%	14,71%	14,19%	13,45%	13,89%	13,80%	13,37%	14,69%	14,23%	14,16%	14,56%	16,48%	
EEC-Eq3	14,84%	16,09%	13,72%	14,29%	15,69%	15,91%	15,86%	13,04%	15,11%	15,50%	15,74%	16,51%	15,91%	14,79%	15,14%	16,62%	16,21%	16,63%	16,63%	16,38%	16,31%	
Euro-Bond	2,16%	2,63%	2,19%	2,15%	2,40%	2,38%	2,19%	2,51%	2,86%	2,66%	2,70%	2,69%	2,38%	2,33%	2,35%	2,33%	2,61%	2,42%	2,49%	2,82%	4,01%	

Turnover	Minimum-Risk							Risk Diversification													CD	
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW	Panel C
GDP-Mix1	2,13%	17,22%	8,68%	6,95%	13,65%	3,93%	15,48%	1,05%	9,27%	4,54%	3,63%	6,32%	1,62%	12,28%	6,12%	11,08%	1,21%	7,93%	8,55%	1,83%	0,00%	
IBGEP-Mix2	0,40%	3,65%	1,90%	1,14%	3,08%	0,89%	4,98%	0,81%	7,28%	3,66%	2,71%	5,25%	0,48%	3,73%	1,93%	4,86%	0,90%	7,65%	9,13%	2,00%	0,00%	
WWA-Mix3	4,04%	33,92%	17,13%	13,69%	30,49%	7,41%	32,63%	0,74%	7,75%	3,43%	3,09%	6,24%	1,23%	12,48%	5,78%	12,03%	0,88%	7,29%	8,13%	1,28%	0,00%	
Euro-Eq1	4,04%	33,92%	17,13%	13,69%	30,49%	7,41%	32,63%	0,74%	7,75%	3,43%	3,09%	6,24%	1,23%	12,48%	5,78%	12,03%	0,88%	7,29%	8,13%	1,28%	0,00%	
WES-Eq2	3,52%	34,79%	16,45%	13,29%	32,23%	7,27%	31,59%	0,55%	5,96%	2,78%	2,30%	4,65%	0,93%	8,91%	4,35%	9,27%	0,63%	5,20%	5,36%	0,91%	0,00%	
EEC-Eq3	3,08%	29,64%	14,14%	10,92%	25,27%	5,94%	25,20%	0,87%	9,55%	4,27%	3,20%	5,87%	1,39%	12,89%	5,96%	11,40%	0,91%	6,25%	7,14%	1,64%	0,00%	
Euro-Bond	8,87%	47,09%	34,18%	20,59%	48,08%	11,84%	47,41%	0,47%	4,49%	2,14%	1,89%	4,24%	0,75%	6,43%	3,11%	7,26%	0,57%	5,10%	5,55%	0,82%	0,00%	

Mean Herfindhal Index	Minimum-Risk							Risk Diversification													CD	
	MinV-ML	MinV-EWMA	MinV-Shr	MinV-Kend	MinV-HFB	MinCVaR-Hist	MinCVaR-HFB	RP-ML	RP-EWMA	RP-Shr	RP-Kend	RP-HFB	NRP-ML	NRP-EWMA	NRP-Shr	NRP-HFB	NCVaRERC-Hist-05	NCVaRERC-HFB-05	CVaRERC-HFB-05	CVaRERC-Hist-05	EW	Panel D
GDP-Mix1	17,37%	19,70%	14,86%	9,96%	17,74%	21,31%	17,67%	89,28%	88,63%	90,60%	89,91%	90,58%	45,22%	40,96%	43,87%	46,63%	86,56%	95,50%	98,85%	88,96%	85,37%	
IBGEP-Mix2	52,00%	44,37%	50,76%	55,94%	57,55%	57,14%	61,22%	94,59%	93,60%	95,55%	95,39%	97,26%	81,47%	70,56%	80,30%	86,86%	93,39%	97,93%	99,93%	94,77%	82,52%	
WWA-Mix3	35,98%	40,43%	38,17%	46,53%	29,22%	35,34%	42,38%	96,92%	95,37%	96,90%	96,66%	97,51%	83,19%	80,61%	84,00%	84,21%	96,37%	94,21%	98,01%	96,66%	96,06%	
Euro-Eq1	64,52%	62,33%	63,94%	77,75%	74,36%	70,59%	71,79%	98,99%	98,89%	99,13%	99,19%	99,66%	97,80%	97,15%	97,89%	98,22%	98,82%	99,61%	100,00%	98,96%	98,26%	
WES-Eq2	44,63%	52,94%	48,20%	76,66%	51,12%	39,65%	55,82%	99,60%	99,49%	99,65%	99,68%	99,87%	98,71%	98,44%	98,77%	98,79%	99,39%	99,82%	100,00%	99,63%	98,79%	
EEC-Eq3	55,95%	61,47%	57,46%	76,70%	64,07%	51,62%	66,37%	99,45%	98,87%	99,50%	99,56%	99,80%	97,75%	96,19%	97,69%	98,06%	99,35%	99,77%	99,52%	99,41%	98,69%	
Euro-Bond	54,74%	37,95%	51,28%	75,77%	34,86%	57,34%	34,46%	96,31%	95,41%	96,52%	96,93%	99,34%	96,42%	96,66%	96,58%	97,57%	97,41%	99,42%	100,00%	96,51%	89,55%	

Table 2.4: *Panel A:* out-of-sample Maximum Drawdown. *Panel B:* out-of-sample Ulcer index. *Panel C:* average Turnover. *Panel D:* average normalized Herfindhal index.

2.2.2.5 Strengths and weaknesses of the models considered

In this section we summarize some strengths and weaknesses of the several analyzed strategies, showing their properties both in terms of performance and of computational burden for portfolio construction.

Equally Weighted portfolio

Strengths:

- The EW portfolio is easy to implement and does not require any optimization approach;
- it does not use any in-sample information, avoiding possible estimation errors;
- the EW portfolio has no turnover (we do not consider price adjustment turnover);
- it produces good out-of-sample expected returns;
- the EW portfolio presents medium performance in terms of risk diversification w.r.t. CVaR5%.

Weaknesses:

- The EW strategy generates portfolios with very high out-of-sample risk both in terms of volatility and of Maximum Drawdown.

Minimum Variance portfolios

Strengths:

- MinV portfolios show low out-of-sample risk;
- MinV approaches tend to not invest in all assets;
- when using ML estimators, the MinV portfolio has low turnover;
- MinV approaches generate portfolios with good out-of-sample Sharpe ratio.

Weaknesses:

- The MinV strategy could invest in few assets, so it could be poorly diversified in terms of capital;
- MinV-EWMA and MinV-HFB portfolios present high turnover on 6 data sets out of 7.

Minimum CVaR portfolios

Strengths:

- MinCVaR strategies generate portfolios with low out-of-sample risk;
- MinCVaR-Hist portfolios show low turnover;
- MinCVaR approaches tend to not invest in all assets;

Weaknesses:

- The MinCVaR strategy, like almost all *Minimum-Risk* portfolios, shows poor risk diversification.

NCVaRERC portfolios
Strengths:

- The NCVaRERC portfolio is easy to implement and does not require any optimization approach;
- to determine the NCVaRERC portfolio, one only needs the marginal distribution of the assets and not the joint one;
- NCVaRERC portfolios present good risk diversification w.r.t. CVaR5%.

Weaknesses:

- the turnover of NCVaRERC portfolios strictly depends on the risk approach (historical or simulated) used to estimate the intrinsic CVaRs of the assets;
- the NCVaRERC portfolio, by construction, always contains all assets.

Risk Parity portfolios
Strengths:

- RP-ML portfolios show a good out-of-sample performance in terms of Sharpe ratio;
- the RP-ML approach produces portfolios with low risk both in terms of volatility and of Maximum Drawdown;
- RP portfolios present good risk diversification w.r.t. CVaR5%;
- RP portfolios show low turnover.

Weaknesses:

- The RP approach, by construction, always contains all assets, also very risky ones.

CVaRERC portfolio

Strengths:

- By construction, CVaRERC portfolios have the best risk diversification w.r.t. CVaR5%, equally splitting risk among all assets;
- CVaRERC portfolios have a good out-of-sample expected return.

Weaknesses:

- the turnover of CVaRERC portfolios strictly depends on the risk approach (historical or simulated) used to estimate the total CVaR and the contribution of each asset to CVaR;
- the existence of a CVaRERC solution is not always guaranteed.

Finally, to better visualize the behavior and the trend of the portfolio strategies considered, we show in Figure 2.1 the cumulative out-of-sample portfolio returns defined as in (2.13) for a subset of the real-world data sets listed in Table 2.2 (GDP-Mix1, WWA-Mix3, WES-Eq2, Euro-Bond). For reasons of space and clarity, we only plot some *Minimum-Risk* and *Risk-Diversification* portfolios together with the benchmark, i.e., the Equally Weighted portfolio.

The results obtained on the mixed assets universes (GDP-Mix1, WWA-Mix3) clearly highlight the different behavior in terms of risk of the EW portfolio w.r.t. the other portfolio strategies. Indeed, it is interesting to note that on GDP-Mix1 the drawdown, which characterizes the period 2008-2009, is significantly reduced for the *Minimum-Risk* and *Risk-Diversification* portfolios, showing lines of cumulative out-of-sample returns more regular than that of the EW portfolio. A similar behavior can be observed on the WWA-Mix3 data set, where the deepest drawdowns (2003 and 2009) of the EW portfolio are completely avoided by the other models. Furthermore, the smooth trend of the *Minimum-Risk* and *Risk-Diversification* curves visually show the high values of Sharpe Ratios, obtained by these strategies.

Regarding the WES-Eq2 data set, we notice that the *Risk-Diversification* portfolios tend to track the EW portfolio, while the *Minimum-Risk* portfolios show large differences w.r.t. the benchmark between 1999 and 2002. However, it seems that the latter strategies are preferable when the market is bearish.

Referring to the Euro-Bond data set, all the performance curves of the analyzed portfolios show a similar behavior up to 2010, i.e., as long as this investment universe is characterized by a single risk source (the interest rate). However, when a new risk factor (the credit risk) appears, the portfolios obtained by the models tend to have a better performance w.r.t. the EW portfolio.

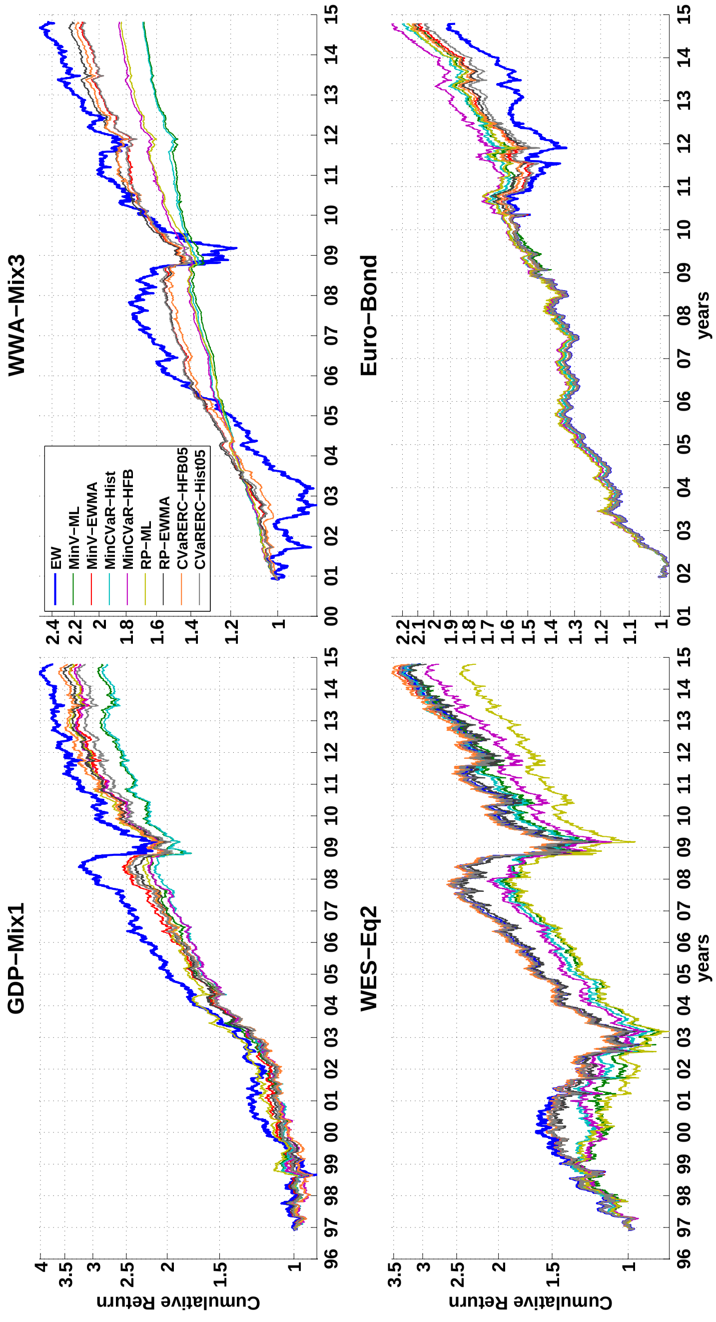

Figure 2.1: Cumulative out-of-sample portfolio returns for mixed assets universes (GDP-Mix1, WWA-Mix3), for Euro government bonds (Euro-Bond), and for an assets universe of equities alone (WES-Eq2).

2.3 Conclusions

In this chapter we have investigated on several portfolio selection models that avoid the use of expected returns as inputs. The Equal Risk Contribution strategy, due to the fact that it has the final goal to spread the risk among all asset, does not deal with the expected return estimation. For this reason, then, we proposed to solve the ERC problem as a convex optimization problem. From a practical viewpoint we have compared the out-of-sample performance of three different categories of portfolio selection models, namely *Minimum-Risk*, *Risk-Diversification* and *Capital-Diversification*. The latter class of models, represented by the Equally Weighted portfolio, is considered the benchmark. For the first two classes of models, short selling and leverage are not allowed thus making the feasible portfolios of the *Minimum-Risk* and of the *Risk-Diversification* models as similar as possible to the benchmark.

The analysis is performed on seven different real-world data sets, which consist of equities, bonds and mixed assets, each with different sources of risk.

We observe that the out-of-sample Sharpe ratio of many strategies is often higher than that of the Equally Weighted portfolio, mainly due to its poor performance in terms of volatility. Indeed, the *Minimum-Risk* models generally have very good Sharpe ratio in almost all assets universes, while the *Risk-Diversification* models tend to have an intermediate position with Sharpe ratios between those of the *Minimum-Risk* and of the *Capital-Diversification* strategies. In addition, we examine the statistical significance of the out-of-sample Sharpe ratio using the robust test proposed by Ledoit and Wolf (2008) (see [67]). We find that on equity markets the Sharpe ratios obtained with the models are not statistically different from the benchmark the EW portfolio, while statistically significant differences are realized in the other investment universes. These empirical findings highlight that when one invests in a market with a single source of risk, there is no clear dominance between the *Minimum-Risk*, the *Risk-Diversification* portfolios, and the EW one. On the other hand, when considering investment universes with multiple sources of risk, the *Minimum-Risk* and the *Risk-Diversification* strategies tend to have better performance than the Equally Weighted one. A relevant instance concerns the Euro-Bond data set. Indeed, as long as this investment universe is characterized by a single source of risk, namely the interest rate risk (up to 2010), the portfolios analyzed are quite similar in performance. But when a new risk factor (the credit risk) pops up, the performance of the portfolios obtained by the models tend to become different from that of the EW portfolio.

However, with regard to out-of-sample expected returns, the EW portfolio performs well on several markets, but it is characterized by high risk both in terms of volatility and of Maximum Drawdown and Ulcer Index. Therefore, the EW portfolio could be an advisable strategy but only with an adequate risk policy. On the other hand, the *Minimum-Risk* and the *Risk-Diversification* models, by construction, tend to have a risk policy that yields damped drawdowns w.r.t. those generated by the EW portfolio, but at the same time show moderate gains. Indeed, although there is no constraint on portfolio expected returns, the *Minimum-Risk* and the *Risk-Diversification* models show positive annual expected returns for all data sets.

The turnover generated by the models has low values for all those models that use historical scenarios to estimate inputs, and in particular those that employ ML estimators.

Generally, one cannot find a clear dominance of a model over all others. Each tested model responds to different requirements that could be related to diverse investors attitudes. On one hand, as expected, the *Minimum-Risk* models are advisable for risk averse investors, avoiding as much as possible any shock represented by deep drawdowns. On the other hand, the *Risk-Diversification* strategies seem to be appropriate for investors mildly averse to total portfolio risk. Specifically, these investors tend to be willing to waive a bit of safety and of return to achieve a more balanced portfolio in terms of risk. Finally, although the Equally-Weighted approach embodies the concept of high diversification, it generates portfolios with very high out-of-sample risk both in terms of volatility and of Maximum Drawdown, but with good out-of-sample expected returns. Therefore, according to our findings, such *Capital-Diversification* model seems to be advisable for sufficiently risk-loving investors, who try to maximize return without worrying about periods of deep drawdowns.

Chapter 3

Using CVaR-deviation for minimum risk and Equal Risk Contribution approaches

In this Chapter we introduce the CVaR-deviation as in Rockafellar et al. (see [97]) as a deviation measure. This particular statistic has the characteristic of being always positive and is able to come across the CVaR drawback. We extend the Equal Risk Contribution problem to CVaR-deviation and we arrived to the conclusion that we can solve it in the same way we proposed for CVaR.

Once we have used many risk measures we put in place a supervised experiment in order to evaluate the effect of the estimation error on the optimal portfolio weights. Kondor et al. (2007) (see [61]) note that covariance matrix also is afflicted by estimation error that leads to optimized portfolio weights very far from the theoretical ones. We replicate their methodology adding CVaR-deviation for unconstrained portfolios weights (long-short portfolios). We extend the test to long only portfolios considering the euclidean distance between theoretical and optimized portfolios as a measure of the effect that estimation error produces on portfolio weights. Within this we agree with Kondor et al since we obtain their results. With respect to CVaR-deviation we find that the effect of the estimation error is less than CVaR that leads to more stable portfolios.

We extend the analysis to long only portfolios in order to consider also Equal Risk Contribution portfolios. In this case, due to the fact that we set a limited interval for portfolio weights ($0 \leq w_i \leq 1$), we see that the effect of estimation error on portfolio weights is extremely low with respect to the previous case. We also note that the effect of estimation error is lower on ERC portfolios.

In the financial industry it is common to use historical data in order to optimize portfolios. Each practitioner is aware that there is huge estimation error when the number of securities is close to the time series length. In this chapter we also propose a way of increasing the number of observations without increasing the time series' length. We use the Filtered Bootstrap procedure to simulate hundred thousand of possible scenarios, doing so we squeeze the ratio N/T that is crucial point to maintain under control the effect of estimation error.

3.1 CVaR-deviation

A functional $\mathcal{D} : \mathcal{L} \to [0, \infty]$ is called a deviation measure in the extended sense if it satisfies:

D1: $\mathcal{D}(C) = 0$ for constant C, but $\mathcal{D}(X) > 0$ for nonconstant X;

D2: $\mathcal{D}((1-\lambda)X + \lambda(X') \leq (1-\lambda)\mathcal{D}(X) + \lambda\mathcal{D}(X')$ for $\lambda \in]0, 1[$ (convexity);

D3: $\mathcal{D}(X) \leq d$ when $\left\|X^k - X\right\| \to 0$ with $D(X^k) \leq d$ (closedness).

A functional $\mathcal{D} : \mathcal{L} \to [0, \infty]$ is called a deviation measure in the basic sense if it satisfies axioms D1, D2, D3 and D4:

D4: $\mathcal{D}(\lambda X) = \lambda D(X)$ (positive homogeneity).

A deviation measure in extended or basic sens is also coherent if it additionally satisfies D5:

D5: $\mathcal{D}(X) \leq sup(X - E[X])$ (upper range boundedness).

Examples of deviation measures in the basic sense:

- Standard Deviation
- Standard Semideviations
- Mean Absolute Deviation

The Value-at-Risk deviation measure

$$VaR_{\varepsilon}^{\Delta}(x) = VaR_{\varepsilon}(R_P(x) - E[R_P(x)])$$

$\varepsilon - VaR$ does not satisfy convexity axioms D2, it is not a deviation measure, while CVaR-deviation defined as

$$CVaR_{\varepsilon}^{\Delta}(x) = CVaR_{\varepsilon}(R_P(x) - E[R_P(x)]) \tag{3.1}$$

is a coherent deviation measure in basic sense.

Rockafellar et al. (see [97]) showed the existence of a one-to-one correspondence between deviation measures in the extended sense and averse risk measures in the extended sense:

$$\mathcal{D}(X) = \mathcal{R}(X - E[X])$$

$$\mathcal{R}(X) = \mathcal{D}(X) + E[X])$$

$\mathcal{R}$ is coherent if end only if $\mathcal{D}$ is coherent and $\mathcal{R}$ is positive homogeneous if and only if $\mathcal{D}$ is positive homogeneous.

3.2 Finding CVaR-deviation ERC portfolios

As described in section 2.1.3.3 the CVaR Equal Risk Contribution (CVaRERC) portfolio is characterized by the same requirement of obtaining a portfolio composition that achieves equal total risk contribution among all assets, where the risk is measured by CVaR.

However, unlike volatility, CVaR may be positive or negative, and in the latter case CVaR indicates a gain. Since the Equal Risk Contribution approach makes sense only in the case of positive risk, we decided to apply it only when the minimum CVaR is positive. In the very few cases where the optimal value of Problem (2.2) is negative, instead of the ERC approach we considered the minimum risk one.

An alternative risk measure, but related to CVaR, is CVaR-deviation that, by definition is greater or equal to 0 (see [97] and [102]). The ERC portfolio based on CVaR-deviation can be obtained as in the case of CVaR. Indeed, CVaR-deviation is a homogeneous function of degree 1. Therefore we have

$$CVaR_{\varepsilon}^{\Delta}(x) = \sum_{i=1}^{n} x_i \frac{\partial CVaR_{\varepsilon}^{\Delta}(x)}{\partial x_i} = \sum_{i=1}^{n} TRC_i^{CVaR^{\Delta}}(x),$$

where $TRC_i^{CVaR^{\Delta}}(x) = x_i \frac{\partial CVaR_{\varepsilon}^{\Delta}(x)}{\partial x_i}$ is the total risk contribution of asset i. Furthermore, we have

$$TRC_i^{CVaR^{\Delta}}(x) = x_i(E\left[-r_i | R_P \leq -VaR_{\varepsilon}(x)\right] - \mu_i) \tag{3.2}$$

(see [19] and [103]). Thus, the CVaR-deviation ERC portfolio can be obtained by solving the following system of equations and inequalities:

$$\begin{cases} TRC_i^{CVaR^{\Delta}}(x) = \lambda & \forall i = 1, ..., n \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 & i = 1, \dots, n \end{cases} \tag{3.3}$$

Similar to (2.9), we can find the CVaR-deviation ERC portfolio by solving the convex optimization problem

$$\begin{aligned} \min \quad & CVaR_{\varepsilon}^{\Delta}(y) \\ s.t. \quad & \\ & \sum_{i=1}^{n} \ln y_i \geq c \end{aligned} \tag{3.4}$$

where c is an arbitrary constant.

It is straightforward to highlight that the differences in portfolio weights using CVaR or CVaR-deviation as risk measure. The former is sensible to expected returns while the latter is not. The reason is: if for a single asset i we increase the expected return his CVaR is less than before (it is going to be less riskier than before) and than its portfolio weight calculated using ERCCVaR will increases. The same situation will not change the portfolio weights in the case of ERCCVaR-deviation.

3.3 Sensitivity to estimation errors of the optimal portfolios

In this section, we follow the procedure presented by Kondor et al. (2007) [61]. Therefore we consider standard normal market where the covariance matrix is the identity matrix. Let N the number of asset in portfolio and let T the available number of observation. In this test we focus on some values of the ratio N/T. In particular we set the value of $T = 500$ as is common in financial industry. We investigate some particular values of $N = \{25, 50, 75, 100, 125, 150, 200, 250, 375, 425\}$. Doing so the $\inf N/T = 0.05$ and $\sup N/T = 0.85$. In order to obtain negligible the estimation error one should set $T \to \infty$. Due to the fact that process is not feasible, in order to obtain a measure the effect of the estimation errors, we replicate any optimization procedure 100 times.

Kondor et al. (2007) ([61]) propose as a stability measure the relative sub-optimality which in the case of minimum variance portfolio is

$$q_0^2 = \frac{w'^{*}\Sigma^{(0)}w^{*}}{w'^{(0)*}\Sigma^{(0)}w^{(0)*}} \tag{3.5}$$

where $w^{(0)*}$ are the weights of the portfolio selected using $\Sigma^{(0)}$ (the true covariance matrix, i.e, the identity matrix), and w^* is the optimal portfolio obtained using the empirical covariance matrix Σ, that is perturbed w.r.t. $\Sigma^{(0)}$. The square root of the denominator in Equation 3.5 is the true risk (standard deviation) of the portfolio, while the square root of the numerator is the risk we run when using the weights derived from the empirical covariance matrix. Since the weights w^* are optimal under the empirical covariance matrix which is different from $\Sigma^{(0)}$, the numerator of Equation (3.5) will always be larger than the denominator, q_0 is always larger than 1 and goes to 1 only when $T \to +\infty$.

Considering Minimum Variance optimization $\bar{q}_0$ has a closed form solution that is

$$\bar{q}_0 = \frac{1}{\sqrt{1 - \frac{N}{T}}} \tag{3.6}$$

In the same framework, we also examine the stability of the minimum risk portfolio using Conditional VaR and Conditional VaR Deviation as risk measure. To evaluate the noise sensitivity of CVaR and CVaR Deviation, we use as stability measure the relative sub-optimality that in this case is:

$$q_{0,CVaR}^2 = \frac{\sum_i (w_i^*)^2}{\sum_i (w_i^{(0)*})^2} \tag{3.7}$$

Now, since in a standard normal market the true optimal weights coincides to those of the EW portfolio (i.e., $w_i^{(0)*} = 1/N$), we have that

$$q_{0,CVaR}^2 = N \sum_i (w_i^*)^2. \tag{3.8}$$

In section 3.3.1 we examine the case in which $w_i \in \mathbb{R}$ and in section 3.3.2 the case in which $w_i \in [0, 1]$.

3.3.1 Long-short optimal portfolios

In this section we present the stability analysis for long-short minimum risk portfolios, as in [61]. We calculate the average of q_0 on 100 experiment replications, by varying N and T (see section 3.3). We perform the stability analysis for CVaR and CVaR-deviation, setting the confidence level

$\varepsilon = 0.3, 0.2, 0.1, 0.05, 0.01$, thus focusing on the 'bad' tail of returns distribution. In Figures 3.1, 3.2, 3.3, 3.4 and 3.5 we show the $\bar{q}_0$ values as function of the ratio N/T with respect to the selected ε.

We note that the empirical values of $\bar{q}_0$ considering the minimum variance portfolio are quite similar to the theoretical one. Minimum CVaR portfolio is pretty stable for $N/T < 0.4$ with exception of $\varepsilon = 0.3$ in which is stable for $N/T < 0.3$. It also shows a higher effect of estimation error with respect to Minimum Variance. Minimum CVaR Deviation has intermediate effect of estimation error between Minimum Variance and Minimum CVaR. It is unstable with high values of ε and when the ratio $N/T > 0.5$

In Figures 3.6 and 3.7 we report the $\bar{q}_0$, calculated for Minimum CVaR and Minimun CVaR Deviation, as a function of $\beta = (1 - \varepsilon)$ for the ratio $N/T = 0.25$ and $N/T = 0.05$ respectively. We find a sort of stability from values of $0.7 \leq \beta \leq 0.9$ while for $\beta > 0.9$ the $\bar{q}_0$ is slightly increasing.

In order to validate the previous statement, we put in place more extensive tests considering the ratio $N/T = 0.25$ and $N/T = 0.05$. We set $\varepsilon \in [0.01, 0.5]$ and we show the calculated $\bar{q}_0$ in the Figures 3.8 and 3.9. For both experiments we set $N = 50$. Considering the ratio $N/T = 0.25$ we can recognize a similar shape that Kondor et al. (2007) (see [61]) shows in Figure 16. On the Minimum CVaR, the effect of the estimation error has a minimum near to 0.78 while Minimun CVaR deviation shows that the effect estimation error increases as β increases (ε decreases). Considering the ratio $N/T = 0.05$, as expected, the effect of the estimation error is extremely small with respect to the previous case. Both models have the same shape as before with the exception that Minimum CVaR is not afflicted by huge estimation error when $\beta = 0.5$.

3.3.2 Long only optimal portfolios

In the previous section, we notice that CVaR and CVaR Deviation minimization suffer the effet of the estimation error due to the increasing N/T ratio. In order to evaluate how portfolio constraints mitigate the effect of the estimation error, we decided to introduce the lower and the upper bound for portfolio weight. In particular $0 \leq w_i \leq 1$. We utilize the same values of ε, N and T as in section 3.3.1.

In Figures 3.10, 3.11, 3.12, 3.13 and 3.14 we show the $\bar{q}_0$ values as function of the ratio N/T with respect to ε.

Introducing boundaries for portfolio weights we eliminate the bad effect of the estimation error. The variability of portfolio weights remains under control for all optimization procedures. Minimum CVaR and Minimum CVaR

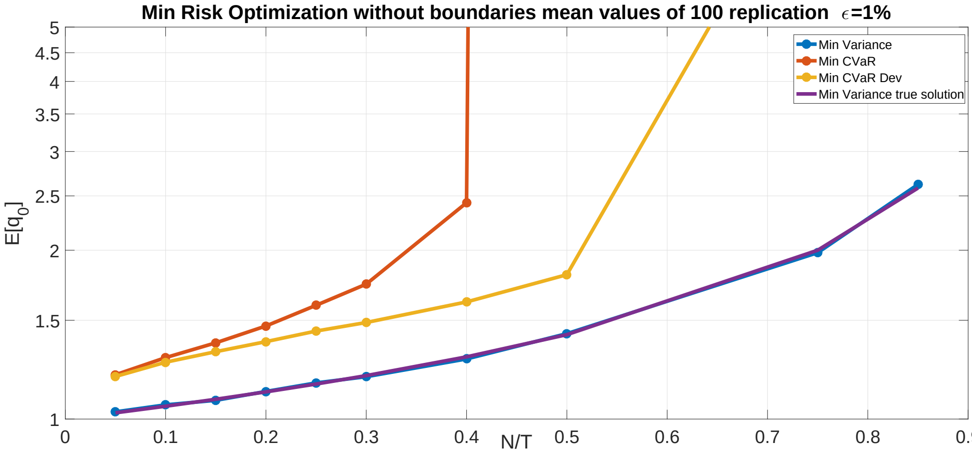

Figure 3.1: Minimum Risk Optimization without boundaries: q_0 mean value of 100 replications $\varepsilon = 1\%$.

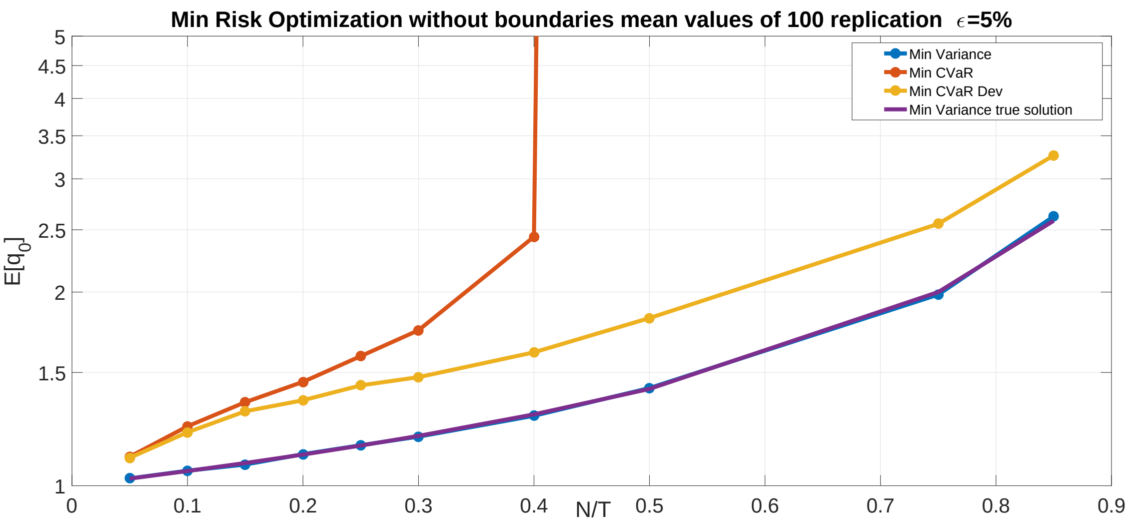

Figure 3.2: Minimum Risk Optimization without boundaries: q_0 mean value of 100 replications $\varepsilon = 5\%$.

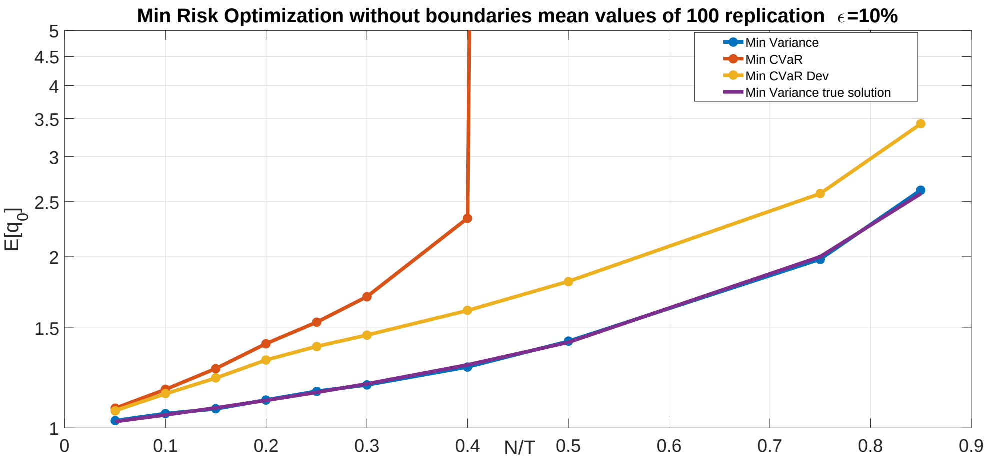

Figure 3.3: Minimum Risk Optimization without boundaries: q_0 mean value of 100 replications $\varepsilon = 10\%$.

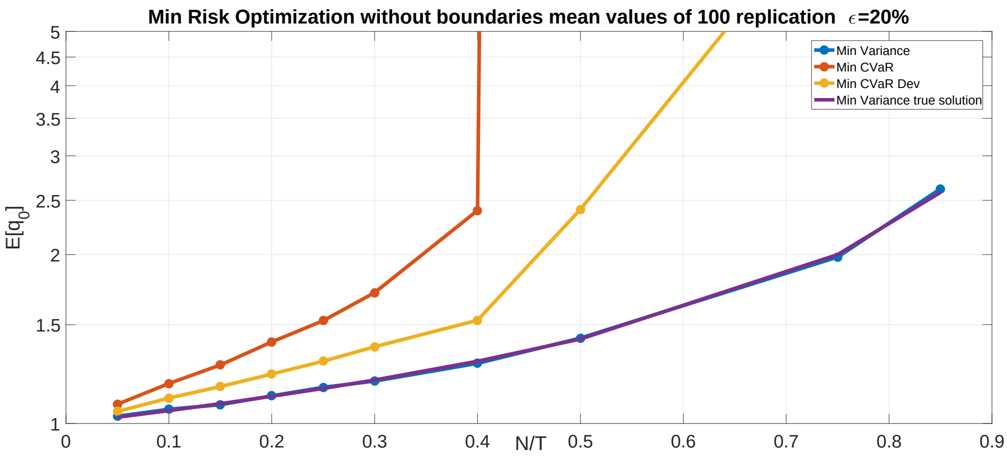

Figure 3.4: Minimum Risk Optimization without boundaries: q_0 mean value of 100 replications $\varepsilon = 20\%$.

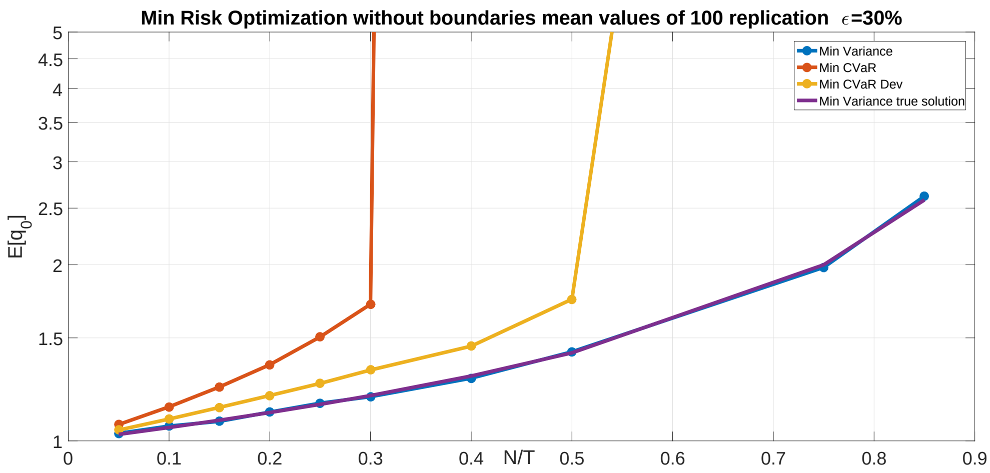

Figure 3.5: Minimum Risk Optimization without boundaries: q_0 mean value of 100 replications $\varepsilon = 30\%$.

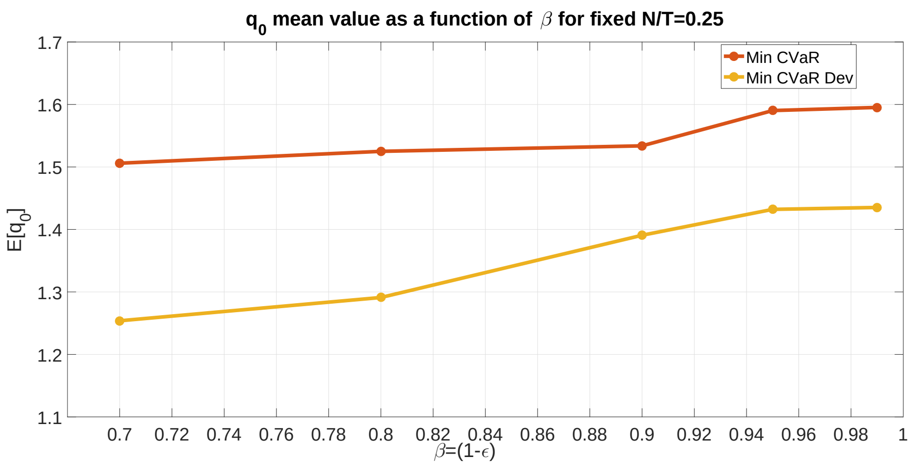

Figure 3.6: $\bar{q}_0$ as a function of β for fixed $N/T = 0.25$: unbounded weights

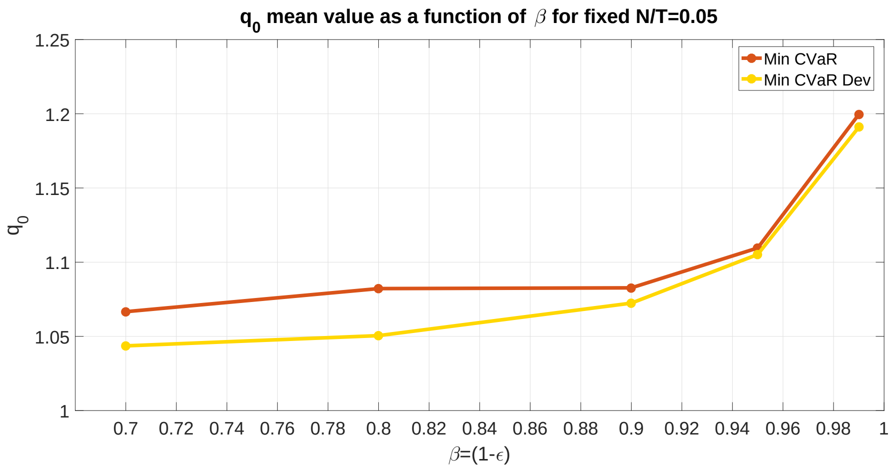

Figure 3.7: $\bar{q}_0$ as a function of β for fixed $N/T = 0.05$: unbounded weights

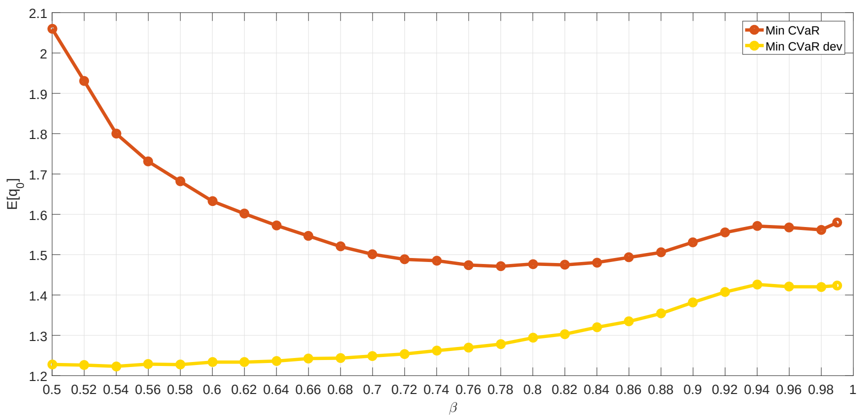

Figure 3.8: $\bar{q}_0$ as a function of β for fixed $N/T = 0.25$: unbounded weights

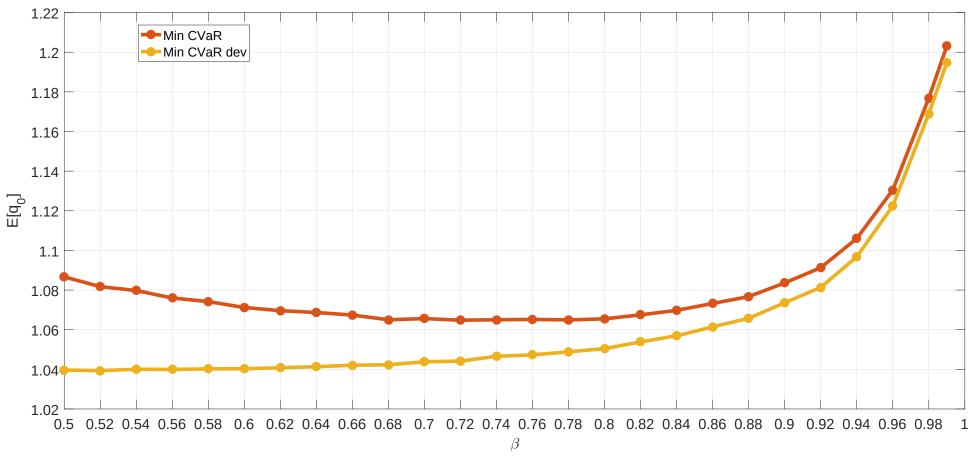

Figure 3.9: $\bar{q}_0$ as a function of β for fixed $N/T = 0.05$: unbounded weights

Deviation show that the effect of the estimation error is still present and it is higher with respect to minimum variance when $N/T \leq 0.5$. When the ratio grows up the effect of the estimation error on minimum variance increases faster than the other models.

In Figures 3.15 we report the $\bar{q}_0$, calculated for Minimum Variance. Minimum CVaR and Minimun CVaR Deviation as a function of $\beta = (1 - \varepsilon)$ for the ratio $N/T = 0.25$ and $N/T = 0.05$ respectively. We find a sort of stability for all portfolio optimization procedures. The Minimum Variance portfolio is the more stable while the other two procedures have an oscillating shape around the value 1.25 (Min CVaR Deviation) and 1.33 (Min CVaR). In Figure 3.16 we recognize the same path discovered for unconstrained portfolio weights.

As observed by Kondor et al. (2007) [61] *A remark on the role of the simplifying assumptions above is in order. As the instability of portfolio selection is an information-deficit catastrophe, we do not believe that the use of real market data, or non-stationary time series, or fat-tailed distributions, or the introduction of a risk free asset and a constraint on expected return would qualitatively modify our conclusions. This is not at all true of the last assumption, the lack of a constraint on short selling. It is evident that a ban on short selling (or any other set of constraints that would render the domain over which we seek an optimum finite) would automatically eliminate the possibility of a divergence. It would then seem that our results refer to a completely unrealistic case. We insist, however, that it is useful to consider this unrealistic case first, because it helps understand the root of the instability and identify the strong residual fluctuations that reflect this instability even after the constraints are reintroduced.* We find exactly what the authors state.

3.4 Evalutate Estimation Error on ERC portfolios

In this section, in order to evaluate the effect of the estimation error on portfolio weights, we define a new measure instead of the q_0 seen in the previous section. Our goal is to identify how far the optimized portfolio is from the theoretical one. A common measure of distance can be useful in order to achieve our objective. The Euclidean distance between vectors is defined as the norm of the difference of the portfolio weights. Let w^* be the optimized portfolio under the perturbed scenario and $w^{(0)*}$ the theoretical portfolio under the unit covariance matrix, and the two are points in the space $\mathbb{R}^n$, then $\left\| w^* - w^{(0)*} \right\|$ represent the distance between the optimized portfolio and the true portfolio

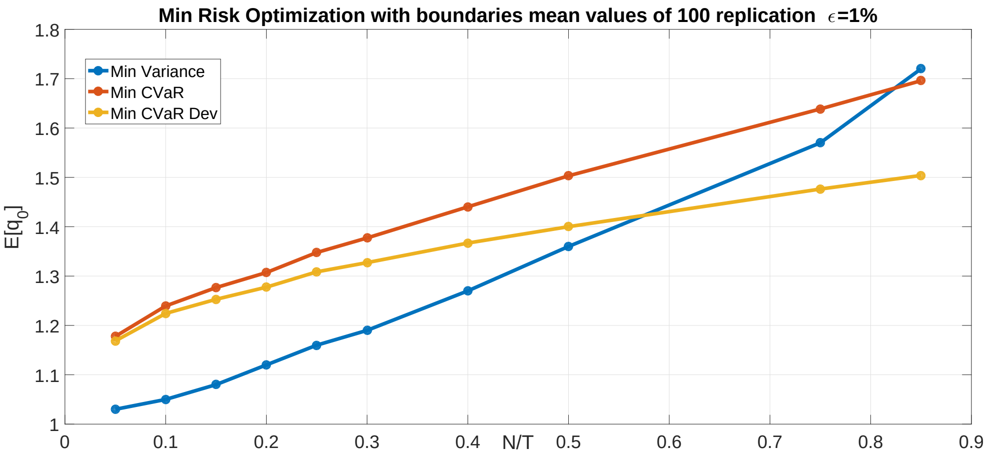

Figure 3.10: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 1\%$.

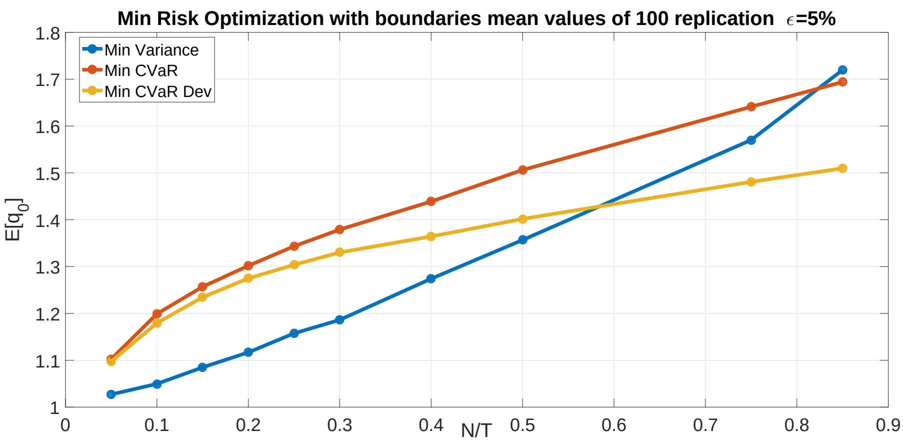

Figure 3.11: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 5\%$.

