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A STABLE SET FOR ABSTRACT GAMES

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Abstract

We introduce a new solution concept for models of coalition formation, called the myopic stable set. The myopic stable set is defined for a very general class of abstract games and allows for an infinite state space. We show that the myopic stable set exists and is non-empty. Under minor continuity conditions, we also demonstrate uniqueness. Furthermore, the myopic stable set is a superset of the core and of the set of pure strategy Nash equilibria in noncooperative games. Additionally, the myopic stable set generalizes and unifies various results from more specific environments. In particular, the myopic stable set coincides with the coalition structure core in coalition function form games if the coalition structure core is nonempty; with the set of stable matchings in the standard one-to-one matching model; with the set of pairwise stable networks and closed cycles in models of network formation; and with the set of pure strategy Nash equilibria in finite supermodular games, finite potential games, and aggregative games. We illustrate the versatility of our concept by characterizing the myopic stable set in a model of Bertrand competition with asymmetric costs, for which the literature so far has not been able to fully characterize the set of all (mixed) Nash equilibria.

Keywords: Abstract Games, Coalition Formation, Stability, Nash Equilibrium.

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Contents

Introduction

Game theory provides general mathematical techniques for analyzing situations in which two or more individuals make decisions that will influence one another's welfare.

In particularly, game theory can be defined as the study of mathematical models of conflict and cooperation between intelligent rational decision-makers (Myerson, 1991). This definition suggests that game theory is divided into two branches, called the non-cooperative and cooperative branches.

The two branches of game theory differ in how they formalize interdependence among the players. In the non-cooperative theory, a game is a detailed model of all the moves available to the players. By contrast, the cooperative theory abstracts away from this level of detail, and describes only the outcomes that result when the players come together in different combinations.

It is interesting to note that, in principle, any social phenomenon can be studied using both the approaches. An example is given by the game theoretic analysis of coalition formation. It is well known, in fact, that coalition formation theory embodies many different settings such as normal form games, networks, TU games and matching models (see Ray, 2007; Ray and Vohra, 2014).

In this paper, we consider a general class of *abstract game* which covers all of these settings and many more.

An abstract game is a framework to model strategic interaction between individuals or groups. An abstract game is the most flexible representation form of a strategic interaction. In fact, many specific settings such as TU games, networks models, one to one matching models and normal form games can be considered particular cases of abstract game.

The prodromic version of abstract game, also called social environment, can be founded in *games in effectiveness form* (Rosenthal, 1972) and in the framework of *social situation* (Greenberg, 1990).

To define an abstract game (Chwe, 1994), we need four ingredients: a finite set of individuals, a preference relation for each individual, a state space and an effectivity correspondence that model the feasible transitions from a state to another.

We impose very little structure on the state space and on the effectivity correspondence. In particular, the state space is only required to be a non-empty and compact metric space. **In contrast to most of the literature, we allow the state space to be infinite**. The state space can encode different information in different settings. For instance, the network structure, the set of matchings or the set of outcomes in a characteristic function games. The effectivity correspondence is closely linked to the idea of coalition formation. Given two states, it specifies which individuals or coalitions are able to change a state into another state. Moreover, the effectivity correspondence can be structured in many different ways. This allows to formalize the interdependence the individuals adopting either a non cooperative approach or a cooperative approach.

For this general class of abstract game with infinite state space, we define a solution concept, the *Myopic Stable Set* . The Myopic Stable Set extend the idea of *Pairwise Myopic Stability* (Herings et al., 2009) from finite network to a general class of abstract game that allow for infinite state space.

The Myopic Stable Set has to satisfy three conditions, (i)*myopic deterrence of external deviations*, (ii)*myopic external stability* and (iii)*minimality*. Roughly speaking, the first condition requires that for any state in the set, there is no profitable deviations from a state in the set to a state outside the set. The second condition makes sure that from any state outside the set there exists a sequence of profitable deviations which approaches the set. By the last condition, the Myopic Stable Set is the minimal set which satisfies the first two conditions.

The notion of dominance is myopic in the sense that agents (or coalitions) do not predict how their decision to change the current state to another one will lead to further changes by other coalitions. Such a notion is natural in very complex abstarct games where the number of possible states and possible actions is overwhelmingly large and agents have little information about the possible actions other agents may take or the incentives of other agents. The myopic stable set thereby distinguishes our approach from the ones in the literature that focus on farsightedness (see among others, Chwe, 1994; Xue, 1998; Herings, Mauleon, and Vannetelbosch, 2004, 2009, 2014; Dutta, Ghosal, and Ray, 2005; Page, Wooders, and Kamat, 2005; Page and Wooders, 2009; Ray and Vohra, 2015). On the other hand, our analysis is more in line with myopic concepts like the Core and the von Neumann-Morgenstern Stable Set. As we will see in the application to normal-form games, it is also intimately connected to the notion of Nash equilibrium.

In Theorem 1, we show that each abstract game contains at least one Myopic Stable Set. Since the state space can be infinite, we have to introduce a notion of asymptotic dominance that allows us to apply Zorn's Lemma. Under a slightly stronger continuity assumption, the Myopic Stable Set is also unique by Theorem 2. Our existence and uniqueness results differ from most of the literature, where even for more special settings, well-known solution concepts do not have these desirable properties. For instance, the Core could be empty (Bondareva, 1963; Scarf, 1967; Shapley, 1967), the von Neumann Morgenstern Stable Set might fail to exist (Lucas, 1968) and is also not always unique (Lucas, 1992) and the set of pure strategy nash equilibria could be empty.

We also provide several additional results that show more insights about the structure of an MSS. For finite state spaces, we fully characterize the MSS as the union of all closed cycles, i.e., subsets which are closed under coalitional better replies. For infinite spaces, the union of all closed cycles is found to be a subset of the MSS. This result is helpful in applications and in the comparison to other solution concepts. For instance, any state in the core is a closed cycle and is therefore included in the MSS. Next we define a generalization of the weak improvement property

(Friedman and Mezzetti, 2001) to abstract games and we show that, under weak continuity conditions, the weak improvement property characterizes the collection of abstract games for which the MSS coincides with the core.

We demonstrate the versatility of these results by analyzing the relationship between the MSS and other solution concepts in more specific social environments. In particular, we show that the MSS coincides with the coalition structure core for TU games with coalition structure (Kóczy and Lauwers, 2004) whenever the coalition structure core is non-empty; with the set of stable matchings in the one-to-one matching model by Gale and Shapley (1962); with the set of the set of pairwise stable networks and closed cycles in models of network formation (Jackson and Watts, 2002), and the set of pure strategy Nash equilibria in finite supermodular games (Bulow, Geanakoplos, and Klemperer, 1985), finite potential games (Monderer and Shapley, 1996), and aggregative games (Selten, 1970). Finally, we illustrate the versatility of our results by characterizing the MSS in a model of Bertrand competition with asymmetric costs. This model is characterized by discontinuous payoff functions and has no pure-strategy Nash equilibrium. Although Blume (2003) has shown the existence of a mixed-strategy Nash equilibrium, the literature has, so far, not been able to characterize the complete class of (mixed) equilibria for this game.

The thesis is structured as follows: Chapter 1 is a review of the mathematical tools used in the proofs of our main results. Chapter 2 provides the primitives of our general framework of abstract game and discusses how it translates to settings with more structure. In Chapter 3 we introduce Myopic Stable Set and we prove that, under mild conditions, it exists and it is non-empty and unique. Chapter 4 explores properties and predictions of the Myopic Stable Set in different settings and relates it to other stability concepts. Chapter 5 concludes.

Chapter 1

Preliminaries and basic terminology

This chapter collects an essential list of mathematical definitions and theorems that we used in the proofs of our main results.

1.1 Topological Spaces and Metrics

A topological space is a pair (X, τ) where X is a non-empty set and τ is a *topology* on X defined as follows:

Definition 1 (Topology)**.** A *topology* τ on a set X is a collection of subset of X which satisfies the following conditions:

1. $\emptyset$, $X \in \tau$
2. if $U \in \tau$ and $V \in \tau$ then $U \cap V \in \tau$, with $U, V \subseteq X$
3. for every index set I: if $U_i \in \tau$ for every $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$

Any element of τ is called an open set of X.

A topological basis of τ is a collection $\mathbb{B}$ of open sets in τ such that all the other open sets can be written as unions of the elements of $\mathbb{B}$.

A metric space is a pair (X, d) where X is a set and d is a metric defined as follow:

Definition 2. A metric d is a map

$$d : X \times X \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto d(x, y)$$

$$d(x, y) \in \mathbb{R}$$

which satisfies the following properties:

1. *Non-negativity*: for all $x, y \in X$, $d(x, y) \geq 0$.
2. *Identity of Indiscernibles*: for all $x, y \in X$, $d(x, y) = 0$ iff $x = y$.
3. *Symmetry*: for all $x, y \in X$, $d(x, y) = d(y, x)$.
4. *Triangle Inequality*: for all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

Given a metric space (X, d), and $x \in X$ and a $\epsilon > 0$ we define $B_\epsilon(x) := \{y \in X : d(x, y) < \epsilon\}$ the ϵ-open ball of x.

A set $U \subseteq X$ is open in the topology induced by d if for every $x \in U$ there exists an $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq U$.

If the collection of set $\mathbb{B} := \{B_\epsilon(x) : x \in X, \epsilon > 0\}$ is a basis for τ we say that the metric d on X generate the topology τ on X.

Definition 3 (Metrizable Topological Space)**.** A topological space $(X, \tau)$is metrizable if there exists a metric d on X which generate the topology τ on X.

Definition 4 (Open Set)**.** A subset U of a metric space (X, d) is *open* if for each $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$.

Definition 5 (Closed Set)**.** A subset C of a metric space (X, d) is *closed* if its complement, $C^c := X \backslash U$, is open.

1.2 Sequences and Nets

Definition 6 (Sequence)**.** A sequence in X is a function $x : \mathbb{N} \longrightarrow X$. For every $n \in \mathbb{N}$, we usually denote $x(n)$ by x_n.

Definition 7 (Subsequence)**.** A subsequence of (x_n) is a sequence of the form (x_{n_r}) where (n_r) is a strictly increasing sequence of natural numbers.

Definition 8 (Convergent Sequence)**.** A sequence (x_n) is said convergent in X if it approaches some limit. Formally, a sequence (x_n) converge to the limit y if and only if for every $\epsilon > 0$ there exists a positive integer N such that $d(x_n, y) < \epsilon$ for every $n > N$.

Note that a sequence is indexed by a countable linearly ordered set $\mathbb{N}$. We introduce the notion of *net* that is obtained defined a sequence on an more general index set.

In particular, the index set of a net is a *directed set* $\mathbb{D}$ which is a set D equipped with a direction $\succeq$ that is a binary relation on D with the property that each pair has an upper bound. Formally:

Definition 9 (Directed Set)**.** A directed set $\mathbb{D}$ is a pair $(D, \succeq)$ if $\succeq \subseteq D \times D$ is a binary relation on D such which satisfies :

1. Reflexivity: $\forall x \in \mathbb{D}$, $x \succeq x$
2. Transitivity: $\forall x, y, z \in \mathbb{D}$, $x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$

3. Upwards Direction: $\forall x, y \in \mathbb{D}, \exists z \in \mathbb{D} \quad s.t. \quad x \succeq z \wedge y \succeq z$

Definition 10 (Net)**.** A net in X is a function $x : \mathbb{D} \longrightarrow X$. For every $d \in \mathbb{D}$, we usually denote $x(d)$ by x_d.

Definition 11 (Subnet)**.** A subnet of $(x_d)_{d\in\mathbb{D}}$ is a net of the form $(y_\lambda)_{\lambda\in\Lambda}$ if there exists a function $\phi : \Lambda \longrightarrow \mathbb{D}$ such that:

- $y_\lambda = x_{\phi_\lambda}$ for each $\lambda \in \Lambda$, where ϕ_λ stands for $\phi(\lambda)$
- for each $d_0 \in \mathbb{D}$ there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $\phi_\lambda \geq d_0$

Definition 12 (Convergent Net)**.** A net (x_d) is said convergent in X if it approach some limit. Formally, a net (x_d) converge to the limit y if only if for every $\epsilon > 0$ there exists an index $d' \in \mathbb{D}$ such that $d(x, y) < \epsilon$ for every $d > d'$.

1.3 Compactness

Definition 13 (Open Cover)**.** Let (X, τ) be a topological space and let G be a subset of X. An open cover of G is a collection $\mathbb{C}$ of element of τ such that $G \subseteq \bigcup_{U\in\mathbb{C}} U$.

Definition 14 (Compactness)**.** X is compact if for every open cover $\mathbb{C}$ of X there is a finite subset $\mathbb{F}$ of $\mathbb{C}$ such that $\mathbb{F}$ is also an open cover of X.

- **Theorem 2.31** (Aliprantis and Border, 2006)

  For a topological space X the following are equivalent:

  1. X is compact
  2. Every net in X has a subnet converging in X

1.4 Order Theory and Zorn's Lemma

Definition 15 (Partially Ordered Set)**.** A partially ordered set is a pair
$(P, \succeq)$ where P is a set and $\succeq P \times P$ is a binary relation satisfying the following properties:

- Reflexivity: $\forall x \in P,\ x \succeq x$
- Transitivity: $\forall x, y, z \in P,\ x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$

Given the relation $\supseteq$ we say that x is a maximal element of $(P, \supseteq)$ if for all $y \in (P, \supseteq)$ if $y \supseteq x$ then $y = x$.

Definition 16 (Totally Ordered Set)**.** A totally ordered set is a pair
$(T, \succeq)$ where T is a set and $\succeq \subseteq T \times T$ is a binary relation satisfying the following properties:

- Completeness: $\forall x \in T,\ x \succeq y \vee y \succeq x$
- Reflexivity: $\forall x \in X,\ x \succeq x$
- Transitivity: $\forall x, y, z \in T,\ x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$

Given a partially ordered Set $(P, \succeq)$, a chain of $(P, \succeq)$ is a proper subset $T \subset P$ such that $(T, \succeq)$ is a totally ordered set

Given the relation $\supseteq$ and a chain of $(P, \supseteq)$, $x \in P$ is an upper bound of the chain $(T, \supseteq)$ if does not exist any $y \in T$ such that $y \supseteq x$.

- **Zorn's Lemma**

  If a partially ordered set $(P, \succeq)$ has the property that every chain has an upper bound, then the set P contains at least one maximal element.

Chapter 2

Model Primitives and Specific Settings

Let N be a finite set of individuals $\{1, ..., n\}$.

A coalition S is defined as a subset of N. Denote the set of all possible non-empty coalitions by $\mathcal{N} := 2^N \backslash \{\emptyset\}$.

Let the state space be a metric space (X, d). We assume (X, d) to have the following properties:

Assumption 1 (Non-emptiness of X)**.** X is non-empty.

Assumption 2 (Compactness of X)**.** X is compact.

For all states $x \in X$ and $y \in X$, the *effectivity correspondence* is defined in the following way

$$E : X \times X \longrightarrow \mathcal{N}$$

$$(x, y) \longmapsto E(x, y)$$

$$E(x, y) \subseteq \mathcal{N}$$

The effectivity correspondence specifies the collection of coalitions $E(x, y) \subseteq \mathcal{N}$ that can change the state x into the state y. If $E(x, y) = \emptyset$ then no coalition can change from x to y.

