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Pseudoeffective cones in 2-Fano varieties and remarks on the Voisin map

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Contents

Résumé

Cette thèse est composée de deux parties distinctes.

Partie 1. Dans la première partie, on étudie le cône des classes pseudoeffectives des variétés k-Fano. Toutes nos variétés sont algébriques, irréductibles, réduites et définies sur le corps des nombres complexes. X est une variété de dimension n, l'anneau R indique un des anneaux $\mathbb{Z}$, $\mathbb{Q}$ ou $\mathbb{R}$. On note $M \otimes_{\mathbb{Z}} R$ avec M_R pour chaque $\mathbb{Z}$-module M.

Définition 0.0.1. Un k-cycle sur X est une somme finie $\mathbb{Z}$-linéaire $\sum a_i V_i$ de sous-variétés de dimension k. Le groupe de tous les k-cycles est notée $Z_k(X)$.

On peut définir sur $Z_k(X)$ plusieurs relations d'équivalence.

- Équivalence rationnelle (Definition 2.1.3), $\mathrm{Rat}_k(X)$ est le groupe de tous les k-cycles rationnellement équivalents à zéro.
- Équivalence algébrique (Definition 2.1.6), $\mathrm{Alg}_k(X)$ est le groupe de tous les k-cycles algébriquement équivalents à zéro. Le groupe des k-cycles modulo l'équivalence rationnelle sur X est le groupe de Chow
$$A_k(X) = {}^{Z_k(X)}\!/_{\mathrm{Rat}_k(X)}.$$
- Équivalence homologique (Definition 2.1.7), $\mathrm{Hom}_k(X)$ est le groupe de tous les k-cycles homologiquement équivalents à zéro.
- Équivalence numérique (Definition 2.1.12), $\mathrm{Num}_k(X)$ est le groupe de tous les k-cycles numériquement équivalents à zéro. Le groupe des k-cycles modulo l'équivalence numérique sur X est le groupe quotient
$$N_k(X) = {}^{Z_k(X)}\!/_{\mathrm{Num}_k(X)}.$$

Les relations d'équivalence définies sur $Z_k(X)$ satisfont la chaîne d'inclusions suivante

$$\mathrm{Rat}_k(X) \subseteq \mathrm{Alg}_k(X) \subseteq \mathrm{Hom}_k(X) \subseteq \mathrm{Num}_k(X) \subseteq Z_k(X),$$

impliquant l'existence du diagramme suivant.

$$\begin{tikzcd} A_k(X) \arrow[r, two heads] & {}^{Z_k(X)}\!/_{\mathrm{Alg}_k(X)} \arrow[r, two heads] & {}^{Z_k(X)}\!/_{\mathrm{Hom}_k(X)} \arrow[r, two heads, "\pi_k"] \arrow[d, hook] & N_k(X) \\ & & H_{2k}(X, \mathbb{Z}) & \end{tikzcd} \tag{0.0.1}$$

Soit E un fibré vectoriel sur X. On peut démontrer qu'il y a un unique ensemble d'applications sur les groupes de Chow

$$c_i(E) \cdot \_ : A_k(X) \to A_{k-i}(X),$$

et sont appelées classes de Chern. Les principales propriétés des classes de Chern sont énoncées dans Theorem 2.1.8. Chaque polynôme en classes de Chern à coefficients dans R définit une action sur la somme directe $\oplus_k A_k(X)_R$. En particulier, si P est un polynôme homogène de degré d, $c_i(E)$ est de degré i et $\alpha \in A_k(X)_R$, alors $P \cdot \alpha \in A_{k-d}(X)_R$.

Les caractères de Chern sont des exemples intéressants de polynômes homogènes à coefficients rationnels. On peut les calculer en utilisant la formule suivante (cf. [Ful98, Example 15.1.2(b)]), où $c_i = c_i(E)$.

$$ch_k(E) = \frac{1}{k!}\det\begin{pmatrix} c_1 & 2c_2 & 3c_3 & \dots & kc_k \\ 1 & c_1 & 2c_2 & \dots & \vdots \\ & 1 & \ddots & \ddots & 3c_3 \\ & & \ddots & c_1 & 2c_2 \\ & & & 1 & c_1 \end{pmatrix}$$

Le polynôme $ch_k(E)$ est homogène de degré k. Les premièrs caractères de Chern de E sont

$$\begin{aligned} ch_1(E) &= c_1(E) \\ ch_2(E) &= \frac{1}{2}(c_1^2(E) - 2c_2(E)) \\ ch_3(E) &= \frac{1}{6}(c_1^3(E) - 4c_1(E)c_2(E) + 3c_3(E)) \end{aligned}$$

Si X est lisse et T_X est le fibré tangent, les classes de Chern et les caractères de Chern de T_X sont respectivement notés $c_k(X)$ et $ch_k(X)$. On définit le cône des classes effectives et pseudoeffectives comme suit.

Définition 0.0.2. Un k-cycle $\sum r_i V_i \in Z_k(X)_R$ sur X est effectif si $r_i \geq 0$ pour chaque i. Une classe $\alpha \in N_k(X)_R$ est effective si $\alpha = [\sum r_i V_i]$ pour chaque k-cycle effectif $\sum r_i V_i$. On note $\mathrm{Eff}_k(X)$, le cône des classes de k-cycles effectifs en $N_k(X)_{\mathbb{R}}$, et par $\overline{\mathrm{Eff}}_k(X)$ son adhérence. Les classes pseudoeffectives sont les classes de $\overline{\mathrm{Eff}}_k(X)$.

On sait que si X est lisse, il existe une forme bilinéaire symétrique non-dégénérée :

$$\_ \cdot \_ : N_k(X)_{\mathbb{R}} \otimes N_{n-k}(X)_{\mathbb{R}} \to \mathbb{R}.$$

Définition 0.0.3. Soit X une variété lisse. Une classe $\alpha \in N_k(X)_{\mathbb{R}}$ est positive si $\alpha \cdot \beta > 0$ pour chaque $\beta \in \overline{\mathrm{Eff}}_{n-k}(X) \backslash \{0\}$, elle est nef si $\alpha \cdot \beta \geq 0$ pour chaque $\beta \in \overline{\mathrm{Eff}}_{n-k}(X)$.

On peut à présent définir l'objet de la première partie de cette thèse.

Définition 0.0.4. Une variété est k-Fano si $ch_s(X)$ est positif pour chaque $1 \leq s \leq k$, elle est k-Fano faible pour $k > 1$ si X est $(k-1)$-Fano et $ch_k(X)$ est nef.

Cette définition généralise la définition de variété de Fano ; en fait les variétés de Fano sont exactement les 1-Fano par le Théorème de Kleiman (Theorem 2.1.20). Les variétés 2-Fano ont été introduites par de Jong et Starr dans [**dJS06**, **dJS07**], où ils ont démontré que par le point général de chaque variété 2-Fano avec pseudo-indice au moins égal à 3, il passe une surface rationnelle. Cela généralise un résultat classique, c'est-à-dire que chaque variété de Fano est uniréglée (par le point général il passe une courbe rationnelle). Ensuite, Araujo et Castravet ont démontré dans [**AC12**, Theorem 1.5(3)] que (avec des hypothèses supplémentaires) le point général d'une variété 3-Fano est traversé par une variété rationnelle de dimension trois. De plus, elles ont classifié les variétés 2-Fano avec indice au moins égal à $n-2$ ([**AC13**]).

Nous sommes intéressés par les cônes des classes pseudoeffectives des variétés k-Fano. En fait nous avons la suivante.

CONJECTURE. *Soit X une variété k-Fano. Alors le cône $\overline{\mathrm{Eff}}_k(X)$ est polyhédral.*

Pour le 1-Fano nous savons que la conjecture est vraie. En fait, on a le corollaire suivant du Théorème du Cône de Mori (Theorem 2.1.25).

COROLLAIRE 0.0.5. *Soit X une variété de Fano. Alors le cône $\overline{\mathrm{Eff}}_1(X)$ est polyhédral.*

Il est donc naturel de se demander si ce résultat aussi peut se généralises aux k-Fano, de même que l'existence de sous-variétés rationnelles.

En général, si X est de Fano nous sommes incapables de dire si $\overline{\mathrm{Eff}}_k(X)$ est polyhédral dans les cas $k \neq 0, 1, n$. Un exemple par Tschinkel (énoncé dans [**DELV11**, Example 6.10]) montre qu'une variété de Fano peut avoir $\overline{\mathrm{Eff}}_2(X)$ qui n'est pas polyhédral (Proposition 2.5.4).

La stratégie adoptée dans la plupart de la thèse est d'utiliser le lemme suivant (Lemma 2.1.29).

LEMME 0.0.6. *Soit X une variété projective. Alors*

(1) Si $\mathrm{rk}A_k(X) = 1$ ou $b_{2k}(X) = 1$, alors $\overline{\mathrm{Eff}}_k(X)$ est un rayon.
(2) Si $\mathrm{rk}A_k(X) = 2$ ou $b_{2k}(X) = 2$, alors $\overline{\mathrm{Eff}}_k(X)$ est un rayon ou il est engendré par deux rayons.

La démonstration est une application du diagramme (0.0.1). En fait, le cône $\overline{\mathrm{Eff}}_k(X)$ est un cône dans un espace vectoriel de dimension au plus 2, c'est donc polyhédral.

Nous calculons donc les nombres de Betti pour certaines variétés k-Fano. Enfin, nous utilisons la classification de Araujo et Castravet [**AC13**, Theorem 3] (Theorem 2.5.2) pour démontrer le suivant.

THÉORÈME 0.0.7. *Soit X une variété k-Fano avec indice $i_X \geq n-2$. Alors le cône $\overline{\mathrm{Eff}}_k(X)$ est polyhédral pour $k = 2, 3$.*

Avant de démontrer ce théorème, on considère certains cas particuliers.

PROPOSITION 0.0.8. *Soit X une intersection complète lisse dans un espace projectif à poids. Alors $b_{2k}(X) = 1$ pour chaque $k \neq \frac{n}{2}$, et donc le cône $\overline{\mathrm{Eff}}_k(X)$ est polyhédral.*

Si X est dans un espace projectif, k-Fano faible et $1 \leq s \leq k$, alors $b_{2s}(X) \leq 2$, et donc le cône $\overline{\mathrm{Eff}}_k(X)$ est polyhédral.

La conjecture est donc vraie pour les intersections complètes dans un espace projectif. On considère maintenant les intersections complètes dans trois familles des variétés homogènes : $G(r,s)$, $OG(r,s)$ e $SG(r,s)$. Soit G un groupe algébrique réductible sur $\mathbb{C}$, un sous-groupe P de G est dit parabolique si G/P est une variété projective. Une variété homogène est une variété pour laquelle il existe une action transitive d'un groupe algébrique. Pour les variétés homogènes, il est bien connu que tous les cônes de classes pseudoeffectives sontpolyhédrales. On peut démontrer qu'une variété homogène est de Fano si et seulement si elle est un produit $G_1/P_1 \times \ldots \times G_k/P_k$, ou P_i est un sous-groupe parabolique du groupe simple G_i. Pour les Grassmanniennes on a le résultat suivante.

PROPOSITION 0.0.9. *Soit X une variété 2-Fano faible, intersection complète dans $G(r,s)$. Alors, $b_4(X) \leq 2$. En particulier le cône $\overline{\mathrm{Eff}}_2(X)$ est polyhédral.*

Pour les Grassmanniennes orthogonales on démontre que :

PROPOSITION 0.0.10. *Soit s, r des entiers positifs tels que $2 \leq r \leq \left[\frac{s}{2}\right]$, e $\left[\frac{s}{2}\right] - r \neq 1, 2$ si s est pair. Si $s \neq 2r$ (respectivement, $s = 2r$), X est une 2-Fano faible, intersection complète dans une composante connexe de $OG(r,s)$ sous le plongement de Plücker (respectivement, demi-spin), X est très général si $X \subseteq OG(2,7)$, alors le cône $\overline{\mathrm{Eff}}_2(X)$ est polyhédral.*

Afin de prouver cela, nous utilisons à la fois [Spa96, Theorem 2] et

LEMME 0.0.11. *Soit X une composante connexe de $OG(r,s)$, $2 \leq r \leq m = \left[\frac{s}{2}\right]$. Alors*

$$b_4(X) = \begin{cases} 1 & \text{si } r = m \\ 3 & \text{si } 1 \leq m - r \leq 2,\ s \text{ pair} \\ 2 & \text{autrement} \end{cases}$$

Pour les intersections complètes dans les Grassmanniennes symplectiques, nous avons le résultat suivant.

PROPOSITION 0.0.12. *Soit X une 2-Fano faible, intersection complète dans $SG(r,s)$. Alors, $b_4(X) \leq 2$. En particulier le cône $\overline{\mathrm{Eff}}_2(X)$ est polyhédral.*

Ces propositions ont été obtenues en utilisant divers résultats dans [AC13], dans laquel sont données plusieurs conditions sur une variété X pour être 2-Fano faible. Nous répondons également à [AC13, Question 39 and 40] ; en fait, nous démontrons le résultat suivant

THÉORÈME 0.0.13. *Soit $Y = G(2,5)$ o $G(2,6)$, et soit X une intersection complète de type $(1,1)$ en Y avec le plongement de Plücker. Alors X n'est pas 2-Fano faible.*

Pour prouver ce théorème, nous avons utilisé [dA15, Corollary 5.1] (impliquant que si X est général alors ce n'est pas 2-Fano), et [Ott15, Proposition 3]. Ce théorème complète la classification des variétés 2-Fano faibles avec indice $i_X \geq n-2$ dans [AC13, Theorem 4]. Dans la démonstration, nous avons utilisé le suivant.

LEMME 0.0.14. *Soit X_{11} une intersection complète de type $(1,1)$ en $G(2,5)$ avec le plongement de Plücker. Alors $b_4(X) = 2$.*

Pour démontrer ce lemme, nous avons calculé le diamant de Hodge de X_{11}. Puisque le théorème de Sommese (Theorem 2.4.2) ne suffit pas pour déterminer le diamant entier, nous avons également utilisé [Sno86].

Partie 2. Dans la seconde partie, nous étudions le lieu d'indétermination d'une application rationnelle introduite dans [Voi16, Proposition 4.8]. Le lieu d'indétermination est le plus grand sous-ensemble fermé constitué des points où l'application n'est pas définie. Nous pouvons facilement vérifier que si

$$\begin{tikzcd} X \arrow[r, dashed, "h"] \arrow[d, dashed, "f"'] & W \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

est un diagramme commutatif avec f et h des applications rationnelles et g un morphisme, alors on a $\mathrm{Ind}(h) \subseteq \mathrm{Ind}(f)$. De plus, par un résultat de Hironaka [Hir64, I.Question (E) p.140], pour chaque application rationnelle $f : X \dashrightarrow Y$ entre variétés lisses, il y a une résolution des indéterminations. C'est-à-dire, un diagramme commutatif

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{f}"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

ou $\widetilde{X}$ est lisse et π est une application rationnelle et un isomorphisme en dehors de $\mathrm{Ind}(f)$.

Définition 0.0.15. Soit X une variété de Kähler compacte. On dit que X est une variété hyperkählérienne si X est simplement connexe et $H^0(X, \wedge^2\Omega_X)$ est engendré par une 2-forme non dégénérée en tout point de X.

Par un résultat classique, la première classe de Chern d'une variété avec métrique de Ricci plate est nulle. De plus, pour la décomposition de Bogomolov [Bog74, Bea83], ces variétés ont un revêtement étale fini donné par le produit des tores complexes, des variétés de Calabi-Yau et des variétés hyperkählériennes. En dimension 2, les seuls exemples des variétés hyperkählériennes sont les surfaces $K3$. Beauville a démontré dans [Bea83, Théorèmes 3 and 4] que pour chaque $n \geq 0$, le schéma de Hilbert $S^{[n]}$ paramétrant les sous-schémas de longueur n d'une surface $K3$ S, et la variété de Kummer généralisée $K^n(T)$ associée à un tore complexe T de dimension 2, sont des variétés hyperkählériennes. Une variété hyperkählérienne obtenue à partir d'une déformation de $S^{[n]}$ (respectivement, d'une variante de Kummer généralisée) est dite de type $K3^{[n]}$ (respectivement, de type $K^n(T)$). Ces variétés sont particulièrement intéressantes car elles permettent de construire des variétés hyperkählériennes complexes de toute dimension paire. Par la suite O'Grady dans [O'G03, O'G99] a construit deux nouveaux exemples en dimension 6 et 10 de variétés hyperkählériennes qui ne sont pas des déformations de types connus. Dans la deuxième partie de la thèse, nous utilisons deux exemples particuliers de variétés hyperkählériennes de type $K3^{[n]}$. Le premier exemple est celui fourni par Beauville et Donagi.

Théorème 0.0.16 ([BD85]). *Soit $Y \subseteq \mathbb{P}^5$ une hypersurface cubique lisse de dimension 4. Soit F la variété de Fano de Y. Alors*

(1) La variété F est hyperkählérienne de type $K3^{[2]}$.
(2) Si Y est pfaffienne, alors F est isomorphe à $S^{[2]}$ pour une surface $K3$ S.

Remarque 0.0.17. Cette variété a été utilisée pour donner un exemple de classe nef mais pas pseudoeffective. En fait, en général, il n'est pas vrai que $\mathrm{Nef}_k(X) \subseteq \overline{\mathrm{Eff}}_k(X)$ pour $2 \leq k \leq \dim X - 2$. Le premier exemple de ce phénomène a été donné dans [DELV11]. Ottem a montré que si la cubique lisse Y est très générale, alors $\overline{\mathrm{Eff}}_2(F) \subsetneq \mathrm{Nef}_2(F)$. De plus, la deuxième classe de Chern $c_2(F)$ est nef mais elle n'est pas effective [Ott15, Theorem 1].

Un autre exemple de variété hyperkählérienne a été donné récemment par C. Lehn, M. Lehn, Sorger et van Straten dans [LLSvS17]. Ils ont observé que si $F_3(Y)$ est une variété compacte des cubiques rationnelles dans Y, où Y est une cubique de dimension 4 lisse qui ne contient pas de plan, alors il y a une fibration en $\mathbb{P}^2$

$$\phi : F_3(Y) \to Z',$$

où Z' est une variété lisse. Il y a aussi un diviseur sur Z' qui peut être contracté, la contraction obtenue $\sigma : Z' \to Z$ produit la variété hyperkählérienne Z, qui est de type $K3^{[4]}$ (cf. [**AL17**, Corollary] ou [**Leh15**, Corollary 6.3]). Il y a un diagramme commutatif

$$\begin{tikzcd} F_3(Y) \arrow[r] \arrow[d, "\phi"'] & G(4,6) \\ Z' \arrow[ur, "g"'] & \end{tikzcd}$$

où $F_3(Y) \to G(4,6)$ est le morphisme qui envoie chaque cubique rationnelle sur l'espace $\mathbb{P}^3$ qui la contient. Le morphisme g est fini de degré 72 sur un ouvert de Z'.

Voisin a construit une application rationnelle de degré 6 dans [**Voi16**, Proposition 4.8]

$$\psi : F \times F \dashrightarrow Z.$$

De manière synthétique, l'application ψ envoie une paire de droites non concourantes $(l, l') \in F \times F$ sur la classe d'une courbe rationnelle normale de degré 3 contenue dans le système linéaire $|L - L' - K_{S_{l,l'}}|$ de la surface cubique $S_{l,l'} := \langle L, L' \rangle \cap Y$. Cela nous donne un diviseur uniréglé sur Z. En fait, si on a une résolution des indéterminations

$$\begin{tikzcd} \widetilde{F \times F} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\psi}"] & \\ F \times F \arrow[r, dashed, "\psi"] & Z \end{tikzcd}$$

l'image par $\widetilde{\psi}$ du diviseur exceptionnel de π est un diviseur uniréglé de Z. Notre objectif est de calculer le lieu d'indétermination de ψ. Nous allons d'abord montrer

Proposition 0.0.18. *Il existe une application rationnelle $\psi' : F \times F \dashrightarrow Z'$ de degré 6 tel que* $\mathrm{Ind}(\psi') = I$.

L'application ψ' est définie en tout point de l'ouvert

$$U_{ADE} := \left\{(l, l') \in F \times F | (l, l') \notin I, S_{l,l'} := \left\langle L, L' \right\rangle \cap Y \text{ a des singularités } ADE\right\}$$

et la composition $\sigma \circ \psi' : F \times F \dashrightarrow Z$ coïncide avec l'application de Voisin. Enfin on montre que.

Théorème 0.0.19. *Le lieu d'indétermination de l'application de Voisin ψ est la variété I des droites concourantes dans Y.*

Riassunto

Questa tesi è divisa in due parti distinte.

Parte 1. Nella prima parte studiamo il cono delle classi pseudoeffettive delle varietà k-Fano. Indichiamo con X una varietà (cioè, uno schema algebrico complesso ridotto e irriducibile) di dimensione n. L'anello R indica uno qualunque tra $\mathbb{Z}$, $\mathbb{Q}$ oppure $\mathbb{R}$. Indichiamo $M \otimes_{\mathbb{Z}} R$ con M_R per ogni $\mathbb{Z}$-modulo M.

Definizione 0.0.1. Un k-ciclo su X è una somma finita $\mathbb{Z}$-lineare $\sum a_i V_i$ di sottovarietà di dimensione k. Il gruppo di tutti i k-cicli è indicato con $Z_k(X)$.

Possiamo definire su $Z_k(X)$ varie relazioni di equivalenza.

- Equivalenza razionale (Definition 2.1.3), $\mathrm{Rat}_k(X)$ è il gruppo di k-cicli razionalmente equivalenti a zero.
- Equivalenza algebrica (Definition 2.1.6), $\mathrm{Alg}_k(X)$ è il gruppo di k-cicli algebricamente equivalenti a zero. Il gruppo dei k-cicli modulo equivalenza razionale su X è il gruppo di Chow
$$A_k(X) = {}^{Z_k(X)}/_{\mathrm{Rat}_k(X)}.$$
- Equivalenza omologica (Definition 2.1.7), $\mathrm{Hom}_k(X)$ è il gruppo di k-cicli omologicamente equivalenti a zero.
- Equivalenza numerica (Definition 2.1.12), $\mathrm{Num}_k(X)$ è il gruppo di k-cicli numericamente equivalenti a zero. Il gruppo dei k-cicli modulo equivalenza numerica su X è il gruppo quoziente
$$N_k(X) = {}^{Z_k(X)}/_{\mathrm{Num}_k(X)}.$$

Le relazioni di equivalenza definite su $Z_k(X)$ soddisfano la seguente catena di inclusioni

$$\mathrm{Rat}_k(X) \subseteq \mathrm{Alg}_k(X) \subseteq \mathrm{Hom}_k(X) \subseteq \mathrm{Num}_k(X) \subseteq Z_k(X),$$

che implica l'esistenza del seguente diagramma.

$$\begin{tikzcd} A_k(X) \arrow[r, two heads] & {}^{Z_k(X)}/_{\mathrm{Alg}_k(X)} \arrow[r, two heads] & {}^{Z_k(X)}/_{\mathrm{Hom}_k(X)} \arrow[r, two heads, "\pi_k"] \arrow[d, hook] & N_k(X) \\ & & H_{2k}(X, \mathbb{Z}) & \end{tikzcd} \tag{0.0.2}$$

Sia E un fibrato vettoriale su X. Sui gruppi di Chow si dimostra l'esistenza di un unico insieme di applicazioni
$$c_i(E) \cdot \_ : A_k(X) \to A_{k-i}(X),$$
dette classi di Chern. Le principali proprietà delle classi di Chern sono enunciate in Theorem 2.1.8. Ogni polinomio nelle classi di Chern a coefficienti in R agisce sulla somma diretta $\oplus_k A_k(X)_R$. In particolare, se P è un polinomio omogeneo di grado d, dove $c_i(E)$ ha grado i e $\alpha \in A_k(X)_R$, allora $P \cdot \alpha \in A_{k-d}(X)_R$.

