Orientations, break Divisors and compactified Jacobians

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Für Anne und Thomas,
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CHAPTER 0

Overview

Our main object of study will be the combinatorics of $\overline{P}^g_X$ and $\overline{P}^g_g$ as constructed by Caporaso in [**14**]. They are in an appropriate sense compactifications of the degree g Picard variety of a stable curve and the universal Jacobian, respectively. There are other compactifications than $\overline{P}^g_X$, but the one of [**14**] plays a somewhat special role as it always is of Néron type as shown in [**17**]. This in particular implies that it is a compactification of the Néron model of the Jacobian for every regular one parameter smoothing of X. The compactification $\overline{P}^g_g$ of the universal degree g Jacobian is the only one we are aware of.

These compactifications are constructed by admitting not only line bundles of certain fixed multidegrees (what we will call balanced ones) on a Deligne-Mumford stable curve X, but also certain line bundles on partial normalizations of X. This leads to a stratification of $\overline{P}^d_X$ given by the set of nodes that are normalized and the multidegree on this partial normalization. After some preliminaries in Chapters 1 and 2, we will study the combinatorics of these stratifications in the degree g case in Chapter 3. To this end, we first give two equivalent combinatorial descriptions of balanced degree g multidegrees:

(1) As given by classes of rooted 1-orientations
(2) As break divisors.

The connection of these descriptions among themselves will follow from [**4**]. As a consequence we get that the compactifications constructed by Simpson in [**29**] coincide in degree g, independent of the chosen polarization. By passing to the residual, this implies that also the compactifications in degree $g-2$ coincide. Together with the case of degree $g-1$ (see e.g. [**2**]) this means that all Simpson compactifications give an up to isomorphism unique compactification in degrees $g-2$, $g-1$ and g. While completing this document, we learned that Jifeng Shen independently has reached similar conclusions in his PhD thesis.

Next we use the description given by the orientations to construct an indexing poset for the stratification of $\overline{P}^g_X$, that is a poset that encodes the containment relations among strata. This will be the set of all rooted 1-orientations on connected spanning subgraphs of G_X, the dual graph of X. In particular we describe it in independent combinatorial terms, i.e. without recurring to the containment relations themselves. The phenomenon of such stratifications of moduli spaces seems to be widespread. In particular the moduli space of stable curves, $\overline{M}_g$, admits a well-known stratification by dual graphs, the partial order being given by edge contractions. We will proceed to describe a similar indexing set also for the universal compactified Picard variety in degree g, $\overline{P}^g_g$, that is compatible with the indexing set of the stratification of $\overline{M}_g$.

The starting point for the constructions of Chapter 3 is twofold: First, it was known that a multidegree is balanced in degree $g-1$ if and only if it is the divisor associated to an orientation. In algebraic geometry this observation was made in [**10**, Lemma 2.1]. Using the 'Basic Inequality' of [**14**] it is also a consequence of a result in graph theory, known as Hakimi's Theorem (originally in [**24**], for a formulation in our framework see [**4**, Theorem 4.8]). Furthermore a multidegree of degree $g-1$ is strictly balanced if it is given by a totally cyclic orientation (cf. [**2**] Prop. 3.6). Thus it is natural to ask, whether balanced multidegrees of different total degree may also be described in terms of orientations.

The second departing point is [**14**, Proposition 5.1], which gives a description of when one stratum is contained in the closure of another stratum in $\overline{P}^d_X$. Combinatorially, this characterization amounts to edge removals on the dual graph of the curve and decreasing the multidegree on one of the vertices adjacent to the removed edges. In some sense this is the 'local' combinatorial ingredient and the constructions in Chapter 3 can be seen as globalizing this proposition in the degree g case. In particular this makes clear how orientations fit well into the description of the containment relations among strata: the vertex on which we decrease the multidegree will be determined by the orientation of the edge removed.

In Chapter 4 we take the perspective of tropical geometry. By the work done in [**27**] and [**4**] each divisor on a metric graph Γ has a unique break divisor linearly equivalent to it. The initial observation for us is that by results in [**4**] this leads to a polyhedral decomposition of $\mathrm{Pic}^g(\Gamma)$, which we will call ABKS-decomposition. We will reformulate the partially ordered sets constructed in Chapter 3 using orientations in terms of break divisors. We will then view the face decomposition of the ABKS-decomposition as a stratification of $\mathrm{Pic}^g(\Gamma)$. The picture obtained in this way then will exhibit the strata inversal phenomena encountered in passing from the algebro-geometric to the tropical side, already known for the case of algebraic and tropical curves (see e.g. [**18**]). The transition between $\overline{P}^g_X$ and $\mathrm{Pic}^g(\Gamma)$ is not as symmetric as in the curve case, as the compactified Jacobians decompose into a compact ('geometric') and non-compact ('combinatorial') part and the combinatorics of $\mathrm{Pic}^g(\Gamma)$ naturally only recovers information about the non-compact part. For example, the dimension of $\overline{P}^g_X$ is g, whereas $\mathrm{Pic}^g(\Gamma)$ has dimension $b_1(G)$, the first Betti number of the underlying graph.

Next we invert the direction: instead of studying containment relations among strata of an already existing space, we construct a tropical analogue of the universal compactified Jacobian by using the poset structure as gluing information. This involves again giving a reinterpretation of the involved posets in terms of break divisors and results in a space, $(P^g_g)^{trop}$, parametrizing break divisors on stable metric graphs of fixed genus greater than two up to automorphisms.

We can summarize the results of Chapters 3 and 4 in the following commutative diagram, whose entries will be defined there:

$$\begin{tikzcd}
\overline{P}^g_g \arrow[rrr, two heads] \arrow[dddd, two heads] & & & \overline{\mathcal{P}\Sigma}_g \arrow[r, leftrightarrow] \arrow[dddd, two heads] & \overline{\mathcal{P}\Sigma}^*_g \arrow[dddd, two heads] & & & (P^g_g)^{trop} \arrow[lll, two heads] \arrow[dddd, two heads] \\
& \frac{\overline{P}^g_X}{\mathrm{Aut}(X)} \arrow[ul, hook] \arrow[r, two heads] \arrow[d] & \overline{\mathcal{P}\Sigma(G_X)} \arrow[ur, hook] \arrow[rrr, leftrightarrow] \arrow[d] & & & \overline{\mathcal{P}\Sigma(G_X)}^* \arrow[ul, hook] \arrow[d] & \frac{\mathrm{Pic}^g(\Gamma)}{\mathrm{Aut}(\Gamma)} \arrow[ur, hook] \arrow[l, two heads] \arrow[d] & \\
& {[X]} \arrow[r, mapsto] \arrow[ddl, mapsto] & {[G_X]} \arrow[rrr, leftrightarrow] \arrow[ddr, mapsto] & & & {[G_X]} \arrow[ddl, mapsto] & \Gamma \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& & & & & & & \\
\overline{M}_g \arrow[rrr, two heads] & & & \mathcal{SG}_g \arrow[r, leftrightarrow] & \mathcal{SG}^*_g & & & (M_g)^{trop} \arrow[lll, two heads]
\end{tikzcd}$$

In Chapter 5 we will illustrate some consequences of this for the geometry of $\overline{P}^g_X$. First we will show, that it is connected through codimension one. Connectedness in codimension one is a topological property of stratifications. While it is trivially satisfied for $\overline{M}_g$ and $\overline{P}^g_g$ as both have a unique maximal dimensional stratum, for $\overline{P}^d_X$ with X reducible it is not immediate. Then we will calculate the number of strata of fixed codimension in terms of the number of maximal dimensional strata. Denoting by $c(i)$ the number of codimension i strata we obtain

$$c(i) = c(0)\binom{b_1(G_X)}{i}.$$

This in particular gives the symmetry $c(i) = c(b_1(G_X) - i)$ and a description of the Euler characteristic of the boundary complex in terms of the number of spanning trees of G_X. We will conclude Chapter 5 by constructing a surjective map $\overline{P}^g_X \to \overline{P}^{g-1}_X$ that contracts some of the maximal dimensional strata of $\overline{P}^g_X$. The existence of such a map is less obvious than it may seem: while there certainly are always maps $\mathrm{Pic}^g(X) \to \mathrm{Pic}^{g-1}(X)$, which are as in the smooth case given by subtracting a fixed point, the construction of $\overline{P}^d_X$ is based on a choice of connected components of $\mathrm{Pic}^d(X)$. Now subtracting a point may or may not preserve this choice. We do not know whether in general there are maps $\overline{P}^d_X \to \overline{P}^{d-1}_X$ and they certainly will not all be surjective.

Finally, Chapter 6 changes the focus from $\overline{P}^g_X$ to fixed balanced line bundles of degree g and we will discuss some implications of the combinatorics for the question when such a line bundle satisfies the Clifford inequality. Even in the restrained setting of balanced degree g line bundles considered, this is a surprisingly complicated question. We feel that at its heart it is combinatorial and connected to gluing over isolated singular base points. We first show that as a consequence of the results in [**10**] and [**15**], the general balanced degree g line bundle in each stratum satisfies the Clifford inequality. If the stratum is isomorphic to a stratum of $\overline{P}^{g-1}_X$ under the map $\overline{P}^g_X \to \overline{P}^{g-1}_X$ constructed in Chapter 5, the locus of line bundles not satisfying the Clifford inequality will be contained in the pullback of the Theta divisor of the corresponding stratum of $\overline{P}^{g-1}_X$. In the last section we use the description by orientations to obtain a sufficient condition under which a balanced degree g line bundle satisfies the Clifford inequality in terms of isolated singular base points.

CHAPTER 1

Combinatorial Preliminaries

This chapter reviews the framework for the combinatorial side of the picture. We begin by fixing some notation about graphs, the most basic objects on the combinatorial side. We will then introduce additional structure on a graph: First a weight, or genus, of the vertices, second a length on the edges. These two we will view as part of the data giving the graph. Finally, on a fixed graph we will consider divisors and orientations, recalling some results on how they are related to each other.

1.1. Graphs and divisors on them

1.1.1. Finite graphs. We will take a *graph* G to be a finite multigraph, i.e. allowing loops and multiple edges if not otherwise specified. That is, a graph G consists of a set of vertices, $V(G)$, and a set of edges, $E(G)$, with every edge adjacent to two not necessarily different vertices. For an edge $e \in E(G)$ we will write $e = v_1v_2$ if $v_1, v_2 \in V(G)$ are the two vertices e is adjacent to. We will call v_1 and v_2 the ends of e. Sometimes we will view an edge e as consisting of two half-edges, each adjacent to a unique vertex.

If not stated otherwise, we will assume the graphs we consider to be connected.

A *subgraph* H of G is the datum of subsets $V(H) \subset V(G)$ and $E(H) \subset E(G)$ that themselves form a graph, i.e. for every $e = v_1v_2 \in E(H)$ also $v_1, v_2 \in V(H)$. It is called an *induced subgraph* if for every $e = v_1v_2 \in E(G)$ with $v_1, v_2 \in V(H)$, also $e \in E(H)$. A subgraph is called *spanning* if $V(H) = V(G)$. Thus the only spanning induced subgraph is G itself.

For a subset $Z \subset V(G)$ of vertices we will denote by $[Z]$ the induced subgraph defined by Z. For a subgraph H we also will write $[H] = [V(H)]$ for the corresponding induced subgraph. For a subset $S \subset E(G)$ of edges we will denote by $G - S$ the spanning subgraph with edges $E(G) \setminus S$. If G is connected then $S \subset E(G)$ will be called disconnecting if $G - S$ is disconnected.

A *cut* $S \subset E(G)$ is a set of edges such that $G - S$ has more connected components than $G - S'$ for any proper subset $S' \subset S$. Alternatively, a cut may be described as follows: For a subgraph H of G, denote by H^c the *complement* of H, i.e. $H^c = [V(G) \setminus V(H)]$. Denote by (H, H^c) the set of edges with one end in H and one in H^c, i.e. $(H, H^c) = E(G) \setminus (E([H]) \cup E(H^c))$. Then (H, H^c) is a cut (the cut defined by H) and every cut can be given in this way. For $Z \subset V(G)$ we will abbreviate $([Z], [Z]^c)$ as (Z, Z^c), the induced cut by Z.

For a subgraph H of G we will set $G-H$ to be the graph with vertices $V(G)\setminus V(H)$ and edges $E(G)\setminus(E([H])\cup([H],[H]^c))$. For a subset $V'\subset V(G)$ we set $G-V'=G-[V']$.

For a vertex v we set $\delta(v)$ to be the number of edges adjacent to v, where loops based at v are counted twice ($\delta(v)$ is sometimes also denoted as $\mathrm{val}(v)$ or $\deg(v)$). Thus if we view an edge as consisting of two half-edges, $\delta(v)$ counts the number of adjacent half-edges. It is called the valency of v. If we index the vertices as $v_i\in V(G)$ we usually abbreviate $\delta(v_i)$ as δ_i. For an induced subgraph H we set $\delta_H=|(H,H^c)|$.

A *path* in G is a connected subgraph P with vertices $v_0,\dots,v_k$ such that $v_i\neq v_j$ and v_i has valency one in P for $i=0,k$ and two otherwise. We will assume that the indexing of the v_i is such that v_i and v_{i+1} are connected by a unique edge in P and will write v_iv_{i+1} for this edge.

A map or *morphism of graphs* $\phi:G\to G'$ consists of two maps $V(G)\to V(G')$ and $E(G)\to E(G')\cup V(G')$ such that if $e=v_1v_2$ is mapped to $e'=v_1'v_2'$ also v_i is mapped to v_i'. An *automorphism of a graph* G is a bijection between the half-edges that induces a morphism $G\to G$.

Remark 1.1.1. The two maps $V(G)\to V(G)$ and $E(G)\to E(G)$ associated to an automorphism of G do not determine the automorphism if G has loops. Consider for example the graph G with one vertex and one loop based at that vertex. Then G has a non-trivial automorphism that interchanges the to half-edges of the loop, which however induces the identity on $V(G)$ and $E(G)$.

For a subset of edges $S\subset E(G)$ we will often consider the map $\gamma:G\to G'$ contracting the edges in S. Let $V_S\subset V(G)$ be the set of vertices that have at least one edge of S adjacent to them. Then G' has vertices $V(G)\setminus V_S$ plus one vertex for each connected component of $[V_S]$. It has edges $E(G)\setminus S$ where an edge $e=v_1v_2$ will be adjacent to v_i in G' if v_i is not in V_S and otherwise to the vertex corresponding to the connected component of $[V_S]$ containing v_i. Then γ maps a vertex v_i to v_i if v_i is not in V_S and otherwise to the vertex corresponding to the connected component of $[V_S]$ containing v_i. It maps en edge e to e if e is not in S and otherwise to the vertex v_i corresponding to the connected component of $[V_S]$ containing e. By construction this is a morphism of graphs. For the graph G' obtained by contracting S in G we will write G/S.

1.1.2. Connectivity. For graphs one usually considers two types of connectivity: *vertex- and edge connectedness.* We will only need the second notion here:

Definition 1.1.2. A graph G is called k-edge connected if for any $S\subset E(G)$ with $|S|=k-1$ the subgraph $G-S$ is connected.

In particular, 1-edge connected graphs are just connected graphs.

A *bridge* of G is an edge e such that $G-e$ has more connected components than G. Thus a graph is 2-edge connected if it is connected and without bridges. We will denote by $G_{br}\subset E(G)$ the set of bridges of G.

Every graph admits by definition a decomposition $G - G_{br} = G_1 \sqcup \cdots \sqcup G_k$ where the G_i are 2-edge connected subgraphs, which we will call the *2-edge connected components of* G.

1.1.3. Weighted graphs. Usually we will assume graphs to be *weighted* (or to be more precise, vertex weighted). This means that the graph is endowed with a weight or genus function $\underline{g} : V(G) \to \mathbb{N}$ from the set of vertices of the graph to the natural numbers with zero. We sometimes will write $g(v)$ for $\underline{g}(v)$.

Two weighted graphs $(G_1, \underline{g}_1)$ and $(G_2, \underline{g}_2)$ will be called *isomorphic* if there is a bijective morphism of graphs $f : G_1 \to G_2$ such that $\underline{g}_1(v) = \underline{g}_2(f(v))$ for every $v \in V(G_1)$. An *automorphism* of $(G, \underline{g})$ is an automorphism of G that induces an isomorphism of weighted graphs $G \to G$.

Anticipating that graphs here will be dual graphs of algebraic curves, this notion is to encode the geometric genus of the irreducible components of the curve. As such, it is somewhat foreign to the combinatorial theory. In fact, to recover a combinatorial behaviour similar to the algebro geometric one, in applications one often has to replace the genus by inserting an appropriate number of loops (see e.g. for the rank [**3**]). The main advantage of this notion is that with it the genus of a graph is preserved under edge contractions.

For a vertex v_i we usually abbreviate $\underline{g}(v_i)$ as g_i. From now on, if not stated otherwise, we will assume all graphs to be weighted (possibly with the trivial weight $\underline{0}$, which has weight zero on each vertex).

The *first Betti number* of a graph will be denoted by $b_1(G)$ or just b_1 if the graph is clear from context, i.e. $b_1(G) = |E(G)| - |V(G)| + 1$. By $g(G)$ or just g we will denote the *genus* of the weighted graph G, that is

$$g = b_1(G) + \Sigma_{v_i \in V(G)} g_i = |E(G)| - |V(G)| + 1 + \Sigma_{v_i \in V(G)} g_i.$$

Thus b_1 is the genus of G with trivial weights. For a subgraph H we will denote by g_H the genus of H.

From now on, if not stated otherwise, we will assume $g(G) \geq 2$.

REMARK 1.1.3. If G is not connected, its genus may be negative. In some contexts it is preferred to define the genus as the number we defined plus the number of connected components of G to always get a non-negative number. As we will only be concerned with connected graphs, this distinction will make no difference.

The following is an easy observation:

LEMMA 1.1.4. *Let H be an induced subgraph of G. Then $g(G) = g_H + g_{H^c} + \delta_H - 1$*

PROOF. If the weights of G are trivial, we have $g_H + g_{H^c} + \delta_H - 1 = |E(H)| - |V(H)| + 1 + |E(H^c)| - |V(H^c)| + 1 + \delta_H - 1 = |E(G)| - |V(G)| + 1 = g(G)$, since $V(G) = V(H) \cup V(H^c)$ and $E(G) = E(H) \cup E(H^c) \cup (H, H^c)$ because H is an induced subgraph. If the weights are not trivial, it is immediate to see that the claim still holds.

□

Definition 1.1.5. A graph is stable, if it is connected and every vertex of weight zero has valency at least three. We will denote by $\mathcal{SG}_g$ the set of all stable graphs of genus g.

This definition is originally motivated by the theory of algebraic curves as we will see later. Recall that we assume $g \geq 2$. In particular we do not want to consider the single vertex of weight one to be stable.

A *tree* T is a connected graph with $b_1(T) = 0$. A *spanning tree* T of G is a spanning subgraph that is a tree. A *cycle* is a connected graph with $b_1(T) = 1$ and no disconnecting edges. A cycle in G is a subgraph of G that is a cycle.

1.1.4. Metric graphs. The next additional structure we want to consider is assigning real-valued lengths to the edges of the graph. In particular when taking this viewpoint, we will no longer be in a discrete setting but can continuously vary the edge lengths.

Definition 1.1.6. A metrized graph is a graph together with a length function $l : E(G) \to \mathbb{R}_+$.

Definition 1.1.7. A metric graph is a compact connected metric space, that locally around each point p is isometric to a star shaped set.

Recall that a star shaped set of radius r is a set of the form

$$S(r,n) = \{z \in \mathbb{C} | z = te^{(k2\pi i)/n}, 0 \leq t \leq r, k \in \mathbb{Z}\}$$

with $r \in \mathbb{R}$ and $1 \leq n \in \mathbb{N}$. The integer n is the *valency* (as it denotes the number of line segments emanating from the center). In particular, every point of a metric graph has a valency.

From a metrized graph one can construct a metric graph by associating to an edge e the line segment $[0, l(e)]$ and then glue two line segments if they are adjacent to the same vertex at the corresponding point. Then by construction the valency of the metric graph coincides with the valency of the vertex in the metrized graph. We will usually denote by Γ the metric graph associated to the metrized graph G and think of it as a geometric realization of G. In the other direction G is usually called a *model* for Γ.

Notice that the model of Γ is not unique. Namely let G' be obtained from G by replacing an edge e with a new vertex and two edges e_1 and e_2 adjacent to that vertex. Choose a length function l' on G' such that it agrees with l on edges different from e and $l(e) = l'(e_1) + l'(e_2)$. Then G' will also be a model for Γ. A graph G' obtained by repeating this operation is usually called a *refinement of* G. The point is, that the metric graph does not keep track of a specified set of vertices. Thus while every model of Γ needs to contain points of valency different from two as vertices, vertices of valency two are not distinguishable from points in the interior of edges. From this viewpoint metric graphs are limits of metrized graphs under refinement (see e.g. [**8**] for details).

REMARK 1.1.8. In the literature there are at least two differing notations concerning these topics: Some authors call what we called a metrized graph already a metric graph (e.g. [**13**], [**1**]). Using the other convention our metrized graphs are called weighted graphs, meaning edge-weighted graphs (e.g. [**8**], [**4**]). Usually the first is preferred if one is interested in (vertex-)weighted graphs and the second if one wants to consider interior points of edges as divisors on the graph. As we will deal with both these aspects, we chose the notation above.

1.1.5. Divisors on graphs. For a graph G we will denote by $\mathrm{Div}(G)$ the free $\mathbb{Z}$-module generated by elements of $V(G)$. It is called the *group of divisors* and its elements are *divisors* on G. That is, a divisor is a formal sum $\underline{d} = \Sigma_{v_i \in V(G)} \underline{d}_i v_i$ with $\underline{d}_i \in \mathbb{Z}$. If we do not index the vertices, we will denote by $\underline{d}_v$ the coefficient of $\underline{d}$ at the vertex v. Motivated by the geometric applications we sometimes will call divisors *multidegrees* and freely switch between the two names.

For a metric graph Γ we will set $\mathrm{Div}(\Gamma)$ analogously to be the free $\mathbb{Z}$-module generated by points of Γ. Thus a divisor on Γ is a (finite) formal sum $\underline{d} = \Sigma_{v_i \in \Gamma} \underline{d}_i v_i$ with $\underline{d}_i \in \mathbb{Z}$. The *support* $\mathrm{supp}(\underline{d})$ of a divisor $\underline{d}$ on Γ is the set $\{p \in \Gamma |\, \underline{d}_p \neq 0\}$.

The *degree of a divisor* is defined as $|\underline{d}| = \Sigma_{v_i \in \mathrm{supp}(\underline{d})} \underline{d}_i \in \mathbb{Z}$. We set $\mathrm{Div}^k(G) \subset \mathrm{Div}(G)$ and $\mathrm{Div}^k(\Gamma) \subset \mathrm{Div}(\Gamma)$ to be the set of divisors of degree k. Notice that $\mathrm{Div}^0(G)$ and $\mathrm{Div}^0(\Gamma)$ are groups under addition, whereas $\mathrm{Div}^k(G)$ and $\mathrm{Div}^k(\Gamma)$ are $\mathrm{Div}^0(G)$ and $\mathrm{Div}^0(\Gamma)$ torsors, respectively. Furthermore, $\mathrm{Div}^k(G)$ and $\mathrm{Div}^j(G)$ are isomorphic as $\mathrm{Div}^0(G)$-torsors and the same for Γ.

If G is a model for Γ, there is a natural inclusion $\mathrm{Div}^k(G) \to \mathrm{Div}^k(\Gamma)$ by viewing vertices of G as points of Γ.

A divisor $\underline{d}$ is called *effective* if $d_i \geq 0$ for all i. We will denote by $\mathrm{Div}^k_+(G) \subset \mathrm{Div}^k(G)$ and $\mathrm{Div}^k_+(\Gamma) \subset \mathrm{Div}^k(\Gamma)$ the subsets of effective divisors.

We say a divisor $\underline{e}$ *dominates* $\underline{d}$, in short $\underline{e} \geq \underline{d}$, if $\underline{e} - \underline{d}$ is effective.

EXAMPLE 1.1.9. The genus function $\underline{g}$ may be viewed as an effective divisor on G. For a metrized graph G we may consider $\underline{g}$ also as a divisor on Γ. Notice that for the genus of G we may write $g(G) = b_1(G) + |\underline{g}|$. This dichotomy of parts of the genus coming from the underlying unweighted graph and from the weights on it plays an important conceptual role in the geometric setting we will be interested in later on. The former corresponds in some sense to combinatorial and the latter to geometric data.

Assume that $g \geq 2$, G is connected and G contains no vertices of valency one and weight zero. Let G be a model of Γ for some length function on G. Then the pair $(\Gamma, \underline{g})$ uniquely determines a stable model of Γ, that is a stable metrized graph G' with weights $\underline{g}$ that is a model of Γ. Indeed, the vertex set of G' will consist of all points of Γ that either have valency at least three or on which $\underline{g}$ is non-zero. One checks that since $g \geq 2$ this set is not empty and since G contains no vertices of valency one and weight zero, it will indeed give a graph. By construction and the assumptions on G, G' is stable. Any metrized graph that is a model of Γ and has weights $\underline{g}$ needs to at least contain these vertices. Any additional vertex would be of weight zero and valency two and thus the resulting model would not be stable.

We will view $(\Gamma, \underline{g})$ as a weighted metric graph and call it stable if it contains no vertices of valency one and weight zero.

1.1.6. Linear Equivalence. We next recall the notion of *linear equivalence*, an equivalence relation among divisors.

For any induced subgraph H of G we define $\underline{t}_H$ to have the following values on vertices: On a vertex v not contained in H, $\underline{t}_H(V)$ is the number of edges e with $e = vv'$ where $v' \in H$. On a vertex v contained in H, $\underline{t}_H(V)$ is minus the number of edges e with $e = vv'$ where $v' \notin H$. Notice that $|\underline{t}_H| = 0$.

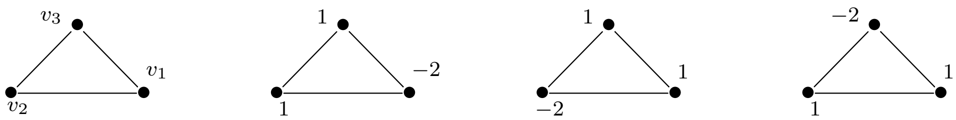


FIGURE 1. The divisors $\underline{t}_{v_1}$, $\underline{t}_{v_2}$ and $\underline{t}_{v_3}$ for the cycle with three vertices v_1, v_2, v_3.

Then we set $\mathrm{Prin}(G) \subset \mathrm{Div}^0(G)$ to be the subgroup generated by $\{t_{[V']} | \, V' \subset V(G)\}$ and will call its elements *principal divisors*.

DEFINITION 1.1.10. Two divisors $\underline{d}, \underline{e} \in \mathrm{Div}(G)$ are said to be linearly equivalent if $\underline{d} - \underline{e} \in \mathrm{Prin}(G)$. In this case we write $\underline{e} \sim \underline{d}$.

Note that if $\underline{e} \sim \underline{d}$ then $|\underline{e}| = |\underline{d}|$ as every principal divisor has degree zero.

We set $\mathrm{Pic}(G) = \mathrm{Div}(G)/\sim$ and $\mathrm{Pic}^k(G) = \mathrm{Div}^k(G)/\sim$. The elements of $\mathrm{Pic}(G)$ will sometimes be called the degree classes of G.

There is another description of this equivalence, known as *chip-firing*:

A chip-firing move based at $v \in V(G)$ or short firing v consists in replacing a divisor $\underline{d}$ by a divisor $\underline{d}'$ in the following way: $\underline{d}'_v = \underline{d}_v - \delta_v$ and $\underline{d}'_w = \underline{d}_w + v \cdot w$ where $v \cdot w$ denotes the number of edges adjacent to both v and w. For a subset of vertices $V' \in V(G)$ firing V' will mean firing every vertex contained in V'. Note that a chip-firing move preserves the degree of $\underline{d}$ and the firing moves commute with each other.

Then $\underline{d} \sim \underline{e}$ if and only if $\underline{d}$ can be obtained from $\underline{e}$ by a series of chip-firing moves. This is an easy consequence of Remark 2.1 in [**20**]. There it is shown that for every $\underline{t} \in \mathrm{Prin}(G)$ there is a decomposition $V(G) = Z_1 \sqcup \cdots \sqcup Z_n$ such that $\underline{t} = \Sigma_i i t_{[Z_i]}$. If the difference of $\underline{d}$ and $\underline{e}$ is $\underline{t}$, firing the set Z_i i times gives the required sequence of chip-firing moves. Conversely one can reconstruct the Z_i from a series of chip-firing moves in the obvious manner.

For a metric graph Γ set $R(\Gamma)$ to be the group under addition of continuous piecewise affine functions $f : \Gamma \to \mathbb{R}$ with integer slopes, usually called the *space of tropical rational* *functions*. By piecewise affine we mean that restricting f to any of the line segments corresponding to an edge e, we get a piecewise affine function $[0, l(e)] \to \mathbb{R}$, for which we in particular require that the set of points on $[0, l(e)]$ over which f is not affine is finite (See [**27**] for a more conceptual treatment in the tropical framework). For every point p of Γ, we set $\sigma_p(f)$ to be the sum of the incoming slopes of f at p. Notice that at points where f is affine, $\sigma_p(f) = 0$. Then we define the *divisor associated to* f as

$$div(f) = \Sigma_{p \in \Gamma} \sigma_p(f) p.$$

Divisors on Γ of this form are called *principal* and they always have degree zero (see [**27**] Cor. 4.3).

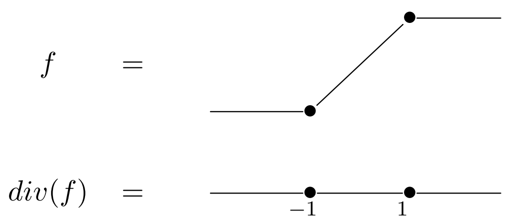


FIGURE 2. The local picture of the graph of a piecewise affine function and its associated divisor. The middle piece is assumed to have slope one and f has a simple pole and a simple zero at the points where it is not affine.

We will set $\mathrm{Prin}(\Gamma) = \{div(f) | f \in R(\Gamma)\} \subset \mathrm{Div}^0(\Gamma)$ and as before set $\underline{d} \sim \underline{e}$ if $\underline{d} - \underline{e} \in \mathrm{Prin}(\Gamma)$. Also analogous to the discrete case we have $\mathrm{Pic}(\Gamma) = \mathrm{Div}(\Gamma)/\sim$ and $\mathrm{Pic}^k(\Gamma) = \mathrm{Div}^k(\Gamma)/\sim$.

REMARK 1.1.11. The similarity in terminology between discrete and metric graphs is motivated by the following observation: To a (non-metrized) graph G we can associate its *regular realization*, the metric graph Γ obtained by setting all edge-lengths to one. Then for $\underline{d}, \underline{e} \in \mathrm{Div}(G)$ with $\underline{d} \sim \underline{e}$ we have under the inclusion $\mathrm{Div}(G) \to \mathrm{Div}(\Gamma)$ also $\underline{d} \sim \underline{e}$ viewed as elements of $\mathrm{Div}(\Gamma)$. Indeed, the $\underline{t}_{V'}$ for $V' \subset V(G)$ can be expressed as $div(f)$ for an $f \in R(\Gamma)$ as follows: Set f to be zero on $[V']$ and one on $[V']^c$. On edges in $([V'], [V']^c)$ extend the values at the vertices linearly (i.e. f has slope one on these edges). Then by definition $\underline{t}_{V'} = div(f)$ under the inclusion $\mathrm{Div}(G) \to \mathrm{Div}(\Gamma)$. Thus $\mathrm{Div}(G) \to \mathrm{Div}(\Gamma)$ descends to a map $\mathrm{Pic}(G) \to \mathrm{Pic}(\Gamma)$ which turns out to be injective as well (See [**8**] Cor. 3.3).

Let G be a graph. The canonical divisor $\underline{\omega}_G$ on G is set to be $(\underline{\omega}_G)_v = 2\underline{g}_v - 2 + \delta(v)$. One calculates that $|\underline{\omega}| = 2g - 2$.

1.1.7. Divisors and edge contractions. Let G be a discrete graph and $\gamma : G \to G/S$ the contraction of $S \subset E(G)$.

DEFINITION 1.1.12. For a divisor $\underline{d} \in \mathrm{Div}(G)$ we define a divisor $\gamma_* \underline{d} \in \mathrm{Div}(G/S)$ by setting $\gamma_* \underline{d}_v = \Sigma_{v_i \in \gamma^{-1}(v)} \underline{d}_{v_i}$.

We will think of $\gamma_* \underline{d}$ as the *push-forward of* $\underline{d}$ under γ. This gives a degree preserving, surjective group homomorphism $\gamma_* : \mathrm{Div}(G) \to \mathrm{Div}(G/S)$.

We next want to recall *weighted contractions*:

DEFINITION 1.1.13. If G is a graph weighted by $\underline{g}$, the weighted contraction of $S \subset E(G)$ is the weighted graph G/S with weights $\gamma_* \underline{g} + \underline{l}_\gamma$ where γ is the contraction map of S and $\underline{l}_\gamma$ is a divisor with value at a vertex v given as the first Betti number of the subgraph $\gamma^{-1}(v)$.

If G is weighted we will usually omit mentioning that the contraction is weighted and always consider G/S as weighted graph if not stated otherwise.

One easily checks the following:

REMARK 1.1.14. If G is a weighted graph and $S \subset E(G)$, then $g(G) = g(G/S)$. If G is stable, then so is G/S.

REMARK 1.1.15. We want to mention, that linear equivalence and edge contractions are not compatible: If $\underline{d}$ and $\underline{e}$ are divisors on G, we have

- $\underline{d} \sim \underline{e} \not\Rightarrow \gamma_* \underline{d} \sim \gamma_* \underline{e}$
- $\gamma_* \underline{d} \sim \gamma_* \underline{e} \not\Rightarrow \underline{d} \sim \underline{e}$

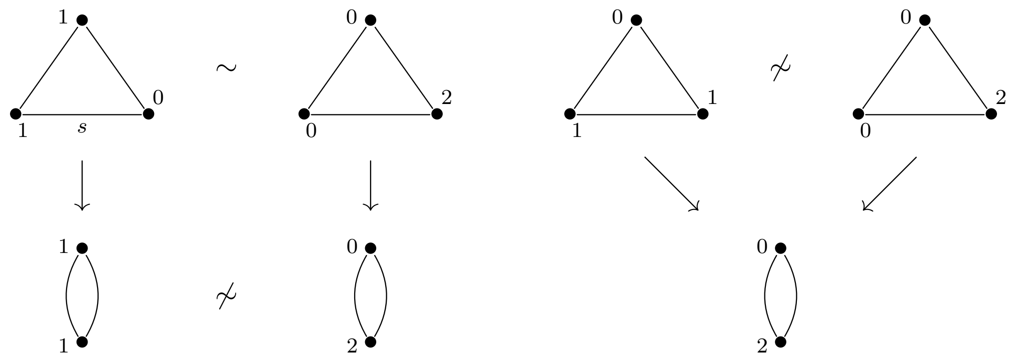


FIGURE 3. An example illustrating the incompatibility of edge contractions and linear equivalence. Here the edge s gets contracted and the numbers on the vertices specify the divisors.

Thus the induced map by an edge contraction $\gamma_* : \mathrm{Div}(G) \to \mathrm{Div}(G/S)$ does not descend to a map from $\mathrm{Pic}(G)$ to $\mathrm{Pic}(G/S)$.

1.1.8. Break divisors. We next recall a particular class of degree g divisors, called *break divisors*, that will play a central role later on. This is an adaption of the definition given in [**27**] to the case of weighted graphs.

Definition 1.1.16. Let Γ be a metric graph and G a model of Γ with weights $\underline{g}$. A divisor $\underline{d} \in \mathrm{Div}(\Gamma)$ is called a break divisor, if there is a spanning tree T of G s.t.

$$\underline{d} = \underline{g} + \Sigma_{e_i \in E(G)\setminus E(T), p_i \in e_i}(p_i).$$

We call the above sum a presentation of $\underline{d}$ by T as a break divisor.

For a weighted graph G, we say that $\underline{d} \in \mathrm{Div}(G)$ is a break divisor if it is a break divisor for the regular realization Γ of G. Recall that the regular realization of G is the metrized graph obtained from G by setting all edge lengths to be one. Thus a break divisor on G is a break divisor on Γ that is supported on vertices of G.

A break divisor is always effective and of degree g. It is effective by construction and for any spanning tree T we have $|E(G) \setminus E(T)| = b_1(G)$. Thus for a break divisor $\underline{d}$ we have $|\underline{d}| = b_1(G) + |\underline{g}| = g(G)$.