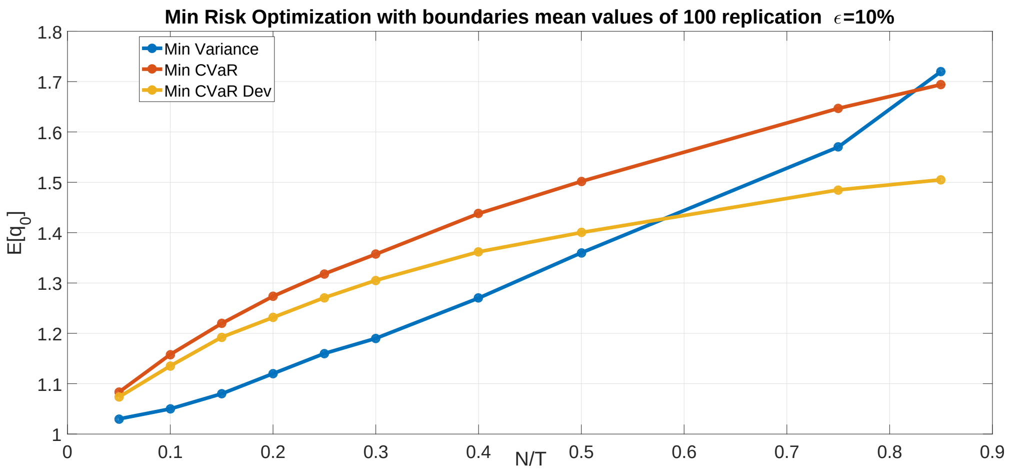

Figure 3.12: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 10\%$.

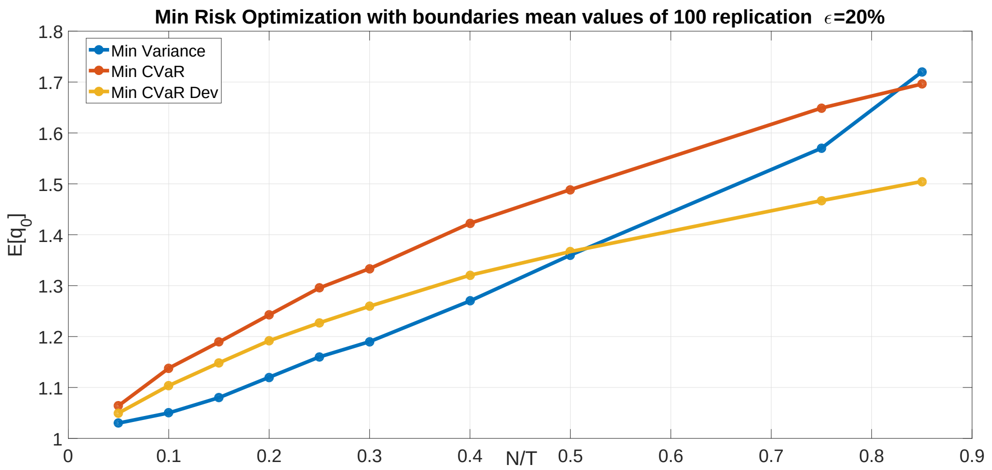

Figure 3.13: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 20\%$.

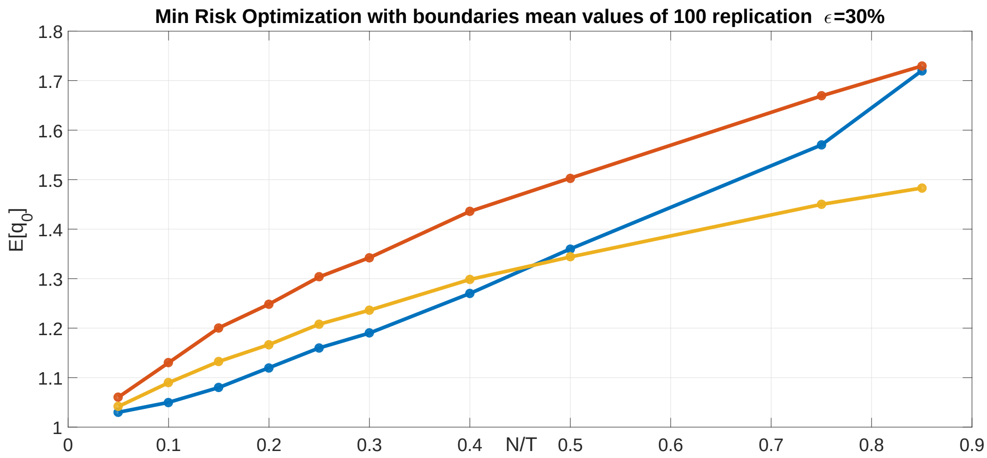

Figure 3.14: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 30\%$.

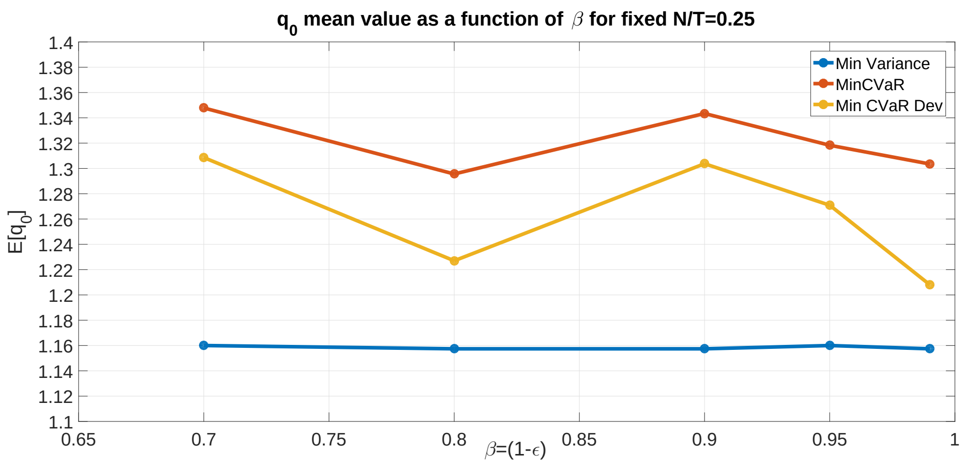

Figure 3.15: $\bar{q}_0$ as a function of β for fixed $N/T = 0.25$: bounded weights

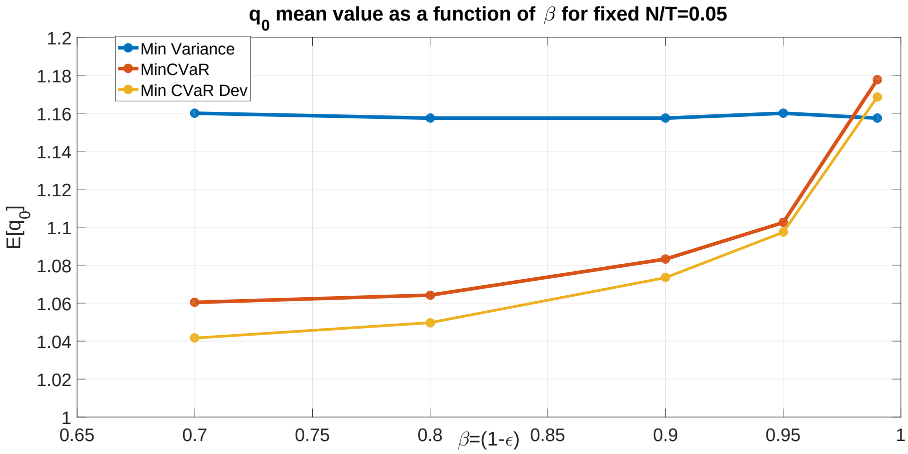

Figure 3.16: $\bar{q}_0$ as a function of β for fixed $N/T = 0.05$: bounded weights

composition. In this section, we only consider bounded portfolios in which $0 \leq w^* \leq 1$, due to the fact that for ERC portfolios we require positive portfolio weights and no leverage.

In figures 3.17, 3.18, 3.19 and 3.20 we analyze the extreme scenario where $\varepsilon = 1\%$. The reference portfolio, the minimum variance, shows a quite stable dispersion over all the considered ratios N/T assuming mean value around 0.045. ERC portfolio has a maximum of variability among portfolio weights when $0.15 \leq N/T \leq 0.30$. Minimum CVaR and Minimum CVaR-deviation portfolios have high variability for small values of the ratio N/T that quickly decreases when N/T increases. The high variability when N/T is close to 0 is due to the fact that we are analyzing few scenarios (5 out of 500) and it brings the optimization procedure to consider extreme cases. In fact, any portfolio can generate a mean distance above the one generated by the minimum variance portfolio.

In figures 3.21, 3.22, 3.23 and 3.24 we analyze the scenario where $\varepsilon = 5\%$. In this case we have the same pattern seen before for Min CVaR and Min CVaR-deviation while ERC portfolios have a stable pattern close to the minimum variance. We are analyzing 25 out of 500 scenarios and the extreme scenarios are quite mitigated.

In figures 3.25, 3.26, 3.27 and 3.28 we analyze the scenario where $\varepsilon = 10\%$. In this case we have the same pattern seen before for Min CVaR and Min CVaR-deviation but both distances diminish considerably in value. ERC portfolios have a stable pattern below to minimum variance. We are analyzing 50 out of 500 scenarios and the extreme scenarios are very well mitigated.

In figures 3.29, 3.30, 3.31, 3.32 3.33, 3.34, 3.35 and 3.36 we analyze the scenario where $\varepsilon = 20\%$ and $\varepsilon = 30\%$. We notice a stability of the recorded distance in which the minimum CVaR and minimum CVaR deviation are above the minimum variance and the ERC portfolios are below the minimum variance. The effect of extreme scenarios in the tail are completely mitigated.

In conclusion, when we use 500 realized scenarios as in the typical way to optimize portfolios using historical returns we notice that few extreme scenarios are able to produce different values of portfolio weight due to the effect of estimation error. This happens mainly when the value of ε is small.

We put in place a new test to consider this effect by setting a fixed ratio of $N/T = 0.15$ and replicating the analysis above, sampling from an independent standard Normal distribution, using $T = 100, 1000, 10000$. As a consequence we need a number of asset in portfolio N equal to $15, 150, 1500$, respectively. We replicate 100 times the experiment considering $\varepsilon = 5\%$. We expect to record high variability in the distance when $T = 100$ and we also expect high values of it. In figure 3.37 we show the result for $T = 100$, there is a huge variability and the mean distance is not below 0.06 for all portfolios. In figure 3.38 we see that the variability decreases and the distance mean value is always below 0.065. In figure 3.39 we replicate only 10 times the procedure using the last values[1]. Although we replicate the experiment only 10 times we notice that distance mean value is below 0.02 for each model. Min CVaR and Min CVaR-deviation have the same shape observed in figure 3.21. This conducts to the conclusion that the effect of the estimation error on the portfolio weights still remain high.

Figure 3.40 shows the distance mean values on an x log scale. The ERC portfolios are very close to the minimum variance while the Min CVaR and Min CVaR-deviation are always above it. A second consideration is that if we want to reduce the estimation error we must increase the number of data taken into account: for each increment of an order of magnitude, the mean distance value is reduced by a factor of 1/3.

Summarizing, fixing the level of confidence level $\varepsilon = 5\%$ and the ratio $N/T = 15\%$ and increasing the number of observations T, we notice that the effect of the estimation error tends to quickly diminish. In real world when we have more than 1000 assets in the portfolio (it happens in a bank trade book) we need more than 10000 points to avoid a huge effect of the estimation error. In the next chapter, we analyze a procedure that allows to overcome this issue.

[1] The computational time is roughly 33 hours using a workstation CPU Intel Xeon v5 3.30GHz, RAM 16GB, Windows XP 7 Professional 64 bit.

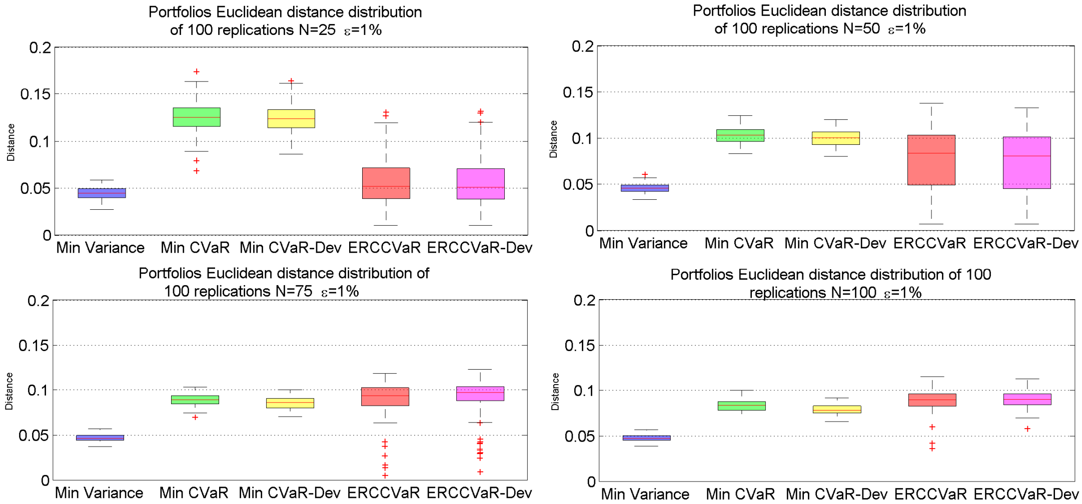

Figure 3.17: Euclidean distance distribution of 100 replications $\varepsilon = 1\%$, $N = 25, 50, 75, 100$

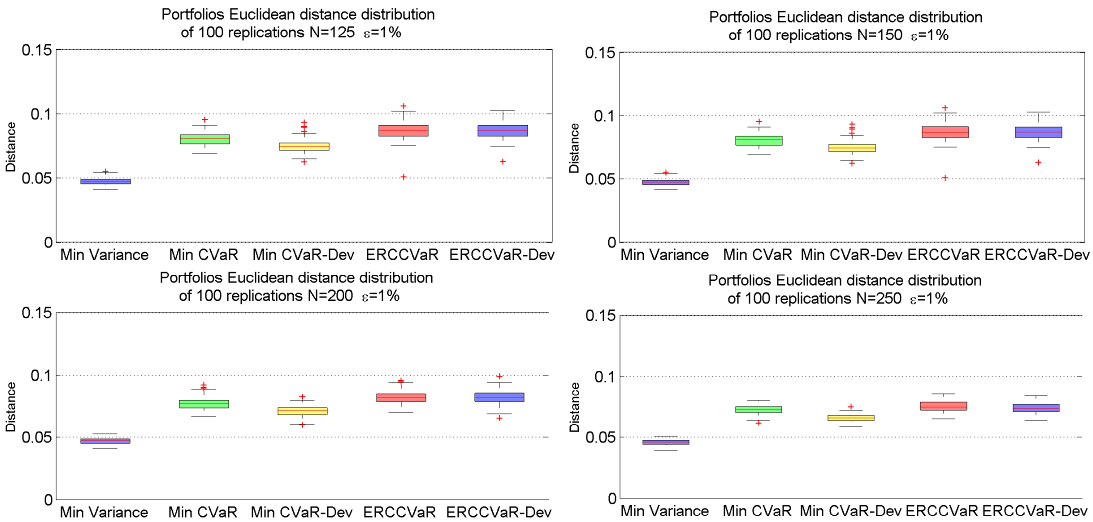

Figure 3.18: Euclidean distance distribution of 100 replications $\varepsilon = 1\%$,$N = 125, 150, 200, 250$

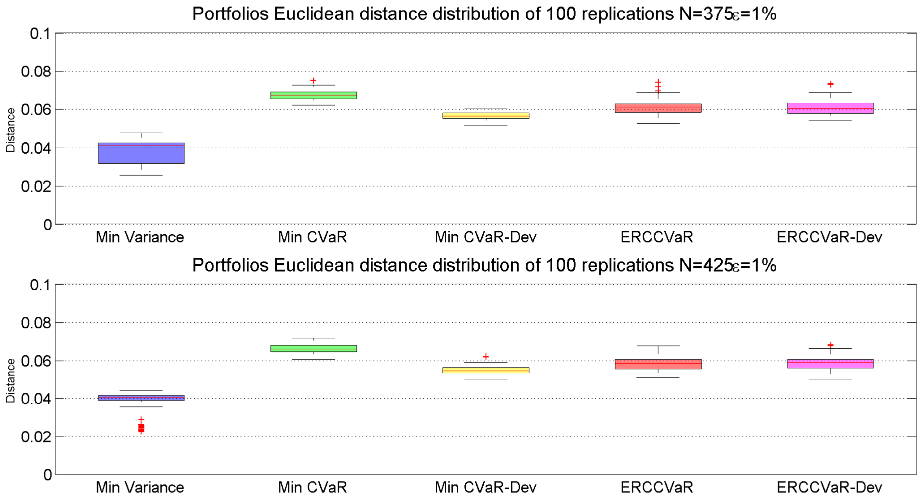

Figure 3.19: Euclidean distance distribution of 100 replications $\varepsilon = 1\%$,$N = 375, 425$

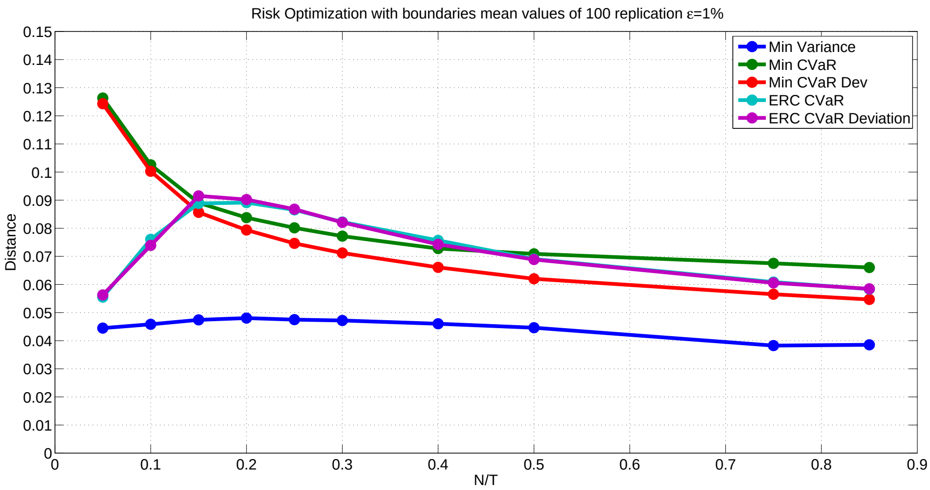

Figure 3.20: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 1\%$.

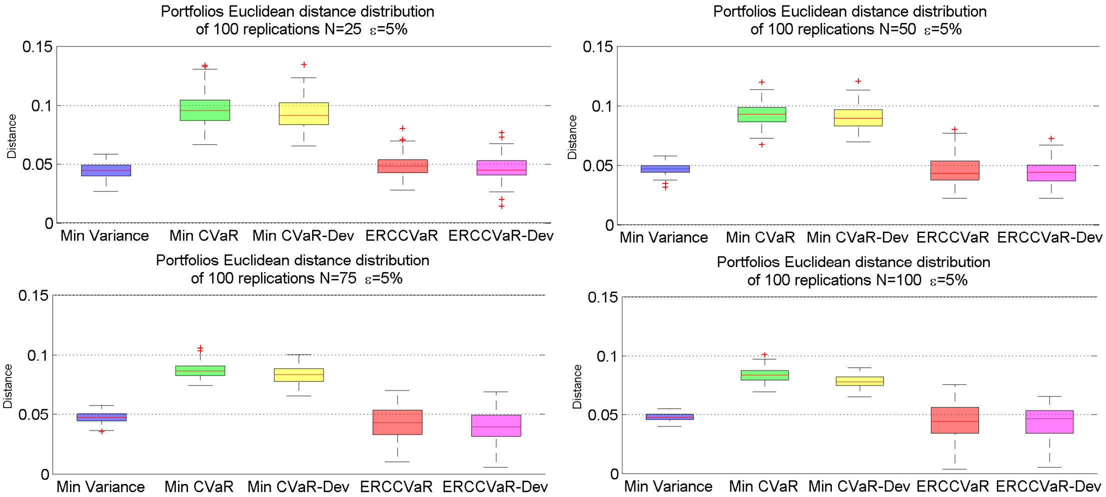

Figure 3.21: Euclidean distance distribution of 100 replications $\varepsilon = 5\%$, $N = 25, 50, 75, 100$

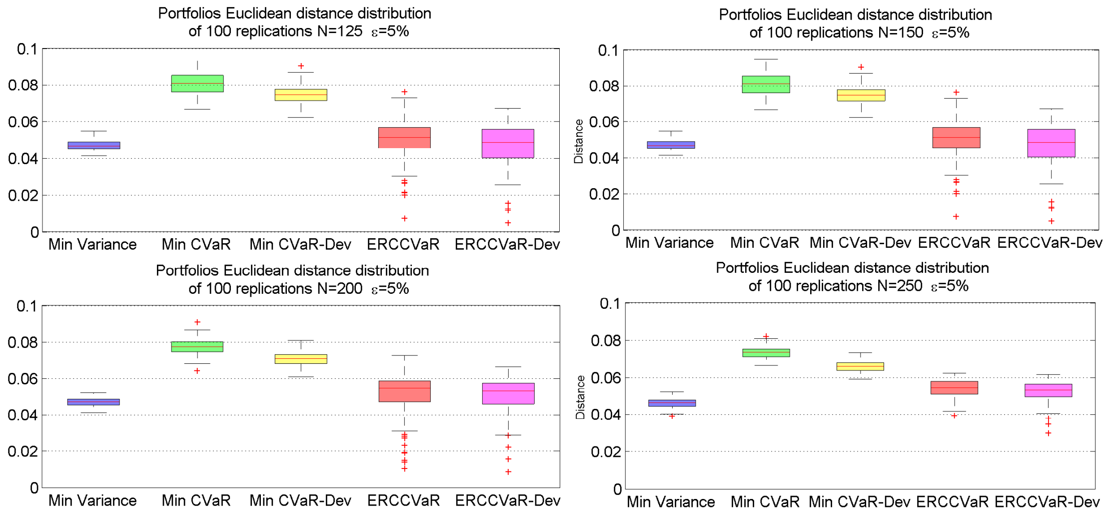

Figure 3.22: Euclidean distance distribution of 100 replications $\varepsilon = 5\%$,$N = 125, 150, 200, 250$

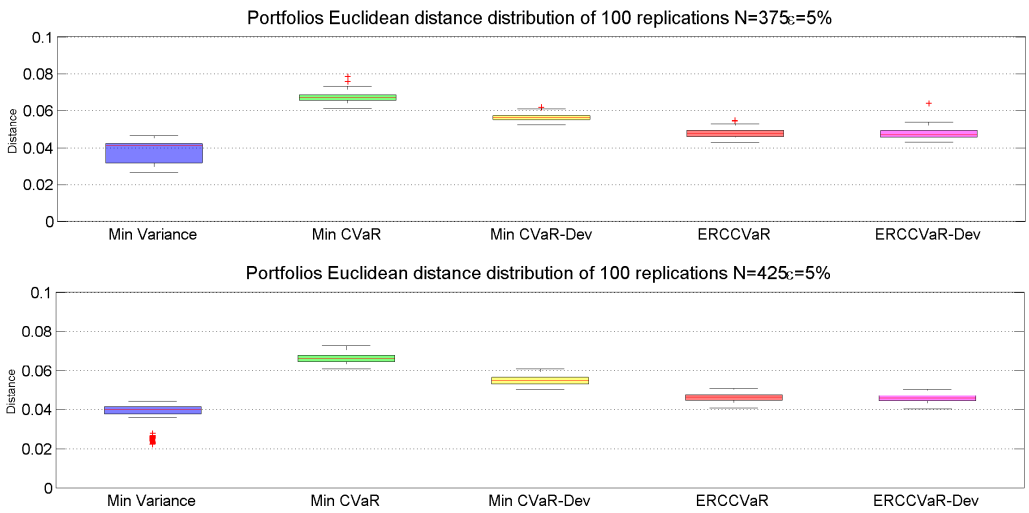

Figure 3.23: Euclidean distance distribution of 100 replications $\varepsilon = 5\%, N = 375, 425$

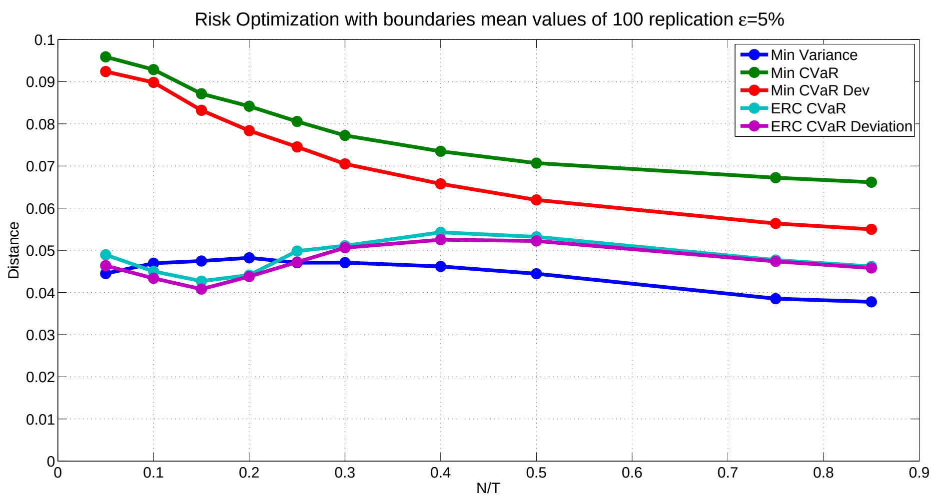

Figure 3.24: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 5\%$.

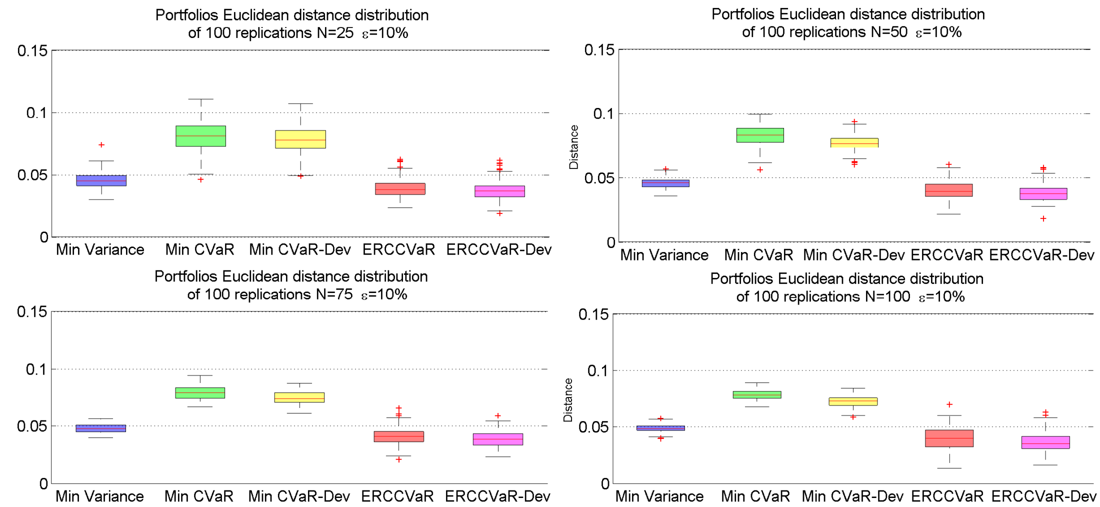

Figure 3.25: Euclidean distance distribution of 100 replications $\varepsilon = 10\%$, $N = 25, 50, 75, 100$

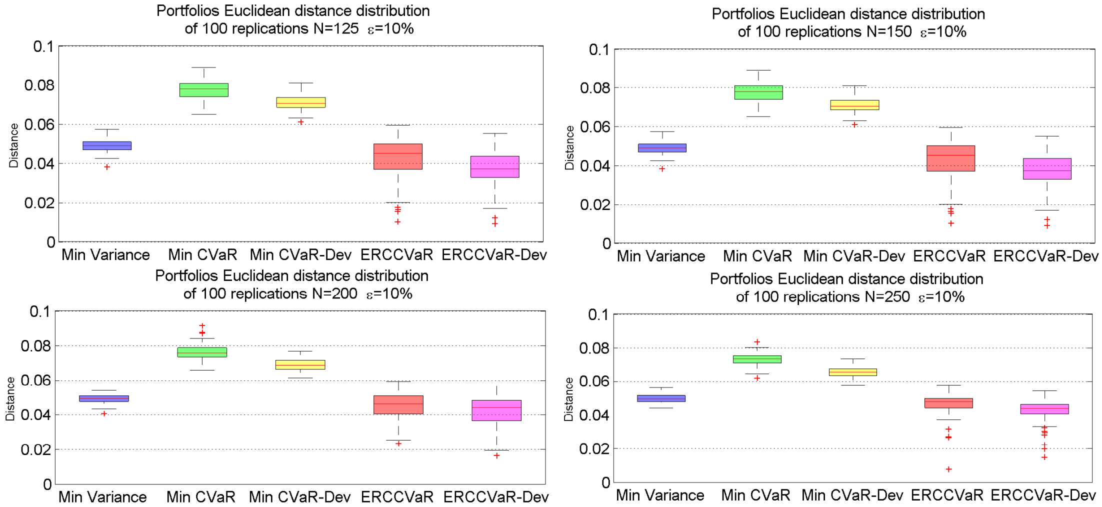

Figure 3.26: Euclidean distance distribution of 100 replications $\varepsilon = 10\%$,$N = 125, 150, 200, 250$

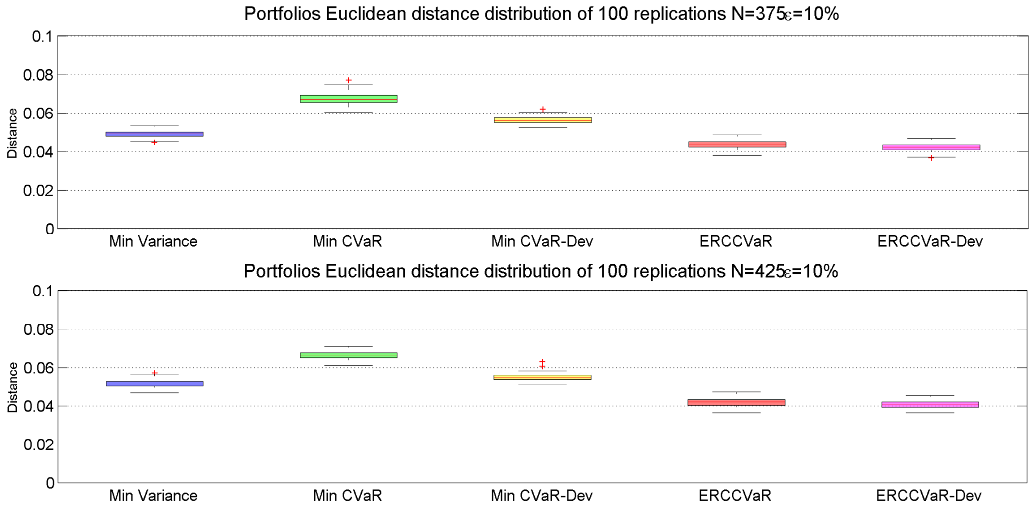

Figure 3.27: Euclidean distance distribution of 100 replications $\varepsilon = 10\%$,$N = 375, 425$

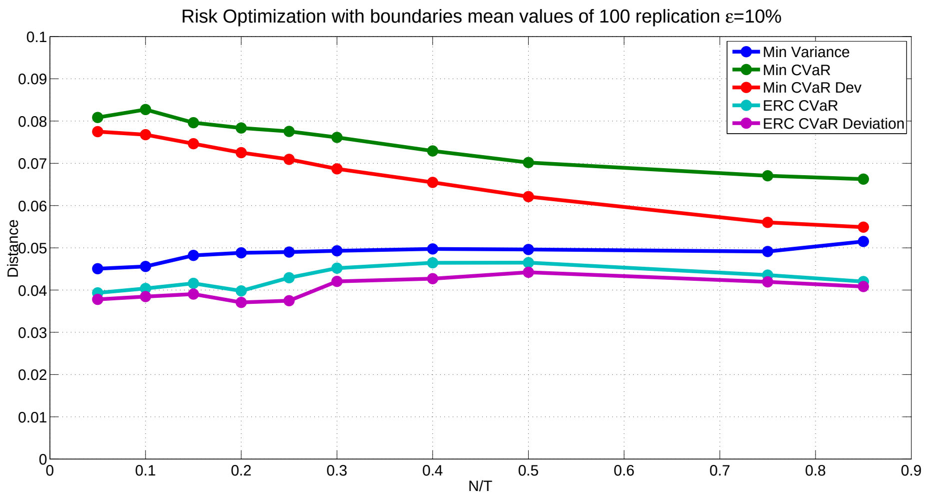

Figure 3.28: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 10\%$.

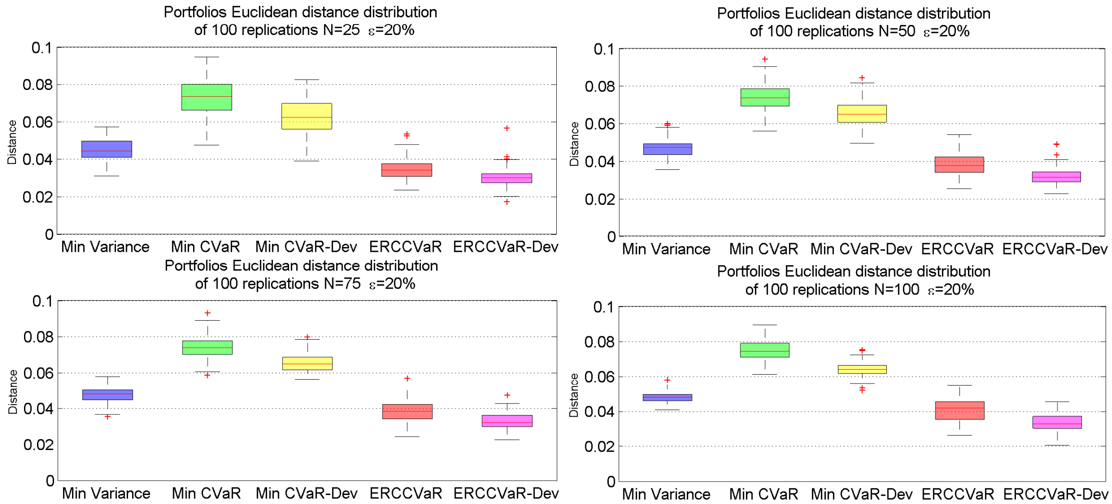

Figure 3.29: Euclidean distance distribution of 100 replications $\varepsilon = 20\%$, $N = 25, 50, 75, 100$

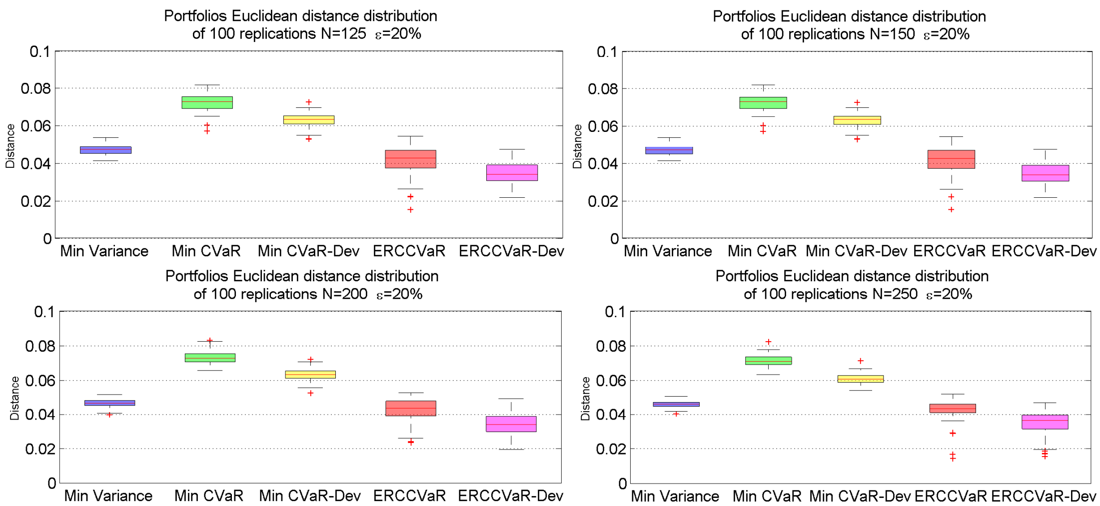

Figure 3.30: Euclidean distance distribution of 100 replications $\varepsilon = 20\%$,$N = 125, 150, 200, 250$

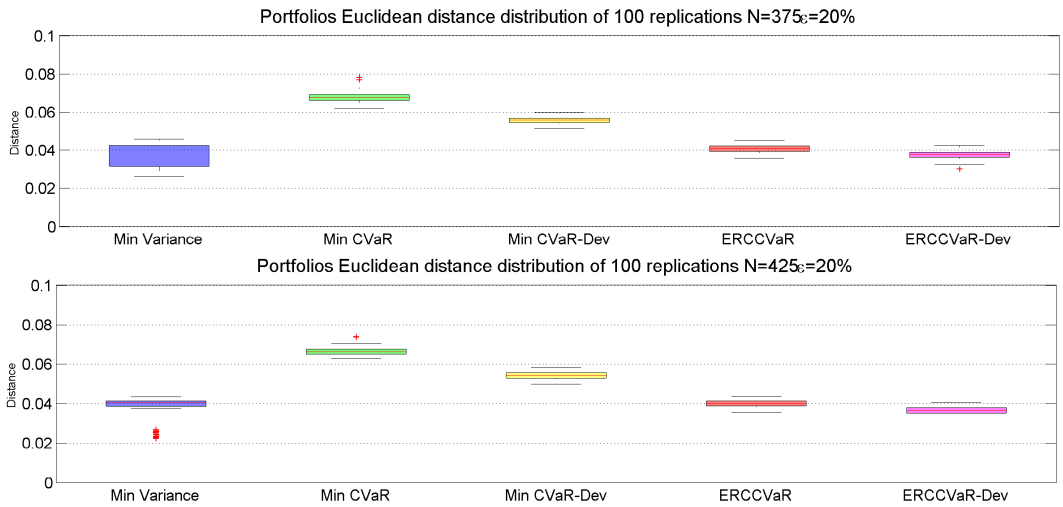

Figure 3.31: Euclidean distance distribution of 100 replications $\varepsilon = 20\%$,$N = 375, 425$

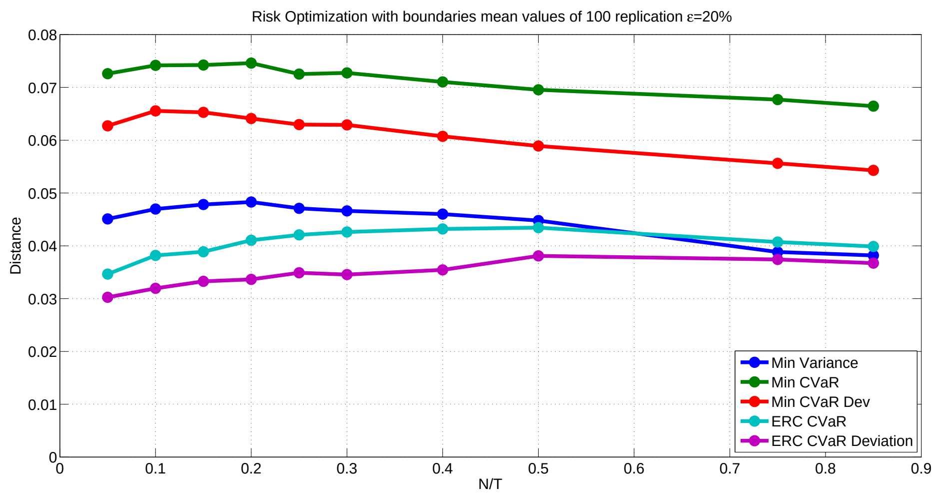

Figure 3.32: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 20\%$.

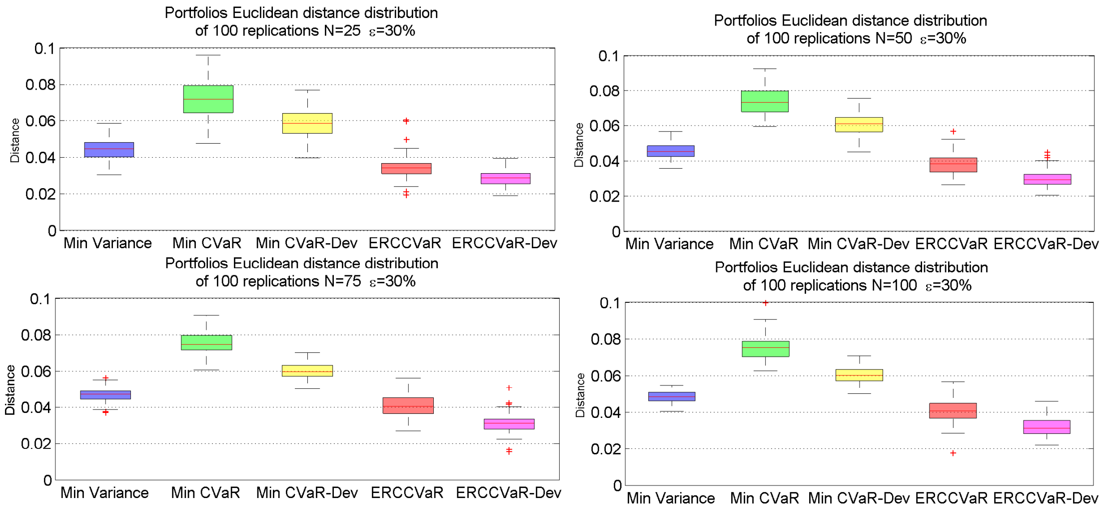

Figure 3.33: Euclidean distance distribution of 100 replications $\varepsilon = 30\%$, $N = 25, 50, 75, 100$

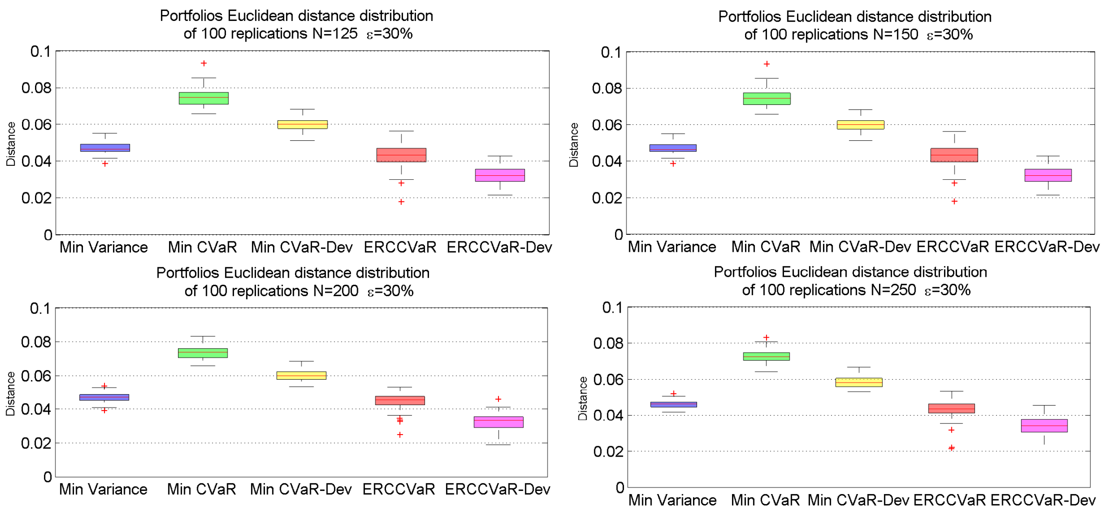

Figure 3.34: Euclidean distance distribution of 100 replications $\varepsilon = 30\%$,$N = 125, 150, 200, 250$

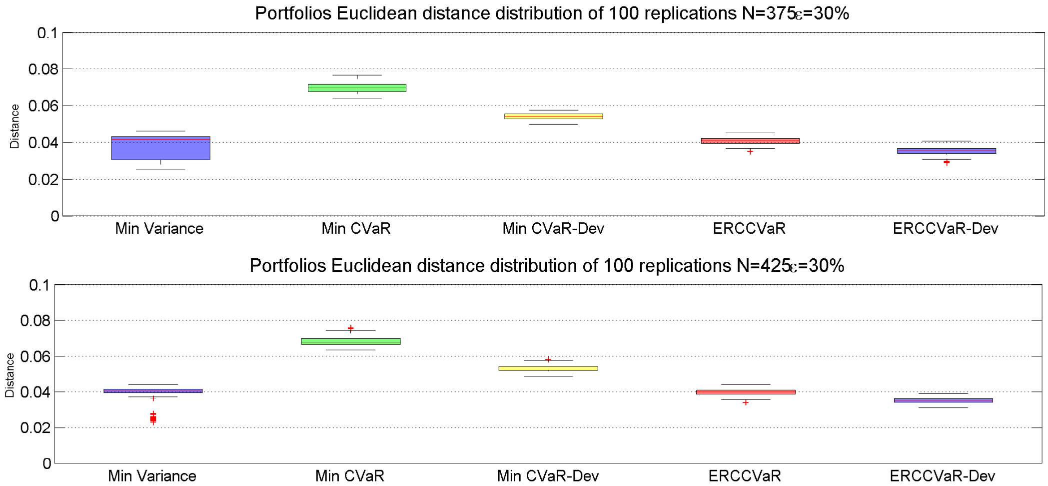

Figure 3.35: Euclidean distance distribution of 100 replications $\varepsilon = 30\%$,$N = 375, 425$

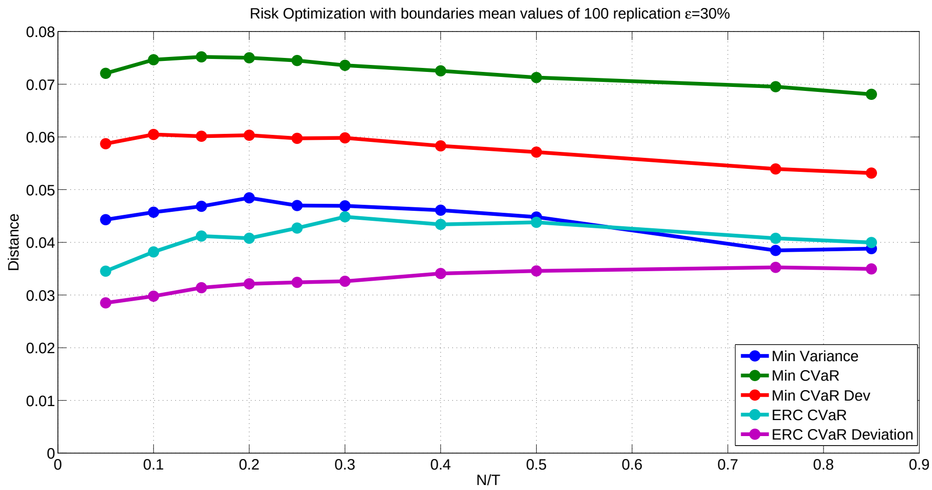

Figure 3.36: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $\varepsilon = 30\%$.

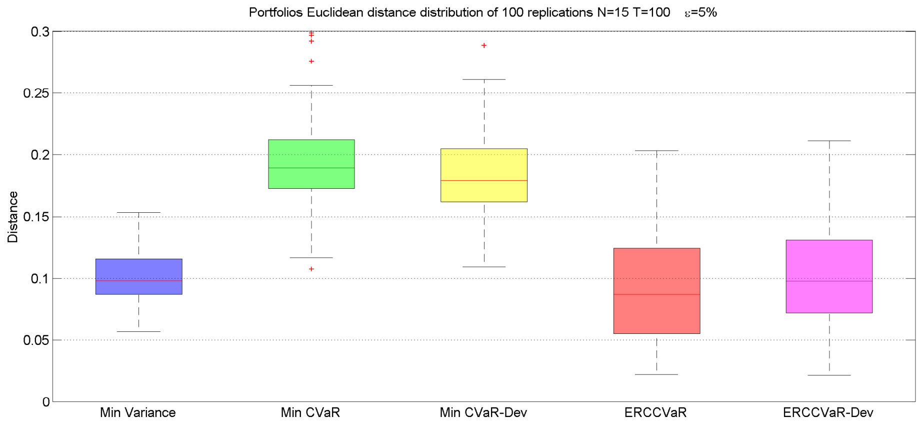

Figure 3.37: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $N = 15$, $T = 100$, $\varepsilon = 5\%$.

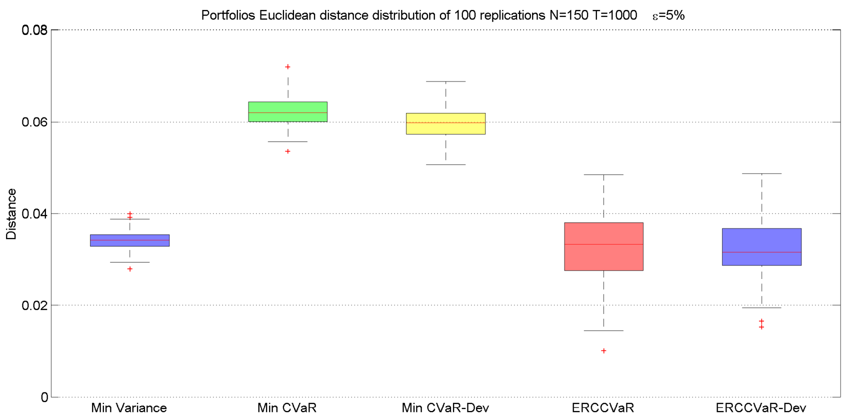

Figure 3.38: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 100 replications $N = 150$, $T = 1000$, $\varepsilon = 5\%$.