I caratteri di Chern sono tra i più interessanti esempi di polinomi omogenei a coefficienti razionali. Si possono calcolare utilizzando la seguente formula chiusa (vedi [**Ful98**, Example 15.1.2(b)]), in cui $c_i = c_i(E)$.

$$ch_k(E) = \frac{1}{k!}\det\begin{pmatrix} c_1 & 2c_2 & 3c_3 & \dots & kc_k \\ 1 & c_1 & 2c_2 & \dots & \vdots \\ & 1 & \ddots & \ddots & 3c_3 \\ & & \ddots & c_1 & 2c_2 \\ & & & 1 & c_1 \end{pmatrix}$$

Il polinomio $ch_k(E)$ è omogeneo di grado k. I primi caratteri di Chern di E sono

$$\begin{aligned} ch_1(E) &= c_1(E) \\ ch_2(E) &= \frac{1}{2}(c_1^2(E) - 2c_2(E)) \\ ch_3(E) &= \frac{1}{6}(c_1^3(E) - 4c_1(E)c_2(E) + 3c_3(E)) \end{aligned}$$

Se X è liscia e T_X è il fibrato tangente, le classi di Chern e i caratteri di Chern di T_X sono indicati con, rispettivamente, $c_k(X)$ e $ch_k(X)$. Definiamo adesso il cono delle classi effettive e pseudoeffettive.

Definizione 0.0.2. Un k-ciclo $\sum r_i V_i \in Z_k(X)_R$ su X è effettivo se $r_i \geq 0$ per ogni i. Una classe $\alpha \in N_k(X)_R$ è effettiva se $\alpha = [\sum r_i V_i]$ per qualche k-ciclo effettivo $\sum r_i V_i$. Indichiamo con $\mathrm{Eff}_k(X)$ il cono generato dalle classi di k-cicli effettivi in $N_k(X)_{\mathbb{R}}$, e con $\overline{\mathrm{Eff}}_k(X)$ la sua chiusura. Le classi pseudoeffettive sono le classi di $\overline{\mathrm{Eff}}_k(X)$.

Sappiamo che se X è liscia, abbiamo una forma bilineare simmetria non degenere:

$$\_ \cdot \_ : N_k(X)_{\mathbb{R}} \otimes N_{n-k}(X)_{\mathbb{R}} \to \mathbb{R}.$$

Definizione 0.0.3. Sia X una varietà liscia. Un ciclo $\alpha \in N_k(X)_{\mathbb{R}}$ è positivo se $\alpha \cdot \beta > 0$ per ogni $\beta \in \overline{\mathrm{Eff}}_{n-k}(X) \setminus \{0\}$, ed è nef se $\alpha \cdot \beta \geq 0$ per ogni $\beta \in \overline{\mathrm{Eff}}_{n-k}(X)$.

Definiamo adesso l'oggetto centrale della prima parte della tesi.

Definizione 0.0.4. Sia X una varietà liscia. Allora X si dice k-Fano se $ch_s(X)$ è positivo per ogni $1 \leq s \leq k$, ed è k-Fano debole per $k > 1$ se X è $(k-1)$-Fano e $ch_k(X)$ è nef.

Questa definizione generalizza la definizione di varietà Fano, infatti le varietà Fano sono esattamente le 1-Fano per il criterio di Kleiman (Theorem 2.1.20). Le 2-Fano sono state introdotte da de Jong e Starr in [**dJS06**, **dJS07**], dove hanno dimostrato che in una varietà 2-Fano con pseudo-indice almeno 3, passa una superficie razionale nel punto generale. In questo modo si migliora un risultato classico, e cioè che una varietà Fano è rigata (cioè passa una curva razionale per il punto generale). Successivamente, Araujo e Castravet hanno dimostrato in [**AC12**, Theorem 1.5(3)] che (sotto opportune ipotesi) in una 3-Fano debole passa una 3-varietà razionale per il punto generale. Inoltre hanno classificato le varietà 2-Fano con indice almeno $n-2$ ([**AC13**]).

Noi saremo interessati ai coni di classi pseudoeffettive delle varietà k-Fano. Infatti abbiamo la seguente.

Congettura. *Sia X una varietà k-Fano. Allora $\overline{\mathrm{Eff}}_k(X)$ è poliedrale.*

Nel caso delle 1-Fano, è già noto che la congettura è vera. Infatti si ha la seguente applicazione del Teorema del Cono di Mori (Theorem 2.1.25).

COROLLARIO 0.0.5. *Sia X una varietà Fano. Allora $\overline{\mathrm{Eff}}_1(X)$ è poliedrale.*

Quindi è lecito chiedersi se anche questo corollario può essere generalizzato per le k-Fano, proprio come è stata generalizzata l'esistenza di sottovarietà razionali.

In generale, se X è Fano non possiamo dire nulla sulla poliedralità di $\overline{\mathrm{Eff}}_k(X)$ nel caso $k \neq 0, 1, n$. Infatti un esempio di Tschinkel (enunciato in [**DELV11**, Example 6.10]) dimostra che una Fano può avere $\overline{\mathrm{Eff}}_2(X)$ non poliedrale (si veda Proposition 2.5.4).

La strategia adottata in gran parte della tesi è quella di utilizzare il seguente lemma (Lemma 2.1.29).

LEMMA 0.0.6. *Sia X una varietà proiettiva. Allora*

(1) Se $\mathrm{rk}A_k(X) = 1$ oppure $b_{2k}(X) = 1$, allora $\overline{\mathrm{Eff}}_k(X)$ è una semiretta.

(2) Se $\mathrm{rk}A_k(X) = 2$ oppure $b_{2k}(X) = 2$, allora $\overline{\mathrm{Eff}}_k(X)$ è una semiretta oppure è generato da due raggi estremali.

La dimostrazione segue dal diagramma (0.0.2). Infatti si ha che $\overline{\mathrm{Eff}}_k(X)$ è un cono in uno spazio vettoriale di dimensione al più 2, quindi è poliedrale.

Dunque calcoliamo i numeri di Betti per più varietà k-Fano possibile. Infine utilizziamo la classificazione di Araujo e Castravet [**AC13**, Theorem 3] (vedi Theorem 2.5.2) per dimostrare il seguente.

TEOREMA 0.0.7. *Sia X una varietà k-Fano con indice $i_X \leq n-2$. Allora $\overline{\mathrm{Eff}}_k(X)$ è poliedrale per $k = 2, 3$.*

Prima di dimostrare il teorema, analizziamo alcuni casi particolari.

PROPOSIZIONE 0.0.8. *Sia X una intersezione completa liscia in uno spazio proiettivo pesato. Allora $b_{2k}(X) = 1$ per ogni $k \neq \frac{n}{2}$. In particolare $\overline{\mathrm{Eff}}_k(X)$ è poliedrale.*

Se X è in uno spazio proiettivo, k-Fano debole e $1 \leq s \leq k$, allora $b_{2s}(X) \leq 2$. In particolare $\overline{\mathrm{Eff}}_s(X)$ è poliedrale.

Dunque, la congettura è vera nel caso delle intersezioni complete in uno spazio proiettivo. Adesso analizzeremo le intersezioni complete in tre famiglie di varietà omogenee: $G(r, s)$, $OG(r, s)$ e $SG(r, s)$. Dato un gruppo algebrico riduttivo G su $\mathbb{C}$, i sottogruppi P tali che G/P è una varietà proiettiva vengono chiamati parabolici. Una varietà omogenea è una varietà per cui esiste un gruppo algebrico che agisce transitivamente. Per le varietà omogenee è già noto che tutti i coni di classi pseudoeffettive sono poliedrali. Si dimostra che una varietà omogenea è Fano se e solo è un prodotto $G_1/P_1 \times ... \times G_k/P_k$, con P_i sottogruppo parabolico del gruppo semplice G_i. Per le Grassmanniane usuali abbiamo la seguente.

PROPOSIZIONE 0.0.9. *Sia X una 2-Fano debole intersezione completa in $G(r, s)$. Allora, $b_4(X) \leq 2$. In particolare $\overline{\mathrm{Eff}}_2(X)$ è poliedrale.*

Per le Grassmanniane ortogonali dimostriamo che.

PROPOSIZIONE 0.0.10. *Siano s, r interi positivi tali che $2 \leq r \leq \left[\frac{s}{2}\right]$, e $\left[\frac{s}{2}\right] - r \neq 1, 2$ se s è pari. Sia $s \neq 2r$ (rispettivamente, $s = 2r$), X una 2-Fano debole intersezione completa in una componente connessa in $OG(r, s)$ con la mappa di Plücker (rispettivamente, half-spinor), X molto generale se $X \subseteq OG(2, 7)$. Allora $\overline{\mathrm{Eff}}_2(X)$ è poliedrale.*

Per questa proposizione utilizziamo sia [**Spa96**, Theorem 2], che il seguente.

LEMMA 0.0.11. *Sia X una componente connessa di $OG(r,s)$, $2 \leq r \leq m = \left[\frac{s}{2}\right]$. Allora*

$$b_4(X) = \begin{cases} 1 & r = m \\ 3 & 1 \leq m - r \leq 2,\, s \text{ pari} \\ 2 & \text{altrimenti.} \end{cases}$$

Per le intersezioni complete nelle Grassmanniane simplettiche abbiamo il seguente risultato.

PROPOSIZIONE 0.0.12. *Sia X una 2-Fano debole intersezione completa in $SG(r,s)$. Allora, $b_4(X) \leq 2$. In particolare $\overline{\mathrm{Eff}}_2(X)$ è poliedrale.*

Queste proposizioni sono state ottenute utilizzando vari risultati in [**AC13**], dove vengono date varie condizioni su una varietà X per essere 2-Fano debole. Inoltre rispondiamo a [**AC13**, Question 39 and 40], infatti dimostriamo il seguente.

TEOREMA 0.0.13. *Sia $Y = G(2,5)$ o $G(2,6)$, e sia X intersezione completa di tipo $(1,1)$ in Y con la mappa di Plücker. Allora X non è 2-Fano debole.*

Per provare il teorema si è utilizzato sia [**dA15**, Corollary 5.1] (che implica che se X è generale allora non è 2-Fano), sia [**Ott15**, Proposition 3]. Questo teorema completa la classificazione delle 2-Fano deboli di indice $i_X \geq n-2$ in [**AC13**, Theorem 4]. Nella dimostrazione abbiamo utilizzato il seguente.

LEMMA 0.0.14. *Sia X_{11} intersezione completa liscia di tipo $(1,1)$ in $G(2,5)$ con la mappa di Plücker. Allora $b_4(X) = 2$.*

Per dimostrare questo lemma abbiamo calcolato il diamante di Hodge di X_{11}. Poiché il Teorema di Sommese (Theorem 2.4.2) non è sufficiente per determinare tutto il diamante, abbiamo utilizzato anche [**Sno86**].

Parte 2. Nella seconda parte studiamo il luogo di indeterminazione della mappa razionale introdotta in [**Voi16**, Proposition 4.8]. Data una mappa razionale, il luogo delle indeterminazioni è il sottoschema chiuso composto dai punti in cui la mappa non è definita. Facilmente possiamo verificare che dato un diagramma commutativo

$$\begin{tikzcd} X \arrow[r, dashed, "h"] \arrow[d, dashed, "f"'] & W \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

con f e h mappe razioni e g un morfismo, abbiamo che $\mathrm{Ind}(h) \subseteq \mathrm{Ind}(f)$. Inoltre per il risultato di Hironaka [**Hir64**, I.Question (E) p.140], per ogni $f : X \dashrightarrow Y$ mappa razionale tra varietà lisce, esiste una risoluzione delle indeterminazioni. Cioè esiste un diagramma commutativo

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{f}"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

dove $\widetilde{X}$ è liscia e π è una mappa birazionale che è un isomorfismo fuori da $\mathrm{Ind}(f)$.

DEFINIZIONE 0.0.15. Sia X una varietà compatta di Kähler. Diciamo che X è hyperKähler, se è semplicemente connessa e lo spazio delle 2 forme olomorfe globali $H^0(X, \wedge^2\Omega_X)$ è generato da una forma simplettica ovunque non degenere.

Per un risultato classico, la prima classe di Chern di una varietà con flusso di Ricci piatto è nulla. Inoltre per la decomposizione di Bogomolov [Bog74, Bea83] queste varietà hanno un ricoprimento étale finito dato dal prodotto di un toro complesso, di una varietà Calabi-Yau e di una varietà hyperKähler. In dimensione 2 gli unici esempi di varietà hyperKähler sono le superfici $K3$. Beauville ha dimostrato in [Bea83, Théorèmes 3 and 4] che per ogni $n \geq 0$, sia lo schema di Hilbert dei punti $S^{[n]}$ dove S è una superficie $K3$, sia la varietà di Kummer generalizzata K^nA associata ad una superficie Abeliana A, sono varietà hyperKähler. Una varietà hyperKähler che si ottiene da una deformazione di $S^{[n]}$ (rispettivamente, di una varietà Kummer generalizzata) è detta di tipo $K3^{[n]}$ (rispettivamente, di tipo K^nA). Queste varietà sono particolarmente interessanti perché permettono di costruire varietà hyperKähler di ogni dimensione complessa pari. Successivamente O'Grady in [O'G03, O'G99] ha costruito due nuovi esempi in dimensione 6 e 10 di varietà hyperKähler che non sono deformazioni di tipo conosciuto. Nella seconda parte della tesi usiamo due particolari esempi di varietà hyperKähler di tipo $K3^{[n]}$. Il primo esempio è quello fornito da Beauville e Donagi.

TEOREMA 0.0.16 ([BD85]). *Sia $Y \subseteq \mathbb{P}^5$ una ipersuperficie cubica liscia di dimensione 4. Sia F la sua varietà di Fano. Allora*

(1) La varietà F è hyperKähler di tipo $K3^{[2]}$.
(2) Se Y è Pfaffiana, allora F è isomorfa a $S^{[2]}$ per qualche superficie $K3$ S.

OSSERVAZIONE 0.0.17. Questa varietà è stata utilizzata per dare un esempio di classe nef ma non pseudoeffettiva. Infatti, in generale non è vero che $\mathrm{Nef}_k(X) \subseteq \overline{\mathrm{Eff}}_k(X)$ per $2 \leq k \leq \dim X - 2$. Il primo esempio di questo fenomeno è stato dato in [DELV11]. Ottem ha mostrato che se la cubica liscia Y è molto generale, allora $\overline{\mathrm{Eff}}_2(F) \subsetneq \mathrm{Nef}_2(F)$. Inoltre, la seconda classe di Chern $c_2(F)$ è nef ma non è effettiva [Ott15, Theorem 1].

L'altro esempio è stato dato più recentemente da C. Lehn, M. Lehn, Sorger e van Straten in [LLSvS17]. Essi hanno osservato che se $F_3(Y)$ è una compattificazione dello spazio delle cubiche razionali in Y, dove Y è una cubica liscia di dimensione 4 che non contiene un piano, allora esiste una fibrazione in $\mathbb{P}^2$

$$\phi : F_3(Y) \to Z',$$

dove Z' è una varietà liscia. Inoltre esiste un divisore di Z' che può essere contratto, la relativa contrazione $\sigma : Z' \to Z$ produce la varietà hyperKähler Z, che è di tipo $K3^{[4]}$ (si veda [AL17, Corollary] oppure [Leh15, Corollary 6.3]). Esiste un diagramma commutativo

$$\begin{tikzcd} F_3(Y) \arrow[r] \arrow[d, "\phi"'] & G(4,6) \\ Z' \arrow[ur, "g"'] & \end{tikzcd}$$

dove $F_3(Y) \to G(4,6)$ è la mappa che associa ad ogni cubica razionale l'unico spazio $\mathbb{P}^3$ che la contiene. Il morfismo g è finito di grado 72 su un aperto di Z'.

Voisin ha costruito in [Voi16, Proposition 4.8] una mappa razionale di grado 6

$$\psi : F \times F \dashrightarrow Z.$$

In breve, la mappa ψ manda coppie di rette non incidenti $(l,l') \in F \times F$ nella classe di una curva razionale normale di grado 3 contenuta nel sistema lineare $|L - L' - K_{S_{l,l'}}|$ della superficie cubica $S_{l,l'} := \langle L, L' \rangle \cap Y$. Questo ci fornisce un divisore rigato su Z. Infatti data una risoluzione delle indeterminazioni

$$\begin{tikzcd} \widetilde{F \times F} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\psi}"] & \\ F \times F \arrow[r, dashed, "\psi"] & Z \end{tikzcd}$$

l'immagine tramite $\widetilde{\psi}$ del divisore eccezionale di π è un divisore rigato di Z. Il nostro obiettivo è calcolare il luogo delle indeterminazioni di ψ. Prima dimostreremo la seguente.

Proposizione 0.0.18. *Esiste una mappa razionale $\psi' : F \times F \dashrightarrow Z'$ dominante, di grado 6 tale che* $\mathrm{Ind}(\psi') = I$.

La mappa ψ' è definita in tutti i punti dell'aperto

$$U_{ADE} := \left\{ (l,l') \in F \times F | (l,l') \notin I, S_{l,l'} := \left\langle L, L' \right\rangle \cap Y \,\text{ha singolarità}\, ADE \right\}$$

e la composizione $\sigma \circ \psi' : F \times F \dashrightarrow Z$ coincide con la mappa di Voisin. Infine dimostreremo che:

Teorema 0.0.19. *Il luogo delle indeterminazioni della mappa di Voisin ψ è la varietà I delle rette incidenti in Y.*

CHAPTER 1

Introduction

This thesis deals with two different problems, studied respectively in Chapter 2 and Chapter 3.

1.1. Introduction to Chapter 2

The study of cones of curves or divisors on smooth complex projective varieties X is a classical subject in Algebraic Geometry and is still an active research topic. However, little is known when we pass to higher dimensions. For example it is a classical result that the cone of nef divisors is contained in the cone of pseudoeffective divisors, but in general $\mathrm{Nef}_k(X) \subseteq \overline{\mathrm{Eff}}_k(X)$ is not true. These phenomena can appear only if $\dim X \geq 4$ and very few examples are known. In particular [**DELV11**] gives two examples of such varieties. Furthermore [**Ott15**] proves that if X is the variety of lines of a very general cubic fourfold in $\mathbb{P}^5$, then the cone of pseudoeffective 2-cycles on X is strictly contained in the cone of nef 2-cycles.

The central subject of the first part will be the k-Fano varieties.

Definition 1.1.1. A smooth Fano variety X is k-Fano if the s^{th} Chern character $ch_s(X)$ is positive (see Definition 2.3.1) for $1 \leq s \leq k$, and weak k-Fano for $k > 1$ if X is $(k-1)$-Fano and $ch_k(X)$ is nef.

There is a large interest in studying varieties with positive Chern characters. For example varieties with positive $ch_1(X)$ are Fano, hence uniruled, that is there is a rational curve through a general point. Fano varieties with positive second Chern character were introduced by J. de Jong and J. Starr in [**dJS06, dJS07**]. They proved a (higher dimensional) analogue of this result: weak 2-Fano varieties of pseudo-index at least 3 have a rational surface through a general point. Furthermore if X is weak 3-Fano then there is a rational threefold through a general point of X (under some hypothesis on the polarized minimal family of rational curves through a general point of X, [**AC12**, Theorem 1.5(3)]).

Another problem concerns how the geometry of the cones of pseudoeffective k-cycles depends on the positivity of the Chern characters $ch_s(X)$. Mori's Cone Theorem resolves this problem for $k = 1$: the positivity of $ch_1(X)$ implies the polyhedrality of the cone of pseudoeffective 1-cycles and the extremal rays are spanned by classes of rational curves. By Kleiman's Theorem, a variety with positive $ch_1(X)$ is just a Fano variety, that is with $c_1(X)$ ample, but this is not enough, in general, for the polyhedrality of cones of pseudoeffective k-cycles for $k > 1$: Tschinkel showed a Fano variety where $\overline{\mathrm{Eff}}_2(X)$ has infinitely many extremal rays (see Example 2.3.8). Therefore more positivity is needed in order to obtain polyhedrality of cones of pseudoeffective k-cycles for $k > 1$.

In this paper we investigate a possible way of generalizing Mori's result:

CONJECTURE 1.1.2. *If X is k-Fano, then $\overline{\mathrm{Eff}}_k(X)$ is a polyhedral cone.*

The computing of the fourth Betti number is enough to show the polyhedrality of some of the cones of 2-cycles for a large class of varieties: complete intersections in weighted projective spaces, rational homogeneous varieties and most complete intersections in them, etc. This allows us to test the conjecture for many 2-Fano varieties, and in particular we prove that it holds for del Pezzo and Mukai varieties. Using the classification of Araujo-Castravet, we also prove the following.

THEOREM 1.1.3 (= Theorem 2.5.3). *Let X be a n-dimensional 2-Fano variety with $i_X \geq n-2$. Then $\overline{\mathrm{Eff}}_2(X)$ is polyhedral. Also, $\overline{\mathrm{Eff}}_3(X)$ is polyhedral with the possible exception of the complete intersection of type $(2,2)$ in $\mathbb{P}^8$.*

In particular, Conjecture 1.1.2 is true for any n-dimensional k-Fano variety with $i_X \geq n-2$ and $k = 2, 3$.

Let X be a complete intersection in $G(2,5)$ or $G(2,6)$ with two hyperplanes under the Plücker embedding. Araujo and Castravet proved that X is not 2-Fano, but questioned if it is weak 2-Fano [**AC13**, Proposition 32 and Questions 39,41]. In [**dA15**, Corollary 5.1] it is proved that a general such X is not 2-Fano by showing that there exists an effective surface S such that $[i(S)] = \sigma_{1,1}^{\vee}$, where i is the inclusion. In this circumstance we can prove that all the smooth complete intersections of this type are not weak 2-Fano, and this completes the classification given in [**AC13**, Theorem 3 and 4].

THEOREM 1.1.4 (= Theorem 2.4.9). *Let $Y = G(2,5)$ or $G(2,6)$, let X be a smooth complete intersection of type $(1,1)$ in Y under the Plücker embedding. Then X is not weak 2-Fano.*

These ideas can be improved in three very promising directions: to generalize Tschinkel's example to higher dimensions, to prove the conjecture for some Fano 4-folds of index 1, and to use minimal families of rational curves to prove the conjecture for other 2-Fano's.

1.2. Introduction to Chapter 3

It is a classical result that a manifold with a Ricci flat metric has trivial first Chern class, and by Bogomolov's decomposition [**Bog74, Bea83**] such manifolds have a finite étale cover given by the product of a Torus, Calabi-Yau varieties and hyperKähler varieties. HyperKähler manifolds are interesting in their own and they are the subject of an intensive research. The first examples are $K3$ surfaces, and Beauville proved in [**Bea83**, Théorèmes 3 and 4] that for any $n \geq 0$, the Hilbert schemes of points $X^{[n]}$, where X is a $K3$ surface or the generalized Kummer varieties K^nA associated to an Abelian surface A, are hyperKähler varieties. Any hyperKähler variety that is a deformation of $X^{[n]}$ where X is a $K3$ surface (respectively, a generalized Kummer variety) is called $K3^{[n]}$-type (respectively, K^nA-type). These examples are particularly interesting because they permit to construct hyperKähler varieties of any even complex dimension. Later O'Grady in [**O'G03, O'G99**] constructed two new examples in dimension 6 and 10 of hyperKähler varieties that are not deformation of known types. There are few explicit complete families of hyperKähler of $K3^{[n]}$-type. Beauville and Donagi proved in [**BD85**, Proposition 1] that the variety of lines $F(Y)$ of a smooth cubic fourfold $Y \subseteq \mathbb{P}^5$ is an hyperKähler variety of $K3^{[2]}$-type. Another example was given much more recently by C. Lehn, M. Lehn, Sorger and van Straten in [LLSvS17]. They observed that if $F_3(Y)$ is a compactification of the space of rational cubic curves in Y, then $F_3(Y)$ is a $\mathbb{P}^2$-fibration based on a smooth variety $Z'(Y)$. Moreover there is a divisor of $Z'(Y)$ that can be contracted and this contraction produces a hyperKähler variety $Z(Y)$. This variety is of $K3^{[4]}$-type by [AL17, Corollary] or [Leh15, Corollary 6.3].

On the other hand the study of k-cycles on smooth complex projective varieties is a classical subject and it is very interesting on hyperKähler manifolds with respect to several regards. Indeed, as we said before, in general it is not true that $\mathrm{Nef}_k(X) \subseteq \overline{\mathrm{Eff}}_k(X)$ for $2 \leq k \leq \dim X - 2$. The first example of such phenomenon was given in [DELV11], but another example is a hyperKähler variety. Indeed, Ottem has shown that if the cubic Y is very general, then $\overline{\mathrm{Eff}}_2(F(Y)) \subsetneq \mathrm{Nef}_2(F(Y))$, and the second Chern class $c_2(F(Y))$ is nef but it is not effective [Ott15, Theorem 1].