Note that neither does a spanning tree give a unique break divisor, nor is there a unique spanning tree associated to a break divisor. Furthermore there may be different presentations as a break divisor even for a fixed spanning tree.

We will denote by $\Sigma(G) \subset \mathrm{Div}^g(G)$ and $\Sigma(\Gamma) \subset \mathrm{Div}^g(\Gamma)$ the sets of break divisors on G and Γ, respectively. If G is a model of Γ, there is an inclusion $\Sigma(G) \hookrightarrow \Sigma(\Gamma)$ whose image are break divisors of Γ supported on vertices of G.

Lemma 1.1.17. *Let Γ be a metric graph with model G. Then $\Sigma(\Gamma) \neq \emptyset$ if and only if Γ (or equivalently G) is connected.*

Proof. If G is not connected, there clearly exists no break divisor on G, as then G has no spanning tree. If it is connected on the other hand, we can choose a spanning tree T and arbitrarily choose a vertex p_i adjacent to the edge e_i for every edge e_i in $E(G) \setminus E(T)$. Then $\underline{g} + \Sigma_i(p_i)$ by construction is a break divisor.

□

Definition 1.1.18. Let G be a model of Γ and $\underline{d} \in \Sigma(\Gamma)$. We will denote by $T_G(\underline{d}) = \{T_1, ..., T_k\}$, or for short $T(\underline{d})$, the set of spanning trees of G that give a presentation of $\underline{d}$ as a break divisors.

Example 1.1.19. The two break divisors $\underline{d}_1, \underline{d}_2 \in \Sigma(G)$ associated to the spanning tree T. In this case $|T(\underline{d}_i)| = 2$.

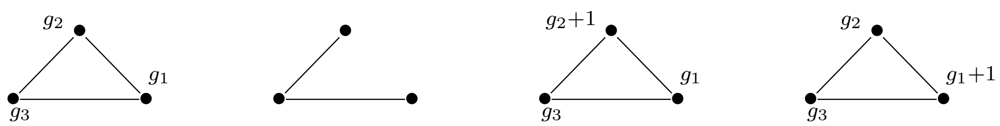


Figure 4. The divisor $\underline{g}$ on G, T and the two break divisors $\underline{d}_1$ and $\underline{d}_2$ associated to T

Adapting the results of [**27**] and [**4**] to the weighted case, each equivalence class in $\mathrm{Pic}(G)$ and $\mathrm{Pic}(\Gamma)$ contains a unique break divisor:

PROPOSITION 1.1.20. *Let G be a connected graph and $\underline{d} \in \mathrm{Div}^g(G)$. Then there exists a unique break divisor $\underline{d}' \in \Sigma(G)$ with $\underline{d}' \sim \underline{d}$.*

Let Γ be a connected metric graph with model G and $\underline{d} \in \mathrm{Div}^g(\Gamma)$. Then there exists a unique break divisor $\underline{d}' \in \Sigma(\Gamma)$ with $\underline{d}' \sim \underline{d}$.

PROOF. If G is weightless and loopless, this is the content of [**4**][Theorem 1.1 and 1.3.] (for the second claim see also [**27**]). The statement then is readily reduced to this situation:

Suppose G is loopless but weighted. For $\underline{d} \in \mathrm{Div}^g(G)$ we can consider $\underline{d} - \underline{g} \in \mathrm{Div}^{b_1(G)}(G)$. Let G^0 be the graph that has the same underlying graph as G but weight $\underline{0}$. Then $b_1(G) = g(G^0)$. Thus by the above cited results we have that there is a unique $\underline{d}' \sim \underline{d} - \underline{g}$ with $\underline{d}' \in \Sigma(G^0)$. Then $\underline{d}' + \underline{g} \in \Sigma(G)$ and $\underline{d}' + \underline{g} \sim \underline{d}$ as desired. This is unique since if there were $\underline{d}'' \neq \underline{d}'$ with $\underline{d}'' \in \Sigma(G)$ and $\underline{d}'' \sim \underline{d}$, we would have $\underline{d}'' - \underline{g} \neq \underline{d}' - \underline{g}$, $\underline{d}'' - \underline{g} \in \Sigma(G^0)$ and $\underline{d}'' - \underline{g} \sim \underline{d} - \underline{g}$, a contradiction to the results in the weightless case. The argument works analogous for Γ with G a loopless but weighted model of Γ.

Suppose now G is any weighted graph. Let G^{ll} be the graph obtained from G by contracting all loops of G (recall that this also changes the weights of G). Let $\gamma : G \to G^{ll}$ be the contraction map. Now given $\underline{d} \in \mathrm{Div}^g(G)$, consider $\gamma_* \underline{d} \in \mathrm{Div}(G^{ll})$. By what we showed above, there is a unique $\underline{d}' \in \Sigma(G^{ll})$ with $\underline{d}' \sim \gamma_* \underline{d}$. As γ only contracts loops and thus in particular $V(G) = V(G^{ll})$ we can view $\underline{d}'$ as a divisor on G. Then $\underline{d} \sim \underline{d}'$ because every cut of G^{ll} is also a cut of G. Let $T \in T_{G^{ll}}(\underline{d}')$ be a spanning tree of G^{ll} that gives a presentation of $\underline{d}'$ as a break divisor. Since γ only contracts loops, we may view T also as a spanning tree of G. Then one easily checks that T gives a presentation of $\underline{d}'$ also as a break divisor on G, i.e. $\underline{d}' \in \Sigma(G)$. By similar arguments as before, one reduces the uniqueness of $\underline{d}'$ on G to the uniqueness of $\underline{d}'$ on G^{ll}.

If Γ is a metric graph with model G we can refine G as follows: at every loop we insert a vertex of valency two and weight zero. We set the lengths of the two new edges at each loop so that they sum to the length of the loop. In this way we obtain a model of Γ with no loops and the claim follows from the cases we discussed before.

□

This in particular means that restricting the maps $\mathrm{Div}^g(G) \to \mathrm{Pic}^g(G)$ to $\Sigma(G)$ gives a bijection between $\Sigma(G)$ and $\mathrm{Pic}(G)$. Similarily one gets a bijection $\Sigma(\Gamma) \to \mathrm{Pic}^g(\Gamma)$.

We have $\mathrm{Div}^k_+(\Gamma) = \Gamma^k / S_k$ where the symmetric group acts by interchanging the coordinates in Γ^k. Of course Γ^k is a topological space with the product topology of Γ, hence we endow $\mathrm{Div}^k_+(\Gamma)$ with the quotient topology. As we saw, every break divisor is effective and of degree g, thus $\Sigma(\Gamma) \subset \mathrm{Div}^g_+(\Gamma)$ and we can endow $\Sigma(\Gamma)$ with the induced topology.

On the other hand we can consider the set of effective divisors up to linear equivalence, $(\mathrm{Div}_+^k(\Gamma)/\sim)$, and endow it with the quotient topology. We have $(\mathrm{Div}_+^k(\Gamma)/\sim) \subset \mathrm{Pic}^k(\Gamma)$. By Proposition 1.1.20, $(\mathrm{Div}_+^g(\Gamma)/\sim) = \mathrm{Pic}^g(\Gamma)$ as every equivalence class has an effective representative (namely the break divisor in the class). This gives a topology on $\mathrm{Pic}^g(\Gamma)$.

PROPOSITION 1.1.21 ([**4**], section 3). *The bijection* $\Sigma(\Gamma) \to \mathrm{Pic}^g(\Gamma)$ *is a homeomorphism.*

1.2. Orientations

1.2.1. Orientations on graphs. An *orientation* of an edge $e = v_1v_2 \in E(G)$ is the assignment of a direction to e. If e is directed from v_1 to v_2, we will denote v_1 the *source* of e and v_2 its *target*. An orientation O on G is the assignment of a an orientation to each edge. A *generalized orientation* of an edge e consists in assigning e either an orientation or biorienting it, in which case v_1 and v_2 are both target and source of e. A generalized orientation O on G is the assignment of a generalized orientation to each edge of G.

DEFINITION 1.2.1. A k-orientation O on G for $k \in \mathbb{N}$ is a generalized orientation on G that has exactly k bioriented edges.

Thus an orientation on G is a 0-orientation and we will keep calling 0-orientations just orientations. A graph together with a generalized orientation will be called an oriented graph.

A *directed cut* of an oriented graph will be a cut (H, H^c) of the graph, such that all edges are oriented away from H (in particular we will not allow bioriented edges in (H, H^c)). If (H, H^c) is empty (i.e. H and H^c span different connected components of G), we will consider it by convention to be a directed cut. A *directed cycle* of an oriented graph will denote a cycle of the graph, such that none of its edges are bioriented and the cycle considered as an oriented subgraph contains no directed cuts.

The *condensed graph*, $c(G)$, of the oriented graph G is defined as the oriented graph obtained from G by contracting all edges not contained in any directed cut. In particular we have a contraction map $\gamma : G \to c(G)$. Note that even though we suppress this in the notation, $c(G)$ clearly depends on the orientation on G.

A path $v_1 \dots v_n$ will be called directed, if the edge v_iv_{i+1} is oriented from v_i to v_{i+1} or is bioriented for every i. We will call the path simply directed path if none of the edges are bioriented. We say a vertex v_i is *reachable* from a vertex v_j, if there is a directed path from v_j to v_i.

REMARK 1.2.2. In a 0-orientation, every edge belongs either to a directed cycle or a directed cut, but not both. Indeed, if an edge $e = v_iv_j$ is part of a directed cut (H, H^c) from H to H^c, none of the vertices in H are reachable from any of the vertices in H^c, so it cannot be contained in a directed cycle. If e is not contained in a directed cut, consider the set of vertices A reachable from v_j. By definition, (A, A^c) forms a directed cut, which by assumption cannot contain e. Thus $v_i \in A$, and e together with a directed path from v_j to v_i forms a directed cycle containing e.

Thus for orientations we get a decomposition $E(G) = G_{cyc} \sqcup G_{cut}$, where G_{cyc} denotes the edges contained in directed cycles and G_{cut} the edges contained in directed cuts. Note that this is no longer true for generalized orientations (see **[7]** for a detailed discussion of the situation in this case).

An orientation will be called:

- *acyclic* if $G_{cyc} = \emptyset$.
- *totally cyclic* if $G_{cyc} = E(G)$.
- *strongly connected* if it is totally cyclic and G is connected (since every edge is contained in a cycle, we get that G is 2-edge connected).

Note that we define these terms only on orientations and not generalized orientations. An acyclic/totally cyclic/strongly connected orientation will thus always be a 0-orientation.

REMARK 1.2.3. On an acyclic orientation reachability gives a partial order: we set $v_j \leq v_i$ for two vertices if v_i is reachable from v_j. Indeed, for reflexivity we set per definition $v_i \leq v_i$. Transitivity is clear, since if there is a directed path from v_i to v_j and one from v_j to v_k, combining these two paths gives a directed path from v_i to v_k. Finally, for the antisymmetry we need that the orientation is acyclic: if $v_i \leq v_j$ and $v_j \leq v_i$, the two directed paths would form a directed cycle unless $v_i = v_j$. Since the orientation does not contain directed cycles, in fact $v_i = v_j$.

If we have a generalized orientation O on G, and a contraction $\gamma : G \to G/S_0 = G'$, we can consider the orientation $\gamma_* O$ on G' which orients every edge of G' as it is oriented by O on G under the inclusion $E(G') \to E(G') \cup S_0 = E(G)$. Note however that if O is a k-orientation, then $\gamma_* O$ is not necessarily also a k-orientation. In fact one easily sees, that if l is the number of bioriented edges of S_0 in O, then $\gamma_* O$ is a $(k-l)$-orientation.

Similarily, for a subgraph H of G we let $O_{|H}$ be the restriction of O to H under the inclusion $E(H) \to E(G)$. Then again $O_{|H}$ is a $(k-l)$-orientation if we let l be the number of bioriented edges of $E(G) \setminus E(H)$ in O.

EXAMPLE 1.2.4. For any graph G and generalized orientation O on G, the condensed graph $c(G)$ will be endowed with the acyclic orientation $\gamma_* O$: it is a 0-orientation because all bioriented edges get contracted as they by definition are not contained in a directed cut. It is acyclic because all edges of $c(G)$ are contained in a directed cut and thus not in a directed cycle. In particular, $V(c(G))$ is always a poset ordered by reachability as described in the previous remark. If O is a 0-orientation consider its strongly connected components, i.e. the connected components of $G - G_{cut}$. Then the vertices of $c(G)$ correspond to these components.

Let O be an orientation on G. We will denote by $\underline{t}_v^O$ the number of half-edges that have the vertex v as target. We here view each edge as consisting of two half-edges; the only case in which this distinction becomes relevant is if there is a bioriented loop adjacent to v. This edge then will contribute two incoming half edges to $\underline{t}_v^O$. For a subgraph H of G we will denote by $t^O(H)$ the number of half-edges not contained in H having a vertex in H as target. For $Z \subset V(G)$ we will abbreviate $t^O([Z])$ by $t^O(Z)$.

A vertex v will be called a *source* if $\underline{t}_v^O = 0$ (i.e. it has no incoming edges) and a *sink* if $\underline{t}_v^O = \delta(v)$ (i.e. all edges adjacent to v are incoming). We will call an orientation *rooted* or *v-rooted*, if there is a vertex v such that every vertex is reachable from v.

EXAMPLE 1.2.5. (1) A strongly connected orientation on a graph is rooted at every vertex.
(2) An orientation on a disconnected graph is never rooted, as there can be no vertex from which every other vertex is reachable.
(3) An acyclic orientation is rooted if and only if it has a unique source (note that every acyclic orientation contains at least one source). This follows directly from the observation, that sources are minimal elements w.r.t. the partial ordering given on acyclic orientations by reachability.

DEFINITION 1.2.6. Let O be a 1-orientation on G with bioriented edge e and set as before $\gamma : G \to c(G)$. Then O will be called rooted, if $\gamma_* O$ is a $\gamma(e)$-rooted orientation on $c(G)$.

CONVENTION 1.2.7. We will understand the empty orientation on a single vertex to be both a rooted 1 and a rooted 0-orientation.

Note that by the example preceding the definition and since $\gamma_* O$ is acyclic, this is the same as requiring that $\gamma(e)$ is the unique source of $\gamma_* O$.

This definition may seem somewhat artificial, but it will be central later on. The next lemma gives alternative characterizations and in particular shows that informally speaking one may view a rooted 1-orientation as a sort of e-rooted orientation where e is the bioriented edge.

LEMMA 1.2.8. *Let O be a 1-orientation on G with bioriented edge e. Then the following are equivalent:*

(1) O is rooted.
(2) For every $Z \subsetneq V(G)$ such that $e \in [Z]$ the cut (Z, Z^c) contains an edge directed away from $[Z]$.
(3) The contraction of e, $\gamma_e : G \to G/e$, induces a v_e-rooted orientation $(\gamma_e)_* O$ on G/e, where $v_e = \gamma(e)$.
(4) Let $e = v_1 v_2$. Then for every vertex v there is a directed path from v_1 and v_2 to v.

PROOF. (1) $\Rightarrow$ (2): Let $Z \subset V(G)$ such that $e \in [Z]$. If the cut (Z, Z^c) is not directed, it contains an edge directed away from Z. If (Z, Z^c) is a directed cut but is empty, then G is disconnected. Then also $c(G)$ is disconnected and thus with $\gamma : G \to c(G)$, $\gamma_* O$ has at least two sources and thus O cannot be rooted. If (Z, Z^c) is a cut directed away from $[Z]$, it contains an edge directed away from $[Z]$. If (Z, Z^c) were a cut directed towards Z, it also would be a directed cut $(\gamma(Z), \gamma(Z^c))$ of $c(G)$ directed towards $\gamma(Z)$. By assumption, $\gamma_* O$ is $\gamma(e)$-rooted on $c(G)$. This is a contradiction, as none of the vertices in $\gamma(Z^c)$ are reachable from $\gamma(e)$.

(2) $\Rightarrow$ (3): Denote by $A \subset V(G/e)$ the set of all vertices reachable from v_e. Then (A, A^c) defines a cut of $V(G/e)$ which lifts to the cut $(\gamma_e^{-1}(A), \gamma_e^{-1}(A^c))$ of G. Since

$e \in [\gamma^{-1}(A)]$ we have by assumption, that if $[\gamma^{-1}(A)] \neq G$ this cut contains an edge directed away from $[\gamma^{-1}(A)]$. This by construction of A however is not possible, as the endpoint of such an edge clearly would be reachable from v_e. Thus $A = V(G/e)$ which proves the claim.

(3) $\Rightarrow$ (4): Any directed path from v_e in G/e to a vertex v lifts to a directed path either from v_1 or v_2 to v. As the edge between v_1 and v_2 is bioriented, this in fact gives a directed path from both v_1 and v_2 to v. By assumption there exists a directed path from v_e to any vertex v in G/e and thus the claim follows.

(4) $\Rightarrow$ (1): Let $\gamma : G \to c(G)$ as before. Any directed path in G gets mapped under γ to a (possibly empty) directed path in $c(G)$, thus by assumption every vertex of $c(G)$ is reachable from $\gamma(e)$.

□

Note that another way of formulating the second characterization is, that for every $Z \subset V(G)$ with $e \in [Z]$ we have $t^O(Z^c) > 0$.

1.2.2. Divisors associated to orientations. We can associate to a generalized orientation a divisor $\underline{d}^O$ by setting for every vertex v:

$$(\underline{d}^O)_v = \underline{t}^O_v + g(v) - 1.$$

In other words, viewing $\underline{t}^O_v$ as the value of a divisor $\underline{t}^O$ at v on G, we get $\underline{d}^O = \underline{t}^O + \underline{g} - \underline{1}$.

Remark 1.2.9. One easily calculates that if O is a k-orientation, we get $|\underline{d}_O| = g(G) - 1 + k$. In particular 0-orientations give degree $g-1$ divisors and 1-orientations give degree g divisors.

Convention 1.2.10. We will define as an exception to the above definition, that the divisor associated to the empty orientation on a graph with a single vertex has value g on that vertex instead of $g-1$. This is because we later will work only with degree g divisors and this convention is needed for consistency.

Recall that $t^O(Z)$ for $Z \subset V(G)$ denoted the number of half-edges having as target a vertex in Z and not lying in $[Z]$ (which as $[Z]$ is an induced subgraph is the same as the number of edges with this property). Denoting by $b(Z)$ the number of bioriented edges of $O_{|[Z]}$, we get by two direct calculations that

$$t^O(Z) = \sum_{z \in Z} \underline{t}^O_z - |E(G[Z])| - b(Z) \tag{1}$$

and

$$|\underline{d}^O_Z| = g(Z) - 1 + b(Z) + t^O(Z). \tag{2}$$

We will consider two equivalence relations on orientations. For the first one we will consider two orientations as equivalent if they induce the same divisor. One easily checks that this indeed gives an equivalence relation.

Definition 1.2.11. Let O and O' be two generalized orientations. We set $O \sim_{cyc} O'$ if $\underline{d}^O = \underline{d}^{O'}$. We will denote by $\overline{O}$ the class of O with respect to this equivalence relation.

There is an explicit description of when two orientations are equivalent established in [**23**]: A *cycle reversal* of an orientation O consists of reversing the direction of every edge contained in a directed cycle. Then $O \sim_{cyc} O'$ if and only if O' can be obtained from O by a series of cycle reversals.

In [**6**] this is extended to generalized orientations by adding the following move:[1] Let v be a vertex and e_1, e_2 edges adjacent to v, such that e_1 is bioriented and e_2 is not oriented towards v. An *edge pivot* then consists in orienting e_1 away from v and biorienting e_2 (thus, informally speaking, moving the towards v oriented half-edge from e_1 to e_2). We then have for generalized orientations O and O' that $O \sim_{cyc} O'$ if and only if O' can be obtained from O by a series of cycle reversals and edge pivots.

Example 1.2.12. To illustrate the edge pivots, consider a bioriented edge moving along a simple directed path by a sequence of edge pivots. In [**6**] this move is called a Jacob's ladder cascade. In this example, the path has length two, but this principle of course works for any length of the path:

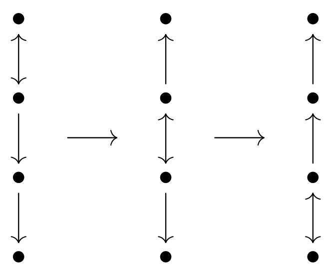


Figure 5. A Jacob's ladder cascade with a bioriented edge.

Next we want to define another equivalence relation, encoding when the induced divisors are linearly equivalent:

Definition 1.2.13. Let O and O' be two generalized orientations. We set $O \sim O'$ if $\underline{d}^O \sim \underline{d}^{O'}$.

Thus $O \sim_{cyc} O'$ implies $O \sim O'$. We again have a description in terms of explicit moves on an orientation:

Let O be a generalized orientation. A *cut reversal* consists in inverting the orientation of every edge contained in some directed cut of O. Then we have $O \sim O'$ if and only if

[1] In [**6**] the orientations are in fact generalized not by biorienting edges, but by allowing unoriented edges. This however directly translates to the situation here by taking the residual. Compare also [**7**] for a unifying framework.

O' is obtained from O by a series of edge pivots, cycle reversals and cut reversals. Thus summarizing for the two equivalence relations:

FACT 1.2.14 (cf. [**23**], [**6**]). *For two generalized orientations O and O' on a graph G we have:*

(1) $\underline{d}^O = \underline{d}^{O'} \Leftrightarrow O \sim_{cyc} O' \Leftrightarrow O'$ is obtained from O by a series of edge pivots and cycle reversals.
(2) $\underline{d}^O \sim \underline{d}^{O'} \Leftrightarrow O \sim O' \Leftrightarrow O'$ is obtained from O by a series of edge pivots, cycle reversals and cut reversals.

We will call a divisor $\underline{d}$ *orientable*, if there is a generalized orientation O s.t. $\underline{d} = \underline{d}^O$. That is, if the divisor can be given as the divisor associated to an orientation.

REMARK 1.2.15. From this description it is immediately clear that if an edge e is contained in a directed cut for some 0-orientation O, it will also be contained in a directed cut for any orientation $O' \sim O$. Thus two orientable divisors that are linearly equivalent have the same decomposition $E(G) = G_{cyc} \sqcup G_{cut}$. In particular, if a divisor is given by a totally cyclic/acyclic orientation, any other orientation giving the divisor will also be totally cyclic/acyclic. Furthermore, reachability between vertices is preserved under cycle reversals, thus being v-rooted is a property shared by orientations that give the same divisor.

Not every divisor is orientable. Clearly if $\underline{d}_v < -1$ or $\underline{d}_v > \delta(v) - 1 + g(v)$ for some vertex v, $\underline{d}$ will not be orientable. In the case of $d = g-1$ it is known that every divisor is linearly equivalent to a divisor associated to a 0-orientation. We will see an interpretation of this fact in the next section, namely that balanced divisors are orientable. In general there are also degree classes that do not contain any orientable divisor. As an easy example consider the following:

EXAMPLE 1.2.16. Let G be a graph with two vertices and n edges between them. Set furthermore $g_1 = g_2 = 0$. Let $\underline{d}$ be given by $d_1 = -2$ and $d_2 = n-2$. Then any divisor linearly equivalent to $\underline{d}$ has $d_i \leq -2$ for some i. It is worthwhile to note, that in this case $g = n-1$ and $|\underline{d}| = n-4 = g-3$; by results discussed below, any divisor with $|\underline{d}| \in \{g-2, g-1, g\}$ is linearly equivalent to an orientable divisor.

In the following we will deal with rooted 1-orientations. We set

DEFINITION 1.2.17. Let $\mathcal{O}^1(G)$ and $\overline{\mathcal{O}}^1(G)$ be the set of rooted 1-orientations and classes w.r.t. $\sim_{cyc}$ of rooted 1-orientations on G, respectively.

After having introduced divisors associated to orientations and equivalence, we can add one more characterization for a 1-orientation being rooted.

DEFINITION 1.2.18. Let O be a 1-orientation. We will say its bioriented edge is freely moving, if for every $e' \in E(G)$ there is a 1-orientation O' with $O \sim_{cyc} O'$ and e' is the bioriented edge of O'.

LEMMA 1.2.19. *Let O be a 1-orientation. Then the following are equivalent:*

(1) *The orientation O is rooted.*
(2) *The bioriented edge of O is freely moving.*
(3) *For every $Z \subset V(G)$ we have $|\underline{d}_Z^O| > g_Z - 1$.*

Proof. (1) $\Rightarrow$ (2): Suppose O is rooted and let $e = v_1v_2$ be its bioriented edge. Let $e' = v_3v_4$ be any other edge and suppose e' is oriented away from v_3. Then by Lemma 1.2.8 (4), there is a directed path from v_1 to v_3. This path extends to a path to v_4 along e'. Performing a Jacob's ladder cascade as in Example 1.2.12 along this path, gives an orientation O' whose bioriented edge is e'.

(1) $\Leftrightarrow$ (3): We have $|\underline{d}_Z^O| = g(Z) - 1 + b(Z) + t^O(Z)$ by (2) which is bigger than $g_Z - 1$ if and only if $b(Z) = 1$, i.e. $e \in [Z]$, or $t^O(Z) > 0$, i.e. there is an edge in the cut (Z, Z^c) directed towards $[Z]$. This is a reformulation of 1.2.8 (2).

(2) $\Rightarrow$ (3): Suppose the bioriented edge of O is freely moving and let $Z \subset V(G)$. For this to be the case G clearly has to be connected. If $Z \neq V(G)$, we can by assumption choose $e' \in (Z, Z^c)$ and $O' \sim_{cyc} O$ such that e' is the bioriented edge of O'. Then $t_Z^{O'} > 0$ and thus $|\underline{d}_Z^{O'}| > g_Z - 1$. Since $\underline{d}^O = \underline{d}^{O'}$ this implies the claim.

$\square$

1.2.3. Rooted orientations and break divisors. Next we recall the connection between orientations and break divisors established in [**4**]:

Proposition 1.2.20 ([**4**], Lemma 3.3 combined with Theorem 1.3). *Let G be a graph and fix $v \in V(G)$. The map $\mathrm{Div}^g(G) \to \mathrm{Div}^{g-1}(G)$ sending $\underline{d}$ to $\underline{d} - (v)$ induces a bijection between break divisors and divisors given by a v-rooted orientation.*

Thus we get in our setting:

Corollary 1.2.21. *The map $\overline{O} \to \underline{d}^O$ gives a bijection between $\overline{\mathcal{O}}^1(G)$ and $\Sigma(G)$.*

Proof. Suppose $\underline{d} \in \Sigma(G)$. Then for $v \in V(G)$ the divisor $\underline{d} - (v)$ is given by a v rooted 0-orientation O by Proposition 1.2.20. As O is v-rooted, it has to have at least one outgoing edge e adjacent to v. Setting O' to be the 1-orientation obtained from O by biorienting e, we get $\underline{d} = \underline{d}^O + (v) = \underline{d}^{O'}$. Furthermore by Lemma 1.2.8 (4), O' will be a rooted 1-orientation.

Conversely, starting with a rooted 1-orientation O, we can assume by Lemma 1.2.19 (2), that the bioriented edge of O is adjacent to v. Replacing the bioriented edge with an edge directed away from v gives, again by Lemma 1.2.8 (4), a v-rooted orientation O'. Thus by Proposition 1.2.20 we get that $\underline{d}^O = \underline{d}^{O'} + (v)$ is a break divisor.

$\square$

Definition 1.2.22. Let O be a 1-orientation on G. An arborescence of O is a spanning tree T of G such that $O_{|T}$ is a rooted 1-orientation on T. We will denote by $T(O)$ the set of arborescences of O.

Spelling this out, an arborescence is a spanning tree that contains the bioriented edge and such that all of its other edges are directed away from the bioriented one. Arborescences play a similar role as presentations of break divisors:

LEMMA 1.2.23. *Let $\underline{d}^O \in \Sigma(G)$ be a break divisor given by the class of a rooted 1-orientation O. Then for every presentation of $\underline{d}^O$ by a spanning tree T there is $O' \in \overline{O}$ such that T is an arborescence of O'. Conversely, any arborescence T of an orientation $O' \in \overline{O}$ gives rise to a presentation of $\underline{d}^O$ as a break divisor by T viewed as a spanning tree.*

PROOF. Suppose $\underline{d}^O$ is a break divisor with a presentation by a spanning tree T, i.e. $\underline{d}^O = \underline{g} + \Sigma_{e_i \in E(G)\setminus E(T), p_i \in e_i}(p_i)$. Let O' be the following orientation: On T fix an arbitrary rooted 1-orientation and orient any edge e_i in $E(G) \setminus E(T)$ towards the vertex p_i. Then by construction we have $\underline{d}^{O'} = \underline{d}^O$ and T is an arborescence of O'. Conversely, starting from an arborescence T, choose p_i to be the vertex towards which the edge e_i in $E(G) \setminus E(T)$ is oriented.

□

LEMMA 1.2.24. *A 1-orientation O on G is rooted if and only if it contains an arborescence.*

PROOF. Suppose O contains an arborescence. We saw in the proof of Lemma 1.2.23, that every arborescence of O gives rise to a presentation of $\underline{d}^O$ as a break divisor. Thus O is rooted by Corollary 1.2.21.

Conversely, if O is a rooted 1-orientation construct an arborescence as follows: Start with the bioriented edge $e_0 = v_1v_2$. Then the cut defined by $Z_1 = \{v_1, v_2\}$ by assumption contains an edge e_1 directed away from Z_1. Let v_3 be the vertex this edge is directed towards and $Z_2 = Z_1 \cup \{v_3\}$. Then the cut defined by Z_2 contains an edge e_2 directed away from Z_2. Repeating this procedure, we eventually get $Z_k = V(G)$ and the e_i define an arborescence.

□

LEMMA 1.2.25. *Let O be a rooted 1-orientation on G with bioriented edge $e = v_1v_2$ and $P = v_1v_2\ldots v_k$ a directed path in G. Then there is an arborescence T of O containing P.*

PROOF. Note that P is an arborescence of $O_{|G-(V(G)\setminus V(P))}$. If $P \subset G$ already contains all vertices, it is an arborescence itself. So let v be a vertex not contained in P. Since O is rooted, there is a simply directed path $P' = v_1' \ldots v_n'$ from either v_1 or v_2 to v. Then there is a unique v_i' such that for the subpath $P'' = v_i' \ldots v_n'$ we have $E(P'') \cap E(P) = \emptyset$ and $V(P'') \cap V(P) = \{v_i'\}$. The union $P'' \cup P$ then will be a an arborescence of $O_{|G-(V(G)\setminus V(P\cup P''))}$. Replacing P by $P'' \cup P$ and repeating this procedure for every vertex not contained in P yields the desired arborescence.

□

CHAPTER 2

Algebro-geometric and tropical moduli spaces

In this chapter we will review some notions about the moduli spaces we will be interested in. We will mainly be concerned with the algebro-geometric moduli spaces of curves and line bundles on them, but will also briefly review the construction of the moduli space of tropical curves, as we later will mirror this procedure to construct $(P_g^g)^{trop}$. Our account here will do no justice to the scope of the theories of these concepts and is not meant as an introduction but more to fix some notation and background.

2.1. Moduli spaces of algebraic and tropical curves

2.1.1. Algebraic curves. The theory of algebraic curves and their moduli is vast and we cannot even begin to give an overview here. We will only recall some of the classical results about the existence and compactification of the moduli space of curves. For any details, we refer to [**5**] and [**25**].

By *curves* we will mean projective, connected algebraic curves over an algebraically closed base field of characteristic zero. A *nodal* curve will be a curve with only ordinary double points as singularities. From now on, we will assume all our curves to be nodal without necessarily mentioning it. By the genus $g = g(X)$ of a curve X we will mean the arithmetic genus if not otherwise specified. We will always assume that $g \geq 2$.

To a curve X one can associate its dual graph, G_X. If the curve X is clear from context, we will sometimes write G for G_X. Vertices of G_X correspond to irreducible components of X and edges to nodes in which the irreducible components intersect (in particular, loops correspond to nodes of irreducible components). The weight of G_X on a vertex v is the geometric genus of the irreducible component of X it corresponds to. We usually shall write $X = \bigcup_{v \in V(G_X)} C_v$ for the irreducible components of X and use the same symbols for edges of G_X and nodes of X.

The genus of the curve equals the genus of its dual graph, in symbols $g(X) = g(G_X)$.

For an irreducible component C_v of X we will write $C_v^c = \overline{(X \setminus C_v)}$.

Definition 2.1.1. A curve X will be called stable, if it satisfies one of the following equivalent conditions:

- It has finitely many automorphisms.
- The dualizing sheaf ω_X is ample.
- Every irreducible component C_v of (arithmetic) genus zero satisfies $|C_v \cap C_v^c| \geq 3$.

From the third characterization it is immediately clear, that X is stable if and only if G_X is stable.

A curve is called *semi-stable*, if every irreducible component C_v of genus zero satisfies $|C_v \cap C_v^c| \geq 2$. The components of genus zero that have $|C_v \cap C_v^c| = 2$ will be called *exceptional components*. A curve is called *quasi-stable*, if it is semi-stable and no two exceptional components intersect each other.

Every semistable curve has a *stable model*, obtained by contracting all exceptional components.

For a subset $S \subset E(G_X)$, we will denote by $\nu : X_S^\nu \to X$ the partial normalization of X at nodes corresponding to edges in S. The dual graph of X_S^ν is $G - S$. We will denote by $\hat{X}_S$ the curve obtained from X_S^ν by attaching for every $e \in S$ a smooth rational component to the two smooth branch points $\nu_S^{-1}(e)$. These additional components will be exceptional in the sense defined above. We have that $g(X) = g(\hat{X}_S)$ and if X is stable, then X_S^ν is quasi-stable. We will denote by $\hat{G}_S$ the dual graph of $\hat{X}_S$, which is obtained from G_X by inserting a weight zero and valency two vertex in each edge of S. In particular $\hat{G}_S$ is a refinement of G_X.

For a line bundle L on X we define a divisor $\underline{\deg}(L)$ on G_X by setting $\underline{\deg}(L)_v = \deg(L_{|C_v})$. Thus we get a map $\underline{\deg} : \mathrm{Pic}(X) \to \mathrm{Div}(G_X)$ with $\deg(L) = |\underline{\deg}(L)|$.

We will denote by M_g the *moduli space of smooth genus g curves*. It coarsely represents the functor that associates to a base scheme flat families of smooth genus g curves. The space M_g is not compact, as smooth curves may degenerate in flat families to singular ones. It is a classical result of **[21]** that M_g can be compactified in a modular way by stable curves. We will denote by $\overline{M_g}$ this compactification, the so called Deligne-Mumford compactification. It coarsely represents the functor that associates to a base scheme flat families of stable genus g curves.

2.1.2. The moduli space of tropical curves. A (abstract) *tropical curve* is a metrized weighted graph. A tropical curve is *stable* if the underlying weighted graph is stable.

We will sketch the construction of the moduli space of tropical curves, M_g^{trop}, following **[1]**. We will however not describe the concepts of tropical geometry as we will not use them later on. For a fixed stable graph G of genus g set $\sigma_G^o \subset \mathbb{R}^{|E(G)|}$ to be the open cone

$$\sigma_G^o = (\mathbb{R}_{\geq 0})^{|E(G)|}.$$

Then to each point of σ_G^o we can associate the metrized graph with underlying graph G and edge length given by the coordinates of the point. To get a space parametrizing tropical curves with underlying graph G we have to account for the automorphisms $\mathrm{Aut}(G)$ of G. Note that any such automorphism acts on σ_G^o by interchanging the coordinates according to the edges they correspond to. Thus we next set:

$$M_G^{trop} = \sigma_G^o / \mathrm{Aut}(G).$$

This in general is no longer homeomorphic to an open cone.

Recall that we denote by $\mathcal{SG}_g$ the set of all stable graphs of genus g. As a set the space parametrizing genus g stable tropical curves thus is given as

$$M_g^{trop} = \bigsqcup_{G \in \mathcal{SG}_g} M_G^{trop}.$$

To add a topology on this space, we consider the closure $\sigma_G \subset \mathbb{R}^{|E(G)|}$ of σ_G^o. That is, we now allow the coordinates corresponding to the edges of G to be zero. This we want to view as changing the underlying graph by contracting all edges of length zero. More precisely, if $\gamma : G \to H$ is the map contracting the edges $S \subset E(G)$, we get an inclusion

$$i_\gamma : \sigma_H \to \sigma_G.$$

This is done as follows: Via $\gamma : G \to H$ we can view $E(H)$ as a subset of $E(G)$ and consider the inclusion $i : \mathbb{R}^{|E(H)|} \to \mathbb{R}^{|E(G)|}$ that sets all coordinates corresponding to edges in $E(G) \setminus E(H) = S$ to zero. Then i_γ is the restriction of i to σ_H. Note that i_γ maps σ_H bijectively to a face of σ_G.