For each individual $i \in N$, we define a preference relation $\succeq_i \subseteq X \times X$ as a binary relation over the state space X which satisfies the following properties:

- Completeness: $\forall x \in X$, $x \succeq_i y \vee y \succeq_i x$
- Reflexivity: $\forall x \in X$, $x \succeq_i x$
- Transitivity: $\forall x, y, z \in X$, $x \succeq_i y \wedge y \succeq_i z \Rightarrow x \succeq_i z$

For all $i \in N$, the vector $\{\succeq_i\}_{i \in N}$ represents all individual preference relations $\succeq_i$.

Finally, we denote an *abstract game* by

$$\Gamma := (N, X, E, \{\succeq_i\}_{i \in N})$$

The state space (X, d) can be used to encode many information of a particular application. To illustrate the generality of our setting, we provide four specific models that have been studied extensively in the literature: TU games with coalition structure, one-to-one matching models, networks, and non-cooperative normal-form games. For each of these examples we specify the abstract game, i.e., the set of players N, the state space (X, d), the preferences $\{\succeq_i\}_{i \in N}$, and the effectivity correspondence E.

2.1 Specific Setting 1: Tu Games with coalition structure

A *TU game with coalition structure* is a tuple (N, v, π) where $N = \{1, ...n\}$ is the set of players and v is the the *characteristic function* v of the following form:

$$v : 2^N \longrightarrow \mathbb{R}$$

$$S \longmapsto v(S)$$

$$v(S) \in \mathbb{R}$$

In words, v assigns to each coalition a number $v(S)$, that is the coalitional value of S.

A coalition structure, or partition structure, is a partition $\pi := \{S^1, ..., S^k, ..., S^m\}$ of N which satisfies the following properties:

- $S^k \neq \emptyset$ for all k
- $\cup_{S^k \in \pi} S^k = N$
- $S^k \cap S^l = \emptyset$ for all $k \neq l$

The collection of all coalition structures is denoted by Π.

In a TU game with coalition structure a vector $u = (u_1, ..., u_n) \in \mathbb{R}^n$ lists the payoff of each individual under two conditions:

- **Individual rationality**: $u_i \geq v(\{i\}) \ \ \forall i \in N$
- **Feasibility**: $\sum_{i \in S} u_i = v(S)$.

The outcome of a TU game with coalition structure is a pair (π, u) with $\pi \in \Pi$ and $u \in \mathbb{R}^n$.

To induce an abstract game by a TU game with coalition structure, first we define preferences $\succeq_i$ over the state space X by setting $x \succeq_i y$ if and only if $u_i(x) \geq u_i(y)$, i.e., the payoff for individual i in state x is at least as high as the payoff for individual i in state y.

Lemma 1. The function $u(x)$ is continuous, i.e. if $x^i \to x$ then $u(x^i) \to u(x)$

Proof. The result follows from the continuity of the projection. □

Moreover, we define the state space in terms of the set of all possible outcomes. Formally,

$$X := \left\{ (\pi, u) \in \Pi \times \mathbb{R}^n \middle| \forall i \in N : u_i \geq v(\{i\}) \text{ and } \forall S \in \pi : \sum_{i \in S} u_i = v(S) \right\}$$

where $\pi(x)$ denotes the projection of x onto its first component (the partition) and $u(x)$ denotes the projection of x onto its second component (the payoff vector). Moreover, given $x \in X$ and $S \in \mathcal{N}$ we denote $\pi_S(x)$ the restriction of $\pi(x)$ to the coalition $S \subset N$ and $u_S(x) := (u_i(x))_{i \in S}$ the restriction of $u(x)$ to the coalition S.

We can define the metric on X in the following way,

$$d(x,y) = \mathbb{1}[\pi(x) \neq \pi(y)] + \|u(x) - u(y)\|_{\infty},$$

where $\mathbb{1}[.]$ is the indicator function that equals one if the condition between parenthesis is true and 0 otherwise and $\|.\|_{\infty}$ is the maximum norm.

Formally, the indicator function is:

$$\mathbb{1} : \Pi(x) \times \Pi(y) \longrightarrow \{0,1\}$$

such that

$$\mathbb{1}(\pi(x), \pi(y)) = \begin{cases} \pi(x) = \pi(y) & 0 \\ \pi(x) \neq \pi(y) & 1 \end{cases}$$

and the maximum norm is:

$$d(u(x), u(y)) := max(\mid u_i(x) - u_i(y) \mid)$$

For each pair (x, y), the effectivity correspondence $E : X \times X \longrightarrow \mathcal{N}$ specifies which coalitions can change a state into another. Thus, in this setting, a coalition formation process is a sequence of state transformations.

In a coalition formation process we can discriminate between three groups of players. The leaving players, the unaffected players and the residual players. The leaving players T are the players that decide to leave their coalition(s) to create one or more alternative groups. This event induces a change from a state say x to a new state y. The collection of coalitions that is unaffected by this change is denoted $\mathcal{U}(x, S)$ and the set of players that is unaffected is denoted by $U(x, S)$. The set $\mathcal{U}(x, S)$ contains all coalitions $T \in \pi(x)$ that are disjoint from S. Formally,

$$\mathcal{U}(x, S) = \{T \in \pi(x) | S \cap T = \emptyset\},$$
$$U(x, S) = \cup_{T \in \mathcal{U}(x,S)} T.$$

Moreover, we call the players in the set $N \setminus (S \cup U(x, S))$ the residual players.

In literature there exist many different assumptions on the effect of the leaving players on the entire partition structure and viceversa.

In principle, it is possible to capture different assumption just imposing more structure on the effectivity correspondence.

As an example, it is possible to consider the so called *coalitional sovereignty* (see Ray and Vohra, 2015) which means that a deviating coalition of players does not have the power to influence of agents outside the coalition.

Coalitional sovereignty is then defined as follows:

1. **Non-interference:** For every $x, y \in X$, if $S \in E(x,y)$ and $T \in \mathcal{U}(x,S)$, then $S \in \pi(y), T \in \pi(y)$, and $u_T(x) = u_T(y)$.
2. **Full support:** For every $x \in X$, every $S \in \mathcal{N}$, and every $u \in \mathbb{R}^S$ such that for all $i \in S : u_i \geq v(\{i\})$ and $\sum_{i \in S} u_i = v(S)$, there is a state $y \in X$ such that $u_S(y) = u$ and $S \in E(x,y)$.

Non Interference states that if a coalition remains together, all individuals in the coalition keep the same payoff. *Full Support* guarantees that any coalition can achieve any feasible payoff from their full support.

In a coalition formation process, one of the most controversial issues concerns the assumption on the behavior of the residual players and their power to influence the leaving players. (see for example Shubik, 1962; Hart and Kurz, 1983; Konishi and Ray, 1997; Ray and Vohra, 2014).

One notable and hypothetical requirements is represented by the γ-model (Hart and Kurz, 1983) which prescribes that the reaction of the residuals is to divide themselves into singletons. This assumption is justified by the the idea that a coalition ca be viewed as a result of a unanimous agreement among its members. In our setting, the γ-model can be formalized as follow:

(3) **γ-model**

For all $x, y \in X$ and $S \in E(x,y)$, if $i \in N \setminus (S \cup U(x,S))$, then $\{i\} \in \pi(y)$.

2.2 Specific Setting 2: One-to-One Matching

We consider the two sided one-to-one matching model from Gale and Shapley (1962). The model consists of a set N of individuals partitioned in two subgroups, (M, W), and a strict preference relation $\succ$ over the set $M \times W$ which gives to each player a complete and transitive preference ordering.

A matching is an injection of the form:

$$\mu : M \cup W \longrightarrow M \cup W$$

$$m \longmapsto \mu(m)$$

$$w \longmapsto \mu(w)$$

satisfying the following properties:

1. For every $m \in M, \mu(m) \in W \cup \{m\}$.
2. For every $w \in W, \mu(w) \in M \cup \{w\}$.

In this setting, the state space X consists of the set of all possible one-to-one matchings, typically denoted by $\mathcal{M}$.

In particular we consider the following discrete metric:

$$d(\mu, \mu') = \mathbb{1}_{\{\mu \neq \mu'\}}$$

Given a matching $\mu \in X$, a player $i \in N$ is said to be unmatched if $\mu(i) = i$. Consequently, a coalition $S \subseteq N$ is said to be unmatched if $\mu(S) = S$.

Each $m \in M$ has a complete transitive strict preference relation $\succ_m$ over the set $W \cup \{m\}$ and each $w \in W$ has a complete transitive strict preference relation $\succ_w$ over the set $M \cup \{w\}$. We assume that the preferences of the individuals over the set X are induced by their preference over their match, i.e., m prefers matching μ over μ' if $\mu(m) \succ_m \mu'(m)$ and w prefers μ over μ' if $\mu(w) \succ_w \mu'(w)$.

Imposing restrictions on the effectivity correspondence allows us to study the consequences of different hypothesis on the matching process. We introduce two common restrictions from the literature on matching.

The first restriction requires that every (non-single) individual, m or w, is allowed to break his link with his current partner. Doing this makes the individual and their former partners single. Moreover all the other individuals remain with the same partner. Formally:

(1) For all $i \in N$ and $\mu \in X$ with $\mu(i) \neq i$, we have $\{i\} \in E(\mu, \mu')$ where $\mu' \in X$ is such that
   - (i) $\mu'(i) = i$,
   - (ii) $\mu'(\mu(i)) = \mu(i)$, and
   - (iii) for every $j \in N \setminus \{i, \mu(i)\}$ we have $\mu'(j) = \mu(j)$.

The second restriction requires than any m and w that are currently not matched to each other can deviate by creating a link and thereby leaving their former partners single. Moreover all the other individuals remain with the same partner. Formally:

(2) For all $m' \in M$, $w' \in W$, and $\mu \in X$ with $\mu(m') \neq w'$, we have that $\{m', w'\} \in E(\mu, \mu')$, where $\mu' \in X$ is such that

   (i) $\mu'(m') = w'$,

   (ii) $\mu(m') \in W$ implies $\mu'(\mu(m')) = \mu(m')$

   (iii) $\mu(w') \in M$ implies $\mu'(\mu(w')) = \mu(w')$,

   (iv) for every $j \in N \setminus \{m', w', \mu(m'), \mu(w')\}$ it holds that $\mu'(j) = \mu(j)$.

Observe that these two conditions respect the γ-model of coalitional sovereignty.

2.3 Specific Setting 3: Networks

We consider the model of social and economic networks by Jackson and Wolinsky (1996).

A network is defined as a tuple $g := (N, \mathcal{E})$. $N = \{1, ..., n\}$ is the finite a set of players and $\mathcal{E}$ is the set of the undirected edges, where an undirected edge is a set of two distinct individuals. The set of all possible links between the players is denoted by $g^N := \{i, j | i, j \in N, i \neq g\}$. Moreover, the set of all possible network is denoted by $\mathbb{G} := \{g | g \subseteq g^N\}$.

Two players are $i, j \in N$ are linked in g if and only if $i, j \in \mathcal{E}$. We often abuse notation and we write $ij \in g$ to indicate that i and j are connected under the network g. As the edges are undirected, given some $i, j \in N$ and $g \in \mathbb{G}$, $ij \in g$ is equivalent to $ji \in g$. For any network g let $N(g) = \{i \in N | \exists j \in N$ such that $ij \in g\}$ be the set of individuals who have at least one link in g.

Let $g + ij$ be the network obtained from network g by adding the link ij and let $g - ij$ be the network obtained by deleting the link ij from g.

We define a value function v such that:

$$v : \mathbb{G} \longrightarrow \mathbb{R}$$

$$g \longmapsto v(g)$$

$$v(g) \in \mathbb{R}$$

where $v(g)$ is the worth of the network g.

Let V be the set of all value functions. Given V, an allocation rule is a map Y such that:

$$Y : \mathbb{G} \times V \longrightarrow \mathbb{R}^n$$

$$(g, v) \longmapsto Y(g, v)$$

$$Y(g, v) \in \mathbb{R}^n$$

thus $Y_i(g) \in \mathbb{R}$ denotes the single payoff for every $i \in g$.

A network problem is given by $(N, \mathbb{G}, (Y_i)_{i \in N})$.

Such a network can be represented within our general framework of abstract game identifying the state space X with the set of all possible networks $\mathbb{G}$

Moreover, we endow the state space X with the following metric

$$d(g, g') = \mathbb{1}_{\{g \neq g'\}}$$

where $\mathbb{1}_{\{g \neq g'\}}$ is the discrete metric.

Every agent $i \in N$ has a preference relation $\succeq_i$ over the set X of all possible networks defined by $x \succeq_i x'$ if $Y_i(x) \geq Y_i(x')$.

We follow Jackson and Wolinsky (1996) by considering deviations by coalitions of size one or two and by assuming link-deletion to be one-sided and link addition to be two-sided. One-sided link deletion allows every player to delete one of its links.

(1) For all individuals $i \in N$, all networks $g \in X$, and all links $ij \in g$, $\{i\} \in E(g, g - ij)$.

Two sided link addition allows any two players that are currently not-linked can change the network by forming a link between themselves.

(2) For all individuals $i, j \in N$, all networks $g \in X$ with $ij \notin g$, we have $\{i, j\} \in E(g, g + ij)$.

It is straightforward to adjust the effectivity correspondence to incorporate models of network formation where more than one link at a time can be changed by coalitions of arbitrary size (Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) or where link formation is one-sided (Bala and Goyal, 2000) into our framework. We refer to Page and Wooders (2009) for a more extensive discussion of alternative rules of network formation.

2.4 Specific Setting 4: Normal Form Games

A normal form game is a triple

$$G := (N, (S_i)_{i\in N}, (u_i)_{i\in N})$$

where N is the set of players and $(S_i)_{i\in N}$ is the set of strategies for all $i \in N$. The utility function u_i is a real value function such that:

$$u_i : S \longrightarrow \mathbb{R}$$

$$s \longmapsto u_i(s)$$

$$u_i(s) \in \mathbb{R}$$

where $S := \times_{i\in N} S_i$ is the set of all strategy profile that can be chosen by the various players.

In particular, for each $i \in N$ and for every $s_i \in S_i$, we denote by (s_i, s_{-i}) the strategy profile where s_i is the strategy of player i and s_{-i} is the list of strategy of every player except i, i.e. $s_{-i} := (s_j)_{j\in N\setminus\{i\}}$

We can represent a normal form game by a general framework of abstract game as follows. The state space consists of all strategy profiles, i.e., $X = S$.

Furthermore, we consider the following product metric:

$$d(s, s') = \sum_{i=1}^{|N|} d_i(s_i, s_i')$$

For each individual $i \in N$, the preference relation $\succeq_i$ is represented by $u_i : S_i \longrightarrow \mathbb{R}$ if $s_i \succeq_i s_i' \Leftrightarrow u_i(s_i) \geq u_i(s_i')$ for all $s_i, s_i' \in X$.

To define the effectivity correspondence, first note that in such a non-cooperative game, each coalition is a singleton $\{i\}$. Each coalition $\{i\}$ can change a state from $s = (s_i, s_{-i})$ to state s', i.e., $\{i\} \in E(s, s')$ if and only if $s' = (s_i', s_{-i})$ for some $s_i' \in S_i$.

In this example we stick to the standard interpretation of a normal-form game where only individuals can deviate. By adjusting the effectivity correspondence, our framework can easily accommodate deviations by groups of players as for instance considered in the concept of strong Nash equilibrium introduced in Aumann (1959).