It is known, due to Mumford's Theorem [Mum68, Theorem], that for a projective hyperKähler variety X of dimension $2n$ the kernel of the cycle map $cl : A^{2n}(X) \to H^{4n}(X)$ is infinite-dimensional (see [Voi03, III.10] for more details). Nevertheless Beauville conjectured in [Bea07] that

Conjecture 1.2.1 (Beauville). *Let X be a projective hyperKähler manifold. Then the cycle class map is injective on the subalgebra of $A^*(X)$ generated by divisors.*

See [Voi16] for an introduction to these topics. On the other hand Shen and Vial in [SV16] used a codimension 2 algebraic cycle to give evidence to the existence of a certain decomposition for the Chow ring of $F(Y)$ for a very general cubic fourfold Y.

Voisin constructed in [Voi16, Proposition 4.8] a degree 6 rational map

$$\psi : F(Y) \times F(Y) \dashrightarrow Z(Y).$$

Roughly speaking, the map ψ sends pairs of non-incident lines $(l, l') \in F(Y) \times F(Y)$ to the (class of the) degree 3 rational normal curve in the linear system $|L - L' - K_{S_{l,l'}}|$ of the cubic surface $S_{l,l'} := \langle L, L' \rangle \cap Y$. The Chapter 3 is devoted to the study of the indeterminacy locus of this map. In particular we prove the following.

Theorem 1.2.2 (= Theorem 3.4.4). *The indeterminacy locus of the Voisin map ψ is the variety I of the intersecting lines in Y.*

We hope that the explicit description of $\mathrm{Ind}(\psi)$ will contribute to the study of $c_2(Z(Y))$, the study of algebraic cycles on $Z(Y)$ and to other aspects of the geometry of $Z(Y)$. We hope to return to these topics in a future work.

1.3. Notations

A variety is a reduced and irreducible algebraic scheme over $\mathbb{C}$. Unless otherwise stated, X is a variety of dimension n.

CHAPTER 2

Betti numbers and pseudoeffective cone in 2-Fano Varieties

> No way of thinking or doing, however ancient, can be trusted without proof.
>
> Henry David Thoreau, Walden.

2.1. General facts about cycles

In this chapter the ring R is either $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, while the notation M_R means $M \otimes_{\mathbb{Z}} R$ for any $\mathbb{Z}$-module M.

DEFINITION 2.1.1. A ***k-cycle*** on X is a finite $\mathbb{Z}$-linear formal sum $\sum a_i V_i$ of subvarieties of dimension k. The group of all k-cycles is indicated by $Z_k(X)$.

In this paragraph we introduce three well-known equivalence relations on $Z_k(X)$.

Let $\mathbb{C}(X)$ be the field of rational functions on X. For any codimension 1 subvariety $Y \subseteq X$, since the ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring, there exists a group homomorphism

$$\begin{array}{cccc} ord_Y : & \mathbb{C}(X)^* & \longrightarrow & \mathbb{Z} \\ & f & \mapsto & ord_Y(f) \end{array}$$

where $ord_Y(f)$ is the integer such that $f = u \cdot t^{ord_Y(f)}$ where u is a unit in $\mathcal{O}_{X,Y}$ and t is the generator of the maximal ideal. In other words, $ord_Y(f)$ is the order of vanishing of f along Y.

DEFINITION 2.1.2. Let r be a rational function on a subvariety $Y \subseteq X$ of dimension $k+1$. The k-cycle on X defined by r is

$$[div(r)] := \sum_{W \subseteq Y} ord_W(r) W$$

where the sum is taken over all the codimension 1 subvarieties of Y.

DEFINITION 2.1.3. A k-cycle α is ***rationally equivalent*** to zero, if there are a finite number of $(k+1)$-dimensional subvarieties Y_i of X, and $r_i \in \mathbb{C}(Y_i)^*$, such that

$$\alpha = \sum [div(r_i)].$$

The subgroup of all the k-cycles rationally equivalent to zero is indicated by $\mathrm{Rat}_k(X)$, the group of k-cycles modulo rational equivalence on X is the ***Chow group***

$$A_k(X) = {}^{Z_k(X)}/_{\mathrm{Rat}_k(X)}.$$

REMARK 2.1.4. When X is smooth and projective variety, by [Ful98, Chapter 8] the direct sum $A_*(X) := \oplus_k A_k(X)$ has a ring structure, graded by codimension. The ring $A_*(X)$ is called Chow ring and its product operation is called intersection product.

Remark 2.1.5. There is a remarkable morphism of group

$$\deg : A_0(X)_R \to R$$

that maps any Chow class of a point to 1.

Let $H_i(X, R)$ and $H^i(X, R)$ be the singular homology and cohomology groups of X for $1 \leq i \leq 2n$ and coefficients in R. By [Ful98, p.372] there exists a cycle map $cl : Z_k(X) \to H_{2k}(X, \mathbb{Z})$ that is a homomorphism of groups and maps each subvariety of X to its homology class.

Definition 2.1.6. A k-cycle α is ***algebraically equivalent*** to zero, if there is a non-singular variety B, a cycle $\alpha' \in Z_{k+\dim B}(X \times B)$, and points $b_1, b_2 \in B$ such that α is the difference of the specializations of α' at b_1 and b_2, that is

$$\alpha = \alpha'_{b_1} - \alpha'_{b_2}.$$

The subgroup of all the k-cycles algebraically equivalent to zero is indicated by $\mathrm{Alg}_k(X)$.

Definition 2.1.7. A k-cycle α is ***homologically equivalent*** to zero, if $cl(\alpha) = 0$. The subgroup of all the k-cycles homologically equivalent to zero is indicated by $\mathrm{Hom}_k(X)$.

Now we want to define the numerical equivalence. First, let us do a digression on vector bundles.

Let E be a vector bundle on an algebraic scheme X. By [Ful98, p.47], there exists a unique set of Chern class operations

$$c_i(E) \cdot \_ : A_k(X) \to A_{k-i}(X),$$

with the following properties.

Theorem 2.1.8 ([Ful98, Theorem 3.2]). *The Chern classes of an algebraic scheme X satisfy the following.*

(1) (Vanishing) For all bundles E on X, all $i > \mathrm{rk}(E)$,

$$c_i(E) = 0.$$

(2) (Commutativity) For all bundles E, F on X, integers i,j, and classes α on X,

$$c_i(E) \cdot (c_j(F) \cdot \alpha) = c_j(F) \cdot (c_i(E) \cdot \alpha).$$

(3) (Projection formula) Let E be a vector bundle on X, $f : X' \to X$ a proper morphism. Then

$$f_*(c_i(f^*E) \cdot \alpha) = c_i(E) \cdot f_*(\alpha)$$

for all classes α on X', all i.

(4) (Pull-back) Let E be a vector bundle on X, $f : X' \to X$ a flat morphism. Then

$$c_i(f^*E) \cdot f^*\alpha = f^*(c_i(E) \cdot \alpha)$$

for all classes α on X, all i.

(5) *(Whitney sum) For any exact sequence*

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles on X*, then*

$$c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'').$$

(6) *(Normalization) If* E *is a line bundle on a variety* X*,* D *a Cartier divisor on* X *with* $\mathcal{O}_X(D) \cong E$*, then*

$$c_1(E) \cdot [X] = [D].$$

In the contest of point 6 (Normalization), we denote by $c_1(E)$ the divisor $c_1(E) \cdot [X]$, if no confusion is possible.

REMARK 2.1.9 ([**Ful98**, Remark 3.2.2]). The commutative law implies that any polynomial in the Chern classes with coefficient in R operates on the direct sum $\oplus_k A_k(X)_R$. In particular, if P is any homogeneous polynomial of weight d, where $c_i(E)$ has weight i, and $\alpha \in A_k(X)_R$, then $P \cdot \alpha \in A_{k-d}(X)_R$.

EXAMPLE 2.1.10. The Chern characters are very interesting examples of homogeneous polynomials in the Chern classes with $\mathbb{Q}$-coefficients. In [**Ful98**, Example 15.1.2(b)] we have the following closed formula, where $c_i = c_i(E)$

$$ch_k(E) = \frac{1}{k!} \det \begin{pmatrix} c_1 & 2c_2 & 3c_3 & \dots & kc_k \\ 1 & c_1 & 2c_2 & \dots & \vdots \\ & 1 & \ddots & \ddots & 3c_3 \\ & & \ddots & c_1 & 2c_2 \\ & & & 1 & c_1 \end{pmatrix}$$

The polynomial $ch_k(E)$ is homogeneous of weight k. The first Chern characters of E are

$$\begin{aligned} ch_1(E) &= c_1(E) \\ ch_2(E) &= \frac{1}{2}(c_1^2(E) - 2c_2(E)) \\ ch_3(E) &= \frac{1}{6}(c_1^3(E) - 4c_1(E)c_2(E) + 3c_3(E)) \end{aligned}$$

REMARK 2.1.11. If X is a smooth variety and T_X is its tangent bundle, the Chern classes and the Chern characters of T_X are usually indicated, respectively, by $c_k(X)$ and $ch_k(X)$.

DEFINITION 2.1.12. A k-cycle α is ***numerically equivalent*** to zero, if $\deg(P \cdot [\alpha]) = 0$ for any weight k homogeneous polynomial P in Chern classes of vector bundles on X. The subgroup of all the k-cycles numerically equivalent to zero is indicated by $\mathrm{Num}_k(X)$, the group of k-cycles modulo numerical equivalence on X is the factor group

$$N_k(X) = {}^{Z_k(X)}/_{\mathrm{Num}_k(X)}.$$

REMARK 2.1.13. There is a chain of inclusions [Ful98, p.374]

$$\mathrm{Rat}_k(X) \subseteq \mathrm{Alg}_k(X) \subseteq \mathrm{Hom}_k(X) \subseteq \mathrm{Num}_k(X) \subseteq Z_k(X)$$

that gives rise to a diagram

$$\begin{tikzcd} A_k(X) \arrow[r, two heads] & Z_k(X)/\mathrm{Alg}_k(X) \arrow[r, two heads] & Z_k(X)/\mathrm{Hom}_k(X) \arrow[r, two heads, "\pi_k"] \arrow[d, hook] & N_k(X) \\ & & H_{2k}(X,\mathbb{Z}) & \end{tikzcd} \tag{2.1.1}$$

We set

$$\pi_{k,\mathbb{R}} : Z_k(X)/\mathrm{Hom}_k(X) \otimes \mathbb{R} \twoheadrightarrow N_k(X)_{\mathbb{R}} \tag{2.1.2}$$

the tensor product of π_k and $id_{\mathbb{R}}$.

DEFINITION 2.1.14. A k-cycle $\sum r_i V_i \in Z_k(X)_R$ on X is ***effective*** if $r_i \geq 0$ for all i. A class $\alpha \in N_k(X)_R$ is effective if $\alpha = [\sum r_i V_i]$ for some effective k-cycle $\sum r_i V_i$.

2.1.0.1. *Cones of cycles.* A cone is a subset of a real vector space that is stable under multiplication by positive scalars.

DEFINITION 2.1.15. The ***effective cone*** $\mathrm{Eff}_k(X)$ is the cone generated by the effective classes in $N_k(X)_{\mathbb{R}}$. The closure $\overline{\mathrm{Eff}}_k(X)$ of the effective cone is the ***pseudoeffective cone***, its classes are called ***pseudoeffective classes***.

It is useful to consider the abstract dual notions of $N_k(X)$.

DEFINITION 2.1.16. Let X be a projective variety. The ***numerical dual groups*** are

$$N^k(X) = \frac{\text{homogeneous Chern polynomials P of weight } k}{\text{Chern polynomials P such that } \deg(\mathrm{P}\cdot[\alpha]) = 0 \text{ for all } \alpha \in N_k(X)}$$

The numerical class groups $N^k(X)$ are also denoted by $N_k(X)^\vee$. The dual class group $N^1(X)$ is called Néron-Severi group and it has the following equivalent definition.

DEFINITION 2.1.17 ([Laz04a, Definition 1.1.15]). Let X be a complete algebraic scheme over $\mathbb{C}$. A Cartier divisor D on X is ***numerically equivalent*** to zero if

$$D\cdot\alpha := \deg\left(c_1(\mathcal{O}_X(D))\cdot\alpha\right) = 0$$

for all 1-cycles classes α. The subgroup of divisors numerically equivalent to zero is indicated by $\mathrm{Num}(X)$, the group of divisors modulo numerical equivalence on X is the Néron-Severi group

$$N^1(X) = \mathrm{Div}(X)/\mathrm{Num}(X).$$

REMARK 2.1.18. There exists a natural map

$$\varphi : N^k(X)_{\mathbb{Q}} \to N_{n-k}(X)_{\mathbb{Q}}$$

by setting

$$\varphi([P]) = P\cdot[X].$$

By linearity of the intersection product, $N_k(X)$ is torsion free. When X is smooth and projective, φ is an isomorphism [Ful98, Example 19.1.5], and gives a perfect pairing

$$N_k(X)_{\mathbb{Q}} \otimes N_{n-k}(X)_{\mathbb{Q}} \to \mathbb{Q}.$$

DEFINITION 2.1.19. Let X be a projective variety. An element $\beta \in N^k(X)$ is ***nef*** if $\beta \cdot \alpha \geq 0$ for any effective $\alpha \in \mathrm{Eff}_k(X)$. The ***nef cone*** $\mathrm{Nef}^k(X)$ is the cone generated by classes of nef classes in $N^k(X)_{\mathbb{R}}$.

When X is smooth and projective, we define $\mathrm{Nef}_{n-k}(X)$ to be the image of $\mathrm{Nef}^k(X)$ under the isomorphism described in Remark 2.1.18. Two very important theorems by Kleiman explain the relation between ample and nef divisors.

THEOREM 2.1.20 (Kleiman's criterion for amplitude). *Let X be a projective variety, and let D be an $\mathbb{R}$-divisor on X. Then D is ample if and only if $D \cdot \alpha > 0$ for every $\alpha \in \overline{\mathrm{Eff}}_1(X)\backslash\{0\}$. Equivalently, choose any norm $\|\cdot\|$ on $N_1(X)_{\mathbb{R}}$, and denote by S the "unit sphere" of classes in $N_1(X)_{\mathbb{R}}$ of length 1. Then D is ample if and only if $D \cdot \alpha > 0$ for every $\alpha \in \overline{\mathrm{Eff}}_1(X)\backslash\{0\} \cap S$.*

PROOF. See [Laz04a, Theorem 1.4.29]. □

THEOREM 2.1.21 (Kleiman's Theorem). *Let X be a projective variety, let $\mathrm{Amp}(X) \subseteq N^1(X)_{\mathbb{R}}$ be the cone generated by the ample divisors. Then $\mathrm{Nef}^1(X)$ is the closure of $\mathrm{Amp}(X)$, and $\mathrm{Amp}(X)$ is the interior of $\mathrm{Nef}^1(X)$.*

PROOF. See [Laz04a, Theorem 1.4.23]. □

Let X be a smooth projective variety. Since every ample divisor D has a multiple with a section, the $(n-1)$-cycle $D \cdot [X]$ is effective. From the relation $\mathrm{Nef}^1(X) = \overline{\mathrm{Amp}}(X)$, using the isomorphism φ, it follows that

$$\mathrm{Nef}_{n-1}(X) \subseteq \overline{\mathrm{Eff}}_{n-1}(X).$$

This relation easily implies $\mathrm{Nef}_1(X) \subseteq \overline{\mathrm{Eff}}_1(X)$. Anyway, in higher codimension the picture is more complicated, indeed:

THEOREM 2.1.22 ([DELV11, Theorem B]). *Let E be an elliptic curve having complex multiplication, and set*

$$X = E \times \ldots \times E \ (n\,\text{times}).$$

Then for $k = 0, 1, n-1, n$, we have

$$\overline{\mathrm{Eff}}_k(X) = \mathrm{Nef}_k(X),$$

while in any other case

$$\overline{\mathrm{Eff}}_k(X) \subsetneqq \mathrm{Nef}_k(X).$$

THEOREM 2.1.23 ([DELV11, Theorem B]). *Let A be a very general principally polarized abelian surface, and let $X = A \times A$. Then $\overline{\mathrm{Eff}}_2(X) \subsetneqq \mathrm{Nef}_2(X)$.*

THEOREM 2.1.24 ([Ott15, Theorem 1]). *Let $Y \subseteq \mathbb{P}^5$ be a very general cubic fourfold and let F be its variety of lines. Then $\overline{\mathrm{Eff}}_2(F) \subsetneqq \mathrm{Nef}_2(F)$. In fact, $c_2(F)$ is positive on every surface, but has no effective multiple.*

I do not know any other example of variety X such that $\mathrm{Nef}_k(X) \not\subseteq \overline{\mathrm{Eff}}_k(X)$ for $k \neq 1, n-1$. Clearly such variety must have dimension at least four.

In this first part we will be mainly interested on the shape of the cones $\overline{\mathrm{Eff}}_k(X)$. The cone $\overline{\mathrm{Eff}}_1(X)$ is called Mori's Cone, and it is indicated also by $\overline{NE}(X)$. The following is the celebrated Mori's Cone Theorem and it gives us a lot of information on the shape of $\overline{\mathrm{Eff}}_1(X)$.

THEOREM 2.1.25 ([Mor82]). *Let X be a smooth projective variety, and suppose that K_X fails to be nef. Given any divisor D on X, let*

$$\begin{aligned} D_{\geq 0} &= \{\alpha \in N_1(X)/D \cdot \alpha \geq 0\} \\ \overline{\mathrm{Eff}}_1(X)_{D \geq 0} &= \overline{\mathrm{Eff}}_1(X) \cap D_{\geq 0} \end{aligned}$$

(1) There are countably many rational curves $C_i \subseteq X$, with

$$0 \leq -(C_i \cdot K_X) \leq n+1$$

which together with $\overline{\mathrm{Eff}}_1(X)_{K_X \geq 0}$ generate $\overline{\mathrm{Eff}}_1(X)$, i.e

$$\overline{\mathrm{Eff}}_1(X) = \overline{\mathrm{Eff}}_1(X)_{K_X \geq 0} + \sum_i \mathbb{R}_+ \cdot [C_i].$$

(2) Fix an ample divisor H. Then given any $\epsilon > 0$, there are only finitely many of these curves - say $C_1, ..., C_t$ - whose classes lie in the region $-(K_X + \epsilon H)_{\geq 0}$. Therefore

$$\overline{\mathrm{Eff}}_1(X) = \overline{\mathrm{Eff}}_1(X)_{(K_X + \epsilon H)_{\geq 0}} + \sum_{i=1}^{t} \mathbb{R}_+ \cdot [C_i].$$

PROOF. See [Laz04a, Theorem 1.5.33]. □

REMARK 2.1.26. People are primarily interested in the cone $\overline{\mathrm{Eff}}_1(X)$ for its use in the Mori Minimal Model Program. Mori's idea is that each extremal ray in the K_X-negative part of the cone defines a map $X \to X_1$ that contract the locus of the ray. The goal is repeating this process up to getting a variety X_m with K_{X_m} nef. Anyway, there are several problems to handle with: the image of it could be too singular, or the process could not be finite. If Mori's Cone is polyhedral (at least in the K_X-negative part), then there is a finite number of contractions. It follows that the structure of the variety X can be fully understood.

In this thesis we study the polyhedrality of $\overline{\mathrm{Eff}}_k(X)$ for $k = 1, 2, ..., n$.

DEFINITION 2.1.27. A ***Fano variety*** is a smooth projective variety such that the anti-canonical divisor is ample.

There are cases where the polyhedrality of some pseudoeffective cone is particularly simple to determinate. The following corollary show that Mori's Cone of Fano varieties is polyhedral.

COROLLARY 2.1.28. *Let X be a Fano variety. Then* $\overline{\mathrm{Eff}}_1(X)$ is polyhedral.

PROOF. Let $H = -K_X$, and $\epsilon = \frac{1}{2}$. Since $-K_X$ is ample, using Theorem 2.1.20, we have

$$\overline{\mathrm{Eff}}_1(X)_{(K_X+\epsilon H)_{\geq 0}} = \overline{\mathrm{Eff}}_1(X)_{\frac{1}{2}K_X{}_{\geq 0}} = \{0\}\,.$$

Now apply statement 2 of the Cone Theorem. □

LEMMA 2.1.29. *Let X be a projective variety. Then*

(1) If either $\mathrm{rk}A_k(X) = 1$ or $b_{2k}(X) = 1$, then $\overline{\mathrm{Eff}}_k(X)$ is a half-line.
(2) If either $\mathrm{rk}A_k(X) = 2$ or $b_{2k}(X) = 2$, then $\overline{\mathrm{Eff}}_k(X)$ is either a half-line or it is spanned by two extremal rays.

PROOF. In the first case, by diagram (2.1.1), we have a surjection $\mathbb{Z} \twoheadrightarrow N_k(X)$ and, as $N_k(X)$ is torsion-free, it must be $N_k(X) \cong \mathbb{Z}$. In the second case, again by diagram (2.1.1), there is a surjection $\mathbb{Z}^2 \twoheadrightarrow N_k(X)$ and then either $N_k(X) \cong \mathbb{Z}$ or $N_k(X) \cong \mathbb{Z}^2$. Since $\overline{\mathrm{Eff}}_k(X)$ generates $N_k(X)_{\mathbb{R}}$, it is either a half-line or it is spanned by two extremal rays, depending on the rank of $N_k(X)_{\mathbb{R}}$. □

EXAMPLE 2.1.30. A smooth cubic surface $S \subseteq \mathbb{P}^3$ is a Fano variety ad it is a classical object in Algebraic Geometry. The surface S is isomorphic to the blow-up of $\mathbb{P}^2$ centered at six points in general position [**Har77**, Corollary II.4.7]. Mori's Cone of S is a polyhedral cone in a 7-dimensional vector space, and it is spanned by 27 extremal rays spanned by the classes of the 27 lines of S [**Deb01**, p.149].

A classical tool in the study of cohomology of varieties is the following.

THEOREM 2.1.31 (Kodaira Vanishing Theorem). *Let X be a smooth projective variety, and let A be an ample divisor on X. Then $H^i(X, K_X + A) = 0$ for $i > 0$.*

PROOF. See [**Laz04a**, Theorem 4.2.1]. □

2.2. Grassmannians and Schubert cycles

2.2.1. Rational homogeneous varieties. Let us give a brief introduction to the rational homogeneous varieties. This is a family of varieties including the projective space, the Grassmann variety and many other classical objects. Let $GL_n(\Bbbk)$ be the group of invertible square matrices with coefficients in a field $\Bbbk$.

DEFINITION 2.2.1 ([**Pro07**, p.170]). A subgroup $H \subseteq GL_n(\Bbbk)$ is called a ***linear group***. A Zariski closed subgroup $H \subseteq GL_n(\Bbbk)$ is a ***linear algebraic group***.

DEFINITION 2.2.2 ([**Pro07**, p.188]). A linear algebraic group is called ***reductive*** if it does not contain any closed unipotent normal subgroup. A linear algebraic group is called ***semisimple*** if it is connected and its solvable radical is trivial.

DEFINITION 2.2.3 ([**Pro07**, p.190]). A subgroup of an algebraic group is called a ***maximal torus*** if it is a closed subgroup, a torus as an algebraic group, and maximal with respect to this property. A subgroup of an algebraic group is called a ***Borel subgroup*** if it is closed, connected and solvable, and maximal with respect to this property.

It is a consequence of [**Pro07**, Lemma p.207] that any maximal torus is contained in some Borel subgroup of G. Furthermore, we have the following.

THEOREM 2.2.4 ([Pro07, p.190]). *All maximal tori are conjugate. All Borel subgroups are conjugate.*

Let G be a reductive linear algebraic group defined over $\Bbbk = \mathbb{C}$, T a maximal torus of G. Let $\mathfrak{g}$ be the Lie algebra of G. It is well known that there exists a root space decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where Φ is the root system of $\mathfrak{g}$ with respect to T, and $\mathfrak{g}_\alpha$ denotes the root subspace of α. Choose a subset $\Phi^+ \subseteq \Phi$ of positive roots, and let $\Delta \subseteq \Phi^+$ be a base of the root system. The Lie subalgebra

$$\mathfrak{b} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha,$$

is maximal solvable [Pro07, Theorem p.341]. Then it is the Lie algebra of a Borel subgroup B containing T by [Pro07, Definition-Proposition 2 p.356].

DEFINITION 2.2.5 ([Pro07, p.354]). A subgroup P with the property that G/P is projective (i.e., compact) is called a ***parabolic subgroup***.