An automorphism $\phi_i \in \mathrm{Aut}(G)$ acts on σ_G in the same manner as on σ_G^o by interchanging the coordinates.

The *moduli space of tropical curves*, as a topological space, then is the colimit

$$M_g^{trop} = \varinjlim(\sigma_G, i_\gamma, \phi_i).$$

That is, we glue the cones σ_G according to the inclusions i_γ and identify points having the same underlying graph if they differ by an automorphism of the graph. Then we have natural maps $\sigma_G \to M_g^{trop}$ that induce natural inclusions $\sigma_G^o/\mathrm{Aut}(G) = M_G^{trop} \to M_g^{trop}$ and recover the above mentioned decomposition $M_g^{trop} = \bigsqcup_{G \in \mathcal{SG}_g} M_G^{trop}$.

For the additional structure of a generalized cone complex that M_g^{trop} can be endowed with, see [**1**].

2.2. Compactified Jacobians

2.2.1. The Picard variety of a nodal curve. Let X be an algebraic curve and denote by $\mathrm{Pic}(X)$ the Picard scheme of X parametrizing line bundles on X up to isomorphism. Then we have a decomposition

$$\mathrm{Pic}(X) = \bigsqcup_{\underline{d} \in \mathrm{Div}(G_X)} \mathrm{Pic}^{\underline{d}}(X),$$

where $\mathrm{Pic}^{\underline{d}}(X)$ parametrizes line bundles of multidegree $\underline{d}$. The $\mathrm{Pic}^{\underline{d}}(X)$ are the connected components of $\mathrm{Pic}(X)$ and are all isomorphic. In particular, they are isomorphic to the generalized Jacobian $\mathrm{Pic}^{\underline{0}}(X)$ parametrizing line bundles of degree zero on each irreducible component. Both $\mathrm{Pic}^{\underline{0}}(X)$ and $\mathrm{Pic}(X)$ are group schemes on which the group structure is given by the tensor product of the line bundles. The neutral element is the structure sheaf of X, $\mathcal{O}_X$. The $\mathrm{Pic}^{\underline{d}}(X)$ are $\mathrm{Pic}^{\underline{0}}(X)$-torsors.

There are two ways in which $\mathrm{Pic}(X)$ is not convenient to work with:

First, $\mathrm{Pic}^{\underline{0}}(X)$ is not compact and thus also $\mathrm{Pic}^{\underline{d}}(X)$ and $\mathrm{Pic}(X)$ are not. In fact, we have a short exact sequence

$$0 \to (k^*)^{b_1(G_X)} \to \mathrm{Pic}^{\underline{0}}(X) \to \mathrm{Pic}^{\underline{0}}(X^\nu_{E(G_X)}) \to 0.$$

Recall that $X^\nu_{E(G_X)}$ is the normalization of X. The map $\mathrm{Pic}^{\underline{0}}(X) \to \mathrm{Pic}^{\underline{0}}(X^\nu_{E(G_X)})$ is given by taking the pullback of a line bundle along $\nu : X^\nu_{E(G_X)} \to X$. Its kernel consists of the different ways to glue the structure sheaf of $X^\nu_{E(G_X)}$ over the nodes of X and one checks that up to isomorphism this amounts to giving $b_1(G_X)$ elements of k^*. Now $\mathrm{Pic}(X^\nu_{E(G_X)})$ is the product of Picard schemes of smooth curves and thus compact. On the other hand, the torus $(k^*)^{b_1(G_X)}$ clearly is not compact if $b_1(G_X) \neq 0$. Thus the generalized Jacobian $\mathrm{Pic}^{\underline{0}}(X)$ is compact if and only if G_X is a tree (i.e. $b_1(G_X) = 0$). Curves that have this property for this reason are called of *compact type*.

REMARK 2.2.1. Informally speaking, the dichotomy embodied in the two parts, $(k^*)^{b_1(G_X)}$ and $\mathrm{Pic}(X^\nu_{E(G_X)})$, is at the core of a general theme concerning line bundles on nodal curves. The former encodes how the line bundle is glued over the nodes, information readily described by the combinatorics of the dual graph. The latter encodes the geometry of the normalization of the irreducible components thus reducing to the case of smooth curves. This in many ways is the non-combinatorial part. At one extreme we encountered the curves of compact type, where all the information of a line bundle on the curve can be read off its pullback to the normalization. The other extreme are curves whose components all have geometric genus zero, i.e. the dual graph has trivial weights. As the Picard groups of the normalization of the irreducible components in this case are trivial, all information about a line bundle is encoded in the gluing over the nodes.

Second, in the relative setting one can consider the relative Picard scheme. That is, for a family of curves we get a scheme over the base, whose fiber over a point is the Picard scheme of the corresponding curve in the family. The relative Picard scheme then in general is not separated. Consider for example a family of curves $\pi : \mathcal{X} \to B$ over a one dimensional base B with regular total space and such that for $b \neq b_0 \in B$ the fiber over b is a smooth curve X_b and over b_0 it is a nodal curve X_0 with more than one irreducible component. Let $\mathcal{L} \to \mathcal{X}$ be a line bundle relative to this family, i.e. restricting to line bundles on the X_b and X_0. Then the irreducible components C_v of X_0 are Cartier divisors on $\mathcal{X}$ and we can twist $\mathcal{L}$ by the associated line bundle. Then $\mathcal{L}_{|\pi^{-1}(B\setminus\{b_0\})} = (\mathcal{L}\otimes\mathcal{O}_{\mathcal{X}}(C_v))_{|\pi^{-1}(B\setminus\{b_0\})}$ but they clearly differ on the central fiber. That is, given a line bundle on $\pi^{-1}(B \setminus \{b_0\})$, there is no unique extension of it to all of $\mathcal{X}$. As it turns out, this phenomenon is modelled directly by linear equivalence of the corresponding divisors on the dual graph: If two line bundles L and L' on X_0 are the extension of the same line bundle on $\pi^{-1}(B \setminus \{b_0\})$ we have $\underline{\deg}(L) \sim \underline{\deg}(L')$.

Addressing both problems thus typically consists in two steps, usually summarized as compactifying the Picard variety: First, choosing classes $\mathrm{Pic}^d(G_X)$ and for each class a unique representative $\underline{d} \in \mathrm{Div}^d(G_X)$. The starting point for constructing the compactified Picard variety then is $\bigsqcup \mathrm{Pic}^{\underline{d}}(X)$, which is separated in the relative setting and coincides with the smooth locus of the compactification. Second, one needs to compactify $\bigsqcup \mathrm{Pic}^{\underline{d}}(X)$, preferably in a modular way.

2.2.2. A compactification. There are three constructions of compactifications we are aware of: the one constructed by Caporaso ([**14**]), Simpson's construction of admissible sheaves applied to curves ([**29**]) and the one constructed by Oda and Seshadri ([**28**]). We refer to [**26**] for a detailed comparison. For a fixed curve the constructions increase in generality in this order, i.e. the previous one is a special case of the following one. The most general one, constructed in [**28**] (see also [**2**] and [**26**]), gives in each degree an infinite number of compactifications and a finite number of compactifications up to isomorphism. Furthermore different degrees allow for the same compactifications.[1] Only the first two however are known to work in the relative setting over families. We will work with the compactification constructed in [**14**], a special case of the above whose isomorphism class depends only on the degree. As far as we know, this is the only compactification for which a compactified universal Picard variety has been constructed (also in [**14**]). Following the construction there, we will work with equivalence classes of line bundles on quasistable curves instead of torsion free rank one sheaves on stable curves; see [**2**], Lemma 1.9 for a precise statement of the equivalence of these two approaches.

Recall that we set $\underline{\omega}_G \in \mathrm{Div}(G)$ to be the canonical divisor on G. For $Z \subset V(G)$ denote by $\underline{\omega}_Z$ the restriction of $\underline{\omega}_G$ to $[Z]$ (Note that this is not the same as $\underline{\omega}_{[Z]}$ where we view $[Z]$ as an abstract graph and not a subgraph of G).

Definition 2.2.2. Let $\underline{d} \in \mathrm{Div}(G)$ be a divisor on a graph G. We will say $\underline{d}$ is balanced, if

$$(|\underline{\omega}_Z||\underline{d}|)/(2g-2) - \delta_Z/2 \leq |\underline{d}_Z|$$

for every proper subset of vertices $Z \subset V(G)$. It is strictly balanced if strict inequality holds. A line bundle will be called (strictly) balanced, if its multidegree is.

We will abbreviate $m_d(Z) = (|\underline{\omega}_Z||\underline{d}|)/(2g-2) - \delta_Z/2$ and set $M_d(Z) = (|\underline{\omega}_Z||\underline{d}|)/(2g-2) + \delta_Z/2$. Note that $m_d(Z) \leq |\underline{d}_Z| \Leftrightarrow |\underline{d}_{Z^c}| \leq M_d(Z^c)$, thus our definition indeed coincides with the basic inequality of [**14**], which requires $m_d(Z) \leq |\underline{d}_Z| \leq M_d(Z)$. The smooth locus of $\overline{P}^d_X$ then consists of the disjoint union of the $\mathrm{Pic}^{\underline{d}}(X)$ with $\underline{d}$ strictly balanced.

To compactify, one considers line bundles not only on X, but also on quasistable curves having X as stable model.

Definition 2.2.3. Let $\hat{L}_S \in \mathrm{Pic}(\hat{X}_S)$ with X a stable curve (and thus $\hat{X}_S$ quasistable). Then $\hat{L}_S$ and its multidegree $\underline{\deg}(\hat{L}_S) \in \mathrm{Div}(\hat{G}_S)$ is called stably balanced, if

(1) $\hat{L}_S$ has degree one on each exceptional component of $\hat{X}_S$.
(2) $\underline{\deg}(\hat{L}_S)_{|G-S}$ is strictly balanced.

We will consider two stably balanced line bundles $\hat{L}_S$ and $\hat{L}'_{S'}$ to be equivalent, if $S = S'$ and their restrictions to X^ν_S are isomorphic.

[1]Note that the claim in [**2**] that the compactifications of [**28**] and [**29**] coincide is not true in degree $g-1$. The Simpson compactification is unique in this degree, but there are as many compactifications as in any other degree for the Oda Sheshadri construction. See [**26**] for details.

Fact 2.2.4 ([**14**]). *Let X be a stable curve. The compactified Jacobian $\overline{P}^d_X$ is a coarse moduli space for equivalence classes of stably balanced line bundles on quasistable curves having X as stable model.*

To a strictly balanced divisor $\underline{d}_S$ on $G - S$ we can associate by definition a stably balanced divisor $\hat{\underline{d}}_S$ on $\hat{G}_S$ under the inclusion $G - S \to \hat{G}_S$ by setting the values on exceptional components of $\hat{G}_S$ to be one.

Note furthermore, that every stably balanced divisor is balanced on $\hat{G}_S$. Subcurves where it is not strictly balanced are the complements of unions of exceptional components of $\hat{G}_S$.

We recall from [**14**], that the compactified Jacobians $\overline{P}^d_X$ glue over $\overline{M_g}$ in the sense that there is a proper scheme $\overline{P}^d_g$ and a projective morphism

$$\psi_{g,b} : \overline{P}^d_g \to \overline{M_g}$$

whose fiber over $[X] \in \overline{M_g}$ is $\overline{P}^d_X/\mathrm{Aut}(X)$.

2.3. Stratifications of moduli spaces

After introducing the framework of stratifications by partially ordered sets, we recall in this section some known results about the stratifications of the moduli spaces we are interested in.

2.3.1. Stratification by a partially ordered sets.

A pair $(\mathcal{P}, \leq)$ or just $\mathcal{P}$ is a *partially ordered set* or short *poset* if $\leq$ is a binary relation on the set $\mathcal{P}$ that is reflexive, antisymmetric and transitive. For $p_1, p_2 \in \mathcal{P}$ we will write $p_1 < p_2$ if $p_1 \leq p_2$ and $p_1 \neq p_2$. We will say p_2 *covers* p_1 if $p_1 < p_2$ and there is no $p \in \mathcal{P}$ such that $p_1 < p < p_2$.

We set the *dual poset* $\mathcal{P}^* = (\mathcal{P}, \leq^*)$ of $(\mathcal{P}, \leq)$ to be the partial order defined on $\mathcal{P}$ by inverting the order given by $\leq$. That is $p_1 \leq p_2 \Leftrightarrow p_2 \leq^* p_1$.

A *morphism of posets* is a map $\mu : (\mathcal{P}, \leq) \to (\mathcal{P}', \leq')$ that respects the partial ordering. More explicitly, if $p_1 \leq p_2$ we need to have $\mu(p_1) \leq' \mu(p_2)$. An *isomorphism of posets* is a bijective morphism of posets whose inverse is also a morphism of posets.

A *quotient of posets* or short just a quotient is a surjective morphism of posets $\mu : (\mathcal{P}, \leq) \to (\mathcal{P}', \leq')$ such that if $p_1' \leq' p_2'$ in $\mathcal{P}'$ there exists for every $p_1 \in \mu^{-1}(p_1')$ a $p_2 \in \mu^{-1}(p_2')$ with $p_1 \leq p_2$.

A *rank function* on a poset $\mathcal{P}$ is a morphism of posets $\rho : \mathcal{P} \to \mathbb{N}$ that preserves the covering relation, meaning that if p_2 covers p_1 in $\mathcal{P}$, then $\rho(p_2)$ covers $\rho(p_1)$ in $\mathbb{N}$, i.e. $\rho(p_2) = \rho(p_1) + 1$. Here we view $\mathbb{N}$ as a poset endowed with the usual partial order. A poset together with a rank function is called a *graded poset*.

Definition 2.3.1. For a finite poset the dual rank function $\rho*$ of ρ on $\mathcal{P}^*$ will be defined as

$$\rho^*(p) = max_{p_i \in \mathcal{P}}(\rho(p_i)) - \rho(p).$$

This indeed is a rank function: if p_2 covers p_1 in $\mathcal{P}^*$, then p_1 covers p_2 in $\mathcal{P}$. Thus $\rho^*(p_2) = max_{p_i \in \mathcal{P}}(\rho(p_i)) - \rho(p_2) = max_{p_i \in \mathcal{P}}(\rho(p_i)) - \rho(p_1) + 1 = \rho^*(p_1) + 1$.

Suppose we have a topological space M together with a decomposition $M = M_1 \sqcup \cdots \sqcup M_n$ into disjoint, locally closed subspaces M_i. We will denote by $M^{strat} = \{M_1, \dots, M_n\}$ the *set of strata* and endow it with a partial order $\leq$ by setting $M_i \leq M_j$ if M_i is contained in the closure of M_j, in symbols $M_i \subset \overline{M_j}$. We check that this gives indeed a poset: Clearly $\leq$ is reflexive and transitive. Antisymmetry is not as obvious and indeed requires that the M_i are disjoint and locally closed:

Suppose $M_i \leq M_j$ and $M_j \leq M_i$. Since $M_i \subset \overline{M_j}$ and $M_j \subset \overline{M_i}$, we get by definition of the closure that $\overline{M_i} = \overline{M_j}$. In particular, M_i is dense in $\overline{M_j}$. Since M_i is locally closed, we can write it as $M_i = U_i \cap W_i$ with U_i open and W_i closed. Then $M_i \subset \overline{M_j} = \overline{M_i} \subset W_i$ and we get $M_i = U_i \cap \overline{M_j}$. Thus M_i is open in $\overline{M_j}$. Summarizing we get that both M_i and M_j are open and dense in $\overline{M_j}$. In particular their intersection is not empty but since we assumed $M_i \cap M_j = \emptyset$ for $i \neq j$ this only leaves the possibility $i = j$.

Definition 2.3.2. With notation as above, a decomposition $M = M_1 \sqcup \cdots \sqcup M_n$ of a topological space M is called a stratification by a partially ordered set $\mathcal{P}$, if the following hold:

(1) The M_i are locally closed.
(2) If $M_i \cap \overline{M_j} \neq \emptyset$ then $M_i \leq M_j$ in M^{strat}.
(3) There is an isomorphism of posets $s : M^{strat} \to \mathcal{P}$.

Thus spelling out the last condition, we need to have $M_i \subset \overline{M_j} \Leftrightarrow s(M_i) \leq s(M_j)$.

If M is stratified by $\mathcal{P}$ we will call the M_i the *strata* of this stratification. The isomorphism $s : M^{strat} \to \mathcal{P}$ induces a map $s : M \to \mathcal{P}$ that we will denote by the same name and which is given by mapping a point of M to the image of the stratum it is contained in. We will use the two notions interchangeably, as one determines the other.

If we have a notion of dimension for M and the M_i, we can impose a more restrictive condition:

Definition 2.3.3. A graded stratification of a topological space M by a poset $\mathcal{P}$ is a stratification $s : M^{strat} \to \mathcal{P}$ by $\mathcal{P}$ such that the M_i are equidimensional and $\rho : \mathcal{P} \to \mathbb{N}, p \to \dim(s^{-1}(p))$ is a rank function for $\mathcal{P}$.

In this case we will also say that M is *stratified by the graded poset* $(\mathcal{P}, \rho)$ or just $\mathcal{P}$ if the grading is clear from context.

Notice that a stratification may indeed fail to be graded, namely if the closure of a stratum contains only strata of codimension two or more.

We will call a stratification *algebraic*, if M and the M_i are algebraic varieties and the M_i are irreducible. In this case, we will view M and the M_i endowed with the Zariski topology, thus in particular the M_i are by definition locally closed iff they are quasi-projective.

2.3.2. The stratifications of $\overline{M_g}$ and M_g^{trop}. The first stratification we want to recall is probably the oldest one considered, the one on the moduli space of stable genus g curves, $\overline{M_g}$. As with all the algebro-geometric moduli spaces we will consider, the stratification mainly describes the structure of the boundary of a compactification, in this case $\overline{M_g} \setminus M_g$. That is to say, the whole open subset M_g forms a single maximal dimensional stratum.

Recall that we denoted by $\mathcal{SG}_g$ the set of stable graphs of genus g.

DEFINITION 2.3.4. We define a partial order on $\mathcal{SG}_g$ by setting $G \leq H$ if for some $S \subset E(G)$, contracting S in G gives H. That is we have a contraction map $\gamma : G \to H$.

One easily checks, that this indeed is a partial order.

Note that $\mathcal{SG}_g$ is finite for any g. The unique maximal element is the graph with no edges and a single vertex, on which it has weight g. The minimal elements are stable graphs with valency three and weight zero at each vertex.

Next we define a rank function on $\mathcal{SG}_g$ by setting $\rho_{\mathcal{SG}_g}(G) = 3g - 3 - |E(G)|$. This indeed preserves the covering relation, since if H covers G, then H has to be obtained from G by contracting a single edge.

Then we can define subsets M_G of $\overline{M_g}$ by setting

$$M_G = \{[X] \in \overline{M_g} | G_X = G\}$$

. This gives a decomposition $\overline{M_g} = \bigsqcup_{G \in \mathcal{SG}_g} M_G$.

For the following see e.g. [**18**][Thm. 4.7]

FACT 2.3.5. *The above decomposition is an algebraic stratification by the graded poset* $(\mathcal{SG}_g, \rho_{\mathcal{SG}_g})$ *where* $s : \overline{M_g}^{strat} \to \mathcal{SG}_g$ *is given by* $M_G \to G$.

Recall that the dual poset $\mathcal{P}^*$ of a poset $\mathcal{P}$ is obtained from $\mathcal{P}$ by inversing the partial order. Thus the dual poset $\mathcal{SG}_g^*$ of $\mathcal{SG}_g$ is the set of stable genus g graphs with the following partial order: $G \leq H$ if there is $S \subset E(H)$ such that contracting S in H gives a map $\gamma : H \to G$.

As the maximum of $\rho_{\mathcal{SG}_g}$ on $\mathcal{SG}_g$ is $3g - 3$, the dual rank functions $\rho^*_{\mathcal{SG}_g}$ of $\rho_{\mathcal{S}g}$ on $\mathcal{SG}_g^*$ is given as $\rho^*_{\mathcal{SG}_g}(G) = |E(G)|$.

Then we have a similar decomposition of the moduli space of tropical curves, the one we already encountered in its construction: $M_g^{trop} = \bigsqcup_{G \in \mathcal{SG}_g} M_G^{trop}$ where M_G^{trop} is the set of metrized graphs whose underlying weighted graph is G. See again [**18**][Thm. 4.7] for the analogous statement in this case:

FACT 2.3.6. *The above decomposition is a stratification by the graded poset* $(\mathcal{SG}_g^*, \rho^*_{\mathcal{SG}_g})$ *where* $s^* : (M_g^{trop})^{strat} \to \mathcal{SG}_g$ *is given by* $M_G^{trop} \to G$.

Thus we get the following diagram which seems to be characteristic for the relation between stratifications of algebro-geometric moduli spaces and their tropical counterparts:

$$\begin{tikzcd} \overline{M_g} \arrow[d, "s"'] & M_g^{trop} \arrow[d, "s^*"'] \\ (\mathcal{SG}_g, \rho_{\mathcal{SG}_g}) \arrow[r, leftrightarrow] & (\mathcal{SG}_g^*, \rho^*_{\mathcal{SG}_g}) \end{tikzcd}$$

Here the vertical arrows are stratifications by graded posets and the horizontal map is an order reversing bijection.

2.3.3. The strata of compactified Jacobians. The starting point for our investigation is twofold. First we want to recall a nice description of degree $g-1$ balanced line bundles:

FACT 2.3.7 ([**10**], Lemma 2.1). *Let X be a curve and $\underline{d} \in \mathrm{Div}^{g-1}(G)$. Then the following are equivalent:*

(1) $\underline{d}$ is balanced.
(2) $\underline{d}$ is orientable.
(3) There is $L \in \mathrm{Pic}^{\underline{d}}(X)$ with $h^0(X,L) = 0$.

The equivalence between the first two characterizations extends to the strictly balanced case:

FACT 2.3.8 ([**2**], Proposition 3.6). *A divisor $\underline{d} \in \mathrm{Div}^{g-1}(G)$ is strictly balanced, if and only if it is given by a totally cyclic orientation.*

Taking into account that in the basic inequality we have in the degree $g-1$ case $m_{g-1}(Z) = g_Z - 1$, these facts were independently known in graph theory: for the balanced case as Hakimi's theorem (originally in [**24**], for a formulation in our framework see [**4**, Theorem 4.8]) and for the strictly balanced case see [**11**, Lemma 1].

Our aim is to study how the description in terms of orientations extends to the degree g case.

REMARK/PROBLEM 2.3.9. It would of course also be interesting to know, whether the description of being balanced as the existence of L with $h^0(X,L) = 0$ can be extended. In the degree g case, every line bundle L has at least $h^0(X,L) = 1$ by Riemann-Roch. This however is not sufficient to characterize balanced line bundles. We will not pursue this further, but touch about related questions in the last chapter.

The second point of departure is the decomposition

$$\overline{P}^g_X = \bigsqcup P_X^{\underline{d}_S}$$

which follows from the modular interpretation of $\overline{P}^g_g$ given in [**14**]. Here $P_X^{\underline{d}_S}$ denotes the subset of stably balanced line bundles defined on $\hat{X}_S$ whose restriction to X_S^ν has multidegree $\underline{d}_S$. Equivalently, stably balanced line bundles defined on $\hat{X}_S$ whose degree is $\hat{\underline{d}}_S$. One of the aims of this text is to give a combinatorial indexing set for this decomposition and show that it is a stratification.

We can extend the decomposition to the compactified universal Jacobian $\overline{P}^g_g = \bigcup P^{\underline{d}_S}_G$ by letting $P^{\underline{d}_S}_G$ be the set of line bundles defined on $\hat{X}_S$ whose restriction to X^ν_S has multidegree $\underline{d}_S$ for some curve X with dual graph G. Things become more complicated here, as the decomposition no longer is disjoint: automorphisms of the curve that induce a non-trivial automorphism of the dual graph will identify strata over that curve (recall that the fiber of $\overline{P}^g_g$ over $[X] \in \overline{M}_g$ is $\overline{P}^g_X/\mathrm{Aut}(X)$).

We recall an important result in this direction obtained in **[14]** reformulated in the language used here, which gives a description in the case of a fixed curve:

FACT 2.3.10 (**[14]**, Proposition 5.1). *Let $P^{\underline{d}_S}_X$ and $P^{\underline{e}_T}_X$ be two strata of $\overline{P}^d_X$. Then $P^{\underline{d}_S} \subset \overline{P^{\underline{e}_T}}$ if and only if $T \subset S$ and the edges in $S \setminus T$ can be oriented so that, denoting by t_v the number of edges in $S \setminus T$ with target a vertex v, we have*

$$(\underline{e}_T)_v = (\underline{d}_S)_v + t_v.$$

CHAPTER 3

The stratifications of $\overline{P}^g_X$ and $\overline{P}^g_g$

3.1. Break divisors are balanced multidegrees

We need to collect some observations about the basic inequality first, before we can proceed to consider orientations. First recall that by Lemma 6.3 of [**14**], if $gcd(d-g+1, 2g-2)=1$, then $m_d(Z)\notin\mathbb{Z}$ for every $Z\subset V(G)$. This in particular is always the case if $d=g$. Recall furthermore that $m_{g-1}(Z)=g_Z-1\in\mathbb{Z}$.

LEMMA 3.1.1. *Let X be a quasistable curve and $Z\subset V(G)$, such that Z^c is not a union of exceptional components. Then $m_{d-1}(Z)<m_d(Z)<m_{d-1}(Z)+1$.*
If $d=g$, the claim also holds if Z^c is a union of exceptional components.

PROOF. The first inequality is clear, so what remains to show is that $m_d(Z)<m_{d-1}(Z)+1$. Assume that $m_d(Z)\geq m_{d-1}(Z)+1$. We thus need to have:

$$\begin{aligned}
& (|\underline{\omega}_Z|d)/(2g-2)-\delta_Z/2 \geq (|\underline{\omega}_Z|(d-1))/(2g-2)-\delta_Z/2+1\\
\Leftrightarrow\quad & |\underline{\omega}_Z|/(2g-2)\geq 1\\
\Leftrightarrow\quad & |\underline{\omega}_Z|\geq 2g-2\\
\Leftrightarrow\quad & |\underline{\omega}_Z|\geq |\underline{\omega}_Z|+|\underline{\omega}_{Z^c}|
\end{aligned}$$

Now since X is connected and quasistable, the only possibility to have $|\underline{\omega}_{Z^c}|=2g_{Z^c}-2+\delta_{Z^c}=0$ is if Z^c is the union of exceptional components.

In case $d=g$, because $m_g(Z)\notin\mathbb{Z}$ and $m_{g-1}(Z)\in\mathbb{Z}$, we can assume $m_g(Z)>m_{g-1}(Z)+1$ and thus replace all the inequalities in the above argument by strict inequalities. ☐

In particular for $d=g$, every balanced line bundle of degree g satisfies the inequalities for being (not necessarily strictly) balanced of degree $g-1$. Furthermore $\underline{d}$ is balanced if and only if $g_Z-1<|\underline{d}_Z|$ for every $Z\subset V(G)$.

LEMMA 3.1.2. *Given a stably balanced multidegree $\underline{d}$ of total degree g on a quasistable curve X, the divisor $\underline{d}'=\underline{d}-(v)$ is balanced of degree $g-1$.*

PROOF. For any subcurve $Z\subset V(G)$ such that $v\in Z$, we have $|\underline{d}'_Z|=|\underline{d}_Z|-1\geq m_g-1>m_{g-1}-1$. Since m_{g-1} is integer, this gives $|\underline{d}'_Z|\geq m_{g-1}$. If $v\notin Z$, i.e. $|\underline{d}'_Z|=|\underline{d}_Z|$, we have $|\underline{d}'_Z|=|\underline{d}_Z|\geq m_g>m_{g-1}$.

☐

Lemma 3.1.2 does not hold true for arbitrary degrees. The following example gives a balanced multidegree, such that decreasing it by one on any vertex gives a divisor that is not balanced.

EXAMPLE 3.1.3. Let G be a tree on three vertices of weights g_1, g_2 and g_3 where the vertex v_2 has valency two and the other two valency one. Then the multidegree $\underline{d} = (g_1, g_2 - 1, g_3)$ is balanced of degree $g - 1 = g_1 + g_2 + g_3 - 1$. None of the multidegrees $(g_1 - 1, g_2 - 1, g_3)$, $(g_1, g_2 - 2, g_3)$ and $(g_1, g_2 - 1, g_3 - 1)$ however are balanced of degree $g - 2$.

In the above example there is a linearly equivalent balanced multidegree $\underline{d}' \sim \underline{d}$ on which we can decrease the degree on some vertices, e.g. $\underline{d}' = (g_1 - 1, g_2 + 1, g_3 - 1)$. The following result shows that in case of $d = g - 1$ this is always the case:

LEMMA 3.1.4. *Given a balanced multidegree $\underline{d}$ of total degree $g - 1$ on a quasistable curve X and a component C_v of X, there exists a unique balanced multidegree $\underline{d}'$, such that $\underline{d} \sim \underline{d}'$ and $\underline{d}' - (v)$ is balanced of degree $g - 2$. In particular, if $\underline{d}$ is strictly balanced, $\underline{d} - (v)$ itself is balanced of degree $g - 2$.*

PROOF. Suppose $|(\underline{d} - (v))_Z| < m_{g-2}(Z)$ for some $Z \in V(G)$. Since $|\underline{d}_Z| - 1 < m_{g-2} < m_{g-1}$ and $m_{g-1} \in \mathbb{Z}$, we actually need to have $|\underline{d}_Z| = m_{g-1}$ and $v \in Z$. As $\underline{d}$ is balanced, it is given by an orientation and a straightforward calculation shows, that since $|\underline{d}_Z| = m_{g-1}$, (Z, Z^c) is a directed cut oriented away from Z. Then there exists a unique divisor $\underline{d}' \sim \underline{d}$ such that $\underline{d}'$ is given by an orientation O, in which v is reachable from every other vertex. Indeed, v being reachable from every other vertex is equivalent to $\underline{\omega_G} - \underline{d}'$ being v-connected, thus the claim follows from [**4**] Theorem 1.2. As $\underline{d}'$ is orientable, it is balanced of degree $g - 1$ and no directed cut can be oriented away from the component containing v. Thus $\underline{d}' - (v)$ is balanced.

The second claim follows from the fact that a strictly balanced divisor is the unique balanced representative in its degree class.

□

LEMMA 3.1.5. *Given a balanced multidegree $\underline{d}$ of total degree $g - 1$ on a quasistable curve X and a component C_v of X, there exists a unique balanced multidegree $\underline{d}'$, such that $\underline{d} \sim \underline{d}'$ and $\underline{d}' + (v)$ is balanced of degree g. In particular, if $\underline{d}$ is strictly balanced, $\underline{d} + (v)$ itself is balanced of degree g.*

PROOF. A divisor $\underline{d}$ is balanced if and only if $\underline{\omega_G} - \underline{d}$ is balanced. Thus the lemma follows from the previous one.

□

We are now ready to express being balanced of degree g in terms of orientations. First, as a consequence of lemma 3.1.2, we have:

LEMMA 3.1.6. *Let $\underline{d}$ be a balanced multidegree on a quasistable curve X of total degree g. Then $\underline{d}$ is orientable, i.e. there is a 1-orientation O on G_X such that $\underline{d} = \underline{d}^O$.*

PROOF. Fix a vertex v. By Lemma 3.1.2, the multidegree $\underline{d}' = \underline{d} - (v)$ is balanced. Thus we have $\underline{d}' = \underline{d}^{O'}$ for some 0-orientation O'. Since $\underline{d}_v \leq g(v) - 1 + \delta_v$ by Lemma 3.1.1 and hence $\underline{d}'_v \leq g(v) - 2 + \delta_v$, there is always an outgoing edge e in O' at v. Biorienting e gives an orientation O as desired.

□

REMARK 3.1.7. The choices of v and e determine the orientation we obtain. Since two orientations obtained for different such choices give the same divisor, they are equivalent under cycle reversals and edge pivots.

LEMMA 3.1.8. *Let $\underline{d}$ be a balanced multidegree with $|\underline{d}| = g$ and given by some 1-orientation O. Let (Z, Z^c) be a directed cut in O, oriented away from Z. Then the bioriented edge e is contained in $[Z]$.*

PROOF. Suppose $e \in [Z^c]$. Then $|\underline{d}_Z| = g_Z - 1$ since $O_{|Z}$ gives a 0-orientation on $[Z]$ and there are no edges directed towards Z in (Z, Z^c). On the other hand, $m_{g-1}(Z) = g_Z - 1$ and thus $|\underline{d}_Z| = m_{g-1}(Z) < m_g(Z)$. Thus we get a contradiction to $\underline{d}$ being balanced.

□

Opposed to the case of $d = g - 1$, not every 1-orientation of degree g gives a balanced divisor.

PROPOSITION 3.1.9. *Let $\underline{d} \in \mathrm{Div}^g(G)$. Then the following are equivalent:*

(1) $\underline{d}$ is balanced.
(2) $\underline{d}$ is given by a rooted 1-orientation.
(3) $\underline{d}$ is a break divisor.

PROOF. (1) $\Rightarrow$ (2): Suppose $\underline{d}$ is balanced. By Lemma 3.1.6 it can be given by an orientation O with a unique bioriented edge e and by Lemma 3.1.8 O satisfies the characterization of being rooted given in Lemma 1.2.8 (2).

(2) $\Rightarrow$ (1): If we are given a rooted 1-orientation O its associated divisor $\underline{d}^O$ will be balanced by Lemma 3.1.1 and Lemma 1.2.19 (3).

We already established (2) $\Leftrightarrow$ (3) as a consequence of [**4**] in Corollary 1.2.21.

□

Thus Theorems 1.2 and 1.3 of [**4**] and the previous proposition reprove the fact shown in Proposition 4.1 of [**14**] that in degree g there exists a unique balanced representative in each degree class:

COROLLARY 3.1.10. *Let $\underline{d} \in \mathrm{Div}^g(G)$. Then $\exists! \underline{d}'$ such that $\underline{d} \sim \underline{d}'$ and $\underline{d}'$ balanced.*

The above considerations have implications for different degree g compactified Jacobians, namely those constructed by Simpson in [**29**]. They depend on the additional datum of an ample line bundle L on X and taking L to be the dualizing sheaf of X gives the compactification we are considering. We will denote them by $\overline{P}_{X,L}^d$. One replaces the basic inequality by the following inequality for every $Z \subset V(G)$:

$$\frac{|\omega_Z|}{2} + \frac{|\underline{\deg}(L)_Z|}{|\underline{\deg}(L)|}(d-g+1) - \delta_Z/2 \leq |\underline{d}_Z|.$$

Then $\overline{P}_{X,L}^d$ parametrizes stably balanced line bundles on curves $\hat{X}_S$ with respect to this inequality and up to equivalence defined as before. It is immediate from this inequality, that for $d = g-1$ the space $\overline{P}_{X,L}^{g-1}$ does not depend on L and thus there in particular is a natural isomorphism $\overline{P}_{X,L}^{g-1} \cong \overline{P}_X^{g-1}$. Denoting by $m_{d,L}(Z) = \frac{|\omega_Z|}{2} + \frac{|\underline{\deg}(L)_Z|}{|\underline{\deg}(L)|}(d-g+1) - \delta_Z/2$ we still have the result of Lemma 3.1.1:

LEMMA 3.1.11. *Let X be a curve and $Z \subset V(G)$ with $Z \neq V(G)$. Then $m_{d-1,L}(Z) < m_{d,L}(Z) < m_{d-1,L}(Z) + 1$.*

PROOF. Since L is ample, we have $\underline{\deg}(L) > \underline{0}$. The first part of the inequality follows from

$$|\underline{\deg}(L)_Z|/|\underline{\deg}(L)| > 0.$$

But we also get

$$m_{d,L}(Z) - m_{d-1,L}(Z) = |\underline{\deg}(L)_Z|/|\underline{\deg}(L)|,$$

for which we have by $\underline{\deg}(L) > \underline{0}$ and $Z \neq V(G)$ that $|\underline{\deg}(L)_Z|/|\underline{\deg}(L)| < 1$.

□

PROPOSITION 3.1.12. *For any curve X and ample line bundle L on X we have $\overline{P}_{X,L}^g \cong \overline{P}_X^g$*

PROOF. Note that to prove that stable divisors are break divisors, we only used Lemma 3.1.1 and the definition of $\overline{P}_X^{g-1}$. Thus one can use exactly the same arguments to show that a degree g divisor is balanced with respect to L iff it is given by a rooted 1-orientation iff it is a break divisor.

□

As we remarked above, there are however other known compactifications in both degree $g-1$ and g (we again refer to **[26]** for a detailed discussion).