Chapter 3

Solution Concept and General Properties

In this section, first we introduce our notion of *asymptotic dominance* then we use this new notion to define our solution concept, the *Myopic Stable Set*.

In the second part of this section, we establish existence, non-emptiness and uniqueness of the solution concept.

We conclude providing some general characterizations.

3.1 Asymptotically Dominance

Intuitively, given two states x and y we say that a state y dominates x if there is a coalition which can move from y to x and each member of the coalition is better off. This intuition is formalized in the following definition:

Definition 17 (Dominance)**.** A state $y \in X$ dominates $x \in X$ under E, $y \succ x$, if there exist a coalition $S \in \mathcal{N}$ such that $S \in E(x, y)$ and $y \succ_i x$ for every $i \in S$.

Furthermore, let us define the dominance correspondence

$$f : X \longrightarrow X$$

$$x \longmapsto f(x)$$

$$f(x) \subseteq X$$

by

$$f(x) := \{x\} \cup \{y \in X | y \succ x\}.$$

In words, $f(x)$ is the subset of X which contains all states that dominate x and x itself.

We define a composition f^2 of two dominance correspondences by

$$f^2(x) := f(f(x)) := \{z \in X | \exists y \in f(x) : z \in f(y)\}.$$

Thus $f^2(x)$ is a particular subset of X which contains all the states that dominate x by a composition of two dominance correspondences.

Extending the same idea, we define f^k a subset of X which contains all states that dominate x by a composition of dominance correspondences of length $k \in \mathbb{N}$, i.e. $y \in f^k(x)$ if there is a $z \in X$ such that $y \in f(z)$ and $z \in f^{k-1}(x)$. Observe that for all $k, t \in \mathbb{N}$ if $k \leq t$, then $f^k(x) \subseteq f^t(x)$.

We define the set of all states that can be reached from x by a finite number of dominations by $f^{\mathbb{N}}(x)$, where

$$f^{\mathbb{N}}(x) := \bigcup_{k \in \mathbb{N}} f^k(x)$$

Differing from most of the previous literature, we allow for infinite state spaces.

Thus, for our solution concept, we impose asymptotic dominance which is slightly weaker than dominance.

In words, a state y asymptotically dominates x if starting from y one can get arbitrary close to x in a finite number of steps.

The next defines formally our notion of asymptotic dominance:

Definition 18 (Asymptotic Dominance)**.** A state $y \in X$ asymptotically dominates $x \in X$ under E, if for all $\epsilon > 0$ there exists an $k \in \mathbb{N}$, and a $y^* \in f^k(x)$ such that $d(y, y^*) < \epsilon$.

Given $x \in X$, we define $f^\infty(x)$ as the set of all the states $y \in X$ that asymptotically dominate x. Formally,

$$f^\infty(x) := \{y \in X | \forall \epsilon > 0 \ \exists k \in \mathbb{N}, y^* \in f^k(x) \ \text{ s.t. } \ d(y, y^*) < \epsilon\}.$$

The definition implies that $f^\infty(x)$ is the closure of $(f^{\mathbb{N}}(x))$:

$$f^\infty(x) := cl(f^{\mathbb{N}}(x))$$

Hence, given $x \in X$, $f^\infty(x)$ is the smaller subset of X containing $f^{\mathbb{N}}(x)$.

3.2 The Myopic Stable Set

Let Ω be the collection of all abstract games $\Gamma := (N, X, E, \{\succeq_i\}_{i\in N})$

Definition 19 (Set-valued solution concept)**.** We define a set-valued solution concept for abstract game as a correspondence Φ such that

$$\Phi : \Omega \longrightarrow 2^X$$
$$(N, X, E, \{\succeq_i\}_{i\in N}) \longmapsto \Phi(N, X, E, \{\succeq_i\}_{i\in N})$$
$$\Phi(N, X, E, \{\succeq_i\}_{i\in N}) \subseteq 2^X$$

In words, Φ assigns to each abstract game $(N, X, E, \{\succeq_i\}_{i\in N}) \in \Omega$ a collection of subsets of X.

A solution of an abstract game, denoted by $\varphi(N, X, E, \{\succeq_i\}_{i\in N}) \subseteq X$, is an element of $\Phi(N, X, E, \{\succeq_i\}_{i\in N}) \subseteq 2^X$, i.e. $\varphi(N, X, E, \{\succeq_i\}_{i\in N}) \in \Phi(N, X, E, \{\succeq_i\}_{i\in N})$.

We now define our solution concept, the Myopic Stable Set:

Definition 20 (Myopic Stable Set)**.** The set $M \subseteq X$ is a Myopic Stable Set (MSS) if it closed and it satisfies the following three conditions:

[1] *Deterrence of external deviations:* For every state $x \in M$ and every state $y \in X \setminus M$, we have $y \notin f(x)$.

[2] *Myopic External Stability*: For every $y \in X \backslash M$ we have that $f^\infty(y) \cap M \neq \emptyset$.

[3] *Minimality*: $\nexists M' \subsetneq M$ such that M' satisfies Conditions [1] and [2].

For any state inside the Myopic Stable Set, Condition [1] requires that there is no transformation to a state outside the set which is preferred by all members of the switching coalition. By Condition [2], any state inside the Myopic Stable Set can be reached from a point outside the set by a combination of dominance correspondences. Furthermore, Condition [2] implies that if M exists then it is non-empty. Condition [3] requires M to be the minimal subset of X which satisfies Conditions [1] and [2].

Denoting by Φ_{MSS} the particular rule the assigns to each $\Gamma := (N, X, E, (\succeq_i)_{i\in N})$ a collection of Myopic Stable Sets, we say that $\Phi_{MSS}(N, X, E, \{\succeq_i\}_{i\in N})$ is the collection of all Myopic Stable Sets of a given $\Gamma := (N, X, E, \{\succeq_i\}_{i\in N})$.

For finite state spaces, it does not matter if one uses $f^{\mathbb{N}}$ or f^∞ in the definition of external stability. On the other hand, for infinite state spaces, the asymptotic dominance relation f^∞ is the natural extension of $f^{\mathbb{N}}$.

Also when the state space is infinite, the Myopic Stable Set might fail to exist if one uses $f^{\mathbb{N}}$ instead of f^∞ in the definition of external stability.

The following example highlights why a standard dominance criterion fail with a infinite state space and why the asymptotic dominance is essential.

Example 1. Consider the abstract game

$$\Gamma = (\{1\}, (X, d), \succeq_1),$$

where,

$$X = \left\{ \frac{1}{k} \middle| k \in \mathbb{N} \right\} \cup \{0\},$$

and d is the usual metric on X, $d(x, y) = |x - y|$. As such, X is closed and limited. Preferences $\succeq_1$ are defined by $x \succeq_1 y$ if and only if $x = y$ or $y > x \geq 0$. The effectivity correspondence E is defined by setting $E(1/k, 1/(k+1)) = \{1\}$ for every $k \in \mathbb{N}$ and $E(x, y) = \emptyset$ otherwise. It follows that

$$f(1/k) = \{1/k, 1/(k+1)\}.$$

Observe that $0 \in f^{\infty}(x)$ for every $x \in X$ and that $f(0) = \{0\}$. It now follows easily that $\{0\}$ is an MSS.

Suppose we replace the requirement of external stability by the stronger notion that for all states $x \notin M$, $f^{\mathbb{N}}(x) \cap M \neq \emptyset$. Since, for every $k \in \mathbb{N}$, $0 \notin f^{\mathbb{N}}(1/k)$, the set $\{0\}$ does not satisfy external stability according to this stronger notion. Actually, we can show that there is no closed set satisfying this stronger notion of external stability together with deterrence of external deviations and minimality. Towards a contradiction, assume that the closed set $M \subseteq X$ satisfies these properties. Given that $M \neq \{0\}$ and M is non-empty, there is $k \in \mathbb{N}$ such that $1/k \in M$. Moreover, let k be the smallest such number. It is possible to verify that the closed set $M' := M \setminus \{1/k\}$ satisfies deterrence of external deviations. and the stronger notion of external stability. Now, since the closed set M' is a proper subset of M, M violates the minimality property.

3.3 Existence

In this section we prove that the Myopic Stable Set always exists.

First we provide the definition of Quasi Myopic Stable Set (QMSS):

Definition 21 (Quasi Myopic Stable Set)**.** A set $M \subseteq X$ is a Quasi Myopic Stable Set iff it is closed and satisfies conditions [1] and [2].

Theorem 1 (Existence)**.** For any abstract game $\Gamma := (N, X, E, \{\succeq_i\}_{i\in})$, there exists at least one Myopic Stable Set.

Proof. First observe that X is a Quasi Myopic Stable Set. Indeed, since it is compact, it is closed. Moreover it trivially satisfies deterrence of external deviations and external stability.

Let $\mathcal{Z}$ be the collection of all Quasi Myopic Stable Sets that are contained in X. Observe that $\mathcal{Z}$ is non-empty, given that $X \in \mathcal{Z}$.

We will use Zorn's lemma to show the existence of a minimal element in $\mathcal{Z}$.

Let $(\mathcal{Z}, \supseteq)$ be a partially ordered set. We say that $Z^\beta \in \mathcal{Z}$ is a minimal element if $Z^\alpha \supseteq Z^\beta$ implies $Z^\alpha = Z^\beta$.

Let I be an index set and $\mathcal{S} := \{Z^\alpha | \alpha \in I\}$ be a decreasing chain in $\mathcal{Z}$, i.e. $Z^\alpha \supseteq Z^\beta \supseteq ...$

Moreover, let $\triangleright$ and order on I such that for $\alpha, \beta \in I$, we write $\alpha \triangleright \beta$ if $Z^\alpha \subseteq Z^\beta$.

We say that $Z^\beta \in \mathcal{Z}$ is a lower bound of $\mathcal{S}$ if $Z^\beta \subseteq Z^\alpha$ for every $Z^\alpha \in \mathcal{S}$.

In order to apply Zorn's Lemma, we have to show that $\mathcal{S}$ has a lower bound. Let $M = \cap_{\alpha\in I} Z^\alpha$. Clearly M is a lower bound of $\mathcal{S}$. First of all, observe that M is closed as it is defined as an intersection of closed sets. Hence, we proceed by showing that $M \in \mathcal{Z}$, i.e. we have to show that M satisfies condition [1] and [2].

Deterrence of external deviations: Let $x \in M$ and $y \notin M$ be given. Then there is $\alpha \in I$ such that $y \notin Z^\alpha$, since otherwise $y \in Z^\alpha$ for all $\alpha \in I$, which means that $y \in M$. Since $x \in Z^\alpha$ and Z^α satisfies deterrence of external deviations, we obtain $y \notin f(x)$ as was to be shown.

External stability: Consider some $y \notin M$. Then there is $\alpha \in I$ such that $y \notin Z^\alpha$. As $\mathcal{S}$ is a chain, it follows that for all $\beta \triangleright \alpha$, we have $y \notin Z^\beta$.

For every $\beta \triangleright \alpha$, there is $x^\beta \in Z^\beta$ such that $x^\beta \in f^\infty(y)$, since Z^β satisfies external stability. This defines a net $\{x^\beta\}_{\beta \triangleright \alpha}$. Given that X is compact, it follows by Theorem 2.31 of Aliprantis and Border (2006) that this net has a convergent subnet, say $\{x^{\beta'}\}_{\beta' \in I'}$, where $I' \subseteq I$ is such that for all $\beta \in I$, there is a $\beta' \in I'$ such that $\beta' \triangleright \beta$. Let $\overline{x}$ be the limit of this convergent subnet. We split the remaining part of the proof in two steps. First, we show that $\overline{x} \in M$. Second, we show that $\overline{x} \in f^\infty(y)$.

Step 1: $\overline{x} \in M$: Towards a contradiction, suppose that $\overline{x} \notin M$. Then, there exists $\gamma \in I$ such that $\overline{x} \notin Z^\gamma$. In particular, given that Z^γ is a closed set, there is $\varepsilon > 0$ such that $B_\varepsilon(\overline{x}) \cap Z^\gamma = \emptyset$. Since $\mathcal{S}$ is a chain, we have that $B_\varepsilon(\overline{x}) \cap Z^\delta = \emptyset$ for all $\delta \triangleright \gamma$. Since $\overline{x}$ is the limit of the subnet $\{x^{\beta'}\}_{\beta' \in I'}$, there is $\gamma' \in I'$ such that $\gamma' \triangleright \gamma$ and $x^{\gamma'} \in B_\varepsilon(\overline{x})$. Then we have $x^{\gamma'} \in Z^{\gamma'}$, $x^{\gamma'} \in B_\varepsilon(\overline{x})$, and $B_\varepsilon(\overline{x}) \cap Z^{\gamma'} = \emptyset$, a contradiction. We conclude that $\overline{x} \in M$.

Step 2: $\overline{x} \in f^{\infty}(y)$: We need to show that for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ and $x \in f^k(y)$ such that $d(x, \overline{x}) < \varepsilon$.

Let some $\varepsilon > 0$ be given. The subnet $\{x^{\beta'}\}_{\beta' \in I'}$ converges to $\overline{x}$. As such, there exists $\gamma' \in I'$ such that $d(x^{\gamma'}, \overline{x}) < \varepsilon/2$. In addition, $x^{\gamma'} \in f^{\infty}(y)$, so there is $k \in \mathbb{N}$ and $x \in f^k(y)$ such that $d(x, x^{\gamma'}) < \varepsilon/2$. Then, by the triangle inequality, it holds that

$$d(x, \overline{x}) \leq d(x, x^{\gamma'}) + d(x^{\gamma'}, \overline{x}) < \varepsilon.$$

Together with $x \in f^k(y)$, this concludes the proof, i.e., $\overline{x} \in f^{\infty}(y)$. □

Having established existence of an MSS, in the next section we are going to analyze the cardinality of such sets.

3.4 Uniqueness

In this section, we establish uniqueness of the Myopic Stable Set under slightly stronger assumptions. In particular, we impose the following additional continuity assumption on the dominance correspondence.

The first lemma derives a property of the MSS that will be used frequently in the following proofs.

Lemma 2. Let Γ be an abstract game and let M be a myopic stable set of Γ. For all $x, y \in X$, if $x \in M$ and $y \in f^{\infty}(x)$ then $y \in M$.

Proof. Let $x \in M$ and $y \in f^{\infty}(x)$ and assume, towards a contradiction, that $y \notin M$. Given that M is closed, there is $\varepsilon > 0$ such that $B_{\varepsilon}(y) \cap M = \emptyset$. Also, by definition, there is $k \in \mathbb{N}$ and $z \in f^k(x)$ such that $z \in B_{\varepsilon}(y)$, i.e. $z \notin M$. Since $z \in f^k(x)$, there is a sequence $z^0, z^1, \ldots, z^k$ of length k such that

$$z^0 = x, z^1 \in f(z^0), \ldots, z^k = z \in f(z^{k-1}).$$

Let $k' \in \{1, \ldots, k\}$ be such that $z^{k'}$ is the first element in this sequence with the property that $z^{k'} \notin M$. Given that $z^0 = x \in M$ and $z^k = z \notin M$, such an element exists. It holds that $z^{k'-1} \in M$, $z^{k'} \in f(z^{k'-1})$, and $z^{k'} \notin M$. This contradicts deterrence of external deviations for M. □

The following lemma shows that any two myopic stable sets cannot be disjoint.