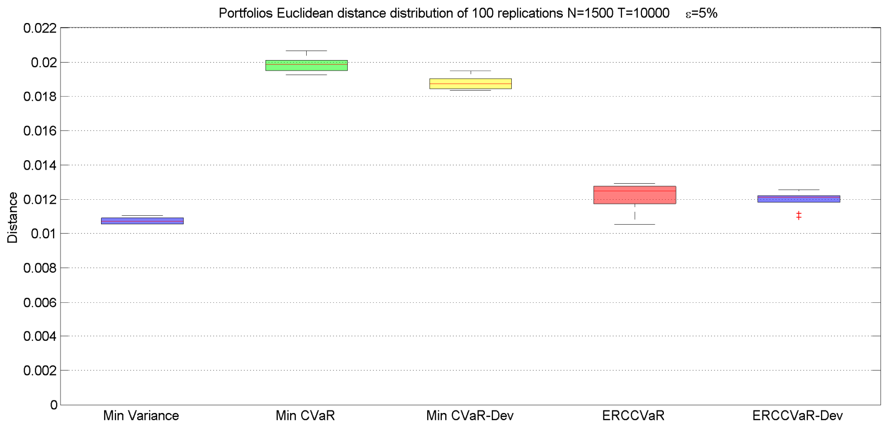

Figure 3.39: Minimum Risk and Equally Risk Contribution Optimization with boundaries: q_0 mean value of 10 replications $N = 1500$, $T = 10000$, $\varepsilon = 5\%$.

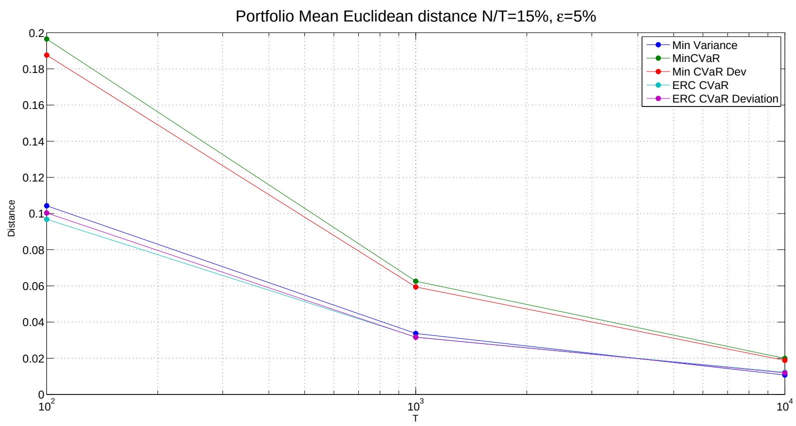

Figure 3.40: Minimum Risk and Equally Risk Contribution Optimization with boundaries: mean value of Euclidean distance $N/T = 15\%$, $\varepsilon = 5\%$.

Chapter 4

How to model the future assets' returns

In the previous chapter, we showed how the ratio N/T generates noise into the estimation procedure. It is common in practice to use historical prices as input for portfolio optimization. As we see, a large number of asset requires a very long prices time series. In order to overcome this problem, we use a simulation procedure we describe in the following sections. It requires only the last 500 daily returns to estimate parameter and to simulate future assets paths. We are used to simulating 20 days ahead sampling from the returns distribution standardized residuals. Since we use a bootstrap procedure on a sample of size 500 and we generate 20 days ahead we may potentially have $T = 500^{20}$ different monthly returns. Hereafter we use a different length of $T = 10^3, 10^4, 10^5, 10^6$ with the goal to reduce as much as possible the ratio N/T.

In order to estimate CVaR for each asset, we apply a really powerful procedure called *Historical Filtered Bootstrap*. The Historical Filtered Bootstrap VaR (HFBVaR) approach (see [8], [24], [113] and [78]) is a mixed procedure in which one represents the market returns using, for instance, an autoregressive moving average generalized autoregressive conditional heteroschedasticity (ARMA-GARCH or ARMA-EGARCH or ARMA-GJRGARCH) model to filter the time series. This approach leads to compute the empirical standardized residuals from data without assuming on them any specific probability distribution. Below we give a step-by-step description of the HFBVaR procedure.

1. We filter the time series of each asset by a univariate ARMA-GARCH model. More precisely, for the observed returns of the asset k we find the Maximum Likelihood estimators of the following AR(1)-StudT-GARCH(1,1)

model:

$$\begin{aligned} \text{AR(1)} &: \quad r_{t,k} = a_k + b_k r_{t-1,k} + \eta_{t,k} \\ \text{StudT-GARCH(1,1)} &: \quad \sigma^2_{t,k} = \alpha_k + \beta_k \sigma^2_{t-1,k} + \gamma_k \eta^2_{t-1,k} \\ & \quad\ \ \eta_{t,k} = \sigma_{t,k} z_{t,k} \end{aligned}$$

where $z_{t,k} = \sqrt{\frac{\nu_k - 2}{\nu_k}} T_{\nu_k}$, T_{ν_k} follows a Student-T distribution with ν_k degrees of freedom, and $\hat{\theta} = \{a_k, b_k, \alpha_k, \beta_k, \gamma_k, \nu_k\}$ are Maximum Likelihood estimators (see [108] and reference therein) obtained on 500 daily data.

2. Using the set of estimators $\hat{\theta} = \{a_k, b_k, \alpha_k, \beta_k, \gamma_k, \nu_k\}$ for all n assets available in the market, we compute from data the standardize residuals $\hat{z}_{t,k}$ with $t = 1, \ldots, T$ and $k = 1, \ldots, n$, i.e., we divide the empirical residuals $\hat{\eta}_{t,k}$ by their estimated volatilities $\hat{\sigma}_{t,k}$.

3. We bootstrap in a parallel fashion the matrix of the empirical standardized residuals $\hat{Z} = \left\{\hat{z}_{t,k}\right\}$ with $t = 1, \ldots, T$ and $k = 1, \ldots, n$. More precisely, we randomly sample with replacement the rows of the matrix $\hat{Z}$, thus allowing to capture the multivariate shocks of the entire system.

4. The bootstrapped standardized residuals $\hat{Z}^{boot} = \left\{\hat{z}^{boot}_{s,k}\right\}$, with $s = 1, \ldots, S$ and $k = 1, \ldots, n$, are then used as multivariate innovations in the (univariate) AR(1)-StudT-GARCH(1,1) models to simulate the one-day-ahead returns. In our empirical analysis we employ S[1] bootstrapped scenarios.

5. Finally, the S scenarios are used to estimate the portfolio CVaR at confidence level $1 - \varepsilon$.

Note that although AR(1)-StudT-GARCH(1,1) estimations are performed on univariate cases, the dependence structure among the assets is captured by the parallel bootstrap procedure on the standardized residuals $\hat{Z}$. In other words, through this approach of sampling we are able to generate scenarios with historical common shocks. However, for more details see [1], [31] and [43].

[1]The aim of the paper is also to show how the choice of S influence the risk estimation.

4.1 How many scenarios do we use in Historical Filtered Bootstrap?

4.1.1 Static Analysis

Using the procedure above we evaluate 100 times the assets CVaR one month horizon (20 days) using $S = 1K$, $S = 10K$, $S = 100K$ and $S = 1M$ scenarios. We investigate two different asset universes, the first one is a multiasset universe as in Cesarone and Colucci (2017) (see [32]). and the second one is composed only of equity indices.

We expect that increasing the number of scenarios the CVaR estimation will be more stable. We report the summary statistic in Table 4.1 for the multiasset universe and in 4.2 for equity universe. We note that the standard deviation of the CVaR fastens decreases giving stability to the estimated value. The estimated CVaR mean is quite similar among different lengths of the scenarios so the mean is unbiased with respect to S. For each run, we also calculate Minimum CVaR portfolio, Minimum CVaR deviation portfolio, ERCCVaR portfolio and ERCCVaR deviation portfolio. We collect asset weights in order to figure out how variable is the portfolio composition when the number of scenarios increase.

All experiments are executed on a workstation with Intel Xeon CPU (E3-1225 v5, 3.3 GHz, 16 Gb RAM) under MS Windows 7 Professional. In order to generate $S = 1M$ scenarios for a single asset the running time is roughly 5 seconds while using $S = 10K$ the running time is only 0.20 seconds. The time necessary to evaluate the complete multivariate distribution is $5 \cdot n$, (when $S = 1M$) where n is the number of assets. In order to find the solution to the Problem 2.4 the running time is roughly 1 minute using $S = 1M$ and with respect to the Problem 2.8 the running time is roughly 2 minutes using $S = 1M$. The time fasten decreases to 1 second considering only $S = 10K$.

4.1.1.1 Multiasset Universe Results

The multi-asset universe is the one that authors in Cesarone and Colucci (2017) (see [32]) call Worldwide Asset. It is composed of 28 assets among bond, equity, and commodities[2].

In Figure 4.1 we show the box plot of the single asset CVaR estimation, the distribution of the CVaR estimate is too large when we use $1K$ scenarios

[2]The complete description is available here
http://host.uniroma3.it/docenti/cesarone/Store/DescriptionOfDatSets.pdf

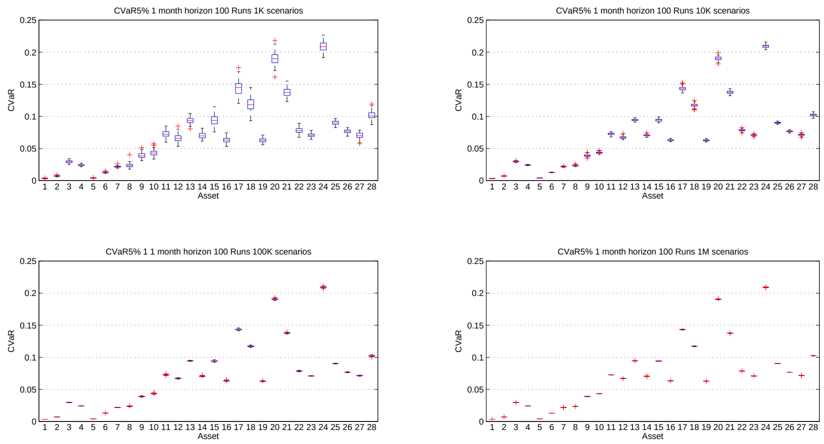

Figure 4.1: CVaR Estimation, Multiasset Universe

while using $10K$ scenarios we observe a better estimate. When we would have a complete precision on the CVaR estimate we have to consider to evaluate $1M$ scenarios. A good compromise between computational time and precision is $100K$ scenarios.

In Figure 4.2 and in Figure 4.3 we show the mean-CVaR plane where the mean is calculated as the $CVaR100\%$ or simply as the scenarios average value. We note that using $1M$ scenarios the points on the plane are near one to each other (for each portfolio construction procedure) while using $1K$ scenarios points are quite dispersed. This is due to the high variability in the CVaR estimation.

In Figure 4.4 and in Figure 4.5 we report the box plot of asset weights obtained using Min CVaR and Min CVaR deviation. The two procedure build up very concentrated portfolios (two assets dominate the portfolio) and as the number of scenarios increases the asset weight is much more stable.

In Figure 4.6 and in Figure 4.7 we report the box plot of asset weights obtained using ERCCVaR and ERCCVaR deviation. First of all, we note that we have more variability in less risky asset weights when we use $1M$ scenarios. This could be the consequence of few extreme scenarios that create variability on asset weights. In this case, the choice of $100K$ scenario could be a good compromise between the stability of the solution and computational time.

4.1.1.2 Equity Asset Universe Results

The Equity Asset universe is composed of 29 equity indices that cover countries and worldwide sectors. The asset considered are listed in the following bullet

Asset	CVaR Mean				CVaR Standard Deviation			
	1K	10K	100K	1M	1K	10K	100K	1M
1	0.335	0.335	0.336	0.336	0.028	0.011	0.003	0.001
2	0.710	0.709	0.711	0.711	0.061	0.022	0.007	0.002
3	2.937	2.955	2.958	2.957	0.210	0.069	0.020	0.005
4	2.423	2.421	2.424	2.423	0.121	0.043	0.011	0.004
5	0.398	0.399	0.401	0.401	0.023	0.008	0.002	0.001
6	1.278	1.278	1.281	1.282	0.084	0.028	0.008	0.002
7	2.162	2.165	2.169	2.169	0.123	0.044	0.013	0.003
8	2.369	2.348	2.355	2.352	0.315	0.086	0.030	0.009
9	3.903	3.898	3.894	3.897	0.435	0.160	0.048	0.014
10	4.316	4.324	4.325	4.325	0.461	0.135	0.042	0.014
11	7.256	7.278	7.272	7.265	0.496	0.161	0.048	0.014
12	6.657	6.717	6.707	6.710	0.592	0.195	0.057	0.020
13	9.350	9.435	9.441	9.443	0.461	0.150	0.043	0.014
14	6.999	7.043	7.037	7.035	0.460	0.139	0.046	0.014
15	9.414	9.398	9.408	9.409	0.801	0.198	0.080	0.022
16	6.294	6.311	6.326	6.324	0.413	0.116	0.045	0.014
17	14.420	14.337	14.327	14.315	1.097	0.307	0.097	0.032
18	11.811	11.737	11.715	11.706	1.001	0.288	0.089	0.031
19	6.259	6.263	6.267	6.264	0.352	0.117	0.035	0.011
20	18.998	19.023	19.042	19.029	0.936	0.288	0.097	0.031
21	13.765	13.750	13.768	13.757	0.676	0.212	0.064	0.019
22	7.774	7.841	7.842	7.843	0.441	0.146	0.050	0.014
23	7.065	7.085	7.091	7.093	0.285	0.091	0.027	0.010
24	20.870	20.926	20.903	20.901	0.774	0.252	0.082	0.026
25	8.992	9.011	9.012	9.010	0.334	0.103	0.030	0.011
26	7.663	7.668	7.666	7.661	0.281	0.107	0.034	0.010
27	7.040	7.122	7.145	7.146	0.434	0.140	0.042	0.011
28	10.187	10.206	10.235	10.240	0.589	0.217	0.067	0.020

Table 4.1: CVaR Mean and Standard Deviation (in percentage) estimated for each number of scenarios estimated 100 times, Multiasset Universe.

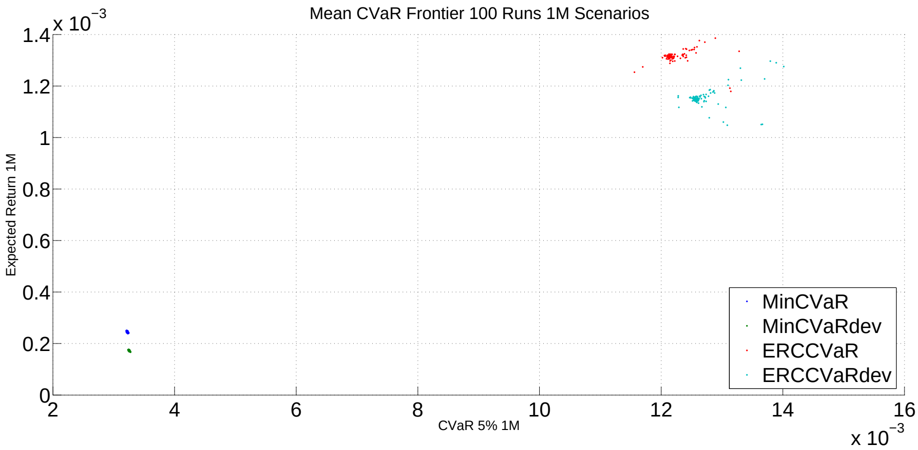

Figure 4.2: Mean CVaR frontier 1 million scenarios, Multiasset Universe

Asset	CVaR Mean				CVaR Standard Deviation			
	1K	10K	100K	1M	1K	10K	100K	1M
1	15.116	15.133	15.102	15.115	0.956	0.300	0.086	0.027
2	12.413	12.329	12.347	12.351	0.900	0.255	0.092	0.027
3	9.417	9.441	9.416	9.421	0.508	0.190	0.055	0.017
4	8.410	8.427	8.408	8.416	0.453	0.165	0.050	0.017
5	8.852	8.917	8.924	8.937	0.467	0.179	0.061	0.021
6	8.932	8.960	8.954	8.962	0.551	0.215	0.067	0.022
7	6.479	6.515	6.505	6.506	0.284	0.113	0.031	0.010
8	8.239	8.126	8.141	8.146	0.659	0.196	0.063	0.019
9	8.592	8.628	8.594	8.600	0.442	0.143	0.052	0.016
10	10.454	10.467	10.445	10.451	0.614	0.211	0.065	0.020
11	7.663	7.689	7.708	7.701	0.487	0.128	0.049	0.015
12	7.455	7.507	7.508	7.500	0.614	0.201	0.070	0.022
13	12.574	12.636	12.673	12.661	0.752	0.220	0.077	0.026
14	8.829	8.857	8.881	8.878	0.451	0.125	0.041	0.013
15	7.392	7.409	7.422	7.415	0.479	0.126	0.047	0.015
16	11.504	11.608	11.620	11.618	0.834	0.242	0.080	0.023
17	11.193	11.303	11.315	11.317	0.788	0.225	0.072	0.021
18	10.846	10.876	10.889	10.892	0.423	0.149	0.045	0.017
19	9.903	9.884	9.890	9.893	0.670	0.226	0.081	0.022
20	9.551	9.665	9.699	9.704	0.652	0.214	0.067	0.020
21	9.213	9.263	9.273	9.277	0.532	0.150	0.059	0.017
22	10.571	10.531	10.532	10.533	0.439	0.119	0.044	0.014
23	8.353	8.363	8.363	8.368	0.492	0.170	0.045	0.017
24	7.802	7.845	7.855	7.854	0.439	0.113	0.040	0.013
25	8.138	8.171	8.189	8.193	0.412	0.131	0.041	0.014
26	13.468	13.539	13.543	13.546	0.738	0.242	0.070	0.021
27	9.370	9.345	9.352	9.362	0.496	0.151	0.054	0.017
28	9.265	9.276	9.295	9.294	0.560	0.176	0.059	0.020
29	7.479	7.465	7.477	7.480	0.319	0.095	0.034	0.012

Table 4.2: CVaR Mean and Standard Deviation (in percentage) estimated for each number of scenarios estimated 100 times, Equity Asset Universe.

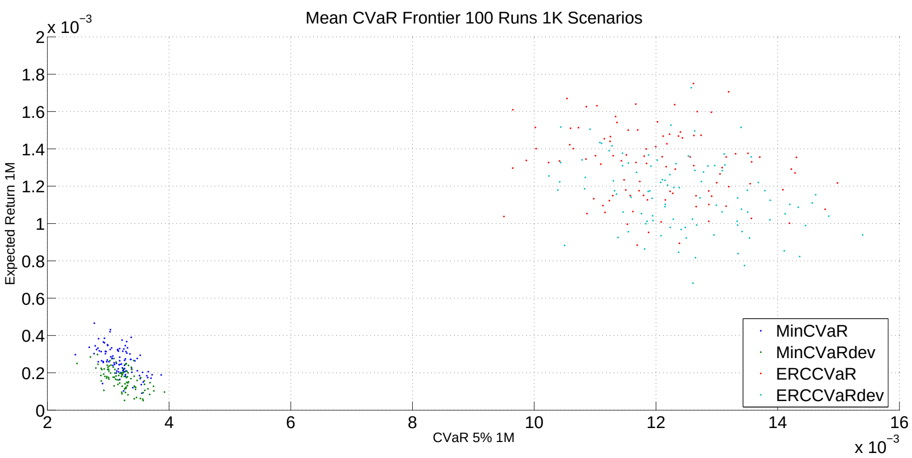

Figure 4.3: Mean CVaR frontier 1000 scenarios, Multiasset Universe

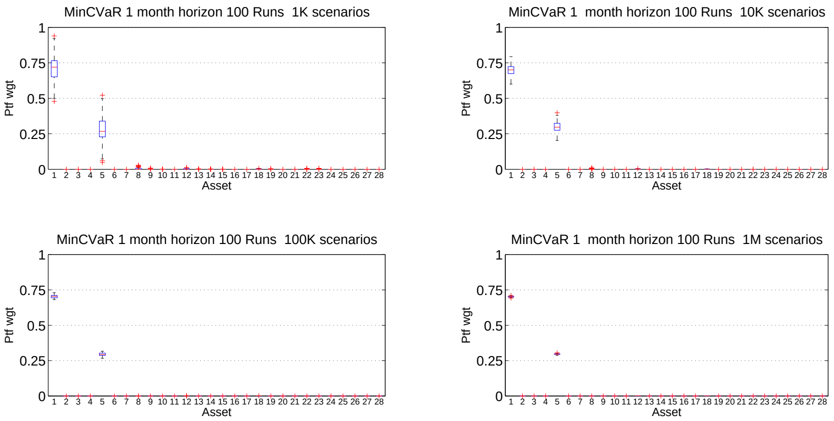

Figure 4.4: Min CVaR portfolio: boxplot of asset weights, Multiasset Universe

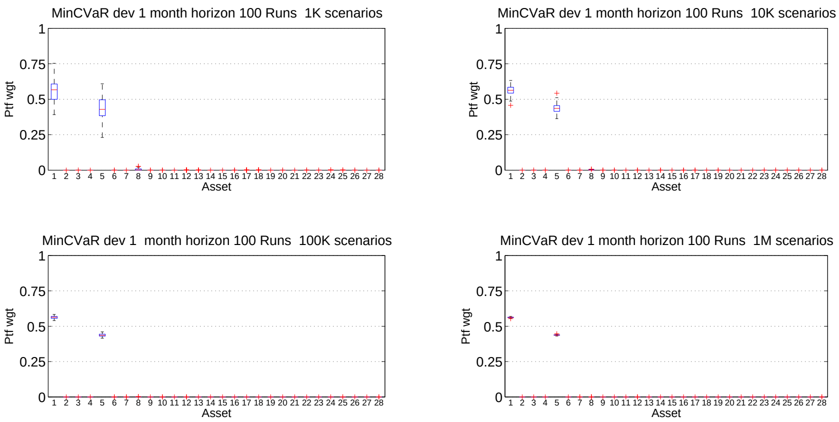

Figure 4.5: Min CVaR deviation portfolio: boxplot of asset weights, Multiasset Universe

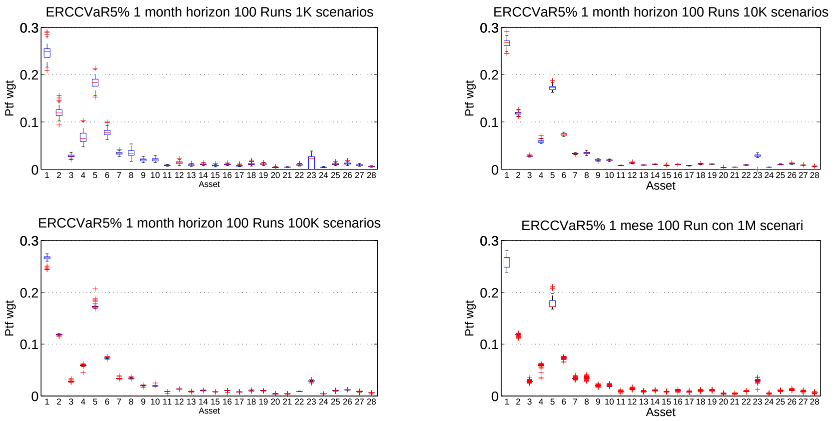

Figure 4.6: ERC CVaR portfolio: boxplot of asset weights, Multiasset Universe

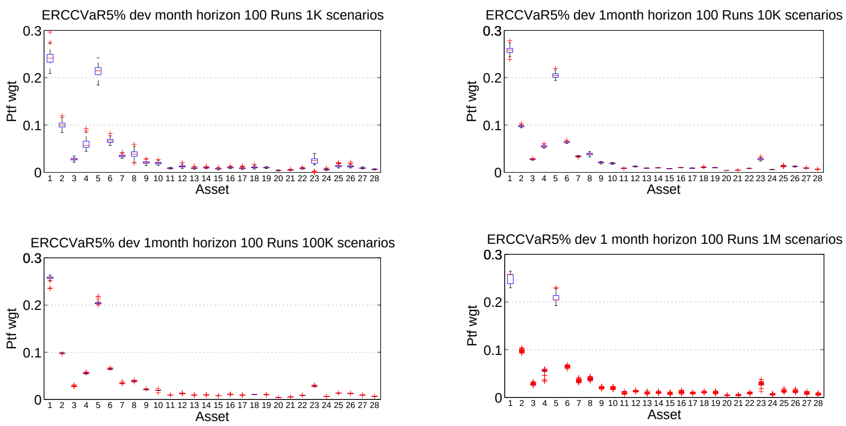

Figure 4.7: ERC CVaR deviation portfolio: boxplot of asset weights, Multiasset Universe

points, when the complete time series is not available a proxy is utilized in order to fill the whole history.

1. MSCI Italy Net TR LOC Bloomberg Ticker: NDDLIT Index
2. FTSE Italy Mid Cap Bloomberg Ticker: ITMCN index
3. Eurostoxx 50 EUR Net TR Bloomberg Ticker: SX5T index
4. Stoxx 50 EUR Net TR Bloomberg Ticker: SX5R index
5. MSCI Eur Min Vol NR Euro Bloomberg Ticker: MAEUVOE index
6. SciBeta Extended Dev Europe Multi-Beta Multi-Strat ERC-4F EUR Net Return Bloomberg Ticker: SBRERHMN index
7. DAX30 Bloomberg Ticker: DAX index
8. MSCI UK EUR Hdg Net Bloomberg Ticker: M0UKHEUR index
9. MSCI Sw 25/35 Hed EUR Bloomberg Ticker: M0CH35HE index
10. CAC 40 NR Bloomberg Ticker: NCAC index
11. S&P 500 Net TR Bloomberg Ticker: SPTR500N index
12. S&P 500 EUR Hdg NTR Bloomberg Ticker: SPXUXEN index
13. NASDAQ 100 STOCK INDX Bloomberg Ticker: NDX index
14. Russ2000Net Return Bloomberg Ticker: RU20N30U index
15. DJ Indus Avg Net TR Bloomberg Ticker: DJINR index
16. MSCI Jpn 100% Hdg TR EUR Bloomberg Ticker: MXJPHEUR index
17. MSCI Daily TR Net Japan Bloomberg Ticker: NDDUJN index
18. S&P/ASX 200 Net Tot Ret Bloomberg Ticker: ASN51 index
19. MSCI Daily TR Net Europe Small Cap Bloomberg Ticker: NCUDE15 index
20. MSCI Daily TR Net Japann Small Cap Bloomberg Ticker: NCUAJN index
21. MSCI Daily TR World Net Bloomberg Ticker: NDWUCSTA index

22. MSCI World Energy Sector Bloomberg Ticker: NDWUENR index
23. FTSE Global Core Infrastructure NT Bloomberg Ticker: FGCIITU index
24. MSCI Daily TR World Net Bloomberg Ticker: NDWUHC index
25. S&P Gb Water USD NTR Bloomberg Ticker: SPGTAQNT index
26. ISE Cyber Security Bloomberg Ticker: HURNTR index
27. FTSE EPRA/NAREIT Developed Dividend+ Net Total Return USD Bloomberg Ticker: TENGDNU index
28. LPX MM Listed Private Equity EUR TR Bloomberg Ticker: LPXMMITR index
29. FTSE EPRA/NAREIT Dev Asia Dvd+ NR USD Bloomberg Ticker: TENADNU index

In Figure 4.8 we show the box plot of the single asset CVaR estimation, also in this case we notice that the distribution of the CVaR estimate is too large when we use $1K$ scenarios and the precision increases as S increases.

In Figure 4.9 and in Figure 4.10 we show the mean-CVaR plane where the mean is calculated as the $CVaR100\%$ or simply as the scenarios average value. We note that using $1M$ scenarios the points on the plane are close to each other (for each portfolio construction procedure) while using $1K$ scenarios points are quite dispersed.

In Figure 4.11 and in Figure 4.12 we report the box plot of asset weights obtained using Min CVaR and Min CVaR deviation. The two procedure build up very concentrated portfolios (four assets dominate the portfolio) and as the number of scenarios increases the asset weight is much more stable.

In Figure 4.13 and in Figure 4.14 we report the box plot of asset weights obtained using ERCCVaR and ERCCVaR deviation. We notice very little difference between portfolio weights using $S = 100K$ and $S = 1M$ scenarios.

4.1.2 Dynamic Analysis

Our goal is to study the performance of all models across two data sets. Our analysis relies on a rolling time windows approach. More specifically, given a daily frequency data set of asset returns with T outcomes, we consider an in-sample time window of $M = 500$ days. Then, we evaluate the portfolio performance in the following 20 days (out-of-sample), during which no rebalances

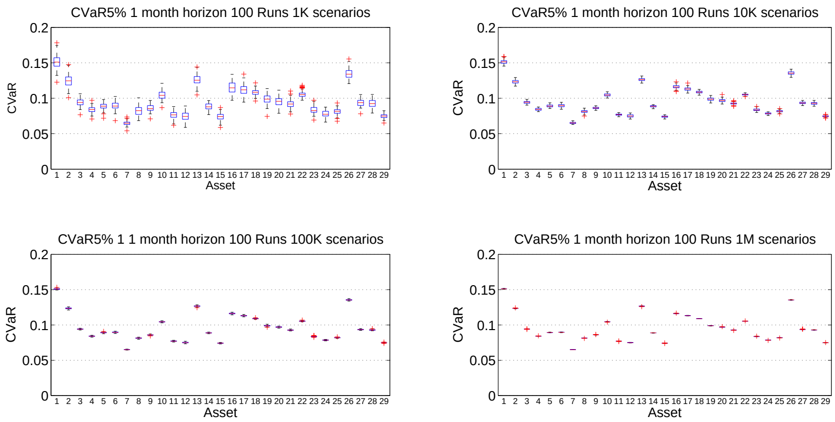

Figure 4.8: CVaR Estimation, Equity Universe

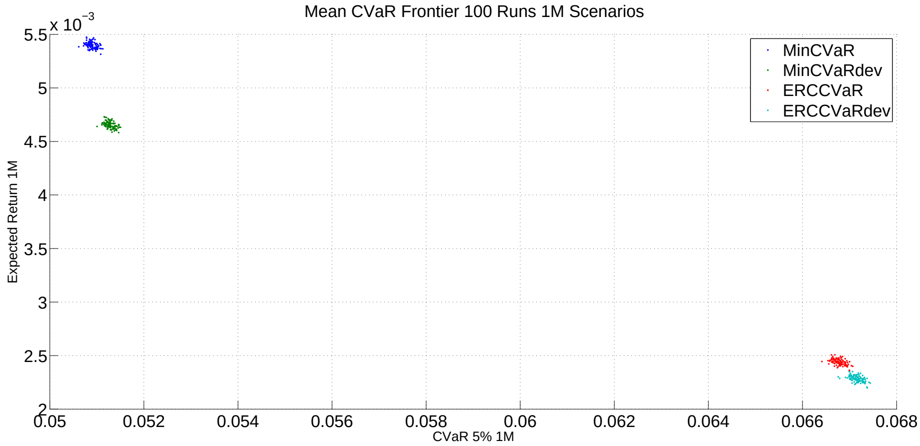

Figure 4.9: Mean CVaR frontier 1 million scenarios, Equity Universe

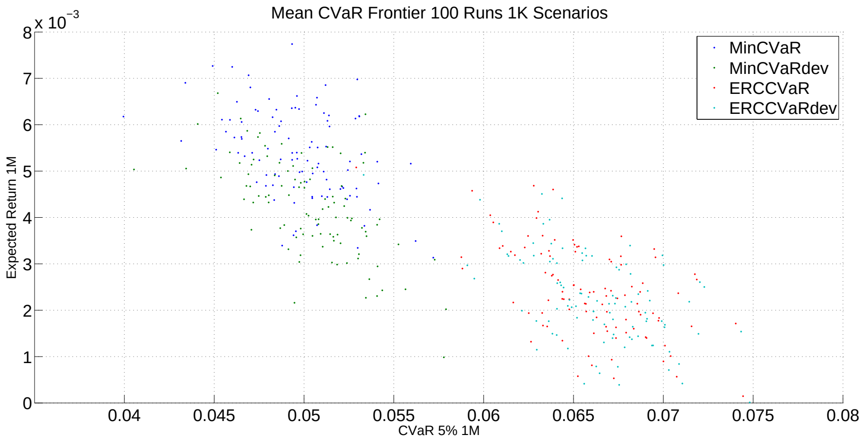

Figure 4.10: Mean CVaR frontier 1000 scenarios, Equity Universe

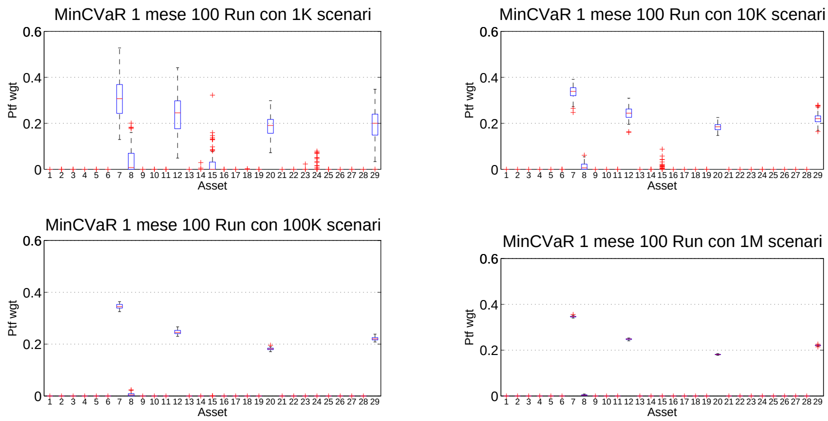

Figure 4.11: Min CVaR portfolio: boxplot of asset weights, Equity Universe

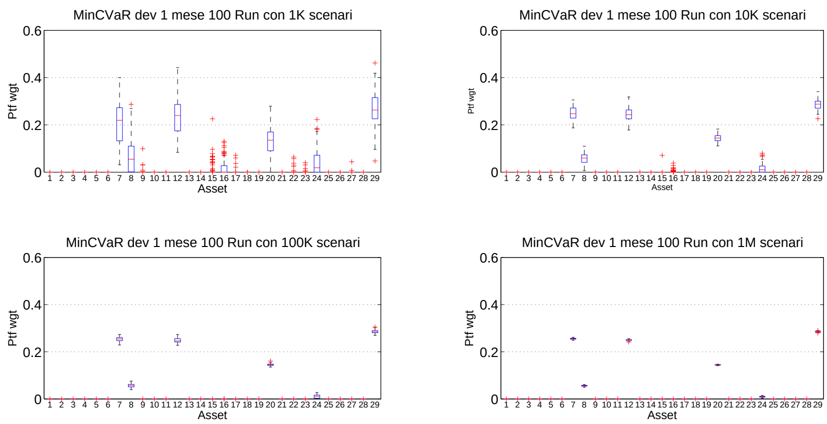

Figure 4.12: Min CVaR deviation portfolio: boxplot of asset weights, Equity Universe

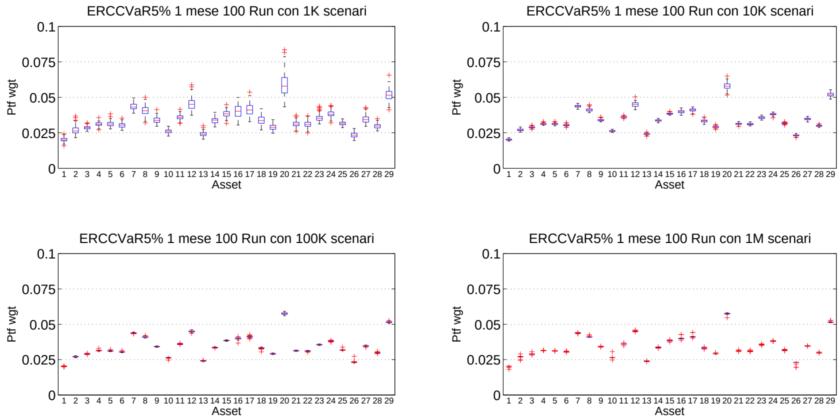

Figure 4.13: ERC CVaR portfolio: boxplot of asset weights, Equity Universe

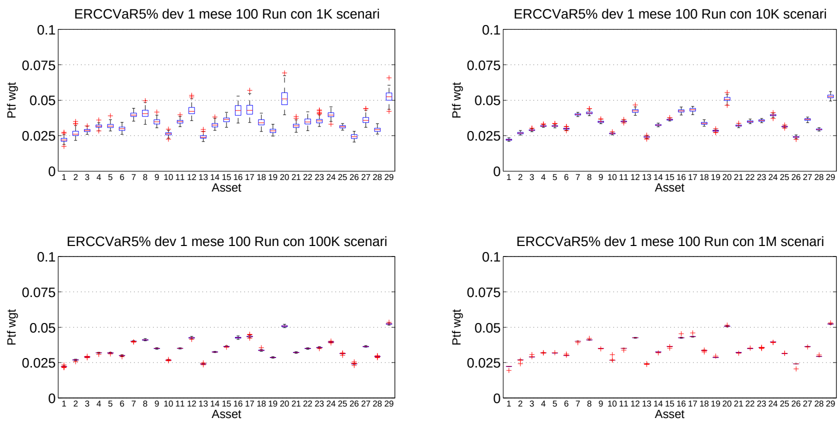

Figure 4.14: ERC CVaR deviation portfolio: boxplot of asset weights, Equity Universe

are allowed. After this, we shift the mentioned in-sample window by 20 days in order to cover the out-of-sample period, we recompute the optimal portfolio w.r.t. the new in-sample window and repeat. Then, for each portfolio strategy the rolling time windows approach generates $T - M$ daily out-of-sample portfolio returns on which we show some figures about performances, risk, portfolio weights and distance between portfolio weight.

Due to our finding, in this case, we use a number of scenarios equal to $S = 10K$, $S = 1M$. The choice to discard $1K$ is due to the fact that we notice a huge variability in CVaR estimation that produce high dispersion in the Mean-CVaR plane and we notice that the computational time of $10K$ scenarios is fast enough to be considered as the benchmark.

In particular, we calculate on each rebalance time and using both numbers of scenarios the four portfolios (MinCVaR, MinCvaRDev, ERCCVaR, and ERCCVaRDev), we collect portfolio weights and we evaluate the portfolios $CVaR5\%$ 1 month horizon (20 days).

In the end, we have the complete sequences of portfolio weights, CVaR and portfolio performances. In order to determine the differences between models when we use few or many scenarios, we calculate the Euclidean distance of portfolio weights. The Euclidean distance between vectors is defined as the norm of the difference of the portfolio weights. Let x_{1M} and x_{10K} points in the space $\mathbb{R}^n$ then $\|(x_t^{1M} - x_t^{10K})\|$ represent the distance at time t. When the solution of the problems is the same using different number of scenarios we expect that the distance is equal to 0, so when the distance is greater than 0 we record some differences that are higher as distance increases. We also report the $CVaR5\%$ (calculated using $1M$ scenarios) of all portfolios to show how the two scenarios size afflict the portfolio risk. If the portfolio risk is quite similar we can say that reducing the number of scenarios does not afflict the portfolio risk. The same happens in evaluating portfolio performances if they are indistinguishable so the little difference in portfolio weights does not change the final portfolio performance.

4.1.2.1 Multi-asset Universe Results

The ratio N/T assumes the values $N/T = 0.0028$ and $N/T = 2.8e^{-}5$. The analyzed time period is from January 1^{st} 1999 to August 4^{th} 2017. The first 500 observation are utilized to estimate the first portfolio. In figure 4.15 we report the portfolios Euclidean distance over time. The portfolio distance calculated on the two Minimum-Risk strategies is shown on top of the figure while on the bottom we report the portfolio distance of the Risk Diversification strate-

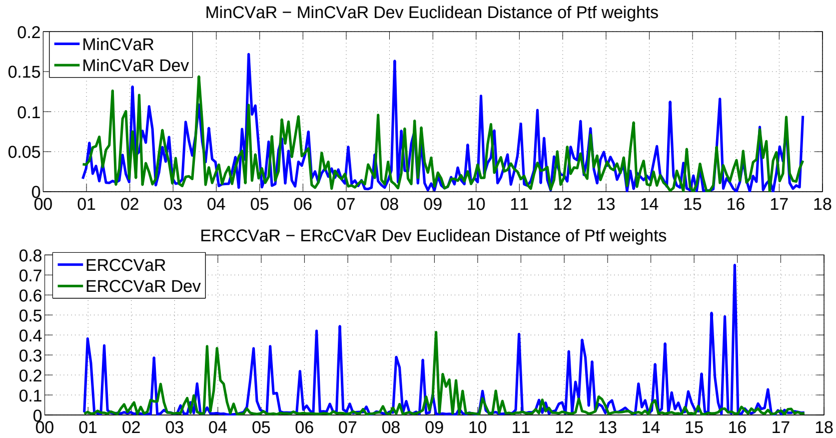

Figure 4.15: Euclidean distance in portfolio weights using $S = 1M$ and $S = 10K$

gies. We observe that CVaR Deviation does not suffer too much the number of scenarios, we note that the distance between the portfolios built by using $10K$ or $1M$ scenarios is often under 0.1. The farthest portfolio, over time, is ERCCVaR, it suffers the number of scenarios that in many cases increase the distance. Portfolio weights over time are reported in Figure 4.16 and 4.17. Graphically is very hard to recognize differences between portfolios. In Figure 4.18 we show the calculated CVaR using $1M$ scenarios. At any time and for each optimized portfolio (both using $10K$ or $1M$ scenarios) we record the CVaR. In this case, we note that there are not sensible differences in Minimum Risk CVaR, while in the Risk Diversification strategies we highlight few differences over time. From a risk manager point of view, the differences are quite small. If the differences in weights and risk are huge we should expect that portfolio performances diverge. In Figure 4.19 we cannot distinguish the portfolio performances among Minimum Risk strategies. Among Risk Diversification strategies the ERCCVaR tends to have slightly different performances while ERCCVaRDev models have similar performances. In Figure 4.20 we show how different models perform over time. In this case, the best performer is ERCCVaR Deviation and the worst is Minimum CVaR Deviation.

In conclusion, when we use $S = 1M$ increasing computational effort, we can not recognize any improvement in portfolio optimization nor in portfolio performances.

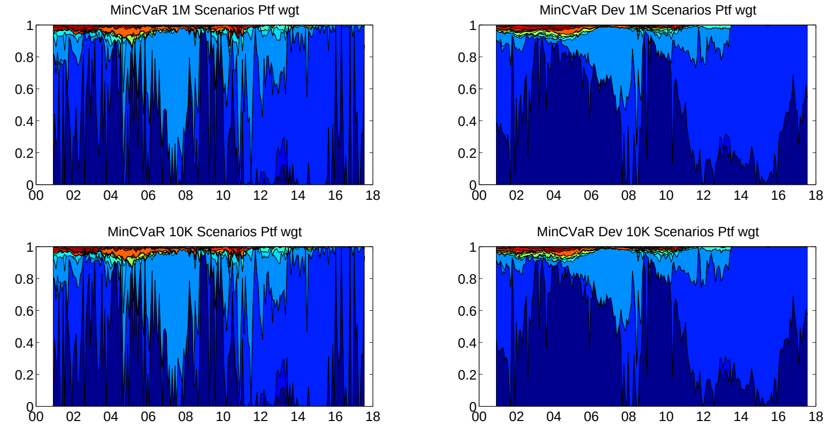

Figure 4.16: Minimum Risk Strategies: portfolio weights over time

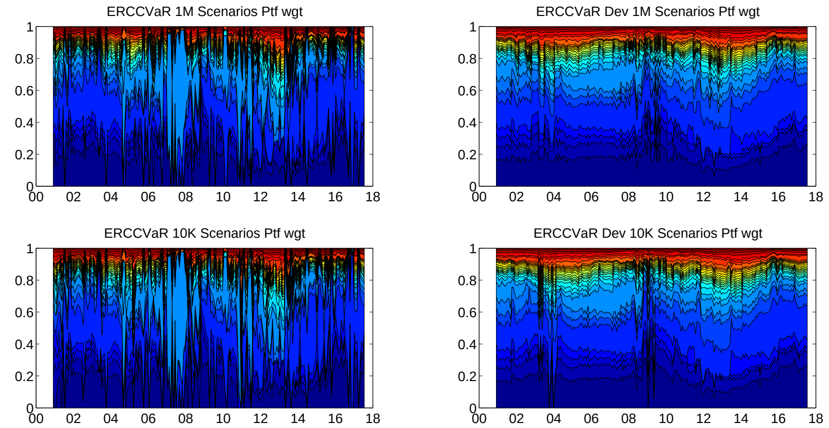

Figure 4.17: Equal Risk Contribution Strategies: portfolio weights over time

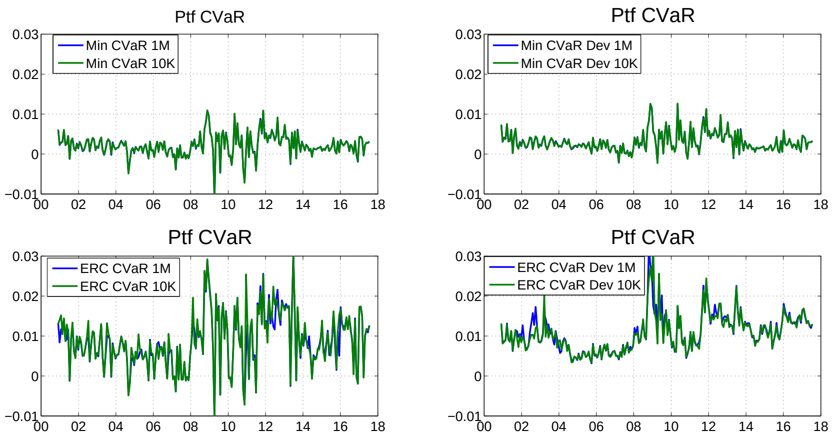

Figure 4.18: Portfolio CVaR over time

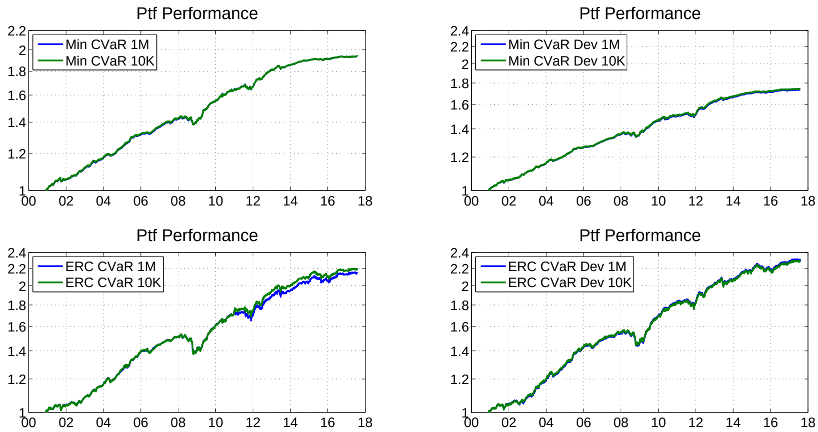

Figure 4.19: Portfolio performance over time

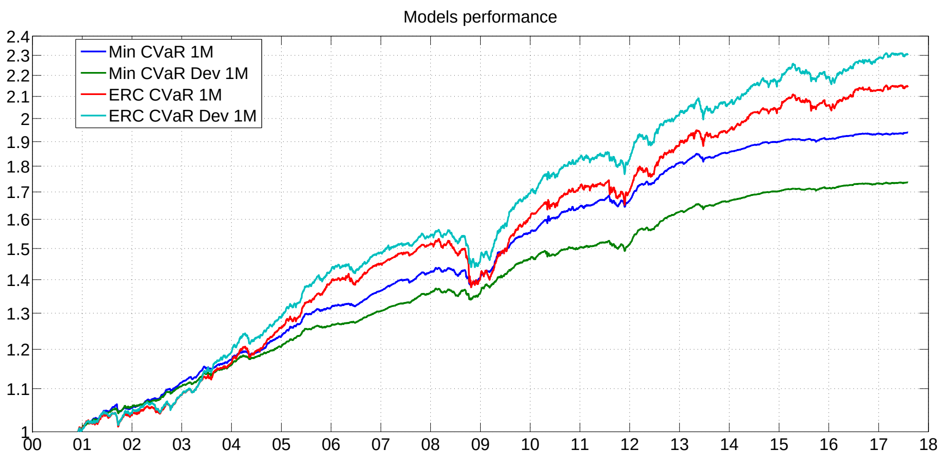

Figure 4.20: Portfolio performance over time: models comparison

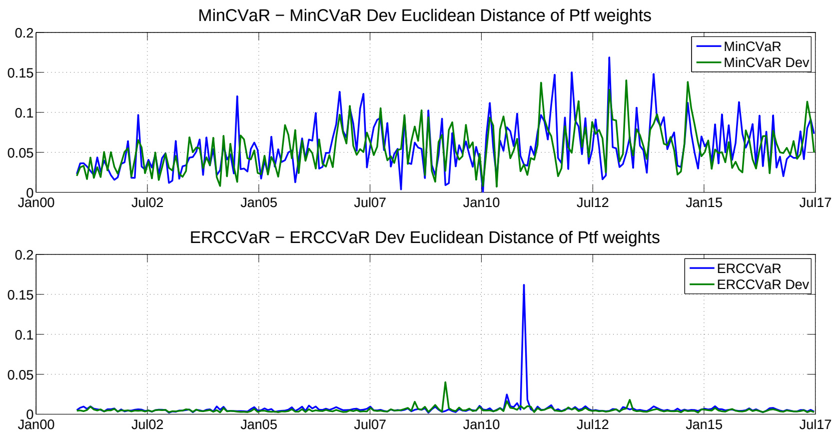

Figure 4.21: Euclidean distance in portfolio weights using $S = 1M$ and $S = 10K$

4.1.2.2 Equity Asset Universe Results

The Equity Asset universe is composed of 29 equity indices that cover countries and worldwide sectors. The analyzed time period is from January 1^{st} 1999 to June 30^{th} 2017. The first 500 observation are utilized to estimate the first portfolio. The ratio N/T assumes the values $N/T = 0.0029$ and $N/T = 2.9e^{-}5$. In figure 4.21 we report the portfolios Euclidean distance over time. The portfolio distance calculated on the two Minimum-Risk strategies is shown on top of the figure while on the bottom we report the portfolio distance of the Risk Diversification strategies. We observe that Risk Diversification strategies do not suffer the difference in the number of scenarios, we note that the distance between the portfolios built by using $10K$ or $1M$ scenarios is close to 0, while the minimum risk strategies show the distance often less than 0.1.