The considerations of this section readily translate to a description of balanced degree $g-2$ divisors in terms of orientations by taking the residual with respect to the dualizing sheaf. One obtains that a divisor is balanced of degree $g-2$ iff it is given by a (-1)-orientation such that every cut contains an edge directed towards the component containing the unoriented edge. Here by (-1)-orientation we mean an orientation that has a single unoriented edge. We will not spell out the corresponding statements for this case, but the combinatorics work analogously to the degree g case.

REMARK/PROBLEM 3.1.13. Thus all balanced divisors in degree $g-2, g-1$ and g are given by generalized orientations (in the more broad sense of **[7]**, i.e. allowing in the $g-2$ case unoriented edges). It is easy to find balanced divisors of degree $g-3$, that have $\underline{d}_v < -1$ and thus cannot be given as divisors associated to an orientation. Is there a way to find a similar combinatorial description also in these cases? Or if not, is there a conceptual reason, why $g-2, g-1$ and g allow for these descriptions?

3.2. The stratification of $\overline{P}^g_X$

Recall that we have a decomposition $\overline{P}^g_X = \bigsqcup P^{\underline{d}_S}_X$ where $\underline{d}_S$ is a strictly balanced divisor on $G - S$. Note that since $|\hat{\underline{d}}_S| = g(G)$, we have $|\underline{d}_S| = g(G) - |S| = g(G-S)$. Thus in particular every balanced divisor $\underline{d}_S$ will be strictly balanced. The first step to define a poset consists in identifying those S, for which there actually exist balanced divisors on $G - S$:

LEMMA 3.2.1. *There exists a balanced* 1*-orientation on a graph G iff G is connected.*

PROOF. This follows from Lemma 1.1.17 and Corollary 1.2.21.

□

DEFINITION 3.2.2. Let G be connected. We set

$$\mathcal{C}(G) = \{S \subset E(G) | S \text{ not disconnecting}\}.$$

Equivalently one may view $\mathcal{C}(G)$ as the set of connected spanning subgraphs of G. We endow $\mathcal{C}(G)$ with a partial order, given by reverse inclusion: $S \leq T$ if $T \subset S$. Again, equivalently the partial order is the one on spanning connected subgraphs by inclusion.

We will adopt the following convention: An orientation that has as subscript a subset $S \subset E(G)$ will be defined on $G - S$. If there is no subscript, it will be defined on all of G. Furthermore if $\underline{d}_S = \underline{d}^{O_S}$ for a balanced divisor $\underline{d}_S$ on $G - S$, we will write $P^{O_S}_X$ for the stratum $P^{\underline{d}_S}_X$ of $\overline{P}^g_X$.

Recall that we defined $\mathcal{O}^1(G)$ to be the set of rooted 1-orientations on G and $\overline{\mathcal{O}}^1(G)$ to be the set of classes of rooted 1-orientations on G up to cycle reversals and edge pivots (i.e. giving the same divisor).

We introduce, for a fixed graph G, the set of all rooted 1-orientations on all spanning connected subgraphs of G.

$$\mathcal{OP}^1(G) := \bigsqcup_{S \in \mathcal{C}(G)} \mathcal{O}^1(G-S).$$

We define a poset structure on $\mathcal{OP}^1(G)$ by setting:

DEFINITION 3.2.3. Let G be a graph and let $S, T \in \mathcal{C}(G)$. Given two rooted 1-orientations O_S on $G - S$ and O_T on $G - T$ we set

$$O_S \leq O_T \quad \text{if} \quad S \leq T \quad \text{and} \quad (O_T)_{|G-S} = O_S.$$

It is easy to check that this is a partial order.

Finally, we consider orientations up to equivalence:

$$\overline{\mathcal{OP}}^1(G) := \bigsqcup_{S \in \mathcal{C}(G)} \overline{\mathcal{O}}^1(G-S).$$

Our next goal is to define a poset structure on $\overline{\mathcal{OP}}^1(G)$ modelled after fact 2.3.10.

LEMMA 3.2.4. *Let $S, T \in \mathcal{C}(G)$ with $T \subset S$. Let $O_S \in \mathcal{O}^1(G-S)$. Then there exists $O_T \in \mathcal{O}^1(G-T)$ such that $O_T \geq O_S$.*

Moreover, if $O_S \sim_{cyc} O'_S$ for some $O'_S \in \mathcal{O}^1(G-S)$, there exists $O'_T \in \mathcal{O}^1(G-T)$ such that $O'_T \geq O'_S$ and $O'_T \sim_{cyc} O_T$.

PROOF. Up to replacing G with $G-T$ we can assume $T = \emptyset$. We may view O_S as a partial orientation of G and have to give an orientation to the edges in S. We claim, that any choice will give a rooted 1-orientation. Indeed, every arborescence of O_S will still be an arborescence on G, thus the orientation obtained by orienting edges in S in any way contains an arborescence and thus is rooted by Lemma 1.2.24.

The second claim follows, since every cycle reversal and edge pivot of O_S on $G-S$ can be done on G as well. □

REMARK 3.2.5. Note that in the other direction, not every cycle reversal on an orientation O of G can be performed on the orientation $O_{|G-S}$ on $G-S$. In particular if O and O' induce the same divisor on G, $O_{|G-S}$ and $O'_{|G-S}$ may induce different divisors on $G-S$.

DEFINITION/PROPOSITION 3.2.6. The set $\overline{\mathcal{OP}}^1(G)$ is partially ordered with respect to the following relation. For $\overline{O}_S$ and $\overline{O}_T$ we set $\overline{O}_S \leq \overline{O}_T$ if $S \leq T$ and if one of the two equivalent conditions below holds.

(i) There exist $O'_S \in \overline{O}_S$ and $O'_T \in \overline{O}_T$ such that $(O'_T)_{|G-S} = O'_S$.
(ii) For every $O'_S \in \overline{O}_S$ there exists $O'_T \in \overline{O}_T$ such that $(O'_T)_{|G-S} = O'_S$.

Moreover, the forgetful map sending $\overline{O}_S$ to S,

$$\overline{\mathcal{OP}}^1(G) \to \mathcal{C}(G)$$

is a quotient of posets, and the following

$$\rho_{\overline{\mathcal{OP}}^1(G)}(\overline{O}_S) \to g(G-S) = g(G) - |S|$$

is a rank function on $\overline{\mathcal{OP}}^1(G)$.

PROOF. Lemma 3.2.4 yields that (i) implies (ii), and the converse is obvious. Now, condition (ii) ensures that we have a quotient. The two forgetful maps are onto by 3.2.1, and are quotients by Lemma 3.2.4. The rest of the statement is clear. □

REMARK 3.2.7. If $\overline{O}_S \leq \overline{O}_T$ then $\underline{d}^{O_S} \leq \underline{d}^{O_T}$, but the converse is not true. See Figure 1, where all vertices have weight 1, $T = \emptyset$ and S consists of the bottom edge on the right of the first graph.

$\underline{d}^{O_T}$ $\underline{d}^{O_S}$

FIGURE 1. $\underline{d}^{O_S} \leq \underline{d}^{O_T}$ but $\overline{O}_S \not\leq \overline{O}_T$

We assemble the observations, we made so far:

THEOREM 3.2.8. *Let X be a stable curve of genus g and G its dual graph. Then the following is a graded algebraic stratification of $\overline{P}^g_X$ by $\overline{\mathcal{OP}}^1(G)$*

$$\overline{P}^g_X = \bigsqcup_{\overline{O_S} \in \overline{\mathcal{OP}}^1(G)} P_X^{O_S}, \tag{3}$$

and we have natural isomorphisms for every $\overline{O_S} \in \overline{\mathcal{OP}}^1(G)$

$$P_X^{O_S} \cong \operatorname{Pic}^{\underline{d}^{O_S}}(X_S^\nu). \tag{4}$$

PROOF. As before we denote by $P_X^{\underline{d}_S} \subset \overline{P}^g_X$ the set of equivalence classes of stably balanced line bundles on $\hat{X}_S$ whose restriction to X_S^ν has degree $\underline{d}_S$, for $\underline{d}_S$ a stable divisor on $G - S$. Then, as already mentioned, by [**14**] we have

$$\overline{P}^g_X = \bigsqcup_{\substack{S \subset E \\ \underline{d}_S \in \Sigma(G-S)}} P_X^{\underline{d}_S}. \tag{5}$$

Now, as noted above, we have $P_X^{O_S} = P_X^{\underline{d}_S}$ for a unique class $\overline{O_S} \in \mathcal{O}^b(G - S)$ such that $\underline{d}_S = \underline{d}^{O_S}$. Moreover, every $\underline{d}_S \in \Sigma^b(G - S)$ is obtained in this way, for every $S \subset E$. Hence (5) yields (3).

Also, we have $P_X^{\underline{d}_S} \cong \operatorname{Pic}^{\underline{d}_S}(X_S^\nu)$, which implies (4). From this it immediately follows that the $P_X^{O_S}$ are irreducible and quasiprojective.

Next, we have to show the following

$$P_X^{O_S} \subset \overline{P_X^{O_T}} \quad \Leftrightarrow \quad \overline{O_S} \leq \overline{O_T}.$$

Recall that by Fact 2.3.10 we have $P_X^{\underline{d}_S} \subset \overline{P_X^{\underline{d}_T}}$ if and only if $T \subset S$ and the edges in $S \setminus T$ can be oriented so that, denoting by t_v the number of edges in $S \setminus T$ with target a vertex v, we have

$$(\underline{d}_T)_v = (\underline{d}_S)_v + t_v.$$

Assume $P_X^{\underline{d}_S} \subset \overline{P_X^{\underline{d}_T}}$ and denote by O'_T the orientation on $G-T$ which extends O_S to $S \setminus T$ by the orientation we just defined (where $\overline{O}_S \in \overline{\mathcal{O}}(G-S)$ is such that $\underline{d}^{O_S} = \underline{d}_S$, by the previous part). Of course $O_S \leq O'_T$ and, as $\underline{d}^{O_T} = \underline{d}_T$ for some $\overline{O}_T \in \overline{\mathcal{O}}(G-T)$, we have

$$\underline{d}^{O_T} = \underline{d}_T = \underline{d}^{O'_T},$$

hence $O'_T \sim O_T$. We conclude that $\overline{O_S} \leq \overline{O_T}$. The converse follows directly from [**14**, Prop. 5.1] as stated above and the definition of $\overline{O_S} \leq \overline{O_T}$.

Finally, we need to show the stratification (5) is graded. Recall that the generalized Jacobian of X_S^ν is an irreducible variety of dimension $g(G-S)$, hence so is $\mathrm{Pic}^{\underline{d}_S}(X_S^\nu)$, hence so is $P_X^{O_S}$. Recalling that

$$\overline{\mathcal{OP}}^1(G) \to \mathbb{N}; \qquad \overline{O_S} \mapsto g(G-S)$$

is a rank function for $\overline{\mathcal{OP}}^1(G)$, we are done. $\square$

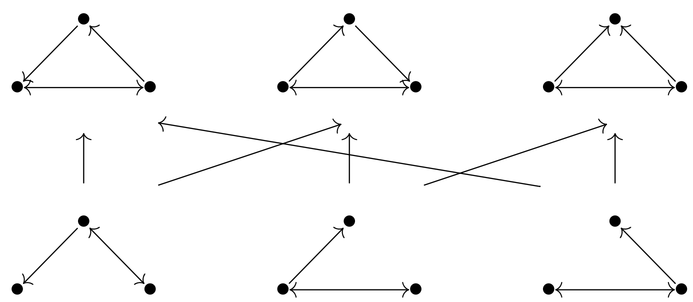


FIGURE 2. The strata of $\overline{P}^g_X$ where G_X is a cycle on three vertices. Each row lists the strata of fixed codimension (zero and one) and vertical arrows depict containment relations.

A somewhat different question is to explicitly describe the strata contained in the closure of a given stratum. The following two examples illustrate the problem:

EXAMPLE 3.2.9. Let G be oriented by O as in Figure 3 with edges e_1, e_2 and e_3. Up to equivalence there are two rooted 1-orientations on G and O is one of them. Now removing e_2 does not give a rooted 1-orientation, as the newly created directed cut points towards the bioriented edge. However Replacing O by O'' by a cycle reversal and then removing the edge e_2 gives a rooted 1-orientation $O'_{\{e_2\}}$. Thus the stratum corresponding to $\overline{O}$ contains the stratum corresponding to $\overline{O}'_{\{e_2\}}$.

EXAMPLE 3.2.10. Let G be oriented by O as in Figure 4, a rooted 1-orientation. Then removing e_2 does not give a rooted 1-orientation and there is no representative in $\overline{O}$ for which it does. Thus the stratum corresponding to $\overline{O}$ does not contain the stratum corresponding to $\overline{O}'_{\{e_2\}}$.

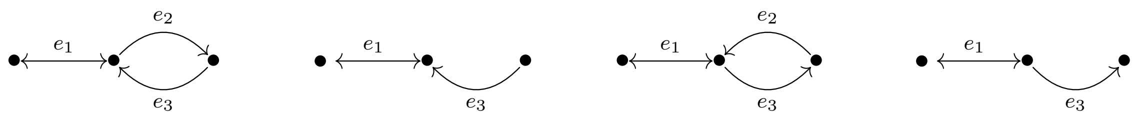


FIGURE 3. (G, O), $(G - e_2, O_{|G-e_2})$, (G, O'') and $(G - e_2, O'_{\{e_2\}})$.

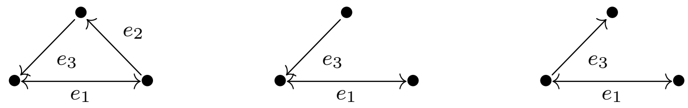


FIGURE 4. (G, O), $(G - e_2, O_{|G-e_2})$ and $(G - e_2, O'_{\{e_2\}})$

We will answer the question in a very special case that will be used later on and then give a necessary and sufficient condition in terms of arborescences:

LEMMA 3.2.11. *Suppose O and O' are two 1-orientations on G with $O \neq O'$, $O_{|G-e} = O'_{|G-e}$ and e is not bioriented in either O or O' for some $e \in E(G)$. Then if both O and O' are rooted, so is their induced orientation on $G - e$.*

PROOF. Let $Z \subset V(G)$. We may view (Z, Z^c) as a cut of both $G - e$ and G. By Lemma 1.2.8 (2) we need to show that if Z contains the bioriented edge, then (Z, Z^c) is not a cut directed towards $[Z]$ in $O_{|G-e}$. Thus assume (Z, Z^c) is a directed cut in $O_{|G-e}$. Now viewing (Z, Z^c) as a cut in G we have two cases: If $e \notin (Z, Z^c)$ it still will be a directed cut in both O and O' and thus by assumption directed away from $[Z]$. If $e \in (Z, Z^c)$, it is a directed cut in either O or O', say O. Thus in O it is directed away from $[Z]$. In both cases we get that (Z, Z^c) is directed away from $[Z]$ in $O_{|G-e}$.

□

Recall that if O is rooted, we denoted by $T(O)$ the set of arborescences of O.

LEMMA 3.2.12. *Let O be a rooted 1-orientation on G and $e \in E(G)$. Then the induced orientation $O_{|G-e}$ on $G - e$ is rooted if and only if there is $T \in T(O)$ with $e \notin T$.*

PROOF. If such a T exists, it will also be an arborescence of $O_{|G-e}$, which implies $O_{|G-e}$ being rooted by Lemma 1.2.24.

Conversely, if $O_{|G-e}$ is rooted, again by Lemma 1.2.24, it contains an arborescence T which in turn will also be an arborescence of O with $e \notin T$. □

3.3. Extending the poset to the universal setting

DEFINITION 3.3.1. For any $\gamma : G \to H = G/S_0$ and any $S \in \mathcal{C}(G)$ set

$$\gamma_* S := S \setminus S_0.$$

We clearly have that $\gamma_* S \in \mathcal{C}(H)$ and if $S' \in \mathcal{C}(G)$ is such that $S \leq S'$, then $\gamma_* S \leq \gamma_* S'$.

Let $\gamma : G \to H = G/S_0$ be a contraction. For $S \subset E$ and a generalized orientation O_S on $G - S$ we can extend O_S to a generalized orientation O on G by orienting edges in S arbitrarily. Then we set

$$\gamma_* O_S := (\gamma_* O)_{|H - \gamma_* S}. \tag{6}$$

Since $E(H - \gamma_* S) = E(G - (S \cup S_0)) \subset E(G - S)$, this definition does not depend on the extension O chosen. Note that while γ induces a contraction $\gamma' : G - S \to (G - S)/\gamma_* S$, its image $(G - S)/\gamma_* S$ may not be a subgraph of H, as γ may contract edges of G not contained in S and in this way identify vertices that are distinct in $(G - S)/\gamma_* S$. There is however always a canonical bijection $E(H - \gamma_* S) \to E(G - S) \setminus S_0$ and $\gamma_* O_S$ orients edges of $H - \gamma^* S$ as they are oriented by O_S under this map.

As a final piece of notation, to γ and $S \subset E$ we associate the divisor $\underline{c}^{\gamma, S}$ on H such that for any $v \in V(H)$

$$\underline{c}_v^{\gamma, S} := |\{e \in S_0 \cap S : \ \gamma(e) = v\}|. \tag{7}$$

If $S = E(G)$ we write $\underline{c}^{\gamma} = \underline{c}^{\gamma, E(G)}$. Of course, $\underline{c}_v^{\gamma, S} \geq 0$ and equality holds if and only if $S \cap S_0 = \emptyset$.

PROPOSITION 3.3.2. *Let G be a graph, $S \subset E(G)$, and O_S a 1-orientation on $G - S$. Let $\gamma : G \to H = G/S_0$ be a contraction such that the bioriented edge of O_S is not in S_0.*

Then $\gamma_* O_S$ is a 1-orientation on $H - \gamma_* S$ and the following hold.

(a) If $O_S \in \mathcal{O}^1(G - S)$ then $\gamma_* O_S \in \mathcal{O}^1(H - \gamma_* S)$.
(b) $\gamma_* \underline{d}^{O_S} = \underline{d}^{\gamma_* O_S} - \underline{c}^{\gamma, S}$.
(c) Let O'_S be a 1-orientation on $G - S$ such that the bioriented edge of O'_S is not contained in S_0. If $O'_S \sim_{cyc} O_S$ then $\gamma_* O_S \sim_{cyc} \gamma_* O'_S$.
(d) Let O_T be a 1-orientation on $G - T$. If $O_S \leq O_T$ then $\gamma_* O_S \leq \gamma_* O_T$.

PROOF. As the bioriented edge is not contained in S_0, $\gamma_* O_S$ is a 1-orientation on $H - \gamma_* S$.

(a). We need to show $\gamma_* O_S$ is rooted. Every directed cut of $H - \gamma_* S$ is a directed cut of $G - S$ under the inclusion $E(H - \gamma_*) \to E(G - S)$, from which this claim easily follows.

(b). For any $v \in V(H)$ set $Z_v = \gamma^{-1}(v)$, which is a connected subgraph of G. We have $g(Z_v) = \sum_{z \in V(Z_v)} \big(g(z) - 1\big) + |E(Z_v)| + 1$, hence

$$(\gamma_* \underline{d}^{O_S})_v = \sum_{z \in V(Z_v)} \big(g(z) - 1 + \underline{t}_z^{O_S}\big) = g(Z_v) - 1 - |E(Z_v)| + \sum_{z \in V(Z_v)} \underline{t}_z^{O_S}.$$

Let $t^{O_S}(Z_v)$ be the number of edges with target in Z_v and not contained in it. As every edge of Z_v lies in S_0,

$$|E(Z_v)| = \sum_{z \in V(Z_v)} \underline{t}_z^{O_S} - t^{O_S}(Z_v) + \underline{c}_v^{\gamma, S}.$$

Therefore

$$(\gamma_*\underline{d}^{O_S})_v = g(Z_v) - 1 + t^{O_S}(Z_v) - \underline{c}_v^{\gamma,S}. \tag{8}$$

On the other hand we have

$$(\underline{d}^{\gamma_* O_S})_v = g(v) - 1 + \underline{t}_v^{\gamma_* O_S} = g(Z_v) - 1 + t^{O_S}(Z_v) \tag{9}$$

Indeed, by the definition of (weighted) contractions, $g(v) = g(Z_v)$ and, clearly, the number of O_S-incoming edges at Z_v equals the number of γ_*O_S-incoming edges at v. Comparing (8) and (9) yields (b).

(c). By hypothesis, $\underline{d}^{O_S} = \underline{d}^{O'_S}$, hence $\underline{t}^{O_S} = \underline{t}^{O'_S}$. As the subgraphs Z_v do not contain the bioriented edge of either O_S or O'_S, for any $v \in V(H)$ we have $t^{O_S}(Z_v) = \Sigma_{v' \in Z_v} \underline{t}_{v'}^{O_S} - |E(Z_v)| = \Sigma_{v' \in Z_v} \underline{t}_{v'}^{O'_S} - |E(Z_v)| = t^{O'_S}(Z_v)$. Combining with (9) we get $\underline{d}^{\gamma_* O_S} = \underline{d}^{\gamma_* O'_S}$, and we are done.

(d). By assumption we have $S \leq T$ and $(O_T)_{|G-S} = O_S$. We obviously have $\gamma_* S \leq \gamma_* T$. Next, as $H - \gamma_* S \subset H - \gamma_* T$

$$(\gamma_* O_T)_{|H-\gamma_* S} = (O_T)_{|H-\gamma_* S} = (O_{T\ |G-S})_{|H-\gamma_* S} = O_{S\ |H-\gamma_* S} = \gamma_* O_S.$$

□

EXAMPLE 3.3.3. In the picture we have $S = S_0 = \{e\}$.

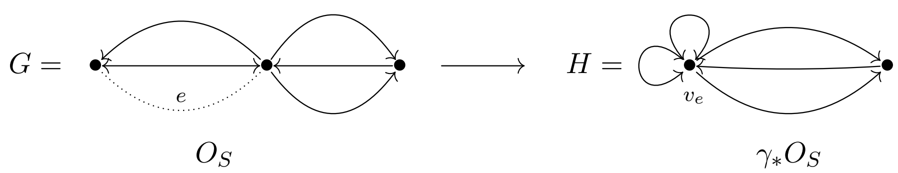


FIGURE 5. Case $S = S_0$

Assume all vertices of G have weight 1, so that v_e has weight 2 in H. We have, ordering the vertices from left to right,

$$\underline{t}^{O_S} = \underline{d}^{O_S} = (2,2,2), \qquad \underline{t}^{\gamma_* O_S} = (4,2), \qquad \underline{d}^{\gamma_* O_S} = (5,2),$$

and

$$\gamma_* \underline{d}^{O_S} = (4,2).$$

In particular $\underline{d}^{\gamma_* O_S} > \gamma_* \underline{d}^{O_S}$.

From the previous result we immediately get:

COROLLARY 3.3.4. *Let $\gamma : G \to H = G/S_0$ be a contraction with $S_0 \neq E(G)$.*

Then we have a morphism of posets

$$\overline{\gamma}_* : \overline{\mathcal{OP}}^1(G) \to \overline{\mathcal{OP}}^1(H)$$

defined as follows: For $\overline{O}_S \in \overline{\mathcal{OP}}^1$ by Lemma 1.2.19 there is $O'_S \sim_{cyc} O_S$ such that the bioriented edge of O'_S is not contained in S_0. Set $\overline{\gamma_*}(\overline{O}_S) = \overline{\gamma_* O'_S}$.

If $S_0 = E(G)$ we set the image of $\overline{\gamma}_*$ to be the trivial orientation, and thus also in this case $\overline{\gamma}_*$ trivially is a morphism of posets.

We are now ready to define the poset of rooted 1-orientations on connected spanning subgraphs of genus g graphs.

Definition 3.3.5. Set

$$\overline{\mathcal{OP}}^1_g := \{(G, \overline{O}_S) : \ G \in \mathcal{SG}_g, \ \overline{O}_S \in \overline{\mathcal{O}}^1(G-S)\}.$$

Let $(H, \overline{O}_T), (G, \overline{O}_S) \in \overline{\mathcal{OP}}^1_g$; we set $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ if $G \leq H$ in $\mathcal{SG}_g$ and if there exists a contraction $\gamma : G \to H$ such that $\overline{\gamma}_* \overline{O}_S \leq \overline{O}_T$ in $\overline{\mathcal{OP}}^1(H)$.

Recall that $\overline{\gamma}_* \overline{O}_S \leq \overline{O}_T$ implies in particular $\gamma_* S \leq T$. Hence $H - \gamma_* S \subset H - T$ and $O'_T \in \overline{O}_T$ can be restricted to $H - \gamma_* S$. By Definition 3.2.3, we require there is $O'_T \in \overline{O}_T$ such that this restriction is equal to $O_S \in \overline{\gamma}_* \overline{O}_S$.

The definition is illustrated in the picture below.

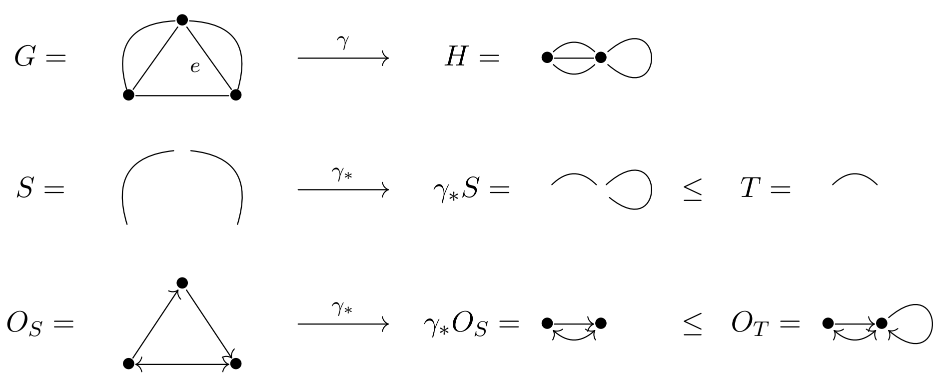


Figure 6. An example of the partial order on $\overline{\mathcal{OP}}^1_g$: $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ with $\gamma : G \to H$ contracting e. The orientations O_S, $\gamma_* O_S$ and O_T are living on $G - S$, $H - \gamma_* S$ and $H - T$, respectively.

Proposition 3.3.6. *$\overline{\mathcal{OP}}^1_g$ is a poset such that the inclusion $\overline{\mathcal{OP}}^1(G) \hookrightarrow \overline{\mathcal{OP}}^1_g$ is a morphism of posets for every $G \in \mathcal{SG}_g$. Furthermore we can define a rank function on $\overline{\mathcal{OP}}^1_g$ by setting:*

$$\rho_{\overline{\mathcal{OP}}^1_g} : \overline{\mathcal{OP}}^1_g \to \mathbb{N}; \qquad (G, \overline{O}_S) \mapsto 3g - 3 - |E(G)| + g(G - S).$$

Proof. One easily checks that this indeed gives a partial order. That $\overline{\mathcal{OP}}^1(G) \to \overline{\mathcal{OP}}^1_g$ is a morphism of posets is clear from the definition of the partial order on $\overline{\mathcal{OP}}^1_g$, which for $G = H$ is identical to the one on $\overline{\mathcal{OP}}^1(G)$.

For the last claim, let $(H, \overline{O}_T), (G, \overline{O}_S) \in \overline{\mathcal{OP}}^1_g$ such that $(H, \overline{O}_T)$ covers $(G, \overline{O}_S)$.

If $G \neq H$, by definition there is a contraction $\gamma : G \to H$. If this contracts more than one edge, it factors as $\gamma = (\gamma_2 \circ \gamma_1) : G \to G' \to H$. Then $(G, \overline{O}_S) \leq (G', \overline{\gamma_1}_*(\overline{O}_S)) \leq (H, \overline{O}_T)$, contradicting the assumption. Thus $|E(G)| = |E(H)| + 1$. By definition, $(H, \overline{\gamma}_*(\overline{O}_S)) \leq (H, \overline{O}_T)$ in $\overline{\mathcal{OP}}^1(H)$. If this is not an equality, we get

$$(G, \overline{O}_S) \leq (H, \overline{\gamma}_*(\overline{O}_S)) \leq (H, \overline{O}_T),$$

again a contradiction. Thus in particular $\gamma_* S = T$ and $g(G - S) = g(G) - |S| = g(H) - |\gamma_* S| - |S \cap S_0|$. We get

$$\rho_{\overline{\mathcal{OP}}_g^1}((H, \overline{O}_T)) - \rho_{\overline{\mathcal{OP}}_g^1}((G, \overline{O}_S)) = |E(G)| - |E(H)| + |\gamma_* S| + |S \cap S_0| - |S| = 1,$$

which proves the claim.

If on the other hand $G = H$, this follows from $\rho_{\overline{\mathcal{OP}}^1(H)}(\overline{O}_T) = g(G - T)$ being a rank function as was shown in 3.2.6.

□

If we didn't have to account for automorphisms, we would be finished defining the poset at this point. However, we will have to identify some of the elements of $\overline{\mathcal{OP}}_g^1$ to get an indexing set for a stratification of $\overline{P}_g^g$.

An automorphism $\sigma \in \mathrm{Aut}(G)$ acts on $\overline{\mathcal{OP}}^1(G)$ as follows: for $S \in \mathcal{C}(G)$ we set $\sigma(S) \subset E(G)$ to be the set of edges that are images of edges in S under σ. Clearly also $\sigma(S) \in \mathcal{C}(G)$. Then we let $\sigma(O_S)$ be the orientation defined on $G - \sigma(S)$, where edges are oriented as their preimage under σ is oriented in O_S. To be more precise, we may view the part of σ that is a bijection $E(G) \to E(G)$ as a bijection on half edges, where the half edge e^1 adjacent to a unique vertex v and contained in an edge e is mapped to the half edge contained in $\sigma(e)$ adjacent to $\sigma(v)$. This viewpoint lets us define $\sigma(O_S)$ in a well-defined manner.

This action preserves equivalence of orientations, thus we get a map

$$\sigma : \overline{\mathcal{OP}}^1(G) \to \overline{\mathcal{OP}}^1(G), \overline{O}_S \to \sigma(\overline{O}_S) = \overline{\sigma(O_S)}.$$

It also clearly preserves both the partial order and rank function on $\overline{\mathcal{OP}}^1(G)$. We set

$$[\mathcal{OP}^1(G)] = \overline{\mathcal{OP}}^1(G)/\mathrm{Aut}(G)$$

and define a partial order on $[\mathcal{OP}^1(G)]$ by setting $[O_S] \leq [O_T]$ if there is $\overline{O}'_S \in [O_S]$ and $\overline{O}'_T \in [O_T]$ such that $\overline{O}'_S \leq \overline{O}'_T$ in $\overline{\mathcal{OP}}^1(G)$.

DEFINITION 3.3.7. For $(H, \overline{O}_T), (G, \overline{O}_S) \in \overline{\mathcal{OP}}_g^1$ we set $(H, \overline{O}_T) \sim (G, \overline{O}_S)$ if $G = H$ and there is a $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(\overline{O}_T) = \overline{O}_S$.

This clearly is an equivalence relation on $\overline{\mathcal{OP}}_g^1$.

DEFINITION/PROPOSITION 3.3.8. We set $[\mathcal{OP}_g^1] = \overline{\mathcal{OP}}_g^1/\sim$ and endow it with the following partial order: $(H, [O_T]) \leq (G, [O_S])$ in $[\mathcal{OP}_g^1]$ if there is $\overline{O}'_T \in [O_T]$ and $\overline{O}'_S \in [O_S]$ such that $(H, \overline{O}'_T) \leq (G, \overline{O}'_S)$ in $\overline{\mathcal{OP}}_g^1$.

Then the inclusion $[\mathcal{OP}^1(G)] \to [\mathcal{OP}^1_g]$ is a morphism of posets and the quotient map $\overline{\mathcal{OP}}^1_g \to [\mathcal{OP}^1_g]$ is a quotient of posets. Furthermore

$$\rho_{[\mathcal{OP}^1_g]}((G,[O_S])) = 3g-3-|E(G)|+g(G-S)$$

is a rank function.

Proof. Different elements that are identified by $\sim$ are not comparable in $\overline{\mathcal{OP}}^1_g$ thus $[\mathcal{OP}^1_g]$ inherits the partial order from $\overline{\mathcal{OP}}^1_g$.

It immediately follows from the definition, that $[\mathcal{OP}^1(G)] \to [\mathcal{OP}^1_g]$ is a morphism of posets.

The quotient map clearly is surjective. Suppose $(H,[O_T]) \leq (G,[O_S])$. By definition, there is $\overline{O}'_T \in [O_T]$ and $\overline{O}'_S \in [O_S]$ with $(H,\overline{O}'_T) \leq (G,\overline{O}'_S)$. Now for any $\overline{O}''_T \in [O_T]$ there is $\sigma \in \mathrm{Aut}(H)$ such that $\sigma(\overline{O}'_T) = \overline{O}''_T$. If $\gamma : H \to G$ is the contraction giving $(H,\overline{O}'_T) \leq (G,\overline{O}'_S)$, we can define a contraction $\gamma' = (\gamma \circ \sigma^{-1})$. Then $\overline{\gamma}'_*(\overline{O}''_T) = \overline{O}'_S$: A half edge of G is oriented in $\overline{\gamma}'_*(\overline{O}''_T)$ as the corresponding half edge of $\sigma^{-1}(O''_T) = O'_T$ on H as it is for O'_S by assumption. Thus $(H,\overline{O}''_T) \leq (G,\overline{O}'_S)$ and we showed that we indeed have a quotient of posets.

The claim about the rank function follows from the fact that it is a quotient of posets and the observation that equivalent elements of $\overline{\mathcal{OP}}^1_g$ have the same rank.

$\square$

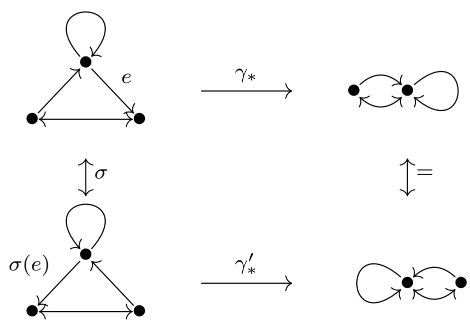


Figure 7. An illustration of the notions in the proof of 3.3.8. Here γ contracts e and σ is the reflection along the vertical line through the top point of the triangle.

3.4. The stratification of $\overline{P}_g^g$

3.4.1. Specialization of polarized curves. We will be interested in (flat, projective) families of curves over a one-dimensional nonsingular base, specializing to a given curve X. Up to shrinking the family near X one can assume that away from X the family is topologically trivial, i.e. that every fiber different from X has the same dual graph of some fixed curve Y. We shall represent such a specialization as follows

$$Y \rightsquigarrow X$$

and refer to it as a *specialization from* Y *to* X. Notice that, as X has only nodes as singularities, the same holds for Y. We shall usually denote by H the dual graph of Y.

Now, suppose our curves X and Y are "polarized", i.e. endowed with a line bundle, $L \in \mathrm{Pic}(X)$ and $M \in \mathrm{Pic}(Y)$. We say that (Y, M) specializes to (X, L), and write

$$(Y, M) \rightsquigarrow (X, L)$$

if there is a specialization of Y to X under which M specializes to L.

REMARK 3.4.1. Let us define all of the above more rigorously. The family under which Y specializes to X is a projective morphism $f : \mathcal{X} \to B$ where B is a smooth, connected, one-dimensional variety with a point b_0 such that $f^{-1}(b_0) \cong X$, and the restriction of f away from b_0 is locally trivial, moreover $f^{-1}(b) \cong Y$ for some $b \neq b_0$. For the polarized version, to say that M specializes to L means that $\mathcal{X}$ is endowed with a line bundle whose restriction to Y is M and whose restriction to X is L. By [**12**, prop. 4, subsect. 8.1], working up to étale base change this is equivalent to saying there is a section, $\sigma : B \to \mathrm{Pic}_{\mathcal{X}/B}$ of the Picard scheme of the family, $\mathrm{Pic}_{\mathcal{X}/B} \to B$, such that $\sigma(b_0) = L$ and $\sigma(b) = M$.

PROPOSITION 3.4.2. *Let X and Y be two nodal curves and G and H their respective dual graphs. Let $L \in \mathrm{Pic}(X)$ and $M \in \mathrm{Pic}(Y)$. If (Y, M) specializes to (X, L) there exists a contraction $\gamma : G \to H$ such that*

$$\gamma_* \underline{\deg}(L) = \underline{\deg}(M).$$

In the opposite direction, we have the following.