Lemma 3. Let Γ be an abstract game and let M_1 and M_2 be two myopic stable sets of Γ. Then $M_1 \cap M_2 \neq \emptyset$.

Proof. Consider a state $x_1 \in M_1$. If $x_1 \in M_2$, then we are done. Otherwise, by external stability of M_2 we know that there is $x_2 \in M_2$ such that $x_2 \in f^{\infty}(x_1)$. Lemma 2 tells us that $x_2 \in M_1$, so $x_2 \in M_1 \cap M_2$. □

The following example shows that uniqueness of an MSS cannot be demonstrated without any additional assumptions.

Example 2. Consider the abstract game $\Gamma = (\{1\}, (X, d), E, \succeq_1)$, where

$$X = \{0, 1/2, 1\} \cup \left\{ \tfrac{1}{k} \mid k \in \mathbb{N} \setminus \{1, 2\} \right\} \cup \left\{ 1 - \tfrac{1}{k} \mid k \in \mathbb{N} \setminus \{1, 2\} \right\},$$

and the metric is $d(x, y) = |x - y|$.

The effectivity correspondence is such that the individual can move from both states 0 and 1 to state 1/2 and, for every $k \in \mathbb{N} \setminus \{1, 2\}$, from state $1 - 1/k$ to state $1/k$ and from state $1/k$ to state $1 - 1/(k + 1)$. The individual cannot make any other moves. The preferences of the individual are such that

$$\tfrac{2}{3} \prec_1 \tfrac{1}{3} \prec_1 \tfrac{3}{4} \prec_1 \tfrac{1}{4} \prec_1 \tfrac{4}{5} \prec_1 \tfrac{1}{5} \prec_1 \cdots \prec_1 1 \prec_1 0 \prec_1 \tfrac{1}{2}.$$

Now, we claim that both $\{0, 1/2\}$ and $\{1/2, 1\}$ are myopic stable sets. It is easy to see that they both satisfy deterrence of external deviations (notice that the individual cannot move from 1 to 0). For external stability, observe that for every $k \in \mathbb{N} \setminus \{1, 2\}$ it holds that $0, 1 \in f^{\infty}(1/k)$ and $0, 1 \in f^{\infty}(1-1/k)$. Moreover, it holds that $1/2 \in f(0) = f^{\infty}(0)$ and $1/2 \in f(1) = f^{\infty}(1)$. Finally, for minimality, the sets $\{0\}$ and $\{1\}$ violate deterrence of external deviations since $1/2 \in f(0)$ and $1/2 \in f(1)$. The set $\{1/2\}$ violates external stability as $1/2 \notin f^{\infty}(x)$ for any $x \in X$ different from 0, 1/2 and 1.

Although Example 2 shows that the MSS is not necessarily unique, we can restore uniqueness by imposing the following mild continuity assumption on the dominance correspondence f.

Definition 22 (Lower Hemi-continuity of f)**.** The dominance correspondence $f : X \to X$ is *lower hemi-continuous* if for every sequence $\{x^k\}_{k\in\mathbb{N}}$ in X such that $x^k \to x$ and for every $y \in f(x)$ there is a sequence $\{y^k\}_{k\in\mathbb{N}}$ in X such that for all k, $y^k \in f(x^k)$ and $y^k \to y$.

In words, if there is a sequence of states converging to x and y dominates x, then it is possible to find a sequence of states that converges to y such that each element in this sequence dominates the corresponding element of the sequence that converges to x. Later on, we will show that this condition is always satisfied if preferences are continuous and some continuity condition on the effectivity relation is satisfied. The following technical lemma is helpful in proving uniqueness of an MSS.

Lemma 4. If the dominance correspondence $f : X \to X$ is lower hemi-continuous, then the asymptotic dominance correspondence $f^{\infty} : X \to X$ is transitive.

Proof. Let $x, y, z \in X$ be such that $y \in f^{\infty}(x)$ and $z \in f^{\infty}(y)$. We have to show that $z \in f^{\infty}(x)$, so we need to show that for every $\varepsilon > 0$, there is $k' \in \mathbb{N}$ and $z' \in f^{k'}(x)$ such that $d(z', z) < \varepsilon$.

By assumption, $z \in f^{\infty}(y)$, so there is $k \in \mathbb{N}$ and $z_1 \in f^k(y)$ such that $d(z_1, z) < \varepsilon/2$. In addition, as $y \in f^{\infty}(x)$, we know that for every $\ell \in \mathbb{N}$ there is $k_\ell \in \mathbb{N}$ and $y^\ell \in f^{k_\ell}(x)$ such that $d(y^\ell, y) < 1/\ell$. This generates a sequence $\{y^\ell\}_{\ell\in\mathbb{N}}$ that converges to y, i.e., $y^\ell \to y$.

Note that f^k is lower hemi-continuous, since it is a composition of k lower hemi-continuous correspondences. Given lower hemi-continuity of f^k and the fact that $z_1 \in f^k(y)$, we know that there is a sequence $\{z_2^\ell\}_{\ell\in\mathbb{N}}$ such that $z_2^\ell \to z_1$ and $z_2^\ell \in f^k(y^\ell)$. Now, we have that $y^\ell \in f^{k_\ell}(x)$ and $z_2^\ell \in f^k(y^\ell)$, which gives $z_2^\ell \in f^{k+k_\ell}(x)$.

Take ℓ large enough such that $d(z_2^\ell, z_1) < \varepsilon/2$. Conclude that $z_2^\ell \in f^{k+k_\ell}(x)$ and

$$d(z_2^\ell, z) \leq d(z_2^\ell, z_1) + d(z_1, z) < \varepsilon.$$

This completes the proof. □

We are now ready to establish the uniqueness of an MSS whenever the dominance correspondence f is lower hemi-continuous.

Theorem 2. Let Γ be an abstract game such that the corresponding dominance correspondence f is lower hemi-continuous. Then Γ has a unique MSS.

Proof. Suppose not, then, by Theorem 1 and Lemma 3, there exists an MSS M_1 and an MSS M_2 such that $M_1 \neq M_2$ and their intersection $M_3 = M_1 \cap M_2$ is non-empty. Let us show that M_3 is a QMSS, contradicting the minimality of M_1 and M_2, and establishing the uniqueness of the MSS. First of all, notice that M_3, being the intersection of two closed sets, is also closed.

For deterrence of external deviations, let $x \in M_3$ and, towards a contradiction, suppose that $y \in f(x)$ and $y \notin M_3$. Then given that $x \in M_1$ and M_1 satisfies deterrence of external deviations, it must be that $y \in M_1$. Also given that $x \in M_2$ and M_2 satisfies deterrence of external deviations, it must be that $y \in M_2$. This implies that $y \in M_1 \cap M_2 = M_3$, a contradiction. Consequently, M_3 satisfies deterrence of external deviations.

For external stability, take any $y \notin M_3$. There are three cases to consider.

Case 1: $\boldsymbol{y \in M_1 \setminus M_3}$: Then, by external stability of M_2, there is $x \in M_2$ such that $x \in f^\infty(y)$. By Lemma 2, we have that $x \in M_1$. This means that $x \in M_2 \cap M_1 = M_3$ what we needed to show.
Case 2: $\boldsymbol{y \in M_2 \setminus M_3}$: The proof is symmetric to Case 1 with M_1 and M_2 interchanged.

Case 3: $\boldsymbol{y \in X \setminus (M_1 \cup M_2)}$: We know, by external stability of M_1, that there is $x \in M_1$ such that $x \in f^\infty(y)$. If $x \in M_3$, we are done. If not, we know from Case 1 above that there is $z \in M_3$ such that $z \in f^\infty(x)$. It follows from $x \in f^\infty(y)$ and $z \in f^\infty(x)$ that $z \in f^\infty(y)$ by Lemma 4. □

The continuity condition of Theorem 2 is trivially satisfied when the state space X is finite. As such, for all applications with a finite state space, we have uniqueness of the MSS.

The dominance correspondence f is defined in terms of the individual preference relations $(\succeq_i)_{i \in N}$ and the effectivity correspondence E. It might therefore be difficult to verify lower hemi-continuity of f directly. We therefore provide conditions on the primitives of a social environment that imply lower hemi-continuity of f. As a first condition, we impose continuity of the preferences.

Definition 23 (Continuity of Preferences)**.** The preference relation $\succeq_i$ of individual $i \in N$ is *continuous* if for any two sequences $\{x^k\}_{k \in \mathbb{N}}$ and $\{y^k\}_{k \in \mathbb{N}}$ in X with $x^k \to x$ and $y^k \to y$ and, for every $k \in \mathbb{N}$, $x^k \succeq_i y^k$, it holds that $x \succeq_i y$.

Our second condition is lower hemi-continuity of the effectivity correspondence E. Towards this end, consider, for every $S \in \mathcal{N}$, the correspondence $G_S : X \to X$ defined by

$$G_S(x) = \{x\} \cup \{y \in X \mid S \in E(x, y)\}, \quad x \in X,$$

which associates to every state $x \in X$ the set of states coalition S can move to together with state x itself.

Definition 24 (Lower Hemi-continuity of E)**.** The effectivity correspondence E is *lower hemi-continuous* if for every coalition $S \in \mathcal{N}$ the correspondence $G_S : X \to X$ is lower hemi-continuous, i.e., for every sequence $\{x^k\}_{k\in\mathbb{N}}$ in X such that $x^k \to x$ and for every $y \in G_S(x)$ there is a sequence $\{y^k\}_{k\in\mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$.

Theorem 3 shows that continuity of preferences and lower hemi-continuity of E is sufficient for the dominance correspondence f to be lower hemi-continuous.

Theorem 3. Let Γ be an abstract game such that the preferences $(\succeq_i)_{i\in N}$ are continuous and the effectivity correspondence E is lower hemi-continuous. Then the dominance correspondence f is lower hemi-continuous.

Proof. Let $x, y \in X$ and sequences $\{x^k\}_{k\in\mathbb{N}}$ and $\{y^k\}_{k\in\mathbb{N}}$ in X be given. Let us first show that if individual $i \in N$ strictly prefers y to x, $y \succ_i x$, then there is a number $\ell \in \mathbb{N}$ such that for all $k \geq \ell$, $y^k \succ_i x^k$. Suppose not, then for every $\ell \in \mathbb{N}$ we can find $k_\ell \geq \ell$ such that $x^{k_\ell} \succeq_i y^{k_\ell}$. This creates sequences $\{x^{k_\ell}\}_{\ell\in\mathbb{N}}$, $\{y^{k_\ell}\}_{\ell\in\mathbb{N}}$ in X with $x^{k_\ell} \to x$ and $y^{k_\ell} \to y$ such that $x^{k_\ell} \succeq_i y^{k_\ell}$. By continuity of $\succeq_i$, $x \succeq_i y$, a contradiction.

Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in X such that $x^k \to x \in X$ and consider some $y \in f(x)$. Then either $y = x$ or $y \neq x$ and there is a coalition S such that $S \in E(x, y)$ and $y \succ_i x$ for all $i \in S$.

If $y = x$, take the sequence $\{y^k\}_{k\in\mathbb{N}}$ in X defined by $y^k = x^k$. We immediately have that, for every $k \in \mathbb{N}$, $y^k \in f(x^k)$ and $y^k \to y$.

If $y \neq x$ and there is a coalition S such that $S \in E(x, y)$ and $y \succ_i x$ for all $i \in S$, we need to show that there is a sequence $\{y^k\}_{k\in\mathbb{N}}$ such that for all k, $y^k \in f(x^k)$ and $y^k \to y$. By lower hemi-continuity of the correspondence G_S, we know that there is a sequence $\{y^k\}_{k\in\mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$. By the first paragraph of the proof, we know that for every $i \in N$ there is $\ell_i \in \mathbb{N}$ such that $y^k \succ_i x^k$ for all $k \geq \ell_i$. Let $\ell = \max_{i\in S} \ell_i$. Then, for every $k \geq \ell$ and every $i \in S$, $y^k \succ_i x^k$ and $S \in E(x^k, y^k)$, which shows that $y^k \in f(x^k)$. The sequence $\{z^k\}_{k\in\mathbb{N}}$ defined by $z^k = x^k$ if $k < \ell$ and $z^k = y^k$ if $k \geq \ell$ therefore has all the desired properties. □

Combining Theorem 2 and Lemma 3 directly yields the following corollary which gives a sufficient condition on the primitives of the model to obtain a unique MSS.

Corollary 1. Let Γ be an abstract game such that the preferences $(\succeq_i)_{i\in N}$ are continuous and the effectivity correspondence E is lower hemi-continuous. Then there is a unique MSS.

3.5 Characterization and Structure

In this section, first we introduce the notion of *closed cycle*. Then we characterize the Myopic Stable Set by proving that the set of all closed cycles is contained in the Myopic Stable Set if the state space is infinite and that it coincide with Myopic Stable Set if the state space is finite.

Furthermore, we characterize the Core in term of closed cycle and then we show that the Myopic Stable Set is a superset of the Core.

We conclude the section introducing the definition of *weak improvement property* and by showing that the Core coincides with the Myopic Stable Set if the abstract game exhibits this property.

Definition 25 (Closed Cycle)**.** A *closed cycle* of an abstract game Γ is a set $C \subseteq X$ such that for every $x \in C$ it holds that $f^{\infty}(x) = C$.

Intuitively, a closed cycle is a subset of X which is closed under the asymptotic dominance correspondence f^{∞}. We denote the union of all closed cycles by CC, so CC contains all the states that are part of some closed cycle. The following result characterizes the MSS for finite abstract games as the union of all closed cycles and shows that this union is a subset of the MSS for abstract game with an infinite state space.

Theorem 4. Let Γ be an abstract game and M be an MSS of Γ. It holds that $\text{CC} \subseteq M$. If X is finite, we have $\text{CC} = M$.

Proof. Towards a contradiction, suppose there is a closed cycle C which is not a subset of M. Let $x \in C$ and $x \notin M$. By external stability there is $y \in M$ such that $y \in f^{\infty}(x)$. As $x \in C$, we also have that $x \in f^{\infty}(y)$. By Lemma 2, it follows that $x \in M$, a contradiction. Since the choice of C was arbitrary, we have shown that $\text{CC} \subseteq M$.

We show next that if X is finite, then $\text{CC} = M$. Since $\text{CC} \subseteq M$, we only need to show that CC is a QMSS. The set CC satisfies deterrence of external deviations, since for all $x \in \text{CC}$, $f(x) \subseteq f^{\infty}(x) \subseteq \text{CC}$. It remains to verify external stability of CC, i.e., for every state $x \notin \text{CC}$, $f^{\infty}(x) \cap \text{CC} \neq \emptyset$.

Let $x \notin \text{CC}$ and define $Y = f^{\infty}(x)$. Note that Y is non-empty since$x \in f(x)$, finite and that $f^{\infty}(y) \subseteq Y$ for every $y \in Y$. Let us represent the set Y and the dominance relation f on Y by a finite directed graph D, i.e., (i) Y are the vertices of D and (ii) D has an arc from y to z if and only if $z \in f(y)$. By contracting each strongly connected component of D to a single vertex, we obtain a directed acyclic graph, which is called the condensation of D. As the condensation is finite and acyclic, it has a maximal element, say c. Observe that c represents a closed cycle C, so $Y \cap \text{CC} \neq \emptyset$. □

A *sink* is a closed cycle which consists of only one state, i.e., $f(x) = x$. The union of all sinks is called the core.