REMARK 2.2.6. In general, a parabolic subgroup needs to be a normal subgroup. Then G/P has not a structure of group.

THEOREM 2.2.7 ([Pro07, p.355]). *For an algebraic subgroup $H \subseteq G$ the following two conditions are equivalent:*

(1) H is maximal connected solvable (a Borel subgroup).
(2) H is minimal parabolic.

As a corollary, we get that there exists a bijective correspondence between subsets of Δ and parabolic subgroups of G containing B. Indeed, let $\Delta' \subseteq \Delta$ be a subset of simple roots. Then Δ' defines a Lie subalgebra

$$\mathfrak{p}_{\Delta'} := \mathfrak{b} \oplus \bigoplus_{\alpha \in \Phi^+ \cap \Phi_{\Delta'}} \mathfrak{g}_{-\alpha},$$

where $\Phi_{\Delta'}$ is the root subsystem generated by the simple roots not in Δ' [Pro07, Remark p.352]. Let $P_{\Delta'}$ be the subgroup with Lie algebra $\mathfrak{p}_{\Delta'}$. The subgroup $P_{\Delta'}$ is parabolic [Pro07, Corollary of the proof p.355]. On the other hand, any parabolic subgroup containing B is equal to $P_{\Delta'}$ for some $\Delta' \subseteq \Delta$ [Pro07, Corollary of the proof p.355].

DEFINITION 2.2.8. A projective variety X is ***homogeneous*** if there exists a group variety which acts on X transitively.

DEFINITION 2.2.9. A ***homogeneous rational variety*** is a homogeneous variety that it is rational.

Since the action of G on G/P by left multiplication is transitive, G/P is a homogeneous variety. There are only finitely many rational homogeneous varieties (up to isomorphism) of fixed dimension.

THEOREM 2.2.10 ([BR62]). *Let X be a projective homogeneous variety. Then*

(1) *X is isomorphic to a direct product $A \times R$, where A is an abelian variety and R is a homogeneous rational variety.*
(2) *If X is rational, then X is isomorphic to a product $G_1/P_1 \times ... \times G_k/P_k$, where for all $i \leq k$, G_i is simple and $P_i \subseteq G_i$ is a parabolic subgroup.*

Furthermore, the rational part of a homogeneous variety is Fano [BH58]. Denote by S the corresponding set of reflections in the Weyl group W of Φ. Then the pair (W, S) is a Coxeter system, i.e. it satisfies the following.

DEFINITION 2.2.11 ([Bou68, Chapitre IV, Définition 3]). We say that (W, S) is a ***Coxeter system*** if it satisfies the following condition: for all $s, s' \in S$, let $m(s, s')$ be the order of ss', let I be the set of pairs (s, s') such that $m(s, s')$ is finite. The generator set S and the relations $(ss')^{m(s,s')} = 1$ for (s, s') in I are a presentation of the group W.

Let $l : W \to \mathbb{N}_0$ be the length function relative to the system S of generators of W. Furthermore, we fix a subset Θ of S and denote by W_Θ the subgroup of W generated by Θ, and by P a parabolic subgroup of G associated to Θ [Pro07, Theorem p.362]. Let w_0 (respectively, w_θ) be the unique element of maximal length of W (respectively, W_Θ). A simple calculation shows that $\dim G/P = l(w_0) - l(w_\theta)$. The element w_0 and w_θ are characterized by the property [Bou68, Chapitre IV, Exercise 22]

$$l(ww_0) = l(w_0) - l(w),\ \forall w \in W \tag{2.2.1}$$

$$l(ww_\theta) = l(w_\theta) - l(w),\ \forall w \in W_\Theta \tag{2.2.2}$$

that imply immediately $w_0^2 = 1$ and $w_\theta^2 = 1$. It follows that, for every $w \in W$

$$l(w_0 w) = l((w_0 w)^{-1}) = l(w^{-1} w_0^{-1}) = l(w^{-1} w_0) = l(w_0) - l(w^{-1}) = l(w_0) - l(w).$$

Furthermore, set $W^\Theta = \{w \in W / l(ws) = l(w) + 1\ \forall s \in \Theta\}$. We have, for every $(w, \bar{w}) \in W^\Theta \times W_\Theta$,

$$l(w\bar{w}) = l(w) + l(\bar{w}). \tag{2.2.3}$$

PROPOSITION 2.2.12. *Let X be a smooth projective n-dimensional variety and let G be an affine group which acts transitively on X. Suppose that, for every $k = 1, ..., n-1$, there exists a finite family of subvarieties $\{\Omega_a\}_{a \in I_k}$ of dimension k such that*

(1) *$\langle \{[\Omega_a] / a \in I_k\} \rangle = H_{2k}(X, \mathbb{Z})$ or $A_k(X)$, and*
(2) *$\forall a \in I_k, \exists b \in I_{n-k}$ such that $\Omega_c \cdot \Omega_b = \delta_{a,c}\ \forall c \in I_k$.*

Then $\mathrm{Nef}_k(X) = \overline{\mathrm{Eff}}_k(X) = \mathrm{Eff}_k(X)$ is polyhedral and simplicial.

PROOF. We will suppose that the classes of the subvarieties $\{\Omega_a\}_{a \in I_k}$ generate $H_{2k}(X, \mathbb{Z})$, the case $A_k(X)$ being similar. Let ω_a be the class of Ω_a in $N_k(X)$. Let $\gamma \in \mathrm{Nef}_k(X)$. By (2.1.2) there is a class $\beta \in {}^{Z_k(X)}/_{\mathrm{Hom}_k(X)} \otimes \mathbb{R} \subseteq H_{2k}(X, \mathbb{R})$ such that $\pi_{k,\mathbb{R}}(\beta) = \gamma$. By (1) we have that $\beta = \sum_{a \in I_k} \gamma_a [\Omega_a]$ and then $\gamma = \sum \gamma_a \pi_k([\Omega_a]) = \sum \gamma_a \omega_a$. Let $a \in I_k$ and let $b \in I_{n-k}$ be as in (2). Then $\gamma \cdot \omega_b = \gamma_a \geq 0$ because γ is nef and ω_b is effective. Therefore $\gamma \in \mathrm{Eff}_k(X)$, then $\mathrm{Nef}_k(X) \subseteq \mathrm{Eff}_k(X)$. Furthermore, from $\omega_c \cdot \omega_a = \delta_{a,c}$ it follows that the system $\{\omega_a\}_{a \in I_k}$ is linearly independent. Let A a subvariety of X of dimension k, and let B be a subvariety of X of codimension k. By Kleiman's Theorem [Kle74] there is an element $g \in G$ such that gA is rationally equivalent to A and generically transverse to B. Then $A \cdot B = (gA) \cdot B = \#((gA) \cap B) \geq 0$, so

$\mathrm{Eff}_k(X) \subseteq \mathrm{Nef}_k(X)$. It is clear that $\mathrm{Nef}_k(X)$ is generated by $\{\omega_a / a \in I_k\}$. Since $\mathrm{Nef}_k(X)$ is closed and, as seen above, generated by the ω_a, we get that $\mathrm{Nef}_k(X) = \overline{\mathrm{Eff}}_k(X)$ is polyhedral. $\square$

PROPOSITION 2.2.13. *Let X be a rational homogeneous variety. Then* $\mathrm{Nef}_k(X) = \overline{\mathrm{Eff}}_k(X) = \mathrm{Eff}_k(X)$ *is polyhedral.*

PROOF. The description of the Chow ring of any rational homogeneous variety given in [Köc91, Corollary(1.5)] is

$$A_*(X) = \bigoplus_{w \in W^\Theta} \mathbb{Z}[X_w],$$

where X_w is the closure of the set BwP/P, with dimension $l(w)$ [Köc91, Proposition(1.3)]. Let $I_k = \{w \in W^\Theta / l(w) = k\}$. Given $w \in W^\Theta$ we claim that $w_0 w w_\theta \in I_{\dim X - k}$. Indeed for all $s \in \Theta$, using (2.2.1) and (2.2.3), we have

$$\begin{aligned} l(w_0 w w_\theta s) &= l(w_0) - l(w w_\theta s) = l(w_0) - l(w) - l(w_\theta s) \\ &= l(w_0) - l(w) - l(w_\theta) + l(s) = l(w_0) - l(w w_\theta) + 1 = l(w_0 w w_\theta) + 1. \end{aligned}$$

Similarly we can prove that $l(w_0 w w_\theta) = l(w_0) - l(w_\theta) - l(w)$. Now given $w \in I_k$ we have, by [Köc91, Proposition(1.4)], that (2) of Proposition 2.2.12 is satisfied. $\square$

The pseudoeffective cone is also polyhedral in the case when the action of G on X has finitely many orbits, see [FMSS95, Corollary p.2] and [Li15].

2.2.2. Grassmann variety. Consider the group $SL_s(\mathbb{C}) := {}^{GL_s(\mathbb{C})}/_{\mathbb{C}^*}$, the root space of its Lie algebra is of type A_{s-1}. Let $\{\alpha_1, ..., \alpha_{s-1}\}$ be the simple roots. If we choose any root α_r, it can be proved that the parabolic subgroup P_{α_r} is the group of block matrices

$$P_{\alpha_r} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in SL_s(\mathbb{C}) / A \in GL_r(\mathbb{C}), C \in GL_{s-r}(\mathbb{C}) \right\}.$$

DEFINITION 2.2.14. The ***Grassmann*** variety is the rational homogeneous variety $G(r,s) := SL_s(\mathbb{C})/P_{\alpha_r}$.

The Grassmann variety can also be defined as the variety that parametrizes r-dimensional vector subspaces in a $\mathbb{C}^s$. Let us see for example the case $G(1,2)$, where we expect to recover the usual projective line.

EXAMPLE 2.2.15. The elements of $SL_2(\mathbb{C})/P_{\alpha_1}$ are cosets $\begin{pmatrix} x & y \\ z & t \end{pmatrix} P_{\alpha_1}$ where a generic element is

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} ax & bx + cy \\ az & bz + ct \end{pmatrix},$$

then it is obvious that

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} P_{\alpha_1} = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix} P_{\alpha_1} \iff [x : z] = [x' : z'] \in \mathbb{P}^1.$$

Since $G(1, s+1) \cong \mathbb{P}^s$, and by duality it can proved that $G(r,s) \cong G(s-r,s)$, we will usually take integers r and s such that $2 \le r \le \frac{s}{2}$. Note that the dimension of $G(r,s)$ is $r(s-r)$. We can define some tautological bundle over $G(r,s)$. The first one is $\mathbb{C}^s \times G(r,s)$, that is the trivial vector bundle of rank s over $G(r,s)$. We have also the universal subbundle $S \to G(r,s)$, that is the subbundle of $\mathbb{C}^s \times G(r,s)$ whose fiber of at each point $[W] \in G(r,s)$ is the vector space $W \subseteq \mathbb{C}^s$. Finally there is an exact sequence

$$0 \to S \to \mathbb{C}^s \times G(r,s) \to Q \to 0 \tag{2.2.4}$$

where Q is called universal quotient bundle. It can be checked that the cotangent bundle of $G(r,s)$ is

$$\Omega_{G(r,s)} = S \otimes Q^\vee.$$

If we apply Whitney sum (Theorem 2.1.8) and [Har77, A.3.(C4)] to the exact sequence (2.2.4), we have

$$-K_{G(r,s)} = -c_1(\det \Omega_{G(r,s)}) = -c_1(\Omega_{G(r,s)}) = s\, c_1(\det S^\vee). \tag{2.2.5}$$

The line bundle $\det S^\vee$ is the Plücker line bundle, it defines an embedding called Plücker embedding:

$$G(r,s) \to \mathbb{P}(\wedge^r \mathbb{C}^s).$$

The Plücker embedding simply sends a k-plane $[W] = \langle w_1, ..., w_r\rangle$ to the multivector $w_1 \wedge ... \wedge w_r$. Further details are in [GH78, p.209]. Then $\det S^\vee$ is very ample, so by (2.2.5) we get that $G(r,s)$ is Fano. Let us understand now the closed subvarieties of $G(r,s)$. Given any flag of linear subspaces $F_\bullet$

$$0 \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F_s \subseteq \mathbb{C}^s$$

and a sequence of integers $\lambda = \{\lambda_i\}$ such that $0 \le \lambda_r \le ... \le \lambda_1$ and $|\lambda| = \sum \lambda_i$, the space $\Omega_{(\lambda)}(F_\bullet)$ is defined as the closure of the locus

$$\{[W] \in G(r,s)/\dim(W \cap F_{s-r+i-\lambda_i}) = i \,\mathrm{for}\, 1 \le \mathrm{i} \le \mathrm{r}\}\,.$$

The following is known as Ehresmann's Theorem [Ehr34].

THEOREM 2.2.16. *The integral homology of the Grassmannian $G(r,s)$ has no torsion and is freely generated as a group by the cycles $[\Omega_{(\lambda)}(F_\bullet)]$ in real codimension $2|\lambda|$, where $\lambda = \{\lambda_i\}$ ranges over all nonincreasing sequences of integers between 0 and $s-r$. In particular, all cohomology in $G(r,s)$ is algebraic.*

PROOF. See [GH78, Proposition pp. 195, 196]. □

REMARK 2.2.17. Since the cohomology of $G(r,s)$ is algebraic, its Hodge Diamond is vertical. That is, $h^{p,q}(G(r,s)) = 0 \iff p \ne q$ [GH78, p,163].

We set $\sigma_{(\lambda)} = [\Omega_{(\lambda)}(F_\bullet)] \in H^{2|\lambda|}(X, \mathbb{Z})$. By Ehresmann's Theorem, two cycles $\sigma_{(\lambda)}$ and $\sigma_{(\lambda')}$ are different if and only if $\lambda \ne \lambda'$, i.e. the cycles $\sigma_{(\lambda)}$ does not depend on the flag. The direct sum

$$H^*(G(r,s), \mathbb{Z}) = \bigoplus_{0 \le k \le r(s-r)} H^{2k}(G(r,s), \mathbb{Z})$$

has a structure of ring. This means that a product $\sigma_{(\lambda)} \cdot \sigma_{(\lambda')}$ is a cycle

$$\sum_{\mu} \gamma_\mu \sigma_{(\mu)} \in H^{2(|\lambda|+|\lambda'|)}(G(r,s),\mathbb{Z}),$$

where μ ranges over all nonincreasing sequences of r integers between 0 and $s-r$ such that $|\mu| = |\lambda| + |\lambda'|$, and γ_μ is an integer. The set of rules used to determinate γ_μ is called Schubert Calculus, and it is the best tool for answering many combinatorial questions coming from geometry.

DEFINITION 2.2.18. The cycles $\sigma_{(\lambda)}$ are called Schubert cycles.

In general, the computations made to determinate each γ_μ are very complicated, but the following theorem can be useful at least in some particular cases.

THEOREM 2.2.19 (Pieri's Formula). *Let $\lambda = (a, 0, ..., 0)$ and λ' be two nonincreasing sequences of r integers between 0 and $s-r$. Then*

$$\sigma_{(\lambda)} \cdot \sigma_{(\lambda')} = \sum_{\substack{\lambda'_i \leq \mu_i \leq \lambda'_{i-1} \\ |\mu| = a + |\lambda'|}} \sigma_{(\mu)}.$$

REMARK 2.2.20. Even if Pieri's formula works in a very particular situation, it is enough to compute all the product rule of the ring $H^*(G(r,s),\mathbb{Z})$. This is because the Schubert cycles $\sigma_{(\lambda)}$ where $\lambda = (a, 0, ..., 0)$ generate the ring $H^*(G(r,s),\mathbb{Z})$ [**GH78**, p.205]. We will denote such cycles simply by σ_a.

LEMMA 2.2.21. *The first even Betti numbers of $G(r,s)$ are: $b_0(G(r,s)) = 1$, $b_2(G(r,s)) = 1$, and $b_4(G(r,s)) = 2$.*

PROOF. It is enough to count the number of sequences λ with the prescribed value of $|\lambda|$. Remember that each sequence is nonincreasing, i.e. $0 \leq \lambda_r \leq ... \leq \lambda_1$. We have

$$\sum \lambda_i = 0 \iff \lambda = (0, ..., 0)$$
$$\sum \lambda_i = 1 \iff \lambda = (1, 0, ..., 0),$$

and

$$\sum \lambda_i = 2 \iff \begin{cases} \lambda = (1, 1, 0, ..., 0) \\ \text{or} \\ \lambda = (2, 0, ..., 0). \end{cases}$$

□

This lemma implies that $H^4(G(r,s),\mathbb{Z})$ is generated by the Schubert cycles σ_2 and $\sigma_{1,1}$. Then the ring $H^{r(s-r)-4}(G(r,s),\mathbb{Z})$ is generated by two Schubert cycles, indicated by $\sigma_2^\vee$ and $\sigma_{1,1}^\vee$, such that

$$\sigma_2 \cdot \sigma_2^\vee = \sigma_1 \cdot \sigma_1^\vee = 1$$
$$\sigma_1 \cdot \sigma_2^\vee = \sigma_2 \cdot \sigma_1^\vee = 0.$$

From now on, we will use implicitly the natural identification

$$\mathbb{Z} \cong H^{2r(s-r)}(G(r,s),\mathbb{Z}),$$

given by the generator $\sigma_{(s-r,\dots,s-r)}$.

EXAMPLE 2.2.22. Let us compute some products of cycles which will be useful later. Using Pieri's formula for the cycles of $G(2,5)$, we have:

$$\begin{aligned}(\sigma_2^2)\cdot\sigma_1^2 &= (\sigma_{3,1}+\sigma_{2,2})\cdot\sigma_1^2 = 2\\ (\sigma_{1,1}^2)\cdot\sigma_1^2 &= \sigma_{2,2}\cdot\sigma_1^2 = 1\\ (\sigma_2\cdot\sigma_{1,1})\cdot\sigma_1^2 &= \sigma_{3,1}\cdot\sigma_1^2 = 1.\end{aligned}$$

While in the case $G(2,6)$ we have

$$\begin{aligned}(\sigma_4\cdot\sigma_2)\cdot\sigma_1^2 &= \sigma_{4,2}\cdot\sigma_1^2 = \sigma_{4,3}\cdot\sigma_1 = 1\\ (\sigma_{2,2}\cdot\sigma_2)\cdot\sigma_1^2 &= \sigma_{4,2}\cdot\sigma_1^2 = \sigma_{4,3}\cdot\sigma_1 = 1\\ (\sigma_4\cdot\sigma_{1,1})\cdot\sigma_1^2 &= 0\cdot\sigma_1^2 = 0\\ (\sigma_{2,2}\cdot\sigma_{1,1})\cdot\sigma_1^2 &= (\sigma_{2,2}\cdot(\sigma_1^2-\sigma_2))\cdot\sigma_1^2 = (\sigma_{2,2}\cdot\sigma_1^2-\sigma_{2,2}\cdot\sigma_2)\cdot\sigma_1^2\\ &= (\sigma_{3,2}\cdot\sigma_1-\sigma_{4,2})\cdot\sigma_1^2 = (\sigma_{4,2}+\sigma_{3,3}-\sigma_{4,2})\cdot\sigma_1^2\\ &= (\sigma_{3,3})\cdot\sigma_1^2 = \sigma_{4,3}\cdot\sigma_1 = 1.\end{aligned}$$

REMARK 2.2.23. It can be shown that the the Plücker line bundle generates $\mathrm{Pic}(G(r,s))$. Then due to (2.2.5), we have $-K_{G(r,s)} = \mathcal{O}(s)$ where $\mathcal{O}(1)$ gives the Plücker embedding.

REMARK 2.2.24. By Serre duality

$$\chi(\Omega^p_{G(2,5)}(-m)) = \chi(\Omega^{6-p}_{G(2,5)}(m))$$

Then we have

$$\chi(\Omega_{G(2,5)}(-m)) = \chi(\Omega^5_{G(2,5)}(m)) = 0 \text{ for } m = 1,2,3, \tag{2.2.6}$$

because all the groups $H^p(G(2,5),\Omega^5_{G(2,5)}(m))$ are zero by [Sno86, Theorem p. 171(3)]. Furthermore,

$$\chi(\Omega^2_{G(2,5)}(-m)) = \chi(\Omega^4_{G(2,5)}(m)) = 0 \text{ for } m = 1,2 \tag{2.2.7}$$

because

$$H^p(G(2,5),\Omega^4_{G(2,5)}(m)) = 0\,\forall p\geq 0$$

by [Sno86, Theorem p.p. 165,169]. By Lemma 2.2.21 and Remark 2.2.17 we have

$$\chi(\Omega_{G(2,5)}) = -1 \text{ and } \chi(\Omega^2_{G(2,5)}) = 2. \tag{2.2.8}$$

REMARK 2.2.25. In order to compute the values of $\chi(\Omega^p_{G(2,s)}(m))$ we can use the explicit formulas given in [Xu11, Chapter 4] and get the same results of Remark 2.2.24. Roughly speaking: for any s consider a sequence of r integer $\lambda = (\lambda_1,\dots,\lambda_r)$, set

$$k_\lambda = \prod_{1\leq i<j\leq r}\frac{\lambda_i-\lambda_j+j-i}{j-i}$$

then [Xu11, p. 39]

$$\begin{aligned}\chi(T_{G(2,s)}(m)) &= k_{(m+2,m+1,\underbrace{1,\ldots,1}_{s-3},0)}\\ \chi(\wedge^2 T_{G(2,s)}(m)) &= k_{(m+3,m+1,\underbrace{1,\ldots,1}_{s-4},0,0)} + k_{(m+3,m+3,\underbrace{2,\ldots,2}_{s-3},0)}.\end{aligned}$$

In our case, the computations give us

$$\begin{aligned}\chi(T_{G(2,5)}(m)) &= \frac{1}{24}(m+1)(m+2)(m+3)(m+4)^2(m+6)\\ \chi(\wedge^2 T_{G(2,5)}(m)) &= \frac{1}{24}(m+2)(m+4)(m+6)(m+7)(m+3)^2\\ &\quad +\frac{1}{16}(m+1)(m+4)^2(m+6)(m+7)(m+3).\end{aligned}$$

By Serre duality

$$\chi(\Omega^{\alpha}_{G(2,s)}(-m)) = \chi(\wedge^{\alpha} T_{G(2,s)}(m-s)),$$

then we get (2.2.6), (2.2.7) and (2.2.8).

2.2.3. Orthogonal Grassmann variety. We are now going to define two types of rational homogeneous varieties: orthogonal Grassmannians $OG(r,s)$ and symplectic Grassmannians $SG(r,s)$. By [Ehr34] the (co)homology of $OG(r,s)$ and $SG(r,s)$ is algebraic and freely generated by the closure of certain loci defined by flags. In particular, we have definitions of Schubert cycles and Schubert calculus analogous to those of $G(r,s)$.

Consider the group $SO_s(\mathbb{C})$ of orthogonal matrices, where s is an odd (respectively, even) number and $m = [\frac{s}{2}]$. The root space of its algebra Lie is of type B_m (respectively, D_m). Let $\{\alpha_1, \ldots, \alpha_m\}$ be the simple roots.

Definition 2.2.26. The ***orthogonal Grassmann*** variety is the rational homogeneous variety $OG(r,s) := SO_s(\mathbb{C})/P_{\alpha_r}$.

The orthogonal Grassmann variety can also be defined as the variety that parametrizes r-dimensional vector subspaces of $\mathbb{C}^s$, that are isotropic with respect to a non-degenerate symmetric bilinear form ω on $\mathbb{C}^s$. The scheme $OG(r,2m)$ has two isomorphic connected components if $r = m$ or $m-1$. In these two cases, we will denote by $OG_+(r,2m)$ a connected component of $OG(r,2m)$.

Remark 2.2.27. The variety $OG(r,s)$ can be embedded in $G(r,s)$ as the zero locus of a section of the bundle $Sym^2 S^\vee$. Indeed, since any symmetric bilinear form ω on $\mathbb{C}^s$ is an element of $Sym^2(\mathbb{C}^s)^\vee$, any such ω is a global section of $Sym^2(\mathbb{C}^s \times G(r,s))^\vee$. By (2.2.4), we have the following surjective map of vector bundles

$$Sym^2(\mathbb{C}^s \times G(r,s))^\vee \twoheadrightarrow Sym^2 S^\vee \tag{2.2.9}$$

which can be described point-wise by

$$(\omega, [V]) \mapsto (\omega_{|V}, [V]).$$

Thus, the composition of ω with (2.2.9) is a global section ϖ of $Sym^2 S^\vee$, with zero-locus given by the set of isotropic subspaces of $\mathbb{C}^s$ with respect to ω. If we take ω non-degenerate, the zero locus of ϖ has the expected dimension $\frac{r(2s-3r-1)}{2}$.