PROPOSITION 3.4.3. *Let $\gamma : G \to H$ be a contraction between two graphs. Then for any curve X dual to G and for any $L \in \mathrm{Pic}(X)$ there exist a curve Y dual to H and a line bundle $M \in \mathrm{Pic}(Y)$ such that $\gamma_* \underline{\deg}(L) = \underline{\deg}(M)$ and such that (Y, M) specializes to (X, L).*

PROOF. We prove Propositions 3.4.2 and 3.4.3 together as their proofs are closely related. They extend [**18**, Thm 4.7 (2)] to polarized, not necessarily stable, curves.

To prove Proposition 3.4.2, assume (Y, M) specializes to (X, L). Under such a specialization every node of Y specializes to a node of X and different nodes specialize to different nodes. Hence we partition $E(G) = S_0 \sqcup T$ so that S_0 is the set of nodes of X which are not specializations of nodes of Y. We let $\gamma : G \to G/S_0$, and, arguing as for [**18**, Thm 4.7], we have $G/S_0 = H$.

For any vertex $w \in V(H)$ we write $D_w \subset Y$ for the irreducible component corresponding to w. As shown in loc.cit., the specialization from Y to X induces a specialization

$$D_w \rightsquigarrow \cup_{\gamma(v)=w} C_v \subset X.$$

Now, M specializes to L and hence $M_{|D_w}$ specializes to the restriction of L to $\cup_{\gamma(v)=w}C_v$. Therefore

$$\underline{\deg}(M)_w = \deg_{D_w} M = \deg L_{|\cup_{\gamma(v)=w}C_v} = \sum_{\gamma(v)=w} \underline{\deg}(L)_v = \gamma_*\underline{\deg}(L)_w.$$

This proves Proposition 3.4.2.

Now let $\gamma : G \to G/S_0 = H$ be a contraction, for some $S_0 \in E(G)$; write $E(G) = S_0 \sqcup T$ so that T is identified with $E(H)$. Let X be a curve dual to G and let X^ν_T be its normalization at T, so that $G-T$ is the dual graph of X^ν_T. The curve X^ν_T is endowed with $|T|$ pairs of marked smooth points, namely the branches over the nodes in T. Observe that the connected components of X^ν_T are in bijection with the connected components of $H-T$, and hence with the vertices of H. We can therefore decompose X^ν_T as follows

$$X^\nu_T = \sqcup_{w\in V(H)} Z_w$$

with Z_w a connected nodal curve whose genus, $g(Z_w)$, is equal to the weight of w as a vertex in H. Therefore we can find a family of smooth curves of genus $g(Z_w)$ specializing to Z_w, i.e. we have a smooth curve, W_w, specializing to Z_w. Considering the union for $w \in V(H)$ we get a specialization

$$\sqcup_{w\in V(H)} W_w \rightsquigarrow \sqcup_{w\in V(H)} Z_w = X^\nu_T.$$

Now, up to étale cover, such a specialization can be endowed with $|T|$ pairs of sections specializing to the $|T|$ pairs of branch points of X^ν_T. By gluing together each such pair of sections we get a specialization to our X from a curve, Y, whose dual graph is H.

Clearly, the contraction $\gamma : G \to H$ corresponds to this specialization from Y to X.

Now, using the notation of Remark 3.4.1, let $f : \mathcal{X} \to B$ be a family under which Y specializes to X, and consider its relative Picard scheme, $\mathrm{Pic}_{\mathcal{X}/B} \to B$. Its fiber over b_0 is $\mathrm{Pic}(X)$ and its fiber over b is $\mathrm{Pic}(Y)$. Write $\underline{d} = \deg L$; we claim that, in the relative Picard scheme, $\mathrm{Pic}^{\underline{d}}(X)$ is the specialization of $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$. Indeed, $\mathrm{Pic}^{\underline{d}}(X)$ must be the specialization of some connected component of $\mathrm{Pic}(Y)$ (even if this Picard scheme were not separated, every connected component of its fiber over b_0 is the specialization of some connected component of the general fiber), and this component is necessarily $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$ by the same computation we used to prove Proposition 3.4.2.

Now, as $\mathrm{Pic}^{\underline{d}}(X)$ is the specialization of $\mathrm{Pic}^{\gamma_*\underline{d}}(Y)$, any $L \in \mathrm{Pic}^{\underline{d}}(X)$ is the specialization of some $M \in \mathrm{Pic}^{\gamma_*\underline{d}}(Y)$, and we are done. □

3.4.2. Combinatorics of the compactified universal Jacobian. Recall that the compactified universal Jacobian comes together with a morphism

$$\psi_{g,b} : \overline{P}^g_g \to \overline{M}_g.$$

Furthermore $\psi_{g,b}$ is a projective morphism whose fiber over $X \in \overline{M}_g$ is $\overline{P}^g_X/\mathrm{Aut}(X)$.

We also mention again the following

FACT 3.4.4. *The following is a graded stratification of $\overline{M}_g$ by $\mathcal{SG}_g$:*

$$\overline{M}_g = \bigsqcup_{G \in \mathcal{SG}_g} M_G,$$

where M_G is the locus of curves having G as dual graph.

Now set

$$P^g_G := \psi^{-1}_{g,b}(M_G).$$

Hence

$$P^g_G = \bigsqcup_{X \in M_G} \overline{P}^g_X/\mathrm{Aut}(X).$$

COROLLARY 3.4.5. *Let $G, H \in \mathcal{SG}_g$. Then*

$$P^g_G \subset \overline{P^g_H} \qquad \textit{if and only if} \qquad G \leq H.$$

PROOF. It suffices to use Fact 3.4.4 and the fact that $\psi_{g,b} : \overline{P}^g_g \to \overline{M}_g$ is a projective morphism. $\square$

By Theorem 3.2.8, $\overline{P}^g_X$ is stratified as follows:

$$\overline{P}^g_X = \bigsqcup_{\overline{O_S} \in \overline{\mathcal{OP}}^1(G)} P^{O_S}_X.$$

Let π be the projection $\overline{P}^g_X \to \overline{P}^g_X/\mathrm{Aut}(X)$ (we suppress X in the notation). For $\overline{O}_S \in \overline{\mathcal{OP}}^1(G)$ we set

$$P^{O_S}_G = \bigsqcup_{X \in M_G} \pi(P^{O_S}_X).$$

The aim is to view the $P^{O_S}_G$ as strata of a stratification of $\overline{P}^g_g$. The problem is that while the $P^{O_S}_X$ are disjoint in $\overline{P}^g_X$, their image under π might coincide (see the examples below). To remedy this we set for any $[O_S] \in [\mathcal{OP}^1_g]$:

$$P^{[O_S]}_G = \bigsqcup_{X \in M_G} \left(\bigcup_{\overline{O}'_S \in [O_S]} \pi(P^{O'_S}_X) \right).$$

EXAMPLE 3.4.6. Let G be the graph on two vertices of weight zero and three edges between them. Let $\sigma \in \mathrm{Aut}(G)$ be an automorphism of G. In this case M_G is a point corresponding to the curve X that consists of two rational components glued along three points. Then there is always a $\sigma_X \in \mathrm{Aut}(X)$ that induces σ on G. There are three maximal dimensional strata of $\overline{P}^g_X$, corresponding to divisors $(1,1)$, $(2,0)$ and $(0,2)$. The last two strata get identified under the automorphism that interchanges the vertices and fixes the edges (interchanging only the half-edges). Thus if $P^{O_S}_X$ and $P^{O'_S}_X$ are the corresponding strata in $\overline{P}^g_X$, which are disjoint, we will have $[O_S] = [O'_S]$ and $\pi(P^{O_S}_X) = \pi(P^{O'_S}_X)$. Furthermore, all strata of codimension one and two will be identified, respectively.

The above example is an extreme case; in general only some curves in M_G will have automorphisms that realize a given automorphism of the graph, especially if the weights are non-trivial:

EXAMPLE 3.4.7. Let G be the graph with two vertices of weight one and two edges between them. Let X be a curve in M_G. The two maximal dimensional strata of $\overline{P}^g_X$ correspond to the divisors $(2,1)$ and $(1,2)$. These divisors are mapped to each other by the automorphism σ of G that interchanges the vertices and fixes the edges. Now for a general choice of X, σ will not be induced by an automorphism of X; in particular if the two genus one components corresponding to the vertices are not isomorphic, there is no chance of interchanging them. If they are however isomorphic and glued along points that get mapped to each other by this isomorphism, there is an automorphism realizing σ. Thus if $P_X^{O_S}$ and $P_X^{O_S'}$ are the corresponding strata in $\overline{P}^g_X$, $\pi(P_X^{O_S})$ and $\pi(P_X^{O_S})$ will be disjoint for most points $X \in M_G$ but intersect over some of them. For the codimension one strata we are in a situation as in the above example: the automorphism of G that fixes the vertices but interchanges the edges is induced by an automorphism of X for any $X \in M_G$. Indeed, any pair of points on a genus one curve can be interchanged by an automorphism (the involution of the corresponding g^1_2), in particular the two gluing points on each component.

REMARK/PROBLEM 3.4.8. Are the $P_G^{[O_S]}$ connected? If it were true, that every automorphism of a graph G can be realized by an automorphism of a curve whose dual graph is G, the answer clearly would be yes. This however is not the case. Consider for example the graph G on two vertices of weight zero and five edges between them. Let σ be the automorphism that interchanges two of the edges. Then there is no curve X with dual graph G and automorphism σ_X such that σ_X induces σ on G. Indeed, any such morphism on each of the rational components would have to be the identity on the three nodes that are not interchanged, and thus the identity on the whole component. This on the other hand does not suffice to establish that the $P_G^{[O_S]}$ are not connected.

We have the following result in this direction:

LEMMA 3.4.9. *Let $\sigma \in \mathrm{Aut}(G)$ for a stable graph G such that σ does not fix any half edges. Then theres is $X \in M_G$ with $\sigma_X \in \mathrm{Aut}(X)$ that induces σ.*

PROOF. We construct X explicitly. Start with a disjoint union $\bigsqcup_{v \in V(G)} C_v$ such that if $g(v) = g(v')$ the corresponding components C_v and $C_{v'}$ are isomorphic under an isomorphism $\phi_{vv'}$. Choose them furthermore in such a way, that each component C_v has an automorphism ϕ_v.

Start with some component C_v and choose a point $p_1 \in C_v$ that is not a fixed point of ϕ_v. We want to view p_1 as corresponding to a half edge e_1^1 contained in some edge $e_1 = vv'$. If $v = v'$, set $p_1' = \phi_v(p_1)$. If $v \neq v'$ and $\sigma(e) = e$, we need to have $\sigma(v) = v'$. In this case set $p_1' = \phi_{vv'}(p_1)$. If $v \neq v'$ and $\sigma(e) \neq e$, choose a point $p_1' \in C_{v'}$ that is not a fixed point of $\phi_{v'}$.

If σ fixes v but does not fix e, set $p_2 = \phi_v(p_1)$. If σ does not fix v and $\sigma(v) = v'' \neq v'$ set $p_2 = \phi_{vv''}(p_1)$. If σ fixes e, skip the definition of p_2. Similarily, if σ fixes v' and does not fix e set $p_2' = \phi_{v'}(p_2)$ and if $\sigma(v') = v''' \notin \{v, v'\}$ set $p_2' = \phi_{vv'''}(p_2)$. If σ fixes e, skip the definition of p_2'.

If G has only edges e and $\sigma(e)$ stop here. Otherwise choose a new half edge and a point p_3 on the component corresponding to the half edge that is not fixed by the automorphism of the component and continue as for p_1 in defining p_3' and possibly p_4 and p_4'. After repeating this finitely many times, this process will leave no edges and we defined $|E(G)|$ pairs of points.

Now construct X by gluing the points p_i and p_i'. Then X will have dual graph G as p_{2k} and p_{2k}' lie on components corresponding to vertices connected by some edge e of G while p_{2k+1} and p_{2k+1}' lie on components corresponding to vertices connected by $\sigma(e)$. Now define an automorphism as follows: A point $p \in C_v$ gets mapped to $\phi_v(p)$ if $\sigma(v) = v$ and to $\phi_{v\sigma(v)}(p)$ otherwise. By construction of the gluing points this indeed will be an automorphism of X and clearly induces σ.

$\square$

REMARK 3.4.10. If σ fixes a half edge of G adjacent to some $v \in V(G)$, we would need the automorphism ϕ_v to fix the point corresponding to the half edge. Thus we would need to be able to choose ϕ_v with enough fixed points, which in general is not possible (cf. the example in 3.4.8).

THEOREM 3.4.11. *The decomposition*

$$\overline{P}_g^g = \bigsqcup_{(G,[O_S]) \in [\mathcal{OP}_g^1]} P_G^{[O_S]}$$

is a graded stratification of $\overline{P}_g^g$ *by* $([\mathcal{OP}_g^1], \rho_{[\mathcal{OP}_g^1]})$.

PROOF. We have

$$\overline{P}_g^g = \bigsqcup_{G \in \mathcal{SG}_g} \left(\bigsqcup_{[O_S] \in [\mathcal{OP}^1(G)]} P_G^{[O_S]} \right) = \bigsqcup_{(G,[O_S]) \in [\mathcal{OP}_g^1]} P_G^{[O_S]}.$$

Indeed, the only thing that might not be clear is that the union $\bigsqcup_{[O_S] \in [\mathcal{OP}^1(G)]} P_G^{[O_S]}$ is disjoint. To see this, recall that any two strata $P_X^{O_S}$ and $P_X^{O_T}$ are disjoint in $\overline{P}_X^g$. Let $\pi : \overline{P}_X^g \to \overline{P}_X^g/\mathrm{Aut}(X)$. Since automorphisms of X map strata to strata in $\overline{P}_X^g$, the images $\pi(P_X^{O_S})$ and $\pi(P_X^{O_T})$ are no longer disjoint if and only if there is an automorphism σ_X of X identifying them. Then one easily checks that the induced automorphism σ on G identifies $\overline{O}_S$ and $\overline{O}_T$ in $\overline{\mathcal{OP}}^1(G)$.

We show that the $P_G^{[O_S]}$ are locally closed: Consider the universal curve over M_G, written $\mathcal{X}_G \to M_G$. This is the topologically trivial family of curves having G as dual graph. The nodes of S define sections of $\mathcal{X}_G \to M_G$ and we let $\mathcal{X}_S^\nu \to M_G$ be the partial normalization of these sections. That is the fiber of $\hat{\mathcal{X}}_S \to M_G$ over X is X_S^ν with dual graph $G - S$. Then there is a surjective map

$$\mathrm{Pic}^{d^{O_S}}_{\mathcal{X}_S^\nu/M_G} \to P_G^{O_S}$$

that maps the connected component $\mathrm{Pic}^{d^{O_S}}_{\mathcal{X}_S^\nu/M_G}$ of the relative Picard scheme of $\mathcal{X}_S^\nu \to M_G$ to $P_G^{O_S}$. This is not an isomorphism: for a curve $X \in M_G$ there may be an automorphism $\sigma \in \mathrm{Aut}(X)$ with $\sigma(P_X^{O_S}) = P_X^{O_S}$. Points differing by σ will get identified in $P_G^{O_S}$. The map exhibits $P_G^{O_S}$ as a quotient of $\mathrm{Pic}^{d^{O_S}}_{\mathcal{X}_S^\nu/M_G}$ by a relative automorphism group. Since $\mathrm{Pic}^{d^{O_S}}_{\mathcal{X}_S^\nu/M_G}$ is quasi-projective, this implies that so is $P_G^{O_S}$. Now $P_G^{[O_S]}$, the union of finitely many $P_g^{O_S}$, is locally closed because for $\overline{O}_S, \overline{O}'_S \in [O_S]$ we have $P_G^{O_S} \cap (\overline{P_G^{O'_S}} \setminus P_G^{O'_S}) = \emptyset$, which will be a consequence of the proof of Proposition 3.4.14.

Since $\mathrm{Aut}(X)$ is a finite group we get (for any $X \in M_G$)

$$\dim P_G^{[O_S]} = \dim M_G + \dim P_X^{O_S} = 3g - 3 - |E(G)| + g(G - S)$$

and we know that the right hand side is a rank function on $[\mathcal{OP}_g^1]$, by Proposition 3.3.8.

To complete the proof we must show that we have a stratification in the sense of Definition 2.3.2. After a small lemma, we will do that in the next two propositions. Recall that for a divisor $\underline{d}_S$ on $G-S$ we defined $\hat{\underline{d}}_S$ on $\hat{G}_S$ by setting the values on the exceptional components to be one. We also defined the divisor $\underline{c}_v^{\gamma,S} = |\{e \in S \cap S_0 |\, \gamma(e) = v\}|$ and set $\underline{c}^\gamma = \underline{c}^{\gamma,E(G)}$.

Lemma 3.4.12. *Let $\underline{d}_S$ be a stable divisor on $G - S$. Then $\widehat{\underline{d}_S}$ is stably balanced and we have a surjective map*

$$\mathrm{Pic}^{\widehat{\underline{d}_S}}(\hat{X}^S) \to \mathrm{Pic}^{\underline{d}_S}(X_S^\nu); \qquad \hat{L} \mapsto \hat{L}_{|X_S^\nu}.$$

For any contraction $\delta : \hat{G}_S \to G$ we have

$$\delta_* \widehat{\underline{d}_S} = \underline{d}_S + \underline{c}^\delta.$$

Proof. A divisor on $G - S$ is also a divisor on G, so the first part follows trivially by definition. We have $\underline{c}_v^\delta = 0$ if $\delta^{-1}(v) = v$, and $\underline{c}_v^\delta = 1$ otherwise. Since the value of $\widehat{\underline{d}_S}$ on exceptional vertices is 1 we have

$$(\delta_* \widehat{\underline{d}_S})_v = \begin{cases} (\widehat{\underline{d}_S})_v & \text{if } \delta^{-1}(v) = v \\ (\widehat{\underline{d}_S})_v + 1 & \text{otherwise.} \end{cases}$$

Hence $(\delta_* \widehat{\underline{d}_S} - \underline{c}^\delta)_v = (\widehat{\underline{d}_S})_v = (\underline{d}_S)_v$. $\square$

Proposition 3.4.13. *Let $(G, \overline{O}_S), (H, \overline{O}_T) \in \overline{\mathcal{OP}}_g^1$. If $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ then $P_G^{O_S} \subset \overline{P_H^{O_T}}$.*

(Equivalently: let $\gamma : G \to H$ be a contraction; fix $\overline{O}_S \in \overline{\mathcal{OP}}^1(G)$ and $\overline{O}_T \in \overline{\mathcal{OP}}^1(H)$ such that $\overline{\gamma}_* \overline{O}_S \leq \overline{O_T}$. Then $P_G^{O_S} \subset \overline{P_H^{O_T}}$.)

PROOF. We must show that for every $X \in M_G$ we have $P_X^{O_S}/\mathrm{Aut}(X) \subset \overline{P_H^{O_T}}$. By hypothesis, for any $X \in M_G$ we can fix a family of curves in M_H specializing to X. Let Y be a curve dual to H and let

$$Y \rightsquigarrow X$$

be a specialization from Y to X such that γ is the associated contraction. We have the associated specialization of compactified Picard varieties:

$$\overline{P}_Y^g \rightsquigarrow \overline{P}_X^g.$$

Observe that $\overline{\gamma}_*\overline{O}_S \in \overline{\mathcal{OP}}^1(H)$, hence $\underline{d}^{\gamma_* O_S}$ is stable, and $P_Y^{\gamma_* O_S}$ parametrizes stably balanced line bundles on $\hat{Y}_R$ of degree $\widehat{\underline{d}^{\gamma_* O_S}}$, where $R = \gamma_* S$. We begin by showing that $P_Y^{\gamma_* O_S}$ specializes to $P_X^{O_S}$. To the contraction γ we naturally associate the contraction

$$\hat{\gamma} : \hat{G}_S \to \hat{H}_R = \hat{G}_S/\hat{S}_0$$

(where $\hat{S}_0 = \delta_E^{-1}(S_0)$ for some choice of contraction $\delta : \hat{G}_S \to G$). Now consider $\widehat{\underline{d}^{O_S}}$ and $\widehat{\underline{d}^{\gamma_* O_S}}$ on $\hat{G}_S$ and $\hat{H}_R$, respectively. We claim

$$\widehat{\underline{d}^{\gamma_* O_S}} = \hat{\gamma}_* \widehat{\underline{d}^{O_S}}. \tag{10}$$

Let $v \in V(\hat{H}_R)$. If $v = v_e$ for $e \in R$ then v_e is also an exceptional vertex of $\hat{G}_S$ mapped to v_e by $\hat{\gamma}$. Hence both divisors appearing in (10) have value 1 on v_e. Now suppose $v \in V(H)$, then, by 3.3.2,

$$(\widehat{\underline{d}^{\gamma_* O_S}})_v = (\underline{d}^{\gamma_* O_S})_v = (\gamma_* \underline{d}^{O_S})_v + \underline{c}_v^{\gamma,S} = \sum_{z \in \gamma_V^{-1}(v)} \underline{d}_z^{O_S} + \underline{c}_v^{\gamma,S} = (\hat{\gamma}_* \widehat{\underline{d}^{O_S}})_v$$

where the last equality follows as $\underline{c}_v^{\gamma,S}$ is equal to the number of exceptional vertices of $\hat{G}_S$ that are mapped to v by $\hat{\gamma}$. (10) is proved.

We can now apply 3.4.3, to obtain that any line bundle $\hat{L} \in \mathrm{Pic}(\hat{X}_S)$ such that $\underline{\deg}\hat{L} = \widehat{\underline{d}^{O_S}}$ is obtained as specialization of a line bundle $\hat{M} \in \mathrm{Pic}(\hat{Y}_R)$ such that

$$\underline{\deg}\hat{M} = \hat{\gamma}_* \underline{\deg}\hat{L} = \hat{\gamma}_* \widehat{\underline{d}^{O_S}} = \widehat{\underline{d}^{\gamma_* O_S}}.$$

This proves that $P_Y^{\gamma_* O_S}$ specializes to $P_X^{O_S}$. Thus we get

$$P_X^{O_S} \subset \overline{P_H^{\gamma_* O_S}}.$$

Now, by Theorem 3.2.8 and the hypothesis $\overline{\gamma}_*\overline{O}_S \leq \overline{O_T}$ we have

$$P_Y^{\gamma_* O_S} \subset \overline{P_Y^{O_T}}.$$

As $\overline{P}_g^g$ is a coarse moduli space, the degeneration of polarized curves $\mathcal{X} \to B$ we constructed gives a map $B \to \overline{P_H^{O_T}}$. By what we showed above we get

$P_X^{O_S}/\mathrm{Aut}(X) \subset \overline{P_H^{\gamma_* O_S}} \subset \overline{P_H^{O_T}}$. The Proposition is proved $\square$

PROPOSITION 3.4.14. *Let $(G,[O_S])$ and $(H,[O_T])$ be in $[\mathcal{OP}_g^1]$. The following are equivalent*

(a) $P_G^{[O_S]} \cap \overline{P_H^{[O_T]}} \neq \emptyset$.
(b) There are $\overline{O}_S \in [O_S]$ *and* $\overline{O}_T$ *in*$[O_T]$ *and a contraction* $\gamma : G \to H$ *such that* $\overline{\gamma}_* \overline{O}_S \leq \overline{O_T}$.
(c) $P_G^{[O_S]} \subset \overline{P_H^{[O_T]}}$

PROOF. (a) $\Rightarrow$(b). By hypothesis, we have a specialization of $(\hat{Y}_T, \hat{M})$ to $(\hat{X}_S, \hat{L})$ where X and Y are curves dual to G and H respectively, and $\hat{L}$ and $\hat{M}$ are stably balanced line bundles on $\hat{X}_S$ and $\hat{Y}_T$ such that $\underline{\deg}_{X_S^\nu} \hat{L} = \underline{d}^{O_S}$ and $\underline{\deg}_{Y_T^\nu} \hat{M} = \underline{d}^{O_T}$ for some $\overline{O}_S \in [O_S]$ and $\overline{O}_T \in [O_T]$.

We denote by $\hat{G}_S$ and $\hat{H}_T$ the dual graphs of $\hat{X}_S$ and $\hat{Y}_T$. By 3.4.2, the above specialization is associated to a contraction

$$\hat{\gamma} : \hat{G}_S \to \hat{H}_T.$$

such that $\hat{\gamma}_* \underline{\deg} \hat{L} = \underline{\deg} \hat{M}$. Now, every exceptional component of $\hat{Y}_T$ specializes to an exceptional component of $\hat{X}_S$, hence we have a specialization of Y to X and the associated contraction

$$\gamma : G \to H = G/S_0.$$

We have an inclusion $T \subset S$ induced by $E(H) \subset E(G)$.

Denote by $\hat{O}_S$ the orientations on $\hat{G}_S$ obtained from O_S by orienting all edges adjacent to exceptional components towards the exceptional component. Then the degree of $\underline{d}^{\hat{O}_S}$ on each exceptional component will be one and $\underline{d}^{\hat{O}_S} = \widehat{(\underline{d}^{O_S})}$. Define $\hat{O}_T$ analogously.

We first assume $T = \emptyset$, then $\hat{H}_T = H$, $O = O_T$ and we have a commutative diagram

$$\begin{tikzcd}
\hat{G}_S \arrow[r, "\hat{\gamma}"] \arrow[d, "\delta"'] & H \\
G \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

Here δ is given as follows: Every exceptional vertex v_e in $\hat{G}_S$ has two adjacent edges h_e and j_e, both oriented towards v_e in $\hat{O}_S$. Defining δ amounts to choosing one of the two for every exceptional vertex. If $e \in S_0$, we can contract any of the two, as $\hat{\gamma}$ contracts both. If $e \notin S_0$, choose the one contracted by $\hat{\gamma}$. This choice clearly makes the diagram commutative. Let O' on G be $\delta_* \hat{O}_S$. Since $\hat{\gamma}_* \underline{\deg} \hat{L} = \underline{\deg} \hat{M}$, i.e. $\hat{\gamma}_*(\underline{d}^{\hat{O}_S}) = \underline{d}^O$, and the above diagram is commutative we get $\gamma_* O' = \gamma_*(\delta_* \hat{O}_S) = \hat{\gamma}_*(\hat{O}_S) \sim_{cyc} O$ by Proposition 3.3.2 (b). On the other hand, by definition $O'_{|G-S} = O_S$, i.e. $O_S \leq O'$, and thus by Proposition 3.3.2 (d) $\gamma_* O_S \leq \gamma_* O' \sim_{cyc} O$, which proves the claim in case $H = \emptyset$.

In general, we have $T \subset S$ and, of course, $T \cap S_0 = \emptyset$. Therefore the restriction of γ to $G - T$ is

$$\gamma_{|G-T} : G - T \to \frac{G-T}{S_0} = H - T.$$

Write $G' = G - T$, $H' = H - T$ and $\gamma' = \gamma_{|G-T}$. Then write $O' = O_T$ and $O'_{S'} = (O_S)_{|G'}$ with $S' = S \setminus T$. By the previous case $\overline{\gamma'_* O'_{S'}} \leq \overline{O'}$, i.e.

$$\gamma'_* O'_{S'} \sim_{cyc} O'_{|H' - \gamma'_* S'}. \tag{11}$$

Now, O_S is defined on $G - S \subset G - T$, hence

$$\gamma'_* O'_{S'} = (\gamma_{|G-T})_* (O_S)_{|G-T} = \gamma_* O_S.$$

Also, as O_T is defined on $H' = H - T$, we have

$$O'_{|H'-\gamma'_* S'} = ((O_T)_{|H-T})_{|H-T-\gamma'_* S'} = (O_T)_{H - \gamma_* S}$$

($\gamma'_* S' \cup T = S' \setminus S'_0 \cup T = S \setminus S_0 = \gamma_* S$ as $T \cap S_0 = \emptyset$). Combining with (11) gives $\gamma_* O_S \sim (O_T)_{H-\gamma_* S}$ and we are done with the implication (a) $\Rightarrow$(b).

(b) $\Rightarrow$(c). Recall that $P_G^{[O_S]} = \bigcup_{\overline{O}'_S \in [O_S]} P_G^{O'_S}$ and $P_H^{[O_T]} = \bigcup_{\overline{O}'_T \in [O_T]} P_H^{O'_T}$. Now for any $\overline{O}'_S \in [O_S]$, there is $\overline{O}'_T \in [O_T]$ such that $(G, \overline{O}'_S) \leq (H, \overline{O}'_T)$ by assumption and Proposition 3.3.8. By Proposition 3.4.13 we get $P_G^{O'_S} \subset \overline{P_H^{O'_T}}$ and thus $P_G^{[O_S]} \subset \overline{P_H^{[O_T]}}$.

(c) $\Rightarrow$(a). Obvious. $\square$

Theorem 3.4.11 is now proved. $\square$

Remark/Problem 3.4.15. As we pointed out, the strata $P_G^{[O_S]}$ have the disadvantage of in general not being irreducible and possibly disconnected (cf. 3.4.8). Thus the stratification is not what we called an algebraic stratification. This boils down to the fact that the $P_G^{O_S}$, while locally closed and irreducible, do not give a stratification, as they are not disjoint. One could remedy this by working only over curves that have no automorphisms. In this case one would need to define a subposet of $\overline{\mathcal{OP}}_g^1$, as in some strata every curve has an automorphism inducing a non-trivial automorphism of the graph; but everything should work out as before.

CHAPTER 4

Tropical aspects

The purpose of this section is to exhibit a stratification of spaces of break divisors on metric graphs dual to the one constructed in the previous section for $\overline{P}_X^g$ and $\overline{P}_g^g$. We will establish the usual strata inversion occurring in this kind of setting. As a by-product, we will give posets $\mathcal{P}\Sigma(G)$, $\mathcal{P}\Sigma_g$ and $\overline{\mathcal{P}\Sigma}_g$ isomorphic to $\overline{\mathcal{OP}}^1(G)$, $\overline{\mathcal{OP}}_g^1$ and $[\mathcal{OP}_g^1]$, respectively, rephrasing the partial order relations in terms of break divisors.

4.1. The stratification of $\mathrm{Pic}^g(\Gamma)$

4.1.1. The poset of break divisors on subgraphs. Let Γ be a metric graph and G a model of Γ. In [**4**] a polyhedral decomposition of $\mathrm{Pic}^g(\Gamma)$ associated to G was constructed, which we will refer to as the ABKS-decomposition. We want to view the associated face decomposition of $\mathrm{Pic}^g(\Gamma)$ as a stratification by a partially ordered set. The indexing set will be a poset isomorphic to $\overline{\mathcal{OP}}^1(G)$ and thus we will establish the strata inversion phenomenon encountered in the transition from the algebro-geometric moduli space to the tropical one.

Recall that we denoted by $\Sigma(G) \subset \mathrm{Div}^g(G)$ respectively $\Sigma(\Gamma) \subset \mathrm{Div}^g(\Gamma)$ the set of break divisors. We will identify $\Sigma(\Gamma)$ and $\mathrm{Pic}^g(\Gamma)$ (cf. section 1.1.8). We defined $\mathcal{C}(G) = \{S \subset E(G) \,|\, G - S \text{ is connected}\}$ for any connected graph G. We endowed $\mathcal{C}(G)$ with a partial order, given by reverse inclusion: $S \leq T$ if $T \subset S$. As before we denote by $\mathcal{C}(G)^*$ the dual poset obtained by reversing the partial order.

Next set

$$\mathcal{P}\Sigma(G) = \{\underline{d}_S \,|\, S \in \mathcal{C}(G), \underline{d}_S \in \Sigma(G - S)\}.$$

Thus $\mathcal{P}\Sigma(G)$ is the set of break divisors on connected subgraphs of G (it is also more generally the set of break divisors on any subgraph of G, as with our definition there are no break divisors on non-connected graphs).

To define a partial order on $\mathcal{P}\Sigma(G)$, observe the following:

Definition 4.1.1. Let $\underline{d} \in \Sigma(G)$ be a break divisor. Let $S \subset E(G)$ and suppose there is $T \in T(\underline{d})$ with $E(T) \cap S = \emptyset$. Fix a presentation $\underline{d} = \underline{g} + \Sigma_{e_i \in E(G) \setminus E(T), p_i \in e_i}(p_i)$. This presentation then induces a break divisor on $G - S$ by setting

$$\underline{d} - S = \underline{g} + \Sigma_{e_i \in E(G-S) \setminus E(T), p_i \in e_i}(p_i).$$

Remark 4.1.2. The notation $\underline{d} - S$ is somewhat inhomogeneous, as we subtract edges from a divisor. However the more natural $\underline{d}_{|G-S}$ is already reserved for the restriction of

$\underline{d}$ to $G-S$. This is not the same as $\underline{d}-S$. In fact we have $|\underline{d}_{|G-S}| = |\underline{d}| = g(G)$ whereas $|\underline{d}-S| = |\underline{d}| - |S| = g(G-S)$.

Similarily, if we have a metric graph Γ with model G, then for $S \subset E(G)$ we can define $\Gamma - S$ as the metric graph associated to the metrized graph $G-S$. Then for any break divisor $\underline{d} \in \Sigma(\Gamma)$ and a presentation of $\underline{d}$ we define $\underline{d} - S \in \Sigma(\Gamma - S)$ analogously to the discrete case.

The divisor $\underline{d}-S$ is indeed a break divisor, as by assumption T is a spanning tree on $G-S$ and the definition gives an explicit presentation as break divisor.

Note that even though we suppress the chosen presentation in the notation, the induced break divisor does depend on it, even for a fixed spanning tree. Consider the following example:

Example 4.1.3. Let G be the graph with two vertices v_1 and v_2, three edges e_1, e_2 and e_3 joining them and trivial weights. With notation as above, take T to be $G-\{e_1,e_2\}$, $S=\{e_1\}$ and $\underline{d} = v_1+v_2$. The presentation as a break divisor is p_1+p_2 with p_1 adjacent to e_1 and p_2 adjacent to e_2. As both vertices are adjacent to both edges, we are free to choose any combination that gives $\underline{d}$. If we choose the presentation $p_1 = v_1$ and $p_2 = v_2$, we get $\underline{d}-S = v_2$, whereas if we choose $p_1 = v_2$ and $p_2 = v_1$ we get $\underline{d}-S = v_1$.

Remark 4.1.4. Recall that there is a bijection between rooted orientations O and break divisors $\underline{d}^O$. If $\underline{d}^O$ has a presentation by a spanning tree T, we saw that this will be an arborescence of O. In particular if $S \cap E(T) = \emptyset$, the induced orientation $O_{|G-S}$ on $G-S$ will again be rooted. Then by definition

$$\underline{d}^O - S = \underline{d}^{O_{|G-S}}.$$

The ambiguity we encountered in the above example then comes from the fact that we can reverse cycles in O without changing $\underline{d}^O$ but possibly getting a different induced orientation on $G-S$.

This sets us up to define a partial order on $\mathcal{P}\Sigma(G)$. To remain consistent with the notation that the (dual) poset is encoding containment relations on the tropical side, we will call the so obtained poset $\mathcal{P}\Sigma(G)^*$. We will denote elements of $\mathcal{P}\Sigma(G)$ by $\underline{d}_S$ to indicate a break divisor defined on $G-S$.

Definition 4.1.5. Let $\underline{d}_S, \underline{e}_R \in \mathcal{P}\Sigma(G)$. Set $\underline{d}_S \leq \underline{e}_R$ if $S \leq R$ in $\mathcal{C}(G)^*$, i.e. if $S \subset R$, and there exists a presentation of $\underline{d}_S$ such that $\underline{e}_R = \underline{d}_S - (R \setminus S)$. We will denote the poset obtained in this way by $\mathcal{P}\Sigma(G)^*$.

One easily checks, that this indeed gives a partial ordering.

Lemma 4.1.6. *The map* $\phi : \overline{\mathcal{OP}}^1(G)^* \to \mathcal{P}\Sigma(G)^*$ *that sends* $\overline{O}_S$ *to* $\underline{d}_S^{O_S}$ *is an isomorphism of posets. Furthermore*

$$\rho^*_{\mathcal{P}\Sigma(G)}(S, \underline{d}) = |S|$$

is the rank function dual to $\rho_{\overline{\mathcal{OP}}^1(G)}$ *under this isomorphism.*

PROOF. By Corollary 1.2.21 we saw that for $\overline{O}_S \in \overline{\mathcal{OP}}^1(G)$ we have $\underline{d}^{O_S} \in \Sigma(G-S)$, thus the definition of ϕ makes sense. By the same proposition, it is a bijection for a fixed S. The choice of S is for both posets $\overline{\mathcal{OP}}^1(G)^*$ and $\mathcal{P}\Sigma(G)^*$ indexed by $\mathcal{C}(G)$ and thus ϕ is a bijection.