Portfolio weights over time are reported in Figure 4.22 and 4.23. Graphically is very hard to recognize differences between portfolios. In Figure 4.24 we show the calculated CVaR using $1M$ scenarios. Also, in this case, is very hard to catch any sensible difference.

In Figure 4.25 we cannot distinguish the portfolio performances among Risk Diversification strategies. Among Minimum Risk strategies, we saw slightly different performances. In Figure 4.26 we show how different models perform over time. In this case, the best performers are the Minimum Risk strategies. In conclusion, when we use $S = 1M$ increasing computational effort, we can not recognize any improvement in portfolio optimization nor in portfolio performances.

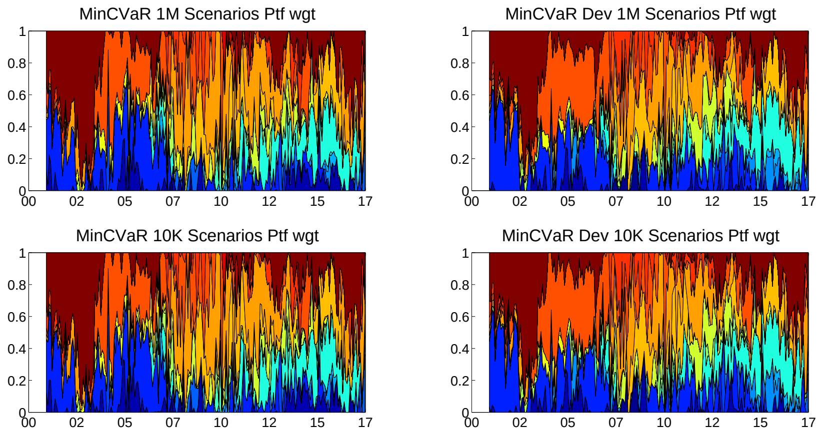

Figure 4.22: Minimum Risk Strategies: portfolio weights over time

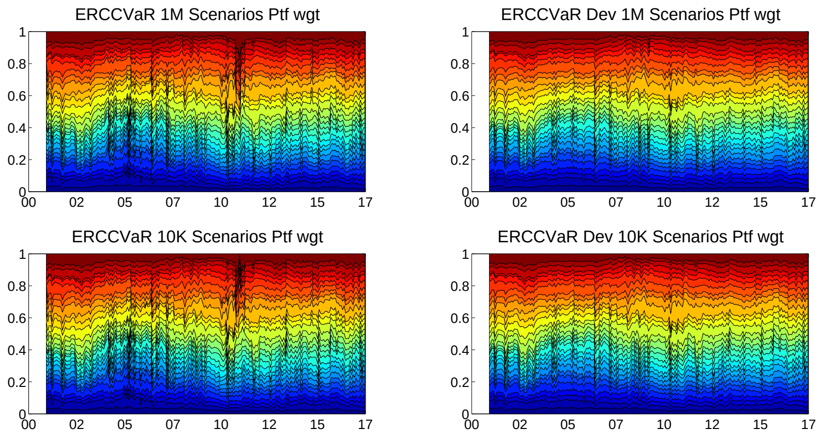

Figure 4.23: Equal Risk Contribution Strategies: portfolio weights over time

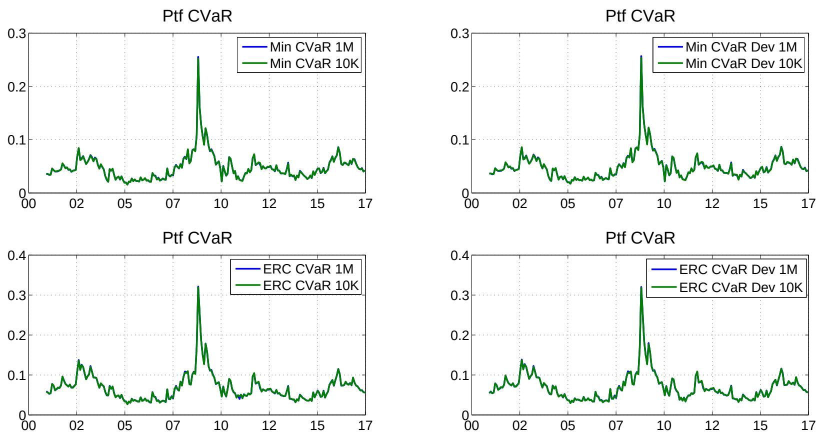

Figure 4.24: Portfolio CVaR over time

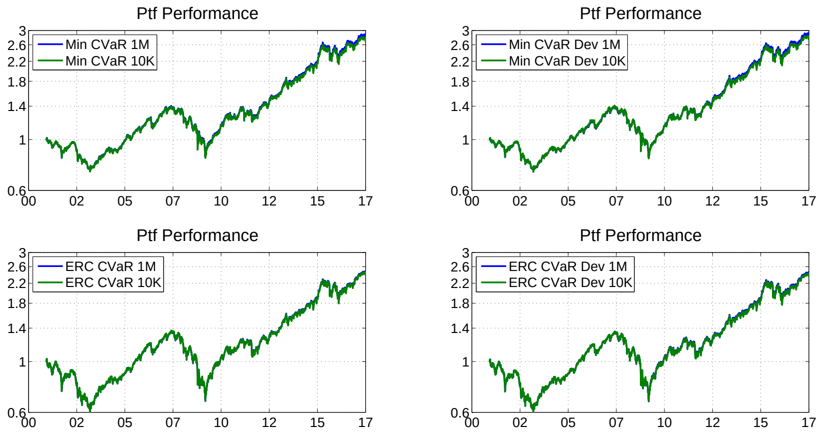

Figure 4.25: Portfolio performance over time

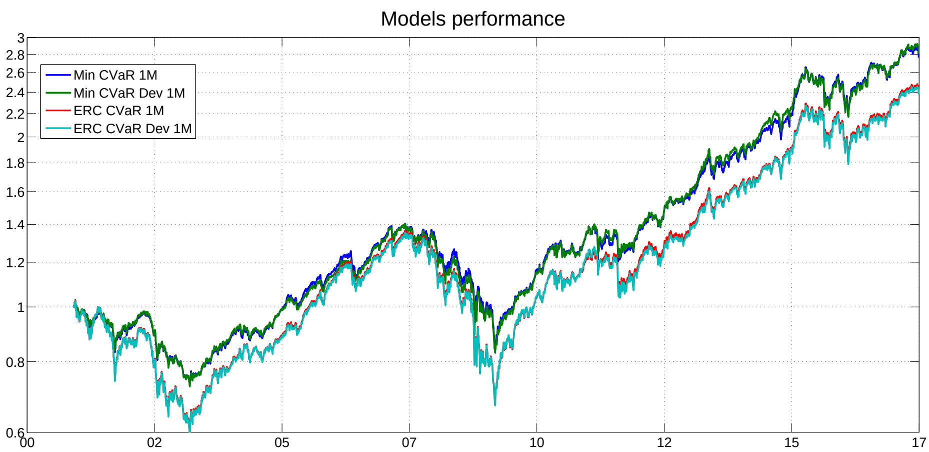

Figure 4.26: Portfolio performance over time: models comparison

4.2 Final considerations

We notice that in the real world the size of assets in the portfolio may be huge let's say over 1000 financial instruments. Considering the results we obtained in order to perform any optimization procedure that limits the estimation error we should maintain the ratio $N/T \leq 0.4$ but it means that we need at least 2500 observation for each asset. If we are dealing with historical daily returns we need 10 years data. To overcome this limitation we propose to use a simulation procedure that generates future possible scenarios for each asset and it needs only the last 500 daily returns. We have analyzed scenarios in which T is minimal and equal to 1000 or 10000, or 100000 up to 1 million. Considering the example above we can always optimize the portfolio increasing the number of the simulated paths.

We have compared the out-of-sample performance of two different categories of portfolio selection models, namely *Minimum-Risk* and *Risk-Diversification*. The two classes of models, short selling, and leverage are not allowed thus making the feasible portfolios of the *Minimum-Risk* and of the *Risk-Diversification* models as similar as possible to the common investor.

The analysis is performed on a static and on dynamic base on two different real-world data sets, which consist of equities and mixed assets, each with different sources of risk.

Our goal is to verify how the number of scenarios used as input for CVaR estimation and for portfolio construction afflicts the CVaR estimation and the optimal solution of the minimum risk and risk diversification problems.

In the static analysis we estimate 100 times the assets CVaR and obviously, we generated the full future returns distribution for each asset. We collected the 100 CVaR and the 100 optimum portfolio weights for all models. We notice, for both cases analyzed, that 1000 scenarios are too low and it are not able to produce a good estimate of the asset CVaR. The variability is very high. Considering 10000 scenarios the computational time is reasonably fast (0.20 seconds to evaluate the single scenario and 1 second to solve each problem) and the accuracy quickly increases (the CVaR standard deviation is reducing to roughly $\frac{1}{3}$ at each order of magnitude). Portfolio weights are more stable and we think that $10K$ scenarios should be considered as a minimum value to use in order to estimate CVaR and to be used as input in an optimization procedure. Using $100K$ and $1M$ estimates became much more accurate but computational time increase. A good compromise between accuracy and computational time may be $100K$.

In the dynamic analysis, we compare the CVaR estimation and optimal solutions using the minimum acceptable number of scenarios ($10K$) and the maximum number of scenarios ($1M$). The aim of this test is to identify the difference between portfolios built by using few scenarios and many scenarios. We discovered that portfolio weights are very close one to each other and the whole portfolio risk, for the four optimization procedures, is similar. Regarding portfolio performances, we can state that there are no important differences when we use $10K$ with respect to $1M$ scenarios.

We believe that $10K$ scenarios are enough when we are dealing with the review portfolio construction procedures. Using $1M$ scenarios is a safe choice but you must take to account the computation effort to calculate the high number of scenarios.

Chapter 5

Shrunk Volatility VaR: a new model to proxy Historical Filtered Bootstrap

A way to be well prepared for managing periods of financial turbulence is to predict market risk, estimated by some risk measures. One may use volatility, value-at-risk (VaR), Conditional VaR, downside volatility or others. However, all these indicators should be monitored in order to have an idea of the market conditions. In financial firms, such as banks and asset management companies, the VaR risk measure is commonly used (see [59]). For instance, banks must periodically report a VaR estimate of the entire business to their own vigilance authority, along with an accurate backtesting procedure that validates the VaR model used for the estimate.

Many models have been developed to foresee market risk (see [1], [23] and [69]), taking into account the following stylized facts that characterize the returns time series: volatility clustering, fat tails and mild skewness (see [44]). Further, VaR models should satisfy two conditions in order to be considered accurate: they should exhibit statistical significance when comparing the observed frequency of VaR violations with those expected and show independence of violations (see [26]).

In this chapter, we propose a naive model to forecast *ex-ante* VaR, using a shrinkage estimator (see [66]) between realized volatility estimated on daily return time series as well as implied volatility extracted from option pricing data. Indeed, several studies highlight that models based on implied volatility produce competitive VaR forecasts (see [51] and [62]). Implied volatility is often indicated as the operator's expectation about future risk, while historical-based volatility simply represents the realized risk up to the estimation time, thus employing a backward-looking approach.

The purpose of this chapter is to compare our model, called shrunk volatility VaR (ShVolVaR), with several prediction strategies, in both the univariate and multivariate cases. In more detail, we first discuss and analyze three simple models to forecast the one-day-ahead VaR, using implied volatility, realized volatility and a shrinkage of both. We then empirically compare their forecasting power with four benchmark VaR models based on the historical simulation (see [16], [54], and [91]), Historical Filtered Bootstrap, extreme value theory Filtered Bootstrap (see [80]) and RiskMetrics (see [82]) over a relatively long time period (at least fourteen years); this depends on the availability of implied volatility values. For these seven models, we evaluate the statistical accuracy of one-day-ahead VaR estimates by means of the following:

- The unconditional coverage (UC) test ([63]), which analyzes the statistical significance of the observed frequency of violations with respect to the expected one;
- the independence (IND) test ([40]) which gauges the independence of violations, namely the absence of violation clustering;
- the conditional coverage (CC) test, which combines these two desirable properties ([41]).

In addition to performing tests on accuracy, we check the practical compliance of the VaR models with respect to specific regulatory rules. More precisely, for backtesting aims, the European regulator, ie, the Committee of European Securities Regulators (CESR; now the European Securities and Markets Authority, ESMA), will accept no more than seven violations of $VaR_{1\%}$ (related to a one-day time horizon) on 250-day rolling time windows (see [35]). Further, the one-day-ahead VaR should satisfy the coverage condition, while no tests are required by ESMA regarding the independence property of VaR violations. From the viewpoint of the regulator, a model that overestimates VaR (ie, the model is conservative) is accepted, even though backtesting shows a high percentage of zero violations. However, from the investors' viewpoint, this means the mismanagement of capital. Conversely, an underestimation of VaR (ie, the model is aggressive) is convenient for the investors, but it is not accepted by the vigilance authority.Therefore, in our backtesting, we highlight the right trade-off between these two different points of view, controlling for both a lack and an excess of violations. In other words, over a period of 250 days, a VaR model should be able to minimize the frequency of violations being absent (the investors' viewpoint) as well as the frequency of more than seven violations occurring (the regulators' viewpoint). Further, to examine the performance of the VaR models, we also perform backtesting based on loss functions that take into account both the regulators' and the investors' viewpoints (see [2] and [27]).

5.1 Models and Tests

Before we introduce the models analyzed in this study to forecast the one-day-ahead Value-at-Risk (VaR), it is useful to specify its mathematical definition. VaR is defined as the maximum loss at a specified confidence level and it is one of the most important risk management tool in the financial industry ([82]).

Let us introduce some notations and assumptions. Since we study the VaR performance of the proposed models both in the univariate and the multivariate frameworks, we use linear returns, so if $p_{t,k}$ is the price of asset k at time t, then $r_{t,k} = \dfrac{p_{t,k} - p_{t-1,k}}{p_{t-1,k}}$ represents its return at time t. Even though for econometric models the returns are usually defined as log-returns, namely $r^{\ln}_{t,k} = \ln p_{t,k} - \ln p_{t-1,k}$, in case of assets portfolios the linear returns are preferred to the logarithmic ones, due to their mathematical tractability. In addition, for small values of $r_{t,k}$, as in this context, it is straightforward to demonstrate that $r_{t,k} \simeq r^{\ln}_{t,k}$.

We denote by $x = (x_1, x_2, \ldots, x_n)^T$ the vector of the assets weights in a portfolio. Thus assuming that n assets are available in an investment universe, the portfolio return at time t $R_t(x) = \displaystyle\sum_{k=1}^{n} x_k r_{t,k}$. Furthermore, the set of feasible portfolios considered in this study satisfy the budget constraint ($\sum_{k=1}^{n} x_k = 1$) and the no short-selling condition ($x_k \geq 0$ for all $k = 1, \ldots, n$).

That being said, VaR_ε is defined as the maximum loss at a given confidence level related to a predefined time horizon. Usually, the confidence levels are 95% and 99%, that is in general equal to $(1 - \varepsilon)100\%$. Hence, $VaR_\varepsilon(x)$ is the value such that the possible portfolio loss $L(x) = -R(x)$ exceeds $VaR_\varepsilon(x)$ with a probability of $\varepsilon 100\%$ (see [5]). In other words, $VaR_\varepsilon(x)$ of a portfolio return distribution is the lower ε-quantile of its distribution with negative sign:

$$VaR_\varepsilon(x) = -F_R^{-1}(\varepsilon, x) \tag{5.1}$$

where $F_R^{-1}(\varepsilon, x) = \inf\{r : F_R(r) > \varepsilon\}$, and F_R^{-1} is the inverse of the portfolio return cumulative distribution function. If R has a multivariate normal distribution with zero means and covariance matrix Σ, then

$$VaR_\varepsilon(x) = \phi^{-1}(\varepsilon)\sigma(x)$$

where $\phi^{-1}(\varepsilon)$ is the $\varepsilon-$quantile of the standard normal distribution, and $\sigma(x) = x^T \Sigma x$.

Below, we briefly describe the Historical Simulation, the RiskMetrics, the Historical Filtered Bootstrap, and the Extreme Value Theory Filtered Bootstrap strategies (see Sections 5.1.1, 5.1.2, 5.1.3, and 5.1.4 respectively), which are considered as benchmarks to estimate the one-day-ahead VaR. In Section 5.1.5 we present our model, called Shrunk Volatility VaR (ShVolVaR), that, as we shall see, includes implicitly other two VaR models. Further, in Section 5.1.6 we briefly report the UC ([63]), the IND ([40]) and CC ([41]) tests, used to verify advisable features that should be satisfied by a risk model (statistical significance when comparing the observed frequency of violations to the expected one, the independence of violations, and both). In Section 5.1.7 we describe the regulator rules to be validated for the acceptance of a VaR model. Finally, to investigate the distance between the *ex-post* portfolio returns and the corresponding *ex-ante* VaR forecasts obtained from the different VaR models, we also introduce a backtesting based on loss functions that take into account both the regulators' and the investors' viewpoint (see [27]).

5.1.1 Historical Simulation VaR model

In the Historical Simulation VaR (HSVaR) model the portfolio return is represented by its empirical distribution, and $-VaR_{\varepsilon}(x)$, related to a predefined time windows, is the ε-quantile of this empirical distribution. Thus considering the observed portfolio returns within a time window of length T, the ε-quantile is the $[\varepsilon T]$-th smallest observation, where $[b]$ rounds the variable b up to the closest integer. On the one hand, the HSVaR model is advantageous because it is easy to implement and it does not depend on parametric assumptions of the portfolio return distribution. On the other hand, this approach presents several drawbacks (see [1], [16], [54], [75] and [91]) mainly due to its strong dependence on observed data and on the length of the learning window. Further, the HSVaR approach, by its nature, tends to react slowly to the market turbulence.

5.1.2 RiskMetrics VaR model

The assumption of the RiskMetrics VaR (RiMeVaR) model is that the returns of a generic asset k follow a random walk with independent and identically distributed (iid) normally distributed changes. More precisely,

$$r_{t,k} = \mu_k + \sigma_{t,k}\eta_{t,k}$$

where $\mu_k = 0$ and $\eta_{t,k} \sim N(0,1)$ is an iid random perturbation. The returns variance $\sigma_{t,k}$ varies with time and can be estimated by the past information. The RiMeVaR model uses the Exponentially Weighted Moving Average (EWMA) approach to predict volatilities and correlations of the portfolio return. More specifically, the volatility forecast of asset k at time $t+1$, given the information available at time t, is

$$\sigma_{t+1|t,k} = \sqrt{\lambda\sigma^2_{t|t-1,k} + (1-\lambda)r^2_{t,k}} \tag{5.2}$$

where $\lambda = 0.94$ for daily data and $\lambda = 0.97$ for monthly data as in RiskMetrics technical documentation ([82]). From (5.2), it is straightforward to recognize the same formulation of the IGARCH(1,1) model. Further, we have that the one-day-ahead correlation between the assets k and j is:

$$\rho_{t+1|t;k,j} = \frac{\sigma_{t+1|t;k,j}}{\sigma_{t+1|t,k}\sigma_{t+1|t,j}}$$

where $\sigma_{t+1|t;k,j}$ is the one-day-ahead covariance forecast between the assets k and j such that $\sigma_{t+1|t;k,j} = \lambda\sigma_{t|t-1;k,j} + (1-\lambda)r_{t,k}r_{t,j}$. Thus, we can define the EWMA covariance matrix as

$$\Sigma^{EWMA}_{t+1|t} = \text{diag}(\sigma_{t+1|t})C^{EWMA}_{t+1|t}\text{diag}(\sigma_{t+1|t})$$

where $\text{diag}(\sigma_{t+1|t})$ is the diagonal matrix with EWMA volatilities of the assets on the diagonal, and $C^{EWMA}_{t+1|t} = \left\{\rho_{t+1|t;k,j}\right\}_{k,j=1,\dots,n}$ is the EWMA correlation matrix. Therefore, portfolio volatility can be written as

$$\sigma_{t+1|t}(x) = \sqrt{x^T\Sigma^{EWMA}_{t+1|t}x}$$

and the one-day-ahead VaR at confidence level $1-\varepsilon$ as

$$VaR_{t+1|t}(\varepsilon, x) = \phi^{-1}(\varepsilon)\sigma_{t+1|t}(x)$$

where $\phi^{-1}(\varepsilon)$ is the $\varepsilon-$quantile of the standard normal distribution.

5.1.3 Historical Filtered Bootstrap VaR model

The Historical Filtered Bootstrap VaR (HFBVaR) approach ([8], [24], [113], [78]) is a mixed procedure, in which one represents the market returns using, for instance, an autoregressive moving average generalized autoregressive conditional heteroscedasticity (ARMA-GARCH) model to filter the time series, and then computes the empirical standardized residuals from data without assuming on them any specific probability distribution on them. Below we give a step-by-step description of HFBVaR procedure.

1. We filter the time series of each asset by an univariate ARMA-GARCH model. More precisely, for the observed returns of the asset k, we find the Maximum Likelihood estimators of the following AR(1)-StudT-GARCH(1,1) model:

$$\begin{aligned} \text{AR(1)} &: \quad r_{t,k} = a_k + b_k r_{t-1,k} + \eta_{t,k} \\ \text{StudT-GARCH(1,1)} &: \quad \sigma^2_{t,k} = \alpha_k + \beta_k \sigma^2_{t-1,k} + \gamma_k \eta^2_{t-1,k} \\ & \quad\ \eta_{t,k} = \sigma_{t,k} z_{t,k} \end{aligned}$$

   where $z_{t,k} = \sqrt{\frac{\nu_k - 2}{\nu_k}} T_{\nu_k}$, T_{ν_k} follows a Student-t distribution with ν_k degrees of freedom, and $\hat{\theta} = \{a_k, b_k, \alpha_k, \beta_k, \gamma_k, \nu_k\}$ are Maximum Likelihood estimators (see, for example, [108]) obtained on 500 daily data.

2. Using the set of estimators $\hat{\theta} = \{a_k, b_k, \alpha_k, \beta_k, \gamma_k, \nu_k\}$ for all n assets available in the market, we compute from data the standardize residuals $\hat{z}_{t,k}$ with $t = 1, \ldots, T$ and $k = 1, \ldots, n$, i.e., we divide the empirical residuals $\hat{\eta}_{t,k}$ by their estimated volatilities $\hat{\sigma}_{t,k}$.

3. We bootstrap in a parallel fashion the matrix of the empirical standardized residuals $\hat{Z} = \left\{\hat{z}_{t,k}\right\}$ with $t = 1, \ldots, T$ and $k = 1, \ldots, n$. More precisely, we randomly sample with replacement the rows of the matrix $\hat{Z}$, thus allowing us to capture the multivariate shocks of the entire system.

4. The bootstrapped standardized residuals $\hat{Z}^{boot} = \left\{\hat{z}^{boot}_{s,k}\right\}$, with $s = 1, \ldots, S$ and $k = 1, \ldots, n$, are then used as multivariate innovations in the (univariate) AR(1)-StudT-GARCH(1,1) models to simulate the one-day-ahead returns. In our empirical analysis we employ $S = 10000$ bootstrapped scenarios.

5. Finally, the S scenarios are used to estimate the one-day-ahead VaR at confidence level $1 - \varepsilon$, $VaR_{t+1|t}(\varepsilon, x)$, as in (5.1).

Note that although AR(1)-StudT-GARCH(1,1) estimations are performed on univariate cases, the dependence structure among the assets is captured by the parallel bootstrap procedure on the standardized residuals $\hat{Z}$. In other words, through this approach of sampling, we are able to generate scenarios with historical common shocks.

5.1.4 Extreme Value Filtered Bootstrap VaR model

The extreme value theory filtered bootstrap VaR (EVTFBVaR) approach consists of a mixed procedure similar to that of HFBVaR; however, the multivariate standardized residuals are modeled using the extreme value theory for the marginal distributions of the assets and a Student-t copula for their interdependence structure. More precisely, as in McNeil and Frey (2000) ([80]) we fit the central part of the marginal distributions by Gaussian Kernels and their tails by generalized Pareto distributions (GPDs). Then, we model the purely joint distribution of innovations by a $t-$copula. To sum up, the EVTFBVaR approach is based on the following steps:

$1'$., $2'$. are identical to 1. and 2. of the HFBVaR model (see Section 5.1.3).

$3'$. For each standardize residual $\hat{z}_{t,k}$ with $t = 1, \ldots, T$ and $k = 1, \ldots, n$, we consider lower and upper thresholds equal to $\gamma-$ and $(1-\gamma)-$quantiles, respectively, with $\gamma = 0.1$. We therefore model the standardized residuals that exceed these thresholds by GPDs, and those between these thresholds by Gaussian Kernels. We denote by $\tilde{F}_{Z_k}$ this composite semi-parametric marginal Cumulative Distribution Function (CDF) corresponding to the standardize residual k.

$4'$. We compute the grade scenarios of the standardize residual k, $\hat{u}_{t,k} = \tilde{F}_{Z_k}(\hat{z}_{t,k})$. We fit the joint grades scenarios $\hat{U} = \left\{\hat{u}_{t,k}\right\}$ through a $t-$copula, estimating its parameters by the Maximum Likelihood (ML) approach.

$5'$. We simulate $S = 10000$ scenarios from a $t-$copula with these ML parameters, $U^{simul} = \left\{u^{simul}_{s,k}\right\}$, with $s = 1, \ldots, S$ and $k = 1, \ldots, n$, that are combined with the marginal CDFs $\tilde{F}_{Z_k}$ with $k = 1, \ldots, n$. Thus applying the inverse of the marginal CDF to the respective grade from the copula, i.e., $z^{simul}_{s,k} = \tilde{F}^{-1}_{Z_k}(u^{simul}_{s,k})$, we obtain the simulated standardized residuals $Z^{simul} = \left\{z^{simul}_{s,k}\right\}$, with $s = 1, \ldots, S$ and $k = 1, \ldots, n$, that are then used as multivariate innovations in the (univariate) AR(1)-StudT-GARCH(1,1) models to simulate the one-day-ahead returns.

$6'$. Finally, as for HFBVaR the S scenarios are used to estimate the one-day-ahead VaR at confidence level $1 - \varepsilon$, $VaR_{t+1|t}(\varepsilon, x)$.

5.1.5 Shrunk Volatility VaR model

We propose here a simple model to forecast *ex-ante* VaR, assuming, as for RiMeVaR model, that the asset returns are normally distributed with zero mean, but that volatility forecast at time $t+1$, given information available at time t, is the shrinkage between realized and implied volatility. The realized volatility $\hat{\sigma}_{t,k}$ is computed as the standard deviation of the index k returns on twenty stock market days (around thirty calendar days), while the implied volatility $\sigma_{t,k}^{impl}$ is obtained from a basket of call and put options with maturity of thirty calendar days in the market index k. In more detail, we compute the daily implied volatility as $\sigma_{t,k}^{imp} = (256)^{-\frac{1}{2}} V_{t,k}^{imp}/100$, where $V_{t,k}^{imp}$ represents the quoted implied volatility (expressed as a percentage) of the market index k. Thus, in a univariate context, we have $r_{t+1|t,k} \sim N(0, \tilde{\sigma}^2_{t+1|t,k}(\alpha))$ with the shrunk volatility

$$\tilde{\sigma}_{t+1|t,k}(\alpha) = (1-\alpha)\hat{\sigma}_{t,k} + \alpha\sigma_{t,k}^{impl} \tag{5.3}$$

where $\alpha \in (0,1)$ and is called *shrinkage parameter*. On the other hand, in a multivariate context we assume $(r_{t+1|t,1}, r_{t+1|t,2}, \dots r_{t+1|t,n}) \sim N(0, \tilde{\Sigma}_{t+1|t}(\alpha))$ with the covariance matrix

$$\tilde{\Sigma}_{t+1|t}(\alpha) = \text{diag}(\tilde{\sigma}_{t+1|t}(\alpha))\hat{C}_{t+1|t}\text{diag}(\tilde{\sigma}_{t+1|t}(\alpha))$$

where $\text{diag}(\tilde{\sigma}_{t+1|t}(\alpha))$ is the diagonal matrix with Shrunk volatilities of the assets on the diagonal, and $\hat{C}_{t+1|t}$ is the sample correlation matrix estimated on 20 days preceding t. Clearly, the portfolio volatility can be written as $\tilde{\sigma}_{t+1|t}(\alpha, x) = \sqrt{x^T\tilde{\Sigma}_{t+1|t}(\alpha)x}$. We then compute the one-day-ahead VaR_ε at $1-\varepsilon$ confidence level for our model, named Shrunk Volatility VaR (ShVolVaR), as follows

$$VaR_{t+1|t}(\varepsilon, \alpha, x) = \phi^{-1}(\varepsilon)\tilde{\sigma}_{t+1|t}(\alpha, x) \tag{5.4}$$

We empirically calibrate the shrinkage parameter α examining the behavior of the ShVolVaR model for different equally-spaced values belonging to the interval $[0,1]$, and choosing the value of α that shows the best results in terms of the accuracy tests, described in the following section.

Note that if $\alpha = 0$, then Model (6.4) coincides with the Realized Volatility VaR (ReVolVaR) model; while if $\alpha = 1$ we have the Implied Volatility VaR (ImVolVaR) model.

In Section 5.2 we will test and compare the ReVolVaR, the ImVolVaR, and the ShVolVaR models with the Historical Simulation VaR (HSVaR), the Historical Filtered Bootstrap VaR (HFBVaR), the Extreme Value Theory Filtered Bootstrap VaR (EVTFBVaR), and the RiskMetrics VaR (RiMeVaR) models that are considered as benchmarks.

5.1.6 Accuracy Tests

In this section we briefly describe the common tests proposed in the literature to evaluate the statistical accuracy of VaR estimates: the UC test (Kupiec, 1995) which analyzes the statistical significance of the observed frequency of violations w.r.t. the expected one; the IND test (Christoffersen, 1998) which gauges the independence of violations, namely the absence of violation clustering; and the CC test which combines these two desirable properties.

Let us denote by $R_t(x)$ the daily *ex-post* portfolio returns with $t = 1, \ldots, T$, and by $VaR_{t|t-1}(\varepsilon)$ the corresponding *ex ante* Value-at-Risk forecasts, where ε is the expected coverage, namely $\mathbf{Pr}_{t-1}(-R_t(x) > VaR_{t|t-1}(\varepsilon)) = \varepsilon$. Let $I_t = \mathbf{1}_{(VaR_{t|t-1}(\varepsilon),+\infty)}(-R_t(x))$ define the random variable hit sequence of $VaR_{t|t-1}(\varepsilon)$ violations, where $\mathbf{1}$ is the indicator function. Note that the hit variable represents only the VaR violations, excluding any information on their size. Assuming that $I_t \sim Bernoulli(\varepsilon)$ is iid., the Unconditional Coverage (UC) test examines the null hypothesis $H_{0,UC}$ that $\varepsilon = \hat{\varepsilon}$, namely that the observed frequency of violations $\hat{\varepsilon}$ is statistical significant with respect to the expected coverage ε. The likelihood function of an iid hit sequence $I_t \sim Bernoulli(\varepsilon)$ with $t = 1, \ldots, T$ and with a known probability ε that 1 occurs, can be written as:

$$L(I, \varepsilon) = \varepsilon^{N_I}(1-\varepsilon)^{T-N_I}$$

where $N_I = \sum_{t=1}^{T} I_t$ is the number of VaR violations. In the case of an iid Bernoulli variable with unknown probability ε that 1 occurs, it can be estimated by means of the maximum likelihood method as $\hat{\varepsilon} = \frac{N_I}{T}$. Thus, we can obtain the likelihood ratio test of unconditional coverage as

$$LR_{UC} = 2[\ln L(I, \hat{\varepsilon}) - \ln L(I, \varepsilon)]$$

where asymptotically $LR_{UC} \sim \chi^2(\nu = 1)$.

As mentioned above, the UC test assumes that I_t with $t = 1, \ldots, T$ are independent, but this property should be explicitly tested. For this purpose, Christoffersen (1998) (see [40]) provides a test for independence, in which the hit sequence $\{I_t\}_{t=1,\ldots,T}$ follows a first-order Markov chain with switching probability matrix

$$\mathbf{\Pi} = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix}$$

where $\pi_{lq} = \mathbf{Pr}(I_t = q | I_{t-1} = l)$, i.e., the probability that the event l in $t-1$ is followed by the event q in t. The Independence (IND) test examines the null hypothesis $H_{0,IND} : \pi_{01} = \pi_{11}$, therefore it investigates on possible violation clustering, namely on eventual repeated deep losses that could cause a bankruptcy. The likelihood function under the hypothesis of the first-order Markov dependence is:

$$L(I;\pi_{01},\pi_{11}) = (1-\pi_{01})^{T_{00}}\pi_{01}^{T_{01}}(1-\pi_{11})^{T_{01}}\pi_{11}^{T_{11}}$$

where T_{lq} represents the number of times that the state l follows the state q. In the case of unknown probabilities π_{01} and π_{11}, they can be estimated as $\hat{\pi}_{01} = \frac{T_{01}}{T_{00}+T_{01}}$ and $\hat{\pi}_{11} = \frac{T_{11}}{T_{10}+T_{11}}$. Therefore, the likelihood ratio for the IND test, under the null hypothesis that $\hat{\pi}_{01} = \hat{\pi}_{11} = \hat{\varepsilon}$, can be written as

$$LR_{IND} = 2[\ln L(I,\hat{\pi}_{01},\hat{\pi}_{11}) - \ln L(I,\hat{\varepsilon})]$$

where $\hat{\varepsilon} = \frac{T_{01}+T_{11}}{T} = \frac{N_I}{T}$, and asymptotically $LR_{IND} \sim \chi^2(\nu = 1)$.

As shown in Christoffersen (1998) [40], these two tests can be combined, determining the so-called CC test, where the null hypothesis $H_{0,CC} : \hat{\pi}_{01} = \hat{\pi}_{11} = \varepsilon$. Clearly, if one of the null hypotheses $H_{0,UC}$ and $H_{0,IND}$ is rejected, even $H_{0,CC}$ will tend to be rejected. For the CC test under the null hypothesis $H_{0,CC}$, the likelihood ratio is

$$LR_{CC} = 2[\ln L(I,\hat{\pi}_{01},\hat{\pi}_{11}) - \ln L(I,\varepsilon)]$$

where asymptotically $LR_{CC} \sim \chi^2(\nu = 2)$.

5.1.7 Regulator Backtesting Procedure

The Undertaking for Collective Investments in Transferable Securities (UCITS) mutual funds, under the ESMA's guidelines ([35]), have to be related to a VaR model with significance level ε =1%. This $VaR_{1\%}$ model, in turn, have to be validated according to specific rules. Indeed, the one-day-ahead $VaR_{1\%}$ forecasts to be accepted have to determine at most four violations over the earlier 250 (stock market) days. If a $VaR_{1\%}$ model presents five, six or seven overshootings[1], risk managers have to declare the violations to the vigilance authority, and by means of a documentation they have to explain and to analyze the causes of the model misspecification. If, instead, a $VaR_{1\%}$ prediction strategy determines eight or more violations, it is not accepted by the regulator. Table 5.1 summarizes the regulator actions corresponding to the $VaR_{1\%}$ model behavior. More precisely, a $VaR_{1\%}$ model is not discarded if the vio-

[1] An overshooting is here a synonym of a VaR violation, a word often used in the financial industry.

Number of violations	Action
$[0, 4]$	The $VaR_{1\%}$ model is accepted; no actions must be done.
$[5, 7]$	Possible crash of the $VaR_{1\%}$ model;
	the causes of the violations must be justified and explained.
$[8, +\infty)$	The $VaR_{1\%}$ model is not accepted, and it must be changed.

Table 5.1: The Regulator actions related to prefixed intervals of violations of $VaR_{1\%}$ forecasts, required by the UCITS funds.

Number of hits	Frequency (%)	p-value (%)
0	0.0	2.5
1	0.4	27.8
2	0.8	74.2
3	1.2	75.8
4	1.6	38.0
5	2.0	16.2
6	2.4	5.9
7	2.8	1.9
8	3.2	0.5

Table 5.2: Number of violations, corresponding frequency on 250 days, and related p-value for the UC test.

lations frequency belongs to the interval $[0.4\%; 2.4\%]$ at 95% confidence level (c.l.), or if the violations frequency belongs to the interval $[0\%; 2.8\%]$ at 99% c.l. (see Table 5.3). The ESMA guidelines require a backtesting procedure at 99% c.l.. Summarizing, a $VaR_{1\%}$ model is considered a good predictive tool up to four overshootings; if during the last 250 business days the number of overshootings for each UCITS exceeds four hits, then the senior management team must be informed. A competent authority may take specific actions, and may apply some limitations for the use of the $VaR_{1\%}$ model, when the overshootings exceed an unacceptable number of violations. Indeed, if the observed hit frequency is higher than 2.80% (seven overshootings), then some kind of action has to be taken in order to reduce the risk model misspecification. On the other hand, if there are no violations, nothing has to be done.

In the empirical analysis, reported in Section 5.2, in order to have an early warning on model performances, we decided to be more restrictive than the ESMA rules adopting a backtesting procedure at 95% c.l.. Therefore, in this work the maximum number of admissible overshootings is 6. Further, from Table 5.3 note that the case of absence of violations (zero hits) is not within the region of acceptance for the UC test at 95% c.l..

5.1.8 Backtesting based on loss functions

Other methodologies used to examine the performance of VaR models are those based on the comparison of loss functions (see, for example, [2] and [27]). Generally, a backtesting based on the loss function focuses on the magnitude of the overshootings, investigating the distance between the *ex-post* portfolio returns and the corresponding *ex-ante* VaR forecasts. Lopez (1999) ([68]) suggests the following general definition for a loss function:

$$L_t = \begin{cases} f(R_t, VaR_{t|t-1}) & \text{if } -R_t > VaR_{t|t-1} \\ g(R_t, VaR_{t|t-1}) & \text{if } -R_t \leq VaR_{t|t-1} \end{cases}$$

Clearly, the preferred model will be the one that minimizes the total loss $L_{tot} = \sum_{t=1}^{T} L_t$ over the backtesting sample of length T.

In this thesis we consider loss functions that take into account both the regulators' and the investors' viewpoint. Obviously, these depend on the specification of $f(R_t, VaR_{t|t-1})$ and $g(R_t, VaR_{t|t-1})$. More precisely, the loss functions of the regulator concentrate on the size of the loss only when an overshooting occurs, and therefore $g(R_t, VaR_{t|t-1}) = 0$. Meanwhile the investors are interested in both reducing the market risk, such as the regulator, and avoiding to set aside of much more money than necessary, and usually $f(R_t, VaR_{t|t-1}) = g(R_t, VaR_{t|t-1})$ ([27]).

However, recently, there has been a long discussion about the use of loss functions in backtesting. The main reason is that the loss functions we are considered are not consistent with VaR, in the sense that their expected value is not minimized by VaR. A paper that had a big influence and initiated an active stream of new research on backtesting was [53], that introduced in the financial community the concepts of consistent scoring function and elicitable functional. A scoring function is consistent for a given statistical functional if the functional can be defined as the minimizer of the expected value of the score; a functional is elicitable if it admits a consistent scoring function. Notice that scores are actually meant as losses, the smaller the better. Consistent scoring functions give a natural way to compare different forecasts of the same risk measure. The simplest elicitable risk measures are VaR, the expectiles, and the couple (VaR, ES). Indeed, VaR is the minimizer of the expected value of a suitable piecwise linear score; expectiles are by definition the minimizers of a suitable piecewise-quadratic score; and it has been recently established by [4] that the couple (VaR, ES) jointly minimizes the expectations of a suitable family of scoring functions (see also [17] and references therein).

For this reason, we decided to apply loss functions only for whom the

	Regulator		Investors
Loss Fun	$f(R_t, VaR_{t\|t-1})$ if $-R_t > VaR_{t\|t-1}$	$g(R_t, VaR_{t\|t-1})$ if $-R_t \leq VaR_{t\|t-1}$	$f(R_t, VaR_{t\|t-1}) = g(R_t, VaR_{t\|t-1})$ $\forall R_t$
Lopez	$1 + (R_t - VaR_{t\|t-1})^2$	0	-
Caporin1	$\left\lvert 1 - \left\lvert \frac{R_t}{VaR_{t\|t-1}} \right\rvert \right\rvert$	0	$\left\lvert 1 - \left\lvert \frac{R_t}{VaR_{t\|t-1}} \right\rvert \right\rvert$
Caporin2	$\frac{\left(\lvert R_t\rvert - \lvert VaR_{t\|t-1}\rvert\right)^2}{\lvert VaR_{t\|t-1}\rvert}$	0	$\frac{\left(\lvert R_t\rvert - \lvert VaR_{t\|t-1}\rvert\right)^2}{\lvert VaR_{t\|t-1}\rvert}$
Caporin3	$\lvert R_t - VaR_{t\|t-1}\rvert$	0	$\lvert R_t - VaR_{t\|t-1}\rvert$

Table 5.3: List of the regulators' and of the investors' loss functions considered.

accuracy test is satisfied. The accuracy tests are validating tests while loss functions are considered selection tests among competing models.

In Table 5.2 we report the list of the different loss functions considered in our empirical analysis.

5.2 Empirical analysis

In this section, we present computational results for seven models: the Implied Volatility VaR (ImVolVaR), the Realized Volatility VaR (ReVolVaR), the Shrunk Volatility VaR (ShVolVaR), the Historical Simulation VaR (HSVaR), the Historical Filtered Bootstrap VaR (HFBVaR), the Extreme Value Theory Filtered Bootstrap VaR (EVTFBVaR), and the RiskMetrics VaR (RiMeVaR) models. The analysis is performed on the S&P 500, Eurostoxx50, Dax, FTSE100 and Topix market indexes, from which, in addition to the index values, are also available the corresponding implied volatilities. The one-day-ahead VaR forecasts obtained by the seven models are validated by considering each individual index and the portfolios of such indexes, namely in both the univariate and multivariate contexts. As reported in Table 6.1, the lengths of the indexes' time series, consisting of daily values obtained from Bloomberg, cover different time windows according to the availability of the implied volatility values. Each data set has the same end date (September 30, 2015), is expressed in local currency and follows its own financial calendar. Further, the start dates of each data set, shown in Table 6.1, refer to the starting points of the VaR forecasts.

In more detail, we adopt a working-days calendar, and on these days we take the prices of all market indexes. If a market is closed on a specific day, eg, for holidays, then for that day we replicate the price with the last available value. Clearly, on that particular day, the index return will be zero. This preprocessing is required to compute a fair analysis of correlations among in-

Market	Index ticker	Implied Volatility ticker	Start date	End date	daily VaR forecasts
SP500	SPTR	VIX	30/01/1990	30/9/2015	6472
Eurostoxx50	SX5T	V2X	01/02/1999	30/9/2015	4265
DAX	DAX	V1X	30/01/1992	30/9/2015	5994
FTSE100	TUKXG	VFTSE	01/02/2000	30/9/2015	3959
Topix	TPXDDVD	VXJ	02/03/1998	30/9/2015	4320

Table 5.4: List of the data sets analyzed.

dexes, ie, to avoid possible lags among the returns time series. Conversely, the realized volatilities are estimated on the original time series without this preprocessing because, unlike the correlation estimates, those of volatilities would be too influenced by this preprocessing.

For completeness, we also list in Table 6.1 the tickers corresponding to each index. In the case of VaR forecast analyses for the portfolios of the five indexes, we consider the maximum time window covered by all data sets, namely from February 1, 2000 to September 30, 2015.

5.2.1 Computational Results

In this section we discuss the main results on the behavior of the VaR models in the univariate (Section 5.2.1.1) and in the multivariate (Section 5.2.1.2) frameworks.

5.2.1.1 Univariate framework

In Tables 5.5 and 5.6 we provide the p-values (expressed as a percentage) of the Unconditional Coverage (UC), the Independence (IND) and the Conditional Coverage (CC) tests. We report in dark-gray the cases in which we can not accept the null hypotheses (i.e., the VaR model is not able to capture the expected frequency of violations (UC), or suffer from dependence of violations (IND), or both (CC)). In light-gray we highlight the cases in which the null hypothesis is accepted at 99% confidence level (c.l.), but rejected at 95% c.l.. The rest of the accepted cases are reported in bold.

Table 5.5 shows that the RiMeVaR model does not suffer from dependence of violations; however, it fails to capture the UC hypothesis, except for the TOPIX index. The HFBVaR model is within the region of acceptance of the UC test at a 99% c.l. for all indexes; when we consider a 95% c.l., however, we observe two rejections (out of five) of the IND hypothesis, and consequently two rejections of the CC hypothesis. HSVaR has the worst performance among the models examined. For all indexes, it suffers from dependence of violations at a 95% c.l.; when a 99% c.l. is considered, the IND hypothesis is rejected four times out of five. Further, the HSVaR model fails the UC test four times out of five, and, accordingly, in only one case out of five can we accept the null hypotheses on the CC test at a 99% c.l.. These results are consistent with the findings of Brandolini and Colucci (2012) ([24]). For the EVTFBVaR model, we observe that the UC and CC tests are always accepted, while the IND hypothesis is rejected only in one case at a 95% c.l.. It is evident from our experiments that the EVTFBVaR and HFBVaR strategies show better results than RiMeVaR and HSVaR.

Table 5.6 shows that the ImVolVaR model ($\alpha = 1$) fails the UC test four times out of five. Further, this model fails the CC test three times at 99% c.l., and only in three cases it does not suffer from dependence of violations. The ReVolVaR model ($\alpha = 0$) presents the worst results; indeed for all indexes the null hypothesis of the UC and CC tests is not accepted.

Index	EVTFBVaR			HSVaR			HFBVaR			RiMeVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
SP500	**97.2**	3.1	**9.9**	0.0	0.0	0.0	**5.03**	2.21	1.07	0.00	**25.17**	0.00
EuroStoxx50	**72.0**	**50.1**	**74.8**	**6.9**	3.9	2.3	**16.42**	**66.74**	**34.65**	0.00	**53.75**	0.00
DAX	**35.8**	**9.2**	**15.9**	0.0	0.0	0.0	**7.83**	**31.88**	**12.91**	0.00	**56.65**	0.00
Ftse100	**56.0**	**33.9**	**53.4**	0.6	0.0	0.0	**14.73**	**63.93**	**31.35**	0.00	**18.79**	0.00
Topix	**73.4**	**6.3**	**16.8**	0.1	0.0	0.0	**6.12**	4.15	2.17	**62.03**	**5.70**	**14.45**

Table 5.5: The p-values (%) of the UC, the IND, and the CC tests for the EVTFBVaR, the HSVaR, the HFBVaR, and the RiMeVaR models.

For sake of brevity, in Table 5.6 we present detailed results only for the ShVolVaR model with $\alpha = \{0.3, 0.5, 0.7\}$. Clearly, low values of α tend to give more importance to the realized volatility, while for high α the one-day-ahead VaR estimation mainly depends on the implied volatility. Considering the ShVolVaR model with $\alpha = 0.3$, we observe three rejections for the UC and the CC tests at 95% confidence level. For $\alpha = 0.7$, the ShVolVaR model presents one rejection for the UC and CC tests at 99% c.l., and only in two cases out of five is the null hypothesis of the IND and CC tests at 95% c.l. accepted. As mentioned above, the best results are obtained by the ShVolVaR model with $\alpha = 0.5$, where therefore the shrunk volatility $\tilde{\sigma}_{k,t}$ in (5.3) can be simply interpreted as the average between the realized and the implied volatility of the asset k.