We now define Schubert cycles in orthogonal Grassmannian. Let us recall the useful notation in [Cos11]. Given a connected component $X \subseteq OG(r,s)$, we will write $s = 2m+1-\epsilon$ with $\epsilon \in \{0,1\}$ and $2 \leq r \leq m$. Let t be an integer such that $0 \leq t \leq r$, and $t \equiv m \ (mod\, 2)$ if $2r = s$. Given a sequence of integers $\lambda = (\lambda_1, ..., \lambda_t)$ of length t such that

$$m - \epsilon \geq \lambda_1 > ... > \lambda_t > -\epsilon.$$

Let $\tilde{\lambda} = (\tilde{\lambda}_{t+1}, ..., \tilde{\lambda}_m)$ be the unique sequence of length $m-t$ such that

- $m - 1 \geq \tilde{\lambda}_{t+1} > ... > \tilde{\lambda}_m \geq 0$,
- $\tilde{\lambda}_j + \lambda_i \neq m - \epsilon$ for every $i = 1,..,t$ and $j = t+1, ..., m$.

The Schubert varieties in X are parametrized by pairs (λ, μ), where μ is any subsequence of $\tilde{\lambda}$ of length $r-t$. Given an isotropic flag of vector subspaces $F_\bullet$

$$0 \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F_m \subseteq F_{m-1}^\perp \subseteq F_{m-2}^\perp \subseteq ... \subseteq F_1^\perp \subseteq \mathbb{C}^s,$$

$\Omega_{(\lambda,\mu)}(F_\bullet)$ is defined as the closure of the locus

$$\begin{gathered}\{[W] \in X / \dim(W \cap F_{m+1-\epsilon-\lambda_i}) = i \text{ for } 1 \leq \text{i} \leq \text{t};\\ \left.\dim(W \cap F_{\mu_j}^\perp) = j \text{ for } t < j \leq r\right\}.\end{gathered}$$

Let us define another sequence λ' of length t' in this way:

- $\lambda' = \lambda$ if either $\epsilon = 0$ or $\epsilon = 1$ and $t \equiv m \ (mod\, 2)$; otherwise
- $\lambda' = \lambda \cup \{b\}$ where $b = \min\{a \in \mathbb{N} / 0 \leq a \leq m-1,\ a \notin \lambda,\ a + \mu_j \neq m - 1\, \forall j = t+1, ..., k\}$.

Let $\tilde{\lambda'}$ be the unique sequence associated to λ' as above. Then the pair (λ, μ) is a subsequence of $(\lambda', \tilde{\lambda'})$. Suppose $(\lambda, \mu) = (\lambda'_{i_1}, ..., \lambda'_{i_t}, \tilde{\lambda'}_{i_{t+1}}, ..., \tilde{\lambda'}_{i_r})$ and let the discrepancy of λ and μ be the non-negative number

$$dis(\lambda, \mu) = \sum_{j=1}^{r} (m - r + j - i_j). \tag{2.2.10}$$

Then the codimension of a Schubert cycle $\Omega_{(\lambda,\mu)}(F_\bullet)$ is (see [Cos11, p.2448])

$$\operatorname{codim}(\Omega_{(\lambda,\mu)}(F_\bullet)) = \sum_{i=1}^{t'} \lambda'_i + dis(\lambda, \mu).$$

Let $\Omega_{(\lambda,\mu)}(F_\bullet)$ be of codimension k and set $\sigma_{(\lambda,\mu)} = \left[\Omega_{(\lambda,\mu)}(F_\bullet)\right] \in H^{2k}(X, \mathbb{Z})$.

The set of all $\sigma_{(\lambda,\mu)}$ of codimension k is a basis of $H^{2k}(X, \mathbb{Z})$ (as we said in the beginning of this subsection). Let us compute some Betti number of $OG(r.s)$.

LEMMA 2.2.28. *Let X be a connected component of $OG(r,s)$, $2 \leq r \leq m = \left[\frac{s}{2}\right]$, we have*

$$b_4(X) = \begin{cases} 1 & r = m \\ 3 & 1 \leq m - r \leq 2,\ s \text{ even} \\ 2 & \text{otherwise.} \end{cases}$$

PROOF. We have to count the number of sequences (λ, μ) such that

$$\sum_{i=1}^{t'} \lambda'_i + dis(\lambda, \mu) = 2.$$

For $1 \leq j \leq r$ let $c_j = m - r + j - i_j$. It can easily be seen that

$$m - r \geq c_1 \geq c_2 \geq .... \geq c_r \geq 0$$

and we can write

$$dis(\lambda, \mu) = \sum_{i=1}^{r} c_j.$$

We are in one of the following cases:

(1) $\sum_{i=1}^{t'} \lambda'_i = 0$ and $dis(\lambda, \mu) = 2$, or
(2) $\sum_{i=1}^{t'} \lambda'_i = 1$ and $dis(\lambda, \mu) = 1$, or
(3) $\sum_{i=1}^{t'} \lambda'_i = 2$ and $dis(\lambda, \mu) = 0$.

Let s be odd. Then

Case (1) t must be 0. If $m - r \geq 1$ then $c_1 = c_2 = 1$, and, if $m - r > 1$, we have also the possibility $c_1 = 2$. These cases correspond to

$$(\lambda, \mu) = \begin{cases} (\emptyset, (r, r-1, r-3, ...,)) \\ (\emptyset, (r+1, r-2, r-3, ...,)). \end{cases}$$

Case (2) Only one possibility if $m - r = 1$, that is $\lambda = (1)$ and $c_2 = 1$. This case corresponds to $(\lambda, \mu) = ((1), (r-2, r-3, ...,))$. No other possibilities if $m - r \neq 1$.

Case (3) It must be $\lambda = (2)$, then $i_1 = 1$ and since $c_j = 0\ \forall j \geq 1$, $c_1 = m - r + 1 - 1 = 0$ implies $m = r$. This is the case $(\lambda, \mu) = ((2), (m-1, m-3, ...))$.

Let s be even. If $s = 2r$, then the discrepancy is 0 because $c_j \leq m - r\ \forall j \geq 1$, then it is possible only the case 3, that is

$$(\lambda, \mu) = \begin{cases} ((2), (m-1, m-2, m-4, \ldots)) & m \,\text{odd} \\ ((2,0), (m-2, m-4, \ldots)) & m \,\text{even}. \end{cases}$$

Suppose $m > r$. Let m be even, then

Case (1) It must be $\lambda' = \emptyset$, then $\lambda = \lambda' = \emptyset$ and $\tilde{\lambda} = (m-1, m-2, m-3, m-4, ...)$. If $m - r \geq 1$ then $c_1 = c_2 = 1$, and, if $m - r \geq 2$, we have also the possibility $c_1 = 2$. These cases corresponds to

$$(\lambda, \mu) = \begin{cases} (\emptyset, (r, r-1, r-3, ...,)) \\ (\emptyset, (r+1, r-2, r-3, ...,)). \end{cases}$$

Case (2) It must be $\lambda' = (1, 0)$, then we can have $\lambda = (0)$ or $\lambda = (1)$.

Suppose $\lambda = (0)$, $\tilde{\lambda} = (m-2, m-3, ...)$, and we have to choose a μ such that $b = 1$ in order to have $\lambda' = \lambda \cup \{1\}$ which implies $\tilde{\lambda}' = (m-3, m-4, ...)$. This can happen only if $m - 2 \notin \mu$, that is, it is enough to choose μ as a subsequence of

$(m-3, m-4, \ldots)$. This case implies that $i_1 = 2$, then $c_1 = m-r+1-2 = m-r-1$, then it must be $m - r = 2$. Since $c_j = 0 \ \forall j \geq 2$, that corresponds to the case

$$(\lambda, \mu) = ((0), (m-4, m-5, \ldots,)).$$

Suppose $\lambda = (1)$, $\tilde{\lambda} = (m-1, m-3, \ldots)$, and we have to choose a μ such that $b = 0$ in order to have $\lambda' = \lambda \cup \{0\}$ which implies $\tilde{\lambda}' = (m-3, m-4, \ldots)$. This can happen only if $m - 1 \notin \mu$, that is, it is enough to choose μ as a subsequence of $(m-3, m-4, \ldots)$. This case implies that $i_1 = 1$, then $c_1 = m-r+1-1 = m-r$, then it must be $m - r = 1$. Since $c_j = 0 \ \forall j \geq 2$, that corresponds to the case

$$(\lambda, \mu) = ((1), (m-3, m-4, m-5, \ldots,)).$$

Case (3) It must be $\lambda' = (2, 0)$, then we can have $\lambda = (0)$ or $\lambda = (2)$.

If $\lambda = (2)$, then $c_1 = m - r$, then the discrepancy is not 0.

So $\lambda = (0)$, $\tilde{\lambda} = (m-2, m-3, \ldots)$, $c_j = 0 \ \forall j \geq 1$, and we have to choose a μ such that $b = 2$ in order to have $\lambda' = \lambda \cup \{2\}$ which implies $\tilde{\lambda}' = (m-2, m-4, \ldots)$. This can happen only if $m - 2 \in \mu$ and $m - 3 \notin \mu$. That is, the sequence

$$((0), \mu) = ((0), (\tilde{\lambda}'_{i_1}, \ldots, \tilde{\lambda}'_{i_r}))$$

seen as a subsequence of $((2, 0), (m-2, m-4, \ldots)) = (\lambda', \tilde{\lambda}')$ must satisfy $i_1 = 2$. The condition $c_j = 0$ implies $i_j = m - r + j$, then $i_1 = m - r + 1 = 2$ implies $m - r = 1$. Then, if $m - r = 1$, we have the sequence

$$(\lambda, \mu) = ((0), (m-2, m-4, \ldots)).$$

Let m be odd, then

Case (1) It must be $\lambda' = (0)$, then we can have $\lambda = \lambda' = (0)$ or $\lambda = \emptyset$.

Suppose $\lambda = \lambda' = (0)$, this implies $\tilde{\lambda} = (m-2, m-3, m-4, \ldots)$ and $c_1 = m - r$. Then

-if $m - r \geq 3$, then this case in not possible since the first summand of the discrepancy (which it must be 2) is $m - r$,

-if $m - r = 2$, then $c_j = 0$ for $j \geq 2$, that is $i_j = m - r + j$ for $j \geq 2$, then

$$(\lambda, \mu) = ((0), (\tilde{\lambda}_{m-r+2}, \tilde{\lambda}_{m-r+3}, \ldots,)) = ((0), (r-2, r-3, \ldots)),$$

-if $m - r = 1$, then $c_j = 0$ for $j \geq 3$ and $c_2 = 1$, that is

$$(\lambda, \mu) = ((0), (\tilde{\lambda}_{m-r+1}, \tilde{\lambda}_{m-r+3}, \ldots,)) = ((0), (r-1, r-3, \ldots)).$$

Suppose $\lambda = \emptyset$, $\tilde{\lambda} = (m-1, m-2, \ldots)$, and we have to choose a μ such that $b = 0$ in order to have $\lambda' = \lambda \cup \{0\}$ which implies $\tilde{\lambda}' = (m-2, m-3, m-4, \ldots)$. This can happen only if $m - 1 \notin \mu$, that is, it is enough to choose μ as a subsequence of $(m-2, m-3, m-4, \ldots)$. If $m - r \geq 1$ we have $c_1 = c_2 = 1$, that corresponds to the case

$$(\lambda, \mu) = (\emptyset, (r, r-1, r-3, \ldots,)).$$

But, in order to make $m - 1 \notin \mu$, we must have $r \neq m - 1$, then this case only happen if $m - r \geq 2$. If $m - r \geq 2$, we have also the possibility $c_1 = 2$, that corresponds to the case

$$(\lambda, \mu) = (\emptyset, (r+1, r-2, r-3, \ldots,)).$$

But, in order to make $m-1 \notin \mu$, $r+1 \neq m-1$, then this case only happen if $m-r \geq 3$.

Case (2) It must be $\lambda' = (1)$, then we can have $\lambda = \lambda' = (1)$ or $\lambda = \emptyset$.

Suppose $\lambda = \lambda' = (1)$, then $\tilde{\lambda} = (m-1, m-3, m-4, ...)$, $c_1 = m-r$, and $c_j = 0$ for $j \geq 2$. So, if $m-r=1$, we have the sequence

$$(\lambda, \mu) = ((1), (\tilde{\lambda}_{m-r+2}, \tilde{\lambda}_{m-r+3}, ...,)) = ((1), (m-3, m-4, ...)).$$

Suppose $\lambda = \emptyset$, $\tilde{\lambda} = (m-1, m-2, ...)$, $c_1 = 1$, $c_j = 0\ \forall j \geq 2$, and we have to choose a μ such that $b=1$ in order to have $\lambda' = \lambda \cup \{1\}$ which implies

$$\tilde{\lambda}' = (m-1, m-3, m-4, ...).$$

This can happen only if $m-1 \in \mu$ and $m-2 \notin \mu$. That is, the sequence

$$(\emptyset, \mu) = (\emptyset, (\tilde{\lambda}'_{i_1}, ..., \tilde{\lambda}'_{i_r}))$$

seen as a subsequence of

$$((1), (m-1, m-3, m-4, ...)) = (\lambda', \tilde{\lambda}')$$

must satisfy $i_1 = 2$. The condition $c_1 = 1$ implies $1 = m-r+1-i_1$, then $1 = m-r+1-2$ that is $m-r=2$, while the condition $c_j = 0 \forall j \geq 2$ implies $i_j = m-r+j$. Then, if $m-r=2$, we have the sequence

$$(\lambda, \mu) = ((\emptyset), (m-1, m-4, m-5, ...)).$$

Case (3) It must be $\lambda' = (2)$, then we can have $\lambda = \lambda' = (1)$ or $\lambda = \emptyset$.

If $\lambda = (2)$, then $c_1 = m-r$, then the discrepancy is not 0.

So $\lambda = \emptyset$, $\tilde{\lambda} = (m-1, m-2, ...)$, $c_j = 0\ \forall j \geq 1$, and we have to choose a μ such that $b=2$ in order to have $\lambda' = \lambda \cup \{2\}$ which implies

$$\tilde{\lambda}' = (m-1, m-2, m-4, ...).$$

This can happen only if $m-1, m-2 \in \mu$ and $m-3 \notin \mu$. That is, the sequence

$$(\emptyset, \mu) = (\emptyset, (\tilde{\lambda}'_{i_1}, ..., \tilde{\lambda}'_{i_r}))$$

seen as a subsequence of

$$((2), (m-1, m-2, m-4, ...)) = (\lambda', \tilde{\lambda}')$$

must satisfy $i_1 = 2$ and $i_2 = 3$. The condition $c_j = 0$ implies $i_j = m-r+j$, then $i_1 = m-r+1 = 2$ and $i_2 = m-r+2 = 3$ imply $m-r=1$. Then, if $m-r=1$, we have the sequence

$$(\lambda, \mu) = ((\emptyset), (m-1, m-2, m-4, ...)).$$

□

LEMMA 2.2.29. $b_6(OG_+(r, 2r)) = 2$.

PROOF. We have to calculate the number of Schubert cycles of dimension 6, that is the number of sequences $r-1 \geq \lambda_1 > ... > \lambda_t \geq 0$ such that $\sum_{i=1}^{t} \lambda_i = 3$, $t \equiv r\ (mod\, 2)$. We get

- If r is odd, $\lambda = (3)$ and $\lambda = (2,1,0)$;
- If r is even, $\lambda = (3,0)$ and $\lambda = (2,1)$.

□

2.2.4. Symplectic Grassmann variety. Consider the group $Sp_s(\mathbb{C})$ of symplectic matrices, and $m = \frac{s}{2}$. The root space of its Lie algebra is of type C_m. Let $\{\alpha_1, ..., \alpha_m\}$ be the simple roots.

DEFINITION 2.2.30. The ***symplectic Grassmann*** variety is the rational homogeneous variety $SG(r,s) := Sp_s(\mathbb{C})/P_{\alpha_r}$.

The Symplectic Grassmann variety can also be defined as the variety that parametrizes r-dimensional vector subspaces of $\mathbb{C}^s$, that are isotropic with respect to a non-degenerate skew-symmetric bilinear form σ on $\mathbb{C}^s$.

REMARK 2.2.31. The variety $SG(r,s)$ can be embedded in $G(r,s)$ as the zero locus of a section of the bundle $\wedge^2 S^\vee$ in the same way as Remark 2.2.27.

We now define Schubert cycles in symplectic Grassmannians $SG(r,s)$ with $2 \le r \le m = \frac{s}{2}$. Let us recall the useful notation in [Cos18]. Let t be an integer such that $0 \le t \le r$. Given a sequence of integers $\lambda = (\lambda_1, ..., \lambda_t)$ of length t such that

$$m \ge \lambda_1 > ... > \lambda_t > 0.$$

Let $\tilde{\lambda} = (\tilde{\lambda}_{t+1}, ..., \tilde{\lambda}_m)$ be the unique sequence of length $m - t$ such that

- $m - 1 \ge \tilde{\lambda}_{t+1} > ... > \tilde{\lambda}_m \ge 0$,
- $\tilde{\lambda}_j + \lambda_i \ne m$ for every $i = 1, .., t$ and $j = t+1, ..., m$.

The Schubert varieties in $SG(r,s)$ are parametrized by pairs (λ, μ), where μ is any subsequence of $\tilde{\lambda}$ of length $r - t$. Given an isotropic flag of vector subspaces $F_\bullet$

$$0 \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F_m \subseteq F_{m-1}^\perp \subseteq F_{m-2}^\perp \subseteq ... \subseteq F_1^\perp \subseteq \mathbb{C}^s$$

$\Omega_{(\lambda,\mu)}(F_\bullet)$ is defined as the closure of the locus

$$\{[W] \in SG(r,s) / \dim(W \cap F_{m+1-\lambda_i}) = i \,\mathrm{for}\, 1 \le \mathrm{i} \le \mathrm{t}; \\ \dim(W \cap F_{\mu_j}^\perp) = j \,\mathrm{for}\, t < j \le r\}.$$

Suppose $(\lambda, \mu) = (\lambda_1, ..., \lambda_t, \tilde{\lambda}_{i_{t+1}}, ..., \tilde{\lambda}_{i_r})$, the codimension of $\Omega_{(\lambda,\mu)}(F_\bullet)$ is (see [Cos18, p. 57])

$$\mathrm{codim}(\Omega_{(\lambda,\mu)}(F_\bullet)) = \sum_{i=1}^{t} \lambda_i + dis(\lambda, \mu)$$

Where the discrepancy $dis(\lambda, \mu)$ is defined in (2.2.10).

The set all $\omega_{(\lambda,\mu)} = \left[\Omega_{(\lambda,\mu)}(F_\bullet)\right]$ of codimension k is a basis of $H^{2k}(SG(r,s), \mathbb{Z})$ (as we said in the beginning of the previous subsection). The following lemma can be proved in the same way as in the case of $OG(r, 2m+1)$.

LEMMA 2.2.32. *Let $2 \le r \le m = \frac{s}{2}$, then*

$$b_4(SG(r,s)) = \begin{cases} 2 & m - r \ge 1 \\ 1 & r = m. \end{cases}$$

2.3. Higher Fano varieties

We now deal with the central object of this chapter, the higher Fano manifolds. Let us define positive cycles and recall the definition of nef cycle.

Definition 2.3.1. Let X be a smooth variety. A class $\alpha \in N_k(X)_{\mathbb{R}}$ is ***positive*** if $\alpha \cdot \beta > 0$ for every $\beta \in \overline{\mathrm{Eff}}_{n-k}(X) \backslash \{0\}$, and it is ***nef*** if $\alpha \cdot \beta \geq 0$ for every $\beta \in \overline{\mathrm{Eff}}_{n-k}(X)$. The cone generated by nef classes of k-cycles is $\mathrm{Nef}_k(X)$.

Definition 2.3.2. A smooth Fano variety X is ***k-Fano*** if the s^{th} Chern character $ch_s(X)$ is positive for $1 \leq s \leq k$, and ***weak k-Fano*** for $k > 1$ if X is $(k-1)$-Fano and $ch_k(X)$ is nef.

Proposition 2.3.3. *Let X be a smooth variety . Then X is Fano if and only if X is 1-Fano. In particular, if X is 1-Fano then $\overline{\mathrm{Eff}}_1(X)$ is polyhedral.*

Proof. Using properties of Chern classes [Har77, A.3.(C4)], we have

$$-K_X = c_1(\det \Omega_X^\vee) = c_1(\Omega_X^\vee) = c_1(X).$$

Then the following claim

$$K_X \,\text{is ample} \iff c_1(X) \,\text{is positive},$$

it follows tautologically by Theorem 2.1.20. Then $\overline{\mathrm{Eff}}_1(X)$ is polyhedral by Corollary 2.1.28. □

It is known that Fano varieties are uniruled [Deb01, Theorem 3.4]. That is, there exist a rational curve through a general point. This result was generalized by de Jong and Starr in 2006. Indeed, they proved that if X is weak 2-Fano with pseudo-index is at least 3, then a general point of X is contained in a rational surface [dJS07, Proposition 1.3].

Following [Kol96, p. 109], for a projective scheme X and a general point $x \in X$, there exists a scheme $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves through x. Let H_x be a minimal family of rational curves through x. For example H_x can be an irreducible component of $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves on X, having minimal degree with respect to some fixed ample line bundle on X. By definition of $\mathrm{RatCurves}^n(X, x)$, H_x is normal. Due to [Keb02, Theorem 3.4], there exists a finite morphism $\tau_x : H_x \to \mathbb{P}(T_x X^\vee)$, sending a smooth curve to its tangent direction at x.

Definition 2.3.4. Let X be a smooth projective uniruled variety and let $x \in X$ be general. A ***polarized minimal family*** of rational curves through $x \in X$ is a pair (H_x, L_x), where $L_x := \tau_x^* \mathcal{O}(1)$.

Araujo and Castravet gave a further generalization of de Jong and Starr's result.

Theorem 2.3.5 ([AC12, Theorem 1.5]). *Let X be a smooth projective Fano variety, and let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$, $d = \dim H_x \geq 1$. If X is 2-Fano and $(H_x, L_x) \ncong (\mathbb{P}^d, \mathcal{O}(2))$, $(\mathbb{P}^1, \mathcal{O}(3))$, then there is a generically injective morphism $g : (\mathbb{P}^2, p) \to (X, x)$ mapping lines through p to curves parametrized by H_x. Moreover if X is weak 3-Fano and $d \geq 2$.*

(1) There is a rational 3-fold through x, except possibly if $(H_x, L_x) \cong (\mathbb{P}^2, \mathcal{O}(2))$ and $\tau_x(H_x)$ is singular.

(2) *Let (W_h, M_h) be a polarized minimal family of rational curves through a general point $h \in H_x$. Suppose that $(H_x, L_x) \not\cong (\mathbb{P}^d, \mathcal{O}(2))$ and $(W_h, M_h) \not\cong (\mathbb{P}^k, \mathcal{O}(2))$, $(\mathbb{P}^1, \mathcal{O}(3))$. Then there is a generically injective morphism $h : (\mathbb{P}^3, q) \to (X, x)$ mapping lines through q to curves parametrized by H_x.*

Then, also weak 3-Fano are uniruled except possibly in one case. We think that Proposition 2.3.3 can be generalized to k-Fano, in the same way the uniruled property as been generalized to k-Fano. That is, we think that it is true the following.

CONJECTURE 2.3.6. *If X is k-Fano, then $\overline{\mathrm{Eff}}_k(X)$ is a polyhedral cone.*

REMARK 2.3.7. The conjecture is interesting only for Fano varieties of dimension at least 4. Indeed, if X is Fano, then it is also a Mori Dream Space [BCHM10, Corollary 1.3.2]. Hence $\overline{\mathrm{Eff}}_{n-1}(X)$ is polyhedral [HK00, Proposition 1.11(2)]. We deduce that $\overline{\mathrm{Eff}}_2(X)$ is polyhedral if X is a Fano threefold (see [Cas06] for an introduction to Mori Dream Spaces).

In general, if X is Fano we cannot say that the other cones $\overline{\mathrm{Eff}}_k(X)$ are polyhedral for $k > 1$.