If $\overline{O}_S \leq \overline{O}_R$, we have $S \subset R$ by definition. Let $O_S \in \overline{O}_S$ and $O_R \in \overline{O}_R$ such that $(O_S)_{|G-R} = O_R$. Fixing an arborescence of O_R also gives an arborescence of O_S and thus by Lemma 1.2.23 a presentation of $\underline{d}^{O_S}$ as a break divisor. We then have by the construction of this presentation that $\underline{d}^{O_S} - (R \setminus S) = \underline{d}^{(O_S)_{|G-R}}$. Since we assumed $(O_S)_{|G-R} = O_R$ this gives $\underline{d}^{O_S} - (R \setminus S) = \underline{d}^{O_R}$ and thus $\phi(\overline{O}_S) \leq \phi(\overline{O}_R)$. This shows that ϕ is a morphism of posets.

Suppose conversely that $\underline{d}_S \leq \underline{d}_R$. Then by definition $S \subset R$ and there exists a presentation of $\underline{d}_S \in \Sigma(G-S)$ such that $\underline{d}_R = \underline{d}_S - (R \setminus S) \in \Sigma(G-R)$. Let O_R be any rooted 1-orientation giving $\underline{d}_R$ and extend it to an orientation O_S by orienting the edges in $(R \setminus S)$ according to this presentation of $\underline{d}_S$. That is, orient an edge e towards the vertex that is the contribution of e to the presentation of $\underline{d}_S$. This will still be a rooted 1-orientation, as it contains an arborescence and by construction we have $\underline{d}_S = \underline{d}^{O_S}$. Then also by construction $(O_S)_{|G-R} = O_R$ and hence $\overline{O}_S \leq \overline{O}_R$ in $\overline{\mathcal{OP}}^1(G)^*$. This shows that also the inverse of ϕ is a morphism of posets.

Recall that the dual rank was defined as $\rho^*(p) = max_{p_i \in \mathcal{P}}(\rho(p_i)) - \rho(p)$ and that we set $\rho_{\overline{\mathcal{OP}}^1(G)}(\overline{O}_S) = g - |S|$. Thus $max_{\overline{O}_S \in \overline{\mathcal{OP}}^1(G)}(\rho_{\overline{\mathcal{OP}}^1(G)}(\overline{O}_S)) = g$ and

$$\rho^*_{\overline{\mathcal{OP}}^1(G)}(\overline{O}_S) = g - g + |S| = |S|,$$

which proves the last claim. $\square$

If e is an edge of G, denote by e^o its interior in Γ. That is all the points of $e \subset \Gamma$ that are not vertices of G.

The next lemma illustrates that, informally speaking, the ambiguity in presentation of a break divisor $\underline{d} \in \Sigma(\Gamma)$ comes from the points of $\underline{d}$ supported on vertices of G.

LEMMA 4.1.7. *Let Γ be a metric graph with model G, $S \subset E(G)$ and $\underline{d} \in \Sigma(\Gamma)$. Suppose that for every $e \in S$, $\underline{d}$ has a point $p \in e^o$. Then $\underline{d} - S$ does not depend on the presentation of $\underline{d}$.*

PROOF. Notice first that the statement makes sense, i.e. for any $T \in T(\underline{d})$ we have $E(T) \cap S = \emptyset$. Indeed, by the definition of break divisors $\underline{d}$ can have no points on the interior of edges of T.

Giving a presentation of $\underline{d}$ as a break divisor amounts to giving a bijection ϕ : $\{e_1, \ldots e_{b_1(G)}\} \to \{p_i\}_i$ such that $\phi(e_i) \in e_i$ for some spanning tree T with $\{e_1, \ldots e_{b_1(G)}\} = E(G) \setminus E(T)$ and p_i a collection of points such that $\underline{d} = \underline{g} + \Sigma_{i=1}^{b_1(G)}(p_i)$. We already saw that under the assumptions we have $S \subset \{e_1, \ldots e_{b_1(G)}\}$ for any such set of edges. In this formulation, we get by definition $\underline{d} - S = \underline{d} - \Sigma_{e_i \in S}\phi(e_i)$.

If now p_i lies in the interior of some edge e_i it is not contained in any other edge e_j (as opposed to if it lies on a vertex v of G, it is contained in all edges adjacent to v). Thus in this case any such bijection has to map e_i to p_i. Thus for any two presentations of $\underline{d}$ given by two bijections ϕ and ϕ', we need to have $\phi_{|S} = \phi'_{|S}$. Thus $\underline{d} - S = \underline{d} - \Sigma_{e_i \in S}\phi(e_i)$ is independent of the choice of presentation.

□

4.1.2. Stratifying $\mathrm{Pic}^g(\Gamma)$**.** Let Γ be a metric graph and G a model of Γ. We want to view $\mathrm{Pic}^g(\Gamma)$ as stratified by the graded poset $(\mathcal{P}\Sigma(G), \rho^*)$ and thus need to define the strata.

Definition 4.1.8. Let $S \in \mathcal{C}(G)$ and $\underline{d}_S \in \Sigma(G - S)$. We define $\Sigma_\Gamma^{\underline{d}_S} \subset \Sigma(G)$ as

$$\Sigma_\Gamma^{\underline{d}_S} = \{\underline{e} \in \Sigma(\Gamma) |\, \underline{e} = \underline{d}_S + \Sigma_{e_i \in S, p_i \in e_i^o}(p_i)\}.$$

In other words, we can identify $\Sigma_\Gamma^{\underline{d}_S}$ as

$$\Sigma_\Gamma^{\underline{d}_S} = \Pi_{e_i \in S} e_i^o \times \{\underline{d}_S\} \hookrightarrow \Gamma^g / S^g \cong \mathrm{Div}_+^g(\Gamma),$$

where S^g is the symmetric group (and not the edges in S to the g). Note that in particular every $\underline{e} \in \Sigma_\Gamma^{\underline{d}_S}$ has an interior point in every edge of S and thus by Lemma 4.1.7 the divisor $\underline{e} - S = \underline{d}_S$ does not depend on the presentation of $\underline{e}$ as a break divisor. Informally speaking, the strata keep track of the edges that contain points in their interior plus the break divisor on $G - S$ obtained by removing those edges.

Remark 4.1.9. Note that if $|S| = b_1(G)$, i.e. $G - S = T$ is a spanning tree, $\underline{g}_S = \underline{g}$ is the unique break divisor on $G - S$. This recovers the interiors of the parallelotopes of the ABKS-decomposition, denoted in [**4**] by Σ_T^o. In this case

$$\Sigma_\Gamma^{\underline{g}_S} \cong \Pi_{e_i \in E(G) \setminus E(T)} e_i^o = \Sigma_T^o.$$

At the other extreme, for every $\underline{d}_\emptyset \in \Sigma(G)$, we have $\Sigma_\Gamma^{\underline{d}_\emptyset} = \{\underline{d}_\emptyset\}$, i.e. strata of dimension zero consist of a single break divisor of Γ supported on vertices of G. They are in bijection with the elements of $\mathrm{Pic}^g(G)$ and correspond to minimal elements in the face decomposition of the ABKS-decomposition.

In this way we get a decomposition

$$\mathrm{Pic}^g(\Gamma) = \bigsqcup_{\underline{d}_S \in \Sigma(G-S)} \Sigma_\Gamma^{\underline{d}_S}.$$

Note that this decomposition depends on the model G of Γ and not just Γ.

Next we give a description of the closure of the strata in $\mathrm{Div}_+^g(\Gamma)$.

Lemma 4.1.10. *Denote by* $\overline{\Sigma_\Gamma^{\underline{d}_S}}$ *the closure of* $\Sigma_\Gamma^{\underline{d}_S}$ *in* $\Sigma(\Gamma)$*. Then* $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$ *if and only if* $\underline{e}$ *has a presentation by a spanning tree* $T \in T(\underline{e})$ *such that* $E(T) \cap S = \emptyset$ *and* $\underline{e} - S = \underline{d}_S$ *with respect to it.*

PROOF. We have $\Sigma_\Gamma^{\underline{d}_S} = \Pi_{e_i\in S} e_i^o \times \{\underline{d}_S\}$. Thus its closure is

$$\overline{\Sigma_\Gamma^{\underline{d}_S}} = \Pi_{e_i\in S} e_i \times \{\underline{d}_S\} \subset \Sigma(\Gamma),$$

which is another way of giving the characterization in the statement of the lemma.

□

LEMMA 4.1.11. *Let* $\underline{e} \in \Sigma(\Gamma)$. *Then* $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}} \Rightarrow T(\underline{d}_S) \subset T(\underline{e})$.

PROOF. Let $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$. Since by the previous lemma $\underline{e} = \underline{d}_S + \Sigma_{e_i\in S, p_i\in e_i}(p_i)$, any presentation of $\underline{d}_S$ by $T \in T(\underline{d}_S)$ as a break divisor will give rise to a presentation of $\underline{e}$ by T as a break divisor. In other words $T(\underline{d}_S) \subset T(\underline{e})$. □

DEFINITION 4.1.12. Let $S \in \mathcal{C}(G)$ and T a spanning tree of G with $E(T) \cap S = \emptyset$. We then set $S_T^c = E(G) \setminus (S \cup E(T))$.

Note that since $T \subset G - S_T^c$ also $S_T^c \in \mathcal{C}(G)$. By construction we have a decomposition $E(G) = S \sqcup S_T^c \sqcup E(T)$.

LEMMA 4.1.13. *Let* $S \in \mathcal{C}(G)$ *and* $\underline{d}_S \in \Sigma(G - S)$. *Let* $T \in T(\underline{d}_S)$ *be a spanning tree of* $G - S$ *(and thus also of* G*) that gives* $\underline{d}_S$ *as a break divisor. Then the map* $\phi : \overline{\Sigma_\Gamma^{\underline{d}_S}} \to \overline{\Sigma_{\Gamma - S_T^c}^{\underline{g}_S}}$ *that sends* $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$ *to* $\underline{e} - (\underline{d}_S - \underline{g})$ *is a homeomorphism.*

PROOF. We first check that for $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$ we indeed have $\underline{e} - (\underline{d}_S - \underline{g}) \in \overline{\Sigma_{\Gamma - S_T^c}^{\underline{g}_S}}$. Choosing a presentation for $\underline{e}$ as in Lemma 4.1.10, we have $\underline{e} = \underline{d}_S + \Sigma_{e_i\in S, p_i\in e_i}(p_i)$. Thus $\underline{e} - (\underline{d}_S - \underline{g}) = \underline{g} + \Sigma_{e_i\in S, p_i\in e_i}(p_i)$ which by definition and the characterization of Lemma 4.1.10 lies in $\overline{\Sigma_{\Gamma - S_T^c}^{\underline{g}_S}}$

The map is clearly injective, as for $\underline{e}, \underline{e}' \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$ with $\phi(\underline{e}) = \phi(\underline{e}')$ we have $\underline{e} - (\underline{d}_S - \underline{g}) = \underline{e}' - (\underline{d}_S - \underline{g}) \Rightarrow \underline{e} = \underline{e}'$. It is surjective since if we have $\underline{f} \in \overline{\Sigma_{\Gamma - S_T^c}^{\underline{g}_S}}$, $\underline{f}$ has a presentation by the spanning tree T. Then $\underline{e} = \underline{f} + (\underline{d}_S - \underline{g})$ also has a presentation by T (i.e. in particular by a spanning tree with $T \cap S = \emptyset$) and using the characterization of Lemma 4.1.10 it is immediate that $\underline{e} \in \overline{\Sigma_\Gamma^{\underline{d}_S}}$.

Furthermore we may view ϕ as the projection of $\overline{\Sigma_\Gamma^{\underline{d}_S}} = \Pi_{e_i\in S} e_i \times \{\underline{d}_S\} \subset \Sigma(\Gamma)$ to $\Pi_{e_i\in S} e_i \times \{\underline{g}_S\} = \overline{\Sigma_{\Gamma - S_T^c}^{\underline{g}_S}} \subset \Sigma(\Gamma - S_T^c)$. This clearly is a homeomorphism.

□

Recall that we denoted by $\mathrm{Pic}^g(\Gamma)^{strat} = \{\Sigma_\Gamma^{\underline{d}_S}\}_{\underline{d}_S}$ the set of strata.

PROPOSITION 4.1.14. *The decomposition* $\mathrm{Pic}^g(\Gamma) = \bigsqcup_{\underline{d}_S\in\Sigma(G-S)} \Sigma_\Gamma^{\underline{d}_S}$ *is a stratification of* $\mathrm{Pic}^g(\Gamma) \cong \Sigma(\Gamma)$ *by* $(\mathcal{P}\Sigma(G)^*, \rho^*_{\mathcal{P}\Sigma(G)})$ *under the map* $s^* : \mathrm{Pic}^g(\Gamma)^{strat} \to \mathcal{P}\Sigma(G)^*$ *that sends* $\Sigma_\Gamma^{\underline{d}_S}$ *to* $\underline{d}_S$.

PROOF. A stratum $\Sigma_\Gamma^{\underline{d}_S} = \Pi_{e_i \in S} e_i^o \times \{\underline{d}_S\}$ is clearly open in its closure $\overline{\Sigma_\Gamma^{\underline{d}_S}} = \Pi_{e_i \in S} e_i \times \{\underline{d}_S\}$.

The map s^* is a bijection as the strata of $\operatorname{Pic}^g(\Gamma)$ are indexed by the elements of $\mathcal{P}\Sigma(G)^*$.

Suppose $\Sigma_\Gamma^{\underline{d}_S} \cap \overline{\Sigma_\Gamma^{\underline{d}'_{S'}}} \neq \emptyset$. Let $\underline{e} \in \Sigma_\Gamma^{\underline{d}_S} \cap \overline{\Sigma_\Gamma^{\underline{d}'_{S'}}}$. Then any other divisor in $\Sigma_\Gamma^{\underline{d}_S}$ is obtained by varying the points of $\underline{e}$ that lie in the interior of edges contained in S. Any divisor obtained in such a way will also be contained in $\overline{\Sigma_\Gamma^{\underline{d}'_{S'}}}$, as varying points in the interior of edges, by construction of the strata, does not change the stratum the divisor is contained in.

Suppose next $\Sigma_\Gamma^{\underline{d}_S} \leq \overline{\Sigma_\Gamma^{\underline{d}'_{S'}}}$ in $\operatorname{Pic}^g(\Gamma)^{strat}$, i.e. $\Sigma_\Gamma^{\underline{d}_S} \subset \overline{\Sigma_\Gamma^{\underline{d}'_{S'}}}$. This means

$$\Pi_{e_i \in S} e_i^o \times \{\underline{d}_S\} \subset \Pi_{e_i \in S'} e_i \times \{\underline{d}'_{S'}\}$$

and thus in particular $S \subset S'$. This implies that we can write $\underline{d}_S$ as $\underline{d}_S = \underline{d}'_{S'} + \Sigma_{e_i \in S' \setminus S, p_i \in e_i}(p_i)$. By definition thus $\underline{d}'_{S'} = \underline{d}_S - (S' \setminus S)$ with respect to this presentation and we have $\underline{d}_S \leq \underline{d}'_{S'}$. We conclude that s^* is a morphism of posets.

Conversely, if $\underline{d}_S \leq \underline{d}'_{S'}$ and $\underline{e} \in \Sigma_\Gamma^{\underline{d}_S}$ we have

$$\begin{aligned} \underline{e} &= \underline{d}_S + \Sigma_{e_i \in S, q_i \in e_i^o}(q_i) \\ &= \underline{d}'_{S'} + \Sigma_{e_i \in S' \setminus S, p_i \in e_i}(p_i) + \Sigma_{e_i \in S, q_i \in e_i^o}(q_i) \\ &\in \Pi_{e_i \in S'} e_i \times \{\underline{d}'_{S'}\} = \overline{\Sigma_\Gamma^{\underline{d}'_{S'}}}. \end{aligned}$$

This shows that also the inverse of s^* is a morphism of posets.

Finally, we have $\dim(\Sigma_\Gamma^{\underline{d}_S}) = \dim(\Pi_{e_i \in S} e_i^o \times \{\underline{d}_S\}) = |S| = \rho^*_{\mathcal{P}\Sigma(G)}(\underline{d}_S)$ which is a rank function by Lemma 4.1.6.

$\square$

Combining all these results, we are able to establish the usual strata inversal in transitioning between the algebro-geometric and the tropical picture.

THEOREM 4.1.15. *Let X be a stable curve, G its dual graph and Γ a metric graph with model G for any choice of (positive) lengths on G. Then, with notation as before, we have the following diagram where the vertical arrows are stratifications by graded posets and the horizontal arrow is an order reversing bijection:*

$$\begin{tikzcd} \overline{P}_X^g \arrow[d, "s"'] & \operatorname{Pic}^g(\Gamma) \arrow[d, "s^*"'] \\ (\mathcal{P}\Sigma(G_X), \rho_{\mathcal{P}\Sigma(G_X)}) \arrow[r, leftrightarrow] & (\mathcal{P}\Sigma(G_X)^*, \rho^*_{\mathcal{P}\Sigma(G_X)}) \end{tikzcd}$$

Furthermore, we have $\operatorname{codim}(P_X^{\underline{d}_S}) = \dim(\Sigma_\Gamma^{\underline{d}_S})$.

PROOF. The properties of the vertical maps have been shown in Theorem 3.2.8 and Proposition 4.1.14, where for the first one we also use the isomorphism between $\mathcal{P}\Sigma(G_X)$ and $\overline{\mathcal{OP}}^1(G)$ established in Lemma 4.1.6. The horizontal map is just the map between a poset and its dual.

The claim about the dimensions follows from $\rho^*_{\mathcal{P}\Sigma(G_X)}$ being the dual grading of $\rho_{\mathcal{P}\Sigma(G_X)}$ by Lemma 4.1.6.

$\square$

REMARK 4.1.16. The dual graph G is not metrized and the theorem holds for every metrized graph whose underlying graph is G. A geometrically meaningful way of metrizing is given by the following well-known procedure: Let $\mathcal{X} \to B$ be a regular smoothing over the spectrum of a valuation ring whose generic fiber is smooth and special fiber X. Then one can take as edge lengths the valuations of the smoothing parameter at the nodes. In this case $\mathrm{Pic}^g(\Gamma)$ is in an appropriate sense the tropicalization of the Néron model of the corresponding family of Jacobians (see **[9]** for details). Note however that $\dim(P^g_X) = g$, whereas $\dim(\mathrm{Pic}^g(\Gamma)) = b_1(G)$.

REMARK/PROBLEM 4.1.17. Note also the following interesting duality: we saw that we have an identification $P^{\underline{d}_S}_X \cong P^{\underline{d}_S}_{X^\nu_S} \cong \mathrm{Pic}^{\underline{d}_S}(X^\nu_S)$ where X^ν_S is the partial normalization of X at nodes corresponding to edges in S. Recall that X^ν_S has dual graph $G - S$. On the other hand we saw in Lemma 4.1.13, that for $T \in T(\underline{d}_S)$, we have $\overline{\Sigma^{\underline{d}_S}_\Gamma} \cong \overline{\Sigma^{\underline{g}_S}_{\Gamma - S^c_T}}$. In both cases, one is able to write any stratum as a maximal dimensional stratum of the Jacobian on a related object. Then switching between $\overline{P^{\underline{d}_S}_X}$ and $\overline{\Sigma^{\underline{d}_S}_\Gamma}$ corresponds, after fixing a $T \in T(\underline{d}_S)$, to switching $G - S$ and $G - S^c_T$.

4.2. The tropical universal Picard variety in degree g

The description of $\mathrm{Pic}^g(\Gamma)$ in terms of break divisors allows the construction of a space parametrizing all equivalence classes of divisors on genus g stable metric graphs up to automorphisms, $(P^g_g)^{trop}$. We want to view this as a tropical analogue of the universal degree g compactified Picard variety $\overline{P}^g_g$ on the algebro-geometric side. While we are not yet able to establish that $(P^g_g)^{trop}$ is in an adequate sense the tropicalization of $\overline{P}^g_g$, we will establish a stratification of $(P^g_g)^{trop}$ that exhibits the already encountered strata-reversal phenomena in passing to the tropical side.

4.2.1. Redefining $\overline{\mathcal{OP}}^1_g$ and $[\mathcal{OP}^1_g]$ in terms of break divisors. We start with the indexing poset and define as a set:

$$\mathcal{P}\Sigma_g = \{(G, \underline{d}_S) |\, G \in \mathcal{SG}_g, \underline{d}_S \in \mathcal{P}\Sigma(G)\}.$$

That is, elements of $\mathcal{P}\Sigma_g$ consist of pairs $(G, \underline{d}_S)$ where G is a stable genus g graph, $S \subset E(G)$ is not disconnecting and $\underline{d}_S$ is a break divisor on $G - S$.

Let $\gamma : G \to G/S_0 = H$ be a contraction. Recall that we defined the divisor $\underline{c}^{\gamma,S}$ by $\underline{c}^{\gamma,S}_v = |\{e \in S \cap S_0 |\, \gamma(e) = v\}|$ for $v \in V(H)$. For $\underline{d}_S \in \Sigma(G - S)$ we may view $\underline{d}_S$ as a divisor on G and thus can consider $\gamma_* \underline{d}_S$ on H. We will however usually view $\gamma_* \underline{d}_S$ as a divisor on $H - \gamma_* S = H - (S \setminus S_0)$.

Note that in general $\gamma_* \underline{d}_S \notin \Sigma(H - \gamma_* S)$. We have however:

LEMMA 4.2.1. *Let $\gamma : G \to G/S_0 = H$ be a contraction and $\underline{d}_S \in \Sigma(G - S)$. Then $\gamma_*(\underline{d}_S) + \underline{c}^{\gamma,S} \in \Sigma(H - \gamma_* S)$. In particular if $S = \emptyset$, i.e. $\underline{d} = \underline{d}_\emptyset \in \Sigma(G)$, then $\gamma_* \underline{d}$ is a break divisor on H.*

PROOF. There is a rooted 1-orientation on $G - S$ with no bioriented edge in S_0 such that $\underline{d}_S = \underline{d}^{O_S}$. Then $\gamma_* \underline{d}_S = \gamma_* \underline{d}^{O_S} = \underline{d}^{\gamma_* O_S} - \underline{c}_v^{\gamma,S}$ with $\gamma_* O_S$ a rooted 1-orientation on $H - \gamma_* S$ by Proposition 3.3.2. Thus $\gamma_*(\underline{d}_S) + \underline{c}^{\gamma,S'} = \underline{d}^{\gamma_* O_S}$ is a break divisor on $H - \gamma_* S$.

In case $S = \emptyset$ we have $\underline{c}^{\gamma,S'} = \underline{0}$ which proves the second claim.

□

DEFINITION 4.2.2. We view $\mathcal{P}\Sigma_g^*$ as a poset by setting $(G, \underline{d}_S) \leq (H, \underline{e}_T)$ if

(1) $G \leq H$ in $\mathcal{SG}_g^*$. That is, there is $S_0 \subset E(H)$ such that for the contraction of S_0 we have $\gamma : H \to H/S_0 = G$.
(2) There is a contraction γ such that $\underline{d}_S \leq \gamma_*(\underline{e}_T) + \underline{c}^{\gamma,T}$ in $\mathcal{P}\Sigma(G)^*$. That is, $S \subset \gamma_* T = T \setminus S_0$ and there is a presentation of $\underline{d}_S$ such that $\underline{d}_S - (\gamma_* T \setminus S) = \gamma_*(\underline{e}_T) + \underline{c}^{\gamma,T}$ on $G - \gamma_* T$.

The following proposition implies that this indeed defines a partial order:

LEMMA 4.2.3. *The map $\phi : (\overline{\mathcal{OP}}_g^1)^* \to \mathcal{P}\Sigma_g^*$ that sends $(G, \overline{O}_S)$ to $(G, \underline{d}_S^{O_S})$ is an isomorphism of posets. The function*

$$\rho^*_{\mathcal{P}\Sigma_g}(G, \underline{d}_S) = |E(G)| + |S|$$

is the dual rank function of $\rho_{\overline{\mathcal{OP}}_g^1}$ under this isomorphism.

PROOF. The map is a bijection because, by Lemma 4.1.6, its restriction to $\overline{\mathcal{OP}}^1(G)$ is bijective.

If $(G, \overline{O}_S) \leq (H, \overline{O}_T)$ in $(\overline{\mathcal{OP}}_g^1)^*$, we have by definition $\overline{O}_S \leq \gamma_* \overline{O}_T$ in $\overline{\mathcal{OP}}^1(G)^*$ for some contraction $\gamma : H \to G$. Thus, by Proposition 4.1.6, $\underline{d}^{O_S} \leq \underline{d}^{\gamma_* O_T}$ in $\mathcal{P}\Sigma(G)^*$. By Proposition 3.3.2 we have $\gamma_* \underline{d}^{O_T} = \underline{d}^{\gamma_* O_T} - \underline{c}^{\gamma,T}$. Combining these we get

$$\underline{d}^{O_S} \leq \underline{d}^{\gamma_* O_T} = \gamma_* \underline{d}^{O_T} + \underline{c}^{\gamma,T}$$

in $\mathcal{P}\Sigma(G)^*$. This shows that ϕ is a morphism of posets.

If $(G, \underline{d}_S) \leq (H, \underline{e}_T)$ in $\mathcal{P}\Sigma_g^*$, we have $\underline{d}_S \leq \gamma_*(\underline{e}_T) + \underline{c}^{\gamma,T}$ in $\mathcal{P}\Sigma(G)^*$. Thus for orientations O_S and O_T giving $\underline{d}_S$ and $\underline{e}_T$ this becomes $\underline{d}^{O_S} \leq \gamma_*(\underline{e}^{O_T}) + \underline{c}^{\gamma,T}$. By Proposition 3.3.2 this implies $\underline{d}^{O_S} \leq \underline{e}^{\gamma_* O_T}$ and from Proposition 4.1.6 we get $\overline{O}_S \leq \gamma_* \overline{O}_T$. This shows that the inverse of ϕ is a morphism of posets.

Finally recall that we defined $\rho_{\overline{\mathcal{OP}}_g^1}(G,\overline{O}_S) = 3g-3-|E(G)|+g(G-S)$. Thus the maximum of $\rho_{\overline{\mathcal{OP}}_g^1}$ is $4g-3$ which is attained for G a single vertex with no edges. Hence the dual rank function is

$$\rho^*_{\overline{\mathcal{OP}}_g^1}(G,\overline{O}_S) = 4g-3-3g+3+|E(G)|-g(G-S) = |E(G)|+|S|.$$

This proves the last claim.

□

We proceed by giving a poset isomorphic to $[\mathcal{OP}_g^1]^*$ in terms of break divisors. Recall that $[\mathcal{OP}_g^1]$ was obtained from $\overline{\mathcal{OP}}_g^1$ by identifying elements that differ by an automorphism of a graph.

An automorphism σ of G acts on the elements of $\mathcal{P}\Sigma_g$: For a subset $S \subset E(G)$ we get $\sigma(S) \subset E(G)$ and for a divisor $\underline{d} = \Sigma_i a_i(p_i)$ we get $\sigma(\underline{d}) = \Sigma_i a_i\sigma(p_i)$. Thus we can set

$$\sigma(G,\underline{d}_S) = (G,\sigma(\underline{d}_S)).$$

We set analogously to the case of orientations $(G,\underline{d}_S) \sim (H,\underline{e}_T)$ if $G = H$ and there is $\sigma \in \mathrm{Aut}(G)$ such that $\sigma(\underline{d}_S) = \underline{e}_T$. We define

$$\overline{\mathcal{P}\Sigma}_g = \mathcal{P}\Sigma_g/\sim.$$

Elements of $\overline{\mathcal{P}\Sigma}_g$ will be marked by an overline, e.g. $\overline{(G,\underline{d}_S)}$. We view $\overline{\mathcal{P}\Sigma}_g^*$ as the poset with underlying set $\overline{\mathcal{P}\Sigma}_g$ and $\overline{(G,\underline{d}_S)} \le \overline{(H,\underline{e}_T)}$ if there are $(G,\underline{d}_S) \in \overline{(G,\underline{d}_S)}$ and $(H,\underline{e}_T) \in \overline{(H,\underline{e}_T)}$ with $(G,\underline{d}_S) \le (H,\underline{e}_T)$ in $\mathcal{P}\Sigma_g^*$. One easily checks that this gives a partial order. As equivalent elements have the same rank, $\rho^*_{\mathcal{P}\Sigma_g^*}$ is also a rank function on $\overline{\mathcal{P}\Sigma}_g^*$.

Lemma 4.2.4. *The map $\phi : [\mathcal{OP}_g^1]^* \to \overline{\mathcal{P}\Sigma}_g^*$ that sends $(G,[O_S])$ to $\overline{(G,\underline{d}_S^{O_S})}$ is an isomorphism of posets. The function*

$$\rho^*_{\overline{\mathcal{P}\Sigma}_g}(G,\underline{d}_S) = |E(G)|+|S|$$

is the dual rank function of $\rho_{[\mathcal{OP}_g^1]}$ under this isomorphism.

Proof. The equivalence relations defined on $\overline{\mathcal{OP}}_g^1$ and $\mathcal{P}\Sigma_g$ and the induced partial order are clearly the same under the isomorphism of Lemma 4.2.3. Thus the claim follows from that lemma.

□

4.2.2. Constructing $(P_g^g)^{trop}$. We will construct $(P_g^g)^{trop}$ analogously to the construction of M_g^{trop} by defining cones over some fixed combinatorial data and then specify a way in which to glue. The containment relations will, by construction, mirror the partial order of $\overline{\mathcal{P}\Sigma}_g^*$. For a metric graph Γ with model G we will assume G to be stable if not specified otherwise.

Suppose we have a (non-metrized) graph G, $S \in \mathcal{C}(G)$ and $\underline{d}_S \in \Sigma(G-S)$. Consider tuples $(l_1, \ldots l_{|E(G)|}, d_{i_1}, \ldots, d_{i_{|S|}}) \in \mathbb{R}_{>0}^{|E(G)|+|S|}$ with $\{i_1, \ldots, i_{|S|}\} \subset \{1, \ldots, |E(G)|\}$ and $d_{i_j} \leq l_{i_j}$ for all j. We want to interpret them as divisors on a metric graph Γ with underlying graph G. To that end choose an indexing of the edges of G by $\{1, \ldots, |E(G)|\}$ and a reference 0-orientation O on G. Then we set $\Gamma_{(l_1, \ldots l_{|E(G)|})}$ to be the metric graph with underlying graph G and length l_i on the i-th edge. We set $\underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})}$ to be the effective divisor on Γ given as follows: for every d_{i_j} consider the i_j-th edge e_{i_j} and let v_j be the source of that edge in O. Let p_j be the point of e_{i_j} that has distance d_{i_j} from v_j and set $\underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})} = \Sigma_j (p_j)$. By definition we have $|\underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})}| = |S|$.

EXAMPLE 4.2.5. Let G be the graph on two vertices v_1 and v_2 with two edges e_1 and e_2 between them. Let the reference orientation be given as the directed cut from v_1 to v_2 and $S = \{e_1\}$. Then $(1, 1, 1/3)$ corresponds to the metric graph with edge length 1 on both edges together with the divisor (p_1) where p_1 is the point on e_1 having distance $1/3$ from v_1.

With this we set for $S \in \mathcal{C}(G)$ and $\underline{d}_S \in \Sigma(G-S)$:

$$\Sigma_G^{\underline{d}_S} = \{(l_1, \ldots l_{|E(G)|}, d_{i_1}, \ldots, d_{i_{|S|}}) \,|\, 0 < d_{i_j} < l_{i_j}\} \subset \mathbb{R}^{|E(G)|+|S|}.$$

The set $\Sigma_G^{\underline{d}_S}$ as defined above does not depend on $\underline{d}_S$. However for two different $\underline{d}_S$ we want to have two copies of the same set and thus view $\underline{d}_S$ as an index. The idea behind that is the following:

LEMMA 4.2.6. *Let* $(l_1, \ldots l_{|E(G)|}, d_{i_1}, \ldots, d_{i_{|S|}}) \in \Sigma_G^{\underline{d}_S}$. *Then* $\underline{d}_S + \underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})}$ *is a break divisor on* $\Gamma_{(l_1, \ldots l_{|E(G)|})}$.

PROOF. By definition, we can write $\underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})} = \Sigma_{e_i \in S, p_i \in e_i^o} (p_i)$. Thus any presentation of $\underline{d}_S$ as a break divisor on $G - S$ immediately gives a presentation of $\underline{d}_S + \underline{d}_{(d_{i_1}, \ldots, d_{i_{|S|}})}$ as a break divisor on G.

□

Let $\underline{d} \in \Sigma_\Gamma^{\underline{d}_S}$, i.e. a break divisor on Γ such that $\underline{d} = \underline{d}_S + \Sigma_{e_i \in S, p_i \in e_i^o} (p_i)$. Then we can define an element $\mathrm{cord}(\underline{d})$ in $\Sigma_G^{\underline{d}_S}$ as follows: let the first $|E(G)|$ coordinates of $\mathrm{cord}(\underline{d})$ be given by the edge lengths of Γ. The remaining $|S|$ coordinates correspond to edges e_i in S. Let v_i be the source of e_i in O and $p_i \in e_i^o$ as above. Then let the value of $\mathrm{cord}(\underline{d})$ at e_i be the distance of p_i from v_i.

LEMMA 4.2.7. *Let* Γ *be a metric graph with model* G. *Then the map* $\mathrm{cord} : \Sigma_\Gamma^{\underline{d}_S} \to \Sigma_G^{\underline{d}_S}$ *is an embedding, i.e. an inclusion that is a homeomorphism onto its image.*

PROOF. Let $\pi_S : \Sigma_G^{\underline{d}_S} \to \mathbb{R}^{|E(G)|}$ be the projection to the first $|E(G)|$ coordinates. Then the image of cord is by definition contained in $\pi_S^{-1}(l_1, \ldots l_{|E(G)|})$ where $l_1, \ldots l_{|E(G)|}$ are the edge-lengths of Γ.

Let div : $\pi_S^{-1}(l_1,\dots l_{|E(G)|}) \to \Sigma_\Gamma^{\underline{d}_S}$ be the map $(l_1,\dots l_{|E(G)|}, d_{i_1},\dots,d_{i_{|S|}}) \to \underline{d}_S + \underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}$ on Γ.

Then cord $\circ$ div clearly gives the identity on $\Sigma_\Gamma^{\underline{d}_S}$, thus cord is injective.

We have $\pi_S^{-1}(l_1,\dots l_{|E(G)|}) \cong \Pi_{e_i\in S} e_i^o$ where the e_i are viewed as edges of Γ. On the other hand, $\Sigma_\Gamma^{\underline{d}_S} = \Pi_{e_i\in S} e_i^o \times \{\underline{d}_S\}$. Under these identifications it is clear that cord is a homeomorphism onto its image, $\pi_S^{-1}(l_1,\dots l_{|E(G)|})$.

$\square$

The closure $\overline{\Sigma_G^{\underline{d}_S}}$ of these sets in $\mathbb{R}^{|E(G)|+|S|}$ then is given as

$$\overline{\Sigma_G^{\underline{d}_S}} = \{(l_1,\dots l_{|E(G)|}, d_{i_1},\dots,d_{i_{|S|}}) \,|\, 0 \le d_{i_j} \le l_{i_j}\} \subset \mathbb{R}^{|E(G)|+|S|}.$$

Interpreting these elements as divisors on a metric graph allows two types of new objects: First, analogous to the description in the previous section, letting the d_{i_j} be equal to either zero or to l_{i_j} corresponds to allowing the points on edges in S to move to the vertices adjacent to the edge. Second, analogous to the construction of M_g^{trop}, allowing the edge lengths to go to zero corresponds to contracting the edge.

We first need to make some adjustments if $l_i = 0$ for some i.

For a tuple $(l_1,\dots l_{|E(G)|}, d_{i_1},\dots,d_{i_{|S|}}) \in \mathbb{R}^{|E(G)|+|S|}$ let $S_{(l_1,\dots l_{|E(G)|})} = \{e_i|\, l_i = 0\} \subset E(G)$ be the set of edges on which $l_i = 0$. In this case we cannot define $\Gamma_{(l_1,\dots l_{|E(G)|})}$ as before, since we require the edge lengths on a metric graph to be greater than zero. Instead define it as the metric graph Γ with model $G/S_{(l_1,\dots l_{|E(G)|})}$ and lengths l_i for all $l_i \neq 0$. Here the indexing and reference orientation are the ones inherited from G. Then $\underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}$ will be the divisor on $\Gamma_{(l_1,\dots l_{|E(G)|})}$ that, as before, is given by points at distance d_{i_j} from the source of the i_j-th edge. If $l_{i_j} = 0$ we need to have $d_{i_j} = 0$ and the corresponding point of $\underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}$ will be the vertex of $G/S_{(l_1,\dots l_{|E(G)|})}$ that the edge e_{i_j} gets contracted to. Thus $\underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}$ is a divisor on $\Gamma_{(l_1,\dots l_{|E(G)|})}$.