In Tables 5.7 and 5.8 we report the frequency related to the observed number of VaR violations on the three overshooting intervals required by the ESMA rules (see Section 5.1.7). We can observe that the HSVaR and the RiMeVaR models show a high percentage of times at which the overshootings are more

Index	ReVolVaR ($\alpha = 0$)			ShVolVar ($\alpha = 0.3$)			ShVolVar ($\alpha = 0.5$)			ShVolVar ($\alpha = 0.7$)			ImVolVaR ($\alpha = 1$)		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
SP500	0.0	2.0	0.0	0.6	**49.6**	1.8	**55.1**	**12.9**	**26.5**	0.6	3.7	0.3	0.0	**56.1**	0.0
EuroStoxx50	0.0	**86.6**	0.0	**9.3**	**71.7**	**22.9**	**95.7**	**45.7**	**75.7**	**68.0**	**39.3**	**63.7**	0.6	**57.2**	1.9
DAX	0.0	**44.6**	0.0	0.3	3.7	0.1	**10.1**	**7.4**	**5.3**	**35.8**	1.2	2.8	0.4	0.2	0.0
Ftse100	0.0	0.3	0.0	0.2	**33.3**	0.4	**31.8**	**56.5**	**51.5**	**36.0**	**30.0**	**38.4**	3.3	**17.9**	4.2
Topix	0.0	0.1	0.0	**14.8**	**17.2**	**13.8**	**62.0**	**5.7**	**14.5**	**10.4**	2.6	2.2	0.3	0.8	0.0

Table 5.6: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the ShVolVaR models with $\alpha = \{0, 0.3, 0.5, 0.7, 1\}$.

than six with the only exception of the Topix index for RiMeVaR. The other benchmark models, HFBVaR and EVTFBVaR, have a frequency of violations on the interval $[7, +\infty)$ that, among the different indexes considered, are not greater than 5.1% and 4.1% respectively. The ImVolVaR model, overestimating VaR, performs well according to the regulator rules; indeed, we observe that the frequency of violations on the interval $[7, +\infty)$ is at most 2%. Conversely, for the same overshooting interval the ReVolVaR model perform poorly, showing a frequency of violations always greater than 26.1%. The new proposed model (ShVolVaR with $\alpha = 0.5$) presents a frequency of overshootings that is not greater than 3% on the interval $[7, +\infty)$. Therefore, from the regulators' viewpoint the ShVolVaR model does not require any modification.

The case of zero violations is allowed by the regulator, but from the investors' viewpoint this means the mismanagement of capital. Furthermore, as shown in Table 5.3 the case of zero hits is always rejected from the UC test with 95% confidence level. However, theVaR models that best minimize the frequency of absence of violations over a period of 250 days are ReVolVaR and RiMeVaR, which are aggressive strategies for estimating the one-day-ahead VaR. Meanwhile, the worst model is ImVolVaR, which is a too conservative strategy. For zero hits, the performance of the ShVolVaR model is comparable to HFBVaR and EVTFBVaR; indeed, ShVolVaR shows that the average frequency of violations on the five indexes is equal to 5.6%, while for HFBVaR and EVTFBVaR the average values are 4.2% and 5.3%, respectively. We stress that the only three models analyzed which comply with regulator requests and satisfy the accuracy tests are EVTFBVaR, HFBVaR and ShVolVaR.

Then, following the experimental procedure of Abad at al. (2015) (see [2]) and of Marimoutou et al. (see [76]), we perform a backtesting based on loss functions only for those models that never reject the CC test at 99% c.l.. In Table 5.9 we report the values of the total loss computed using the regulators' loss functions (see Section 5.1.8). Considering the average of total loss for all the

VaR	SP500				Topix			
Model	$[0,4]$	$[5,6]$	$[7,+\infty)$	0 hits	$[0,4]$	$[5,6]$	$[7,+\infty)$	0 hits
ImVolVaR	96.1	1.8	2.0	34.4	99.2	0.8	0.0	29.5
ReVolVaR	38.9	23.5	37.5	0.4	36.2	35.7	28.1	0.0
ShVolVaR	89.8	7.2	3.0	14.4	94.9	5.1	0.0	5.4
HFBVaR	83.9	12.8	3.3	6.1	77.6	19.9	2.5	4.4
RiMeVaR	39.8	28.9	31.3	0.4	94.9	5.1	0.0	5.4
HSVaR	66.8	18.8	14.4	20.6	70.9	9.1	20.0	11.8
EVTFBVaR	92.3	7.7	0.0	5.3	91.2	5.8	3.0	2.0

Table 5.7: Observed frequencies of VaR violations on the three representative overshooting intervals required by the ESMA rules for the SP500 and Topix indexes.

VaR	EuroStoxx50				DAX				Ftse100			
Model	$[0,4]$	$[5,6]$	$[7,+\infty)$	0 hits	$[0,4]$	$[5,6]$	$[7,+\infty)$	0 hits	$[0,4]$	$[5,6]$	$[7,+\infty)$	0 hits
ImVolVaR	96.9	3.1	0.0	19.0	98.9	1.1	0.0	25.1	95.4	4.6	0.0	30.7
ReVolVaR	36.7	37.2	26.1	0.0	32.9	39.3	27.8	1.3	21.8	33.6	44.6	0.0
ShVolVaR	91.5	8.4	0.0	3.2	87.2	12.3	0.5	3.7	88.5	10.8	0.8	1.3
HFBVaR	84.6	15.4	0.0	1.0	83.1	16.8	0.0	3.1	85.0	9.9	5.1	6.6
RiMeVaR	40.5	33.4	26.2	0.0	43.8	42.2	14.0	1.3	34.0	34.3	31.7	0.0
HSVaR	63.1	15.3	21.6	39.7	57.3	17.2	25.5	27.3	57.0	17.5	25.5	34.1
EVTFBVaR	93.9	6.1	0.0	1.6	99.5	0.5	0.0	6.9	91.4	4.6	4.1	10.6

Table 5.8: Observed frequencies of VaR violations on the three representative overshooting intervals required by the ESMA rules for the EuroStoxx50, DAX and Ftse100 indexes.

Portfolio	Lopez			Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
SP500	**60.0070**	*81.0068*	65.0055	16.7095	*21.1220*	**16.3722**	0.2565	*0.2870*	**0.2306**	0.4040	*0.4461*	**0.3541**
Topix	**40.0103**	*56.0101*	41.0082	14.3900	*15.6653*	**12.0426**	0.3108	*0.3366*	**0.2624**	0.4463	*0.4556*	**0.3742**
Eurostoxx 50	**43.0042**	*52.0050*	45.0041	10.1899	*13.0137*	**10.1477**	0.1448	*0.1796*	**0.1401**	0.2800	*0.3419*	**0.2761**
Dax	73.0082	*74.0070*	**53.0056**	*18.5371*	18.0669	**12.8350**	*0.2840*	0.2446	**0.1837**	*0.4715*	0.4660	**0.3431**
Ftse100	46.0026	*49.0028*	**36.0020**	10.1055	*11.8675*	**8.8829**	0.0985	*0.1160*	**0.0812**	0.2441	*0.2619*	**0.2076**
Mean	52.4065	*62.4063*	**48.0051**	13.9864	*15.9471*	**12.0561**	0.2189	*0.2328*	**0.1796**	0.3692	*0.3943*	**0.3110**

Table 5.9: Values of the total loss over the backtesting sample for each index, using Regulator's loss functions.

Portfolio	Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
SP500	4622.9418	**4571.3496**	*4666.3160*	92.9660	**92.7960**	*103.2646*	170.8513	**170.1141**	*182.1709*
Topix	3047.1406	**2986.8866**	*3052.8878*	79.3427	**75.1967**	*81.6495*	146.0535	**140.9367**	*148.4493*
Eurostoxx 50	2974.2073	**2970.8724**	*3010.3411*	**77.7086**	79.1699	*84.3858*	**146.0354**	147.1937	*153.2888*
Dax	**4126.3450**	4175.1117	*4254.6667*	**99.8571**	106.6990	*115.9781*	**193.0378**	200.1147	*211.1495*
Ftse100	2761.9257	**2746.4252**	*2806.4317*	**57.9089**	59.7209	*64.7489*	**109.1349**	110.9604	*116.7976*
Mean	3506.5121	**3490.1291**	*3558.1287*	**81.5566**	82.7165	*90.0054*	**153.0226**	153.8639	*162.3712*

Table 5.10: Values of the total loss over the backtesting sample for each index, using Investors's loss functions.

indexes analyzed, HFBVaR shows the worst results, ShVolVaR has intermediate performance, and EVTFBVaR presents the best results. However, when using investors' loss functions, as reported in Table 5.10, on average EVTFBVaR tends to perform the worst, while ShVolVaR has the best results two times out of three.

We point out, therefore, that our naive model acheives performance results that are as good as the EVTFBVaR and the HFBVaR models, which seem to be among the best methods to estimate VaR (see [1]).

5.2.1.2 Multivariate framework

In this section we report the empirical results for multivariate VaR estimations on twenty portfolios, listed. These are listed in Table 5.11, and are commonly considered as benchmark portfolios in the asset management industry (see, for example, [83] and [106]). The portfolios are composed of a number of indexes, ranging from two to five. Further, we present here only the empirical analysis

Portfolio	Weights (%)				
name	SP500	EuroStoxx50	DAX	Ftse100	Topix
BMK1	50.0	0.0	0.0	25.0	25.0
BMK2	50.0	50.0	0.0	0.0	0.0
BMK3	66.0	34.0	0.0	0.0	0.0
BMK4	0.0	50.0	0.0	0.0	50.0
BMK5	10.0	10.0	10.0	10.0	60.0
BMK6	34.0	33.0	0.0	0.0	33.0
BMK7	0.0	33.0	0.0	33.0	34.0
BMK8	50.0	25.0	0.0	25.0	0.0
BMK9	0.0	35.0	25.0	40.0	0.0
BMK10	0.0	50.0	0.0	50.0	0.0
BMK11	10.0	60.0	10.0	10.0	10.0
BMK12	50.0	0.0	50.0	0.0	0.0
BMK13	25.0	25.0	0.0	25.0	25.0
BMK14	50.0	25.0	0.0	0.0	25.0
BMK15	0.0	0.0	50.0	50.0	0.0
BMK16	12.5	30.0	5.0	30.0	12.5
BMK17	50.0	20.0	5.0	15.0	10.0
BMK18	10.0	25.0	20.0	35.0	10.0
BMK19	20.0	20.0	20.0	20.0	20.0
BMK20	50.0	0.0	0.0	0.0	50.0

Table 5.11: List of the 20 benchmark portfolios weights used for the multivariate VaR estimations analysis.

for those models that in the univariate framework have shown the best results (namely EVTFBVaR, HFBVaR and ShVolVaR with $\alpha = 0.5$).

In Table 5.12 we report the p-values obtained by the EVTFBVaR, the HFBVaR and the ShVolVaR models for the UC test, that is, the accuracy test on which the regulator is mainly interested. Following the convention described in Section 5.2.1.1, we note that for the EVTFBVaR model all the portfolios pass the UC test at 95% c.l.; for ShVolVaR nineteen portfolios out of twenty pass the UC test at 95% c.l., while for HFBVaR only thirteen portfolios out of 20 pass. If we consider the UC test at 99% c.l. (in light-gray), then for all the benchmark portfolios the VaR estimations obtained by EVTFBVaR, ShVolVaR and HFBVaR show statistical significance when we compare the observed frequency of VaR violations with that expected.

Portfolio	UC test → p-value (%)		
name	ShVolVaR	HFBVaR	EVTFBVaR
BMK1	**98.2**	2.4	**76.8**
BMK2	**89.2**	**42.8**	**85.8**
BMK3	**85.8**	**21.5**	**43.5**
BMK4	**73.9**	**34.6**	**42.8**
BMK5	**73.9**	**27.5**	**89.2**
BMK6	**73.9**	3.5	**76.8**
BMK7	**62.6**	**12.5**	**16.5**
BMK8	**62.6**	**42.8**	**89.2**
BMK9	**42.8**	**34.6**	**62.6**
BMK10	**34.6**	**9.3**	**52.2**
BMK11	**34.6**	3.5	**27.5**
BMK12	**27.5**	**27.5**	**89.2**
BMK13	**27.5**	**16.5**	**89.2**
BMK14	**21.5**	3.5	**52.2**
BMK15	**16.5**	**34.6**	**85.8**
BMK16	**16.5**	4.9	**52.2**
BMK17	**16.5**	**34.6**	**43.5**
BMK18	**12.5**	2.4	**21.5**
BMK19	**12.5**	3.5	**73.9**
BMK20	3.5	**16.5**	**98.2**

Table 5.12: The p-values (%) of the UC test obtained by the EVTFBVaR, the HFBVaR, and the ShVolVaR models for the 20 benchmark portfolios.

In Table 5.13, for each benchmark portfolio, we report the frequency (%) related to the number of VaR violations achieved by the EVTFBVaR, HFBVaR and ShVolVaR models on the three representative overshooting intervals. For each interval and each portfolio, the best result is marked in bold. Comparing ShVolVaR and HFBVaR, we can observe that, in the case of VaR violations greater than six, the ShVolVaR model performs better than HFBVaR in eight cases out of twenty, for two benchmark portfolios the frequencies are equal and in ten cases out of twenty HFBVaR presents better results than ShVolVaR. Therefore, there is no apparent relation of dominance between these two ap-

Portfolio	[0, 4]			[5, 6]			[7, +∞)		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
BMK1	89.5	72.4	**91.4**	10.1	22.3	**8.0**	**0.3**	5.4	0.5
BMK2	88.8	89.9	**92.0**	9.1	9.7	**8.0**	2.1	0.3	**0.0**
BMK3	94.1	85.8	**96.0**	5.5	13.1	**4.0**	0.3	1.1	**0.0**
BMK4	**92.6**	91.2	88.8	**7.0**	8.4	10.8	0.4	0.4	0.4
BMK5	**94.0**	90.0	88.7	**6.0**	10.0	10.6	**0.0**	**0.0**	0.7
BMK6	86.1	78.9	**91.0**	13.9	18.1	**8.7**	**0.0**	3.0	0.3
BMK7	**88.6**	83.6	83.3	**9.8**	14.7	13.5	**1.6**	1.7	3.2
BMK8	86.0	90.4	**92.9**	8.6	9.3	**7.1**	5.4	0.3	**0.0**
BMK9	**93.8**	93.5	92.9	**5.5**	6.5	7.1	0.7	**0.0**	**0.0**
BMK10	87.5	82.3	**90.4**	10.2	17.5	8.7	2.3	**0.2**	0.9
BMK11	82.1	78.5	**85.1**	17.2	16.9	**14.9**	0.7	4.7	**0.0**
BMK12	87.9	86.5	**97.2**	9.9	13.2	**2.8**	2.3	0.3	**0.0**
BMK13	80.3	84.3	**98.0**	18.2	15.0	**2.0**	1.4	0.8	**0.0**
BMK14	79.4	82.6	**88.8**	20.0	**9.0**	10.7	0.7	8.4	**0.5**
BMK15	83.7	88.7	**95.1**	14.0	11.1	**4.9**	2.3	0.2	**0.0**
BMK16	80.7	83.8	**94.5**	16.8	15.2	**4.6**	2.5	1.0	**0.9**
BMK17	84.5	88.0	**97.0**	13.0	8.9	**3.0**	2.5	3.1	**0.0**
BMK18	82.1	81.3	**92.3**	14.9	17.4	**7.2**	3.0	1.4	**0.4**
BMK19	81.3	79.2	**97.0**	17.3	18.5	**3.0**	1.4	2.3	**0.0**
BMK20	80.6	86.3	**97.8**	17.9	12.8	**1.5**	1.5	0.9	**0.7**
Mean	86.2	84.8	**92.5**	12.2	13.4	**7.1**	1.6	1.8	**0.4**
Min	79.4	72.4	**83.3**	5.5	6.5	**1.5**	0.0	0.0	0.0
Max	94.1	93.5	**98.0**	20.0	22.3	**14.9**	5.4	8.4	**3.2**

Table 5.13: Observed frequencies of VaR violations on the three representative overshooting intervals required by the ESMA rules for the 20 portfolios listed in Table 5.11.

proaches. The average of the frequencies on the twenty portfolios is below 2% for both models; in very few cases, the ShVolVaR and HFBVaR models require adjustments. In the overshooting interval $[0, 4]$, the average frequency is 86.2% for ShVolVaR and 84.8% for HFBVaR. For the intermediate interval $[5, 6]$, the ShVolVaR and HFBVaR models show average frequencies of 12.2% and 13.4%, respectively. Thus, even in these cases the two approaches seem to perform similarly. The EVTFBVaR approach has the best results for all the intervals considered, and in our experiments it seems to dominate the other two approaches.

We stress here that the ideal model, both from the regulator' and from the investors' viewpoint, should have 100% of the VaR violations on the interval $[1, 4]$. Table 5.14 shows that on the interval $[1, 4]$ the average frequencies of ShVolVaR and of HFBVaR are both equal to 82.4%, and that also the minimum and maximum frequencies are similar, around 70.0% and 92.7%. Again, for the same interval, EVTFBVaR shows its best results with average frequencies of 87.7%, and minimum and maximum frequencies equal to 77% and 94% respectively.

We can conclude that the forecasting power of our naive model is definitely comparable to that of the two best methods of estimating VaR (HFBVaR and

Portfolio	0 hits			$[1, 4]$		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
BMK1	8.8	**2.4**	6.1	80.8	70.0	**85.3**
BMK2	8.4	**3.6**	3.8	80.4	86.3	**88.2**
BMK3	5.7	**3.6**	8.4	*88.5*	82.2	87.6
BMK4	4.9	6.9	**1.0**	87.6	84.3	*87.8*
BMK5	**1.5**	**1.5**	11.9	**92.4**	88.5	76.8
BMK6	7.9	**1.7**	9.6	78.2	77.1	**81.4**
BMK7	4.2	**2.7**	3.2	**84.4**	80.9	80.2
BMK8	**1.9**	3.6	3.8	84.0	86.7	**89.1**
BMK9	1.2	**0.8**	3.4	92.6	**92.7**	89.5
BMK10	**1.2**	**1.2**	3.4	86.3	81.1	**87.0**
BMK11	7.4	**0.8**	5.7	74.7	77.7	**79.5**
BMK12	5.7	**3.6**	5.1	82.2	82.9	**92.1**
BMK13	**2.0**	2.1	4.1	78.4	82.1	**94.0**
BMK14	7.9	**2.2**	2.8	71.5	80.4	**86.0**
BMK15	**0.6**	2.0	3.4	83.1	86.6	**91.7**
BMK16	**0.8**	**0.8**	7.0	80.0	83.0	**87.5**
BMK17	**0.2**	4.4	4.6	84.4	83.6	**92.4**
BMK18	2.6	**0.0**	0.8	79.5	81.3	**91.5**
BMK19	0.8	**0.0**	3.3	80.5	79.2	**93.7**
BMK20	**1.0**	5.5	5.1	79.5	80.8	**92.7**
Mean	3.7	**2.5**	4.8	82.4	82.4	**87.7**
Min	0.2	**0.0**	0.8	71.5	70.0	**76.8**
Max	8.8	**6.9**	11.9	92.6	92.7	**94.0**

Table 5.14: Observed frequencies of the number of VaR violations on the interval $[1, 4]$, and in the case of absence of violations (0 hits) for the 20 benchmark portfolios.

EVTFBVaR), even though EVTFBVaR seems to show the best performance. However, while the VaR forecasts of HFBVaR and EVTFBVaR are based on fairly sophisticated procedures, those of the ShVolVaR model are extremely simple to obtain. Indeed, we stress that in the step $4'$. of the EVTFBVaR approach (see Section 5.1.4), the estimation process of the t-copula parameters is very time-consuming, especially when the number of assets in a portfolio increases. In Table 5.15 we report the values of the total loss, computed using regulators' loss functions, for the twenty benchmark portfolios. As for the univariate case, EVTFBVaR performs better than the other models, regardless of which regulators' loss function is used, while ShVolVaR has the second best results in two cases out of four. However, observing Table 5.16, from the investors' viewpoint ShVolVaR seems to have the best results, while EVTFBVaR tends to perform poorly.

Portfolio	Lopez			Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
BMK 1	41.0024	*56.0014*	**39.0014**	11.8004	*11.8753*	**9.5624**	*0.1150*	0.0889	**0.0776**	*0.2276*	0.1966	**0.1741**
BMK 2	**40.0022**	*46.0020*	42.0015	11.0043	*11.6690*	**9.5196**	0.1040	*0.1063*	**0.0845**	*0.2380*	0.2335	**0.1861**
BMK 3	42.0027	*49.0022*	**36.0017**	10.7298	*11.6358*	**9.2958**	*0.1158*	0.1091	**0.0925**	*0.2366*	0.2359	**0.1778**
BMK 4	**43.0043**	*47.0040*	46.0045	**12.4769**	13.2080	*13.2423*	0.1727	**0.1670**	*0.1883*	**0.3013**	*0.3089*	0.3029
BMK 5	43.0047	*48.0042*	**40.0047**	12.9197	*13.2803*	**12.7542**	0.1909	**0.1844**	*0.2075*	0.3023	*0.3030*	**0.2799**
BMK 6	43.0021	*55.0016*	**39.0019**	*11.8215*	11.0342	**10.5083**	*0.1012*	**0.0898**	0.0963	*0.2387*	0.2080	**0.2037**
BMK 7	**44.0033**	*51.0026*	50.0031	**11.3448**	11.7837	*12.5689*	*0.1382*	**0.1135**	0.1381	**0.2569**	0.2632	*0.2668*
BMK 8	44.0023	*46.0017*	**40.0014**	*11.0168*	10.9811	**9.0389**	*0.1050*	0.0928	**0.0793**	*0.2273*	0.2041	**0.1677**
BMK 9	46.0036	*47.0040*	**44.0033**	10.0879	*11.8892*	**9.9623**	0.1269	*0.1466*	**0.1198**	0.2464	*0.2863*	**0.2404**
BMK 10	47.0032	*52.0036*	**45.0030**	**10.0346**	*12.0697*	10.2231	0.1155	*0.1370*	**0.1103**	*0.2439*	0.2834	**0.2425**
BMK 11	**47.0027**	*55.0029*	48.0026	**9.9611**	*12.2593*	10.7392	0.1082	*0.1268*	**0.1041**	**0.2287**	*0.2649*	0.2406
BMK 12	*48.0025*	48.0022	**40.0016**	*11.1457*	10.9929	**9.2701**	*0.1110*	0.1042	**0.0862**	*0.2397*	0.2345	**0.1888**
BMK 13	48.0020	*50.0015*	**40.0016**	*11.2789*	11.1992	**10.4994**	*0.0956*	0.0872	**0.0851**	*0.2217*	**0.1969**	**0.1969**
BMK 14	49.0021	*55.0013*	**45.0015**	*11.4191*	10.9727	**9.7201**	*0.1001*	0.0823	**0.0806**	*0.2310*	0.1939	**0.1842**
BMK 15	*50.0037*	47.0040	**42.0033**	10.4366	*11.5619*	**9.6015**	0.1277	*0.1416*	**0.1167**	0.2524	*0.2839*	**0.2319**
BMK 16	50.0018	*54.0015*	**45.0015**	10.7664	*11.4956*	**10.4364**	*0.0862*	0.0832	**0.0760**	0.2032	*0.2049*	**0.1894**
BMK 17	*50.0022*	47.0015	**36.0013**	*10.8053*	10.5164	**9.0651**	*0.1028*	0.0837	**0.0785**	*0.2174*	0.1930	**0.1670**
BMK 18	51.0025	*56.0022*	**49.0022**	10.7565	**11.4130**	*10.0838*	*0.1047*	0.1020	**0.0914**	0.2331	*0.2358*	**0.2137**
BMK 19	51.0021	*55.0018*	**43.0018**	11.2734	*11.9365*	**10.7623**	*0.0980*	0.0945	**0.0863**	*0.2310*	0.2299	**0.2138**
BMK 20	*55.0031*	50.0023	**41.0028**	*13.6751*	13.1640	**12.1072**	*0.1362*	**0.1139**	0.1290	*0.2846*	0.2484	**0.2455**
Mean	46.6028	*50.7024*	**42.5023**	11.2377	*11.7469*	**10.4480**	*0.1178*	0.1127	**0.1064**	*0.2431*	0.2405	**0.2157**

Table 5.15: Values of the total loss over the backtesting sample for the 20 benchmark portfolios, using Regulator's loss functions.

To better highlight the forecasting power in the multivariate framework, we also test the ShVolVaR model on 2000 portfolios, uniformly distributed on the unit simplex, ie, portfolios with weights that must sum to 1 without short-sellings. To generate these 2000 portfolios, we employ an algorithm provided by Rubinstein (1982) (see [99]). In Table 5.17 we report some statistics (mean, median, minimum and maximum) of the observed frequencies (also called the empirical coverages) of the VaR violations and of the p-values obtained by the UC test for the ShVolVaR model. The empirical coverages range from 0.91% to 1.35% with the median equal to 1.17%, while the p-values are between 3.37% and 98.28% with the median equal to 27.47%. This means that the UC test at 99% c.l. never rejects the null hypothesis, while at 95% c.l.

Portfolio	Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR	ShVolVaR	HFBVaR	EVTFBVaR
BMK 1	2861.8086	**2843.3050**	*2895.2287*	**47.1939**	48.2835	*51.1557*	**88.4635**	89.6406	*93.0658*
BMK 2	**2880.5416**	2888.7287	*2909.7733*	**61.3640**	63.9055	*65.9134*	**112.6127**	115.2428	*117.6684*
BMK 3	2892.5173	**2889.8671**	*2918.7054*	**60.0871**	62.0754	*65.0732*	**109.6493**	111.6916	*115.1995*
BMK 4	**2862.0488**	2873.1896	*2894.1636*	**58.7096**	60.0447	*61.5921*	**109.0054**	110.4305	*112.2853*
BMK 5	2887.1598	**2857.5478**	*2905.3007*	56.0739	**54.7633**	*58.0770*	102.7530	**101.0832**	*104.9893*
BMK 6	2848.4629	**2847.2812**	*2885.8065*	**50.9912**	51.6217	*54.4267*	**95.4108**	96.0279	*99.4022*
BMK 7	**2862.5540**	2867.4588	*2888.6687*	**55.0583**	56.6075	*57.9690*	**102.5779**	104.2721	*105.8404*
BMK 8	**2882.9738**	2887.9026	*2911.5020*	**56.8424**	59.5752	*61.2658*	**104.3797**	107.3155	*109.3737*
BMK 9	**2861.0194**	2863.4197	*2887.2108*	**66.1469**	67.4476	*70.3007*	**123.0245**	124.0556	*127.5516*
BMK 10	2869.3064	**2868.4500**	*2890.4673*	**65.6539**	66.9954	*69.3758*	**121.6295**	122.7508	*125.6588*
BMK 11	**2859.8384**	2872.1217	*2884.7462*	**63.4489**	65.5773	*67.1863*	**117.8323**	119.9517	*121.8877*
BMK 12	**2868.4947**	2882.1922	*2908.5404*	**61.3271**	64.0797	*66.4918*	**113.0821**	115.9264	*118.8529*
BMK 13	**2855.3365**	2855.8923	*2888.2478*	**50.5587**	51.9650	*53.8853*	**94.4609**	95.9934	*98.2783*
BMK 14	**2856.5063**	2865.9246	*2893.1276*	**50.6278**	53.1376	*54.6959*	**94.5035**	97.2606	*99.1871*
BMK 15	**2855.1820**	2862.6287	*2888.5853*	**64.1659**	66.1049	*68.9156*	**119.7082**	121.5295	*124.9579*
BMK 16	**2860.5910**	2865.3378	*2889.2869*	**50.5081**	51.9308	*53.6850*	**93.9425**	95.3916	*97.4845*
BMK 17	**2873.3879**	2886.8909	*2905.6094*	**53.4088**	56.1323	*57.7742*	**98.5612**	101.5860	*103.5170*
BMK 18	**2857.8482**	2869.3629	*2887.6852*	**58.3338**	60.2353	*62.1683*	**108.6752**	110.6139	*112.9188*
BMK 19	**2849.1685**	2851.0512	*2884.3096*	**53.5703**	54.8537	*57.2044*	**100.2416**	101.5959	*104.3555*
BMK 20	2868.4960	**2842.9578**	*2901.1273*	50.7030	**50.5122**	*53.8039*	94.2041	**93.9590**	*97.8738*
Mean	**2865.6621**	2867.0755	*2895.9046*	**56.7387**	58.2924	*60.5480*	**105.2359**	106.8159	*109.5174*

Table 5.16: Values of the total loss over the backtesting sample for the 20 benchmark portfolios, using Investor's loss functions.

the UC hypothesis is rejected in seventeen cases out of 2000. Table 5.17 also provides some statistics (mean, median, minimum and maximum) of the observed frequencies related to the number of VaR violations on the usual three overshooting intervals required by the UCITS rules. The results obtained by the ShVolVaR model for 2000 randomly generated portfolios show a behavior similar to that achieved from the restricted set of twenty benchmark portfolios. So, even this last analysis confirms that our simple prediction strategy to model VaR shows a very promising performance. Hence, due to its ease of implementation, the ShVolVaR model could be used as a tool for portfolio managers to quickly monitor investment decisions before employing more sophisticated risk management systems.

	Empirical	UC test	Overshooting intervals			
	coverage	p-value	$[0, 4]$	$[5, 6]$	$[7, +\infty)$	0 hits
Mean	1.17	33.32	86.2	12.3	1.5	2.7
Min	0.91	3.47	72.8	1.1	0.0	0.0
Max	1.35	98.24	98.9	25.5	7.3	13.2
Median	1.17	27.47	86.1	12.3	1.3	2.0

Table 5.17: Mean, median, min and max of the realized frequency of the VaR violations and of the p-value obtained by the UC test for the ShVolVaR model on 2000 portfolios uniformly distributed on the unit simplex.

5.3 Conclusions

In this chapter we have proposed a new method to predict VaR, using both variables known in the market (implied volatilities) and variables estimated on data (realized volatilities). The main idea behind our approach is to use a combination of information on both expected future risk and past estimated risk. The forecasting power of our ShVolVaR model is compared with that of several models proposed in the literature, including two of the best methods for estimating VaR, such as the HFBVaR and EVTFBVaR models. All models are tested on both statistical accuracy (by means of the UC, IND and CC tests) and efficiency (by means of the backtesting procedure of the vigilance authority and backtesting based on loss functions). Further, they are validated on both individual assets and the portfolios of such assets, in both the univariate and multivariate frameworks.

Although the ShVolVaR model is based on strong assumptions like those of Risk- Metrics (namely that one-day-ahead returns are normally distributed with zero mean), its forecasting power is comparable to that of the more sophisticated HFBVaR and EVTFBVaR models. Thus, we provide a fast and simple tool that can also be imple- mented on a common spreadsheet, which could be directly integrated with, for instance, data providers.

Talking in practical terms, in this work we have examined the case of a portfolio manager who administers a flexible UCITS fund, aiming to obtain the maximum return with a constraint on risk, measured by VaR. Since the portfolio managers must face transaction costs when they buy or sell assets, they could use our quick forecasting tool as a what-if scenario analysis before they start trading. If the portfolio VaR is within specific risk bounds, the portfolio managers can purchase and sell; otherwise, they should revise their investment. Therefore, the pre-analysis obtained by our model may allow for the control of risk both upstream and downstream of the investment process.

Indeed, it is typical for the asset manager to construct their portfolio, and only afterwards for the risk manager to ensure compliance with the risk limit. So, if the portfolio VaR goes beyond the regulators' limitations, then the portfolio manager has to change their investment strategy, thus leading to carry the trading costs twice.

Chapter 6

Shrunk Volatility VaR: an application on US Balanced Portfolios

The purpose of this chapter is to compare the Shrunk Volatility VaR (ShVolVaR), with several prediction strategies over many balanced portfolios composed by equity and bonds.

More in detail, we generalize the Shrunk Volatility VaR model and we empirically compare its forecasting power with six benchmark VaR models: the Historical Filtered Bootstrap with symmetric and asymmetric conditional variance (see [52] ,[86] and [87]), Extreme Value Theory Historical Filtered Bootstrap with symmetric and asymmetric conditional variance and on RiskMetrics approaches over a relatively long time period (at least thirteen years) that depends on the availability of implied volatility values. For these seven models, we use the same validation procedure seen in chapter 5. We test four VaR confidence level namely 95%, 99%, 99.5% and 99.9%.

6.1 Models and Tests

6.1.1 Historical Filtered Bootstrap VaR model

The model is the one presented in 5.1.3. To highlight the power of asymmetric GARCH models in forecasting VaR we perform the same procedure changing the filter using an AR(1)-StudT-EGARCH(1,1) and AR(1)-StudT-GJRGARCH(1,1) (HFBEVaR and HFBGJRVaR). In this way, the first step the formulation changes and the AR(1)-StudT-EGARCH(1,1) is defined as follows

$$\begin{aligned}
\text{AR(1)} &: \quad r_{k,t} = a_k + b_i r_{k,t-1} + \eta_{k,t} \\
\text{StudT-EGARCH(1,1)} &: \quad \log \sigma^2_{k,t} = \alpha_k + \gamma_k \log \sigma^2_{k,t-1} + \delta_k \left(|z_{k,t-1}| - E(|z_{k,t-1}|)\right) + \beta_i z_{k,t-1} \\
&\quad\ \eta_{k,t} = \sigma_{k,t} z_{k,t}
\end{aligned}$$

and AR(1)-StudT-GJRGARCH(1,1) is defined as follow

$$\begin{aligned}
\text{AR(1)} &: \quad r_{t,k} = a_k + b_k r_{t-1,k} + \eta_{t,k} \\
\text{StudT-GJRGARCH(1,1)} &: \quad \sigma^2_{t,k} = \alpha_k + \beta_k \sigma^2_{t-1,k} + \gamma_k \eta^2_{t-1,k} + L_k I \eta^2_{t-1,k} \\
&\quad\ \eta_{t,k} = \sigma_{t,k} z_{t,k}
\end{aligned}$$

where $I = 1$ if $\eta_{t-1,k} < 0$ or $I = 0$ if $\eta_{t-1,k} > 0$ and $z_{t,k} = \sqrt{\frac{\nu_k - 2}{\nu_k}} T_{\nu_k}$, T_{ν_k} follows a Student-T distribution with ν_k degrees of freedom, and $\hat{\theta} = \{a_k, b_k, \alpha_k, \beta_k, \gamma_k, \nu_k, \delta_k, L_k\}$ are Maximum Likelihood estimators.

6.1.2 Shrunk Volatility VaR model: Generalization

We have seen in the 5.1.5 how to forecast *ex-ante* VaR using a quick tool that avoid to estimate ARMA GARCH models. The main assumption we did is, as for RiMeVaR model, the asset returns are normally distributed with zero means, but that volatility forecast at time $t+1$, given information available at time t is the shrinkage between realized and implied volatility.

Here we generalize ShVolVaR model using a Student-t distribution. We assume that the return of the generic asset k, $r_{t+1|t,k} = z_{t+1|t,k} \tilde{\sigma}_{t+1|t,k}$, where $z_{t+1|t,k} = T_{\nu k} \sqrt{\frac{\nu_k - 2}{\nu_k}}$ and $T_{\nu k}$ follows a T-Student distribution with ν degrees of freedom. We consider

$$\tilde{\sigma}_{t+1|t,k}(\alpha) = (1-\alpha)\hat{\sigma}_{t,k} + \alpha \sigma^{impl}_{t,k}, \tag{6.1}$$

where $\alpha \in [0,1]$ is called *shrinkage parameter.* $\hat{\sigma}_{t,k}$ is computed as the standard deviation of the index k returns on 20 stock market days, and the implied volatility

$$\sigma^{imp}_{t,k} = (256)^{-\frac{1}{2}} V^{imp}_{t,k}/100,$$

where $V^{imp}_{t,k}$ represents the quoted implied volatility of the market index k.

In the multivariate context we assume

$$(r_{t+1|t,1}, r_{t+1|t,2}, \ldots r_{t+1|t,n}) \sim T_\nu(0, \tilde{\Sigma}_{t+1|t}(\alpha)),$$

with the covariance matrix

$$\tilde{\Sigma}_{t+1|t}(\alpha) = \text{diag}(\tilde{\sigma}_{t+1|t}(\alpha))\hat{C}_{t+1|t}\text{diag}(\tilde{\sigma}_{t+1|t}(\alpha)) \tag{6.2}$$

where $\text{diag}(\tilde{\sigma}_{t+1|t}(\alpha))$ is the diagonal matrix with shrunk volatilities of the assets on the diagonal, and $\hat{C}_{t+1|t}$ is the sample correlation matrix estimated on 20 days preceding t. Clearly, the portfolio volatility can be written as

$$\tilde{\sigma}_{t+1|t}(\alpha, x) = \sqrt{x^T\tilde{\Sigma}_{t+1|t}(\alpha)x}. \tag{6.3}$$

The portfolio VaR is as follows,

$$VaR_{\varepsilon}(\alpha, \nu, x) = \tilde{\sigma}_{t+1|t}(\alpha, x)F^{-1}(\nu, \varepsilon), \tag{6.4}$$

where F^{-1} is the inverse CDF of the standard Student-t. Note that, hidden in the ShVolVaR model there are two other models Realized Volatility VaR (ReVolVaR) choosing $\alpha = 0$ and Implied Volatility VaR (ImVolVaR) choosing $\alpha = 1$.

In Section 6.2 we will test and compare the ShVolVaR model with the Historical Filtered Bootstrap VaR (HFBVaR), the Historical Filtered Bootstrap VaR with asymmetric conditional variance (HFBEVaR and HFBGJRVaR), the Extreme Value Filtered Bootstrap VaR models (EVTHFBVaR, EVTHFBGJRVaR) and the RiskMetrics VaR (RiMeVaR), models that are considered as the benchmarks.

6.2 Empirical analysis

In this section, we present computational results for seven models: the Shrunk Volatility VaR (ShVolVaR), the Historical Filtered Bootstrap VaR (HFBVAR) and its others specification with asymmetric conditional volatility (HFBEVaR and HFBGJRVaR), Extreme Value Theory Historical Filtered Bootstrap (EVTHFBVaR and EVTHFBGJRVaR) and the RiskMetrics VaR (RiMeVaR) models. The analysis is performed on the S&P 500 and Bloomberg/Effas US Govt 7-10y Yr TR indices, in which, in addition to the index values, are also available the corresponding implied volatilities. The one-day-ahead VaR forecasts obtained by the seven models are validated by considering 39 balanced portfolios, where equity weight ranges from 2.5% to 97.5% with a step of 2.5%. Our aim is to test the Shrunk Volatility VaR model on an equally spaced combination of the aforementioned indexes. The portfolios composi-

Market	Index ticker	Implied Volatility ticker	Start date	End date	daily VaR forecasts
SP500	SPTR	VIX	28/01/2003	29/1/2016	3393
US Govt 7-10y	US4TR	TYVIX	28/01/2003	29/1/2016	3393

Table 6.1: List of the data sets analyzed.

tions are reported in table 6.2. We estimate daily VaR from January 28, 2003, to January 29, 2016. Both indexes are expressed in local currency.

More in detail, we adopt a working days calendar, and on these days we take the prices of all market indexes. If a market is closed on a specific day, e.g., for holidays, then for that day, we replicate the price with the last available value. Clearly, on that particular day, the index return will be zero. This pre-processing is required to compute a fair analysis of correlations among indexes, i.e., to avoid possible lags among the returns time series. Conversely, the realized volatilities are estimated on the original time series without this pre-processing, because unlike the correlations estimates, those of volatilities would be too influenced by this pre-processing. For completeness, we also list in Table 6.1 the tickers corresponding to each index.

We perform the VaR estimation at four confidence level in particular 95%, 99%, 99.5% and 99.9%. The former two are often utilized in asset management industry (see [35]) while the latter two are utilized in insurance (see [48]) and bank industries (see [11], [12]).

Under the main assumption (conditionally normal returns), we expect the ShVolVaR to produce good results at low confidence levels (i.e. 95% and 99%) and not-so-good results when high confidence levels are considered. This is the main pitfall of the model; the normality assumption fails to capture rare events. It is for this reason we generalize it.

6.2.1 Computational Results

In this section, we report the empirical results for multivariate VaR estimations on thirty-nine balanced portfolios, listed in Table 6.2. It is convenient to recall that low values of α tend to place more importance to the realized volatility, while for high α the one-day-ahead VaR estimation mainly depends on the implied volatility. As before we place much more importance to implied volatility when VaR confidence level increase. We decided to adopt the α proposed in the previous chapter regarding the $VaR_{1\%}$ estimation. Knowing that implied volatility outperforms past volatility in forecasting future volatility (see [39], [84] and [85]) and implied volatilities tend to overestimate future market risk, we increase the value of α as we want to estimate tail risk.

Portfolio	Weights (%)	
name	SP500	US Govt 7-10 Yr
Ptf 1	97.5	2.5
Ptf 2	95.0	5.0
Ptf 3	92.5	7.5
Ptf 4	90.0	10.0
Ptf 5	87.5	12.5
Ptf 6	85.0	15.0
Ptf 7	82.5	17.5
Ptf 8	80.0	20.0
Ptf 9	77.5	22.5
Ptf 10	75.0	25.0
Ptf 11	72.5	27.5
Ptf 12	70.0	30.0
Ptf 13	67.5	32.5
Ptf 14	65.0	35.0
Ptf 15	62.5	37.5
Ptf 16	60.0	40.0
Ptf 17	57.5	42.5
Ptf 18	55.0	45.0
Ptf 19	52.5	47.5
Ptf 20	50.0	50.0
Ptf 21	47.5	52.5
Ptf 22	45.0	55.0
Ptf 23	42.5	57.5
Ptf 24	40.0	60.0
Ptf 25	37.5	62.5
Ptf 26	35.0	65.0
Ptf 27	32.5	67.5
Ptf 28	30.0	70.0
Ptf 29	27.5	72.5
Ptf 30	25.0	75.0
Ptf 31	22.5	77.5
Ptf 32	20.0	80.0
Ptf 33	17.5	82.5
Ptf 34	15.0	85.0
Ptf 35	12.5	87.5
Ptf 36	10.0	90.0
Ptf 37	7.5	92.5
Ptf 38	5.0	95.0
Ptf 39	2.5	97.5

Table 6.2: List of the thirty-nine portfolios weights used for the multivariate VaR estimations analysis.

Summarizing, we adopt $\alpha = 0.275$ and $\alpha = 0.325$ dealing with $VaR_{5\%}$, $\alpha = 0.5$ and $\alpha = 0.6$ dealing with $VaR_{1\%}$, $\alpha = 0.75$ dealing with $VaR_{0.5\%}$ and $\alpha = 1$ dealing with $VaR_{0.1\%}$.

Initially we use $\nu = \infty$ and in Tables 6.3, 6.4, 6.5, and 6.6 we provide the p-values (expressed as a percentage) of the Unconditional Coverage (UC), the Independence (IND) and the Conditional Coverage (CC) tests at confidence levels of 95%, 99%, 99.5% and 99.9%. We report in dark-gray the cases in which we cannot accept the null hypotheses (i.e., the VaR model is not able to capture the expected frequency of violations (UC) or suffer from dependence of violations (IND), or both (CC)). In light-gray we highlight the cases in which the null hypothesis is accepted at 99% confidence level (c.l.) but rejected at 95% c.l.. The rest of the accepted cases have no one highlight.

Table 6.3 shows that the RiMeVaR model does not suffer from dependence of violations; nevertheless it fails to capture the UC hypothesis, in 12 portfolios out 39 at a 99% c.l., and we record 15 rejections of the CC null hypothesis at a 95% c.l.. The HFBVaR model is within the region of acceptance of the UC test at a 99% c.l. for all combinations. Considering the IND hypothesis, we observe 2 rejections (out of 39) at 99% c.l. and 5 rejection at 95%. Consequently, we record 4 rejections of the CC hypothesis at 95% c.l.. The HFBEVaR and HFBGJRVaR do not suffer from dependence but they fail to capture UC and CC hypothesis in more than 20 portfolios. Considering the ShVolVaR model with $\alpha = 0.275$, and with $\alpha = 0.325$ we observe no rejections.

Table 6.4 shows that the RiMeVaR model does not suffer from dependence of violations, but it fails to capture the UC and CC hypothesis in all portfolios. The HFBVaR model is within the region of acceptance of the UC and CC test at 99% c.l. for all combinations. Considering a 95% c.l., we observe 2 rejections (out of 39) of the IND hypothesis. The HFBEVaR exhibits only 2 rejections of the UC test at 95% c.l.. HFBGJRVaR garners 2 rejections (out of 39) of the UC test at 99% c.l. and 10 rejections of the UC test at 95% c.l. and, consequently, the CC is rejected 11 times at 95% c.l.. Considering the ShVolVaR model with $\alpha = 0.5$, the UC and CC tests are rejected at 99% c.l. when equity weight is between 5% and 37.5%. When using an $\alpha = 0.6$ we observe more rejections cases.

Table 6.5 shows that, as before, the RiMeVaR model does not suffer from dependence of violations; however it fails to capture the UC and CC hypothesis. The HFBVaR model is within the region of acceptance of the UC and CC test at 99% c.l. for all combinations. Considering a 95% c.l. we observe 1 rejections (out of 39) of the CC hypothesis due to one rejection during the IND test. HFBGJRVaR has no rejections while HFBEVaR shows 2 rejections of the UC test at 99% c.l. and 5 rejections of the UC test at 95% c.l.. Consequently, for the CC test we record 4 rejections at a 95% c.l. when the equity weight is greater than 90%. Considering the ShVolVaR model with $\alpha = 0.75$, the UC and CC test are rejected at 99% c.l. when the equity weight is between 2.5% and 40%. The IND test is rejected at a 95% c.l. when the equity weight is between 42.5% and 65%,

Table 6.6 shows that the RiMeVaR model suffers from dependence of violations when equity weight is greater than 45% and fails to capture the UC and CC hypothesis. The HFBVaR demonstrates 5 rejections out of 39 for the CC test when the equity weight is between 37.5% and 50%. Considering a 95% c.l., we observe many rejections of the UC hypothesis and, consequently, the CC hypothesis. HFBGJRVaR has only 4 rejections of the UC and CC tests when equity is less than 10%, while HFBEVaR has 10 rejections of the UC test at 99% c.l. and 2 rejections of the UC test at 95% c.l.. Consequently, the CC exhibits 9 rejection at a confidence level of 95% when the equity weight is greater than 67.5%. Considering the ShVolVaR model with $\alpha = 1$, the UC and CC tests are rejected at 99% c.l. when the equity weight is between 2.5% and 37.5%.

As expected, ShVolVaR, performance get worse as confidence level increases. The normality assumption cannot guarantee a perfect VaR forecast. RiMeVaR, which operates under the same assumption, give the worst performances of all the considered models. ShVolVaR increases the quality of VaR estimation whit respect to RiMeVaR. HFB models that assume Student-t innovations better predict rare events. Looking at a more complete set of predictions, we can state that Risk Metrics procedure is able to foresee VaR only at 95% c.l.; after this, then its performances quikly decline. Symmetric HFB offers good VaR estimations for all confidence levels; however, at 99.9% it loses some prediction power. Asymmetric HFB gives its best performances when evaluating extreme VaRs (99.9% and 99.5%), in which case the GJR specification seems to be better than EGARCH specification; the opposite is true when considering the 99% c.l.. ShVolVaR is able to capture markets movements and correctly estimate any confidence level, but only when the equity weight is greater than 55%. When the equity weight decreases, the model becomes less efficient at forecasting VaR over a 95% c.l..

We now focus on 9 balanced portfolios commonly used in the industry namely Ptf 8 (Table 6.7), 12 (Table 6.8), 16 (Table 6.9), 20 (Table 6.10), 24 (Table 6.11), 28 (Table 6.11) and 32 (Table 6.13). First, no model exists that is able to capture any of aforementioned balanced portfolios at all the VaR levels. This could mean that the joint estimation of equity and bond risk is difficult. Second, RiMeVaR is only a good choice when estimating $VaR_{5\%}$.

Let us consider a portfolio manager that has a perfectly capital diversified portfolio (50/50) (see Table 6.10) and has to choose which risk models to use to forecast VaR. No one of the considered models can ensure a perfect forecast of each percentile at 95% c.l.. ShVolVaR may be the best choice at 99% c.l.. HFBVaR may also be useful if the portfolio manager is interested in common VaR activity in the finance industry, while the asymmetric version of HFB would help her to understand rare events. For a more conservative portfolio (table 6.13) in which equity represents the 20% of its makeup, HFBVaR should be utilized, as this allows us to perfectly foresee all of the VaR levels considered. ShVolVaR fails to catch market movements in the tail and should only be utilized for $VaR_{5\%}$. For a more aggressive portfolio (table 6.8) in which equity represents the 70% of its makeup, HFBVaR or HGBGJRVaR should be utilized, as these allow us to perfectly foresee the all the VaR levels considered. ShVolVaR is able to reach the same results of VaR forecasting in such case, so it cab be used as a practical tool to preliminary VaR estimation.