EXAMPLE 2.3.8 (Tschinkel). This example first appeared in [DELV11, Example 6.10], and shows that to be Fano is not enough to have polyhedrality for $\overline{\mathrm{Eff}}_2(X)$. Let $f : X_b \to \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ along the surface Y_b described in [Cut00]. That is, Y_b is a smooth quartic in $\mathbb{P}^3$ such that $\overline{\mathrm{Eff}}_1(Y_b) = \mathrm{Nef}_1(Y_b)$ is a round cone. Then X_b is Fano. But $\overline{\mathrm{Eff}}_2(X_b)$ is not polyhedral since every curve $C \in \mathrm{Eff}_1(Y_b)$ defines an extremal surface $[\pi^{-1}(C)] \in \overline{\mathrm{Eff}}_2(X_b)$, where $f_{|E} = \pi : E \to Y_b$ is the restriction to the exceptional divisor. Anyway, X_b is not weak 2-Fano. We work out some of the details in Proposition 2.5.4.

2.4. Complete Intersections

DEFINITION 2.4.1. Let X be a projective variety. We say that a vector bundle E is (very) ***ample*** over X if the Serre line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a (very) ample line bundle on the projective bundle $\mathbb{P}(E)$.

A complete intersection is the zero-locus of a section of a direct sum $\mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_r)$, where $a_i > 0$ and $\mathcal{O}(1)$ is a very ample line bundle. In this section we want to study the polyhedrality of the cones of pseudoeffective cycles on complete intersections. Since the direct sum of ample line bundles is ample [Laz04b, Proposition 6.1.13(i)], we have that the following theorem can be applied to any complete intersection.

THEOREM 2.4.2 (Sommese's Theorem). *Let X be a non-singular variety, let E be an ample vector bundle over X of rank e, and let Z be the zero-locus of a section of E. Then*

$$H^i(X, Z; \mathbb{Z}) = 0 \quad \text{for} \quad i \leq n - e.$$

In particular the restriction map $H^i(X, \mathbb{Z}) \to H^i(Z, \mathbb{Z})$ is an isomorphism for $i < n - e$, and injective when $i = n - e$.

PROOF. See [Laz04b, Theorem 7.1.1]. $\square$

2.4.1. Complete intersections in weighted projective spaces. Let $w_0, w_1, ..., w_n$ be a set of strictly positive integers and let $\mathbf{w} = (w_0, w_1, ..., w_n)$ be a vector. Consider the action associated to $\mathbf{w}$

$$\begin{array}{ccccc} \mathbb{C}^* & \times & \mathbb{C}^{n+1} & \to & \mathbb{C}^{n+1} \\ \lambda & & (x_0, x_1, ..., x_n) & \mapsto & (\lambda^{w_0} x_0, \lambda^{w_1} x_1, ..., \lambda^{w_n} x_n) \end{array}$$

We define the quotient

$$\mathbb{P}(\mathbf{w}) := {}^{\mathbb{C}^{n+1}\setminus\{0\}}/_{\mathbb{C}^*}$$

The space $\mathbb{P}(\mathbf{w})$ can be seen as the projective variety $\mathrm{Proj}\mathbb{C}[x_0, ..., x_n]$ where $\deg(x_i) = w_i$ (see [**Dim92**, p. 230]). Sommese's Theorem cannot be applied to it, since $\mathbb{P}(\mathbf{w})$ is the quotient of a smooth quasi projective variety by a finite group. Then it has orbifold singularities [**Dim92**, B10]. But we can use the following.

Theorem 2.4.3 (Lefschetz's Theorem over $\mathbb{Q}$). *Let V be a weighted complete intersection in $\mathbb{P}(\mathbf{w})$, then the restriction map*

$$H^i(\mathbb{P}(\mathbf{w}), \mathbb{Q}) \to H^i(V, \mathbb{Q})$$

is an isomorphism for $i < \dim V$, and injective when $i = \dim V$.

Proof. See [**Dim92**, B22]. $\square$

Proposition 2.4.4. *Let X be a n-dimensional smooth complete intersection in a weighted projective space. If $k \neq \frac{n}{2}$ then $b_{2k}(X) = 1$. In particular $\overline{\mathrm{Eff}}_k(X)$ is polyhedral.*

Proof. Recall [**Dim92**, B13] that $\dim H^{2i}(\mathbb{P}(\mathbf{w}), \mathbb{Q}) = 1$ for every $0 \leq i \leq \dim \mathbb{P}(\mathbf{w})$. By Theorem 2.4.3 we have that $H^{2k}(X, \mathbb{Q}) \cong H^{2k}(\mathbb{P}(\mathbf{w}), \mathbb{Q})$ for $2k < n$, then $b_{2k}(X) = 1$ for $k < \frac{n}{2}$. But $b_{2n-2k}(X) = b_{2k}(X)$, then it follows that, for $k \neq \frac{n}{2}$, $b_{2k}(X) = 1$ and by Lemma 2.1.29 that $\overline{\mathrm{Eff}}_k(X)$ is a half-line. $\square$

Furthermore, if X is a k-Fano complete intersection in a projective space, then we can solve Conjecture 2.3.6, even for weak Fano.

Theorem 2.4.5. *Let X be a n-dimensional weak k-Fano complete intersection in a projective space. If $1 \leq s \leq k$, then $b_{2s}(X) \leq 2$. In particular $\overline{\mathrm{Eff}}_s(X)$ is polyhedral.*

Proof. Let X be of type $(d_1, ..., d_c)$ in $\mathbb{P}^{n+c}$, with $d_i \geq 2$ for $1 \leq i \leq c$. By Proposition 2.4.4, we can suppose n even and $s = \frac{n}{2}$. We know from [**AC13**, 3.3.1] that $ch_{\frac{n}{2}}(X)$ is nef if and only if $d_1^{\frac{n}{2}} + ... + d_c^{\frac{n}{2}} \leq n + c + 1$. Since $n \geq 4$, it follows easily that $c = 1$. On the other hand $d_1^{\frac{n}{2}} \leq n + 2$ is possible only for $d_1 = 2$, that is X is an n-dimensional quadric. But $b_n(X) = 2$ [**Rei72**, p.20] and the theorem follows by Lemma 2.1.29. $\square$

2.4.2. Complete intersections in Grassmann varieties. In this subsection we will compute the number $b_4(X)$, where X is a complete intersection in a Grassmannian variety. We first consider the smooth complete intersection of type $(1, 1)$ in $G(2, 5)$, in light of Proposition 2.4.8.

Lemma 2.4.6. *Let X be a smooth complete intersection of type $(1, 1)$ in a Grassmann variety $G(2, 5)$ under the Plücker embedding. Then $b_4(X) = 2$.*

PROOF. By [Laz04b, Example 7.1.5], all rows of the Hodge Diamond of X, except the middle row, are equal to those of the Hodge Diamond of $G = G(2,5)$. Since X is Fano, $h^{0,4}(X) = 0$ then

$$\chi(\Omega_X) = -1 - h^{1,3}(X) \tag{2.4.1}$$
$$\chi(\Omega_X^2) = h^{2,2}(X) \tag{2.4.2}$$
$$b_4(X) = h^{2,2}(X) + 2h^{1,3}(X). \tag{2.4.3}$$

Note that by Serre duality and adjunction formula, for any integer m

$$h^4(\mathcal{O}_X(-m)) = h^0(\mathcal{O}_X(m) \otimes \mathcal{O}_G(2-5)_{|X}) = h^0(\mathcal{O}_X(m-3))$$

then by Theorem 2.1.31, $\chi(\mathcal{O}_X(-1)) = \chi(\mathcal{O}_X(-2)) = 0$. Take the Koszul resolution of the sheaf $\mathcal{O}_X$

$$0 \to \mathcal{O}_G(-2) \to \mathcal{O}_G(-1)^{\oplus 2} \to \mathcal{O}_G \to \mathcal{O}_X \to 0 \tag{2.4.4}$$

and tensor it by Ω_G

$$0 \to \Omega_G(-2) \to \Omega_G(-1)^{\oplus 2} \to \Omega_G \to \Omega_{G|X} \to 0 \tag{2.4.5}$$

then, by Remark 2.2.24,

$$\chi(\Omega_{G|X}) = \chi(\Omega_G(-2)) - 2\chi(\Omega_G(-1)) + \chi(\Omega_G) = -1$$

If we tensor (2.4.5) by $\mathcal{O}_G(-1)$ we have

$$\chi(\Omega_{G|X}(-1)) = \chi(\Omega_G(-3)) - 2\chi(\Omega_G(-2)) + \chi(\Omega_G(-1)) = 0$$

From the canonical sequence

$$0 \to \mathcal{O}_X(-1)^{\oplus 2} \to \Omega_{G|X} \to \Omega_X \to 0 \tag{2.4.6}$$

we get $\chi(\Omega_X) = \chi(\Omega_{G|X}) - 2\chi(\mathcal{O}_X(-1)) = -1$, then $h^{1,3}(X) = 0$ by (2.4.1). If, instead, we tensor (2.4.4) by Ω_G^2, that is

$$0 \to \Omega_G^2(-2) \to \Omega_G^2(-1)^{\oplus 2} \to \Omega_G^2 \to \Omega_{G|X}^2 \to 0$$

we get, by Remark 2.2.24,

$$\chi(\Omega_{G|X}^2) = \chi(\Omega_G^2(-2)) - 2\chi(\Omega_G^2(-1)) + \chi(\Omega_G^2) = 2.$$

By [Har77, Exercise II.5.16d] and (2.4.6) we get

$$\chi(\Omega_X^2) = \chi(\Omega_{G|X}^2) - 2\chi(\Omega_{G|X}(-1)) - 3\chi(\mathcal{O}_X(-2)) = 2.$$

Then by (2.4.2) and (2.4.3) we get $h^{2,2}(X) = 2$ and $b_4(X) = 2$. □

PROPOSITION 2.4.7. *Let X be a n-dimensional weak 2-Fano complete intersection in a Grassmann variety $G(r,s)$ under the Plücker embedding. Then, $b_4(X) \leq 2$. In particular $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.*

Proof. Assume that X is of type $(d_1, ..., d_c)$. If $n > 4$, by Theorem 2.4.2, we have $b_4(X) = b_4(G(r,s)) \leq 2$ and we can apply Lemma 2.1.29. If $n = 4$, using [AC13, Proposition 31], we have the following conditions: $c = r(s-r) - 4$ and $\sum_{i=1}^{c} d_i \leq s - 1$. It is easy to see that this leads to the following cases

$G(r,s)$	Type		$G(r,s)$	Type
$G(2,7)$	$(1,1,1,1,1,1)$		$G(2,5)$	$(1,1)$
$G(3,6)$	$(1,1,1,1,1)$			$(1,2)$
$G(2,6)$	$(1,1,1,1)$			$(1,3)$
	$(1,1,1,2)$			$(2,2)$

None of them is weak 2-Fano by [AC13, Proposition 31 and 32(iv)], and Theorem 2.4.9. □

After Lemma 2.2.21, we introduced the following notation: the group $H^4(G(r,s),\mathbb{Z})$ is generated by $\{\sigma_2, \sigma_{1,1}\}$, while $H^{r(s-r)-4}(G(r,s),\mathbb{Z})$ is generated by a basis $\{\sigma_2^\vee, \sigma_{1,1}^\vee\}$ dual to $\{\sigma_2, \sigma_{1,1}\}$.

Proposition 2.4.8 (from [dA15, Corollary 5.1]). *Let $Y = G(2,5)$ or $G(2,6)$, let X be a* **general** *complete intersection of type $(1,1)$ in Y under the Plücker embedding. Then X is not weak 2-Fano. In particular, there exist a surface in $S \subseteq X$ such that $[i(S)] = \sigma_{1,1}^\vee$ where $i : X \to Y$ is the inclusion.*

Now we can prove the following.

Theorem 2.4.9. *Let $Y = G(2,5)$ or $G(2,6)$, let X be a smooth complete intersection of type $(1,1)$ in Y under the Plücker embedding. Then X is not weak 2-Fano.*

Proof. Let $\mathcal{O}_Y(1)$ be the Plücker line bundle and let

$$\mathcal{U} \subseteq \mathbb{P}(H^0(Y, \mathcal{O}_Y(1))) \times \mathbb{P}(H^0(Y, \mathcal{O}_Y(1)))$$

be the open set parametrizing the smooth complete intersections in Y of bidegree $(1,1)$. For $t \in \mathcal{U}$, we denote by X_t the corresponding variety. Let $\mathcal{X} := \{(x,t) \in Y \times \mathcal{U} : x \in X_t\}$ and consider the family

$$\begin{tikzcd} \mathcal{X} \arrow[r, "pr_1"] \arrow[d, "pr_2"] & Y \\ \mathcal{U} & \end{tikzcd}$$

Suppose $Y = G(2,5)$. Let $i : X_t \to Y$ be the inclusion, the map $i^* : H^4(Y,\mathbb{Z}) \to H^4(X_t,\mathbb{Z})$ is injective with torsion free cokernel by Theorem 2.4.2 and [Laz04b, Example 7.1.2]. Since $b_4(Y) = b_4(X_t) = 2$ by Lemma 2.4.6, we have that $i^* : H^4(Y,\mathbb{Z}) \to H^4(X_t,\mathbb{Z})$ is an isomorphism. By [dA15, Corollary 5.1], for a general t there exists a surface S_t such that $[i(S_t)] = \sigma_{1,1}^\vee$. Then there exist $a_t, b_t \in \mathbb{Z}$ such that $S_t = a_t \sigma_{2|X_t} + b_t \sigma_{1,1|X_t}$. From Example 2.2.22, we know that

$$\begin{aligned} (\sigma_{2|X_t})^2 &= (\sigma_2^2) \cdot \sigma_1^2 = 2 \\ (\sigma_{1,1|X_t})^2 &= (\sigma_{1,1}^2) \cdot \sigma_1^2 = 1 \\ \sigma_{2|X_t} \cdot \sigma_{1,1|X_t} &= (\sigma_2 \cdot \sigma_{1,1}) \cdot \sigma_1^2 = 1. \end{aligned}$$

Using the condition $[i(S_t)] = \sigma_{1,1}^\vee = \sigma_{2,2}$, we have

$$0 = \sigma_{2,2} \cdot \sigma_2 = S_t \cdot \sigma_{2|X_t} = 2a_t + b_t$$
$$1 = \sigma_{2,2} \cdot \sigma_{1,1} = S_t \cdot \sigma_{1,1|X_t} = a_t + b_t$$

then $a_t = -1$ and $b_t = 2$. Let $\mathcal{S} := pr_1^*(-\sigma_2 + 2\sigma_{1,1})$, then the surface $\mathcal{S}_{|X_t}$ is such that $[S_t] = [\mathcal{S}_{|X_t}]$, and since we see that it is effective for a general t, hence it is effective for all[1] t. Let $t \in \mathcal{U}$, then X_t is not weak 2-Fano since using [AC13, Proposition 32]

$$ch_2(X_t) \cdot \mathcal{S}_{|X_t} = \frac{1}{2}(\sigma_{2|X_t} - \sigma_{1,1|X_t}) \cdot (-\sigma_{2|X_t} + 2\sigma_{1,1|X_t}) = -\frac{1}{2}.$$

Suppose $Y = G(2,6)$. By Theorem 2.4.2 we have that $H^4(Y,\mathbb{Z}) \cong H^4(X_t,\mathbb{Z})$, then $b_8(X_t) = b_4(X_t) = 2$. Now consider $i^* : H^8(Y,\mathbb{Z}) \to H^8(X_t,\mathbb{Z})$, where $i : X_t \to Y$ is the inclusion. Using again Example 2.2.22, we have

$$\begin{aligned}
\sigma_{4|X_t} \cdot \sigma_{2|X_t} &= (\sigma_4 \cdot \sigma_2) \cdot \sigma_1^2 = 1\\
\sigma_{2,2|X_t} \cdot \sigma_{2|X_t} &= (\sigma_{2,2} \cdot \sigma_2) \cdot \sigma_1^2 = 1\\
\sigma_{4|X_t} \cdot \sigma_{1,1|X_t} &= (\sigma_4 \cdot \sigma_{1,1}) \cdot \sigma_1^2 = 0\\
\sigma_{2,2|X_t} \cdot \sigma_{1,1|X_t} &= (\sigma_{2,2} \cdot \sigma_{1,1}) \cdot \sigma_1^2 = 1.
\end{aligned}$$

Hence $\sigma_{4|X_t}$ and $\sigma_{2,2|X_t}$ are a basis of $H^8(X_t,\mathbb{Z})$, since that group is torsion free (see Remark 2.4.10). Then $[S_t] = a_t\sigma_{4|X_t} + b_t\sigma_{2,2|X_t}$, where as before S_t is the surface described in [dA15, Corollary 5.1] for general $t \in \mathcal{U}$. Using the condition $[i(S_t)] = \sigma_{1,1}^\vee = \sigma_{3,3}$, we have

$$0 = \sigma_{3,3} \cdot \sigma_2 = S_t \cdot \sigma_{2|X_t} = a_t + b_t$$
$$1 = \sigma_{3,3} \cdot \sigma_{1,1} = S_t \cdot \sigma_{1,1|X_t} = b_t$$

then $a_t = -1$ and $b_t = 1$. Let $\mathcal{S} := pr_1^*(-\sigma_4 + \sigma_{2,2})$, then $[S_t] = [\mathcal{S}_{|X_t}]$, that is $\mathcal{S}_{|X_t}$ is effective for all t. Let $t \in \mathcal{U}$, then X_t is not weak 2-Fano since using [AC13, Proposition 32]

$$ch_2(X_t) \cdot \mathcal{S}_{|X_t} = (\sigma_{2|X_t} - \sigma_{1,1|X_t}) \cdot (-\sigma_{4|X_t} + \sigma_{2,2|X_t}) = -1.$$

□

REMARK 2.4.10. By Theorems 2.4.2 and 2.2.16 we have $H^5(X_t,\mathbb{Z}) = 0$. By [Hat02, Corollary 3.3] $H_4(X_t,\mathbb{Z})$ is torsion free, then also $H^8(X_t,\mathbb{Z})$ is torsion free by Poincaré duality.

REMARK 2.4.11. Let X be a smooth complete intersection of $G(2,5)$ of type $(1,1)$ under the Plücker embedding, let Z be the surface given by the union of lines in X through a general point. In [AC13, Example 30] it is written that Z has homology class equal to $\sigma_2^\vee + \sigma_{1,1}^\vee$. This should be read as $2\sigma_{1,1}^\vee + \sigma_2^\vee$.

[1]This is a well-known fact for experts. A good reference is [Ott15, Proposition 3].

2.4.3. Complete intersection in orthogonal and symplectic Grassmannians. In this subsection we study the cone $\overline{\mathrm{Eff}}_2(X)$ in complete intersections in orthogonal and symplectic Grassmannian varieties.

PROPOSITION 2.4.12. *Let s, r be positive integers such that $2 \leq r \leq \left[\frac{s}{2}\right]$, and $\left[\frac{s}{2}\right] - r \neq 1, 2$ if s is even. Let $s \neq 2r$ (respectively, $s = 2r$), let X be a n-dimensional weak 2-Fano complete intersection in a connected component of the orthogonal Grassmann variety $OG(r, s)$ under the Plücker (respectively, half-spinor) embedding, with X very general if $X \subseteq OG(2, 7)$. Then $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.*

PROOF. Assume that X is of type $(d_1, ..., d_c)$. If $n > 4$, by Theorem 2.4.2 and Lemma 2.2.28, we have $b_4(X) \leq 2$ and we can apply Lemma 2.1.29. Then we have $n = 4$ and $c = \frac{r(2s-3r-1)}{2} - 4$. If $2r = s$, by [AC13, Proposition 34] and Remark 2.4.13, we see that X is weak 2-Fano if and only if either $d_i = 1$ and $c \leq 4$, or X of type (2). Therefore we get $r = 4$ and X of type $(1, 1)$. By [AC13, Proposition 34] we have that $K_X = -c_1(X) = -4H$, where H is the half-spinor embedding. But then, by [KO73, Corollary p.37], X is a smooth quadric in $\mathbb{P}^5$ and then $b_4(X) = 2$ by [Rei72, p.20], so we apply Lemma 2.1.29.

If $2r \neq s$, since $c_1(OG(r, s)) = (s - r - 1)\sigma_1$ we get that $\sum_{i=1}^{c} d_i \leq s - r - 2$. It is easy to see that this leads to the following cases

$OG(r,s)$	Type
$OG(3,7)$	$(1,1)$
$OG(2,7)$	$(1,1,1)$
$OG_+(2,6)$	(1)
	(2)

But $OG(3, 7) \cong OG_+(4, 8)$, then the first case is a quadric. Let X_{111} be the variety $(1, 1, 1)$ in $OG(2, 7)$. This is the variety (b8) in the classification given in [K9̈5]. Indeed, for the reader's convenience, we point out that X_{111} is the zero-locus of a global section of the bundle

$$\left(\wedge^2 S^\vee\right)^{\oplus 3} \oplus Sym^2 S^\vee$$

where $S^\vee$ is $(1, 0; 0, 0, 0, 0, 0)$ in Küchle's notation (see [K9̈5, Section 2.5]). So $h^{1,3}(X_{111}) > 0$ by [K9̈5, Theorem 4.8]. Now apply [Spa96, Theorem 2] to conclude that the space of algebraic cycles of X_{111} is induced by the space of algebraic cycles of $OG(2, 7)$. Then

$$Z_2(X_{111})/\mathrm{Alg}_2(X_{111}) \otimes \mathbb{R}$$

is at most 2-dimensional. Hence $\overline{\mathrm{Eff}}_2(X_{111})$ is polyhedral by (2.1.1) and Lemma 2.1.29. The last two varieties do not satisfy the condition $\left[\frac{s}{2}\right] - r \neq 1, 2$, anyway, they are not weak 2-Fano by [AC13, Example 21]. Indeed, $OG_+(2, 6)$ is the zero section of the bundle $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ [Kuz15, Proposition 2.1], and it can easily be seen that the Plücker embedding is given by the divisor $(1, 1)$, then the two varieties are isomorphic to, respectively, a complete intersection of type $(1, 1)$ and $(1, 2)$ in $\mathbb{P}^3 \times \mathbb{P}^3$ under the embedding given by $\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1)$. □

REMARK 2.4.13. In [AC13, Proposition 34], it is stated that the smooth complete intersection of $OG_+(k, 2k)$ of type $(2,2)$ under the Plücker embedding is a weak 2-Fano variety. This should be read as (2).

PROPOSITION 2.4.14. *Let X be a n-dimensional weak 2-Fano complete intersection in a symplectic Grassmann variety $SG(r,s)$ under the Plücker embedding. Then $b_4(X) \leq 2$. In particular $\overline{\mathrm{Eff}}_2(X)$ is polyhedral.*

PROOF. Assume that X is of type $(d_1, ..., d_c)$. If $n > 4$, by Theorem 2.4.2 and Lemma 2.2.32, we have $b_4(X) = b_4(SG(r,s)) \leq 2$ and we can apply Lemma 2.1.29. If $n = 4$, since $c_1(SG(r,s)) = (s-r+1)\sigma_1$ we have the following conditions: $c = \frac{r(2s-3r+1)}{2} - 4$ and $\sum_{i=1}^{c} d_i \leq s-r$. It is easy to see that this leads to the following cases:

$SG(r,s)$	Type
$SG(3,6)$	$(1,1)$
	$(1,2)$
$SG(2,6)$	$(1,1,1)$
	$(1,1,2)$

The variety $SG(2,6)$ is a section of $\wedge^2 S^\vee = \mathcal{O}_{G(2,6)}(1)$, as we said in Remark 2.2.31. Thus the last two case are, respectively, $(1,1,1,1)$ and $(1,1,1,2)$ in $G(2,6)$. The first two cases are not weak 2-Fano by [AC13, Proposition 36], the last two by [AC13, Proposition 32(i)]. □

2.5. Fano manifolds of dimension n and index $i_X \geq n-2$

DEFINITION 2.5.1. The ***index*** of a Fano variety X is the maximal integer i_X such that $-K_X$ is divisible by i_X in $\mathrm{Pic}(X)$.

Fano varieties of high index have been classified: [KO73] proved that $i_X \leq n+1$, $i_X = n+1$ if and only if $X = \mathbb{P}^n$, and $i_X = n$ if and only if $X \subseteq \mathbb{P}^{n+1}$ is a smooth hyperquadric.

Furthermore the case $i_X = n-1$ (the so called Del Pezzo varieties) has been classified by Fujita in [Fuj82a, Fuj82b], and the case $i_X = n-2$ (the so called Mukai varieties) by Mukai (see [Muk89] and [IP99]).