LEMMA 4.2.8. *With notation as above, let* $(l_1,\dots l_{|E(G)|}, d_{i_1},\dots,d_{i_{|S|}}) \in \overline{\Sigma_G^{\underline{d}_S}}$ *and* $\gamma : G \to G/S_{(l_1,\dots l_{|E(G)|})} = H$. *Then* $\gamma_*\underline{d}_S + \underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})} \in \overline{\Sigma_{\Gamma_{(l_1,\dots l_{|E(G)|})}}^{\gamma_*\underline{d}_S+\underline{c}^{\gamma,S}}}$. *In particular, it is a break divisor on* $\Gamma_{(l_1,\dots l_{|E(G)|})}$.

PROOF. Note that by construction we can write $\underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})} = \underline{c}^{\gamma,S} + \Sigma_{e_i\in\gamma_*S, p_i\in e_i}(p_i)$. By Lemma 4.2.1 $\gamma_*\underline{d}_S + \underline{c}^{\gamma,S} \in \Sigma(H - \gamma_*S)$. Thus using Lemma 4.1.10:

$$\gamma_*\underline{d}_S + \underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})} = \gamma_*\underline{d}_S + \underline{c}^{\gamma,S} + \Sigma_{e_i\in\gamma_*S, p_i\in e_i}(p_i) \in \overline{\Sigma_{\Gamma_{(l_1,\dots l_{|E(G)|})}}^{\gamma_*\underline{d}_S+\underline{c}^{\gamma,S}}}.$$

$\square$

Recall that earlier we considered $\pi_S : \overline{\Sigma_G^{\underline{d}_S}} \to \mathbb{R}^{|E(G)|}$. Analogously we let $\overline{\pi}_S : \overline{\Sigma_G^{\underline{d}_S}} \to \mathbb{R}^{|E(G)|}$ be the projection to the first $|E|$ coordinates. Fix some $(l_1, \dots l_{|E(G)|}) \in \overline{\pi}_S(\overline{\Sigma_G^{\underline{d}_S}})$ and set as before $S_{(l_1,\dots l_{|E(G)|})} = \{e_i | l_i = 0\}$, $\gamma : G \to G/S_{(l_1,\dots l_{|E(G)|})} = H$ and $\Gamma = \Gamma_{(l_1,\dots l_{|E(G)|})}$ with underlying graph H. Then by Lemma 4.2.8 we can extend the map div defined in the proof of Lemma 4.2.7 to a map

$$\overline{\mathrm{div}} : \overline{\pi}_S^{-1}(l_1, \dots l_{|E(G)|}) \to \overline{\Sigma_\Gamma^{\gamma_* \underline{d}_S + \underline{c}^{\gamma,S}}}$$

by setting

$$\overline{\mathrm{div}}((l_1, \dots l_{|E(G)|}, d_{i_1}, \dots, d_{i_{|S|}})) = \gamma_* \underline{d}_S + \underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}.$$

While $\overline{\mathrm{div}}$ remains surjective, it is no longer injective. Indeed, as we now allow the points corresponding to edges in S to be on vertices of G, the presentation of $\gamma_* \underline{d}_S + \underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})}$ is no longer unique after fixing a presentation of $\gamma_* \underline{d}_S + \underline{c}^{\gamma,S}$:

Example 4.2.9. Let G be the graph on two vertices v_1 and v_2 with three edges e_1, e_2 and e_3 between them. Let the reference orientation be given as the directed cut from v_1 to v_2 and $S = \{e_1, e_2\}$. Then for $(1, 1, 1, 0, 1)$ and $(1, 1, 1, 1, 0)$ in $\overline{\pi}_S^{-1}(1, 1, 1)$ we have $\underline{d}_{(1,0)} = \underline{d}_{(0,1)} = (v_1) + (v_2)$.

We introduce an equivalence relation, $\sim$, on $\overline{\Sigma_G^{\underline{d}_S}}$ that turns $\overline{\mathrm{div}}$ by definition injective, i.e. after fixing the l_i, equivalence classes are given by fibers of $\overline{\mathrm{div}}$. We set

$$(l_1, \dots l_{|E(G)|}, d_{i_1}, \dots, d_{i_{|S|}}) \sim (l'_1, \dots l'_{|E(G)|}, d'_{i_1}, \dots, d'_{i_{|S|}})$$

on $\overline{\Sigma_G^{\underline{d}_S}}$ if $l_i = l'_i$ for all i and $\underline{d}_{(d_{i_1},\dots,d_{i_{|S|}})} = \underline{d}_{(d'_{i_1},\dots,d'_{i_{|S|}})}$ as divisors on $\Gamma_{(l_1,\dots l_{|E(G)|})}$.

We then set:

$$(P_G^{\underline{d}_S})^{trop} = \overline{\Sigma_G^{\underline{d}_S}} / \sim .$$

The projection $\overline{\pi}_S : \overline{\Sigma_G^{\underline{d}_S}} \to \mathbb{R}^{|E(G)|}$ is constant on equivalence classes and descends to a projection $\widetilde{\pi}_S : (P_G^{\underline{d}_S})^{trop} \to \mathbb{R}^{|E(G)|}$. With notation as before, by construction the map $\overline{\mathrm{div}}$ descends to a homeomorphism

$$\widetilde{\mathrm{div}} : \widetilde{\pi}_S^{-1}(l_1, \dots l_{|E(G)|}) \to \overline{\Sigma_{\Gamma_{(l_1,\dots l_{|E(G)|})}}^{\gamma_* \underline{d}_S + \underline{c}^{\gamma,S}}}. \tag{12}$$

Thus

$$(P_G^{\underline{d}_S})^{trop} \cong \bigsqcup_{(l_1,\dots l_{|E(G)|}) \in \mathbb{R}_{\geq 0}^{|E(G)|}} \overline{\Sigma_{\Gamma_{(l_1,\dots l_{|E(G)|})}}^{\gamma_* \underline{d}_S + \underline{c}^{\gamma,S}}} \tag{13}$$

for γ contracting $S_{(l_1,\dots l_{|E(G)|})}$ defined as above, varying with the $(l_1, \dots, l_{|E(G)|})$.

Furthermore, the inclusion $\Sigma_G^{\underline{d}_S} \hookrightarrow \overline{\Sigma_G^{\underline{d}_S}}$ descends to an inclusion $\Sigma_G^{\underline{d}_S} \hookrightarrow (P_G^{\underline{d}_S})^{trop}$ since $\sim$ by Lemma 4.2.7 does not identify points in the dense open set $\Sigma_G^{\underline{d}_S}$.

Summarizing, we get:

$$\begin{tikzcd}
\Sigma_G^{\underline{d}_S} \arrow[dd, hook] \arrow[dr, hook] \arrow[drr, "\pi_S"] & & \\
& (P_G^{\underline{d}_S})^{trop} \arrow[r, "\widetilde{\pi}_S"] & \mathbb{R}^{|E(G)|} \\
\overline{\Sigma_G^{\underline{d}_S}} \arrow[ur, two heads] \arrow[urr, "\overline{\pi}_S"'] & &
\end{tikzcd}$$

and for $(l_1, \ldots, l_{|E(G)|}) \in \pi_S(\Sigma_G^{\underline{d}_S})$:

$$\begin{tikzcd}
\pi_S^{-1}((l_1, \ldots, l_{|E(G)|})) \arrow[dd, hook] \arrow[dr, hook] \arrow[drr, hook, "\mathrm{div}"] & & \\
& \widetilde{\pi}_S^{-1}((l_1, \ldots, l_{|E(G)|})) \arrow[r, leftrightarrow, "\widetilde{\mathrm{div}}"] & \overline{\Sigma_{\Gamma_{(l_1,\ldots l_{|E(G)|})}}^{\underline{d}_S}} \\
\overline{\pi}_S^{-1}((l_1, \ldots, l_{|E(G)|})) \arrow[ur, two heads] \arrow[urr, two heads, "\overline{\mathrm{div}}"'] & &
\end{tikzcd}$$

Now fix an indexing of the edges and a reference orientation as before for all graphs $G \in \mathcal{SG}_g$.

For any $(G, \underline{d}_S) \leq (H, \underline{e}_R)$ in $\mathcal{P}\Sigma_g^*$, we get an inclusion

$$i_{G \to H} : (P_G^{\underline{d}_S})^{trop} \to (P_H^{\underline{e}_R})^{trop}.$$

To see this, first suppose that $G = H$ and $\underline{d}_S \leq \underline{e}_R$. Then for any contraction $\gamma : G \to G/S_{(l_1,\ldots l_{|E(G)|})}$ by Proposition 3.3.2 we have $\gamma_* \underline{d}_S + \underline{c}^{\gamma,S} \leq \gamma_* \underline{e}_R + \underline{c}^{\gamma,S}$. Thus by Proposition 4.1.14 we get $\overline{\Sigma_{\Gamma_{(l_1,\ldots l_{|E(G)|})}}^{\gamma_* \underline{d}_S + \underline{c}^{\gamma,S}}} \subset \overline{\Sigma_{\Gamma_{(l_1,\ldots l_{|E(G)|})}}^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,S}}}$ for a fixed $(l_1, \ldots l_{|E(G)|}) \in \mathbb{R}^{|E(G)|}$ and $S_{(l_1,\ldots l_{|E(G)|})}$ and γ defined as before. By (13), this gives an inclusion

$$i_{G \to G} : (P_G^{\underline{d}_S})^{trop} \to (P_G^{\underline{e}_R})^{trop}.$$

Let $\gamma : H \to H/S_0 = G$ be the contraction of $S_0 \subset E(H)$. We then construct an inclusion $i_{G \to H} : (P_G^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,R}})^{trop} \to (P_H^{\underline{e}_R})^{trop}$. Indeed, assume that the indexing of the edges in H is such that under the inclusion $E(G) \subset E(H)$ the first $|E(G)|$ edges of H are those of $E(G)$ and the remaining those of S_0. Also assume that the reference orientation on the edges of G is the same in both G and H (these assumptions are not necessary for the definition, but simplify notation). Then the inclusion maps

$$(l_1, \dots l_{|E(G)|}, d_{i_1}, \dots, d_{i_{|\gamma_* R|}}) \in (P_G^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,R}})^{trop}$$

to

$$(l_1, \dots l_{|E(G)|}, 0, \dots, 0, d_{i_1}, \dots, d_{i_{|\gamma_* R|}}, 0, \dots, 0) \in (P_H^{\underline{e}_R})^{trop}.$$

There are $|S_0|$ zeros after the l_i and $|R \cap S_0|$ zeros after the d_{i_j}.

Combining these two inclusions, for a general inequality $(G, \underline{d}_S) \leq (H, \underline{e}_R)$ in $\mathcal{P}\Sigma_g^*$ we get an inclusion $(P_G^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,R}})^{trop} \to (P_H^{\underline{e}_R})^{trop}$ and, since by the definition of the partial order on $\mathcal{P}\Sigma_g^*$ we have $\underline{d}_S \leq \gamma_* \underline{e}_R + \underline{c}^{\gamma,R}$, another inclusion $(P_G^{\underline{d}_S})^{trop} \to (P_G^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,R}})^{trop}$. We then take $i_{G \to H}$ to be the composition of these two inclusions.

This together with (13) gives a decomposition

$$(P_H^{\underline{e}_R})^{trop} = \bigsqcup_{(G, \underline{d}_S) \leq (H, \underline{e}_R)} \Sigma_G^{\underline{d}_S}. \tag{14}$$

EXAMPLE 4.2.10. Let H be the graph with a single vertex v of weight two and one loop e and G the graph with a single vertex v of weight three. Let $R = \{e\}$ and $\underline{e}_R = 2(v) \in \Sigma(H - R)$. Then $(P_H^{\underline{e}_R})^{trop} = \{(l, d) |\, 0 \leq d \leq l\}/\sim$ where $\sim$ identifies the points $(l, 0)$ with the points (l, l). For $\gamma : H \to G$ contracting e we have $\gamma_* \underline{e}_R + \underline{c}^{\gamma,R} = 3(v)$ and $(P_G^{\gamma_* \underline{e}_R + \underline{c}^{\gamma,R}})^{trop}$ is a single point. Then $i_{G \to H}$ maps this point to $(0, 0)$ in $(P_H^{\underline{e}_R})^{trop}$. Setting $\underline{e}_\emptyset = 3(v)$ we have $(P_H^{\underline{e}_\emptyset})^{trop} = \{(l) |\, 0 \leq l\}$ and a map $i_{H \to H}$ that maps $l \in (P_H^{\underline{e}_\emptyset})^{trop}$ to $(l, 0) = (l, l)$ in $(P_H^{\underline{e}_R})^{trop}$. Thus the decomposition of (14) is $\{(0, 0)\} \cup \{(l, 0) |\, 0 < l\} \cup \{(l, d) |\, 0 < d < l\}$. The fiber of the projection $\mathbb{R}^2 \to \mathbb{R}$ to the first component over some l is $\{d |\, 0 \leq d \leq l\}/\sim$ which is homeomorphic to $\overline{\Sigma_{\Gamma_l}^{\underline{e}_R}} = \Sigma(\Gamma_l)$, illustrating the decomposition of (13).

Finally we have to account for automorphisms. We view an automorphism $\sigma \in \mathrm{Aut}(G)$ as a map of half-edges of the graph. It induces a homeomorphism $i_\sigma : (P_G^{\underline{d}_S})^{trop} \to (P_G^{\sigma(\underline{d}_S)})^{trop}$ as follows: σ acts on the indexing of the edges sending i to the index of the image of the i-th edge. Recall that we fixed a reference orientation O on G. We can consider the orientation $\sigma(O)$ on G defined as before. For an element $(l_1, \dots, l_{|E(G)|}, d_{i_1}, \dots, d_{i_{|S|}}) \in (P_G^{\underline{d}_S})^{trop}$ set

$$\sigma(d_{i_j}) = \begin{cases} d_{\sigma^{-1}(i_j)} & \text{if } e_{i_j} \text{ is oriented in } O \text{ as in } \sigma(O) \\ l_{\sigma^{-1}(i_j)} - d_{\sigma^{-1}(i_j)} & \text{if } e_{i_j} \text{ is oriented in } O \text{ different than in } \sigma(O). \end{cases}$$

We set

$$\sigma((l_1, \dots, l_{|E(G)|}, d_{i_1}, \dots, d_{i_{|S|}})) = (l_{\sigma^{-1}(1)}, \dots, l_{\sigma^{-1}(|E(G)|)}, \sigma(d_{i_1}), \dots, \sigma(d_{i_{|S|}})).$$

By construction we have $\underline{d}_{(\sigma(d_{i_1}), \dots, \sigma(d_{i_{|S|}}))} = \sigma(\underline{d}_{(d_{i_1}, \dots, d_{i_{|S|}})})$ under the isomorphism $\Gamma_{(l_1, \dots, l_{|E(G)|})} \to \Gamma_{(l_{\sigma^{-1}(1)}, \dots, l_{\sigma^{-1}(|E(G)|)})}$ induced by σ.

EXAMPLE 4.2.11. Let H and $\underline{e}_R$ be as in the example 4.2.10 above. Then there is an automorphism of H fixing both v and e but interchanging the half edges of e. Then σ maps $(l,d) \in (P_H^{\underline{e}_R})^{trop}$ to $(l, l-d)$.

EXAMPLE 4.2.12. Let G be the graph with two vertices v_1, v_2, two edges e_1, e_2 between them and weights $\underline{g} = \underline{1} = (v_1) + (v_2)$. Fix as reference orientation the directed cut from v_1 to v_2. Let $S_1 = \{e_1\}$ and $S_2 = \{e_2\}$. Consider $(l,l,d_1) \in (P_G^{\underline{1}_{S_1}})^{trop}$ with $0 < d_1 < l$. Then the automorphism $\sigma_1 \in \mathrm{Aut}(G)$ interchanging the edges of G identifies $(l,l,d_1) \in (P_G^{\underline{1}_{S_1}})^{trop}$ with $(l,l,d_1) \in (P_G^{\underline{1}_{S_2}})^{trop}$. Note that while these elements are given by the same coordinates they are defined in different sets. The automorphism $\sigma_2 \in \mathrm{Aut}(G)$ interchanging the vertices identifies (l,l,d_1) with $(l,l,l-d_1) \in (P_G^{\underline{1}_{S_1}})^{trop}$.

The maps $i_{G\to H}$ and i_σ will be the gluing data for the strata $(P_G^{\underline{d}_S})^{trop}$:

DEFINITION 4.2.13. We set

$$(P_g^g)^{trop} = \varinjlim((P_G^{\underline{d}_S})^{trop}, i_{G\to H}, i_\sigma)$$

where G runs over all stable genus g graphs, $S \in \mathcal{C}(G)$, $\underline{d}_S \in \Sigma(G-S)$ and the maps run over all inclusions $i_{G\to H}$ defined above and automorphisms σ of the graph G.

Let $\mathrm{Aut}(G, \underline{d}_S) \subset \mathrm{Aut}(G)$ be the subgroup of automorphisms that fix S as a set and $\underline{d}_S$ on $G-S$. In other words $\mathrm{Aut}(G,\underline{d}_S)$ is the stabilizer of $\underline{d}_S$ under the action of $\mathrm{Aut}(G)$ on $\mathcal{P}\Sigma(G)$. Elements of $\mathrm{Aut}(G,\underline{d}_S)$ are those whose associated homeomorphism i_σ maps $(P_G^{\underline{d}_S})^{trop}$ to $(P_G^{\underline{d}_S})^{trop}$. Thus we can set

$$[\Sigma_G^{\underline{d}_S}] = \Sigma_G^{\underline{d}_S}/\mathrm{Aut}(G,\underline{d}_S)$$

and the map $\Sigma_G^{\underline{d}_S} \to (P_g^g)^{trop}$ descends to an inclusion $[\Sigma_G^{\underline{d}_S}] \to (P_g^g)^{trop}$. Then (14) descends to a decomposition

$$(P_g^g)^{trop} = \bigsqcup_{\overline{(G,\underline{d}_S)} \in \overline{\mathcal{P}\Sigma}_g} [\Sigma_G^{\underline{d}_S}]. \tag{15}$$

PROPOSITION 4.2.14. *The above decomposition is a graded stratification of* $(P_g^g)^{trop}$ *by* $(\overline{\mathcal{P}\Sigma}_g^*, \rho^*_{\overline{\mathcal{P}\Sigma}_g})$ *under the map* $s^* : ((P_g^g)^{trop})^{strat} \to \overline{\mathcal{P}\Sigma}_g^*$ *that sends* $[\Sigma_G^{\underline{d}_S}]$ *to* $\overline{(G,\underline{d}_S)}$.

PROOF. The map is bijective, since the elements of the decomposition are indexed by elements of the poset.

The $\Sigma_G^{\underline{d}_S}$ are dense and open in $(P_G^{\underline{d}_S})^{trop}$. Thus they are locally closed in any $(P_H^{\underline{e}_R})^{trop}$ using the inclusion $i_{G\to H} : (P_G^{\underline{d}_S})^{trop} \to (P_H^{\underline{e}_R})^{trop}$. The $[\Sigma_G^{\underline{d}_S}]$ are endowed with the quotient topology and thus locally closed as well.

That s^* is a stratification follows by construction and (14).

We have $\dim([\Sigma_G^{\underline{d}_S}]) = \dim(\Sigma_G^{\underline{d}_S}) = |E(G)| + |S| = \rho^*_{\overline{\mathcal{P}\Sigma}_g}(\overline{(G,\underline{d}_S)})$.

$\square$

Since we assume G to be stable, $|E(G)|$ is maximal if G has trivial weights and every vertex valency three. In this case $|E(G)| = 3g - 3$. Choosing S as the complement of a spanning tree on such a graph, we get that maximal dimensional strata have dimension $4g - 3$. Thus, contrary to the relationship between $\mathrm{Pic}^g(\Gamma)$ and $\overline{P}^g_X$, we always have $\dim((P^g_g)^{trop}) = \dim(\overline{P}^g_g) = 4g - 3$. The unique minimal dimensional stratum has dimension 0 and corresponds to G consisting of a single vertex.

THEOREM 4.2.15. *We have the following diagram where the vertical arrows are stratifications by graded posets and the horizontal arrow is an order reversing bijection:*

$$\begin{tikzcd} \overline{P}^g_g \arrow[d, "s"'] & (P^g_g)^{trop} \arrow[d, "s^*"'] \\ (\overline{\mathcal{P}\Sigma}_g, \rho_{\overline{\mathcal{P}\Sigma}_g}) \arrow[r, leftrightarrow] & (\overline{\mathcal{P}\Sigma}^*_g, \rho^*_{\overline{\mathcal{P}\Sigma}_g}) \end{tikzcd}$$

In particular $\mathrm{codim}(P^{\underline{d}_S}_G) = \dim([\Sigma^{\underline{d}_S}_G])$.

PROOF. Combine Theorem 3.4.11, Lemma 4.2.4 and Proposition 4.2.14.

□

Define the forgetful map

$$\psi : (P^g_g)^{trop} \to M^{trop}_g$$

that sends an element $(l_1, \dots, l_{|E(G)|}, d_1, \dots, d_{|S|}) \in [\Sigma^{\underline{d}_S}_G]$ to $[(l_1, \dots, l_{|E(G)|})] \in M^{trop}_G$. Recall that elements $[(l_1, \dots, l_{|E(G)|})]$ of M^{trop}_G are defined up to automorphisms of G, thus the map is well defined.

PROPOSITION 4.2.16. *The fiber of ψ over a point $[(l_1, \dots, l_{|E(G)|})]$ of M^{trop}_G is homeomorphic to* $\mathrm{Pic}^g(\Gamma_{(l_1,\dots,l_{|E(G)|})})/\mathrm{Aut}(\Gamma_{(l_1,\dots,l_{|E(G)|})})$.

PROOF. After possibly passing to another graph by contracting some edges of G we can assume that $l_i \neq 0$. By (12) and the construction of $(P^g_g)^{trop}$ we can identify $\psi^{-1}(l_1, \dots, l_{|E(G)|})$ as the union

$$\psi^{-1}(l_1, \dots, l_{|E(G)|}) = \left(\bigcup_{\underline{d}_S \in \mathcal{P}\Sigma(G)} \overline{\Sigma^{\underline{d}_S}_{\Gamma_{(l_1,\dots l_{|E(G)|})}}} \right) / \mathrm{Aut}(\Gamma_{(l_1,\dots,l_{|E(G)|})}).$$

Now note that the gluing by the $i_{G \to G}$ is along strata and given by the partial order on $\mathcal{P}\Sigma(G)^*$. Thus by Proposition 4.1.14 we have that

$$\psi^{-1}(l_1, \dots, l_{|E(G)|}) \cong \mathrm{Pic}^g(\Gamma_{(l_1,\dots,l_{|E(G)|})})/\mathrm{Aut}(\Gamma_{(l_1,\dots,l_{|E(G)|})}).$$

□

Thus $(P^g_g)^{trop}$ parametrizes break divisors on stable metric graphs of fixed genus up to automorphisms.

We conclude by repeating the commutative diagram from the introduction, summarizing the results so far:

$$\begin{tikzcd}
\overline{P_g^g} \arrow[rrr, two heads] \arrow[ddd, two heads] & & & \overline{\mathcal{P}\Sigma}_g \arrow[r, leftrightarrow] \arrow[ddd, two heads] & \overline{\mathcal{P}\Sigma}_g^* \arrow[ddd, two heads] & & & (P_g^g)^{trop} \arrow[lll, two heads] \arrow[ddd, two heads] \\
 & \frac{\overline{P_X^g}}{\mathrm{Aut}(X)} \arrow[ul, hook] \arrow[r, two heads] \arrow[d] & \overline{\mathcal{P}\Sigma(G_X)} \arrow[urr, hook] \arrow[rrr, leftrightarrow] \arrow[d] & & & \overline{\mathcal{P}\Sigma(G_X)}^* \arrow[ull, hook] \arrow[d] & \frac{\mathrm{Pic}^g(\Gamma)}{\mathrm{Aut}(\Gamma)} \arrow[ur, hook] \arrow[l, two heads] \arrow[d] & \\
 & {[X]} \arrow[r, mapsto] \arrow[dl, mapsto] & {[G_X]} \arrow[rrr, leftrightarrow] \arrow[drr] & & & {[G_X]} \arrow[dll] & \Gamma \arrow[l, mapsto] \arrow[dr, mapsto] & \\
\overline{M}_g \arrow[rrr, two heads] & & & \mathcal{SG}_g \arrow[r, leftrightarrow] & \mathcal{SG}_g^* & & & (M_g)^{trop} \arrow[lll, two heads]
\end{tikzcd}$$

REMARK 4.2.17. We always assumed that the model G of a metric graph Γ is stable. But in an appropriate sense $(P_g^g)^{trop}$ parametrizes break divisors on any metric graph Γ of genus g up to automorphism: Let Γ be a metric graph with model G. If there is a stable model G' giving Γ, i.e. G is semistable, this is clear (note however that the construction above and the resulting decomposition is tied to the existence of a unique stable model). If Γ has no stable model, it needs to have a vertex of valency one on which the divisor $\underline{g}$ is not supported. For any model G of Γ, this corresponds to a weight zero vertex of valency one. Contracting all edges adjacent to such vertices gives a semistable graph G' (its semistable model). Let the metric graph corresponding to G' be Γ'. Then there is a homeomorphism $\Sigma(\Gamma) \cong \Sigma(\Gamma')$. This easily follows from the observation that the edge adjacent to a valency one vertex is a bridge and thus contained in every spanning tree, which implies that a break divisor on Γ cannot have points in the interior of the edge.

CHAPTER 5

Algebro-geometric consequences

5.1. Connectedness through codimension one

First, we want to study a property known as connectedness through codimension one:

Definition 5.1.1. Let $M = \bigsqcup_{p \in \mathcal{P}} M_p$ be a topological space of pure dimension d stratified by a graded poset $(\mathcal{P}, \rho)$. We say M is connected through codimension one (w.r.t. this stratification), if the space

$$\bigsqcup_{\dim(M_p) \geq d-1} M_p$$

is connected.

Remark 5.1.2. If M contains a unique connected stratum of maximal dimension, M obviously is connected through codimension one.

The question of being connected through codimension one has usually been studied for tropical varieties, but can equally be asked for the corresponding algebro-geometric objects. For $\overline{M_g}$ and $\overline{P}_g^g$ this is not an interesting question for the stratifications we have been considering, as both spaces contain a unique stratum of maximal dimension (which in particular is connected). The situation is different for $\overline{P}_X^g$ however, as the stratification in this case in general contains many strata of maximal dimension.

Proposition 5.1.3. *For any stable curve X, $\overline{P}_X^g$ is connected through codimension one.*

Proof. We show this in the following way: Given any two rooted 1-orientations O and O' on G, we construct a sequence of rooted 1-orientations $O = O_1, ..., O_k = O'$ where O_i is obtained from O_{i-1} by reversing the orientation of a single edge e. By construction we then have $(O_i)_{|G-e} = (O_{i-1})_{|G-e}$ and by Lemma 3.2.11 $(O_i)_{|G-e}$ is a rooted 1-orientation. Thus the strata corresponding to O_i and O_{i-1}, $P_X^{O_i}$ and $P_X^{O_{i-1}}$, both contain the codimension one stratum corresponding to $(O_i)_{|G-e}$ in their closure. Showing this for every O and O' implies the claim.

To do this, we first can assume by Lemma 1.2.19, that the same edge $e_1 = v_1v_2$ is bioriented in O and O'. By Lemma 1.2.24, O contains an arborescence T. Let $e_2 \in E(G) \setminus E(T)$ be an edge which is oriented differently in O and O'. Let O_2 be the 1-orientation obtained from O by orienting e as in O'. Then O_2 will still be rooted by Lemma 1.2.24, as T still is an arborescence of O_2. Then let $e_3 \in E(G) \setminus E(T)$ be an edge which is oriented differently in O_2 and O' and repeat the procedure to obtain O_3. Thus after finitely many such steps we reach a rooted 1-orientation O_n, such that all edges in $E(G) \setminus E(T)$ are oriented in O_n as in O'. Now let $e'_{n+1} \in E(T)$ be an edge which is oriented differently in O_n and O' and v the vertex it is directed towards in O_n. Then, since O' is rooted, there is a directed path P' from v_1 to v in O'. P' may not be a directed path in O_n, as it may contain edges of $E(T)$. Let P be the longest subpath of P', starting at v_1, that is directed also in O_n. If $P \neq P'$ let $e_{n+1} \in E(P')$ be the edge following the last vertex of P in P'. If $P = P'$ set $e_{n+1} = e'_{n+1}$. Then e_{n+1} is oriented differently in O_n and O'. By Lemma 1.2.25, we can find an arborescence T_1 of O_n containing P. By construction, $e_{n+1} \notin E(T_1)$ and thus reorienting e_{n+1} as in O' will give a rooted orientation O_{n+1} since T_1 will still be an arborescence. Repeating this procedure will eventually produce the orientation O' and we proved the claim.

□

5.2. Number of strata

Next, we can calculate the number of strata of fixed codimension in $\overline{P}^g_X$. Recall that for a set of non-disconnecting edges $S \subset E(G_X)$ and a break divisor $\underline{d} \in \Sigma(G - S)$, the codimension of the stratum $P^{\underline{d}_S}_X$ in $\overline{P}^g_X$ is $|S|$.

Definition 5.2.1. For a stable curve X let $c(i)$ denote the number of codimension i strata in $\overline{P}^g_X$.

Remark 5.2.2. As the subgraph $G-S$ has to be spanning and connected, the minimal dimensional strata have codimension $b_1(G) = |E(G)|-|V(G)|+1$. In this case $T := G-S$ is a spanning tree. Since on every tree T there is a unique break divisor, $c(b_1(G))$ is equal to the number of spanning trees of G. On the other hand, maximal dimensional strata correspond by Proposition 1.1.20 to classes of divisors on G up to linear equivalence and thus $c(0)$ equals $|\mathrm{Pic}^g(G)|$. It follows from Kirchoff's matrix-tree theorem, that the number of spanning trees equals $|\mathrm{Pic}^g(G)|$ and thus we have $c(0) = c(b_1(G))$.

Proposition 5.2.3. *Let X be a stable curve with dual graph G. Then we have for the number of codimension i strata of $\overline{P}^g_X$:*

$$c(i) = c(0)\binom{b_1(G)}{i}.$$

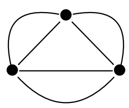
Proof. As mentioned, a codimension i stratum corresponds to a spanning subgraph H with $|E(G)|-|E(H)| = i$ and a break divisor on H. As there is a unique break divisor in each degree class, every break divisor on H corresponds by the previous remark to a spanning tree of H. Since H itself is spanning, this will also be a spanning tree of G. To count the number of strata, we thus can count the number of times a fixed spanning tree T is contained in spanning subgraphs H with $|E(G)|-|E(H)| = i$. But this is easy: Any collection of $b_1(G) - i$ edges not contained in T will give a subgraph as required and conversely, every H containing T is characterized by its edges not contained in T. As there are $b_1(G)$ edges with $e \in E(G)$ but $e \notin E(T)$, there are $\binom{b_1(G)}{b_1(G)-i} = \binom{b_1(G)}{i}$ strata corresponding to the spanning tree T (i.e. strata such that $T \subset H$ and the break divisor on H corresponds to T). Furthermore, if $T_1 \neq T_2$ are two different spanning trees, any stratum corresponding to them will be different: Either $H_1 \neq H_2$ or $H_1 = H_2$, but the multidegrees correspond to different spanning trees. Thus the total number of codimension i strata is given as $c(i) = c(b_1(G))\binom{b_1(G)}{i} = c(0)\binom{b_1(G)}{i}$.

□

In particular, this gives a description of the Euler characteristic of the boundary decomposition of $\overline{P}^g_X$ in terms of the number of spanning trees of the dual graph.

REMARK 5.2.4. Note the interesting symmetry $c(i) = c(b_1(G) - i)$, generalizing the above mentioned $c(0) = c(b_1(G))$. Furthermore $c(i)$ is always a multiple of $c(0)$.

EXAMPLE 5.2.5. Consider a curve X with the following dual graph G:



Then G has twelve spanning trees and we get for the number of codimension i strata:

i	0	1	2	3	4
c(i)	12	48	72	48	12

We want to point out two similar (to us) open questions, concerning the combinatorics of $\overline{P}^d_X$:

PROBLEM 5.2.6. A question in some sense inverse to the problem of this section is the following: A rooted 1-orientation on a connected subgraph of G corresponds to a stratum of $\overline{P}^g_X$. Then what is the number of strata (of fixed dimension) containing this stratum in its boundary? This would be interesting to know, because these combinatorics give a necessary criterion for the singularities at the boundary to be normal crossing. For example, a codimension one stratum will always be contained in the closure of exactly two maximal dimensional strata: adding the removed edge gives two possibilities of orienting it, both ways will give rooted orientations that are different from each other. But already for the question of how many maximal dimensional strata contain a given stratum of codimension two in their closure, the answer depends on the graph and the chosen stratum: the two removed edges can be oriented in four different ways and the obtained orientations will always be rooted. But they may be related to each other by a cycle reversal and thus give the same stratum.

PROBLEM 5.2.7. As we saw, the number of maximal dimensional strata of $\overline{P}^g_X$ equals the number $|\mathrm{Pic}^g(G)|$. This number in turn may be calculated as an evaluation of the so called Tutte polynomial of the graph. On the other hand, the number of maximal dimensional strata of $\overline{P}^{g-1}_X$ equals the number of totally cyclic orientations on $G - G_{br}$, the graph obtained from G by removing all bridges. Now the number of totally cyclic orientations on a connected component G_i of $G - G_{br}$ again is given as an evaluation of the Tutte polynomial of G_i. Thus the number of maximal dimensional strata of $\overline{P}^{g-1}_X$ is a sum of evaluations of the Tutte polynomials of the G_i. Is it then possible to express the number of maximal dimensional strata for any $\overline{P}^d_X$ as the sum of evaluations of the Tutte polynomial of certain subgraphs?

5.3. A map of compactified Picard varieties

Next, we want to point out a consequence of Lemma 3.1.2. It states that if $\underline{d}$ is balanced of degree g, then $\underline{d}-(v)$ is balanced of degree $g-1$ for any vertex v. Fixing such a vertex v allows us to define a map

$$\phi_v : \overline{P}^g_X \to \overline{P}^{g-1}_X.$$

At its heart, ϕ_v maps L to $L(-p)$ for some fixed $p \in C_v$. But while $\underline{d}-(v)$ is always balanced by Lemma 3.1.2, it might not be strictly balanced. In that case L and $\phi_v(L)$ will be defined on different curves: on every edge belonging to a directed cut we need to add a new exceptional component.

To be more precise, fix a smooth point p on the irreducible component C_v of X corresponding to v. Let L be a stably balanced line bundle on $\hat{X}_S$ of multidegree $\hat{\underline{d}}_S$ with $|\hat{\underline{d}}_S| = g$. Recall that $\hat{X}_S$ is obtained from X by inserting an exceptional component at each node contained in S. Then define $\phi_v(L)$ as follows: By Lemma 3.1.2, $\hat{\underline{d}}_S-(v)$ is balanced of degree $g-1$. Thus $\hat{\underline{d}}_S-(v)$ can be given by a 0-orientation O on $\hat{X}_S$ and set $S' = S \cup (G-S)_{cut}$ where $(G-S)_{cut}$ denotes the set of edges of $G-S$ that are contained in a directed cut of $O_{|G-S}$. Then $\phi_v(L)$ will be defined on $\hat{X}_{S'}$. Consider the partial normalization $\nu : X^\nu_{S'} \to X^\nu_S$. For each $e_i \in (G-S)_{cut}$ let C_i be the component towards which e_i is oriented in O and $p_i \in C_i$ the preimage of e_i under ν. Define

$$L' = (\nu^* L_{|X^\nu_S})(-p - \Sigma_i p_i).$$

Note that $\underline{\deg}(L')$ by construction is given by the totally cyclic orientation $O_{|G-S'}$ and thus L' is strictly balanced. We can view $X^\nu_{S'}$ as a subcurve of $\hat{X}_{S'}$, as the latter is constructed by inserting exceptional components at the normalized nodes. Finally let $\phi_v(L)$ be the line bundle on $\hat{X}_{S'}$ that restricts to L' on $X^\nu_{S'}$ and has degree one on each exceptional component (recall that the objects parametrized by $\overline{P}^{g-1}_X$ are only defined up to equivalence, thus different choices for the gluing at exceptional components get identified). Then $\phi_v(L)$ is stably balanced as it has degree one on exceptional components and $\phi_v(L)_{|X^\nu_{S'}} = L'$ is strictly balanced.

We chose to give this explicit description of the construction first. There is a more conceptual way of constructing this map, which in particular shows that it indeed is an algebraic map. We sketch this approach next.