Definition 26 (Core)**.** Let Γ be an abstract game. The *core* of Γ is given by

$$\text{CO} = \{x \in X \mid f(x) = \{x\}\}.$$

It is well-known that the core may be empty for some abstract game. However, if it is not empty, then it is always contained in the myopic stable set by virtue of Theorem 4.

Corollary 2. Let Γ be an abstract game and let M be an MSS. Then we have $\text{CO} \subseteq M$.

The next definition is inspired by the finite analogue for normal-form games as presented in Friedman and Mezzetti (2001).

Definition 27 (Weak (Finite) Improvement Property)**.** An abstract game Γ satisfies the *weak finite improvement property* if for each state $x \in X$, $f^{\mathbb{N}}(x)$ contains a sink and the *weak improvement property* if for each state $x \in X$, $f^{\infty}(x)$ contains a sink.

The following provides a characterization for the MSS in abstract game with the weak improvement property.

Theorem 5. Let Γ be an abstract game and let f be lower hemi-continuous. Then, the MSS of Γ is equal to the core if and only if the abstract game satisfies the weak improvement property.

Proof. Assume that Γ has the weak improvement property. By Corollary 2, $\text{CO} \subseteq M$. We will show that CO is a QMSS. By minimality, it then follows that $\text{CO} = M$.

In order to see that CO is closed let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence in CO, i.e., for all k, $\{x^k\} = f(x^k)$. Now assume that $x^k \to x$ and $x \notin \text{CO}$. This means that there is $y \neq x$ such that $y \in f(x)$. By lower hemi-continuity of f, there should be a sequence $\{y^k\}_{k\in\mathbb{N}}$ such $y^k \in f(x^k)$ and $y^k \to y$. As for all k, $x^k \in \text{CO}$, we have that for all k, $y^k = x^k$ which means that $y^k \to x \neq y$, a contradiction. Deterrence of external deviations is immediate for the core as it is the union of sinks. If the abstract game satisfies the weak improvement property we have that for all $x \notin \text{CO}$, $f^{\infty}(x) \cap \text{CO} \neq \emptyset$ thus the core satisfies external stability.

For the reverse, assume that $\text{CO} = M$. Now, if $x \in M$, it is a sink, so $f^{\infty}(x) = \{x\} \subseteq \text{CO}$. If $x \notin \text{CO}$ we have by external stability of M, that $f^{\infty}(x) \cap M \neq \emptyset$, so $f^{\infty}(x)$ contains a sink. This shows that Γ satisfies the weak improvement property. □

The requirement of lower hemi-continuity of f in Theorem 5 can be weakened to the requirement that CO should be closed.

Chapter 4

Application to Specific Settings

In this section we discuss how our results can be applied to the specific settings presented in chapter 2.

4.1 TU Games with Coalition Structure

We can associate a social environment $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ to each TU game with coalition structure (N, v, π) as in Section 2.1, so we impose the properties of non-interference, full support and the γ-model.

By Theorem 1 we know that there exists at least one non-empty MSS. Let us first show that for TU games with coalition structure, the MSS is also unique. Towards this end, we first show that the preference relations $\succeq_i$ are continuous and that the effectivity correspondence E is lower hemi-continuous.

Lemma 5. Let (N, v, π) be a TU game with coalition structure and let $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ be the induced abstract game as in Section 2.1. Then, for every $i \in N$, the preference relation $\succeq_i$ is continuous and the effectivity correspondence E is lower hemi-continuous.

Proof. Let some $i \in N$ be given. To show continuity of $\succeq_i$, let $\{x^k\}_{k \in \mathbb{N}}$ and $\{y^k\}_{k \in \mathbb{N}}$ be sequences in X such that $x^k \to x$ and $y^k \to y$. Then, by the continuity of u_i, we have that $u_i(x^k) \to u_i(x)$ and $u_i(y^k) \to u_i(y)$. So if $u_i(x^k) \geq u_i(y^k)$ for all $k \in \mathbb{N}$, we obtain $u_i(x) \geq u_i(y)$, which shows that $x \succeq_i y$.

To show lower hemi-continuity of E, let some $S \in \mathcal{N}$, a sequence $\{x^k\}_{k \in \mathbb{N}}$ in X such that $x^k \to x$ and some $y \in G_S(x)$ be given. We show that there is a sequence $\{y^k\}_{k \in \mathbb{N}}$ such that $y^k \in G_S(x^k)$ and $y^k \to y$. If $y = x$, then the choice $y^k = x^k$ would do, so consider the case $y \neq x$,

First of all, there is $k' \in \mathbb{N}$ such that for all $k \geq k'$, $\pi(x^k) = \pi(x)$, so in particular $\mathcal{U}(x^k, S) = \mathcal{U}(x, S)$. For every $k < k'$, we define $y^k = x^k$. For every $k \geq k'$, we define $y^k \in X$ by $\pi(y^k) = \pi(y)$

and

$$u_i(y^k) = \begin{cases} u_i(y), & i \in N \setminus U(x,S), \\ u_i(x^k), & i \in U(x,S). \end{cases}$$

Consider some $k \geq k'$. Since $y \neq x$, it holds that $S \in \pi(y)$ and, for every $i \in N \setminus (S \cup U(x,S))$, we have that i is a residual player and the properties of the γ-model imply that $\{i\} \in \pi(y)$. The same properties hold for $\pi(y^k)$. For every $i \in S$, it holds that $u_i(y^k) = u_i(y)$, so $u_i(y^k) \geq v(\{i\})$ and $\sum_{i \in S} u_i(y^k) = v(S)$. For every $i \in N \setminus (S \cup U(x,S))$, we have that $u_i(y^k) = v(\{i\}) = u_i(y)$. For every $i \in U(x,S)$ it holds that $u_i(y) = u_i(x)$ and $u_i(y^k) = u_i(x^k)$. By coalitional sovereignty, we have that $y^k \in G_S(x^k)$. Using that $x^k \to x$, it follows easily that $y^k \to y$. □

Lemma 5 together with Theorem 2 and Theorem 3 shows uniqueness of the MSS.

Corollary 3. Let (N,v) be a coalition function form game and let Γ be the induced abstract game as in Section 2.1. Then Γ has a unique MSS.

In fact, most other models of coalitional sovereignty will also lead to lower hemi-continuity of E so will also have a unique MSS. However, establishing the lower hemi-continuity of E must be done case by case.

The Coalition Structure Core One of the most prominent set-valued solution concepts for coalition function form games is the coalition structure core.

Definition 28 (Coalition Structure Core)**.** Let (N,v,π) be a TU game with coalition structure and let $\Gamma = (N,(X,d),E,(\succeq_i)_{i \in N})$ be the induced abstract game as in Section 2.1. The *coalition structure core* of (N,v,π) is the set of states $x \in X$ such that for every coalition $S \in \mathcal{N}$

$$\sum_{i \in S} u_i(x) \geq v(S).$$

In words, the coalition structure core gives to the members of each coalition at least the payoff they can obtain by forming that coalition.

Lemma 6. Let (N,v,π) be a TU game with coalition structure and let $\Gamma = (N,(X,d),E,(\succeq_i)_{i \in N})$ be the induced abstract game as in Section 2.1. The coalition structure core of (N,v,π) is equal to the core of Γ.

Proof. Let Y be the coalition structure core. Let $y \in \mathrm{CO}$ and assume $y \notin Y$. Then there is a coalition S such that $\sum_{i \in S} u_i(y) < v(S)$. Since $y \in X$, it holds for all $i \in S$, $u_i(y) \geq v(\{i\})$. Now, let u_S be a vector of payoffs for the members in S such that $\sum_{i \in S} u_i = v(S)$ and for all $i \in S$, $u_i > u_i(y)$. Then, by full support, there exists a state $y' \in X$ such that $S \in E(y,y')$ and $u_S = u_S(y')$. Conclude that $y' \in f(y)$. This contradicts the fact that $y \in \mathrm{CO}$.

For the reverse, let $y \in Y$ and $z \in f(y)$ such that $z \neq y$, i.e., $y \notin \mathrm{CO}$. Then there is

$S \in E(y, z)$ such that $u_S(z) \gg u_S(y)$. Also,

$$v(S) = \sum_{i \in S} u_i(z) > \sum_{i \in S} u_i(y) \geq v(S),$$

where the first equality follows from the definition of the state space and the last inequality from the definition of Y. We have obtained a contradiction. □

Kóczy and Lauwers (2004) define the coalition structure core to be accessible if from any initial state there is a finite sequence of states ending with an element of the coalition structure core and each element in that sequence outsider independently dominates the previous element. The notion of outsider independent domination differs from our notion of a myopic improvement in two ways. First, residual players are not required to become singletons after a move has taken place. Second, improvements for the members of the coalition that moves are not necessarily strict improvements.

The following example illustrates that under the requirement of strict improvements of all members involved in a move, as in our dominance correspondence f, the coalition structure core does not satisfy strong external stability, i.e., it is not the case that for all states $x \in X$, there is a state y in the coalitional structure core such that $y \in f^{\mathbb{N}}(x)$.

Example 3. Let (N, v, π) be a TU game with coalition structure such that $N = \{1, 2, 3\}$, $v(\{1, 2\}) = 1$, and $v(\{2, 3\}) = 1$. All other coalitions have a worth of 0. Here, player 2 can choose to form a coalition with either player 1 or player 3 to form a two-person coalition generating a surplus equal to one. The coalition structure core therefore consists of only two states, y and y', with equal payoffs, $u(y) = u(y') = (0, 1, 0)$, and coalitional structures $\pi(y) = \{\{1, 2\}, \{3\}\}$, and $\pi(y') = \{\{1\}, \{2, 3\}\}$.

Consider an initial state $x^0 \in X$ such that $\pi(x^0) = \{\{1\}, \{2\}, \{3\}\}$ and $u(x^0) = (0, 0, 0)$. Under our notion of a myopic improvement, where all players involved in a move have to gain strictly, a state x^1 belongs to $f(x^0)$ if and only if either $\pi(x^1) = \{\{1, 2\}, \{3\}\}$ and $u(x^1) = (\varepsilon, 1 - \varepsilon, 0)$ for some $\varepsilon \in (0, 1)$ or $\pi(x^1) = \{\{1\}, \{2, 3\}\}$ and $u(x^1) = (0, 1 - \varepsilon, \varepsilon)$ for some $\varepsilon \in (0, 1)$. It follows that x^1 is a state where either player 1 or player 3 receives a payoff of zero and the other two players receive a strictly positive payoff summing up to 1.

Now consider any state x^k such that either player 1 or player 3 receives 0 and the other two players receive a strictly positive payoff summing up to 1. We claim that any state $x^{k+1} \in f(x^k)$ has the same properties. Without loss of generality, assume that $u_3(x^k) = 0$. Let x^{k+1} be an element of $f(x^k)$ different from x^k. Since $u_1(x^k) + u_2(x^k) = 1$, the only coalition that can move is $\{2, 3\}$ and it holds that $\pi(x^{k+1}) = \{\{1\}, \{2, 3\}\}$. Moreover, it must also hold that $u_2(x^{k+1}) > u_2(x^k) > 0$ and $u_3(x^{k+1}) > u_3(x^k) = 0$, which proves the claim. It now follows that for every $k \in \mathbb{N}$, if $x^k \in f^k(x^0)$, then x^k is such that there are two players with a strictly positive payoff. Given this, there is no $k \in \mathbb{N}$ such that x^k belongs to the coalition structure core.

Theorem 6 shows that the MSS coincides with the coalition structure core whenever it is non-empty.

Theorem 6. Let (N, v, π) be a TU game with coalition structure, Γ be the induced abstract game as in Section 2.1, and Y be the coalition structure core of Γ. If Y is non-empty, then the unique MSS of Γ is equal to Y.

Proof. From Lemma 5 we know that f is lower hemi-continuous. Also Lemma 6 shows that Y is equal to the core of Γ. If we can show that Γ satisfies the weak improvement property whenever $Y \neq \emptyset$, then we can use Theorem 5 to establish our proof. Since the proof is trivial when the number of individuals $n = 1$, we assume $n \geq 2$ throughout.

So assume that $Y \neq \emptyset$. We need to show that for all $x^0 \in X$, $f^\infty(x^0) \cap Y \neq \emptyset$. If x^0 in Y, then nothing needs to be shown, so assume that $x^0 \in X \setminus Y$. We need to show that for every $\varepsilon > 0$ there is a number $k' \in \mathbb{N}$, a state $x^{k'} \in f^{k'}(x^0)$, and a state $y \in Y$ such that $d(x^{k'}, y) < \varepsilon$.

Let some $\varepsilon > 0$ be given. Béal, Rémila, and Solal (2013) show that there exists a sequence of states $(x^0, \ldots, x^{k'})$ such that $x^{k'} \in Y$, k' is less than or equal to $(n^2 + 4n)/4$, and, for every $k \in \{1, \ldots, k'\}$,

1. there is $S^k \in \mathcal{N}$ such that $S^k \in E(x^{k-1}, x^k)$,
2. $u_{S^k}(x^{k-1}) < u_{S^k}(x^k)$.

Notice that the inequality in 2. only means that at least one of the players in S^k gets a strictly higher payoff, though not necessarily all of them. Let P^k be the set of partners of the players in S^k at state x^{k-1}, more formally defined as

$$P^k = \cup_{\{S \in \pi(x^{k-1}) | S \cap S^k \neq \emptyset\}} S,$$

so P^k is equal to the moving coalition S^k together with the residual players. Since $S^k \in E(x^{k-1}, x^k)$, it follows that

$$\begin{aligned} u_i(x^k) &= v(\{i\}), & i &\in P^k \setminus S^k, \\ u_i(x^k) &= u_i(x^{k-1}), & i &\in N \setminus P^k. \end{aligned}$$

We define $W^k \subset S^k$ to be the, possibly empty, proper subset of S^k consisting of players that only weakly improve when moving from state x^{k-1} to state x^k, so for every $i \in W^k$ it holds that $u_i(x^{k-1}) = u_i(x^k)$. We define

$$\begin{aligned} \delta &= \min_{k \in \{1,\ldots,k'\}} \min_{i \in S^k \setminus W^k} u_i(x^k) - u_i(x^{k-1}), \\ \varepsilon' &= \min\{\delta, \varepsilon\}, \end{aligned}$$

so δ is the smallest improvement of any of the strictly improving players involved in any move along the sequence. It holds that $\delta > 0$ and therefore that $\varepsilon' > 0$. For $k \in \{0, \ldots, k'\}$, define

$$\nu_k = \frac{n^{2k}}{n^{2k'+1}}.$$

We define $e(W^k) = 0$ if $W^k = \emptyset$ and $e(W^k) = 1$ otherwise. We use the sequence $(x^0, x^1, \ldots, x^{k'})$ of states as constructed by Béal, Rémila, and Solal (2013) to define a new sequence $(\widetilde{x}^0, \widetilde{x}^1, \ldots, \widetilde{x}^{k'})$

of states by setting $\widetilde{x}^0 = x^0$ and, for every $k \in \{1, \ldots, k'\}$,

$$\pi(\widetilde{x}^k) = \pi(x^k),$$

$$\begin{aligned} u_i(\widetilde{x}^k) &= u_i(x^k) + \varepsilon'\nu_k \tfrac{|S^k \setminus W^k|}{|W^k|}, && i \in W^k, \\ u_i(\widetilde{x}^k) &= u_i(x^k) - \varepsilon'\nu_k e(W^k), && i \in S^k \setminus W^k, \\ u_i(\widetilde{x}^k) &= u_i(x^k) = v(\{i\}), && i \in P^k \setminus S^k, \\ u_i(\widetilde{x}^k) &= u_i(\widetilde{x}^{k-1}), && i \in N \setminus P^k. \end{aligned}$$

Notice that the first line does not entail a division by zero, since if $i \in W^k$, then $W^k \neq \emptyset$.