In general, balanced mutual fund managers should use a HFBVaR as risk management system, but they could easily adopt the ShVolVaR (with $\nu = \infty$) to approximate the VaR. When a portfolio is held by an insurance company the risk management tool should be an asymmetric version of HFB and the ShVolVaR (under normality assumption) could be a good proxy if the equity weight is not too low.

6.2.1.1 ShVolVaR $\nu = 20, 16, 10, 8$

In this section we compare some finite values of ν; in particular $\nu = 20, 16, 10, 8$.

In Table 6.14 we report the CDF used in the equation 6.4. We observe that with $\varepsilon = 5\%$ the value of $F^{-1}(\varepsilon)$ are quite similar. However, with $\varepsilon = 0.1\%$ the values of $F^{-1}(\varepsilon)$ increase quickly as degrees of freedom decrease. This feature allows the ShVolVaR model to better capture rare events, while at 95% c.l. we expect no relevant changes in the VaR estimation.

Table 6.15 shows the test results for $VaR_{5\%}$. Using finite values of ν, there are no differences between VaR estimations, and no specifications produce a rejections. Considering $VaR_{1\%}$ (see Table 6.16), we observe that there is no clear advantage to having finite values of ν over infinite. In particular, when the equity weight is high, and we use $\nu = 20, 16, 10, 8$ we observe many independence violations. It seems advisable to use a normal specification when the equity weight is higher than 45%. Conversely, if the equity weight is lower than 45% one should adopt $\nu = 8$. In Table 6.17 we report our results for

Ptfs	ShVolVaR $\alpha = 0.275, \nu = \infty$			ShVolVaR $\alpha = 0.325, \nu = \infty$			RiMeVaR			HFBEVaR			HFBGJRVaR			HFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	79.0	29.0	55.2	96.2	35.7	65.4	0.4	61.8	1.5	73.0	47.9	73.3	56.3	79.3	81.7	51.1	40.0	56.5
Ptf 2	73.0	27.5	51.9	83.7	39.4	68.1	0.9	47.8	2.5	67.2	50.2	73.0	85.0	65.5	88.9	56.3	41.9	61.0
Ptf 3	56.3	41.9	61.0	90.0	37.5	67.0	0.7	45.6	2.0	61.6	52.6	72.2	85.0	65.5	88.9	73.0	48.2	73.6
Ptf 4	91.2	54.9	83.1	83.7	64.6	88.1	0.7	45.6	2.0	61.6	52.6	72.2	83.7	52.7	80.2	56.3	93.4	84.3
Ptf 5	91.2	54.9	83.1	83.7	64.6	88.1	0.7	28.4	1.5	85.0	43.3	72.2	77.6	50.3	76.7	73.0	97.9	94.2
Ptf 6	73.0	48.2	73.6	65.7	72.4	85.2	0.7	45.6	2.0	96.2	36.9	66.8	71.6	47.9	72.9	79.0	95.0	96.3
Ptf 7	91.2	54.9	83.1	60.1	47.5	67.5	0.6	43.5	1.6	83.7	33.0	61.0	65.7	69.5	83.9	61.6	76.5	84.3
Ptf 8	83.7	39.4	68.1	39.9	56.4	59.3	0.7	45.6	2.0	71.6	29.4	53.9	35.6	80.7	63.3	41.7	87.9	71.1
Ptf 9	65.7	45.4	68.4	39.9	56.4	59.3	0.7	45.6	2.0	54.7	24.4	42.3	27.8	75.1	52.8	46.3	87.7	75.5
Ptf 10	54.7	49.6	66.2	27.8	63.7	49.6	0.4	41.5	1.2	35.6	33.1	40.7	18.4	96.9	41.3	56.3	93.4	84.3
Ptf 11	49.5	51.8	64.3	27.8	63.7	49.6	0.6	26.9	1.1	27.8	49.1	43.7	13.6	76.7	31.5	56.3	93.4	84.3
Ptf 12	49.5	51.8	64.3	31.5	35.2	39.2	0.9	30.0	1.8	15.8	40.1	26.0	3.2	99.0	9.9	56.3	93.4	84.3
Ptf 13	49.5	51.8	64.3	24.3	38.9	34.9	1.1	31.6	2.3	15.8	40.1	26.0	2.6	98.1	8.2	67.2	73.7	86.4
Ptf 14	60.1	47.5	67.5	24.3	38.9	34.9	1.3	19.1	2.0	11.6	59.0	25.1	0.8	83.8	2.9	85.0	92.1	97.8
Ptf 15	60.1	47.5	67.5	18.4	42.7	30.2	2.4	22.9	3.8	6.9	51.5	15.4	1.0	86.7	3.6	91.2	82.2	96.9
Ptf 16	49.5	51.8	64.3	21.2	40.8	32.6	5.0	28.6	8.4	4.7	46.8	10.7	0.2	69.9	0.8	37.4	56.2	57.0
Ptf 17	44.5	54.1	62.0	15.8	44.7	27.7	4.2	44.9	9.5	0.6	28.6	1.3	0.1	61.9	0.3	37.4	82.1	65.7
Ptf 18	21.2	68.8	42.4	11.6	48.9	22.8	3.5	42.8	7.9	0.4	25.2	0.8	0.0	42.6	0.0	41.7	84.9	70.7
Ptf 19	60.1	47.5	67.5	21.2	40.8	32.6	2.4	38.8	5.4	0.1	35.4	0.3	0.0	63.1	0.0	20.3	94.7	44.4
Ptf 20	44.5	54.1	62.0	13.6	46.8	25.3	8.4	77.8	21.5	0.1	37.4	0.3	0.0	72.4	0.0	8.4	96.6	22.4
Ptf 21	60.1	47.5	67.5	18.4	42.7	30.2	13.3	86.1	31.9	0.0	73.9	0.0	0.0	83.0	0.0	11.5	90.8	28.6
Ptf 22	54.7	49.6	66.2	15.8	44.7	27.7	37.4	82.1	65.7	0.0	98.5	0.0	0.0	65.1	0.0	26.3	73.7	50.5
Ptf 23	35.6	88.9	64.6	15.8	94.0	36.9	41.7	87.9	71.1	0.0	76.0	0.0	0.0	54.8	0.0	23.2	21.6	22.7
Ptf 24	31.5	77.9	58.0	18.4	66.9	37.7	91.2	25.0	51.3	0.0	67.8	0.0	0.0	47.6	0.0	46.3	1.8	4.7
Ptf 25	21.2	69.6	42.5	8.2	53.9	18.3	79.0	28.3	54.2	0.0	15.6	0.0	0.0	47.6	0.0	79.0	0.9	3.2
Ptf 26	49.5	61.5	69.8	27.8	49.1	43.7	29.7	48.7	45.6	0.0	39.2	0.0	0.0	21.4	0.0	97.5	1.5	5.2
Ptf 27	60.1	95.0	87.0	49.5	51.8	64.3	23.2	76.5	46.7	0.0	31.3	0.0	0.0	8.2	0.0	83.7	2.4	7.7
Ptf 28	65.7	8.1	19.7	49.5	12.4	24.2	61.6	33.7	55.7	0.0	13.4	0.0	0.0	8.9	0.0	65.7	0.7	2.5
Ptf 29	65.7	27.6	50.1	44.5	21.4	34.5	56.3	21.5	39.3	0.0	27.7	0.0	0.0	4.5	0.0	54.7	1.4	4.1
Ptf 30	71.6	8.8	21.8	44.5	5.7	12.1	73.0	30.0	55.1	0.0	29.5	0.0	0.0	10.6	0.0	71.6	2.0	6.1
Ptf 31	65.7	15.5	33.0	60.1	14.4	30.0	67.2	10.4	24.3	0.0	12.4	0.0	0.0	12.4	0.0	44.5	11.4	21.5
Ptf 32	79.0	16.3	36.5	79.0	16.3	36.5	56.3	3.2	8.4	0.0	14.5	0.0	0.0	67.8	0.0	44.5	11.4	21.5
Ptf 33	67.2	18.8	38.5	79.0	28.3	54.2	51.1	13.1	25.8	0.0	37.1	0.0	0.0	65.1	0.0	35.6	9.7	16.5
Ptf 34	90.0	80.7	96.3	83.7	52.7	80.2	33.4	46.4	48.0	0.0	78.8	0.0	0.0	81.6	0.0	31.5	17.4	24.0
Ptf 35	79.0	68.2	88.7	90.0	90.6	98.5	15.4	41.3	25.9	0.0	25.4	0.0	0.0	67.3	0.0	54.7	41.2	59.5
Ptf 36	97.5	86.4	98.5	96.2	83.5	97.8	17.7	82.2	39.2	0.0	90.1	0.0	0.0	18.8	0.0	21.2	69.6	42.5
Ptf 37	79.0	45.6	73.0	90.0	55.2	83.1	17.7	58.6	34.7	0.3	92.2	1.1	0.1	11.7	0.2	35.6	53.9	54.0
Ptf 38	90.0	55.2	83.1	71.6	72.3	87.9	41.7	87.9	71.1	5.7	28.5	9.2	1.0	45.6	2.8	27.8	49.1	43.7
Ptf 39	61.6	68.7	81.3	91.2	82.2	96.9	13.3	60.9	28.4	31.5	31.2	36.2	9.8	85.4	25.0	31.5	91.7	60.0

Table 6.3: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 95%

Ptfs	ShVolVaR $\alpha = 0.5, \nu = \infty$			ShVolVaR $\alpha = 0.6, \nu = \infty$			RiMeVaR			HFBEVaR			HFBGJRVaR			HFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	49.0	7.5	16.1	73.8	3.7	10.7	0.0	46.6	0.0	23.7	31.7	30.1	85.3	39.3	68.2	13.3	12.2	9.8
Ptf 2	30.8	9.2	14.4	49.0	2.8	7.1	0.0	49.0	0.0	23.7	31.7	30.1	72.2	38.0	63.8	9.6	13.3	8.1
Ptf 3	30.8	9.2	14.4	60.9	3.2	8.9	0.0	49.0	0.0	49.0	35.3	51.2	72.2	38.0	63.8	17.9	54.9	33.9
Ptf 4	23.7	10.1	13.0	60.9	3.2	8.9	0.0	49.0	0.0	30.8	49.7	47.2	87.3	42.1	71.4	17.9	54.9	33.9
Ptf 5	23.7	10.1	13.0	49.0	2.8	7.1	0.0	53.9	0.0	49.0	44.7	59.0	73.8	43.5	69.7	23.7	52.3	40.5
Ptf 6	30.8	9.2	14.4	38.4	2.4	5.4	0.0	53.9	0.0	49.0	44.7	59.0	38.4	47.9	53.3	23.7	52.3	40.5
Ptf 7	23.7	10.1	13.0	49.0	2.8	7.1	0.0	49.0	0.0	49.0	44.7	59.0	29.2	49.5	45.5	23.7	52.3	40.5
Ptf 8	30.8	9.2	14.4	60.9	3.2	8.9	0.0	46.6	0.0	60.0	42.3	63.3	29.2	49.5	45.5	17.9	54.9	33.9
Ptf 9	39.2	8.3	15.4	73.8	3.7	10.7	0.0	44.3	0.0	49.0	44.7	59.0	29.2	49.5	45.5	17.9	54.9	33.9
Ptf 10	39.2	8.3	15.4	87.3	4.2	12.5	0.0	46.6	0.0	60.0	42.3	63.3	21.6	51.0	37.4	23.7	52.3	40.5
Ptf 11	49.0	7.5	16.1	98.9	4.8	14.0	0.0	19.4	0.0	85.3	37.6	66.5	29.2	49.5	45.5	30.8	49.7	47.2
Ptf 12	72.2	6.0	16.0	98.9	4.8	14.0	0.0	20.9	0.0	72.2	39.9	65.8	15.4	52.6	29.6	30.8	49.7	47.2
Ptf 13	98.9	4.8	14.0	87.3	4.2	12.5	0.0	20.9	0.0	60.0	42.3	63.3	7.1	55.9	16.5	17.9	11.1	11.4
Ptf 14	85.3	5.4	15.3	87.3	4.2	12.5	0.0	18.1	0.0	72.2	39.9	65.8	4.5	57.5	11.5	13.3	57.6	27.6
Ptf 15	98.9	4.8	14.0	38.4	2.4	5.4	0.0	16.8	0.0	87.3	33.2	61.6	7.1	55.9	16.5	17.9	11.1	11.4
Ptf 16	85.3	0.5	2.0	49.0	0.2	0.7	0.0	39.9	0.0	87.3	33.2	61.6	10.6	54.2	22.6	23.7	10.1	13.0
Ptf 17	85.3	0.5	2.0	29.2	0.1	0.3	0.0	29.8	0.0	98.9	35.4	65.0	10.6	17.9	11.0	30.8	9.2	14.4
Ptf 18	73.8	0.3	1.1	49.0	0.2	0.7	0.0	33.7	0.0	73.8	31.0	56.5	1.7	12.0	1.7	23.7	10.1	13.0
Ptf 19	73.8	0.3	1.1	60.9	0.2	0.9	0.0	31.7	0.0	87.3	33.2	61.6	4.5	14.8	4.8	39.2	8.3	15.4
Ptf 20	98.9	0.4	1.7	49.0	0.2	0.7	0.0	26.3	0.0	73.8	31.0	56.5	1.7	12.0	1.7	30.8	9.2	14.4
Ptf 21	39.2	1.0	2.5	72.2	0.6	2.2	0.0	18.6	0.0	60.9	45.0	65.9	1.7	12.0	1.7	85.3	5.4	15.3
Ptf 22	17.9	1.6	2.2	39.2	1.0	2.5	0.0	18.6	0.0	29.2	49.5	45.5	7.1	16.3	7.4	85.3	37.6	66.5
Ptf 23	4.8	2.7	1.2	72.2	0.6	2.2	0.0	4.9	0.0	38.4	47.9	53.3	7.1	16.3	7.4	98.9	35.4	65.0
Ptf 24	9.6	2.0	1.7	30.8	9.2	14.4	0.0	5.4	0.0	29.2	49.5	45.5	4.5	14.8	4.8	87.3	33.2	61.6
Ptf 25	2.2	3.4	0.8	23.7	10.1	13.0	0.0	6.0	0.0	29.2	49.5	45.5	4.5	14.8	4.8	60.0	6.7	16.3
Ptf 26	6.9	2.3	1.5	17.9	1.6	2.2	0.0	15.9	0.0	15.4	52.6	29.6	2.8	13.4	2.9	49.0	7.5	16.1
Ptf 27	1.0	4.3	0.5	3.3	3.0	1.0	0.0	2.2	0.0	2.8	59.2	7.7	1.7	12.0	1.7	39.2	8.3	15.4
Ptf 28	0.2	5.9	0.2	1.0	4.3	0.5	0.0	13.6	0.0	4.5	57.5	11.5	0.5	64.4	1.8	72.2	6.0	16.0
Ptf 29	0.1	8.0	0.1	0.2	5.9	0.2	0.0	43.0	0.0	15.4	52.6	29.6	0.3	66.1	1.0	98.9	4.8	14.0
Ptf 30	0.0	43.0	0.0	0.0	38.5	0.0	0.0	62.8	0.0	29.2	23.1	28.0	2.8	13.4	2.9	73.8	3.7	10.7
Ptf 31	0.0	17.2	0.0	0.0	12.5	0.0	0.0	28.0	0.0	29.2	23.1	28.0	15.4	19.6	15.7	60.9	28.9	50.1
Ptf 32	0.0	43.0	0.0	0.0	40.7	0.0	0.0	62.8	0.0	21.6	21.3	21.4	15.4	19.6	15.7	85.3	37.6	66.5
Ptf 33	0.0	50.1	0.0	0.0	38.5	0.0	0.0	68.2	0.0	21.6	21.3	21.4	15.4	19.6	15.7	49.0	44.7	59.0
Ptf 34	0.0	47.6	0.0	0.0	45.3	0.0	0.0	55.1	0.0	15.4	19.6	15.7	15.4	19.6	15.7	60.0	42.3	63.3
Ptf 35	0.0	14.8	0.0	0.0	99.3	0.1	0.0	79.0	0.0	38.4	25.0	35.3	15.4	19.6	15.7	49.0	44.7	59.0
Ptf 36	0.0	16.3	0.1	0.0	16.3	0.1	0.0	71.4	0.0	60.0	42.3	63.3	60.9	28.9	50.1	87.3	33.2	61.6
Ptf 37	0.0	96.5	0.1	0.0	96.5	0.1	0.0	79.0	0.0	60.0	42.3	63.3	49.0	26.9	42.8	85.3	37.6	66.5
Ptf 38	0.1	93.7	0.2	0.1	88.1	0.6	0.0	79.0	0.0	72.2	39.9	65.8	73.8	31.0	56.5	98.9	35.4	65.0
Ptf 39	2.2	71.2	6.8	3.3	68.4	9.5	0.0	89.7	0.0	60.0	42.3	63.3	85.3	37.6	66.5	98.9	35.4	65.0

Table 6.4: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99%

Ptfs	ShVolVaR $\alpha = 0.75, \nu = \infty$			RiMeVaR			HFBEVaR			HFBGJRVaR			HFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	12.1	78.9	29.0	0.0	26.6	0.0	1.4	49.5	3.9	62.6	71.5	83.1	10.7	16.3	10.3
Ptf 2	12.1	78.9	29.0	0.0	24.9	0.0	0.8	47.9	2.3	99.2	67.9	91.8	16.3	57.5	32.3
Ptf 3	12.1	78.9	29.0	0.0	24.9	0.0	0.8	47.9	2.3	99.2	67.9	91.8	24.1	59.2	43.5
Ptf 4	20.3	77.0	42.5	0.0	26.6	0.0	1.4	49.5	3.9	80.2	66.1	88.0	16.3	57.5	32.3
Ptf 5	20.3	77.0	42.5	0.0	26.6	0.0	2.5	51.0	6.4	99.2	67.9	91.8	16.3	57.5	32.3
Ptf 6	20.3	77.0	42.5	0.0	24.9	0.0	4.1	52.6	10.2	80.2	66.1	88.0	24.1	59.2	43.5
Ptf 7	20.3	77.0	42.5	0.0	28.4	0.0	4.1	52.6	10.2	99.2	67.9	91.8	10.7	55.9	23.0
Ptf 8	12.1	78.9	29.0	0.0	28.4	0.0	6.8	54.2	15.6	81.4	69.7	90.2	10.7	55.9	23.0
Ptf 9	12.1	78.9	29.0	0.0	24.9	0.0	10.7	55.9	23.0	81.4	69.7	90.2	6.8	54.2	15.6
Ptf 10	12.1	78.9	29.0	0.0	26.6	0.0	10.7	16.3	10.3	31.5	75.2	57.4	16.3	57.5	32.3
Ptf 11	31.5	75.2	57.4	0.0	23.2	0.0	24.1	13.4	16.3	62.6	71.5	83.1	24.1	59.2	43.5
Ptf 12	20.3	77.0	42.5	0.0	23.2	0.0	34.3	12.0	19.1	45.7	73.3	71.6	16.3	57.5	32.3
Ptf 13	20.3	77.0	42.5	0.0	21.6	0.0	34.3	12.0	19.1	31.5	75.2	57.4	24.1	59.2	43.5
Ptf 14	20.3	3.3	4.5	0.0	3.4	0.0	62.6	64.4	79.8	20.3	77.0	42.5	16.3	57.5	32.3
Ptf 15	12.1	2.7	2.6	0.0	3.0	0.0	62.6	64.4	79.8	20.3	77.0	42.5	10.7	55.9	23.0
Ptf 16	31.5	3.9	7.2	0.0	3.4	0.0	47.2	62.6	68.6	6.6	80.8	18.0	16.3	57.5	32.3
Ptf 17	31.5	3.9	7.2	0.0	2.3	0.0	81.4	69.7	90.2	6.6	80.8	18.0	16.3	57.5	32.3
Ptf 18	45.7	4.7	10.5	0.0	2.3	0.0	81.4	69.7	90.2	20.3	77.0	42.5	62.6	64.4	79.8
Ptf 19	62.6	0.1	0.5	0.0	2.0	0.0	31.5	75.2	57.4	31.5	75.2	57.4	34.3	12.0	19.1
Ptf 20	99.2	0.2	1.0	0.0	2.7	0.0	62.6	71.5	83.1	45.7	73.3	71.6	47.2	10.8	21.1
Ptf 21	99.2	0.2	1.0	0.0	1.3	0.0	81.4	69.7	90.2	20.3	77.0	42.5	34.3	12.0	19.1
Ptf 22	99.2	0.2	1.0	0.0	1.0	0.0	99.2	67.9	91.8	6.6	80.8	18.0	47.2	10.8	21.1
Ptf 23	10.7	1.1	1.1	0.0	1.6	0.0	81.4	69.7	90.2	12.1	78.9	29.0	34.3	12.0	19.1
Ptf 24	2.5	1.8	0.5	0.0	0.2	0.0	81.4	69.7	90.2	20.3	77.0	42.5	34.3	12.0	19.1
Ptf 25	6.8	17.9	7.6	0.0	12.2	0.0	99.2	67.9	91.8	31.5	75.2	57.4	47.2	10.8	21.1
Ptf 26	0.4	26.9	0.9	0.0	11.1	0.0	99.2	67.9	91.8	45.7	73.3	71.6	80.2	8.4	21.8
Ptf 27	0.1	31.0	0.3	0.0	13.3	0.0	99.2	67.9	91.8	12.1	78.9	29.0	62.6	5.5	14.1
Ptf 28	0.1	31.0	0.3	0.0	13.3	0.0	31.5	75.2	57.4	12.1	78.9	29.0	62.6	5.5	14.1
Ptf 29	0.0	35.4	0.1	0.0	14.5	0.0	45.7	73.3	71.6	20.3	77.0	42.5	80.2	0.3	1.2
Ptf 30	0.0	47.2	0.0	0.0	65.7	0.0	45.7	73.3	71.6	6.6	80.8	18.0	47.2	10.8	21.1
Ptf 31	0.0	54.9	0.0	0.0	25.0	0.0	62.6	71.5	83.1	20.3	77.0	42.5	62.6	9.6	22.1
Ptf 32	0.0	14.5	0.0	0.0	24.0	0.0	81.4	69.7	90.2	31.5	75.2	57.4	80.2	8.4	21.8
Ptf 33	0.0	13.3	0.0	0.0	19.5	0.0	47.2	62.6	68.6	45.7	73.3	71.6	34.3	12.0	19.1
Ptf 34	0.0	13.3	0.0	0.0	20.3	0.0	62.6	64.4	79.8	62.6	71.5	83.1	47.2	10.8	21.1
Ptf 35	0.0	32.9	0.0	0.0	21.2	0.0	80.2	66.1	88.0	62.6	71.5	83.1	34.3	12.0	19.1
Ptf 36	0.0	36.6	0.0	0.0	19.5	0.0	62.6	64.4	79.8	81.4	69.7	90.2	34.3	12.0	19.1
Ptf 37	0.0	39.3	0.0	0.0	27.1	0.0	47.2	62.6	68.6	81.4	69.7	90.2	47.2	10.8	21.1
Ptf 38	0.0	39.3	0.0	0.0	36.6	0.0	80.2	66.1	88.0	62.6	71.5	83.1	47.2	10.8	21.1
Ptf 39	0.4	46.4	1.3	0.0	40.7	0.1	80.2	66.1	88.0	81.4	69.7	90.2	24.1	59.2	43.5

Table 6.5: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.5%

Ptfs	ShVolVaR $\alpha = 1, \nu = \infty$			RiMeVaR			HFBEVaR			HFBGJRVaR			HFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	74.8	92.3	94.5	0.0	1.1	0.0	0.0	75.2	0.0	41.5	90.3	71.2	3.4	84.6	10.3
Ptf 2	74.8	92.3	94.5	0.0	1.1	0.0	0.0	75.2	0.0	41.5	90.3	71.2	3.4	84.6	10.3
Ptf 3	74.8	92.3	94.5	0.0	0.9	0.0	0.0	75.2	0.0	41.5	90.3	71.2	3.4	84.6	10.3
Ptf 4	74.8	92.3	94.5	0.0	0.7	0.0	0.0	75.2	0.0	41.5	90.3	71.2	1.2	82.7	4.1
Ptf 5	41.5	90.3	71.2	0.0	0.7	0.0	0.0	75.2	0.0	74.8	92.3	94.5	1.2	82.7	4.1
Ptf 6	41.5	90.3	71.2	0.0	0.5	0.0	0.0	75.2	0.0	82.8	94.2	97.4	1.2	82.7	4.1
Ptf 7	41.5	90.3	71.2	0.0	0.5	0.0	0.0	77.0	0.1	82.8	94.2	97.4	1.2	82.7	4.1
Ptf 8	41.5	90.3	71.2	0.0	0.5	0.0	0.0	77.0	0.1	82.8	94.2	97.4	1.2	82.7	4.1
Ptf 9	74.8	92.3	94.5	0.0	0.6	0.0	0.1	78.9	0.5	82.8	94.2	97.4	3.4	84.6	10.3
Ptf 10	74.8	92.3	94.5	0.0	0.5	0.0	0.4	80.8	1.4	82.8	94.2	97.4	1.2	82.7	4.1
Ptf 11	74.8	92.3	94.5	0.0	0.5	0.0	1.2	82.7	4.1	82.8	94.2	97.4	3.4	84.6	10.3
Ptf 12	74.8	92.3	94.5	0.0	0.3	0.0	1.2	82.7	4.1	74.8	92.3	94.5	3.4	84.6	10.3
Ptf 13	74.8	92.3	94.5	0.0	0.4	0.0	1.2	82.7	4.1	74.8	92.3	94.5	3.4	84.6	10.3
Ptf 14	74.8	92.3	94.5	0.0	0.2	0.0	8.7	86.5	22.8	74.8	92.3	94.5	41.5	90.3	71.2
Ptf 15	82.8	94.2	97.4	0.0	0.1	0.0	8.7	86.5	22.8	82.8	94.2	97.4	41.5	90.3	71.2
Ptf 16	82.8	94.2	97.4	0.0	0.2	0.0	20.2	88.4	43.8	82.8	94.2	97.4	8.7	86.5	22.8
Ptf 17	82.8	94.2	97.4	0.0	0.2	0.0	20.2	88.4	43.8	82.8	94.2	97.4	0.4	80.8	1.4
Ptf 18	82.8	94.2	97.4	0.0	0.2	0.0	20.2	88.4	43.8	12.6	98.1	31.0	1.2	82.7	4.1
Ptf 19	74.8	92.3	94.5	0.0	0.4	0.0	41.5	90.3	71.2	12.6	98.1	31.0	3.4	84.6	10.3
Ptf 20	41.5	90.3	71.2	0.0	0.5	0.0	41.5	90.3	71.2	12.6	98.1	31.0	3.4	1.3	0.5
Ptf 21	74.8	92.3	94.5	0.0	0.4	0.0	41.5	90.3	71.2	12.6	98.1	31.0	20.2	0.6	1.1
Ptf 22	74.8	92.3	94.5	0.0	0.5	0.0	41.5	90.3	71.2	12.6	98.1	31.0	8.7	0.9	0.8
Ptf 23	41.5	90.3	71.2	0.0	10.8	0.0	74.8	92.3	94.5	12.6	98.1	31.0	8.7	0.9	0.8
Ptf 24	20.2	88.4	43.8	0.0	13.4	0.0	82.8	94.2	97.4	12.6	98.1	31.0	3.4	1.3	0.5
Ptf 25	3.4	84.6	10.3	0.0	17.9	0.0	82.8	94.2	97.4	41.3	96.1	71.4	0.4	2.1	0.1
Ptf 26	0.4	80.8	1.4	0.0	1.5	0.0	74.8	92.3	94.5	12.6	98.1	31.0	8.7	86.5	22.8
Ptf 27	0.0	77.0	0.1	0.0	1.1	0.0	74.8	92.3	94.5	41.3	96.1	71.4	8.7	86.5	22.8
Ptf 28	0.0	77.0	0.1	0.0	1.3	0.0	41.5	90.3	71.2	82.8	94.2	97.4	8.7	86.5	22.8
Ptf 29	0.0	71.5	0.0	0.0	55.9	0.0	41.5	90.3	71.2	82.8	94.2	97.4	8.7	86.5	22.8
Ptf 30	0.0	66.1	0.0	0.0	57.5	0.0	20.2	88.4	43.8	74.8	92.3	94.5	20.2	88.4	43.8
Ptf 31	0.0	64.4	0.0	0.0	57.5	0.0	20.2	88.4	43.8	41.5	90.3	71.2	8.7	86.5	22.8
Ptf 32	0.0	62.6	0.0	0.0	54.2	0.0	20.2	88.4	43.8	41.5	90.3	71.2	8.7	86.5	22.8
Ptf 33	0.0	59.2	0.0	0.0	54.2	0.0	8.7	86.5	22.8	41.5	90.3	71.2	8.7	86.5	22.8
Ptf 34	0.0	59.2	0.0	0.0	59.2	0.0	8.7	86.5	22.8	20.2	88.4	43.8	1.2	82.7	4.1
Ptf 35	0.0	59.2	0.0	0.0	60.9	0.0	3.4	84.6	10.3	20.2	88.4	43.8	3.4	84.6	10.3
Ptf 36	0.0	64.4	0.0	0.0	60.9	0.0	1.2	82.7	4.1	1.2	82.7	4.1	3.4	84.6	10.3
Ptf 37	0.0	69.7	0.0	0.0	62.6	0.0	1.2	82.7	4.1	1.2	82.7	4.1	1.2	82.7	4.1
Ptf 38	0.0	71.5	0.0	0.0	66.1	0.0	0.4	80.8	1.4	1.2	82.7	4.1	0.4	80.8	1.4
Ptf 39	0.1	78.9	0.5	0.0	69.7	0.0	0.4	80.8	1.4	1.2	82.7	4.1	0.4	80.8	1.4

Table 6.6: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.9%

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu = \infty$	68.1	14.4	29.0	71.2
RiMeVaR	2.0	0.0	0.0	0.0
HFBEVaR	53.9	63.3	15.6	0.1
HFBGJRVaR	63.3	45.5	90.2	97.4
HFBVaR	71.1	33.9	23.0	4.1

Table 6.7: The p-values (%) of the Conditional Coverage tests for the Ptf 8 (equity weight 80%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	64.3	16.0	42.5	94.5
RiMeVaR	1.8	0.0	0.0	0.0
HFBEVaR	26.0	65.8	19.1	4.1
HFBGJRVaR	9.9	29.6	71.6	94.5
HFBVaR	84.3	47.2	32.3	10.3

Table 6.8: The p-values (%) of the Conditional Coverage tests for the Ptf 12 (equity weight 70%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	64.3	2.0	7.2	97.4
RiMeVaR	8.4	0.0	0.0	0.0
HFBEVaR	10.7	61.6	68.6	43.8
HFBGJRVaR	0.8	22.6	18.0	97.4
HFBVaR	57.0	13.0	32.3	22.8

Table 6.9: The p-values (%) of the Conditional Coverage tests for the Ptf 16 (equity weight 60%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	62.0	1.7	1.0	71.2
RiMeVaR	21.5	0.0	0.0	0.0
HFBEVaR	0.3	56.5	83.1	71.2
HFBGJRVaR	0.0	1.7	71.6	31.0
HFBVaR	22.4	14.4	21.1	0.5

Table 6.10: The p-values (%) of the Conditional Coverage tests for the Ptf 20 (equity weight 50%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	58.0	1.7	0.5	43.8
RiMeVaR	51.3	0.0	0.0	0.0
HFBEVaR	0.0	45.5	90.2	97.4
HFBGJRVaR	0.0	4.8	42.5	31.0
HFBVaR	4.7	61.6	19.1	0.5

Table 6.11: The p-values (%) of the Conditional Coverage tests for the Ptf 24 (equity weight 40%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	19.7	0.2	0.3	0.1
RiMeVaR	55.7	0.0	0.0	0.0
HFBEVaR	0.0	11.5	57.4	71.2
HFBGJRVaR	0.0	1.8	29.0	97.4
HFBVaR	2.5	16.0	14.1	22.8

Table 6.12: The p-values (%) of the Conditional Coverage tests for the Ptf 28 (equity weight 30%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

Models	$VaR_{5\%}$	$VaR_{1\%}$	$VaR_{0.5\%}$	$VaR_{0.1\%}$
ShVolVaR $\nu=\infty$	36.5	0.0	0.0	0.0
RiMeVaR	8.4	0.0	0.0	0.0
HFBEVaR	0.0	21.4	90.2	43.8
HFBGJRVaR	0.0	15.7	57.4	71.2
HFBVaR	21.5	66.5	21.8	22.8

Table 6.13: The p-values (%) of the Conditional Coverage tests for the Ptf 32 (equity weight 20%). ShVolVal with $\alpha = 0.275$ ($VaR_{5\%}$), $\alpha = 0.5$ ($VaR_{1\%}$), $\alpha = 0.75$ ($VaR_{0.5\%}$) and $\alpha = 1$ ($VaR_{0.1\%}$.)

$VaR_{0.5\%}$. At this confidence level, we also note no clear advantage to reducing the value of ν. In this case, when the equity weight is high, we note rejections at 95% c.l in the UC test. Also in this case, it seems advisable to use a normal specification when the equity weight is higher than 55%, conversely, if the equity weight is lower than 55%, one should impose $\nu = 8$. It the end, when considering the extremely rare events (see Table 6.18), there is an improvement when normality is discarded in favor of $\nu = 10$. When adopting $\nu = 10, 8$ the results of $VaR_{5\%}$ are not modified and the estimation of $VaR_{0.1\%}$ improves.

6.2.1.2 Backtest based on loss function: results

Following the experimental procedure of Abad at al. (2015) (see [2]) and of Marimoutou et al. (2009) (see [76]), we perform a backtest based on loss functions only for those models that never reject the CC test at 99% c.l. as well as the proposed models with normal and Student-t assumptions. In Tables

ϵ	$\nu=\infty$	$\nu=20$	$\nu=16$	$\nu=10$	$\nu=8$
0.1%	-3.0902	-3.3734	-3.4544	-3.7253	-3.8889
0.5%	-2.5758	-2.6983	-2.7324	-2.8402	-2.9023
1.0%	-2.3263	-2.3978	-2.4178	-2.4758	-2.5059
5.0%	-1.6448	-1.6354	-1.6318	-1.6222	-1.6098

Table 6.14: Cumulative distribution function of standardized Normal and standardized T.

Ptfs	ShVolVaR $\alpha = 0.275, \nu = \infty$			ShVolVaR $\alpha = 0.275, \nu = 20$			ShVolVaR $\alpha = 0.275, \nu = 16$			ShVolVaR $\alpha = 0.275, \nu = 10$			ShVolVaR $\alpha = 0.275, \nu = 8$		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	79.0	29.0	55.2	67.2	26.0	48.5	61.6	24.6	45.0	56.3	23.3	41.5	46.3	38.1	52.0
Ptf 2	73.0	27.5	51.9	73.0	27.5	51.9	67.2	26.0	48.5	46.3	20.8	34.5	41.7	36.2	47.5
Ptf 3	56.3	41.9	61.0	56.3	41.9	61.0	56.3	41.9	61.0	41.7	58.6	62.0	41.7	58.6	62.0
Ptf 4	91.2	54.9	83.1	67.2	46.0	69.6	61.6	44.0	65.4	41.7	58.6	62.0	37.4	56.2	57.0
Ptf 5	91.2	54.9	83.1	91.2	54.9	83.1	85.0	52.6	80.4	56.3	66.1	76.8	41.7	58.6	62.0
Ptf 6	73.0	48.2	73.6	73.0	48.2	73.6	67.2	46.0	69.6	41.7	58.6	62.0	33.4	53.9	51.9
Ptf 7	91.2	54.9	83.1	73.0	48.2	73.6	73.0	48.2	73.6	51.1	63.6	72.1	41.7	58.6	62.0
Ptf 8	83.7	39.4	68.1	97.5	34.0	63.3	85.0	30.6	58.2	61.6	44.0	65.4	56.3	41.9	61.0
Ptf 9	65.7	45.4	68.4	83.7	39.4	68.1	90.0	37.5	67.0	85.0	52.6	80.4	67.2	46.0	69.6
Ptf 10	54.7	49.6	66.2	54.7	49.6	66.2	77.6	41.3	68.7	85.0	52.6	80.4	73.0	74.0	89.2
Ptf 11	49.5	51.8	64.3	54.7	49.6	66.2	54.7	49.6	66.2	96.2	59.7	86.8	73.0	74.0	89.2
Ptf 12	49.5	51.8	64.3	60.1	47.5	67.5	71.6	43.3	68.8	90.0	62.1	87.8	96.2	59.7	86.8
Ptf 13	49.5	51.8	64.3	71.6	69.8	86.8	83.7	64.6	88.1	91.2	54.9	83.1	73.0	48.2	73.6
Ptf 14	60.1	47.5	67.5	83.7	39.4	68.1	83.7	39.4	68.1	85.0	52.6	80.4	56.3	41.9	61.0
Ptf 15	60.1	47.5	67.5	77.6	41.3	68.7	90.0	37.5	67.0	85.0	30.6	58.2	61.6	24.6	45.0
Ptf 16	49.5	51.8	64.3	65.7	45.4	68.4	71.6	43.3	68.8	90.0	37.5	67.0	91.2	32.3	60.9
Ptf 17	44.5	54.1	62.0	54.7	49.6	66.2	60.1	47.5	67.5	77.6	41.3	68.7	83.7	39.4	68.1
Ptf 18	21.2	68.8	42.4	39.9	56.4	59.3	49.5	51.8	64.3	71.6	43.3	68.8	77.6	41.3	68.7
Ptf 19	60.1	47.5	67.5	77.6	41.3	68.7	90.0	37.5	67.0	90.0	37.5	67.0	97.5	34.0	63.3
Ptf 20	44.5	54.1	62.0	65.7	45.4	68.4	71.6	43.3	68.8	90.0	37.5	67.0	85.0	52.6	80.4
Ptf 21	60.1	47.5	67.5	71.6	43.3	68.8	77.6	67.2	87.8	97.5	85.0	98.2	85.0	92.1	97.8
Ptf 22	54.7	49.6	66.2	65.7	72.4	85.2	77.6	96.4	95.9	91.2	89.2	98.5	85.0	92.1	97.8
Ptf 23	35.6	88.9	64.6	35.6	88.9	64.6	35.6	88.9	64.6	49.5	89.3	78.5	97.5	85.0	98.2
Ptf 24	31.5	77.9	58.0	35.6	80.7	63.3	39.9	83.5	68.6	44.5	86.4	73.6	65.7	97.9	90.6
Ptf 25	21.2	69.6	42.5	31.5	77.9	58.0	35.6	80.7	63.3	44.5	86.4	73.6	49.5	89.3	78.5
Ptf 26	49.5	61.5	69.8	54.7	64.1	74.8	54.7	64.1	74.8	77.6	31.2	57.6	77.6	31.2	57.6
Ptf 27	60.1	95.0	87.0	65.7	97.9	90.6	71.6	99.2	93.6	96.2	83.5	97.8	85.0	65.5	88.9
Ptf 28	65.7	8.1	19.7	77.6	9.5	23.9	90.0	11.2	28.1	96.2	12.1	30.0	91.2	14.1	33.6
Ptf 29	65.7	27.6	50.1	65.7	27.6	50.1	65.7	27.6	50.1	83.7	10.3	26.0	90.0	11.2	28.1
Ptf 30	71.6	8.8	21.8	83.7	5.2	14.8	83.7	5.2	14.8	96.2	6.2	17.5	97.5	6.8	18.9
Ptf 31	65.7	15.5	33.0	71.6	16.7	36.0	83.7	10.3	26.0	73.0	17.5	37.6	67.2	18.8	38.5
Ptf 32	79.0	16.3	36.5	73.0	17.5	37.6	67.2	10.4	24.3	33.4	9.8	16.0	26.3	11.5	15.4
Ptf 33	67.2	18.8	38.5	61.6	20.1	39.0	51.1	23.0	39.2	26.3	20.2	23.7	20.3	23.1	21.7
Ptf 34	90.0	80.7	96.3	90.0	80.7	96.3	91.2	62.9	88.4	79.0	68.2	88.7	51.1	57.6	69.0
Ptf 35	79.0	68.2	88.7	46.3	60.2	66.7	41.7	62.8	64.0	29.7	31.3	34.9	20.3	37.1	29.8
Ptf 36	97.5	86.4	98.5	79.0	95.0	96.3	73.0	97.9	94.2	37.4	90.8	66.9	26.3	99.5	53.4
Ptf 37	79.0	45.6	73.0	79.0	45.6	73.0	73.0	47.9	73.3	41.7	62.8	64.0	23.2	76.5	46.7
Ptf 38	90.0	55.2	83.1	85.0	65.5	88.9	67.2	50.2	73.0	33.4	46.4	48.0	23.2	53.6	40.4
Ptf 39	61.6	68.7	81.3	56.3	66.1	76.8	46.3	85.0	75.0	41.7	87.9	71.1	23.2	76.5	46.7

Table 6.15: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 95% for ShVolVaR.

Ptfs	ShVolVaR $\alpha = 0.5, \nu = \infty$			ShVolVaR $\alpha = 0.5, \nu = 20$			ShVolVaR $\alpha = 0.5, \nu = 16$			ShVolVaR $\alpha = 0.5, \nu = 10$			ShVolVaR $\alpha = 0.5, \nu = 8$		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	49.0	7.5	16.1	73.8	3.7	10.7	38.4	2.4	5.4	10.6	1.3	1.2	7.1	1.1	0.8
Ptf 2	30.8	9.2	14.4	60.9	3.2	8.9	21.6	1.8	2.8	10.6	1.3	1.2	4.5	0.9	0.5
Ptf 3	30.8	9.2	14.4	60.9	3.2	8.9	29.2	2.1	4.0	7.1	1.1	0.8	4.5	0.9	0.5
Ptf 4	23.7	10.1	13.0	49.0	2.8	7.1	38.4	2.4	5.4	7.1	1.1	0.8	4.5	0.9	0.5
Ptf 5	23.7	10.1	13.0	87.3	4.2	12.5	38.4	2.4	5.4	7.1	1.1	0.8	4.5	0.9	0.5
Ptf 6	30.8	9.2	14.4	73.8	3.7	10.7	38.4	2.4	5.4	7.1	1.1	0.8	4.5	0.9	0.5
Ptf 7	23.7	10.1	13.0	49.0	2.8	7.1	29.2	2.1	4.0	10.6	1.3	1.2	4.5	0.9	0.5
Ptf 8	30.8	9.2	14.4	49.0	2.8	7.1	38.4	2.4	5.4	10.6	1.3	1.2	4.5	0.9	0.5
Ptf 9	39.2	8.3	15.4	60.9	3.2	8.9	60.9	3.2	8.9	10.6	1.3	1.2	4.5	0.9	0.5
Ptf 10	39.2	8.3	15.4	73.8	3.7	10.7	73.8	3.7	10.7	10.6	1.3	1.2	2.8	0.7	0.3
Ptf 11	49.0	7.5	16.1	73.8	3.7	10.7	73.8	3.7	10.7	7.1	1.1	0.8	2.8	0.7	0.3
Ptf 12	72.2	6.0	16.0	87.3	4.2	12.5	49.0	2.8	7.1	15.4	1.5	1.9	2.8	0.7	0.3
Ptf 13	98.9	4.8	14.0	73.8	3.7	10.7	73.8	3.7	10.7	10.6	1.3	1.2	2.8	0.7	0.3
Ptf 14	85.3	5.4	15.3	73.8	3.7	10.7	60.9	3.2	8.9	15.4	1.5	1.9	2.8	0.7	0.3
Ptf 15	98.9	4.8	14.0	49.0	2.8	7.1	49.0	2.8	7.1	7.1	1.1	0.8	2.8	0.7	0.3
Ptf 16	85.3	0.5	2.0	49.0	0.2	0.7	15.4	1.5	1.9	15.4	1.5	1.9	2.8	0.7	0.3
Ptf 17	85.3	0.5	2.0	29.2	0.1	0.3	29.2	0.1	0.3	29.2	0.1	0.3	2.8	0.7	0.3
Ptf 18	73.8	0.3	1.1	38.4	0.2	0.5	29.2	0.1	0.3	29.2	0.1	0.3	7.1	0.1	0.0
Ptf 19	73.8	0.3	1.1	60.9	0.2	0.9	49.0	0.2	0.7	15.4	0.1	0.1	7.1	0.1	0.0
Ptf 20	98.9	0.4	1.7	49.0	0.2	0.7	38.4	0.2	0.5	29.2	0.1	0.3	7.1	0.1	0.0
Ptf 21	39.2	1.0	2.5	73.8	0.3	1.1	73.8	0.3	1.1	38.4	0.2	0.5	29.2	0.1	0.3
Ptf 22	17.9	1.6	2.2	72.2	0.6	2.2	73.8	0.3	1.1	49.0	0.2	0.7	29.2	0.1	0.3
Ptf 23	4.8	2.7	1.2	98.9	0.4	1.7	98.9	0.4	1.7	49.0	0.2	0.7	38.4	2.4	5.4
Ptf 24	9.6	2.0	1.7	85.3	5.4	15.3	98.9	4.8	14.0	73.8	3.7	10.7	60.9	3.2	8.9
Ptf 25	2.2	3.4	0.8	39.2	8.3	15.4	39.2	8.3	15.4	72.2	6.0	16.0	87.3	4.2	12.5
Ptf 26	6.9	2.3	1.5	17.9	1.6	2.2	17.9	1.6	2.2	30.8	1.2	2.5	49.0	0.9	2.5
Ptf 27	1.0	4.3	0.5	6.9	2.3	1.5	13.3	1.8	2.0	49.0	0.9	2.5	72.2	6.0	16.0
Ptf 28	0.2	5.9	0.2	2.2	3.4	0.8	4.8	15.8	5.2	23.7	10.1	13.0	49.0	7.5	16.1
Ptf 29	0.1	8.0	0.1	1.0	21.6	1.6	1.0	21.6	1.6	1.5	20.0	2.2	3.3	17.1	4.0
Ptf 30	0.0	43.0	0.0	0.0	34.3	0.1	0.1	30.3	0.2	0.6	23.2	1.1	1.0	21.6	1.6
Ptf 31	0.0	17.2	0.0	0.1	8.0	0.1	0.2	85.3	1.0	0.4	82.4	1.5	0.4	82.4	1.5
Ptf 32	0.0	43.0	0.0	0.1	32.3	0.1	0.2	26.6	0.5	0.6	23.2	1.1	2.2	18.6	3.1
Ptf 33	0.0	50.1	0.0	0.1	32.3	0.1	0.1	30.3	0.2	0.6	23.2	1.1	2.2	18.6	3.1
Ptf 34	0.0	47.6	0.0	0.1	32.3	0.1	0.1	28.4	0.4	2.2	24.0	3.7	3.3	25.0	5.3
Ptf 35	0.0	14.8	0.0	0.2	19.5	0.4	0.4	20.3	0.7	1.0	22.1	1.6	3.3	25.0	5.3
Ptf 36	0.0	16.3	0.1	0.2	19.5	0.4	1.0	22.1	1.6	1.0	22.1	1.6	4.8	26.1	7.5
Ptf 37	0.0	96.5	0.1	0.4	82.4	1.5	1.5	23.1	2.5	9.6	28.2	14.1	17.9	30.5	23.9
Ptf 38	0.1	93.7	0.2	1.5	74.0	4.8	3.3	68.4	9.5	13.3	57.6	27.6	49.0	35.3	51.2
Ptf 39	2.2	71.2	6.8	17.9	54.9	33.9	30.8	49.7	47.2	72.2	39.9	65.8	85.3	39.3	68.2

Table 6.16: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99% for ShVolVaR.