Recall that $\overline{P}^g_X$ by **[14]** Proposition 8.2 (2) coarsely represents the functor whose value over a scheme S is the set of families $(\mathcal{L}, \mathcal{X})$ where $\mathcal{X} \to S$ is a family of quasistable curves having X as stable model and $\mathcal{L}$ a line bundle on $\mathcal{X}$ whose restriction to each fiber is stably balanced of degree g up to equivalence. By **[22]**, Theorem 4.4 (see also **[26]** Theorem 7.1), $\overline{P}^g_X$ in fact is a fine moduli space. Thus there exists a universal family

$(\mathcal{P}, \mathcal{X}) \to \overline{P}^g_X$, where $(\mathcal{P}, \mathcal{X})$ is a pair as before over the scheme $\overline{P}^g_X$ and the fiber over a point $[L]$ of $\overline{P}^g_X$ is $L \to \hat{X}_S$. Then a smooth point p on a component C_v defines a section σ_p of $\mathcal{X}$ and we can consider the pair $(\mathcal{P}(-\sigma_p), \mathcal{X})$. This then by Lemma 3.1.2 is a family over $\overline{P}^g_X$ whose fibers are stably balanced line bundles of degree $g-1$ over a quasistable curve that has X as stable model up to equivalence. While $\overline{P}^{g-1}_X$ is not a coarse moduli space for the functor giving such families, any family of this type by **[14]** Proposition 8.2 (1) still gives a map $\phi_v : \overline{P}^g_X \to \overline{P}^{g-1}_X$.

The two constructions of ϕ_v coincide. This follows from the GIT construction of $\overline{P}^d_X$ in **[14]** (cf. the proof of Proposition 5.1 in loc. cit.).

Theorem 5.3.1. *There is a map $\phi_v : \overline{P}^g_X \to \overline{P}^{g-1}_X$ with the following properties:*

(1) It is surjective.
(2) It maps strata surjectively to strata, i.e. $\phi_v(P^{\underline{d}_S}_X) = P^{\underline{d}'_{S'}}_X$.
(3) There is a dense open set U of $\overline{P}^{g-1}_X$ such that $(\phi_v)_{|\phi_v^{-1}(U)}$ is injective. More precisely, for every stratum $P^{\underline{d}'_{S'}}_X$ of $\overline{P}^{g-1}_X$ there exists a stratum $P^{\underline{d}_S}_X$ of $\overline{P}^g_X$ such that $(\phi_v)_{|P^{\underline{d}_S}_X} : P^{\underline{d}_S}_X \to P^{\underline{d}'_{S'}}_X$ is an isomorphism.

Proof. The map ϕ_v was constructed above.

For the first claim let $L \in \overline{P}^{g-1}_X$ be a stably balanced line bundle defined on $\hat{X}_{S'}$. Let $G = G_X$. Recall that $\underline{\deg}(L)$ in this case is given by a 0-orientation O on $\hat{G}_S$ that has every exceptional vertex as a sink and restricts to a totally cyclic orientation on $G - S'$.

If $G - S'$ is connected, since L is stably balanced, $L(p)$ is stably balanced on $\hat{X}_{S'}$ by Lemma 3.1.5. More precisely, Lemma 3.1.5 assures that there is $\underline{d} \sim \underline{\deg}(L_{|X^\nu_{S'}})$ on $G - S'$ with $\underline{d}$ balanced such that $\underline{d} + (v)$ is balanced on $G - S'$. We have $\underline{\deg}(L_{|X^\nu_{S'}}) = \underline{d}^{O_{|G-S'}}$ and $O_{|G-S'}$ contains by assumption no directed cuts. Since $\underline{d}$ is orientable, this implies that in fact $\underline{d} = \underline{\deg}(L_{|X^\nu_{S'}})$. Thus $L(p) \in \overline{P}^g_X$ with $L(p)$ a line bundle on $\hat{X}_{S'}$ and $\phi_v(L(p)) = L$.

If $G - S'$ is not connected, there is a union of cuts $R \subset S'$ such that $G - (S' \setminus R)$ is connected. Set $S = (S' \setminus R)$. Define an orientation O' on $G - S$ as follows: orient edges of $G - S$ contained in $G - S'$ as in $O_{|G-S'}$. Orient each cut $(Z, Z^c) \subset R$ in such a way that it is a directed cut. Lemma 3.1.5 ensures that there is $O'' \sim_{lin} O'$ such that $\underline{d}^{O''} + (v)$ is balanced on $G - S$. Since $O'_{|G-S'} = O_{|G-S'}$ is totally cyclic, O'' is obtained from O' by reversing some of the directed cuts $(Z, Z^c) \subset R$. Let $\nu : X^\nu_{S'} \to X^\nu_S$ be the normalization of the nodes corresponding to edges in R of X^ν_S. For each $e_i \in R$ let C_i be the component towards which e_i is oriented in O'' and $p_i \in C_i$ the preimage of e_i under ν. We may view $X^\nu_{S'}$ as a subcurve of $\hat{X}_{S'}$ and let $L' = L_{|X^\nu_{S'}}(\Sigma_i p_i)$. Finally, let $L'' \in \mathrm{Pic}(X^\nu_S)$ with $\nu^* L'' = L'$ be obtained from L' by gluing over the nodes of R in some way. Then by construction $\underline{\deg}(L'') = \underline{d}^{O''}$ and $L''(p)$ is strictly balanced on X^ν_S since

O'' was chosen as in Lemma 3.1.5. Extending $L''(p)$ to a stably balanced line bundle on $\hat{X}_S$ gives an element of $\overline{P}_X^g$ contained in the fiber of ϕ_v over L.

The second claim is immediate from the construction, as it only regards the multi-degrees involved. Surjectivity follows by a similar argument as above.

For the third claim recall that maximal dimensional strata of $\overline{P}_X^{g-1}$ correspond to divisors given by a totally cyclic orientations on $G - G_{br}$. Thus in this case, with notation as in the proof of the first claim, we have $S = S' = G_{br}$ and in particular no way of choosing S differently. The choice in constructing L'' came from choosing a line bundle L'' with $\nu^* L'' = L'$, which in general is not unique. Since gluing over separating nodes produces isomorphic line bundles for different choices of the gluing data, the construction of L'' in this case actually is unique. Since the union of the maximal dimensional strata forms a dense open subset of $\overline{P}_X^{g-1}$, this proves the first half of the third claim. For the second half, note that we always can choose R to consist only of bridges of $G - S$. Then by the same argument as for the first half, the construction of L'' is unique.

□

EXAMPLE 5.3.2. The strata of $\overline{P}_X^g$ of minimal dimension are given by $S \subset E(G_X)$ such that $G_X - S$ is a spanning tree. In this case every edge of $G_X - S$ is a bridge. Thus by the proof above, ϕ_v maps all these strata isomorphically to the stratum of $\overline{P}_X^{g-1}$ of minimal dimension, which is given by the empty orientation on $G_X - E(G_X)$.

EXAMPLE 5.3.3. Consider X with G_X the cycle on three vertices. We described the strata of $\overline{P}_X^g$ in Figure 3.2. The strata of $\overline{P}_X^{g-1}$ are given by the totally cyclic 0-orientation on G_X and the empty orientation on $G_X \setminus \{e_1, e_2, e_3\}$ where the e_i are the three edges of G_X. Then ϕ_v maps strata as follows:

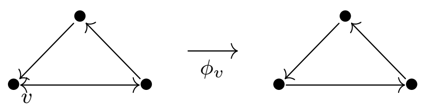


FIGURE 1. One codimension zero stratum of $\overline{P}_X^g$ gets mapped isomorphically to the codimension zero stratum of $\overline{P}_X^{g-1}$.

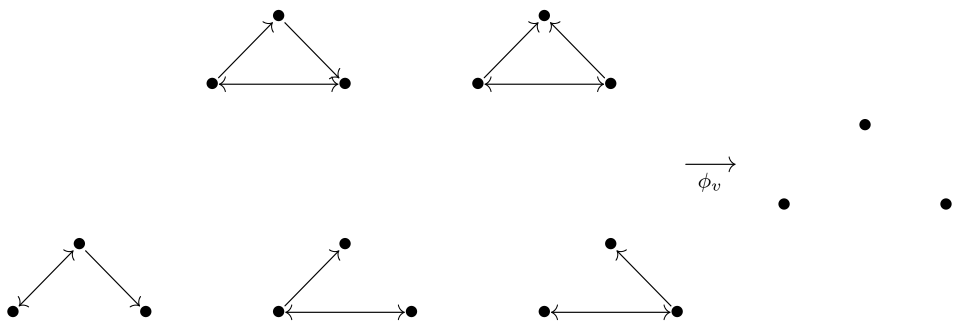


FIGURE 2. Two codimension zero strata of $\overline{P}^g_X$ get contracted to the codimension one stratum of $\overline{P}^{g-1}_X$ and the three codimension one strata of $\overline{P}^g_X$ get mapped isomorphically to the codimension one stratum of $\overline{P}^{g-1}_X$.

PROBLEM 5.3.4. It would be interesting to know, whether the map ϕ_v extends to a map $\overline{P}^g_g \to \overline{P}^{g-1}_g$. The main problem seems to be, that it is not possible to extend the choice of a component C_v and a point v in an algebraic way to different curves. That is, the universal family over $\overline{M}_g$ has no section. This issue could be addressed by considering instead $\overline{M}_{g,n}$ and compactified Jacobians over these spaces.

CHAPTER 6

About the Clifford inequality for stable curves

In this section we will give another application of the combinatorial description of stably balanced multidegrees in degree g. It is of a different flavour compared to the ones discussed in the previous section, as it does not concern the geometry of any of the Jacobians studied, but instead gives a result on the number of sections of a single line bundle on a nodal curve.

6.1. The problem

For smooth curves the Clifford theorem gives an upper bound for the number of sections of a line bundle in the range where it is not calculated explicitly by the Riemann-Roch theorem. The classical result is:

THEOREM 6.1.1. *Let X be a smooth curve and $L \in \mathrm{Pic}^d(X)$ a line bundle on X with $0 \leq d \leq 2g-2$. Then*

$$h^0(X, L) \leq d/2 + 1$$

and equality holds if and only if L is the trivial line bundle, the canonical line bundle or a multiple of a hyperelliptic line bundle.

Here we use the common notation $h^0(X, L) = \dim(H^0(X, L))$ for the dimension of the vector space of global sections of L. In the following we will use G for G_X, the dual graph of X, as we will work on a fixed curve.

It is well known, that this statement fails for line bundles on nodal or more restrictively stable curves. In fact, for any reducible stable curve X and any $0 \leq d \leq 2g-2$ there exist infinitely many $\underline{d}$ with $|\underline{d}| = d$ such that for every line bundle L with $\underline{\deg}(L) = \underline{d}$ one has $h^0(X, L) > d/2+1$ (**[19]**, Prop. 1.7 (4)(b)). Informally speaking, this is because we can choose for $\underline{d}_v$ an arbitrarily small negative number and thus get as many sections as we would like on the rest of the curve.

In the case of a smooth curve, the inequality is a direct consequence of the Riemann-Roch Theorem and the fact that $h^0(X, L) + h^0(X, L') \leq h^0(X, L + L') + 1$ for two line bundles L, L' (more precisely, one in particular needs $h^0(X, L) + h^0(X, K_X - L) \leq h^0(X, K_X) + 1$). While the Riemann-Roch Theorem still holds for nodal curves, the second claim does not. It fails because there no longer is a well behaved notion of a linear system in terms of divisors on the curve associated to a line bundle.

It is a difficult problem to characterize line bundles on nodal curves that satisfy the Clifford inequality. We want however point out a nice result of **[20]**: in each class of

$\mathrm{Pic}^d(G)$ with $0 \leq d \leq 2g-2$ there is a representative $\underline{d}$ such that every line bundle L with $\underline{\deg}(L) = \underline{d}$ satisfies the Clifford inequality. This result is obtained by linking the question to the Baker-Norine rank of the divisor, which satisfies the Clifford inequality. It remains however an open question, whether the representatives $\underline{d}$ can be characterized. Notably the possible candidates given by reduced divisors or balanced divisors turn out not to satisfy this property in general – in both cases there are examples of line bundles not satisfying the Clifford inequality.

The question is interesting for two connected reasons. First, recall that a line bundle L on X is called *smoothable* if there is a one parameter smoothing $\pi : \mathcal{X} \to B$ of X together with a line bundle $\mathcal{L} \to \mathcal{X}$ such that $\mathcal{L}_{|X} = L$ and $h^0(\pi^{-1}(b), \mathcal{L}_{|\pi^{-1}(b)}) = h^0(X, L)$ for every point $b \in B$. In other words, while we always can view L as the limit of line bundles on smooth curves, it is not clear whether we can view it as the limit of line bundles on smooth curves having the same rank. Then satisfying the Clifford inequality gives a necessary condition for L to be smoothable. Second, it is a basic instance of Brill-Noether theory on a nodal curve. Balanced line bundles not satisfying the Clifford inequality are $L \in \overline{P}^d_X$ with $h^0(X, L) > d/2 + 1$ and thus form a Brill-Noether locus.

We will study the behaviour of break divisors in this framework. We will later give an example that not every line bundle whose multidegree is a break divisor satisfies the Clifford inequality. We will however give a sufficient criterion, when a break divisor does satisfy it.

We will work with the following modification of the Clifford Theorem:

DEFINITION 6.1.2. Let $L \in \mathrm{Pic}^d(X)$. We will say that L satisfies Clifford if the following hold:

(1) $h^0(X, L) \leq d/2 + 1$
(2) If $h^0(X, L) = d/2 + 1$, L has no smooth base points.

We will say that L satisfies the Clifford inequality, if only the first condition holds.

Recall that a base point of L is a point p in X on which every section of L vanishes. In particular if p is a smooth point, we have $h^0(X, L) = h^0(X, L(-p))$.

DEFINITION 6.1.3 ([**16**]). For a line bundle L on a curve X, we say that two smooth points $q_1, q_2 \in X$ are a *neutral pair* if q_1 is a base point of $L(-q_2)$ and q_2 a base point of $L(-q_1)$.

This definition is equivalent to requiring

$$h^0(X, L(-q_1)) = h^0(X, L(-q_2)) = h^0(X, L(-q_1 - q_2)).$$

Note that if both q_1 and q_2 are base points of L they form a neutral pair, but not every neutral pair is of this form. If q_1 and q_2 however lie on different connected components of X, them being a neutral pair is equivalent to both of them being base points. Furthermore, if q_1 and q_2 are a neutral pair and one of them is a base point, the other will also be a base point.

Recall that for $S \subset E(G)$ we denote by $\nu : X^\nu_S \to X$ the partial normalization of nodes corresponding to elements of S.

DEFINITION 6.1.4. Let $S \subset E(G)$. For $L \in \operatorname{Pic}(X^\nu_S)$ we will denote the fiber of ν^* over L by $F_L(X)$, i.e.

$$F_L(X) = \{L' \in \operatorname{Pic}(X) | \, \nu^*(L') = L\}.$$

LEMMA 6.1.5 ([**16**], Lemma 1.4). *Let $S = \{e\} \subset E(G)$ and $L \in \operatorname{Pic}(X^\nu_S)$. Let q_1 and q_2 be the preimages of the node normalized by $\nu : X^\nu_S \to X$. Then there is $L' \in F_L(X)$ with $h^0(L') = h^0(L)$ if and only if q_1 and q_2 are a neutral pair of L. If the q_i are not base points of L, then L' is unique in $F_L(X)$.*

6.2. A connection to the Theta divisor

We will first use the map $\phi_v : \overline{P}^g_X \to \overline{P}^{g-1}_X$ constructed in section 5.3 and the Theta divisor on $\overline{P}^{g-1}_X$ to show that a general element of $\overline{P}^g_X$ satisfies Clifford. In fact, we will show the stronger claim that line bundles satisfying Clifford are dense in each stratum of $\overline{P}^g_X$. To this end, we need the following, which might be of independent interest:

PROPOSITION 6.2.1. *Let $L' \in \overline{P}^{g-1}_X$. Then for a general $L \in \phi_v^{-1}(L')$ we have $h^0(\hat{X}_S, L) \leq h^0(\hat{X}_{S'}, L') + 1$. If L is defined on $\hat{X}_{G_{br}}$ (i.e. contained in a maximal dimensional stratum of $\overline{P}^{g-1}_X$), the claim is true for every $L \in \phi_v^{-1}(L')$.*

PROOF. Recall that if $L \in \overline{P}^g_X$ is a stably balanced line bundle on $\hat{X}_S$, then $\phi_v(L)$ will be stably balanced on $\hat{X}_{S'}$ for some $S \subset S' \subset E(G)$. If $S' = S$, then $\phi_v(L) = L(-p)$ and the claim is obvious. The claim in general then follows from successively applying the next lemma to the edges in $S' \setminus S$ (cf. the construction of ϕ_v at the beginning of section 5.3).

□

LEMMA 6.2.2. *Let $S = \{e\} \subset E(G)$ and $L \in \operatorname{Pic}(\hat{X}_S)$ such that the degree of L on the exceptional component C_e is one. Denote by q_1 and q_2 the preimages of the node corresponding to e under the normalization $\nu : X^\nu_S \to X$ and $L^\nu = L_{|X^\nu_S}$. Then for a general $L' \in F_{L^\nu(q_1)}(X)$ we have $h^0(X, L') \leq h^0(\hat{X}_S, L)$. If e is a bridge of G, the claim is true for every $L' \in F_{L^\nu(q_1)}(X)$.*

PROOF. As C_e is a smooth rational component with $|C_e \cap C_e^c| = 2$ and L has degree one on it, we have $h^0(\hat{X}_S, L) = h^0(X^\nu_S, L^\nu)$. Furthermore we have $h^0(X, L') \leq h^0(X^\nu_S, L^\nu(q_1))$ as L' is obtained from $L^\nu(q_1)$ by gluing over a node.

If q_1 is a base point of $L^\nu(q_1)$, we have

$$h^0(\hat{X}_S, L) = h^0(X^\nu_S, L^\nu) = h^0(X^\nu_S, L^\nu(q_1)) \geq h^0(X, L'),$$

which shows the claim.

If q_1 is not a base point of $L^\nu(q_1)$, we have $h^0(X^\nu_S, L^\nu) + 1 = h^0(X^\nu_S, L^\nu(q_1))$. If q_1 and q_2 are not a neutral pair of $L^\nu(q_1)$, we have by Lemma 6.1.5

$$h^0(X, L') < h^0(X^\nu_S, L^\nu(q_1)).$$

Thus

$$h^0(\hat{X}_S, L) + 1 = h^0(X^\nu_S, L^\nu) + 1 = h^0(X^\nu_S, L^\nu(q_1)) > h^0(X, L'),$$

which implies $h^0(X, L') \leq h^0(\hat{X}_S, L)$.

Finally, if q_1 and q_2 are a neutral pair but no base points, by Lemma 6.1.5 there is a unique $L' \in F_{L^\nu(q_1)}(X)$ for which $h^0(X^\nu_S, L^\nu(q_1)) = h^0(X, L')$. For any other choice of L', we still have $h^0(X, L') < h^0(X^\nu_S, L^\nu(q_1))$ and can proceed as before.

For the second claim it is enough to observe that the only case in which the claim was not true for every $L' \in F_{L^\nu(q_1)}(X)$ was if the q_i are not base points of $L^\nu(q_1)$ but form a neutral pair. If e is a bridge, this cannot happen, as in this case the q_i lie on different connected components of X^ν_S. □

REMARK 6.2.3. Note that by what we showed before, L can be obtained as a degeneration of L'. The claim however does not immediately follow from upper semicontinuity, as the L' may form a closed locus on which the number of sections is higher.

The requirement that $L' \in F_{L^\nu(q_1)}(X)$ is general cannot be dropped in the lemma, as the following example shows:

EXAMPLE 6.2.4. Let X be a curve whose dual graph consists of two vertices v_1 and v_2 of weight one each and two edges between them. Denote by q_1, q_2 and p_1, p_2 the pairs of points lying over the nodes of the normalization of X on the components C_1 and C_2, respectively. Let $S = \{e_1\}$ contain one of the edges and let as before $\hat{X}_S$ be the curve obtained by inserting an exceptional component at the node corresponding to e_1. Let $L \in \mathrm{Pic}(\hat{X}_S)$ be the line bundle whose pullback to the normalization is $\mathcal{O}_{C_1}(p_1)$ on C_1, $\mathcal{O}_{C_2}(p)$ on C_2 for some point $p \neq p_2, p \neq q_2$ and the unique degree one line bundle on the exceptional component (thus $\underline{\deg}(L) = (1,1,1)$). Then $h^0(\hat{X}_S, L) = 1$ and q_1 is not a base point of $L^\nu(q_1)$ but q_1 and q_2 are a neutral pair. Thus we can find L' on X with $h^0(X, L') = h^0(X^\nu_S, L^\nu(q_1)) = h^0(X^\nu_S, L^\nu) + 1 = h^0(\hat{X}_S, L) + 1$.

THEOREM 6.2.5. *Let X be a stable curve. Then in each stratum $P_X^{\underline{d}_S}$ of $\overline{P}_X^g$ there is a dense open subset U such that every $L \in U$ satisfies Clifford.*

PROOF. It was shown in [**10**], that there is an intrinsically defined divisor Θ_X on P_X^{g-1}, the so called Theta divisor. This divisor is the image of the Abel map and contains a line bundles L if $h^0(X, L) \geq 1$. Furthermore in [**15**] it was shown that the extension of this divisor to $\overline{P}_X^{g-1}$ restricts to a divisor $\Theta_{\underline{d}'_S}$ on each stratum $P_X^{\underline{d}'_S}$ of $\overline{P}_X^{g-1}$ that is described by the same property: L is contained in $\Theta_{\underline{d}'_S}$ if and only if $h^0(\hat{X}_S, L) \geq 1$. Let $U' = P_X^{\underline{d}'_S} \setminus \Theta_{\underline{d}'_S}$. Then U' is open in $P_X^{\underline{d}'_S}$ and its points are line bundles in $P_X^{\underline{d}'_S}$ with $h^0(\hat{X}_{S'}, L) = 0$. By Theorem 5.3.1, every stratum $P_X^{\underline{d}_S}$ of $\overline{P}_X^g$ is contained in the preimage of some stratum $P_X^{\underline{d}'_S}$ of $\overline{P}_X^{g-1}$ under ϕ_v. Furthermore $\phi_v^{-1}(U') \cap P_X^{\underline{d}_S}$ is a dense open subset of $P_X^{\underline{d}_S}$. By Proposition 6.2.1, a general element L of $\phi_v^{-1}(U') \cap P_X^{\underline{d}_S}$ will then satisfy $h^0(\hat{X}_S, L) \leq 1$ and we take U to be the set of such line bundles. As $\deg(L) = g$ and $g \neq 0$ we have $1 < d/2 + 1$, thus L satisfies Clifford.

□

There is a more direct way to prove the previous result: Recall that $P_X^{\underline{d}_S} \cong \mathrm{Pic}^{\underline{d}_S}(X_S^\nu)$ and consider the isomorphism $f_v : \mathrm{Pic}^{\underline{d}_S}(X_S^\nu) \to \mathrm{Pic}^{\underline{d}_S - (v)}(X_S^\nu)$ that sends an element L to $L(-p)$ for some fixed smooth point $p \in C_v$. If $\underline{d}_S$ is balanced of degree $g(G-S)$ on $G - S$ by Lemma 3.1.2 $\underline{d}_S - (v)$ will be balanced of degree $g - 1$. It is immediate that $h^0(L) \leq h^0(f_v(L)) + 1$ and using the Theta divisor on $\mathrm{Pic}^{\underline{d}_S - (v)}(X_S^\nu)$ we can argue as before (cf. **[10]** Proposition 2.2). This avoids the intricacies in the construction of ϕ_v that arise from $\underline{d}_S - (v)$ in general not being strictly balanced by working with the components of the Picard scheme. We chose the proof via ϕ_v to further study this map.

6.3. A sufficient criterion

In this section we give a condition under which a stably balanced line bundle of degree g satisfies Clifford. First note that not every balanced line bundle of degree g does:

EXAMPLE 6.3.1. Let $X = C_1 \cup C_2 \cup C_3 \cup C_4$ with $|C_1 \cap C_i| = 1$ for $i = 2, 3, 4$, $C_i \cap C_j = \emptyset$ for $i, j = 2, 3, 4$ and $i \neq j$. Set $\underline{g} = (0, 1, 1, 1)$ so that $g(X) = 3$. Let $\underline{d} = (0, 1, 1, 1)$. Note that X is stable and $\underline{d}$ a break divisor. While a general line bundle with $\underline{\deg}(L) = \underline{d}$ satisfies Clifford there is one that does not: if we choose the base point of $L_{|C_i}$ for $i = 2, 3, 4$ to lie over the node with C_1, we get $h^0(X, L) = 3$. Thus L does not satisfy Clifford as $3 > 3/2 + 1$.

REMARK 6.3.2. Note that the above example satisfies the assumptions of Proposition 3.1 in **[16]**, however does not satisfy the claim made there, i.e. does not satisfy Clifford. The problem in the proof of Proposition 3.1 of **[16]** is that at some point of the argument it may happen that one leaves the range of the induction.

DEFINITION 6.3.3. Let L be a line bundle on X. We will say L has *no isolated singular base points* if for every partial normalization $\nu : X_S^\nu \to X$ the pull back $\nu^* L$ has no isolated base points mapped to nodes of X by ν.

Let $\underline{d} \in \Sigma(G)$ be a balanced multidegree of degree g and $L \in \mathrm{Pic}^{\underline{d}}(X)$. As we saw, $\underline{d} = \underline{d}^O$ where O is a rooted 1-orientation. Let $S \subset E(G)$ such that the bioriented edge is not contained in S and $\nu : X_S^\nu \to X$ the partial normalization of X at S. For every $e_i \in S$ let C_i be the component e_i is directed towards in O. Let $p_i \in C_i$ with $\nu(p_i) = e_i$ be the point lying over the node corresponding to e_i and $q_i \in X_S^\nu$ the second preimage of e_i under ν. Set $\widetilde{L}_S = \nu^* L(-\Sigma_{e_i \in S} p_i)$. The next definition is tailored to give a sufficient criterion for satisfying the Clifford inequality in the case of a balanced degree g line bundle:

DEFINITION 6.3.4. Let $\underline{d} \in \Sigma(G)$ be a balanced multidegree of degree g and $L \in \mathrm{Pic}^{\underline{d}}(X)$. We will say L is *well behaved* if there is an orientation O with $\underline{d} = \underline{d}^O$ and an arborescence T of O such that with notation as above for every $S \subset E(G) \setminus E(T)$ the line bundle $\widetilde{L}_S = \nu^* L(-\Sigma_{e_i \in S} p_i)$ has no isolated singular base points and no isolated base points at the q_i.

REMARK 6.3.5. Note that for $\underline{d} = \underline{d}^O \in \Sigma(G)$ and $S \subset E(G) \setminus E(T)$ with T an arborescence of O we have $\underline{\deg}(\widetilde{L}_S) = \underline{d}^{O_{|G-S}}$. Since T is still an arborescence of $O_{|G-S}$ and thus $O_{|G-S}$ a rooted 1-orientation, this implies $\underline{\deg}(\widetilde{L}_S) \in \Sigma(G-S)$ and $\widetilde{L}_S$ is a balanced line bundle on X^ν_S of degree $g(G-S) = g(X^\nu_S)$.

LEMMA 6.3.6. *Let $L \in \operatorname{Pic}(X)$ have no isolated singular base points and $\underline{\deg}(L) \geq \underline{0}$. Suppose there is a component C_v on which every section of L vanishes. Then there is $L' \in \operatorname{Pic}(X)$ with no isolated singular base points, $\underline{\deg}(L) = \underline{\deg}(L')$ and*

$$h^0(X, L) < h^0(X, L').$$

Furthermore if L is well behaved, L' may be chosen well behaved.

PROOF. We can consider the subcurve of X induced by all irreducible components on which all sections of L vanish. Let Z be the connected component of it containing C_v.

If $Z = X$, we have $h^0(X, L) = 0$. Since $\underline{\deg}(L) \geq \underline{0}$, there is L' with $h^0(X, L') \geq 1$, as we can start with $\mathcal{O}_X$ for which we have $h^0(X, \mathcal{O}_X) = 1$ and add smooth points until we get the desired multidegree. As adding smooth points can not decrease the number of sections, we get an L' as desired.

If $Z \neq X$, let $S = (Z, Z^c) \subset E(G)$ be the cut defined by Z and $\nu : X^\nu_S \to X$ the partial normalization at S. By arguments as above, there is a line bundle L_Z on Z with $\underline{\deg}(L_Z) = \underline{\deg}(\nu^* L)_Z$ and $h^0(Z, L_Z) \geq 1$. As we construct L_Z by adding smooth points to $\mathcal{O}_Z$, L_Z has on every irreducible component non-vanishing sections and we may choose the added smooth points in such a way, that L_Z has no base points lying over nodes of S. On the other hand, by construction of Z we have that $L_{Z^c} = (\nu^* L)_{|Z^c}$ has sections, i.e. $h^0(Z^c, L_{Z^c}) \geq 1$. Furthermore since L has no isolated singular base points, L_{Z^c} has no base points at points lying over nodes in S. Choose a section $s \in H^0(Z^c, L_{Z^c})$ that does not vanish at any point lying over nodes in S. Let $s' \in H^0(Z, L_Z)$ be such that it does not vanish at any point lying over a node of S. Now glue L_Z and L_{Z^c} along s and s' to obtain L' on X. By that we mean that for every pair of points $p_1 \in Z$, $p_2 \in Z^c$ lying over a node of S the isomorphism of the fibers of L_Z and L_{Z^c} is given by $s'(p_1)/s(p_2)$. Then $L'_{|Z^c} = L_{|Z^c}$ and by construction (s, s') descends to a section of L' not vanishing on Z, thus $h^0(X, L) < h^0(X, L')$. As L has no isolated singular base points, the section s may be chosen such that also L' has no isolated singular base points. Similarily for L well behaved.

□

PROPOSITION 6.3.7. *Let X be quasistable and $\underline{d} \in \Sigma(G_X)$ a balanced divisor of degree g. Let $L \in \operatorname{Pic}^{\underline{d}}(X)$ be well behaved. Then L satisfies the Clifford inequality.*

We will deal with a special case separately:

LEMMA 6.3.8. *Let X be of compact type, i.e. G_X a tree, and $\underline{d}$ with $0 \leq d_v \leq g(v)$ for all vertices v of G_X. Let $L \in \mathrm{Pic}^{\underline{d}}(X)$ have no isolated singular base points. Then L satisfies Clifford. In particular, if $\underline{d} \in \Sigma(G_X)$ it satisfies Clifford.*

PROOF OF LEMMA 6.3.8. Set $G = G_X$. We prove the claim by induction on the number of vertices of G. If G has a single vertex, the claim is the content of the Clifford Theorem for irreducible curves. So let G be a tree with $n \geq 2$ vertices and $L \in \mathrm{Pic}^{\underline{d}}(X)$ as in the assumptions of the lemma. Choose a leaf v of G, i.e. a vertex such that $G - v$ is connected. Set $X' \subset X$ to be the subcurve obtained by removing the component C_v of X corresponding to v and $L' = L_{|X'}$. Then by induction L' satisfies Clifford on X'.

Let X be obtained by gluing C_v and X' along q_1 and q_2. Then since $|X' \cap C_v| = 1$ we get by Lemma 6.1.5:

$$h^0(X, L) = h^0(X', L') + h^0(C_v, L_{|C_v}) - f,$$

where $f = 0$ if q_1 is a base point of $L_{|C_v}$ and q_2 a base point of L' and $f = 1$ otherwise. By assumption both L' and $L_{|C_v}$ satisfy Clifford.

Suppose $f = 0$: Since L has no isolated singular base points, q_1 is not an isolated base point of $L_{|C_v}$. If q_1 is a non-isolated base point, $L_{|C_v}$ has no sections. Thus $h^0(X, L) = h^0(X', L')$ and the claim follows from induction.

Suppose $f = 1$: With $\underline{d} = \underline{\deg}(L)$ and $\underline{d}' = \underline{\deg}(L')$ we get:

$$h^0(X, L) = h^0(X', L') + h^0(C_v, L_{|C_v}) - 1 \leq |\underline{d}'|/2 + 1 + \underline{d}_v/2 + 1 - 1 = |\underline{d}|/2 + 1.$$

This proves the first part of satisfying Clifford.

Now suppose L has a base point p on an irreducible component C_v. If $\underline{d}_v \geq 1$, $L(-p)$ is in the rangs of the assumptions and by what we proved above satisfies the Clifford inequality. We get:

$$h^0(X, L) = h^0(X, L(-p)) \leq (|\underline{d}| - 1)/2 + 1 < |\underline{d}|/2 + 1.$$

It remains the case $\underline{d}_v = 0$, in which case $L(-p)$ no longer is in the range of the assumption. Notice that as L has degree zero on C_v, every section of L vanishes on all of C_v. Thus by Lemma 6.3.6, there is L' with $\underline{\deg}(L') = \underline{d} = \underline{\deg}(L)$ and without isolated singular base points such that $h^0(X, L) < h^0(XL')$. By what we showed before, $h^0(X, L') \leq |\underline{d}|/2 + 1$ which shows $h^0(X, L) < |\underline{d}|/2 + 1$.

As on a tree every divisor is equivalent to every other divisor of the same total degree, $\Sigma(G_X)$ contains a unique element. This, as we saw, is given by a rooted 1-orientation on the tree, namely $\underline{g} = (g(v_1), \ldots, g(v_n))$. Thus the last claim follows from what we showed above.

□

PROOF OF PROPOSITION 6.3.7. Set $G = G_X$ and let $L \in \mathrm{Pic}^{\underline{d}}(X)$ be well behaved with $\underline{d} \in \Sigma(G)$. We will prove the claim by induction on $b_1(G)$. The induction basis will be the case in which G is a tree, where the claim follows from Lemma 6.3.8 (note that being well behaved implies having no isolated singular base points).

For the induction step, let O be a 1-orientation such that $\underline{d} = \underline{d}^O$. Fix an arborescence T that gives L as a well behaved line bundle. Choose an edge $e \notin T$ and set $G' = G - e$ and $O' = O_{|G'}$. Note that G' is connected since it contains T and we have $b_1(G') = b_1(G) - 1$. Then $\underline{d}' = \underline{d}^{O'} \in \Sigma(G')$ as O' contains T as an arborescence. Let $S = \{e\}$ and $\nu : X_S^\nu \to X$ be the normalization of X at the node q corresponding to e. Let q_1 and q_2 be the two preimages of q under this normalization. Set as before $\widetilde{L}_S = \nu^* L(-q_1)$ where q_1 lies on the irreducible component e is directed towards in O. Then $\widetilde{L}_S$ is well behaved since L is and $\widetilde{L}_S$ is balanced on X_S^ν. Thus by induction $\widetilde{L}_S$ satisfies the Clifford inequality, i.e.

$$h^0(X_S^\nu, \widetilde{L}_S) \leq (|\underline{d}| - 1)/2 + 1.$$

By Lemma 6.1.5 we have

$$h^0(X, L) \leq h^0(X_S^\nu, \widetilde{L}_S) + 1 - f \leq |\underline{d}|/2 + 1/2 + 1 - f,$$

where $f = 0$ if q_1 and q_2 are a neutral pair of $\nu^* L$ and one otherwise.

If $f = 1$, the claim immediately follows. So suppose $f = 0$: Let C be the irreducible component containing q_2. Since $f = 0$ we get that q_2 is a base point of $\widetilde{L}_S$ and since L is well behaved q_2 cannot be an isolated base point. In other words, every section of $\widetilde{L}_S$ vanishes on C. Then by Lemma 6.3.6 there is $L' \in \mathrm{Pic}(X_S^\nu)$ well behaved with $\underline{\deg}(L') = \underline{\deg}(\widetilde{L}_S)$ and $h^0(X_S^\nu, \widetilde{L}_S) < h^0(X_S^\nu, L')$. By induction we have $h^0(X_S^\nu, L') \leq (\underline{d} - 1)/2 + 1$ and thus $h^0(X_S^\nu, \widetilde{L}_S) < (\underline{d} - 1)/2 + 1$. This gives:

$$h^0(X, L) \leq h^0(X_S^\nu, \widetilde{L}_S) + 1 < (\underline{d} - 1)/2 + 2,$$

which implies the claim.

□

Remark/Problem 6.3.9. We are not aware of an example of a balanced line bundle L of degree g that does not satisfy Clifford if one of the following two conditions are satisfied:

(1) G_X is two-edge connected or
(2) L has no isolated singular base points.

The question thus is, whether the assumptions of Proposition 6.3.7 can be weakened to L has no isolated singular base points over bridges of G_X.

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