Compared to the sequence $(x^0, x^1, \ldots, x^{k'})$, the sequence $(\widetilde{x}^0, \widetilde{x}^1, \ldots, \widetilde{x}^{k'})$ is such that each strictly improving player in $S^k \setminus W^k$ donates an amount $\varepsilon'\nu_k/|W^k|$ to each of the players in W^k whenever the latter set is non-empty. It is also important to observe that the fraction ν_k is an n^2 multiple of ν_{k-1} and that $\nu_{k'} = 1/n$.

We show first by induction that, for every $k \in \{0, \ldots, k'\}$, $\widetilde{x}^k \in X$. Obviously, it holds that $\widetilde{x}^0 = x^0 \in X$. Assume that, for some $k \in \{1, \ldots, k'\}$, $\widetilde{x}^{k-1} \in X$. We show that $\widetilde{x}^k \in X$. It holds that

$$\begin{aligned} u_i(\widetilde{x}^k) &> u_i(x^k) \geq v(\{i\}), && i \in W^k, \\ u_i(\widetilde{x}^k) &\geq u_i(x^{k-1}) + \delta - \varepsilon'\nu_k > u_i(x^{k-1}) + \delta - \varepsilon' \geq u_i(x^{k-1}) \geq v(\{i\}), && i \in S^k \setminus W^k, \\ u_i(\widetilde{x}^k) &= v(\{i\}), && i \in P^k \setminus S^k, \\ u_i(\widetilde{x}^k) &= u_i(\widetilde{x}^{k-1}) \geq v(\{i\}), && i \in N \setminus P^k, \end{aligned}$$

where the very last inequality follows from the induction hypothesis. Moreover, for every $S \in \pi(x^k)$, it holds that either $S = S^k$ and $W^k = \emptyset$, so

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in S^k} u_i(x^k) = v(S),$$

or $S = S^k$ and $W^k \neq \emptyset$, so

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in W^k} \left(u_i(x^k) + \varepsilon'\nu_k \frac{|S^k \setminus W^k|}{|W^k|} \right) + \sum_{i \in S^k \setminus W^k} \big(u_i(x^k) - \varepsilon'\nu_k \big) = \sum_{i \in S^k} u_i(x^k) = v(S),$$

or $S = \{i'\}$ with $i' \in P^k \setminus S^k$ and

$$\sum_{i \in S} u_i(\widetilde{x}^k) = u_{i'}(\widetilde{x}^k) = u_{i'}(x^k) = v(\{i'\}) = v(S),$$

or $S \subseteq N \setminus P^k$, so $S \in \pi(\widetilde{x}^{k-1})$, and

$$\sum_{i \in S} u_i(\widetilde{x}^k) = \sum_{i \in S} u_i(\widetilde{x}^{k-1}) = v(S),$$

where the last equality makes use of the induction hypothesis. We have now completed the proof of the fact that for every $k \in \{0, \ldots, k'\}$, $\widetilde{x}^k \in X$.

We show next by induction that, for every $k \in \{0, \ldots, k'\}$, and for every $i \in N$,

$$|u_i(\widetilde{x}^k) - u_i(x^k)| \leq \varepsilon' \nu_k (n-1).$$

Obviously, for every $i \in N$, it holds that $|u_i(\widetilde{x}^0) - u_i(x^0)| = 0 \leq \varepsilon' \nu_0 (n-1)$. Assume that, for some $k \in \{1, \ldots, k'\}$, for every $i \in N$, $|u_i(\widetilde{x}^{k-1}) - u_i(x^{k-1})| \leq \varepsilon' \nu_{k-1}(n-1)$. We show that, for every $i \in N$, $|u_i(\widetilde{x}^k) - u_i(x^k)| \leq \varepsilon' \nu_k (n-1)$. If $i \in W^k$, then $W^k \neq \emptyset$, and the statement follows from the observation that

$$0 \leq u_i(\widetilde{x}^k) - u_i(x^k) = \varepsilon' \nu_k \tfrac{|S^k \setminus W^k|}{|W^k|} \leq \varepsilon' \nu_k (n-1).$$

If $i \in S^k \setminus W^k$, then we have that

$$0 \geq u_i(\widetilde{x}^k) - u_i(x^k) \geq -\varepsilon' \nu_k \geq -\varepsilon' \nu_k (n-1).$$

If $i \in P^k \setminus S^k$, then $|u_i(\widetilde{x}^k) - u_i(x^k)| = 0$. If $i \in N \setminus P^k$, then it holds that

$$|u_i(\widetilde{x}^k) - u_i(x^k)| = |u_i(\widetilde{x}^{k-1}) - u_i(x^{k-1})| \leq \varepsilon' \nu_{k-1}(n-1) < \varepsilon' \nu_k (n-1),$$

where the first inequality makes use of the induction hypothesis and the last inequality of the fact that $\nu_{k-1} < \nu_k$.

Let some $k \in \{1, \ldots, k'\}$ and some $i \in S^k$ be given. We show that $u_i(\widetilde{x}^k) > u_i(\widetilde{x}^{k-1})$. If $i \in W^k$, then it holds that

$$\begin{aligned}
u_i(\widetilde{x}^k) &= u_i(x^k) + \varepsilon' \nu_k \frac{|S^k \setminus W^k|}{|W^k|} \\
&= u_i(x^{k-1}) + \varepsilon' \nu_k \frac{|S^k \setminus W^k|}{|W^k|} \\
&\geq u_i(\tilde{x}^{k-1}) - \varepsilon' \nu_{k-1}(n-1) + \varepsilon' \nu_k \frac{1}{n-1} \\
&> u_i(\widetilde{x}^{k-1}),
\end{aligned}$$

where the strict inequality uses that $\nu_k = n^2 \nu_{k-1}$. If $i \in S^k \setminus W^k$, then it holds that

$$\begin{aligned}
u_i(\widetilde{x}^k) &\geq u_i(x^k) - \varepsilon' \nu_k \\
&\geq u_i(x^{k-1}) + \delta - \varepsilon' \nu_k \\
&\geq u_i(\widetilde{x}^{k-1}) - \varepsilon' \nu_{k-1}(n-1) + \delta - \varepsilon' n^2 \nu_{k-1} \\
&> u_i(\widetilde{x}^{k-1}),
\end{aligned}$$

where the strict inequality uses the facts that $\delta \geq \varepsilon'$ and

$$(n^2 + (n-1))\nu_{k-1} < 2n^2 \nu_{k-1} \leq 2\nu_k \leq 1.$$

Combining the statements proven so far, it follows that $\widetilde{x}^{k'} \in f^{k'}(x^0)$. We complete the proof of the weak improvement property by noting that $x^{k'} \in Y$ by the result of Béal, Rémila, and Solal (2013) and by demonstrating that $d(\widetilde{x}^{k'}, x^{k'}) < \varepsilon$. It follows that $d(\widetilde{x}^{k'}, x^{k'}) < \varepsilon$ since $\pi(\widetilde{x}^{k'}) = \pi(x^{k'})$ and, for every $i \in N$,

$$|u_i(\widetilde{x}^{k'}) - u_i(x^{k'})| \leq \varepsilon' \nu_{k'}(n-1) < \varepsilon' \leq \varepsilon.$$

□

We conclude this section presenting a TU game with a non-empty Core

Example 4. (Ray and Vohra, 2015):

A convex characteristic function game is a game in which the characteristic function is supermodular, i.e. $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$ for every $S \subseteq T$.

Consider the following convex game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	3	3	3	6

In a convex characteristic function game the Core is non-empty (Shapley, 1971).

The Coalition Structure Core of this game is represented by the set

$$A := \{u \in \mathbb{R}^3 | u_1, u_2, u_3 \geq 0; u_1 + u_2 \geq 3; u_2 + u_3 \geq 3; u_1 + u_3 \geq 3; u_1 + u_2 + u_3 = 6)\}$$

It is possible to check that the set A coincides with the MSS of the induced abstract game as defined in section 2.1. Intuitively: A is closed since it is defined as a system of weak linear inequalities; A satisfies deterrence of external deviations since every state in the set is a sink; A satisfies external stability since for every state not in A such that the grand coalition is formed there exists some coalition which can deviate to another state outside A such that the grand coalition is not formed, and for every state outside A such that the grand coalition is not formed there exists a sequence of coalitional deviations which approaches the set A; A is the minimal set which satisfies these conditions.

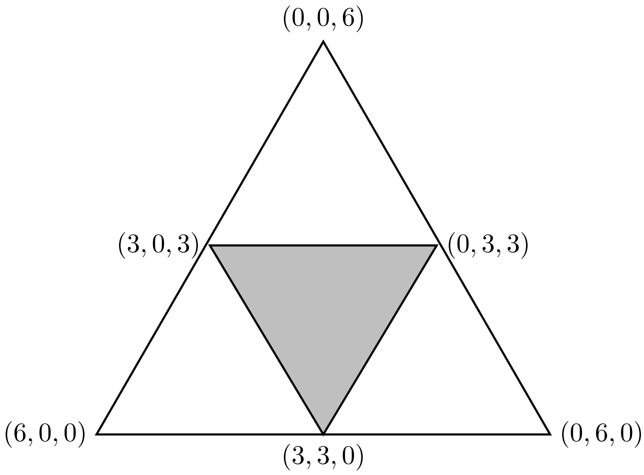


Figure 4.1: convex game

The set of all payoff allocation for the grand coalition is represented by the the convex hull shown in Figure 4.1

4.2 One-to-One Matching

A matching μ' dominates $\mu \in X$ under E, $\mu' \succ \mu$, if there exists a coalition $S \in \mathcal{N}$ such that $\mu(S) = S$ and $\mu' \succ_S \mu$, where $\succ_S := (\succ_i)_{i \in S}$.

Consequently, the dominance correspondence f specifies the relations of dominance among the matching, i.e. $f(\mu)$ is the subset of X which contains all the matching that dominate μ and μ itself.

In particular the Core of a One-to-One matching problem can be defined as follows:

Definition 29 (Core)**.** The Core of a matching problem $(M, N, \succ)$ consists of all undominated matchings:

$$C_M := \{\mu \in X | f(\mu) = \emptyset\}$$

The Core of a matching problem is not empty (Gale and Shapley, 1962).

For our purposes we refer to the following result:

Lemma 7. (Roth and Vande Vate, 1990)

For every $\mu \in X$ we have that $f^{\mathbb{N}}(\mu) \cap C_M \neq \emptyset$.

Since the set of states is finite in this application, it holds that $f^{\mathbb{N}}(\mu) = f^{\infty}(\mu)$. As such, the result of Roth and Vande Vate (1990) can be rephrased as saying that Γ satisfies the weak improvement property as defined in Definition 27. Given that for finite settings f is always lower hemi-continuous, the following result now follows from Theorem 5.

Corollary 4. Let $(M, W, \succ)$ be a matching problem and let Γ be the induced abstract game as in Section 2.2. Then the MSS of Γ is unique and equal to the set of stable matchings.

Herings, Mauleon, and Vannetelbosch (2016) define the level-1 farsighted set for matching problems. It is not hard to see that the MSS for abstract game Γ as in Section 2.2 coincides with the level-1 farsighted set. Corollary 4 is therefore equivalent to Theorem 3 of Herings, Mauleon, and Vannetelbosch (2016) that characterizes the level-1 farsighted set as the core of the matching problem.

Let consider the following example (see Gale and Shapley, 1962)

	w^1	w^2	w^3
m^1	$1,3$	$2,2$	$3,1$
m^2	$3,1$	$1,3$	$2,2$
m^2	$2,2$	$3,1$	$1,3$

Figure 4.2:

Example 5. The above *ranking matrix* the preferences of each player: the first number of each pair in the matrix gives ranking of player $w^i \in W$ by the player $m^i \in M$; the second number of each pair in the matrix gives ranking of player $m^i \in M$ by the player $w^i \in W$. For example: player m^1 ranks w^1 first, w^2 second and w^3 third while player w^1 ranks m^2 first, m^3 second and m^1 third, and so on.

It easy to check that the Core and the Myopic Stable Set correspond to the following set of three matchings:

$$= \{(M^1W^1; M^2W^2; M^3W^3)(M^1W^3; M^2W^1; M^3W^2)(M^1W^2; M^2W^3; M^3W^1)\}$$

4.3 Networks

As in section 2.3, we can associate an abstract game $\Gamma := (N, (X, d), \{\succeq_i\}_{i\in N})$ to each network problem $(N, \mathbb{G}, (Y_i)_{i\in N})$.

Definition 30 (Pairwise Stability)**.** A network g is said to be pairwise stable (Jackson and Wolinsky, 1996) if for every $ij \in g$ it holds that $Y_i(g - ij) \leq Y_i(g)$ and $Y_j(g - ij) \leq Y_j(g)$ and for every $ij \notin g$ it holds that $Y_i(g + ij) > Y_i(g)$ implies $Y_j(g + ij) < Y_j(g)$.

Pairwise stability as defined in Jackson and Wolinsky (1996) is somewhat stronger and also requires that there is no $ij \notin g$ such that $Y_i(g + ij) > Y_i(g)$ and $Y_j(g + ij) = Y_j(g)$. The weaker notion used here is discussed as an alternative in Section 5 of Jackson and Wolinsky (1996) and is also widely used in the literature. For generic network problems, there are no indifferences, so the two definitions are equivalent.

It is not hard to show that a network is pairwise stable if and only if it is in the core of the social environment Γ as defined in Definition 26.

Corollary 2 shows that any pairwise stable network is in the myopic stable set. However, it is not necessarily the case that the MSS only contains the pairwise stable networks.

Consider the binary relation R on X defined by gRg' if $g \in f^{\mathbb{N}}(g')$, i.e., g can be reached from g' by a finite number of dominations. Let I be the symmetric part of R, i.e., gIg' if and only if gRg' and $g'Rg$. Consider the set of equivalence classes $\mathbb{E}$ induced by I. Let us denote the equivalence class of network g by $[g]$, i.e., $g' \in [g]$ if and only if $g'Ig$. For two distinct equivalence classes $[g]$ and $[g']$ write $[g]P[g']$ if gRg'. It is easy to see that $[g]P[g']$ if and only if gRg' and not gRg'.