Araujo and Castravet [AC13, Theorem 3] succeeded to classify 2-Fano Del Pezzo and Mukai varieties. They proved:

THEOREM 2.5.2. *Let X be a 2-Fano variety of dimension $n \geq 3$ and index $i_X \geq n-2$. Then X is isomorphic to one of the following.*

- $\mathbb{P}^n$.
- *Complete intersection in projective spaces:*
  - *Quadric hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with $n > 2$;*
  - *Complete intersections of type $(2,2)$ in $\mathbb{P}^{n+2}$ with $n > 5$;*
  - *Cubic hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with $n > 7$;*
  - *Quartic hypersurfaces $X \subseteq \mathbb{P}^{n+1}$ with $n > 15$;*
  - *Complete intersections of type $(2,3)$ in $\mathbb{P}^{n+2}$ with $n > 11$;*
  - *Complete intersections of type $(2,2,2)$ in $\mathbb{P}^{n+3}$ with $n > 9$.*

- *Complete intersection in weighted projective spaces:*
  - *Degree 4 hypersurfaces in* $\mathbb{P}(2,1,...,1)$ *with* $n > 11$*;*
  - *Degree 6 hypersurfaces in* $\mathbb{P}(3,2,1,...,1)$ *with* $n > 23$*;*
  - *Degree 6 hypersurfaces in* $\mathbb{P}(3,1,...,1)$ *with* $n > 26$*;*
  - *Complete intersections of type* $(2,2)$ *in* $\mathbb{P}(2,1,...,1)$ *with* $n > 14$*.*
- $G(2,5)$*.*
- $OG_+(5,10)$ *and its linear sections of codimension* $c < 4$*.*
- $SG(3,6)$*.*
- G_2/P_2*.*

Here G_2/P_2 is a 5-dimensional homogeneous variety for a group of type G_2.
Using the results in the previous sections we obtain:

THEOREM 2.5.3. *Let* X *be a* n*-dimensional 2-Fano variety with* $i_X \geq n-2$*. Then* $\overline{\mathrm{Eff}}_2(X)$ *is polyhedral. Also,* $\overline{\mathrm{Eff}}_3(X)$ *is polyhedral with the possible exception of the complete intersection of type* $(2,2)$ *in* $\mathbb{P}^8$*.*

In particular, Conjecture 1.1.2 is true for any n*-dimensional* k*-Fano variety with* $i_X \geq n-2$ *and* $k = 2,3$*.*

PROOF. For $\overline{\mathrm{Eff}}_2(X)$: In the case $\mathbb{P}^n$ and its complete intersections, we can invoke Theorem 2.4.5. Since none of the complete intersections in $\mathbb{P}(\mathbf{w})$ of the list has dimension 4, we can use Proposition 2.4.4. Also $G(2,5)$, $OG_+(5,10)$, $SG(3,6)$ and G_2/P_2 are rational homogeneous varieties, then their cone of pseudoeffective 2-cycles is polyhedral by Proposition 2.2.13. Whereas the complete intersections of $OG_+(5,10)$ have polyhedral cone of pseudoeffective 2-cycles by Proposition 2.4.12.

For $\overline{\mathrm{Eff}}_3(X)$: In Theorem 2.5.2, the only complete intersections of dimension 6 in a weighted projective space are the one of type $(2,2)$ in $\mathbb{P}^8$ and the smooth quadric $Q \subseteq \mathbb{P}^7$. The first one is not weak 3-Fano since by [AC13, Equation (3.1)], $ch_3(X) = -\frac{7}{6}h^3_{|X}$ where h is the class of an hyperplane in $\mathbb{P}^8$. Then $h^3_{|X}$ is effective, and $ch_3(X)\cdot h^3_{|X} = -\frac{7}{6}h^6_{|X} < 0$. For the quadric, by [Rei72, p.20] $b_6(Q) = 2$, then $\overline{\mathrm{Eff}}_3(X)$ is polyhedral by Lemma 2.1.29. For the other complete intersections we can use Proposition 2.4.4, whilst for the rational homogeneous varieties we can use Proposition 2.2.13. Also for the complete intersections in $OG_+(5,10)$ we have $b_6(X) = 2$, because $b_6(OG_+(5,10)) = 2$ by Lemma 2.2.29 and we can use Theorem 2.4.2. □

PROPOSITION 2.5.4 ([DELV11, Example 6.10]). *Let* X_b *be the blow-up of* $\mathbb{P}^4$ *along a smooth quartic surface* $Y_b \subseteq \mathbb{P}^3$*. Then*

(1) *The variety* X_b *is Fano.*
(2) *The variety* X_b *is not weak 2-Fano.*

Suppose now that $\overline{\mathrm{Eff}}_1(Y_b) = \mathrm{Nef}_1(Y_b)$ *is a round cone. Then*

(3) *The cone* $\overline{\mathrm{Eff}}_2(X_b)$ *is not polyhedral.*

PROOF. Let $f : X_b \to \mathbb{P}^4$ be the blow-up. It follows that

$$-K_{X_b} \quad = \quad 5f^*H - E,$$

where H is the class of an hyperplane section of $\mathbb{P}^4$, E is the exceptional divisor. Let L_1 be the class of a fiber of the map $\pi : E \to Y_b$. Let L_2 be the class of the strict transform of a line $\overline{L_2}$ contained in the projective space $\mathbb{P}^3 \supseteq Y_b$. Note that L_1 and L_2 are effective. We claim the following.

Claim. The two divisors

$$\begin{aligned} D_1 &= f^*H, \\ D_2 &= 4f^*H - E \end{aligned}$$

are nef and extremal in $\mathrm{Nef}(X_b)$.

Proof of the Claim. The vector space $N^1(X_b)$ is 2-dimensional because it is generated by f^*H and E. So a nef divisor is extremal if and only if it has zero intersection with an effective 1-cycle. Since H is nef in $\mathbb{P}^4$, its pull-back D_1 is nef. Since $D_1 \cdot L_1 = 0$ because L_1 is f-exceptional, we get that D_1 is extremal. Note that D_2 is base-point free, hence nef, by [**Har77**, Example II.7.17.3] since f is the blow up of the base ideal of the linear system of quartics containing Y_b. Furthermore, $\overline{L_2}$ meets Y_b in four points, so

$$D_2 \cdot L_2 = (4f^*H - E) \cdot L_2 = 4H \cdot f_*(L_2) - E \cdot L_2 = 4 - 4 = 0.$$

□

Then $-K_{X_b}$ is the sum of two extremal nef divisors, then it must be an internal point of $\mathrm{Nef}(X_b)$. By Kleiman's theorem, $-K_{X_b}$ is ample, so X_b is Fano. Anyway, X_b is not weak 2-Fano. This is because for any $Z \subseteq \mathbb{P}^n$ such that $codim_{\mathbb{P}^n} Z > 1$, we have that $Bl_Z\mathbb{P}^n$ is not weak 2-Fano [**dJS06**, p.5].

Let us prove the point (3). In [**DELV11**, Example 6.10] we have the following decomposition of $N^2(X_b)$

$$N^2(X_b) = \left\langle f^*(H^2) \right\rangle_{\mathbb{R}} \oplus N^1(Y_b). \tag{2.5.1}$$

Let $\gamma \in \mathrm{Eff}_1(Y_b)$ be an effective cycle. Let us remember the commutative diagram of the blow-up

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "\pi"'] & X_b \arrow[d, "f"] \\ Y_b \arrow[r, "j"'] & \mathbb{P}^4 \end{tikzcd}$$

By Projection Formula [**Ful98**, Proposition 8.3(c)]

$$(i_*\pi^*\gamma)\cdot f^*(H^2) = f_*\left((i_*\pi^*\gamma)\cdot f^*(H^2)\right) = (f_*i_*\pi^*\gamma)\cdot H^2 = (j_*\pi_*\pi^*\gamma)\cdot H^2 = (j_*\gamma)\cdot H^2 = 0$$

the last intersection being zero for dimensional reasons. Then $i_*\pi^*\overline{\mathrm{Eff}}_1(Y_b)$ is in the boundary of $\overline{\mathrm{Eff}}_2(X_b)$. Moreover, the decomposition of (2.5.1) suggests that $i_*\pi^*\overline{\mathrm{Eff}}_1(Y_b)$ injects in $\overline{\mathrm{Eff}}_2(X_b)$. Then $\overline{\mathrm{Eff}}_2(X_b)$ has a (partially) round boundary. □

La science, mon garçon, est faite d'erreurs, mais ce sont des erreurs qu'il est utile de faire, parce qu'elles conduisent peu à peu à la vérité.
Jules Verne, Voyage au centre de la Terre.

CHAPTER 3

The indeterminacy locus of the Voisin map

> Le peu de temps que j'ai eu a été cause de l'un et de l'autre. Je n'ai l'ait celle-ci plus longue que parce que je n'ai pas eu le loisir de la faire plus courte...
>
> Blaise Pascal, Lettre XVI (to the Jesuits).

3.1. General facts about rational maps

DEFINITION 3.1.1. Let $f : X \dashrightarrow Y$ be a rational map. We say that f is defined at a point $x \in X$ if there exists a morphism $f_1 : U_1 \to Y$ with class f such that $x \in U_1$. The ***domain*** of f is the largest open subset $\mathrm{dom}(f) \subseteq X$ of points where f is defined. A point of X is called a point of indeterminacy of f, if it is not contained in $\mathrm{dom}(f)$. We set $\mathrm{Ind}(f)$ to be the closed subscheme of the points of indeterminacy of f. We call $\mathrm{Ind}(f)$ the ***indeterminacy locus*** of f.

If $g : Y \dashrightarrow W$ is a rational map and $f(\mathrm{dom}(f))$ is dense in Y, then it is well-defined the composition

$$g \circ f : X \dashrightarrow W.$$

LEMMA 3.1.2. *Let X, Y and W be varieties sitting inside the following commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "h"] \arrow[d, dashed, "f"'] & W \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

where f and h are rational maps and g is a morphism. Then $\mathrm{Ind}(h) \subseteq \mathrm{Ind}(f)$.

PROOF. Since the composition $g \circ f$ is defined in the domain of f, then h is defined in the domain of f by commutativity. $\square$

DEFINITION 3.1.3. Let $f : X \dashrightarrow Y$ be a rational map. We will denote by $\widetilde{f} : \widetilde{X} \to Y$ a resolution of the indeterminacy of the map f, i.e. a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{f}"] & \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd} \tag{3.1.1}$$

where $\widetilde{X}$ is a non-singular variety and π is a birational morphism that is an isomorphism outside $\mathrm{Ind}(f)$.

The existence of such $\widetilde{X}$ is a consequence of the problem of the elimination of points of indeterminacy. A solution was found by Hironaka [Hir64, I.Question (E) p.140]. Indeed, he proved that for any rational map $f : X \dashrightarrow Y$ of smooth varieties there exists a resolution like in Definition 3.1.3, where π may be obtained as a sequence of blow-ups along smooth subvarieties.

3.2. General facts about the variety of lines

In this subsection we will study the Fano variety of lines on a smooth cubic fourfold. In particular we will be interested in cubic fourfolds not containing a plane. For the reader's convenience, let us prove that they exists.

PROPOSITION 3.2.1. *The general cubic fourfold in $\mathbb{P}^5$ does not contain a plane.*

PROOF. Consider the incidence variety

$$\mathcal{J} := \{(L, Y) \in G(3,6) \times \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))/L \subset Y\}$$

and the projections pr_1 to $G(3,6)$ and pr_2 to $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))$. The dimension of the fiber $pr_1^{-1}(L)$ is given by the dimension of the variety of cubics containing L. From the exact sequence

$$0 \to \mathcal{I}_{L/\mathbb{P}^5}(3) \to \mathcal{O}_{\mathbb{P}^5}(3) \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0$$

since L is a complete intersection, we get

$$h^0(\mathbb{P}^5, \mathcal{I}_{L/\mathbb{P}^5}(3)) = h^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)) - h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) = 46.$$

Then the dimension of $pr_1^{-1}(L) \cong \mathbb{P}^{45}$ is 45. In particular it is never empty, in other words pr_1 is surjective. Then $\mathcal{J}$ is irreducible by [Sha94, Chap. I, 6.4, Theorem 8] and we have

$$\dim \mathcal{J} = \dim G(3,6) + \dim pr_1^{-1}(L) = 9 + 45 = 54.$$

Observe that the dimension of $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(3)))$ is 55, so pr_2 cannot be dominant, which means that the general cubic does not contain a plane. $\square$

In order to introduce the variety of lines on a cubic fourfold, let us give a brief introduction to the Hilbert scheme.

THEOREM 3.2.2. *Let $M = \oplus_{l \in \mathbb{Z}} M_l$ be a finitely generated graded $\mathbb{C}[x_0, ..., x_n]$-module. Then there is a unique polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\dim_{\mathbb{C}} M_l = P_M(l)$ for all $l \gg 0$. Furthermore, $\deg P_M(z) = \dim Z(Ann\, M)$, where Z denotes the zero set in $\mathbb{P}^n$ of a homogeneous ideal. The polynomial P_M is called the Hilbert polynomial of M.*

PROOF. [Har77, Theorem.I.7.5]. $\square$

DEFINITION 3.2.3 ([Har77, p.52]). If $X \subseteq \mathbb{P}^n$ is an algebraic set of dimension r, we define the ***Hilbert polynomial*** of X to be the Hilbert polynomial P_X of its homogeneous coordinate ring. (By the theorem, it is a polynomial of degree r.) We define the degree of X to be $r!$ times the leading coefficient of P_X.

THEOREM 3.2.4 (Grothendieck). *Let $X \subseteq \mathbb{P}^n$ be a projective $\mathbb{C}$-scheme and $P \in \mathbb{Q}[z]$. The functor*

$$\begin{array}{cccc} \mathrm{Hilb}^P(X)(\cdot): & Sch_{\mathbb{C}} & \xrightarrow{\mathrm{Hilb}^P} & Set \\ & B & \longmapsto & \left\{ \begin{array}{c} \text{Subschemes } V \subseteq X \times B \text{ which are proper and} \\ \text{flat over } B \text{ and have Hilbert polynomial } P \end{array} \right\} \end{array}$$

is representable by a pair $(\mathrm{Hilb}^P(X), \mathrm{Univ}^P(X))$, *where* $\mathrm{Hilb}^P(X)$ *is a projective complex scheme.*

PROOF. [Kol96, 1.4 Theorem]. □

This theorem implies that there exists a scheme parametrizing all the subschemes of X with a given Hilbert polynomial. If $V \subseteq X$ is a subscheme, we set $[V]$ to be its point in $\mathrm{Hilb}^P(X)$. Furthermore, there exists a universal family $\mathrm{Univ}^P(X) \subseteq X \times \mathrm{Hilb}^P(X)$. The morphism $\mathrm{Univ}^P(X) \to \mathrm{Hilb}^P(X)$ is flat, the fiber over a point $[V]$ is

$$\left\{(x, [V]) \in X \times \mathrm{Hilb}^P(X)/x \in V\right\}.$$

We are now ready to define the variety of lines on a cubic fourfold. Let $L \subsetneq \mathbb{P}^n$ be a line, that is a degree 1 rational curve. Then by [Har77, Proposition.IV.1.1] its Hilbert polynomial is $P_L(z) = z + 1$.

DEFINITION 3.2.5. Let $Y \subseteq \mathbb{P}^5$ a smooth cubic fourfold. We denote by F the projective scheme $\mathrm{Hilb}^{z+1}(Y)$, and by P the projective scheme $\mathrm{Univ}^{z+1}(Y)$. In particular we have the following universal family

$$\begin{tikzcd} P := \{(x,[L]) \in Y \times F/x \in L\} \arrow[r, "q"] \arrow[d, "p"'] & Y. \\ F & \end{tikzcd} \tag{3.2.1}$$

Given a line $L \subseteq Y$, the point $[L] \in F$ will be indicated with the same letter in lowercase, i.e. $l = [L]$.

We can deduce by [Kol96, 4.3 Theorem] that F is smooth and connected of dimension 4. Furthermore, by what we said in Subsection 2.2.2, we have $\mathrm{Hilb}^{z+1}(\mathbb{P}^n) = G(2, n+1)$. Then F is canonically embedded in $G(2, 6)$.

DEFINITION 3.2.6. We denote by I the subscheme in $F \times F$, endowed with the reduced structure, of intersecting lines, i.e.

$$I := \left\{(l, l') \in F \times F | L \cap L' \neq \emptyset\right\}.$$

Note that I can be defined as $I := (p \times p)(q \times q)^{-1}(\Delta_Y)$, where $\Delta_Y \subseteq Y \times Y$ is the diagonal.

REMARK 3.2.7. Let $x \in Y$ be a point. Let C_x be the subvariety of F parametrizing lines through x. Take a system of coordinates of $\mathbb{P}^5$ such that $x = [0:0:0:0:0:1]$. Then the equation of Y is $x_5^2 q_1 + x_5 q_2 + q_3 = 0$, where the polynomial q_i is homogeneous of degree i. Since Y is smooth, q_1 is not the zero polynomial. The variety C_x can now be seen in $\mathbb{P}^4$ as given by $q_1 = q_2 = q_3 = 0$. Then C_x can be seen as $q_2 = q_3 = 0$ in

$\mathbb{P}^3$. Thus it is connected [**Har77**, Exercise II.8.4c]. Consider the projection to the first component $pr_1 : I \to F$, the fibers are

$$pr_1^{-1}(l) \cong \bigcup_{x \in L} C_x$$

It is known that C_x is a curve for $x \in Y$, except for finitely many $x_i \in Y$ such that C_{x_i} is a surface [**CS09**, Proposition 2.4]. Hence $pr_1^{-1}(l)$ is a surface for all $l \in F$. Moreover, $pr_1^{-1}(l)$ is connected for all $l \in F$. Indeed, $pr_1^{-1}(l)$ is the union of connected subvarieties C_x all of them meeting at the point (l, l). Furthermore, $pr_1^{-1}(l)$ is smooth for general $l \in F$ by [**Voi86**, Section 3 Lemma 1], hence irreducible.

I thank Mingmin Shen for suggesting me the following proof.

LEMMA 3.2.8. *The variety $I \subseteq F \times F$ is irreducible of dimension 6.*

PROOF. Let $J := (q \times q)^{-1}(\Delta_Y)$. Then J is locally defined by four equations in $P \times P$, hence each component of J has dimension at least 6. The map

$$p \times p : J \to I$$

is surjective and only contracts Δ_P to Δ_F. Hence $p \times p$ is birational and each component of I has dimension at least 6. Since $pr_1^{-1}(l)$ is a surface for all $l \in F$ by Remark 3.2.7, each component of I has dimension 6. Moreover, $pr_1^{-1}(l)$ is irreducible for general $l \in F$. It follows that only one component of I maps surjectively over F. Indeed, let I_1 be any irreducible component of I such that $pr_1(I_1) = F$, then we have a surjective map $pr_{1|I_1} : I_1 \to F$ between two irreducibles varieties of dimension, respectively, 6 and 4. The fibers of this map are $pr_{1|I_1}^{-1}(l) = pr_1^{-1}(l) \cap I_1$. Thus, for dimensional reasons, the general fiber of $pr_{1|I_1}$ is a surface, then $pr_1^{-1}(l) \cap I_1$ is a component of $pr_1^{-1}(l)$. As $pr_1^{-1}(l)$ is irreducible for general $l \in F$, it follows that $pr_1^{-1}(l)$ has only one component, then

$$pr_1^{-1}(l) = pr_1^{-1}(l) \cap I_1.$$

We have proved that $pr_1^{-1}(l) \subseteq I_1$ for general $l \in F$, then if I_2 is another irreducible component of I such that $pr_1(I_2) = F$, the same argument implies that $pr_1^{-1}(l) \subseteq I_2$ for general $l \in F$. It follows that $pr_1^{-1}(l) \subseteq I_1 \cap I_2$ for general $l \in F$, then $I_1 = I_2$ as they are both irreducible.

Any other component of I different from I_1 maps over a closed subset of F, then for dimensional reasons this component must have dimension at most 5, then $I = I_1$. It follows that I is irreducible and $\dim I = 6$. □

If X_1, X_2 are subvarieties in the same projective space, we denote by $\langle X_1, X_2 \rangle$ their linear span.

Let $a, b, n \in \mathbb{Z}_{\geq 0}$ such that $a+b+1 < n$. Let $P_1 \subset \mathbb{P}^n$ and $P_2 \subset \mathbb{P}^n$ be linear subspaces of dimension, respectively, a and b. By definition, if $P_1 \cap P_2 = \emptyset$ we set $\dim P_1 \cap P_2 = -1$. It is know by linear algebra that

$$\dim \langle P_1, P_2 \rangle = a + b - \dim P_1 \cap P_2.$$

Let us consider the incidence correspondence

$$\mathcal{I} := \{(P_1, P_2, M) \in G(a+1, n+1) \times G(b+1, n+1) \times G(a+b+2, n+1) / \langle P_1, P_2 \rangle \subseteq M\}$$

with the canonical projections

$$\begin{tikzcd} \mathcal{I} \arrow[r, "p_3"] \arrow[d, "p_1\times p_2"'] & G(a+b+2,n+1) \\ G(a+1,n+1)\times G(b+1,n+1) & \end{tikzcd}$$

LEMMA 3.2.9. *The scheme $\mathcal{I}$ is irreducible.*

PROOF. Consider $M \in G(a+b+2,n+1)$. The fiber of the map

$$p_3 : \mathcal{I} \to G(a+b+2,n+1)$$

over M is

$$p_3^{-1}(M) = \{(P_1,P_2,M) \in G(a+1,n+1)\times G(b+1,n+1)\times G(a+b+2,n+1)/\, \langle P_1,P_2\rangle \subseteq M\}$$

and the natural projection

$$\begin{array}{ccc} p_3^{-1}(M) & \to & G(a+1,M)\times \mathbb{G}(b+1,M) \\ (P_1,P_2,M) & \mapsto & (P_1,P_2) \end{array}$$

is an isomorphism. Then we have the following chain of isomorphisms

$$p_3^{-1}(M) \cong G(a+1,M)\times G(b+1,M) \cong G(a+1,a+b+2)\times G(b+1,a+b+2).$$

Hence $\mathcal{I}$ is irreducible by [Sha94, Chap. I, 6.4, Theorem 8]. $\square$

DEFINITION 3.2.10. Let $I_{\mathbb{G}} := \{(P_1,P_2) \in G(a+1,n+1)\times G(b+1,n+1) | P_1\cap P_2 \neq \emptyset\}$, and set

$$U := G(a+1,n+1)\times G(b+1,n+1)\backslash I_{\mathbb{G}}.$$

If $P_1 \cap P_2 = \emptyset$, then $\langle P_1,P_2\rangle \in G(a+b+2,n+1)$. Then there exists a rational map

$$\rho : G(a+1,n+1)\times G(b+1,n+1) \dashrightarrow G(a+b+2,n+1)$$

such that $\forall (P_1,P_2) \in U$

$$\rho(P_1,P_2) = \langle P_1,P_2\rangle$$

and $\rho_{|U} : U \to G(a+b+2,n+1)$ is a morphism with graph

$$\Gamma_{\rho_{|U}} := \{(P_1,P_2,M) \in U\times G(a+b+2,n+1) | \rho(P_1,P_2) = M\}\,.$$

We denote by Γ_ρ the closure of $\Gamma_{\rho_{|U}}$ inside $G(a+1,n+1)\times G(b+1,n+1)\times G(a+b+2,n+1)$.

LEMMA 3.2.11. $\Gamma_\rho = \mathcal{I}$, *and* $\mathrm{Ind}(\rho) = I_{\mathbb{G}}$.