Ptfs	ShVolVaR $\alpha = 0.75, \nu = \infty$			ShVolVaR $\alpha = 0.75, \nu = 20$			ShVolVaR $\alpha = 0.75, \nu = 16$			ShVolVaR $\alpha = 0.75, \nu = 10$			ShVolVaR $\alpha = 0.75, \nu = 8$		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	12.1	78.9	29.0	1.5	84.6	5.1	1.5	84.6	5.1	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 2	12.1	78.9	29.0	1.5	84.6	5.1	1.5	84.6	5.1	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 3	12.1	78.9	29.0	1.5	84.6	5.1	1.5	84.6	5.1	0.6	86.5	2.3	0.6	86.5	2.3
Ptf 4	20.3	77.0	42.5	3.3	82.7	10.1	1.5	84.6	5.1	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 5	20.3	77.0	42.5	3.3	82.7	10.1	3.3	82.7	10.1	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 6	20.3	77.0	42.5	3.3	82.7	10.1	3.3	82.7	10.1	0.2	88.4	0.9	0.2	88.4	0.9
Ptf 7	20.3	77.0	42.5	3.3	82.7	10.1	3.3	82.7	10.1	0.2	88.4	0.9	0.2	88.4	0.9
Ptf 8	12.1	78.9	29.0	3.3	82.7	10.1	3.3	82.7	10.1	0.2	88.4	0.9	0.2	88.4	0.9
Ptf 9	12.1	78.9	29.0	3.3	82.7	10.1	0.6	86.5	2.3	0.2	88.4	0.9	0.2	88.4	0.9
Ptf 10	12.1	78.9	29.0	1.5	84.6	5.1	0.6	86.5	2.3	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 11	31.5	75.2	57.4	1.5	84.6	5.1	1.5	84.6	5.1	0.6	86.5	2.3	0.2	88.4	0.9
Ptf 12	20.3	77.0	42.5	1.5	84.6	5.1	1.5	84.6	5.1	0.6	86.5	2.3	0.6	86.5	2.3
Ptf 13	20.3	77.0	42.5	3.3	82.7	10.1	3.3	82.7	10.1	0.6	86.5	2.3	0.6	86.5	2.3
Ptf 14	20.3	3.3	4.5	3.3	82.7	10.1	3.3	82.7	10.1	1.5	84.6	5.1	0.6	86.5	2.3
Ptf 15	12.1	2.7	2.6	3.3	82.7	10.1	3.3	82.7	10.1	3.3	82.7	10.1	3.3	82.7	10.1
Ptf 16	31.5	3.9	7.2	3.3	82.7	10.1	3.3	82.7	10.1	3.3	82.7	10.1	3.3	82.7	10.1
Ptf 17	31.5	3.9	7.2	3.3	82.7	10.1	3.3	82.7	10.1	3.3	82.7	10.1	1.5	84.6	5.1
Ptf 18	45.7	4.7	10.5	6.6	80.8	18.0	3.3	82.7	10.1	1.5	84.6	5.1	1.5	84.6	5.1
Ptf 19	62.6	0.1	0.5	6.6	80.8	18.0	3.3	82.7	10.1	1.5	84.6	5.1	1.5	84.6	5.1
Ptf 20	99.2	0.2	1.0	6.6	80.8	18.0	3.3	82.7	10.1	3.3	82.7	10.1	1.5	84.6	5.1
Ptf 21	99.2	0.2	1.0	31.5	75.2	57.4	31.5	75.2	57.4	3.3	82.7	10.1	3.3	82.7	10.1
Ptf 22	99.2	0.2	1.0	31.5	75.2	57.4	31.5	75.2	57.4	12.1	78.9	29.0	12.1	78.9	29.0
Ptf 23	10.7	1.1	1.1	45.7	73.3	71.6	45.7	73.3	71.6	6.6	80.8	18.0	3.3	82.7	10.1
Ptf 24	2.5	1.8	0.5	80.2	66.1	88.0	62.6	71.5	83.1	20.3	77.0	42.5	12.1	78.9	29.0
Ptf 25	6.8	17.9	7.6	24.1	59.2	43.5	34.3	60.9	56.0	62.6	71.5	83.1	20.3	77.0	42.5
Ptf 26	0.4	26.9	0.9	10.7	55.9	23.0	16.3	57.5	32.3	47.2	62.6	68.6	81.4	69.7	90.2
Ptf 27	0.1	31.0	0.3	4.1	52.6	10.2	6.8	54.2	15.6	24.1	59.2	43.5	47.2	62.6	68.6
Ptf 28	0.1	31.0	0.3	0.4	26.9	0.9	0.8	25.0	1.5	6.8	17.9	7.6	24.1	13.4	16.3
Ptf 29	0.0	35.4	0.1	0.2	28.9	0.5	0.4	26.9	0.9	2.5	21.3	3.7	10.7	16.3	10.3
Ptf 30	0.0	47.2	0.0	0.4	26.9	0.9	0.8	25.0	1.5	4.1	19.6	5.4	10.7	55.9	23.0
Ptf 31	0.0	54.9	0.0	0.1	42.1	0.2	0.4	46.4	1.3	4.1	52.6	10.2	24.1	59.2	43.5
Ptf 32	0.0	14.5	0.0	0.0	36.6	0.0	0.1	42.1	0.2	0.8	47.9	2.3	6.8	54.2	15.6
Ptf 33	0.0	13.3	0.0	0.0	8.3	0.0	0.0	7.5	0.0	0.1	43.5	0.4	2.5	51.0	6.4
Ptf 34	0.0	13.3	0.0	0.0	39.3	0.0	0.0	40.7	0.1	0.2	45.0	0.7	0.8	47.9	2.3
Ptf 35	0.0	32.9	0.0	0.0	39.3	0.0	0.0	39.3	0.0	0.2	45.0	0.7	2.5	51.0	6.4
Ptf 36	0.0	36.6	0.0	0.0	40.7	0.1	0.2	45.0	0.7	1.4	49.5	3.9	10.7	55.9	23.0
Ptf 37	0.0	39.3	0.0	0.8	47.9	2.3	1.4	49.5	3.9	10.7	55.9	23.0	62.6	64.4	79.8
Ptf 38	0.0	39.3	0.0	4.1	52.6	10.2	10.7	55.9	23.0	34.3	60.9	56.0	81.4	69.7	90.2
Ptf 39	0.4	46.4	1.3	16.3	57.5	32.3	34.3	60.9	56.0	80.2	66.1	88.0	62.6	71.5	83.1

Table 6.17: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.5% for ShVolVaR.

Ptfs	ShVolVaR $\alpha = 1, \nu = \infty$			ShVolVaR $\alpha = 1, \nu = 20$			ShVolVaR $\alpha = 1, \nu = 16$			ShVolVaR $\alpha = 1, \nu = 10$			ShVolVaR $\alpha = 1, \nu = 8$		
	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 2	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 3	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 4	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 5	41.5	90.3	71.2	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 6	41.5	90.3	71.2	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 7	41.5	90.3	71.2	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 8	41.5	90.3	71.2	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 9	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 10	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 11	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 12	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 13	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 14	74.8	92.3	94.5	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 15	82.8	94.2	97.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	12.6	98.1	31.0
Ptf 16	82.8	94.2	97.4	41.3	96.1	71.4	41.3	96.1	71.4	41.3	96.1	71.4	12.6	98.1	31.0
Ptf 17	82.8	94.2	97.4	41.3	96.1	71.4	41.3	96.1	71.4	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 18	82.8	94.2	97.4	41.3	96.1	71.4	41.3	96.1	71.4	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 19	74.8	92.3	94.5	12.6	98.1	31.0	12.6	98.1	31.0	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 20	41.5	90.3	71.2	41.3	96.1	71.4	12.6	98.1	31.0	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 21	74.8	92.3	94.5	41.3	96.1	71.4	12.6	98.1	31.0	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 22	74.8	92.3	94.5	82.8	94.2	97.4	41.3	96.1	71.4	12.6	98.1	31.0	0.9	100.0	3.4
Ptf 23	41.5	90.3	71.2	74.8	92.3	94.5	82.8	94.2	97.4	12.6	98.1	31.0	0.9	100.0	3.4
Ptf 24	20.2	88.4	43.8	74.8	92.3	94.5	74.8	92.3	94.5	12.6	98.1	31.0	0.9	100.0	3.4
Ptf 25	3.4	84.6	10.3	20.2	88.4	43.8	74.8	92.3	94.5	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 26	0.4	80.8	1.4	8.7	86.5	22.8	41.5	90.3	71.2	12.6	98.1	31.0	12.6	98.1	31.0
Ptf 27	0.0	77.0	0.1	1.2	82.7	4.1	8.7	86.5	22.8	74.8	92.3	94.5	41.3	96.1	71.4
Ptf 28	0.0	77.0	0.1	1.2	82.7	4.1	1.2	82.7	4.1	41.5	90.3	71.2	74.8	92.3	94.5
Ptf 29	0.0	71.5	0.0	1.2	82.7	4.1	1.2	82.7	4.1	8.7	86.5	22.8	41.5	90.3	71.2
Ptf 30	0.0	66.1	0.0	0.1	78.9	0.5	0.4	80.8	1.4	1.2	82.7	4.1	3.4	84.6	10.3
Ptf 31	0.0	64.4	0.0	0.0	75.2	0.0	0.0	77.0	0.1	1.2	82.7	4.1	3.4	84.6	10.3
Ptf 32	0.0	62.6	0.0	0.0	69.7	0.0	0.0	75.2	0.0	1.2	82.7	4.1	1.2	82.7	4.1
Ptf 33	0.0	59.2	0.0	0.0	71.5	0.0	0.0	73.3	0.0	1.2	82.7	4.1	3.4	84.6	10.3
Ptf 34	0.0	59.2	0.0	0.0	71.5	0.0	0.0	75.2	0.0	1.2	82.7	4.1	3.4	84.6	10.3
Ptf 35	0.0	59.2	0.0	0.0	75.2	0.0	0.0	75.2	0.0	3.4	84.6	10.3	41.5	90.3	71.2
Ptf 36	0.0	64.4	0.0	0.0	75.2	0.0	0.4	80.8	1.4	74.8	92.3	94.5	41.3	96.1	71.4
Ptf 37	0.0	69.7	0.0	0.4	80.8	1.4	8.7	86.5	22.8	41.3	96.1	71.4	41.3	96.1	71.4
Ptf 38	0.0	71.5	0.0	8.7	86.5	22.8	20.2	88.4	43.8	41.3	96.1	71.4	12.6	98.1	31.0
Ptf 39	0.1	78.9	0.5	8.7	86.5	22.8	74.8	92.3	94.5	41.3	96.1	71.4	12.6	98.1	31.0

Table 6.18: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.9% for ShVolVaR.

6.19, 6.20, 6.21 and 6.22 we report the values of the total loss computed using Regulators' loss functions (see Section 5.1.8) with respect to the four VaR levels analyzed. We report the best average value recorded in each test in boldface and the worst and in italic. Considering the average of the total loss for all portfolios, ShVolVaR ($\nu = \infty$) offers the best results three out four tests when estimating $VaR_{5\%}$, while having $\nu = 8$ gives the worst results. The opposite is true when we look at the other percentiles. ShVolVaR with finite values of ν offers almost the best results, while normally distributed ShVolVaR presents the worst performances. However, when using investors' loss functions, as reported in Tables 6.23, 6.24, 6.25 and 6.26, on average, ShVolVaR with $\nu = 8, 10$ tends to give the worst performance, while ShVolVaR ($\nu = \infty$) has the best results three times out of four. These results are in line with the evidence in 5. Summarizing, we note a negative trade-off with regards models' results for investors' and regulators' loss functions. If a model performs well for one test, then it performs poorly according to other test. This particular feature is very useful, as it means one can use a particular specification of the ShVolVaR model to reach the goal of satisfying a particular viewpoint (regulator or investor).

Portfolio name	Lopez			Caporin1			Caporin2			Caporin3		
	ShVolVaR $\alpha = 0.275, \nu = \infty$	HFBVaR	ShVolVaR $\alpha = 0.275, \nu = 8$	ShVolVaR $\alpha = 0.275, \nu = \infty$	HFBVaR	ShVolVaR $\alpha = 0.275, \nu = 8$	ShVolVaR $\alpha = 0.275, \nu = \infty$	HFBVaR	ShVolVaR $\alpha = 0.275, \nu = 8$	ShVolVaR $\alpha = 0.275, \nu = \infty$	HFBVaR	ShVolVaR $\alpha = 0.275, \nu = 8$
Ptf 1	173.0124	179.0093	179.0132	61.8920	63.6606	66.9368	0.6583	0.5811	0.7145	0.9691	0.8978	1.0306
Ptf 2	174.0116	178.0087	180.0124	61.6252	63.8820	66.6759	0.6350	0.5646	0.6893	0.9373	0.8701	0.9974
Ptf 3	177.0109	176.0081	180.0116	61.3369	63.9721	66.4087	0.6118	0.5463	0.6643	0.9056	0.8437	0.9643
Ptf 4	171.0102	178.0076	181.0109	61.0812	64.2334	66.1204	0.5888	0.5286	0.6394	0.8746	0.8192	0.9312
Ptf 5	171.0095	178.0071	180.0101	60.8415	64.7579	65.8314	0.5659	0.5108	0.6148	0.8441	0.7966	0.8983
Ptf 6	174.0088	177.0067	182.0094	60.5679	65.4398	65.6024	0.5433	0.4938	0.5903	0.8134	0.7751	0.8664
Ptf 7	171.0082	179.0062	181.0088	60.3710	65.8791	65.3703	0.5209	0.4744	0.5662	0.7837	0.7517	0.8346
Ptf 8	167.0076	184.0058	177.0081	60.2150	66.9089	65.1591	0.4988	0.4624	0.5424	0.7549	0.7309	0.8033
Ptf 9	164.0070	182.0054	176.0075	60.0842	67.7329	64.9567	0.4770	0.4485	0.5189	0.7269	0.7087	0.7723
Ptf 10	162.0065	179.0050	174.0070	59.9399	67.9914	64.7540	0.4556	0.4306	0.4957	0.6990	0.6826	0.7420
Ptf 11	161.0060	176.0046	174.0064	59.7842	68.5221	64.5554	0.4344	0.4151	0.4729	0.6712	0.6582	0.7121
Ptf 12	161.0055	175.0042	169.0059	59.5939	69.3370	64.3649	0.4137	0.4040	0.4505	0.6432	0.6335	0.6827
Ptf 13	161.0050	174.0039	174.0054	59.3655	70.0679	64.1824	0.3935	0.3912	0.4287	0.6155	0.6100	0.6537
Ptf 14	163.0046	173.0036	177.0049	59.1148	69.9362	63.9536	0.3740	0.3757	0.4076	0.5880	0.5841	0.6245
Ptf 15	163.0042	177.0033	176.0045	58.8111	70.3162	63.6405	0.3553	0.3622	0.3872	0.5604	0.5598	0.5952
Ptf 16	161.0038	184.0030	171.0041	58.4303	71.2256	63.1932	0.3374	0.3537	0.3678	0.5327	0.5367	0.5653
Ptf 17	160.0035	183.0028	167.0037	57.9522	71.7728	62.6428	0.3207	0.3422	0.3495	0.5050	0.5140	0.5356
Ptf 18	154.0032	181.0026	166.0034	57.4270	72.0567	62.0662	0.3051	0.3318	0.3324	0.4779	0.4892	0.5066
Ptf 19	163.0029	185.0024	170.0031	57.0601	71.4886	61.8089	0.2909	0.3188	0.3168	0.4527	0.4654	0.4810
Ptf 20	160.0026	188.0022	172.0028	57.0230	71.6057	61.7434	0.2787	0.3102	0.3034	0.4311	0.4430	0.4574
Ptf 21	163.0024	189.0020	172.0026	57.4359	72.2308	62.1935	0.2687	0.3078	0.2924	0.4130	0.4272	0.4379
Ptf 22	162.0022	186.0019	173.0024	57.9368	73.4210	62.7134	0.2612	0.3084	0.2840	0.3956	0.4170	0.4193
Ptf 23	158.0021	177.0019	170.0022	58.6655	74.1605	63.3137	0.2565	0.3062	0.2784	0.3801	0.4055	0.4020
Ptf 24	157.0019	180.0018	164.0021	59.6030	75.6261	64.2367	0.2546	0.3108	0.2760	0.3663	0.3965	0.3869
Ptf 25	154.0018	173.0018	162.0019	60.8049	76.1494	65.4505	0.2559	0.3185	0.2767	0.3543	0.3873	0.3737
Ptf 26	161.0017	173.0017	166.0018	62.8832	76.2033	67.6828	0.2600	0.3212	0.2806	0.3467	0.3770	0.3658
Ptf 27	163.0017	168.0017	172.0018	65.1829	76.2328	70.0956	0.2672	0.3291	0.2877	0.3413	0.3663	0.3596
Ptf 28	164.0016	164.0017	171.0017	67.9858	74.9725	72.9742	0.2774	0.3296	0.2980	0.3396	0.3559	0.3575
Ptf 29	164.0016	161.0017	168.0017	71.0711	74.3143	76.0863	0.2895	0.3318	0.3104	0.3413	0.3532	0.3582
Ptf 30	165.0016	165.0017	170.0017	73.6982	73.4129	78.8174	0.3004	0.3314	0.3216	0.3433	0.3493	0.3600
Ptf 31	164.0016	161.0016	175.0017	75.6996	71.5937	80.8978	0.3062	0.3231	0.3277	0.3466	0.3436	0.3631
Ptf 32	173.0015	158.0016	184.0016	75.8548	69.2844	81.2330	0.3048	0.3141	0.3263	0.3465	0.3373	0.3637
Ptf 33	175.0015	154.0015	186.0016	74.5579	65.5893	79.9505	0.2976	0.2975	0.3188	0.3429	0.3271	0.3607
Ptf 34	168.0014	159.0015	178.0015	72.7532	63.0077	77.9337	0.2863	0.2850	0.3071	0.3406	0.3210	0.3581
Ptf 35	173.0014	156.0014	186.0015	70.4313	61.4188	75.7449	0.2725	0.2715	0.2930	0.3384	0.3227	0.3574
Ptf 36	170.0014	153.0014	184.0014	67.6162	60.0533	72.7890	0.2590	0.2601	0.2791	0.3366	0.3254	0.3558
Ptf 37	173.0013	157.0014	186.0014	64.7333	58.7827	69.8546	0.2466	0.2512	0.2665	0.3356	0.3316	0.3556
Ptf 38	168.0013	157.0015	185.0014	62.1977	57.9791	67.2809	0.2364	0.2470	0.2561	0.3371	0.3387	0.3581
Ptf 39	176.0014	154.0015	185.0014	60.0734	56.5370	65.1472	0.2278	0.2398	0.2476	0.3431	0.3445	0.3655
Mean	**165.6966**	172.5932	*175.4661*	**62.8641**	*68.5066*	67.8042	**0.3638**	0.3674	*0.3944*	0.5250	**0.5179**	*0.5568*

Table 6.19: Values of the total loss over the backtesting sample for each portfolio, using Regulator's loss functions: $VaR_{5\%}$.

Portfolio	Lopez				Caporin1				Caporin2				Caporin3			
name	ShVolVaR	HFBEVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBEVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBEVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBEVaR	HFBVaR	ShVolVaR
	$\alpha = 0.5, \nu = \infty$			$\alpha = 0.5, \nu = 8$	$\alpha = 0.5, \nu = \infty$			$\alpha = 0.5, \nu = 8$	$\alpha = 0.5, \nu = \infty$			$\alpha = 0.5, \nu = 8$	$\alpha = 0.5, \nu = \infty$			$\alpha = 0.5, \nu = 8$
Ptf 1	38.0029	46.0016	48.0015	24.0021	8.0863	13.1245	8.4693	5.2905	0.1032	0.1035	0.0779	0.0720	0.1916	0.1750	0.1608	0.1368
Ptf 2	40.0027	44.0015	47.0014	23.0020	8.0109	12.8329	8.5111	5.2357	0.0986	0.0979	0.0746	0.0686	0.1846	0.1660	0.1557	0.1317
Ptf 3	40.0025	43.0013	45.0013	23.0018	7.9502	12.5364	8.5500	5.1788	0.0939	0.0922	0.0713	0.0652	0.1780	0.1570	0.1504	0.1266
Ptf 4	41.0023	43.0012	44.0012	23.0017	7.8858	12.2147	8.5516	5.1158	0.0894	0.0871	0.0679	0.0618	0.1715	0.1480	0.1445	0.1215
Ptf 5	41.0021	41.0011	44.0011	23.0015	7.8118	11.8584	8.5021	5.0458	0.0848	0.0815	0.0649	0.0585	0.1650	0.1391	0.1384	0.1163
Ptf 6	40.0020	39.0010	43.0010	23.0014	7.7330	11.5614	8.4459	4.9800	0.0804	0.0765	0.0616	0.0552	0.1584	0.1310	0.1330	0.1112
Ptf 7	41.0018	38.0009	42.0009	23.0013	7.6629	11.2538	8.3849	4.9207	0.0760	0.0714	0.0580	0.0519	0.1520	0.1235	0.1278	0.1063
Ptf 8	40.0016	37.0008	42.0008	23.0012	7.6029	10.9001	8.3677	4.8564	0.0718	0.0663	0.0549	0.0488	0.1457	0.1158	0.1232	0.1014
Ptf 9	39.0015	36.0007	43.0008	23.0011	7.5475	10.5245	8.3942	4.7963	0.0676	0.0612	0.0522	0.0457	0.1396	0.1086	0.1184	0.0966
Ptf 10	39.0014	37.0006	40.0007	23.0010	7.4987	10.1548	8.4546	4.7353	0.0635	0.0563	0.0494	0.0427	0.1338	0.1019	0.1148	0.0919
Ptf 11	38.0012	36.0006	40.0006	22.0009	7.4549	9.7788	8.4152	4.6906	0.0596	0.0519	0.0464	0.0398	0.1285	0.0950	0.1105	0.0874
Ptf 12	36.0011	35.0005	39.0006	23.0008	7.4045	9.2875	8.2027	4.6521	0.0559	0.0475	0.0430	0.0370	0.1231	0.0876	0.1048	0.0830
Ptf 13	34.0010	35.0004	42.0005	22.0007	7.3676	8.7748	7.9717	4.6386	0.0524	0.0432	0.0401	0.0344	0.1180	0.0805	0.0991	0.0790
Ptf 14	35.0009	34.0004	43.0005	22.0006	7.3342	8.2205	7.9630	4.6500	0.0491	0.0389	0.0378	0.0319	0.1129	0.0735	0.0953	0.0758
Ptf 15	34.0008	32.0003	42.0004	22.0006	7.2948	7.6618	7.9667	4.6940	0.0460	0.0348	0.0361	0.0295	0.1077	0.0669	0.0918	0.0736
Ptf 16	35.0007	30.0003	41.0004	22.0005	7.3207	7.2870	7.8077	4.7349	0.0431	0.0317	0.0343	0.0273	0.1037	0.0616	0.0875	0.0715
Ptf 17	35.0007	31.0002	39.0004	22.0004	7.3935	6.8395	7.7342	4.7787	0.0406	0.0279	0.0331	0.0253	0.1001	0.0565	0.0838	0.0694
Ptf 18	32.0006	27.0002	40.0003	25.0004	7.5478	6.4791	7.8258	4.9105	0.0384	0.0250	0.0320	0.0236	0.0973	0.0523	0.0814	0.0680
Ptf 19	32.0005	31.0002	39.0003	24.0003	7.7526	6.2681	8.1992	5.0756	0.0366	0.0223	0.0315	0.0222	0.0951	0.0496	0.0805	0.0668
Ptf 20	34.0005	32.0002	38.0003	24.0003	7.9906	6.0643	8.6720	5.2361	0.0353	0.0202	0.0323	0.0212	0.0929	0.0472	0.0805	0.0655
Ptf 21	39.0005	30.0001	35.0003	28.0003	8.4176	5.8749	9.1308	5.4725	0.0345	0.0186	0.0336	0.0206	0.0924	0.0451	0.0801	0.0647
Ptf 22	42.0004	29.0001	36.0003	28.0003	8.9001	5.6318	9.5568	5.8090	0.0345	0.0174	0.0345	0.0206	0.0925	0.0431	0.0803	0.0645
Ptf 23	46.0004	28.0001	37.0003	29.0003	9.5611	5.5383	9.8267	6.3530	0.0354	0.0169	0.0357	0.0213	0.0934	0.0426	0.0798	0.0660
Ptf 24	44.0004	28.0001	34.0003	31.0003	10.4371	5.5114	9.7840	7.0716	0.0376	0.0171	0.0367	0.0229	0.0959	0.0425	0.0774	0.0687
Ptf 25	48.0004	25.0001	34.0003	33.0003	11.5580	5.4966	9.9553	7.8222	0.0410	0.0179	0.0380	0.0256	0.0999	0.0423	0.0780	0.0712
Ptf 26	45.0005	22.0001	34.0003	38.0003	12.8197	5.6236	9.8386	8.8436	0.0459	0.0191	0.0379	0.0295	0.1038	0.0427	0.0770	0.0756
Ptf 27	50.0005	22.0001	35.0003	36.0003	14.2067	5.8047	10.5222	10.0571	0.0520	0.0205	0.0413	0.0344	0.1083	0.0434	0.0825	0.0813
Ptf 28	53.0005	24.0002	35.0003	38.0004	15.6053	6.0170	10.8936	11.1543	0.0586	0.0221	0.0415	0.0399	0.1132	0.0443	0.0835	0.0858
Ptf 29	56.0005	25.0002	33.0003	48.0004	17.2154	6.2549	10.7463	12.2713	0.0655	0.0237	0.0411	0.0454	0.1194	0.0459	0.0816	0.0910
Ptf 30	61.0006	28.0002	31.0003	50.0004	18.6633	6.5648	10.8458	13.2905	0.0716	0.0250	0.0423	0.0503	0.1251	0.0486	0.0804	0.0956
Ptf 31	65.0006	28.0002	31.0003	52.0004	19.6682	7.0755	10.9520	14.1388	0.0758	0.0270	0.0441	0.0538	0.1286	0.0527	0.0787	0.0988
Ptf 32	61.0005	26.0002	36.0003	49.0004	20.2616	7.4737	10.7990	14.7983	0.0767	0.0286	0.0430	0.0543	0.1305	0.0562	0.0768	0.1014
Ptf 33	64.0005	27.0002	36.0003	48.0004	20.1236	7.8104	10.5887	14.6356	0.0731	0.0301	0.0400	0.0511	0.1301	0.0588	0.0766	0.1001
Ptf 34	63.0005	28.0002	35.0003	47.0003	19.0366	7.9442	10.5333	13.6769	0.0665	0.0309	0.0393	0.0457	0.1250	0.0601	0.0783	0.0945
Ptf 35	59.0004	30.0002	36.0003	47.0003	17.4799	8.1184	10.4171	12.3797	0.0593	0.0314	0.0387	0.0400	0.1178	0.0619	0.0790	0.0879
Ptf 36	57.0004	31.0002	34.0003	46.0002	15.9318	8.1671	9.9580	10.9766	0.0524	0.0311	0.0382	0.0346	0.1112	0.0628	0.0772	0.0800
Ptf 37	57.0003	32.0002	32.0003	42.0002	14.3134	8.2595	9.1447	9.6188	0.0465	0.0311	0.0338	0.0304	0.1035	0.0650	0.0731	0.0725
Ptf 38	56.0003	33.0003	34.0002	38.0002	12.8046	8.3227	8.5357	8.4407	0.0417	0.0314	0.0299	0.0269	0.0970	0.0672	0.0706	0.0662
Ptf 39	48.0003	34.0003	33.0002	35.0002	11.3886	8.3181	8.1498	7.5949	0.0378	0.0315	0.0278	0.0240	0.0902	0.0694	0.0705	0.0629
Mean	*45.0779*	32.2312	38.2569	**31.3340**	*11.0524*	8.2931	9.1171	**7.5031**	*0.0588*	0.0413	0.0440	**0.0393**	*0.1225*	**0.0777**	0.0965	0.0874

Table 6.20: Values of the total loss over the backtesting sample for each portfolio, using Regulator's loss functions: $VaR_{1\%}$.

Portfolio	Lopez				Caporin1				Caporin2				Caporin3			
name	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR
	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$
Ptf 1	11.0017	18.0006	25.0009	6.0009	3.0687	4.0683	4.7366	1.7784	0.0482	0.0451	0.0489	0.0255	0.0940	0.0633	0.0917	0.0596
Ptf 2	11.0015	17.0006	25.0008	6.0008	3.0364	3.9742	4.7468	1.7444	0.0458	0.0431	0.0464	0.0240	0.0903	0.0594	0.0893	0.0569
Ptf 3	11.0014	17.0005	25.0007	7.0008	3.0013	3.8634	4.7480	1.7108	0.0434	0.0410	0.0437	0.0226	0.0867	0.0557	0.0863	0.0543
Ptf 4	12.0013	16.0005	25.0007	6.0007	2.9655	3.7353	4.7503	1.6810	0.0411	0.0389	0.0410	0.0213	0.0831	0.0518	0.0832	0.0518
Ptf 5	12.0012	15.0004	24.0006	6.0006	2.9320	3.5957	4.7068	1.6543	0.0387	0.0368	0.0382	0.0199	0.0797	0.0477	0.0789	0.0496
Ptf 6	12.0011	17.0004	24.0005	6.0006	2.8951	3.4945	4.6059	1.6256	0.0364	0.0350	0.0354	0.0185	0.0763	0.0443	0.0735	0.0474
Ptf 7	12.0010	17.0004	26.0004	6.0005	2.8544	3.3560	4.5095	1.5947	0.0341	0.0330	0.0327	0.0172	0.0728	0.0406	0.0682	0.0451
Ptf 8	11.0009	16.0003	26.0004	6.0005	2.8114	3.1597	4.4230	1.5612	0.0319	0.0301	0.0303	0.0159	0.0694	0.0366	0.0631	0.0428
Ptf 9	11.0008	15.0003	27.0003	6.0004	2.7639	2.9675	4.3225	1.5250	0.0297	0.0280	0.0282	0.0146	0.0659	0.0326	0.0580	0.0405
Ptf 10	11.0007	12.0003	25.0003	6.0004	2.7110	2.8538	4.2615	1.4857	0.0276	0.0261	0.0264	0.0133	0.0623	0.0302	0.0536	0.0382
Ptf 11	13.0007	13.0002	22.0003	6.0003	2.6654	2.7439	4.2519	1.4428	0.0256	0.0237	0.0252	0.0121	0.0589	0.0283	0.0511	0.0359
Ptf 12	12.0006	14.0002	22.0003	7.0003	2.6496	2.6990	4.3330	1.4008	0.0236	0.0219	0.0243	0.0109	0.0562	0.0270	0.0496	0.0335
Ptf 13	12.0005	13.0002	22.0002	7.0003	2.6348	2.5868	4.2378	1.3607	0.0217	0.0196	0.0227	0.0097	0.0536	0.0250	0.0468	0.0313
Ptf 14	12.0005	10.0002	23.0002	7.0002	2.6193	2.5321	4.1081	1.3161	0.0199	0.0177	0.0210	0.0086	0.0509	0.0239	0.0442	0.0289
Ptf 15	11.0004	10.0001	24.0002	9.0002	2.6175	2.4383	4.0690	1.2769	0.0182	0.0158	0.0190	0.0076	0.0485	0.0227	0.0431	0.0268
Ptf 16	13.0004	10.0001	23.0002	9.0002	2.6264	2.3571	3.9536	1.2698	0.0166	0.0137	0.0171	0.0066	0.0464	0.0217	0.0417	0.0253
Ptf 17	13.0003	9.0001	23.0001	8.0001	2.6538	2.2566	3.7235	1.2608	0.0151	0.0121	0.0155	0.0058	0.0451	0.0205	0.0393	0.0237
Ptf 18	14.0003	10.0001	20.0001	8.0001	2.6873	2.1558	3.6857	1.2735	0.0139	0.0104	0.0145	0.0051	0.0436	0.0192	0.0384	0.0225
Ptf 19	15.0002	11.0001	21.0001	8.0001	2.7298	2.0148	3.8274	1.2834	0.0129	0.0085	0.0133	0.0045	0.0422	0.0178	0.0396	0.0212
Ptf 20	17.0002	11.0001	22.0001	8.0001	2.8394	1.8668	4.1115	1.2884	0.0121	0.0070	0.0137	0.0041	0.0413	0.0164	0.0413	0.0198
Ptf 21	17.0002	11.0000	23.0001	9.0001	3.0193	1.7749	4.6289	1.3071	0.0118	0.0060	0.0149	0.0038	0.0411	0.0156	0.0442	0.0191
Ptf 22	17.0002	11.0000	22.0001	11.0001	3.2248	1.7723	5.1058	1.3739	0.0118	0.0052	0.0161	0.0038	0.0409	0.0157	0.0465	0.0190
Ptf 23	24.0002	11.0000	23.0002	9.0001	3.5801	1.7408	5.4689	1.5447	0.0122	0.0046	0.0173	0.0041	0.0421	0.0156	0.0478	0.0201
Ptf 24	27.0002	13.0000	22.0002	11.0001	4.1910	1.6960	5.5883	1.7953	0.0134	0.0041	0.0180	0.0048	0.0454	0.0154	0.0481	0.0217
Ptf 25	25.0002	12.0000	21.0002	12.0001	5.0176	1.7240	5.4876	2.2380	0.0157	0.0042	0.0186	0.0058	0.0499	0.0157	0.0468	0.0244
Ptf 26	30.0002	12.0000	19.0002	16.0001	6.0608	1.7479	5.3654	2.7913	0.0189	0.0045	0.0194	0.0075	0.0561	0.0160	0.0451	0.0279
Ptf 27	32.0003	12.0000	16.0002	20.0001	7.2247	1.9241	5.3205	3.4751	0.0233	0.0052	0.0200	0.0101	0.0630	0.0174	0.0439	0.0326
Ptf 28	32.0003	12.0001	15.0002	22.0002	8.3729	2.1492	5.2820	4.2525	0.0286	0.0063	0.0199	0.0135	0.0686	0.0191	0.0425	0.0383
Ptf 29	34.0003	13.0001	19.0001	25.0002	9.5260	2.4007	5.0473	5.1011	0.0345	0.0077	0.0189	0.0173	0.0746	0.0206	0.0399	0.0440
Ptf 30	39.0004	11.0001	20.0001	24.0002	10.4889	2.6892	4.9222	5.8932	0.0400	0.0091	0.0182	0.0211	0.0793	0.0226	0.0388	0.0492
Ptf 31	42.0004	11.0001	19.0001	22.0002	11.3472	2.9407	4.9411	6.5699	0.0441	0.0101	0.0184	0.0239	0.0830	0.0244	0.0387	0.0535
Ptf 32	45.0004	14.0001	19.0001	25.0002	12.3488	3.1650	5.0920	6.9867	0.0454	0.0112	0.0191	0.0242	0.0884	0.0260	0.0401	0.0557
Ptf 33	44.0003	16.0001	21.0002	27.0002	12.3975	3.4333	5.3336	6.8521	0.0428	0.0123	0.0199	0.0219	0.0889	0.0281	0.0425	0.0545
Ptf 34	44.0003	16.0001	20.0002	29.0002	11.5828	3.7143	5.3975	6.1797	0.0374	0.0136	0.0206	0.0183	0.0846	0.0301	0.0432	0.0493
Ptf 35	40.0002	16.0001	21.0002	27.0001	10.3589	3.8585	5.4400	5.3467	0.0316	0.0142	0.0209	0.0145	0.0776	0.0316	0.0438	0.0435
Ptf 36	37.0002	16.0001	22.0002	24.0001	9.0152	4.0351	5.2997	4.3865	0.0263	0.0147	0.0197	0.0115	0.0696	0.0335	0.0437	0.0362
Ptf 37	35.0002	17.0001	21.0002	19.0001	7.6696	4.1052	5.1080	3.6522	0.0223	0.0149	0.0181	0.0096	0.0608	0.0345	0.0433	0.0316
Ptf 38	35.0002	16.0001	21.0002	16.0001	6.6828	4.0677	5.1215	3.0867	0.0193	0.0146	0.0181	0.0079	0.0557	0.0348	0.0444	0.0283
Ptf 39	30.0002	15.0001	23.0002	16.0001	5.7149	4.0486	5.0068	2.6505	0.0168	0.0143	0.0176	0.0066	0.0501	0.0356	0.0444	0.0258
Mean	*22.4364*	13.4874	22.0772	**12.8721**	*5.2202*	2.8643	4.7454	**2.6928**	*0.0270*	0.0175	0.0234	**0.0128**	*0.0638*	**0.0291**	0.0515	0.0367

Table 6.21: Values of the total loss over the backtesting sample for each portfolio, using Regulator's loss functions: $VaR_{0.5\%}$.

Portfolio	Lopez			Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBGJRVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	ShVolVaR
	$\alpha = 1, \nu = \infty$		$\alpha = 1, \nu = 10$	$\alpha = 1, \nu = \infty$		$\alpha = 1, \nu = 10$	$\alpha = 1, \nu = \infty$		S$\alpha = 1, \nu = 10$	$\alpha = 1, \nu = \infty$		$\alpha = 1, \nu = 10$
Ptf 1	4.0006	5.0003	2.0001	1.0281	1.7113	0.4208	0.0141	0.0206	0.0034	0.0393	0.0276	0.0155
Ptf 2	4.0005	5.0003	2.0001	1.0032	1.6571	0.4072	0.0133	0.0194	0.0031	0.0375	0.0261	0.0148
Ptf 3	4.0005	5.0003	2.0001	0.9768	1.5994	0.3929	0.0125	0.0183	0.0028	0.0356	0.0245	0.0141
Ptf 4	4.0005	5.0003	2.0001	0.9485	1.5394	0.3777	0.0117	0.0172	0.0025	0.0338	0.0229	0.0133
Ptf 5	5.0004	5.0002	2.0001	0.9208	1.4734	0.3616	0.0109	0.0161	0.0023	0.0319	0.0213	0.0126
Ptf 6	5.0004	3.0002	2.0001	0.8924	1.4230	0.3445	0.0102	0.0150	0.0020	0.0301	0.0201	0.0118
Ptf 7	5.0004	3.0002	2.0001	0.8619	1.3780	0.3263	0.0094	0.0139	0.0017	0.0283	0.0189	0.0111
Ptf 8	5.0003	3.0002	2.0001	0.8291	1.3322	0.3069	0.0087	0.0128	0.0015	0.0264	0.0178	0.0103
Ptf 9	4.0003	3.0002	2.0000	0.7992	1.2894	0.2861	0.0080	0.0118	0.0013	0.0247	0.0168	0.0095
Ptf 10	4.0003	3.0001	2.0000	0.7714	1.2428	0.2639	0.0073	0.0108	0.0011	0.0232	0.0157	0.0087
Ptf 11	4.0002	3.0001	2.0000	0.7414	1.1927	0.2401	0.0067	0.0098	0.0009	0.0216	0.0146	0.0079
Ptf 12	4.0002	3.0001	2.0000	0.7089	1.1310	0.2146	0.0061	0.0088	0.0007	0.0200	0.0135	0.0071
Ptf 13	4.0002	3.0001	2.0000	0.6737	1.0467	0.1871	0.0054	0.0077	0.0006	0.0184	0.0121	0.0063
Ptf 14	4.0002	3.0001	2.0000	0.6353	0.9626	0.1574	0.0049	0.0065	0.0005	0.0168	0.0109	0.0054
Ptf 15	3.0001	3.0001	2.0000	0.6026	0.8684	0.1254	0.0043	0.0054	0.0004	0.0154	0.0096	0.0046
Ptf 16	3.0001	3.0000	2.0000	0.5680	0.7770	0.0905	0.0038	0.0045	0.0003	0.0141	0.0083	0.0037
Ptf 17	3.0001	3.0000	1.0000	0.5301	0.6638	0.0850	0.0033	0.0036	0.0003	0.0128	0.0069	0.0033
Ptf 18	3.0001	3.0000	1.0000	0.4883	0.5533	0.0808	0.0029	0.0029	0.0002	0.0114	0.0056	0.0029
Ptf 19	4.0001	1.0000	1.0000	0.4521	0.4732	0.0752	0.0025	0.0022	0.0002	0.0102	0.0046	0.0026
Ptf 20	5.0001	1.0000	1.0000	0.4544	0.4073	0.0676	0.0022	0.0016	0.0001	0.0096	0.0038	0.0021
Ptf 21	4.0000	1.0000	1.0000	0.4990	0.3402	0.0573	0.0020	0.0011	0.0001	0.0095	0.0031	0.0017
Ptf 22	4.0000	1.0000	1.0000	0.5590	0.2595	0.0433	0.0018	0.0006	0.0001	0.0095	0.0023	0.0012
Ptf 23	5.0000	1.0000	1.0000	0.6361	0.1790	0.0241	0.0017	0.0003	0.0000	0.0102	0.0015	0.0006
Ptf 24	6.0000	1.0000	1.0000	0.7400	0.0925	0.0149	0.0018	0.0001	0.0000	0.0113	0.0008	0.0002
Ptf 25	8.0000	0.0000	1.0000	0.9865	0.0000	0.0603	0.0021	0.0000	0.0000	0.0135	0.0000	0.0008
Ptf 26	10.0000	0.0000	1.0000	1.3253	0.0000	0.1080	0.0029	0.0000	0.0001	0.0159	0.0000	0.0013
Ptf 27	12.0001	2.0000	4.0000	1.8415	0.0420	0.3194	0.0045	0.0000	0.0004	0.0198	0.0005	0.0036
Ptf 28	12.0001	2.0000	5.0000	2.4751	0.1105	0.6288	0.0070	0.0001	0.0010	0.0248	0.0013	0.0070
Ptf 29	15.0001	2.0000	7.0000	3.1130	0.1851	0.9875	0.0101	0.0002	0.0020	0.0295	0.0021	0.0105
Ptf 30	18.0001	4.0000	9.0000	4.0161	0.3123	1.4283	0.0134	0.0004	0.0032	0.0357	0.0035	0.0151
Ptf 31	19.0002	5.0000	9.0001	4.7237	0.5001	1.7203	0.0161	0.0007	0.0043	0.0406	0.0054	0.0178
Ptf 32	20.0002	5.0000	9.0001	4.9940	0.7047	1.7735	0.0169	0.0012	0.0046	0.0424	0.0074	0.0181
Ptf 33	22.0001	5.0000	9.0000	4.8562	0.8960	1.5403	0.0152	0.0019	0.0039	0.0413	0.0090	0.0156
Ptf 34	22.0001	6.0000	9.0000	4.2329	1.1188	1.2035	0.0121	0.0028	0.0027	0.0366	0.0107	0.0123
Ptf 35	22.0001	6.0000	8.0000	3.5708	1.3270	0.7651	0.0088	0.0039	0.0015	0.0315	0.0122	0.0076
Ptf 36	19.0001	8.0000	4.0000	2.7871	1.4016	0.4419	0.0060	0.0040	0.0010	0.0247	0.0130	0.0041
Ptf 37	16.0000	8.0000	2.0000	2.1881	1.4710	0.3372	0.0046	0.0041	0.0007	0.0200	0.0138	0.0032
Ptf 38	15.0000	8.0000	2.0000	1.7281	1.5032	0.2532	0.0035	0.0040	0.0004	0.0170	0.0145	0.0025
Ptf 39	11.0000	9.0000	2.0000	1.4106	1.4792	0.1576	0.0026	0.0035	0.0002	0.0149	0.0148	0.0015
Mean	*8.7438*	3.5898	**3.1026**	*1.5787*	0.8883	**0.4276**	*0.0072*	0.0061	**0.0013**	*0.0233*	0.0107	**0.0075**

Table 6.22: Values of the total loss over the backtesting sample for each portfolio, using Regulator's loss functions: $VaR_{0.1\%}$.

Portfolio	Caporin1			Caporin2			Caporin3		
name	ShVolVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBVaR	ShVolVaR
	$\alpha = 0.275, \nu = \infty$		$\alpha = 0.275, \nu = 8$	$\alpha = 0.275, \nu = \infty$		$\alpha = 0.275, \nu = 8$	$\alpha = 0.275, \nu = \infty$		$\alpha = 0.275, \nu = 8$
Ptf 1	2,108.37	2,108.94	2,095.10	27.04	28.35	26.25	61.19	62.71	60.12
Ptf 2	2,107.90	2,108.42	2,094.65	26.24	27.47	25.47	59.44	60.85	58.39
Ptf 3	2,107.32	2,107.31	2,094.16	25.45	26.59	24.70	57.68	58.99	56.67
Ptf 4	2,106.87	2,106.23	2,093.65	24.65	25.71	23.93	55.94	57.14	54.96
Ptf 5	2,106.48	2,104.46	2,093.09	23.86	24.82	23.16	54.20	55.28	53.25
Ptf 6	2,105.88	2,102.62	2,092.58	23.07	23.93	22.40	52.47	53.43	51.55
Ptf 7	2,105.42	2,100.09	2,092.00	22.29	23.04	21.64	50.75	51.57	49.86
Ptf 8	2,104.88	2,097.89	2,091.35	21.51	22.17	20.88	49.04	49.74	48.18
Ptf 9	2,104.31	2,094.96	2,090.64	20.73	21.30	20.12	47.35	47.92	46.51
Ptf 10	2,103.54	2,091.31	2,089.73	19.96	20.43	19.37	45.66	46.11	44.85
Ptf 11	2,102.65	2,087.29	2,088.82	19.19	19.58	18.63	43.98	44.32	43.21
Ptf 12	2,101.64	2,083.04	2,087.81	18.44	18.73	17.89	42.32	42.53	41.57
Ptf 13	2,100.17	2,078.55	2,086.43	17.68	17.88	17.16	40.67	40.75	39.96
Ptf 14	2,098.37	2,072.83	2,084.87	16.94	17.05	16.44	39.04	39.00	38.36
Ptf 15	2,095.94	2,068.47	2,082.46	16.20	16.24	15.72	37.43	37.30	36.77
Ptf 16	2,093.06	2,064.16	2,079.42	15.47	15.45	15.01	35.84	35.62	35.21
Ptf 17	2,089.65	2,059.81	2,075.85	14.75	14.68	14.31	34.28	33.98	33.67
Ptf 18	2,085.46	2,054.47	2,071.62	14.04	13.92	13.62	32.74	32.38	32.16
Ptf 19	2,081.08	2,048.85	2,067.39	13.34	13.17	12.94	31.24	30.81	30.68
Ptf 20	2,077.77	2,044.31	2,063.90	12.67	12.47	12.30	29.78	29.29	29.25
Ptf 21	2,075.79	2,042.13	2,062.08	12.03	11.80	11.68	28.36	27.83	27.86
Ptf 22	2,074.55	2,042.06	2,060.72	11.44	11.17	11.10	27.00	26.42	26.53
Ptf 23	2,073.40	2,039.61	2,059.49	10.88	10.59	10.55	25.70	25.08	25.25
Ptf 24	2,072.31	2,044.93	2,058.45	10.35	10.08	10.05	24.47	23.86	24.04
Ptf 25	2,071.62	2,050.27	2,058.02	9.87	9.61	9.58	23.32	22.74	22.91
Ptf 26	2,072.59	2,059.24	2,059.30	9.44	9.22	9.17	22.26	21.77	21.87
Ptf 27	2,072.41	2,067.76	2,059.27	9.05	8.90	8.79	21.32	20.93	20.94
Ptf 28	2,071.97	2,078.14	2,059.14	8.72	8.65	8.47	20.50	20.27	20.14
Ptf 29	2,071.77	2,088.92	2,058.78	8.45	8.48	8.22	19.83	19.78	19.49
Ptf 30	2,068.11	2,096.07	2,055.55	8.24	8.39	8.02	19.33	19.47	18.99
Ptf 31	2,063.96	2,101.13	2,051.39	8.09	8.34	7.87	19.00	19.32	18.67
Ptf 32	2,055.13	2,098.14	2,042.71	7.98	8.33	7.76	18.86	19.33	18.54
Ptf 33	2,043.15	2,090.26	2,030.64	7.92	8.36	7.70	18.92	19.53	18.59
Ptf 34	2,028.71	2,079.92	2,015.34	7.93	8.45	7.71	19.17	19.89	18.84
Ptf 35	2,014.17	2,064.23	2,000.49	8.01	8.54	7.78	19.61	20.35	19.28
Ptf 36	2,001.13	2,049.01	1,987.45	8.17	8.69	7.94	20.23	20.93	19.88
Ptf 37	1,990.86	2,034.18	1,976.57	8.41	8.89	8.16	20.99	21.63	20.63
Ptf 38	1,983.14	2,017.26	1,968.69	8.73	9.11	8.47	21.89	22.37	21.51
Ptf 39	1,979.62	2,005.41	1,965.17	9.13	9.41	8.86	22.91	23.20	22.51
Mean	*2,073.62*	2,072.63	**2,060.12**	14.52	*14.82*	**14.10**	33.71	*33.96*	**33.12**

Table 6.23: Values of the total loss over the backtesting sample for each portfolio, using Investors' loss functions: $VaR_{5\%}$.