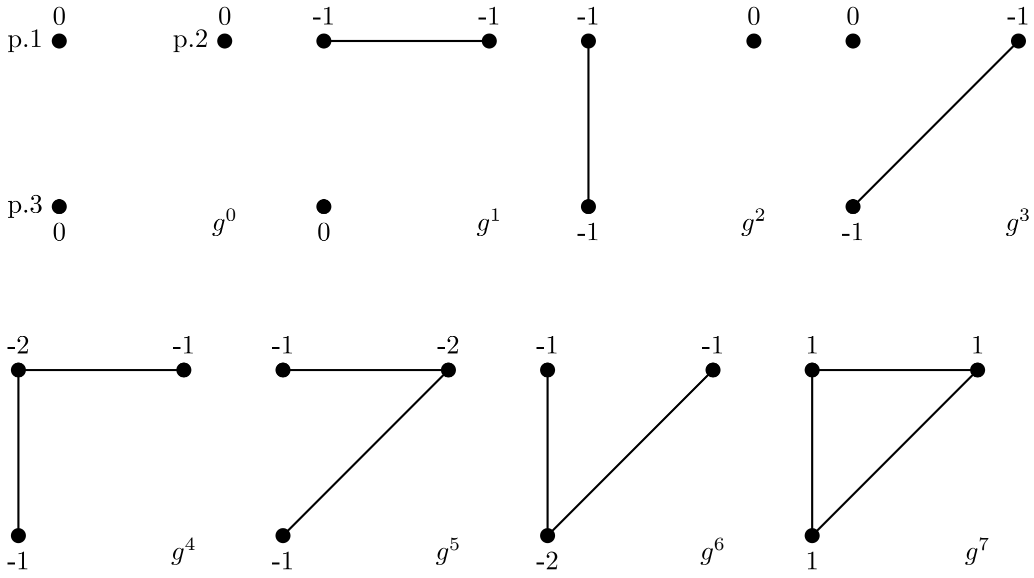
Let V be the collection of maximal elements of $(\mathbb{E}, P)$, i.e., $[g] \in V$ if there is no $[g']$ such that $[g']P[g]$. Since an element of V simply represents a closed cycle as defined in Definition 25, the following result follows from Theorem 4.

Corollary 5. Let $(N, \mathbb{G}, (Y_i)_{i\in N})$ be a network problem and let Γ be the induced abstract game as in Section 2.3. A network g belongs to an MSS M if and only if the equivalence class $[g]$ belongs to V, i.e., $M = \{g \in X | [g] \in V\}$.

Herings, Mauleon, and Vannetelbosch (2009) define the pairwise myopically stable sets for network problems using the weaker notion of dominance corresponding to pairwise stability as defined in Jackson and Wolinsky (1996). It is not hard to see that the MSS for abstract game Γ as in Section 2.3 coincides with the pairwise myopically stable set for generic network problems. For such network problems, Corollary 5 is therefore equivalent to Theorem 1 of Herings, Mauleon, and Vannetelbosch (2009) that characterizes the pairwise myopically stable set as the union of closed cycles. In their paper, a closed cycle is defined in the sense of Jackson and Watts (2002) for network problems. The notion of closed cycle of Definition 25 is the appropriate generalization to social environments.

We conclude this section providing the following example of network formation:

Example 6 (Investment Game)**.** We consider three players. Every player can form a link with another player with a cost of 1. Every player achieves a pay of n if all players have formed a link with all other players and a pay of zero if at least one link is missing.



It easy to check that the Myopic Stable Set of this investment game contains two networks: the empty network and the complete network, i.e. $MSS = \{g_0, g_7\}$.

4.4 Normal Form Games

In Section 2.4, we associated an abstract game $\Gamma = (N, (X, d), E, (\succeq_i)_{i \in N})$ to each normal-form game $G = (N, (S_i)_{i \in N}, (\succeq_i)_{i \in N})$.

A strategy profile $s \in S$ is said to be a pure strategy Nash equilibrium of the game G if, for every $i \in N$, for every $s'_i \in S_i$, it holds that $s \succeq_i (s'_i, s_{-i})$.

It can easily be shown that a strategy profile is a pure strategy Nash equilibrium if and only if it is in the core of the abstract game Γ as defined in Definition 26. Corollary 2 then shows that every pure strategy Nash equilibrium belongs to every MSS. For normal-form games, Theorem 5 reduces to the following result.

Corollary 6. Let G be a normal-form game and let Γ be the induced abstract game as in Section 2.4. The MSS of Γ is equal to the set of pure strategy Nash equilibria if and only if Γ has the weak improvement property.

The following result exploits the fact that many classes of games have the weak improvement property.

Corollary 7. Let G be a normal-form game and let Γ be the induced abstract game as in Section 2.4. Then the MSS is equal to the set of pure strategy Nash equilibria for finite potential games, aggregative games, and finite supermodular games.

Proof. For finite supermodular games, Friedman and Mezzetti (2001) show that the game has the weak finite improvement property which implies the weak improvement property. Monderer and Shapley (1996) establish the weak finite improvement property for potential games. For aggregative games, it is easily verified that E is lower hemi-continuous and preferences are continuous by assumption, so f is lower hemi-continuous by Theorem 3. Dindoš and Mezzetti (2006) show that aggregative games have the weak finite improvement property. The result now follows from Theorem 5. □

As an illustration, consider the two games in Example 7.

Example 7. Game 1 has a unique Nash equilibrium, but does not satisfy the weak finite improvement property. Thus, the MSS may contain strategy profiles which are not Nash equilibria.

Game 1:

		Player 2			
		E	F	G	H
Player 1	A	$4, 1$	$-20, -20$	$1, 4$	$0, 0$
	B	$2, 2$	$4, 1$	$-20, -20$	$0, 0$
	C	$-20, -20$	$2, 2$	$4, 1$	$0, 0$
	D	$0, 0$	$0, 0$	$0, 0$	$1, 1$

The Nash equilibrium of Game 1 is (D, H) and the unique myopic stable set M is given by

$$M = \{(A, E), (A, G), (B, E), (B, F), (C, F), (C, G), (D, H)\}.$$

On the other hand, Game 2 shows that not every strategy profile where each strategy is played with positive probability in a mixed-strategy Nash equilibrium is part of an MSS. In this game there exists a pure strategy Nash equilibrium (B, R) and two mixed strategy Nash equilibria

$$\Big((\tfrac{1}{2}T, \tfrac{1}{2}M, 0B), \quad (\tfrac{1}{2}L, \tfrac{1}{2}C, 0R) \Big), \Big((\tfrac{19}{42}T, \tfrac{1}{6}M, \tfrac{8}{21}B), \quad (\tfrac{8}{21}L, \tfrac{8}{21}C, \tfrac{5}{21}R) \Big).$$

The unique MSS contains only the pure strategy Nash equilibrium (B, R). Of course, if we define the states in the abstract game corresponding to Game 2 to be the mixed strategy profiles, then the mixed Nash equilibria would be part of the MSS.

Game 2:

		Player 2		
		L	C	R
	T	$1,3$	$3,1$	$0,0$
Player 1	M	$3,1$	$1,3$	$0,0$
	B	$0,0$	$\frac{3}{2},\frac{3}{2}$	$4,4$

We conclude this section providing three examples of games which exhibits the weak improvement property.

Example 8. The traveler's dilemma is a symmetric two player game in which the set of strategy for each player is the set of natural numbers between 2 and 100, i.e. $s_i = [2, 100]$ for all $i \in N$. For each player, the utility function is defined as follow:

$$u_i(s_i, s_j) = \begin{cases} s_i + 2, s_i - 2 & s_i < s_j \\ s_i, s_i & s_i = s_j \\ s_j - 2, s_j + 2 & s_i > s_j \end{cases}$$

		Player 2			
		2	...	99	100
	2	$2,2$	...	$4,0$	$4,0$
Player 1	...	...	...	...	...
	99	$0,4$	...	$99,99$	$101,97$
	100	$0,4$	...	$97,101$	$100,100$

Figure 4.3: Traveler's dilemma.

The unique Nash equilibrium of this game is $NE = \{2, 2\}$.

It is possible to check that this equilibrium coincides with the Myopic Stable Set of this game. In fact, for each player, the strategy $\{2, 2\}$ satisfies myopic deterrence of external deviation, i.e. every deviating player obtain a lower utility. Also, it is easy to see that $\{2, 2\}$ is externally stable, i.e. from any other strategy profile there exists a composition of dominance correspondence that terminates in $\{2, 2\}$. Moreover, as $\{2, 2\}$ is a singleton then Minimality is trivially satisfied.

Example 9. Consider the following version of Cournot's model. Two firms, $N = \{1, 2\}$, produce two quantities $q_1, q_2 \geq 0$, of homogeneous goods. We write $Q = q_1 + q_2$ and we define the inverse demand function in the following way:

$$P(Q) = \begin{cases} a - Q & Q < a \\ 0 & Q \geq a \end{cases}$$

For all $i \in \{1, 2\}$ we assume no fixed costs and a constant marginal cost $C_i(q_i) = cq_i$ with $c < a$.

For each firms, the set of strategy is given by the set $S_i = [0, \bar{q}_i]$ for all $i \in N$, where a strategy is a quantity choice, i.e. $q_i \in S_i$.

For each firms, the utility is given by the profit, i.e. $\pi_i(q_i, q_j) = q_i[a - (q_i + q_j) - c]$ for all $i \in N$.

The Nash Equilibrium corresponds to the strategy profile (q_i, q_j) which solve the following program:

$$max_{q_i \in S_i} \pi_i(q_i, q_j^*) = max_{q_i \in S_i} q_i[a - (q_i + q_j^*) - c]$$

that is $NE = \{\frac{a-c}{3}, \frac{a-c}{3}\}$

It is possible to check that the pure strategy Nash equilibrium correspond to the Myopic Stable Set. First notice that every deviation from the Nash equilibrium involve a strategy profile in which the deviating firm obtain a lower profit. To verify external stability assume any strategy profile outside the equilibrium, take for example $s^1 = (\frac{a-c}{2}, 0)$ where firm 1 get the entire market. Firm 2 has the incentive to deviate from s^1 to any s^2 with $q_2 > 0$, that is $s^2 \in f(s^1)$. Consequently, firm 1 will adjust its strategy with respect the new demand available in the market. This involve a deviation to a strategy profile $s^3 \in f(f(s^1))$. Iterating this reasoning there will be a integer n and a composition of dominance correspondence such that $s^n = \{\frac{a-c}{3}, \frac{a-c}{3}\}$ where no firm has the incentive to deviate.

Moreover, as this set is a singleton then condition [3] is trivially satisfied.

Example 10. Next we consider a Bertrand's model. The setting of the Bertand's model differ from the Cournot's model by the fact that the set of strategy consists in the price choices, i.e. $S_i = [0, \bar{p}_i]$ for all $i \in N$.

We assume the following demand function:

$$D(p_i, p_j) = \begin{cases} \frac{(a-p_i)}{b} & p_i < p_j \\ \frac{(a-p_i)}{2b} & p_i = p_j \\ 0 & p_i < p_j \end{cases}$$

For each firms, the utility is given by the profit, i.e.

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c)q_i & p_i < p_j \\ \frac{(p_i - c)q_i}{2} & p_i = p_j \\ 0 & p_i > p_j \end{cases}$$

.

The Nash Equilibrium corresponds to the strategy profile (q_i, q_j) which solve the following program:

$$max_{p_i \in S_i} \pi_i(p_i, p_j^*) = max_{p_i \in S_i}(p_i - c)q_i$$

that is $NE = \{c, c\}$ where the two firm obtain a zero profit.

It is possible to check that the pure strategy Nash equilibrium correspond to the Myopic Stable Set.

First notice that every deviation from the Nash equilibrium involve a strategy profile in which the deviating firm obtain a lower or equal profit. To verify external stability assume any strategy profile outside the equilibrium, s^1 where a firm, say firm 1, get the entire market. At this point, firm 2 has the incentive to deviate from s^1 to any s^2 with $p_2 < p_1$ to get the entire market. Consequently, firm 1 will adjust its strategy with respect the new strategy profile s^1 to reverse the situation. Iterating this reasoning there will be a composition of dominance correspondence which involve the implementation of strategy profile (c, c) where no firm has the incentive to deviate (below (c, c) the two firm have negative payoff).

Chapter 5

An Economic Application

5.1 Asymmetric Bertrand Competition without legal restriction

In this section we consider a Bertrand model with asymmetric marginal costs, i.e. $c_i \neq c_j$. In particular we consider $c_1 < c_2$.

We assume that the firm with the lowest price sells the amount $q_i \geq 0$ at its posted price $p_i \geq 0$ and incurs a cost per unit of $c_i \geq 0$.

If both firms post the same price, output is split equally between the two firms. Profits are therefore given by

$$u_i(p_i, p_j) = \begin{cases} (p_i - c_i)q \text{ if } p_i < p_j, \\ (p_i - c_i)q/2 \text{ if } p_i = p_j, \\ 0 \text{ if } p_i > p_j. \end{cases}$$

This game is interesting for several reasons.

If the set of strategies S is discrete then there are two pure strategies Nash equilibria $(c_2, c_2+\epsilon)$ and $(c_2 - \epsilon, c_2)$ where ϵ is assumed to be the smallest monetary unit. In this case firm 1 get the entire market of the market and the associate profits are $\pi_1 = (c_2 - c_1)(\frac{a-c_2}{b})$ and $\pi_2 = 0$ in the first case and $\pi_1 = (c_2 - \epsilon - c_1)(\frac{a-c_2-\epsilon}{b})$ and $\pi_2 = 0$. The Myopic Stable Set coincides with the two pure strategies Nash equilibria.

If the set of strategies is continuous then there are not Nash equilibria in pure strategy and the Core of the associated abstract game is empty.

Several papers mistakenly claim that this game has no Nash equilibrium. As Blume (2003) noted, there are mixed Nash equilibria in which player 1 chooses $p_1 = c_2$ and player 2 randomizes his prize p_2 continuously on an interval $[c_2, c_2 + \varepsilon]$. On the other hand, the literature has so far not yet been able to determine all mixed strategies of this game.

We consider the state space $X = \{p \in \mathbb{R}^2_+ \mid p_1 \leq \bar{p},\ p_2 \leq \bar{p}\}$ and the metric $d_i(p, p') = |p - p'|$, $i = 1, 2$.

Observe that the payoff functions are not continuous. Given this, the dominance correspondence f is not lower hemi-continuous, so we cannot use Theorem 2 to establish uniqueness of the Myopic Stable Set.

The construction of the Myopic Stable Set, which crucially relies on the fact that the closeness of the set, proceeds in several steps.

Step 1. $P^1 = \{(p_1, p_2) \in X | c_1 \leq p_1 = p_2 \leq c_2\} \subseteq M$.

Towards a contradiction, suppose that $(p_1, p_2) \in X$ satisfies $c_1 < p_1 = p_2 < c_2$ and (p_1, p_2) is not in M. Once the contradiction is obtained, we get the result of Step 1 exploiting the fact that M is closed.

Take any $p_1' \in \mathbb{R}_+$ such that $c_1 < p_1' < p_1$. There are two cases to consider. In case 1, $(p_1', p_2) \in M$. Given that $(p_1, p_2) \notin M$ and M is closed, there is an $\varepsilon' > 0$ such that for every $\varepsilon \in (0, \varepsilon')$ we have $(p_1 - \varepsilon, p_2) \notin M$. However, for ε small enough, $(p_1 - \varepsilon, p_2) \in f(p_1', p_2)$ as firm 1 will find it profitable to deviate to $p_1 - \varepsilon > p_1'$. Since M satisfies deterrence of external deviations, it follows that $(p_1 - \varepsilon, p_2) \in M$, leading to a contradiction. In case 2, we have $(p_1', p_2) \notin M$. By external stability, there must be $(p_1'', p_2'') \in M$ such that $(p_1'', p_2'') \in f^\infty(p_1', p_2)$. At (p_1', p_2), firm 2 makes no sales and has zero profits. Since $p_1' < c_2$, it has no profitable deviation. For firm 1, any $\tilde{p}_1 \in \mathbb{R}_+$ such that $p_1' < \tilde{p}_1 < p_2$ is a profitable deviation, $\tilde{p}_1 = p_2$ may or may not be a profitable deviation, and $\tilde{p}_1 > p_2$ is not a profitable deviation. It is now easy to see that $f^\infty(p_1', p_2) = \{(\tilde{p}_1, p_2) \in \mathbb{R}^2_+ | p_1' \leq \tilde{p}_1 \leq p_2\}$. External stability therefore implies that there is $(p_1'', p_2) \in M$ with $c_1 < p_1'' < p_1$, but then we are back in case 1, and we obtain a contradiction as before. Consequently, it holds that $(p_1, p_2) \in M$.

Step 2. $P^2 = \{(p_1, p_2) \in X | c_1 \leq p_1 \leq c_2,\ p_1 \leq p_2\} \subseteq M$.

Take $(p_1, p_2) \in P^1$ such that $p_2 < c_2$. It follows from Step 1 that $(p_1, p_2) \in M$. It holds that $u_2(p_1, p_2)$ is strictly negative. As such, firm two can gain by increasing p_2 above the value of p_1 as this will give him a profit of zero. By deterrence of external deviations, all these options must also be in M. The result of Step 2 now follows from the requirement that M is closed.

Step 3. $P^3 = \{(p_1, p_2) \in X | c_1 \leq p_1 \leq p_2\} \subseteq M$.

Take $(p_1, p_2) \in P^2$ such that $c_1 < p_1 < p_2$. By Step 2 it holds that $(p_1, p_2) \in M$. Then firm 1 can deviate and can increase profits by choosing p_1' such that, $p_1 < p_1' < p_2$. Since M satisfies deterrence of external deviations, it holds that $(p_1', p_2) \in M$. This shows that we can drop the restriction $p_1 \leq c_2$ from the definition of the set P^2. Using closedness of M we can again change strict inequalities to weak inequalities.

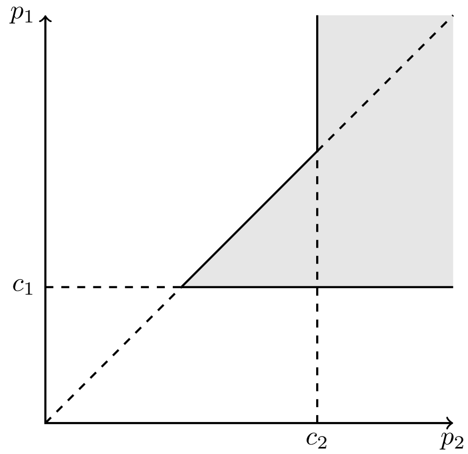
Step 4. $P^4 = P^3 \cup \{(p_1, p_2) \in X | c_2 \leq p_2 \leq p_1\} \subseteq M$.

Take $(p_1, p_2) \in P^3$ such that $c_2 < p_1 < p_2$. By Step 3 it holds that $(p_1, p_2) \in M$. Now firm 2 can deviate and set p_2' such that $c_2 < p_2' < p_1$ and make strictly positive profits. Thus, the set $\{(p_1, p_2) \in X | c_2 \leq p_2 \leq p_1\}$ is a subset of M. The set P^4 is given by the shaded area in the left panel of Figure 5.2.

Step 5. P^4 is the unique MSS.

We have shown that P^4 is contained in any MSS, so we only need to show that P^4 itself is

Figure 5.1: The MSS for the asymmetric Bertrand model.



a QMSS. First, observe that P^4 is closed. Next, $X \setminus P^4$ is given by

$$\underbrace{\{(p_1, p_2) \in X | p_1 < c_1\}}_{P^5} \cup \underbrace{\{(p_1, p_2) \in X | p_1 > p_2, p_2 < c_2\}}_{P^6}.$$

In order to see that P^4 satisfies deterrence of external deviations, observe that firm 1 will never deviate to a point in the set P^5 as this gives zero or negative profits for firm 1 and profits at states in P^4 are non-negative for firm 1. Firm 2 has no possibility to deviate to P^5 from a point in the set P^4. Also, any point in the set P^6 gives firm 2 negative profits. Firm 2 only obtains negative profits at states in P^4 if $p_1 = p_2 < c_2$. However, if firm 2 deviates to $p_2' < p_1$, then his profits would become more negative so firm 2 will never deviate to states in P^6.

It remains to show that P^4 satisfies external stability. If $p_2 \leq p_1 < c_1$, then firm 2 can gain by choosing p_2' such that $p_2' > c_2$. Next firm 1 can gain by choosing p_1' such that $c_2 < p_1' < p_2$. The strategy profile (p_1', p_2') belongs to P^4. If $p_1 < p_2 \leq c_1$, then firm 1 can gain by choosing p_1' such that $p_1' > c_2$. Then firm 2 can gain by choosing p_2' such that $c_2 < p_2' < p_1'$. The strategy profile (p_1', p_2') is in P^4. If $p_1 < c_1 < p_2$, firm 1 can gain by choosing p_1' such that $c_1 < p_1' < p_2$ which leads to the strategy profile (p_1', p_2) in P^4. Hence, external stability holds starting from any state in P^5. For $(p_1, p_2) \in P^6 \setminus P^5$ it holds that $p_1 > p_2$, $c_1 \leq p_1$, and $p_2 < c_2$, so firm 2 can gain by choosing $p_2' = c_2$. The strategy profile (p_1, c_2) belongs to P^4.

5.2 Asymmetric Bertrand Competition under legal restrictions

Let us now consider a slightly different version of Bertrand competition. In many countries, pricing below marginal or average cost is considered to be predatory pricing and thus forbidden by law. We analyze how this restriction influences the MSS. To do so, we adjust the state space and define

$$X = \{(p_1, p_2) \in P | c_1 \leq p_1,\ c_2 \leq p_2\}.$$

The MSS is considerably smaller than in the previous setting. In particular, we will show that it is equal to the set

$$P^* = \left\{ (p_1, c_2) \in P \middle| \frac{c_1 + c_2}{2} \leq p_1 \leq c_2 \right\},$$

which is illustrated in the right panel of Figure 5.2.

Recall that the mixed Nash equilibrium derived by Blume (2003) had $p_1 = c_2$ and p_2 being drawn from an atomless distribution on an interval $[c_2, c_2 + \varepsilon]$. The prediction of the MSS is that prices will be set lower than in the mixed Nash equilibrium.

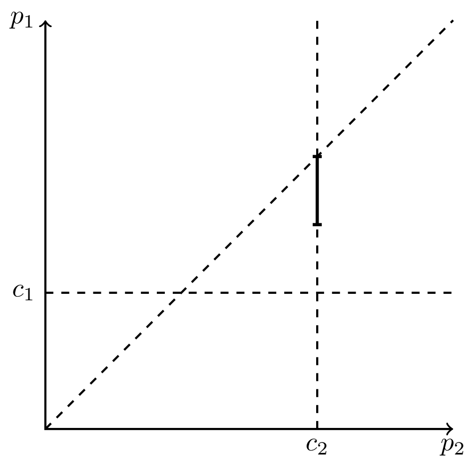
Again, we split the proof into several steps.

Step 1. First we show that P^* is a QMSS. We first establish deterrence of external deviations. For $(p_1, c_2) \in P^*$, profits of firm 1 are non-negative. Thus, setting $p_1 > c_2$ with a payoff of zero is not a profitable deviation for firm 1 from any point in P^*. Note that the payoff of player 1 is increasing in p_1 for $p_1 < c_2$. Thus, a deviation to a $p_1 < (c_1 + c_2)/2$ could only be profitable from the strategy profile (c_2, c_2). This requires $(p_1 - c_1)q > (c_2 - c_1)q/2$ or, equivalently, $p_1 > (c_1 + c_2)/2$, which is not the case. It is easily verified that firm 2 cannot increase its profits by deviating from any $(p_1, c_2) \in P^*$. This shows deterrence of external deviations for P^*.

It remains to verify external stability. Let some state $(p_1, p_2) \in X \setminus P^*$ be given. If $c_2 < p_1 < p_2$, then firm 2 can profitably deviate to $p_2' = (c_2 + p_1)/2$ and firm 1 can profitably deviate in the next step to $p_1' = (c_2 + p_2')/2$ and so forth. It follows that $(c_2, c_2) \in f^\infty(p_1, p_2)$. If $p_1 \leq c_2 < p_2$, then firm 1 can profitably deviate to p_1' such that $c_2 < p_1' < p_2$ and we can continue as in the previous case. If $c_2 < p_2 \leq p_1$, then firm 1 can profitably deviate to p_1' such that $c_2 < p_1' < p_2$ and we can continue as before. If $p_1 \notin [(c_1 + c_2)/2, c_2]$ and $p_2 = c_2$, then firm 1 can profitably deviate to $p_1' = (c_1 + c_2)/2$ to reach a state in P^*. We have covered all states in $X \setminus P^*$ and thereby shown that P^* satisfies external stability.

Step 2. Let M be a QMSS. Let us show that for every $(p_1, c_2) \in P^* \setminus \{(c_2, c_2)\}$, if $(p_1, c_2) \in M$, then $(c_2, c_2) \in M$. Suppose $(c_2, c_2) \notin M$. By closedness of M, there is $\bar{\varepsilon} > 0$ such that, for every $\varepsilon \in (0, \bar{\varepsilon})$, $(c_2 - \varepsilon, c_2) \notin M$. Take $p_1' = \max\{(p_1 + c_2)/2, c_2 - \bar{\varepsilon}/2\}$, then $(p_1', c_2) \in f(p_1, c_2)$, so $(p_1', c_2) \in M$. Given that $p_1' > c_2 - \bar{\varepsilon}$, we obtain a contradiction.

Figure 5.2: The MSS for the asymmetric Bertrand model with legal restriction.



Step 3. Let M be a QMSS. Let us show that if $(c_2, c_2) \in M$, then, for every $(p_1, c_2) \in P^* \setminus \{(c_2, c_2)\}$, we have $(p_1, c_2) \in M$. This follows from the fact that any strategy profile in $(p_1, c_2) \in P^* \setminus \{(c_2, c_2)\}$ with $p_1 > (c_1 + c_2)/2$ offers higher profits for firm 1 compared to (c_2, c_2) and the fact that M is closed.

Step 4. We are now ready to show that P^* is an MSS. First of all, by step 1 it is a QMSS. So if, towards a contradiction, P^* is not an MSS, it should violate minimality. This means that there is a proper subset of P^* that is also a QMSS. This subset either contains (c_2, c_2) or it is a subset of $P^* \setminus \{(c_2, c_2)\}$. If contains (c_2, c_2) then, by Step 3, it should contain $P^* \setminus \{(c_2, c_2)\}$ and therefore be equal to P^*. If it is a subset of $P^* \setminus \{(c_2, c_2)\}$, then by Step 2, it should contain (c_2, c_2), a contradiction.

Step 5. Finally, let us show that the set P^* is the unique MSS. Let M be an MSS. By Lemma 3, it holds that $P^* \cap M \neq \emptyset$. If M contains (c_2, c_2), then, by Step 3, M should also contain $P^* \setminus \{(c_2, c_2)\}$, so $P^* \subseteq M$ and by minimality $P^* = M$. If M contains an element of $P^* \setminus \{(c_2, c_2)\}$, then, by Step 2, it should also contain (c_2, c_2) and, by Step 3, also $P^* \setminus \{(c_2, c_2)\}$. Again, we obtain $P^* \subseteq M$ and by minimality $P^* = M$.

By characterizing the MSS for the asymmetric Bertrand model, we have shown that it is possible to find the MSS in nontrivial non-cooperative games. The difference between the MSS in the two versions of the Bertrand model given above is substantial which emphasizes the great importance of details in this model, i.e., the choice of strategy sets. The underlying origin of this sensitivity is due to the discontinuity in payoff functions. Note that in both cases no pure strategy Nash equilibrium exists, any mixed-strategy Nash equilibrium involves randomizations over a continuous interval, and the literature contains no full characterization of the set of Nash equilibria. The fact that it is not overly complicated to characterize the MSS in such a complex environment further boosts the appeal of the MSS as an equilibrium concept.

Chapter 6

Conclusion

We generalize the concept of Pairwise Myopic Stability by Herings et al. (2009) from finite networks to a general class of abstract games which allows for an infinite state space. The framework of abstract games is general enough to accommodate different models of coalition formation such as TU games with coalition structure, networks and matching models, but also includes non-cooperative games.

The solution concept consists of three intuitive conditions which can be summarized as follows: (i) no coalition has a profitable deviation from a state in the set to a state outside the set, (ii) for any state outside the set, there exists a sequence of improving deviations which converges to the set and (iii) the set is the minimal set satisfying (i) and (ii). Under minimal assumptions (compact and nonempty state space, complete and transitive preferences), the myopic stable set exists and it is nonempty. Moreover, under additional weak continuity assumptions, it is also unique.

We have compared our solution concept to other concepts in several examples. The Myopic Stable Set contains the Core and the set of pure strategy Nash equilibria. It coincides with the Coalition Structure Core in TU games with coalition structure (Kóczy and Lauwers, 2004) if the Coalition Structure Core exists, the set of Stable Matchings in the standard one-to-one matching model (Gale and Shapley, 1962), the set of pairwise stable networks and closed cycles of networks (Jackson and Watts, 2002) and the set of pure strategy Nash equilibria in finite supermodular games (Bulow et al., 1985) and finite potential games (Monderer and Shapley, 1996) and aggregative games (Selten, 1970).

The Myopic Stable Set for abstract games provides an umbrella for well-known solution concepts in many examples. Moreover, it generates novel predictions for many other examples, for instance in a Bertrand duopoly with asymmetric cost functions. Finally, it allows for predictions when other solution concepts fail to exist.

In this setting we assumed that individuals only take care to the immediate consequences of their actions. This behavioral assumption is expressed by the dominance correspondence which characterize our solution concept as ”myopic”.

From this point of view, a natural development is to extend the concept of dominance correspondence including farsightedness (see Chwe, 1994; Xue, 1998; Herings et al., 2009; Ray and Vohra, 2015).

The farsighted dominance criterion captures the idea that individuals take into account the ultimate consequences of their actions.

In particular, we say that a state $y \in X$ farsightedly dominates $x \in X$ under E, if there exist a sequence of states $(x^0, ..., x^n)$ with $x^0 = x$ and $x^n = y$, and coalitions $(S^0, ..., S^{n-1})$ such that:

1. $S^{k-1} \in E(x^{k-1}, x^k), \qquad \forall k \in \{1, ..., n\}$,
2. $u_{S^k}(y) \gg u_{S^k}(x^k), \qquad \forall k \in \{0, ..., n-1\}$.

This kind of extension gives the opportunity to explore many specific settings in which a myopic analysis provides unfeasible results. A typical example is given by coalitional games with positive spillovers (see Bloch 1996, Yi 1997) where the actions of the individuals modify the payoff structure of the entire coalition structure. Observe that this strategic scenario can be represented by an abstract game first, using a partition function (Thrall and Lucas 1963) for modeling the state space; second, considering a proper structure on the effectivity correspondence (see Ray and Vohra 2014).

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