Portfolio name	Caporin1				Caporin2				Caporin3			
	ShVolVaR $\alpha = 0.5, \nu = \infty$	HFBEVaR	HFBVaR	ShVolVaR $\alpha = 0.5, \nu = 8$	ShVolVaR $\alpha = 0.5, \nu = \infty$	HFBEVaR	HFBVaR	ShVolVaR $\alpha = 0.5, \nu = 8$	ShVolVaR $\alpha = 0.5, \nu = \infty$	HFBEVaR	HFBVaR	ShVolVaR $\alpha = 0.5, \nu = 8$
Ptf 1	2,436.40	2,397.36	2,436.97	2,500.73	48.26	50.12	52.06	54.05	87.29	89.39	91.39	93.93
Ptf 2	2,436.34	2,399.34	2,435.61	2,500.71	46.86	48.90	50.48	52.49	84.79	87.10	88.68	91.24
Ptf 3	2,436.21	2,401.47	2,434.07	2,500.58	45.46	47.68	48.92	50.93	82.30	84.81	86.01	88.56
Ptf 4	2,436.01	2,403.63	2,432.51	2,500.40	44.07	46.47	47.37	49.37	79.82	82.53	83.35	85.89
Ptf 5	2,435.73	2,405.92	2,430.70	2,500.13	42.68	45.27	45.80	47.82	77.35	80.26	80.68	83.22
Ptf 6	2,435.32	2,408.53	2,428.79	2,499.75	41.30	44.06	44.21	46.27	74.88	78.00	77.99	80.57
Ptf 7	2,434.83	2,411.27	2,426.74	2,499.29	39.92	42.87	42.59	44.72	72.43	75.76	75.29	77.93
Ptf 8	2,434.20	2,414.29	2,424.57	2,498.65	38.54	41.69	40.99	43.18	69.99	73.54	72.59	75.30
Ptf 9	2,433.44	2,417.70	2,422.90	2,497.90	37.18	40.52	39.41	41.65	67.56	71.34	69.94	72.69
Ptf 10	2,432.47	2,421.41	2,421.32	2,496.95	35.81	39.36	37.88	40.13	65.15	69.16	67.35	70.09
Ptf 11	2,431.33	2,425.48	2,419.61	2,495.88	34.46	38.21	36.38	38.62	62.76	66.99	64.80	67.51
Ptf 12	2,430.05	2,429.83	2,417.97	2,494.61	33.12	37.09	34.93	37.11	60.38	64.85	62.31	64.95
Ptf 13	2,428.46	2,434.44	2,416.23	2,493.07	31.79	35.98	33.51	35.62	58.02	62.74	59.85	62.41
Ptf 14	2,426.54	2,439.38	2,414.89	2,491.28	30.46	34.88	32.10	34.15	55.69	60.66	57.42	59.90
Ptf 15	2,424.06	2,444.37	2,414.06	2,489.01	29.15	33.80	30.73	32.68	53.38	58.61	55.06	57.42
Ptf 16	2,421.19	2,450.68	2,413.30	2,486.22	27.85	32.74	29.39	31.22	51.10	56.60	52.74	54.96
Ptf 17	2,417.98	2,457.40	2,412.74	2,483.01	26.56	31.71	28.08	29.79	48.86	54.64	50.48	52.55
Ptf 18	2,414.10	2,464.51	2,412.92	2,479.30	25.29	30.69	26.79	28.36	46.65	52.72	48.27	50.17
Ptf 19	2,409.53	2,472.33	2,413.72	2,474.91	24.04	29.71	25.55	26.97	44.49	50.85	46.13	47.84
Ptf 20	2,404.70	2,481.27	2,415.14	2,470.14	22.83	28.77	24.39	25.61	42.37	49.04	44.08	45.57
Ptf 21	2,400.10	2,491.21	2,417.23	2,465.15	21.66	27.87	23.29	24.30	40.32	47.29	42.12	43.35
Ptf 22	2,395.52	2,502.17	2,421.43	2,460.22	20.54	27.02	22.27	23.04	38.34	45.61	40.25	41.21
Ptf 23	2,390.97	2,513.85	2,426.16	2,455.15	19.47	26.22	21.33	21.84	36.43	44.00	38.50	39.15
Ptf 24	2,386.19	2,525.94	2,432.84	2,450.08	18.45	25.46	20.49	20.70	34.62	42.47	36.90	37.20
Ptf 25	2,381.68	2,537.94	2,444.14	2,444.70	17.50	24.75	19.81	19.63	32.91	41.03	35.53	35.36
Ptf 26	2,377.06	2,549.72	2,458.64	2,439.78	16.63	24.09	19.35	18.64	31.33	39.68	34.44	33.66
Ptf 27	2,371.45	2,559.99	2,474.21	2,434.04	15.83	23.45	19.02	17.74	29.91	38.45	33.56	32.12
Ptf 28	2,364.52	2,568.14	2,490.66	2,426.97	15.12	22.86	18.83	16.94	28.65	37.33	32.91	30.77
Ptf 29	2,358.36	2,573.70	2,504.95	2,420.19	14.54	22.31	18.74	16.28	27.61	36.33	32.43	29.64
Ptf 30	2,350.89	2,574.98	2,514.49	2,412.16	14.06	21.77	18.66	15.75	26.80	35.47	32.08	28.76
Ptf 31	2,343.16	2,572.15	2,516.73	2,404.57	13.71	21.26	18.54	15.36	26.25	34.76	31.78	28.17
Ptf 32	2,334.20	2,563.28	2,509.80	2,396.91	13.48	20.75	18.33	15.10	25.98	34.21	31.55	27.88
Ptf 33	2,325.00	2,548.38	2,497.69	2,388.76	13.38	20.27	18.04	15.00	26.01	33.83	31.37	27.92
Ptf 34	2,314.86	2,527.57	2,480.49	2,380.20	13.42	19.83	17.76	15.06	26.33	33.63	31.33	28.26
Ptf 35	2,306.16	2,502.48	2,462.33	2,372.54	13.62	19.46	17.67	15.30	26.93	33.60	31.60	28.91
Ptf 36	2,299.05	2,474.99	2,442.05	2,365.99	13.96	19.19	17.72	15.70	27.78	33.77	32.11	29.83
Ptf 37	2,292.63	2,446.49	2,423.09	2,360.72	14.44	19.04	17.93	16.25	28.86	34.13	32.85	30.99
Ptf 38	2,288.78	2,419.39	2,406.60	2,357.71	15.04	19.01	18.33	16.95	30.13	34.67	33.88	32.37
Ptf 39	2,287.82	2,394.99	2,391.44	2,357.48	15.78	19.12	18.83	17.79	31.56	35.35	35.03	33.93
Mean	**2,387.88**	*2,472.51*	2,439.99	2,452.46	**25.80**	*30.88*	28.63	28.93	**47.74**	*53.47*	50.89	51.34

Table 6.24: Values of the total loss over the backtesting sample for each portfolio, using Investors' loss functions: $VaR_{1\%}$.

Portfolio	Caporin1				Caporin2				Caporin3			
name	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR	ShVolVaR	HFBGJRVaR	HFBVaR	ShVolVaR
	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$	$\alpha = 0.75, \nu = \infty$			$\alpha = 0.75, \nu = 8$
Ptf 1	2,571.55	2,503.48	2,516.73	2,662.97	60.09	59.81	61.27	71.55	100.77	100.32	101.74	113.43
Ptf 2	2,571.62	2,505.76	2,515.86	2,663.02	58.36	58.36	59.43	69.50	97.89	97.76	98.76	110.19
Ptf 3	2,571.59	2,508.17	2,515.11	2,662.99	56.64	56.93	57.63	67.45	95.03	95.22	95.82	106.96
Ptf 4	2,571.52	2,510.87	2,514.42	2,662.92	54.92	55.50	55.85	65.40	92.17	92.70	92.90	103.75
Ptf 5	2,571.38	2,513.83	2,513.83	2,662.79	53.21	54.09	54.05	63.36	89.33	90.20	89.98	100.54
Ptf 6	2,571.11	2,517.11	2,512.73	2,662.55	51.50	52.69	52.25	61.32	86.49	87.71	87.05	97.35
Ptf 7	2,570.74	2,520.57	2,511.20	2,662.22	49.79	51.30	50.44	59.29	83.66	85.25	84.12	94.16
Ptf 8	2,570.20	2,524.23	2,509.72	2,661.74	48.09	49.93	48.64	57.27	80.85	82.81	81.21	91.00
Ptf 9	2,569.52	2,528.25	2,508.09	2,661.14	46.39	48.58	46.85	55.26	78.05	80.39	78.31	87.84
Ptf 10	2,568.63	2,532.61	2,506.35	2,660.35	44.71	47.24	45.08	53.25	75.27	77.99	75.45	84.71
Ptf 11	2,567.57	2,537.47	2,505.53	2,659.39	43.03	45.91	43.37	51.26	72.50	75.62	72.67	81.59
Ptf 12	2,566.35	2,542.63	2,505.83	2,658.25	41.36	44.59	41.72	49.27	69.76	73.27	69.96	78.50
Ptf 13	2,564.82	2,548.27	2,506.66	2,656.82	39.71	43.30	40.12	47.30	67.03	70.95	67.30	75.42
Ptf 14	2,562.92	2,554.16	2,507.92	2,655.07	38.06	42.02	38.56	45.35	64.33	68.65	64.71	72.38
Ptf 15	2,560.47	2,560.09	2,509.32	2,652.82	36.43	40.75	37.01	43.41	61.66	66.39	62.15	69.37
Ptf 16	2,557.47	2,566.26	2,510.87	2,650.13	34.80	39.50	35.50	41.48	59.01	64.16	59.65	66.39
Ptf 17	2,554.01	2,572.86	2,511.69	2,646.96	33.20	38.27	34.00	39.57	56.41	61.96	57.16	63.45
Ptf 18	2,549.74	2,579.42	2,512.97	2,643.07	31.61	37.06	32.52	37.69	53.84	59.81	54.73	60.55
Ptf 19	2,544.64	2,586.23	2,513.82	2,638.42	30.04	35.87	31.05	35.83	51.32	57.70	52.34	57.71
Ptf 20	2,539.21	2,593.56	2,516.26	2,633.31	28.51	34.73	29.66	34.01	48.85	55.66	50.04	54.93
Ptf 21	2,533.38	2,601.80	2,520.93	2,627.73	27.03	33.63	28.40	32.25	46.44	53.67	47.89	52.21
Ptf 22	2,527.26	2,610.98	2,526.38	2,621.76	25.60	32.59	27.22	30.55	44.10	51.76	45.86	49.58
Ptf 23	2,520.75	2,620.19	2,534.04	2,615.30	24.23	31.59	26.21	28.91	41.85	49.92	44.02	47.04
Ptf 24	2,513.69	2,629.88	2,544.16	2,608.12	22.91	30.65	25.35	27.34	39.71	48.18	42.39	44.62
Ptf 25	2,506.09	2,639.95	2,556.79	2,600.50	21.67	29.77	24.66	25.86	37.68	46.54	40.98	42.33
Ptf 26	2,498.08	2,650.00	2,572.24	2,592.17	20.52	28.95	24.15	24.48	35.79	45.03	39.81	40.19
Ptf 27	2,488.83	2,659.05	2,587.46	2,582.68	19.45	28.19	23.72	23.20	34.07	43.64	38.80	38.25
Ptf 28	2,478.38	2,666.19	2,600.31	2,572.13	18.49	27.48	23.35	22.06	32.54	42.39	37.92	36.52
Ptf 29	2,467.91	2,671.36	2,610.50	2,561.68	17.68	26.85	23.01	21.09	31.24	41.29	37.15	35.06
Ptf 30	2,456.10	2,672.78	2,617.36	2,550.96	17.00	26.25	22.75	20.28	30.20	40.36	36.59	33.89
Ptf 31	2,444.80	2,670.08	2,621.36	2,540.77	16.49	25.66	22.59	19.68	29.47	39.57	36.26	33.06
Ptf 32	2,434.80	2,661.81	2,618.59	2,531.03	16.13	25.09	22.44	19.26	29.08	38.95	36.09	32.61
Ptf 33	2,425.27	2,648.17	2,612.43	2,522.34	15.96	24.53	22.38	19.07	29.03	38.50	36.18	32.55
Ptf 34	2,415.37	2,629.13	2,601.68	2,514.11	15.97	24.00	22.38	19.11	29.33	38.22	36.47	32.88
Ptf 35	2,407.06	2,605.97	2,586.72	2,507.77	16.18	23.55	22.41	19.39	29.95	38.13	36.89	33.59
Ptf 36	2,400.71	2,579.82	2,567.14	2,502.78	16.58	23.17	22.42	19.90	30.88	38.21	37.38	34.63
Ptf 37	2,395.62	2,551.92	2,545.57	2,499.52	17.15	22.92	22.51	20.60	32.07	38.49	38.03	35.99
Ptf 38	2,393.10	2,524.25	2,519.68	2,497.93	17.89	22.83	22.52	21.50	33.49	38.99	38.65	37.60
Ptf 39	2,392.87	2,499.31	2,495.80	2,498.51	18.78	22.94	22.69	22.58	35.11	39.72	39.43	39.43
Mean	**2,512.18**	2,579.45	2,540.10	*2,606.79*	**31.95**	37.10	34.52	*38.10*	**54.78**	60.41	57.51	*61.60*

Table 6.25: Values of the total loss over the backtesting sample for each portfolio, using Investors' loss functions: $VaR_{0.5\%}$.

6.2.2 ShVolVaR versus EVTVaR

In chapter 5 we notice that Extreme Value Theory Historical Filtered Bootstrap is slightly better than common HFBVaR.

We test both symmetric and asymmetric version of EVTVaR ((EVTHFBVaR and EVTHFBGJRVaR respectively). As the GJR specification has a better VaR prediction with respect to EGARCH we adopt this for the estimation of asymmetric EVTVaR. In order to achieve better VaR estimation at each confidence level, one should use the values of the parameters α and ν, as reported in Table 6.27, in the ShVolVaR model. By doing so, ShVolVaR can achieve the same results as the more commonly VaR models in the finance industry.

We apply the method reported above and compare the best result of our model with those of EVTVaR and HFBVaR. In Tables 6.28, 6.29, 6.30 and 6.31, we report comparisons of the UC, IND and CC tests. For simplicity, we only report the best model among HFB and EVT models.

ShVolVaR is absolutely comparable to the industry models HFB and EVTHFB at 95% c.l.. It performs badly at 99% c.l., where it suffers of dependence of violation when $47.5\% \leq x^{Eqty} \leq 60\%$ and $x^{Eqty} = 35\%$; it produces two rejections

Portfolio name	Caporin1			Caporin2			Caporin3		
	ShVolVaR $\alpha=1,\nu=\infty$	HFBGJRVaR	ShVolVaR $\alpha=1,\nu=10$	ShVolVaR $\alpha=1,\nu=\infty$	HFBGJRVaR	ShVolVaR $\alpha=1,\nu=10$	ShVolVaR $\alpha=1,\nu=\infty$	HFBGJRVaR	ShVolVaR $\alpha=1,\nu=10$
Ptf 1	2,740.84	2,700.29	2,848.18	83.03	101.24	106.31	125.84	144.56	150.55
Ptf 2	2,740.91	2,701.86	2,848.26	80.67	98.72	103.28	122.26	140.83	146.27
Ptf 3	2,740.92	2,703.49	2,848.27	78.30	96.20	100.25	118.69	137.12	142.00
Ptf 4	2,740.87	2,705.25	2,848.25	75.94	93.69	97.23	115.14	133.42	137.74
Ptf 5	2,740.76	2,707.18	2,848.17	73.59	91.19	94.22	111.59	129.73	133.49
Ptf 6	2,740.56	2,709.29	2,848.01	71.24	88.69	91.21	108.05	126.05	129.26
Ptf 7	2,740.26	2,711.64	2,847.78	68.89	86.21	88.21	104.53	122.40	125.04
Ptf 8	2,739.82	2,714.13	2,847.43	66.56	83.73	85.22	101.02	118.76	120.84
Ptf 9	2,739.27	2,716.85	2,846.98	64.23	81.28	82.25	97.53	115.15	116.66
Ptf 10	2,738.55	2,719.64	2,846.38	61.91	78.83	79.28	94.06	111.56	112.50
Ptf 11	2,737.68	2,722.67	2,845.66	59.60	76.40	76.33	90.60	107.99	108.36
Ptf 12	2,736.62	2,726.01	2,844.78	57.31	73.99	73.40	87.17	104.45	104.25
Ptf 13	2,735.28	2,729.68	2,843.67	55.03	71.62	70.48	83.76	100.96	100.17
Ptf 14	2,733.63	2,733.58	2,842.29	52.76	69.27	67.59	80.38	97.51	96.12
Ptf 15	2,731.50	2,737.57	2,840.50	50.51	66.95	64.71	77.03	94.10	92.11
Ptf 16	2,728.88	2,742.18	2,838.31	48.27	64.67	61.85	73.72	90.76	88.14
Ptf 17	2,725.77	2,747.26	2,835.77	46.05	62.44	59.02	70.45	87.48	84.23
Ptf 18	2,721.91	2,752.63	2,832.61	43.86	60.25	56.22	67.22	84.27	80.36
Ptf 19	2,717.32	2,758.54	2,828.83	41.69	58.10	53.46	64.04	81.14	76.56
Ptf 20	2,712.28	2,765.17	2,824.61	39.57	56.02	50.76	60.93	78.08	72.83
Ptf 21	2,706.70	2,772.42	2,819.86	37.50	53.99	48.12	57.89	75.10	69.19
Ptf 22	2,700.58	2,780.27	2,814.63	35.49	52.03	45.55	54.94	72.20	65.65
Ptf 23	2,693.67	2,788.61	2,808.65	33.56	50.14	43.08	52.08	69.42	62.23
Ptf 24	2,685.87	2,797.12	2,801.71	31.69	48.31	40.69	49.34	66.74	58.94
Ptf 25	2,677.31	2,805.85	2,793.92	29.92	46.56	38.43	46.74	64.18	55.82
Ptf 26	2,667.66	2,814.07	2,785.13	28.25	44.86	36.29	44.30	61.73	52.89
Ptf 27	2,656.51	2,821.30	2,775.14	26.70	43.24	34.31	42.06	59.44	50.21
Ptf 28	2,643.88	2,826.43	2,764.10	25.30	41.66	32.52	40.05	57.28	47.80
Ptf 29	2,630.84	2,829.40	2,752.75	24.08	40.14	30.96	38.32	55.28	45.73
Ptf 30	2,617.64	2,828.77	2,740.90	23.05	38.68	29.66	36.92	53.46	44.04
Ptf 31	2,605.01	2,824.39	2,729.63	22.25	37.31	28.65	35.89	51.86	42.80
Ptf 32	2,592.63	2,815.52	2,718.93	21.68	36.04	27.95	35.27	50.54	42.06
Ptf 33	2,581.64	2,802.11	2,709.50	21.37	34.94	27.60	35.09	49.55	41.84
Ptf 34	2,571.80	2,785.39	2,701.86	21.34	34.07	27.61	35.34	48.94	42.16
Ptf 35	2,564.81	2,766.54	2,696.38	21.59	33.52	27.99	36.03	48.78	42.98
Ptf 36	2,559.96	2,747.16	2,692.94	22.12	33.29	28.72	37.11	49.07	44.28
Ptf 37	2,557.03	2,727.96	2,691.52	22.89	33.40	29.75	38.54	49.78	46.00
Ptf 38	2,556.09	2,710.02	2,691.53	23.89	33.77	31.06	40.26	50.84	48.07
Ptf 39	2,557.29	2,693.72	2,692.93	25.09	34.32	32.63	42.24	52.10	50.44
Mean	**2,679.76**	2,754.92	*2,795.81*	**44.02**	*59.74*	56.48	**68.01**	*84.43*	81.30

Table 6.26: Values of the total loss over the backtesting sample for each portfolio, using Investors' loss functions: $VaR_{0.1\%}$.

	Best VaR among industry models	ShVolVaR		
		α	ν	x^{Eqty}
$VaR_{5\%}$	EVTHFBGJRVaR	$\in [0.275, 0.325]$	Any	Any
$VaR_{1\%}$	HFBVaR	0.5	> 200	$\geq 45\%$
	EFBEVaR	0.5	$\simeq 8$	$< 45\%$
$VaR_{0.5\%}$	HFBGJRVaR	0.75	> 200	$\geq 55\%$
		0.75	$\simeq 8$	$< 55\%$
$VaR_{0.1\%}$	HFBGJRVaR	1	$\simeq 10$	Any

Table 6.27: Shrunk Volatility VaR and Industry models VaR: suggested values for α, ν and x^{Eqty} according to the VaR confidence interval.

of the UC test. Looking at more extreme VaR, the ShVolVaR estimations are quite similar to those of EVTHFBGJRVaR (none of the models present rejection at 99% c.l.). HFBGJRVaR seems to be the best model in this particular period and dataset to estimate the tail.

Our last comparison has to do with the regulators' and investors' loss functions. In Table 6.32, we report the average value for the regulators' loss functions calculated on the 39 portfolios. As before, we report in boldface the best results and in italic the worst. Doing so makes it easy to see that EVTHFBGJRVaR has the best result for $VaR_{5\%}$ and $VaR_{1\%}$, while HFVGJRVaR has the best result for $VaR_{0.5\%}$. ShVolVaR si the best model in evaluating $VaR_{0.1\%}$. From the investors' viewpoint (see Table 6.33) ShVolVaR has the best result for $VaR_{5\%}$ and $VaR_{1\%}$ and it demonstrates the best result two out three times for $VaR_{0.5\%}$. EVTHFBVaR has the best results for $VaR_{0.1\%}$ in two out of three times. We end up with the same result we have in the Chapter 5: ShVolVaR is a good model under the investors' loss function, but the opposite is true under a regulators'.

6.3 Conclusions

In this chapter we have generalized the Shrunk Volatility VaR, to estimate balanced portfolios' VaR. The forecasting power of ShVolVaR model is compared with that of the best methods for estimating VaR, such as the HFBVaR and its asymmetric specifications. All models are validated both on both their statistical accuracy and via loss functions framework (by means of the Unconditional Coverage, the Independence, and the Conditional Coverage tests).

Although the ShVolVaR model is based on strong assumptions such as those of RiskMetrics - namely, the one-day-head returns are normally distributed with zero mean - its forecasting power is comparable to that of the more

Ptfs	ShVolVaR			HFBVaR			EVTHFBGJRVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	78.95	29.03	55.17	51.14	39.97	56.54	65.74	45.36	68.44
Ptf 2	72.99	27.51	51.93	56.26	41.93	61.03	77.60	41.32	68.71
Ptf 3	56.26	41.93	61.03	72.99	48.17	73.56	71.59	43.31	68.84
Ptf 4	91.23	54.91	83.07	56.26	93.44	84.28	89.95	62.13	87.81
Ptf 5	91.23	54.91	83.07	72.99	97.91	94.18	71.59	69.77	86.79
Ptf 6	72.99	48.17	73.56	78.95	95.02	96.31	77.60	96.35	95.93
Ptf 7	91.23	54.91	83.07	61.62	76.51	84.34	96.23	87.80	98.72
Ptf 8	83.73	39.39	68.08	41.70	87.91	71.11	77.60	96.35	95.93
Ptf 9	65.74	45.36	68.44	46.28	87.74	75.47	83.73	93.48	97.59
Ptf 10	54.65	49.63	66.15	56.26	93.44	84.28	60.08	75.06	82.91
Ptf 11	49.46	51.84	64.29	56.26	93.44	84.28	54.65	77.76	80.11
Ptf 12	49.46	51.84	64.29	56.26	93.44	84.28	60.08	47.47	67.54
Ptf 13	49.46	51.84	64.29	67.21	73.71	86.42	49.46	51.84	64.29
Ptf 14	60.08	47.47	67.54	85.04	92.13	97.76	54.65	49.63	66.15
Ptf 15	60.08	47.47	67.54	91.23	82.19	96.91	60.08	47.47	67.54
Ptf 16	49.46	51.84	64.29	37.41	56.24	56.96	83.73	64.63	88.13
Ptf 17	44.53	54.11	62.00	37.41	82.11	65.67	96.23	59.67	86.84
Ptf 18	21.22	68.78	42.36	41.70	84.91	70.65	91.23	54.91	83.07
Ptf 19	60.08	47.47	67.54	20.30	94.72	44.37	61.62	96.32	88.10
Ptf 20	44.53	54.11	62.00	8.37	96.62	22.38	67.21	71.33	85.46
Ptf 21	60.08	47.47	67.54	11.45	90.81	28.60	72.99	97.91	94.18
Ptf 22	54.65	49.63	66.15	26.28	73.68	50.49	72.99	70.95	87.90
Ptf 23	35.55	88.86	64.61	23.15	21.59	22.73	83.73	52.69	80.15
Ptf 24	31.51	77.87	58.03	46.28	1.82	4.71	54.65	64.13	74.80
Ptf 25	21.22	69.56	42.54	78.95	0.91	3.23	35.55	80.69	63.33
Ptf 26	49.46	61.50	69.78	97.49	1.49	5.16	15.84	40.08	25.99
Ptf 27	60.08	95.01	87.04	83.73	2.42	7.71	15.84	40.08	25.99
Ptf 28	65.74	8.05	19.68	65.74	0.73	2.48	9.78	34.00	16.11
Ptf 29	65.74	27.64	50.12	54.65	1.39	4.05	6.87	51.50	15.43
Ptf 30	71.59	8.76	21.77	71.59	1.95	6.11	8.22	32.11	13.50
Ptf 31	65.74	15.50	32.97	44.53	11.43	21.49	9.78	18.71	10.64
Ptf 32	78.95	16.34	36.53	44.53	11.43	21.49	8.22	53.94	18.30
Ptf 33	67.21	18.81	38.45	35.55	9.73	16.51	9.78	85.40	24.97
Ptf 34	89.95	80.66	96.28	31.51	17.42	23.98	5.71	49.11	12.91
Ptf 35	78.95	68.22	88.74	54.65	41.17	59.53	9.78	85.40	24.97
Ptf 36	97.49	86.37	98.49	21.22	69.56	42.54	6.87	55.65	16.05
Ptf 37	78.95	45.55	73.04	35.55	53.88	54.02	6.87	55.65	16.05
Ptf 38	89.95	55.16	83.09	27.77	49.05	43.74	6.87	55.65	16.05
Ptf 39	61.62	68.71	81.32	31.51	91.69	60.04	6.87	29.44	11.01

Table 6.28: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 95%

Ptfs	ShVolVaR			HFBVaR			EVTHFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	48.97	7.48	16.11	13.25	12.20	9.75	85.29	5.35	15.25
Ptf 2	30.76	9.18	14.35	9.63	13.34	8.13	73.80	31.01	56.49
Ptf 3	30.76	9.18	14.35	17.88	54.91	33.85	87.33	33.15	61.62
Ptf 4	23.68	10.13	12.97	17.88	54.91	33.85	85.29	37.62	66.45
Ptf 5	23.68	10.13	12.97	23.68	52.30	40.51	85.29	37.62	66.45
Ptf 6	30.76	9.18	14.35	23.68	52.30	40.51	85.29	37.62	66.45
Ptf 7	23.68	10.13	12.97	23.68	52.30	40.51	72.23	39.94	65.82
Ptf 8	30.76	9.18	14.35	17.88	54.91	33.85	60.04	42.32	63.26
Ptf 9	39.19	8.30	15.43	17.88	54.91	33.85	30.76	49.74	47.20
Ptf 10	39.19	8.30	15.43	23.68	52.30	40.51	30.76	49.74	47.20
Ptf 11	48.97	7.48	16.11	30.76	49.74	47.20	17.88	54.91	33.85
Ptf 12	72.23	6.01	16.03	30.76	49.74	47.20	23.68	52.30	40.51
Ptf 13	98.90	4.75	14.03	17.88	11.13	11.40	30.76	9.18	14.35
Ptf 14	85.29	5.35	15.25	13.25	57.55	27.57	30.76	9.18	14.35
Ptf 15	98.90	4.75	14.03	17.88	11.13	11.40	17.88	1.55	2.16
Ptf 16	85.29	0.51	1.95	23.68	10.13	12.97	39.19	8.30	15.43
Ptf 17	85.29	0.51	1.95	30.76	9.18	14.35	48.97	44.74	59.03
Ptf 18	73.80	0.29	1.13	23.68	10.13	12.97	23.68	10.13	12.97
Ptf 19	73.80	0.29	1.13	39.19	8.30	15.43	39.19	8.30	15.43
Ptf 20	98.90	0.43	1.69	30.76	9.18	14.35	60.04	42.32	63.26
Ptf 21	39.19	0.99	2.50	85.29	5.35	15.25	72.23	39.94	65.82
Ptf 22	17.88	1.55	2.16	85.29	37.62	66.45	85.29	37.62	66.45
Ptf 23	38.39	2.44	5.44	98.90	35.36	65.02	87.33	33.15	61.62
Ptf 24	60.91	3.23	8.88	87.33	33.15	61.62	98.90	35.36	65.02
Ptf 25	87.33	4.20	12.49	60.04	6.72	16.32	72.23	39.94	65.82
Ptf 26	48.97	0.85	2.46	48.97	7.48	16.11	17.88	11.13	11.40
Ptf 27	72.23	6.01	16.03	39.19	8.30	15.43	23.68	10.13	12.97
Ptf 28	48.97	7.48	16.11	72.23	6.01	16.03	13.25	12.20	9.75
Ptf 29	3.29	17.14	4.04	98.90	4.75	14.03	30.76	9.18	14.35
Ptf 30	0.95	21.57	1.61	73.80	3.69	10.72	72.23	6.01	16.03
Ptf 31	0.38	82.42	1.48	60.91	28.93	50.05	39.19	8.30	15.43
Ptf 32	2.22	18.55	3.05	85.29	37.62	66.45	23.68	52.30	40.51
Ptf 33	2.22	18.55	3.05	48.97	44.74	59.03	30.76	49.74	47.20
Ptf 34	3.29	25.04	5.32	60.04	42.32	63.26	39.19	47.22	53.53
Ptf 35	3.29	25.04	5.32	48.97	44.74	59.03	39.19	47.22	53.53
Ptf 36	4.80	26.07	7.52	87.33	33.15	61.62	98.90	35.36	65.02
Ptf 37	17.88	30.48	23.93	85.29	37.62	66.45	85.29	37.62	66.45
Ptf 38	48.97	35.34	51.22	98.90	35.36	65.02	98.90	35.36	65.02
Ptf 39	85.29	39.29	68.24	98.90	35.36	65.02	73.80	31.01	56.49

Table 6.29: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99%

Ptfs	ShVolVaR			HFBGJRVaR			EVTHFBGJRVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	12.09	78.91	28.98	62.64	71.51	83.10	81.35	69.70	90.15
Ptf 2	12.09	78.91	28.98	99.22	67.90	91.79	81.35	69.70	90.15
Ptf 3	12.09	78.91	28.98	99.22	67.90	91.79	81.35	69.70	90.15
Ptf 4	20.25	77.04	42.54	80.21	66.12	88.03	62.64	71.51	83.10
Ptf 5	20.25	77.04	42.54	99.22	67.90	91.79	62.64	71.51	83.10
Ptf 6	20.25	77.04	42.54	80.21	66.12	88.03	62.64	71.51	83.10
Ptf 7	20.25	77.04	42.54	99.22	67.90	91.79	45.74	73.34	71.60
Ptf 8	12.09	78.91	28.98	81.35	69.70	90.15	45.74	73.34	71.60
Ptf 9	12.09	78.91	28.98	81.35	69.70	90.15	31.47	75.18	57.38
Ptf 10	12.09	78.91	28.98	31.47	75.18	57.38	31.47	75.18	57.38
Ptf 11	31.47	75.18	57.38	62.64	71.51	83.10	45.74	73.34	71.60
Ptf 12	20.25	77.04	42.54	45.74	73.34	71.60	62.64	71.51	83.10
Ptf 13	20.25	77.04	42.54	31.47	75.18	57.38	62.64	71.51	83.10
Ptf 14	20.25	3.27	4.54	20.25	77.04	42.54	62.64	71.51	83.10
Ptf 15	12.09	2.67	2.58	20.25	77.04	42.54	45.74	73.34	71.60
Ptf 16	31.47	3.94	7.24	6.64	80.79	18.01	31.47	75.18	57.38
Ptf 17	31.47	3.94	7.24	6.64	80.79	18.01	81.35	6.41	17.51
Ptf 18	45.74	4.69	10.54	20.25	77.04	42.54	81.35	6.41	17.51
Ptf 19	1.50	84.58	5.08	31.47	75.18	57.38	62.61	9.55	22.13
Ptf 20	1.50	84.58	5.08	45.74	73.34	71.60	47.18	10.75	21.13
Ptf 21	3.32	82.68	10.12	20.25	77.04	42.54	47.18	10.75	21.13
Ptf 22	12.09	78.91	28.98	6.64	80.79	18.01	34.31	12.02	19.08
Ptf 23	3.32	82.68	10.12	12.09	78.91	28.98	34.31	12.02	19.08
Ptf 24	12.09	78.91	28.98	20.25	77.04	42.54	47.18	10.75	21.13
Ptf 25	20.25	77.04	42.54	31.47	75.18	57.38	34.31	12.02	19.08
Ptf 26	81.35	69.70	90.15	45.74	73.34	71.60	24.08	13.38	16.34
Ptf 27	47.18	62.62	68.56	12.09	78.91	28.98	24.08	13.38	16.34
Ptf 28	24.08	13.38	16.34	12.09	78.91	28.98	24.08	13.38	16.34
Ptf 29	10.69	16.32	10.31	20.25	77.04	42.54	16.32	14.81	13.29
Ptf 30	10.69	55.87	22.97	6.64	80.79	18.01	6.76	17.90	7.63
Ptf 31	24.08	59.20	43.54	20.25	77.04	42.54	4.14	52.62	10.22
Ptf 32	6.76	54.23	15.63	31.47	75.18	57.38	6.76	54.23	15.63
Ptf 33	2.45	51.04	6.42	45.74	73.34	71.60	0.78	24.97	1.50
Ptf 34	0.78	47.94	2.26	62.64	71.51	83.10	1.41	23.10	2.39
Ptf 35	2.45	51.04	6.42	62.64	71.51	83.10	4.14	19.56	5.41
Ptf 36	10.69	55.87	22.97	81.35	69.70	90.15	16.32	57.52	32.33
Ptf 37	62.61	64.36	79.80	81.35	69.70	90.15	16.32	57.52	32.33
Ptf 38	81.35	69.70	90.15	62.64	71.51	83.10	34.31	60.90	55.98
Ptf 39	62.64	71.51	83.10	81.35	69.70	90.15	80.21	66.12	88.03

Table 6.30: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.5%

Ptfs	ShVolVaR			HFBGJRVaR			EVTHFBVaR		
	UC	IND	CC	UC	IND	CC	UC	IND	CC
Ptf 1	41.25	96.13	71.40	41.48	90.33	71.19	3.35	84.58	10.25
Ptf 2	41.25	96.13	71.40	41.48	90.33	71.19	8.69	86.49	22.77
Ptf 3	41.25	96.13	71.40	41.48	90.33	71.19	8.69	86.49	22.77
Ptf 4	41.25	96.13	71.40	41.48	90.33	71.19	8.69	86.49	22.77
Ptf 5	41.25	96.13	71.40	74.81	92.26	94.53	8.69	86.49	22.77
Ptf 6	41.25	96.13	71.40	82.80	94.19	97.41	20.17	88.41	43.79
Ptf 7	41.25	96.13	71.40	82.80	94.19	97.41	41.48	90.33	71.19
Ptf 8	41.25	96.13	71.40	82.80	94.19	97.41	41.48	90.33	71.19
Ptf 9	41.25	96.13	71.40	82.80	94.19	97.41	41.48	90.33	71.19
Ptf 10	41.25	96.13	71.40	82.80	94.19	97.41	41.48	90.33	71.19
Ptf 11	41.25	96.13	71.40	82.80	94.19	97.41	41.48	90.33	71.19
Ptf 12	41.25	96.13	71.40	74.81	92.26	94.53	41.48	90.33	71.19
Ptf 13	41.25	96.13	71.40	74.81	92.26	94.53	74.81	92.26	94.53
Ptf 14	41.25	96.13	71.40	74.81	92.26	94.53	74.81	92.26	94.53
Ptf 15	41.25	96.13	71.40	82.80	94.19	97.41	82.80	94.19	97.41
Ptf 16	41.25	96.13	71.40	82.80	94.19	97.41	74.81	92.26	94.53
Ptf 17	12.59	98.06	30.98	82.80	94.19	97.41	74.81	92.26	94.53
Ptf 18	12.59	98.06	30.98	12.59	98.06	30.98	82.80	94.19	97.41
Ptf 19	12.59	98.06	30.98	12.59	98.06	30.98	82.80	94.19	97.41
Ptf 20	12.59	98.06	30.98	12.59	98.06	30.98	82.80	94.19	97.41
Ptf 21	12.59	98.06	30.98	12.59	98.06	30.98	74.81	92.26	94.53
Ptf 22	12.59	98.06	30.98	12.59	98.06	30.98	82.80	94.19	97.41
Ptf 23	12.59	98.06	30.98	12.59	98.06	30.98	41.25	96.13	71.40
Ptf 24	12.59	98.06	30.98	12.59	98.06	30.98	82.80	94.19	97.41
Ptf 25	12.59	98.06	30.98	41.25	96.13	71.40	74.81	92.26	94.53
Ptf 26	12.59	98.06	30.98	12.59	98.06	30.98	74.81	92.26	94.53
Ptf 27	74.81	92.26	94.53	41.25	96.13	71.40	74.81	92.26	94.53
Ptf 28	41.48	90.33	71.19	82.80	94.19	97.41	74.81	92.26	94.53
Ptf 29	8.69	86.49	22.77	82.80	94.19	97.41	74.81	92.26	94.53
Ptf 30	1.17	82.68	4.06	74.81	92.26	94.53	8.69	86.49	22.77
Ptf 31	1.17	82.68	4.06	41.48	90.33	71.19	3.35	84.58	10.25
Ptf 32	1.17	82.68	4.06	41.48	90.33	71.19	1.17	82.68	4.06
Ptf 33	1.17	82.68	4.06	41.48	90.33	71.19	1.17	82.68	4.06
Ptf 34	1.17	82.68	4.06	20.17	88.41	43.79	3.35	84.58	10.25
Ptf 35	3.35	84.58	10.25	20.17	88.41	43.79	3.35	84.58	10.25
Ptf 36	74.81	92.26	94.53	1.17	82.68	4.06	3.35	84.58	10.25
Ptf 37	41.25	96.13	71.40	1.17	82.68	4.06	1.17	82.68	4.06
Ptf 38	41.25	96.13	71.40	1.17	82.68	4.06	1.17	82.68	4.06
Ptf 39	41.25	96.13	71.40	1.17	82.68	4.06	3.35	84.58	10.25

Table 6.31: The p-values (%) of the Unconditional Coverage, the Independence, and the Conditional Coverage tests for the models with c.l. 99.9%

Test	Model	$VaR_{5\%}$	$VaR_{1\%}$	Model	$VaR_{0.5\%}$	$VaR_{0.1\%}$
LOPEZ	ShVolVaR	165.697	*39.283*	ShVolVaR	15.231	**3.103**
	HFBVaR	*172.593*	38.257	HFBGJRVaR	**13.487**	3.590
	EVTHFBGJRVaR	**160.003**	**37.641**	EVTHFBGJRVaR	18.975	*6.128*
	EVTHFBVaR	161.798	37.590	EVTHFBVaR	*22.052*	5.513
CAPORIN 1	ShVolVaR	62.864	9.054	ShVolVaR	3.288	**0.428**
	HFBVaR	*68.507*	9.117	HFBGJRVaR	**2.864**	0.888
	EVTHFBGJRVaR	**61.621**	**8.985**	EVTHFBGJRVaR	4.521	*1.355*
	EVTHFBVaR	62.594	*9.546*	EVTHFBVaR	*4.844*	1.246
CAPORIN 2	ShVolVaR	0.364	*0.051*	ShVolVaR	0.020	**0.001**
	HFBVaR	0.367	0.044	HFBGJRVaR	**0.018**	0.006
	EVTHFBGJRVaR	**0.310**	**0.043**	EVTHFBGJRVaR	0.023	0.008
	EVTHFBVaR	*0.483*	0.046	EVTHFBVaR	*0.024*	*0.008*
CAPORIN 3	ShVolVaR	*0.525*	*0.110*	ShVolVaR	0.049	**0.007**
	HFBVaR	0.518	0.097	HFBGJRVaR	**0.029**	0.011
	EVTHFBGJRVaR	**0.460**	**0.079**	EVTHFBGJRVaR	0.042	*0.015*
	EVTHFBVaR	0.483	0.090	EVTHFBVaR	*0.049*	0.015

Table 6.32: Values of the average total loss over the backtesting sample for each portfolio, using Regulators' loss functions.

Test	Model	$VaR_{5\%}$	$VaR_{1\%}$	Model	$VaR_{0.5\%}$	$VaR_{0.1\%}$
CAPORIN 1	ShVolVaR	**2,073.62**	**2,415.96**	ShVolVaR	2,564.38	*2,795.81*
	HFBVaR	2,072.63	*2,439.99*	HFBGJRVaR	*2,579.45*	2,754.92
	EVTHFBGJRVaR	2,089.77	2,437.01	EVTHFBGJRVaR	**2,542.24**	**2,714.38**
	EVTHFBVaR	*2,090.56*	2,439.93	EVTHFBVaR	2,545.41	2,718.10
CAPORIN 2	ShVolVaR	**14.52**	**26.61**	ShVolVaR	**34.07**	56.48
	HFBVaR	14.82	28.63	HFBGJRVaR	*37.10*	*59.74*
	EVTHFBGJRVaR	15.26	*29.46*	EVTHFBGJRVaR	35.74	**50.83**
	EVTHFBVaR	*15.30*	29.44	EVTHFBVaR	35.69	50.56
CAPORIN 3	ShVolVaR	**33.71**	**48.69**	ShVolVaR	**57.15**	81.30
	HFBVaR	33.96	50.89	HFBGJRVaR	*60.41*	*84.43*
	EVTHFBGJRVaR	*34.72*	*51.87*	EVTHFBGJRVaR	58.89	75.16
	EVTHFBVaR	34.65	51.79	EVTHFBVaR	58.80	**74.84**

Table 6.33: Values of the average total loss over the backtesting sample for each portfolio, using Investors' loss functions.

sophisticated HFBVaR, HFBEVaR and HFBGJRVaR models, at least at 95% and 99% confidence levels. Adopting 8 or 10 degrees of freedom, its forecasting power becomes close to HFBGJRVaR at 99.5% and 99.9% confidence levels.

For our backtesting based on loss functions, we find results similar to the one we have seen in the chapter 5: ShVolVaR with $\nu \to \infty$ shows good results from the investors' viewpoint, while ShVolVaR with $\nu = 8, 10$ attains the best results according to the regulators' viewpoint. When considering ShVolVaR using the best way of estimating VaR according to the percentile of interest and the equity weight og the portfolio, we note that it is the best model under the investors' viewpoint.

We recall that, due to the fact that very few implied volatility indices are quoted, the use of the ShVolVaR is not always possible. When it is, however, it can be a useful tool in the hand to a portfolio managers. They can easily and quickly estimate the risk of their portfolio quite accurately.

In our empirical analysis, we find that a VaR model which correctly estimates VaR for all ε and for all the balanced portfolios does not exist.

Chapter 7

Conclusions

In this thesis, we have discussed in detail how to optimize a portfolio without estimating expected returns since their estimation is very hard. A new stream of research about portfolio optimization has developed and we want to contribute to it. We presented three different models *Minimum-Risk*, *Risk-Diversification* and *Capital-Diversification* which does not need any optimization procedure. We applied these models to different risk measures such as volatility (that is not a real risk measure due to the fact that it is not coherent), Conditional VaR and Conditional VaR deviation. We formulate the Equally Risk Contribution with respect to CVaR as a nonlinear problem and then we reformulate it as a convex problem in a convex set. Then applying the results in Rockafellar and Uryasev (2000) (see [96]) the convex problem can be linearized. We also extend the ERC problem to the CVaR deviation that cannot be negative when considering CVaR.

We performed a backtest analysis on seven different real-world data sets, which consist of equities, bonds and mixed assets, each with different sources of risk. We cannot find a clear dominance of a model over the others. Each tested model responds to different requirements that could be related to diverse investor attitudes. On one hand, as expected, the *Minimum-Risk* models are advisable for risk averse investors, avoiding as much as possible the shocks represented by deep drawdowns. On the other hand, the *Risk-Diversification* strategies seem to be appropriate for investors mildly adverse to total portfolio risk. *Capital-Diversification* model seems to be advisable for sufficiently risk-loving investors, who try to maximize returns without worrying about periods of deep drawdowns.

We have examined the statistical significance of the out-of-sample Sharpe ratio using the robust test proposed by Ledoit and Wolf (2008) (see [67]). We find that if one invests in a universe with a single source of risk, there is no statistical difference in Sharpe Ratio between the *Minimum-Risk*, the *Risk-Diversification* portfolios, and the EW ones. On the other hand, when considering investment universes with multiple sources of risk, the *Minimum-Risk* and the *Risk-Diversification* strategies tend to have better performance than the Equally Weighted do, in terms of Sharpe Ratio.

We have replicated the experiment as in Kondor et al. (2007) (see [61]) in order to evaluate the effect of the estimation error for each optimization method. We have obtained the same results as in the cited paper for the unbounded portfolios. We have introduced constraints in portfolio weights to verify if it is possible to achieve stability. We have found that the Minimum CVaR-deviation and Minimum CVaR are stable and that the effect of the estimation error is lower in the former with respect to the latter for each value of ε taken into account.

When we wanted to evaluate the effect of the estimation error of ERC procedure we changed the comparison metric and we maintained portfolio constraints. We introduced the Euclidean distance between the portfolios weights. They are considered as points in the $\mathbb{R}^n$ space. The higher is the distance between the optimized portfolio and the theoretical one, the stronger is the effect of the estimation error. We replicate the same experiment 100 times for all values of ε and we find that for any ε the estimation error effect afflicts Min CVaR and Min CVaR-deviation portfolios more than it does with variance and ERC portfolios.

For any small value of ε (so we are considering only a few scenarios in the bad tail of returns distribution) joint with small values of the ratio N/T, Min CVaR and Min CVaR-deviation have shown high values of Euclidean distance, while when the ε increases the effect of the estimation error tends to be much more stable over the N/T ratio. We also notice that when $\varepsilon > 0.1$ the ERC optimization procedure has lower estimation error effect than minimum variance approach does.

We tried to set the ratio $N/T = 15\%$ and to replicate the experiment by increasing the value of T. We have seen that each increment in a magnitude order reduces the effect of the estimation error by a factor of $1/3$. It seems that in order to reduce this effect we should consider at least 10000 data points. If we are dealing with daily data it means that we need a time series of 40 years. In traditional optimization procedure it is common to use historical data but in this case it would be impossible to have a complete time series for all the assets.

To solve this particular and important issue we have introduced the Historical Filtered Bootstrap procedure. It is a Montecarlo simulation based on an ARMA-GARCH estimation procedure. We performed a static analysis to identify the minimum number of simulated scenarios that allows us to reduce as much as possible the effect of the estimation error. We have considered two different asset universes, the first one is a multi asset while the second one is made of equity only. First of all, for all assets, we have estimated 100 times the Conditional VaR and we have recorded a huge variability of the estimate when the number of scenarios S is small ($S = 1000$). The phenomenon quickly diminishes when S increases. Different values of the same parameter influence the estimation as in the case of optimized portfolio weights.

In the dynamic analysis, we have compared the CVaR estimation and optimal solutions using the minimum acceptable number of scenarios ($S = 10K$) and the maximum number of scenarios ($S = 1M$). We discovered that portfolio weights are very close one to each other and the whole portfolio risk, for the four optimization procedures, is similar. Regarding portfolio performances, we can state that there are no important differences when we use $S = 10K$ with respect to $S = 1M$ scenarios.

The results we have obtained confirm that the minimum number of scenarios should be equal to $10,000$ but a good compromise between computational time and estimation accuracy is $100,000$.

At this point, we wanted to validate the Historical Filtered Bootstrap as Risk Management procedure. There are many papers that provide results but they are not updated and they do not consider the big subprime crisis and the euro bond crisis. So we put in place a comparative backtest procedure considering the common test, unconditional coverage, independence and conditional coverage test. We also use the loss function according regulators' and investors' point of view. Doing so we proposed a new method to predict VaR, both using variables known on the market (implied volatilities) and variable estimated on data (realized volatilities). We call this model Shrunk Volatility VaR. We find that its forecasting power is comparable to that of the more sophisticated HFBVaR and EVTFBVaR models that have pretty good results in all test.

These particular results are stable across Equity portfolio and balanced portfolios and the generalized Shrunk Volatility VaR is able to achieve similar results as well as Historical Filtered Bootstrap.

We are ready to answer the questions we announced at the end of the Chapter 1,

1. **Q:** Could we formulate and solve an ERC problem based on a coherent risk measure as CVaR?

A: yes we could. CVaR is a homogeneous function of degree 1 and it admits Euler decomposition. In addition we formulate the ERC problem as a convex problem in a convex set.

2. **Q:** How intense is the effect of estimation errors on the ERCCVaR model?

   A: it is higher with respect to the one we record with the minimum variance approach when $\varepsilon < 5\%$. It is quite comparable when $\varepsilon = 5\%$, and we notice a smaller effect when $\varepsilon > 5\%$. We also recall that ERC procedure always has a smaller effect than Minimum CVaR and Minimum CVaR-deviation do.

3. **Q:** Could we mitigate this effect?

   A: yes we could. We can consider introducing constraints to portfolio weights and reducing the ratio N/T by increasing the amount of data we use to estimate parameters. We can also consider evaluating CVaR, not in the extreme tail, so $\varepsilon > 10\%$.

4. **Q:** Is there any difference between a historical scenario and a simulated scenario?

   A: yes there is. Using a historical scenario we need too many observation data points in order to reduce the estimation error effect, whereas using simulated scenarios (that are simulated using a quite good estimation of the hidden data generating process) we can force the effect of the estimation error to be negligible.

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