PROOF. Let $(P_1,P_2) \in \Gamma_{\rho_{|U}}$, then $\exists! M_{P_1,P_2}$ such that $(P_1,P_2,M_{P_1,P_2}) \in \Gamma_{\rho_{|U}}$, that is $\langle P_1,P_2\rangle = M_{P_1,P_2}$. Then we have $\langle P_1,P_2\rangle \subseteq M_{P_1,P_2}$ so $(P_1,P_2,M_{P_1,P_2}) \in \mathcal{I}$. This implies that $\Gamma_{\rho_{|U}} \subseteq \mathcal{I}$, but $\mathcal{I}$ is closed, then

$$\Gamma_\rho = \overline{\Gamma_{\rho_{|U}}} \subseteq \mathcal{I}.$$

To prove that $\mathrm{Ind}(\psi) = I_{\mathbb{G}}$ we argue as follows. Let U_ρ be the domain of ρ. By [Deb01, 1.39], U_ρ is also the maximal open set such that $(p_1\times p_2)_{|\Gamma_\rho}$ is an isomorphism. Since $U \subseteq U_\rho$, it follows that $\Gamma_{\rho_{|U}}$ is isomorphic to U. Then, as $G(a+1,n+1)\times G(b+1,n+1)$

is irreducible, it follows that U is irreducible, then $\Gamma_{\rho_{|U}}$ is irreducible, then also Γ_ρ is irreducible. It follows that

$$(p_1 \times p_2)^{-1}(U) = \Gamma_{\rho_{|U}}$$

then $\Gamma_{\rho_{|U}}$ is an open subset of $\mathcal{I}$. Then $\dim \mathcal{I} = \dim \Gamma_{\rho_{|U}} = \dim U = \dim \Gamma_\rho$ and, by Lemma 3.2.9 $\mathcal{I} = \Gamma_\rho$. Since the fiber $(p_1 \times p_2)^{-1}(P, P')$ of any point of $I_{\mathbb{G}}$ is not a point, it follows that $U = U_\rho$. □

REMARK 3.2.12. As Γ_ρ is birational to $G(a+1, n+1) \times G(b+1, n+1)$, it follows that Γ_ρ is the blow up of $G(a+1, n+1) \times G(b+1, n+1)$ with respect to a sheaf of ideals supported in $I_{\mathbb{G}}$ [Har77, Theorem II.7.17,Exercise II.7.11c]. Let $X \subseteq G(a+1, n+1) \times G(b+1, n+1)$ be any irreducible subvariety such that $X \nsubseteq I_{\mathbb{G}}$. Then the inclusion induces a rational map $X \dashrightarrow G(a+b+2, n+1)$, and its graph Γ_X is the strict transform of $U \cap X$ in Γ_ρ. By [Har77, Corollary II.7.15], Γ_X is the blow up of X along a sheaf of ideals supported in $X \cap I_{\mathbb{G}}$.

Notice that $(F \times F) \cap I_{\mathbb{G}} \subsetneq F \times F$.

PROPOSITION 3.2.13. *The indeterminacy locus of the restricted rational map*

$$\rho_{|F\times F} : F \times F \dashrightarrow G(4, 6)$$

is the variety I.

PROOF. In the setting of Remark 3.2.12, with $X = F \times F$, the closed subset $X \cap I_{\mathbb{G}} = I$ is of codimension 2 by Lemma 3.2.8. Then $\Gamma_{F\times F}$ is not isomorphic to $F \times F$ exactly at the points of I. It follows that $F \times F \dashrightarrow G(4, 6)$ cannot be extended at any point of I. □

3.3. General facts about the LLSS variety

DEFINITION 3.3.1. Let X be a compact Kähler manifold. We say that X is ***hyperKähler***, if it is simply connected and the space of its global holomorphic two-forms $H^0(X, \wedge^2\Omega_X)$ is spanned by a nowhere degenerate symplectic form.

REMARK 3.3.2. The symplectic form induces an isomorphism between the vector bundle Ω_X and $T_X = \Omega_X^\vee$. It follows that for every $k \geq 0$ we have

$$c_k(\Omega_X) = c_k(X) = (-1)^k c_k(\Omega_X).$$

Then the odd Chern classes are zero, i.e. $c_{2k+1}(X) = 0$.

EXAMPLE 3.3.3. The only examples of hyperKähler varieties in dimension two are the $K3$ surfaces. That is, a surface S such that $c_1(S) = 0$ and $H^1(S, \mathcal{O}_S) = 0$.

Let us see some other example of hyperKähler varieties. Let S be a complex surface. For any positive integer n, we denote by $S^{[n]}$ the Hilbert scheme of dimension 0 and length n subschemes of S. That is, the general point of $S^{[n]}$ represents the union of n general points of X. Let $S^{(n)}$ be the quotient of the direct product $S^{\times n}$ by the action of the symmetric group Σ_n. That is

$$S^{(n)} := {}^{S^{\times n}}/_{\Sigma_n}.$$

It can be seen that there exists a birational morphism $S^{[n]} \to S^{(n)}$.

THEOREM 3.3.4 ([Bea83, Théorème 3]). *Let S be a $K3$ surface. Then $S^{[n]}$ is a hyperKähler variety of dimension $2n$.*

DEFINITION 3.3.5. Let X be a hyperKähler variety. Suppose that X is deformation equivalent to $S^{[n]}$ for some $K3$ surface S. We say that X is of ***$K3^{[n]}$-type***.

If A is an Abelian surface, the operation of A induces a map

$$A^{(n)} \to A.$$

Let $f : A^{[n+1]} \to A$ be the composition of $A^{[n+1]} \to A^{(n+1)}$ and $A^{(n+1)} \to A$. Let K^nA be a generalized Kummer variety, that is $K^nA = f^{-1}(0)$. We have the following.

THEOREM 3.3.6 ([Bea83, Théorème 4]). *The variety K^nA is a hyperKähler variety of dimension $2n$.*

DEFINITION 3.3.7. Let X be a hyperKähler variety. Suppose that X is deformation equivalent to K^nA for some Abelian surface A. We say that X is of ***K^nA-type***.

REMARK 3.3.8. hyperKähler varieties of K^nA-type and $K3^{[n]}$-type are the only families of hyperKähler varieties in any complex even dimension. O'Grady in [O'G03, O'G99] constructed two new examples in dimension 6 and 10 of hyperKähler varieties that are not deformation of known types.

THEOREM 3.3.9 ([BD85]). *Let $Y \subseteq \mathbb{P}^5$ a smooth cubic fourfold. Let F be its variety of lines. Then*

(1) The variety F is hyperKähler of $K3^{[2]}$-type.
(2) If Y is Pfaffian, then F is isomorphic to $S^{[2]}$ for some $K3$ surface S.

We are now going to study the variety Z defined in [LLSvS17]. This variety is constructed using the Hilbert scheme of twisted cubic curves. Let us give a brief introduction to this Hilbert scheme.

DEFINITION 3.3.10. A rational normal curve of degree 3, or ***twisted cubic*** for short, is a smooth curve $C \subseteq \mathbb{P}^3$ that is projectively equivalent to the image of $\mathbb{P}^1$ under the Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$ of degree 3.

Piene and Schlessinger [PS85] showed that $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3) = H_0 \cup H_1$, where H_0 is a 12-dimensional smooth component such that the general point is a rational normal curve, and H_1 is a 15-dimensional smooth component such that the general point is a curve C such that C_{red} is a plane cubic. A generalized twisted cubic is a curve with class in H_0.
We define $\mathrm{Hilb}^{gtc}(\mathbb{P}^3)$ as the H_0 component of $\mathrm{Hilb}^{3z+1}(\mathbb{P}^3)$ and

$$\mathrm{Hilb}^{gtc}(\mathbb{P}^5) := \{[\Gamma] \in \mathrm{Hilb}^{3z+1}(\mathbb{P}^5) / \exists \mathbb{P}^3 \subseteq \mathbb{P}^5, [\Gamma] \in \mathrm{Hilb}^{gtc}(\mathbb{P}^3)\}.$$

DEFINITION 3.3.11. Let $Y \subseteq \mathbb{P}^5$ be a smooth cubic fourfold that does not contain a plane. We set

$$F_3(Y) := \mathrm{Hilb}^{gtc}(\mathbb{P}^5) \cap \mathrm{Hilb}^{3z+1}(Y).$$

By [LLSvS17, Theorem 4.7] $F_3(Y)$ is a smooth variety of dimension 10.
An element $[\Gamma] \in F_3(Y)$ is the class of a one dimensional subscheme Γ of Y, with Hilbert polynomial equal to $3z+1$. The linear span of Γ is a 3-dimensional space $\langle\Gamma\rangle \cong \mathbb{P}^3$, and $[\Gamma] \in \mathrm{Hilb}^{gtc}(\langle\Gamma\rangle)$. Then the span induces a morphism

$$F_3(Y) \to G(4, 6)$$

and, as stated by [LLSvS17, p.113 and Theorem 4.8] and [LLMS17, (1.1.1)], there is a commutative diagram

$$\begin{tikzcd} F_3(Y) \arrow[r] \arrow[d, "\phi"'] & G(4,6) \\ Z' \arrow[ur, "g"'] & \end{tikzcd} \tag{3.3.1}$$

where Z' is a smooth irreducible projective variety [LLSvS17, Theorem 4.8]. The diagram (3.3.1) has the following remarkable properties:

- The morphism ϕ is a $\mathbb{P}^2$-fibration.
- The morphism g is finite on the open subset $g^{-1}(W_{ADE}) =: V_{ADE} \subseteq Z'$ where
$$W_{ADE} := \{P \in G(4,6)/P \cap Y \,\text{has}\, ADE \,\text{singularities}\}.$$
- The degree of g on V_{ADE} is 72 by [LLSvS17, Theorem 2.1 and Table 1].

Moreover, by [LLSvS17, Theorem 4.11 and Proposition 4.5] there exists a divisorial contraction

$$\begin{tikzcd} Z' \arrow[d, "\sigma"'] & \mathbb{P}(T_Y) \arrow[l, hook'] \arrow[d] \\ Z & Y \arrow[l, hook'] \end{tikzcd}$$

making Z' be the blow up of a variety Z over a subvariety canonically isomorphic to Y. Z is an hyperKähler variety [LLSvS17, Theorem 4.19] of $K3^{[4]}$-type [AL17, Corollary] (or also [Leh15, Corollary 6.3]).

Obviously, both Z' and Z depend on Y, so they should be denoted by $Z'(Y)$ and $Z(Y)$. We choose to keep the same notation as [LLSvS17]. So, when no confusion is possible, we will simply write Z' and Z.

3.4. The Voisin map

In [Voi16, Proposition 4.8] Voisin defined a rational map $\psi : F \times F \dashrightarrow Z$ using the following nice geometric argument. Let $(l, l') \in F \times F$ be a general point, that is l and l' are the classes of two disjoint lines L and L' such that the following surface

$$S_{l,l'} := \langle L, L' \rangle \cap Y$$

is smooth. The point (l, l') defines a linear system in $S_{l,l'}$ given by the divisor

$$D_{l,l'} = L - L' - K_{S_{l,l'}}.$$

Since $\mathcal{O}_{S_{l,l'}}(1) = \mathcal{O}_{S_{l,l'}}(-K_{S_{l,l'}})$, in $|D_{l,l'}|$ there is a curve of the form $L \cup C'_x$, where x is any point of L and C'_x is the unique conic such that $\langle x, L' \rangle \cap Y = L' \cup C'_x$. Then any member of this linear system is a generalized twisted cubic [Har82, Section 1.b p. 39] contained in Y. Voisin defines the map ψ by setting $\psi(l, l')$ to be the class in Z of any member of $|D_{l,l'}|$. The degree of the map is obtained as follows. It can be seen that $D_{l,l'}$ defines a morphism $\varphi_{D_{l,l'}} : S_{l,l'} \to \mathbb{P}^2$ that contracts exactly 6 lines. The members of $|D_{l,l'}|$ are pull-back of lines in $\mathbb{P}^2$. The line L is the inverse image of a blown up point, thus it is a component of the pull-back of any line through that point. We can see, by intersection theory in $S_{l,l'}$, that L' is the strict transform of a conic through the other five points. Then we have 6 lines that are components of some rational normal curve in $|D_{l,l'}|$, so we have 6 possible choices of pairs of lines $R, R' \subseteq S_{l,l'}$ such that $|D_{l,l'}| = |D_{r,r'}|$.

We follow a slightly different approach: we construct a degree 6 rational map

$$\psi' : F \times F \dashrightarrow Z',$$

and then we will check that $\psi = \sigma \circ \psi'$.

First of all, we notice that if L is a line and C is a conic in a projective space, with L not contained in the plane defined by C and $L \cap C = \{x\}$, then $L \cup C$ is a limit of rational normal curves [Har82, Section 1.b p. 39]. If both L and C are contained in Y, we have $[L \cup C] \in F_3(Y)$. As already pointed out in [Voi16, Proposition 4.8], there is a rational map

$$\psi_1 : P \times F \dashrightarrow F_3(Y)$$

defined as follows. Let (l, l') be not in I, let $x \in L$ be a point and let C'_x be the unique conic such that

$$\langle x, L' \rangle \cap Y = L' \cup C'_x.$$

Then

$$\psi_1(l, x, l') := [L \cup C'_x].$$

Consider

$$U_{ADE} := \left\{(l, l') \in F \times F | (l, l') \notin I, S_{l,l'} := \langle L, L' \rangle \cap Y \,\text{has}\, ADE \,\text{singularities}\right\}.$$

Pick $(l, l') \in U_{ADE}$. Since the linear span of $L \cup C'_x$ is $\langle L, L' \rangle$ for each $x \in L$, then the image of the curve

$$\Gamma_{l,l'} = \{[L \cup C_x]/x \in L \cong \mathbb{P}^1\}$$

under the span map

$$F_3(Y) \to G(4, 6)$$

is the point $\langle L, L' \rangle$. In other words the curve $\Gamma_{l,l'}$ is contracted by $g \circ \phi$. By construction, $g \circ \phi(\Gamma_{l,l'}) \subset W_{ADE}$ then

$$\phi(\Gamma_{l,l'}) \subset V_{ADE}.$$

The curve $\Gamma_{l,l'}$ must be contracted by ϕ, since $\phi(\Gamma_{l,l'})$ is in the set where g is finite. Let

$$p_1 : P \times F \to F \times F$$

be the canonical $\mathbb{P}^1$-bundle. Then the restricted map

$$\phi \circ \psi_1 : p_1^{-1}(U_{ADE}) \to Z'$$

contracts all the fibers of

$$p_1 : p_1^{-1}(U_{ADE}) \to U_{ADE}$$

Since p_1 is a linear $\mathbb{P}^d$-bundle, we can consider an open set $U \subseteq U_{ADE}$ trivializing p_1 [BS95, Section 3.2]. Then the diagram

$$\begin{tikzcd} p_1^{-1}(U) \cong \mathbb{P}^1 \times U \arrow[r, "\phi\circ\psi_1"] \arrow[d, "p_{1,U}"'] & Z' \\ U & \end{tikzcd} \tag{3.4.1}$$

satisfies the hypothesis of the Rigidity Lemma [GW10, Proposition 16.54]. Indeed: U is reduced, Z' is separated and $\mathbb{P}^1$ is reduced, connected and proper. It follows that there exists a unique morphism $U \to Z'$ making (3.4.1) commutative. We can cover U_{ADE} by trivializing open subsets, repeat that argument and get a map

$$U_{ADE} \to Z'$$

which point-wise is

$$(l,l') \mapsto \phi([L \cup C_x]).$$

Because we know that this map does not depend on the choice of $x \in L$. We have therefore defined a rational map

$$\begin{array}{cccc} \psi': & F \times F & \dashrightarrow & Z' \\ & (l,l') & \mapsto & \phi([L \cup C'_x]), \end{array}$$

for all (l,l') in the open subset U_{ADE} of $F \times F$.

Proposition 3.4.1. *The rational map $\psi' : F \times F \dashrightarrow Z'$ defined above is dominant, has of degree 6 and* $\mathrm{Ind}(\psi') = I$.

Proof. The composition of ψ' with $g : Z' \to G(4,6)$ gives rise to a commutative diagram

$$\begin{tikzcd} F \times F \arrow[r, dashed, "\rho_{|F\times F}"] \arrow[d, dashed, "\psi'"'] & G(4,6) \\ Z' \arrow[ur, "g"'] & \end{tikzcd} \tag{3.4.2}$$

where ρ is the span map. The inclusion $I \subseteq \mathrm{Ind}(\psi')$ is an application of Lemma 3.1.2 and Proposition 3.2.13 to the diagram (3.4.2). To prove the other inclusion, consider the diagram

$$\begin{tikzcd} P \times F \arrow[r, dashed, "\psi_1"] \arrow[d, "p_1"'] & F_3(Y) \arrow[d, "\phi"] \\ F \times F \arrow[r, dashed, "\psi'"] & Z' \end{tikzcd} \tag{3.4.3}$$

that is commutative in $p_1^{-1}(U_{ADE})$. Indeed, if $(l,x,l') \in p_1^{-1}(U_{ADE})$, then we know that ψ_1 is defined in (l,x,l') as $(l,l') \notin I$. Then

$$\phi(\psi_1(l,x,l')) = \phi([L \cup C_x]) = \phi(\Gamma_{l,l'}) = \psi'(l,l') = \psi'(p_1(l,x,l')).$$

Consider a section $s : U \to P \times F$ of p_1 defined in a open subset $U \subseteq F \times F \backslash I$, and let

$$\psi'' := \phi \circ \psi_1 \circ s$$

be the rational map defined on U. By commutativity of (3.4.3), ψ'' coincides with ψ' on $U \cap U_{ADE}$. Since U is arbitrary, it follows that ψ' can be extended to every point of $F \times F \backslash I$. We have then proved that

$$\mathrm{Ind}(\psi') \subseteq I.$$

Let $M \in G(4,6)$, then

$$\rho^{-1}(M) = \left\{(l,l') \in F \times F \backslash I / \left\langle L, L' \right\rangle = M\right\}.$$

In particular, the pairs $(l,l') \in \rho^{-1}(M)$ represent pairs of disjoint lines contained in the cubic surface $M \cap Y$. If M is sufficiently general, then $M \cap Y$ is smooth and contains 27 lines, each of them meeting exactly 10 other lines [GH78, p. 485]. It can easily be seen that there are $27 \cdot (27 - 11) = 432$ pairs of lines contained in $\rho^{-1}(M)$, and therefore ρ is generically finite of degree 432. All the manifolds appearing in the commutative diagram (3.4.2) are 8-dimensional manifolds. Then as ρ is generically finite is also dominant, this implies that also g and ψ' are dominant and generically finite. Since $\deg \rho = \deg \psi' \deg g$ by commutativity of (3.4.2), the degree of ψ' is

$$\deg \psi' = \frac{432}{72} = 6.$$

$\square$

Let ψ be the Voisin map. We will denote by $\widetilde{\psi} : \widetilde{F \times F} \to Z$ a resolution of the indeterminacy of the map ψ as in Definition 3.1.3. Then there is a commutative diagram

$$\begin{tikzcd} \widetilde{F \times F} \arrow[d, "\pi"'] \arrow[dr, "\widetilde{\psi}"] & \\ F \times F \arrow[r, dashed, "\psi"] & Z \end{tikzcd} \tag{3.4.4}$$

where $\widetilde{F \times F}$ is a non-singular variety, and π is a birational morphism that is an isomorphism outside $\mathrm{Ind}(\psi)$. We will denote by E the support of the exceptional divisor of π.

Remark 3.4.2. Notice that we have the following commutative diagram.

$$\begin{tikzcd} P \times F \arrow[rr, dashed, "\psi_1"] \arrow[dd, "p_1"'] & & F_3(Y) \arrow[dl, "\phi"'] \arrow[dd, "\phi\circ\sigma"] \\ & Z' \arrow[dr, "\sigma"] & \\ F \times F \arrow[ur, dashed, "\psi'"] \arrow[rr, dashed, "\psi"] & & Z \end{tikzcd} \tag{3.4.5}$$

Hence we have another definition of the Voisin map in [Voi16, Proposition 4.8] as the composition $\sigma \circ \psi'$.

For the reader's convenience, we collect the following.

Lemma 3.4.3 ([Voi16, Remark 4.10]). *The map $\psi : F \times F \dashrightarrow Z$ is étale of degree 6 where it is defined. Furthermore, the image of the exceptional divisor of the resolution of ψ is a divisor.*

PROOF. The map ψ is dominant of degree 6 because it is the composition of a dominant degree 6 rational map and of a blow up. Let $R_{\tilde{\psi}}$ be the ramification divisor of $\tilde{\psi}$, that is the divisor supported in the subset of points of $\widetilde{F \times F}$ where the induced map $d\tilde{\psi} : T_{\widetilde{F\times F}} \to \tilde{\psi}^* T_Z$ is not an isomorphism. The scheme structure is given locally by the vanishing of the Jacobian determinant $\det d\tilde{\psi}$ [Ful98, Example 3.2.20]. Thus we have the formula

$$K_{\widetilde{F\times F}} = \pi^* K_{F\times F} + E' = \tilde{\psi}^* K_Z + R_{\tilde{\psi}}$$

and since the first Chern class of F and Z is trivial, $E' = R_{\tilde{\psi}}$. This implies that the ramification locus of $\tilde{\psi}$ is $E = \mathrm{Supp} E'$, then the Jacobian matrix is of maximal rank outside E. Let $D = \tilde{\psi}(E)$ be the image of the exceptional divisor. Let $G = \tilde{\psi}^{-1}(D) \supseteq E$ be the inverse image of D. By the properties of the ramification, the maps

$$\begin{tikzcd}
\widetilde{F \times F}\backslash G \arrow[d, "\pi_{\widetilde{F\times F}\backslash G}"'] \arrow[dr, "\tilde{\psi}_{\widetilde{F\times F}\backslash G}"] & \\
F \times F\backslash \pi(G) \arrow[r, "\psi"] & Z\backslash D
\end{tikzcd}$$

do not ramify. Then all of them are étale [Mil80, I Corollary 3.16], in particular $\tilde{\psi}_{|\widetilde{F\times F}\backslash G}$ is a degree 6 topological cover of $Z\backslash D$ [Mil80, III Lemma 3.14]. Since Z is simply connected [LLSvS17, Theorem 4.19], if D is not a divisor then by [God71, Chap. X Théorème 2.3] also $Z\backslash D$ is simply connected, so it cannot have nontrivial topological cover [Ful95, Corollary 13.8]. Hence D must be a divisor and we are done. $\square$

We are now ready for the following.

THEOREM 3.4.4. *The indeterminacy locus of the Voisin map $\psi : F \times F \dashrightarrow Z$ is the variety I of the intersecting lines in Y.*

PROOF. From the commutative diagram

$$\begin{tikzcd}
F \times F \arrow[r, dashed, "\psi"] \arrow[d, dashed, "\psi'"'] & Z \\
Z' \arrow[ur, "\sigma"'] &
\end{tikzcd} \tag{3.4.6}$$

we have the inclusion

$$\mathrm{Ind}(\psi) \subseteq I, \tag{3.4.7}$$

by Lemma 3.1.2 and Proposition 3.4.1. Since $\sigma : Z' \to Z$ is a blow up along $Y \subseteq Z$, then $\sigma_{|Z'\backslash D} : Z'\backslash D \to Z\backslash Y$ is an isomorphism. Then on $Z\backslash Y$ we can compose its inverse with $g : Z' \to G(4,6)$ and get a map

$$g_Z : Z\backslash Y \to G(4,6).$$

Let $\tilde{\psi} : \widetilde{F \times F} \to Z$ be as in diagram (3.4.4), and set

$$W := \pi(\tilde{\psi}^{-1}(Y)). \tag{3.4.8}$$

We point out that outside the support of the exceptional divisor, $\tilde{\psi}$ is finite by Lemma 3.4.3 and the fact that

$$\pi_{\widetilde{F\times F}\backslash E} : \widetilde{F\times F}\backslash E \to F\times F\backslash \mathrm{Ind}(\psi)$$

is an isomorphism. Then there is a commutative diagram of rational maps

$$\begin{tikzcd} F\times F\backslash W \arrow[rr, dashed, "\rho_{|F\times F\backslash W}"] \arrow[d, dashed, "\psi_{|F\times F\backslash W}"'] & & G(4,6) \\ Z\backslash Y \arrow[urr, "g_Z"'] & & \end{tikzcd} \tag{3.4.9}$$

We argue by contradiction. Suppose $\mathrm{Ind}(\psi) \subsetneq I$, and set

$$T := \pi(\pi^{-1}(I)\backslash(\widetilde{\psi}^{-1}(Y)\cap\pi^{-1}(I))).$$

Claim. The set T is dense in I.

Proof of the Claim. By Lemma 3.2.8 and by our assumption $\mathrm{Ind}(\psi) \subsetneq I$, $I\backslash\mathrm{Ind}(\psi)$ is a dense open subset of I. Consider the set

$$\widetilde{I} = \pi^{-1}(I\backslash\mathrm{Ind}(\psi))$$

that is dense in the strict transform of I, since π is an isomorphism outside $\mathrm{Ind}(\psi)$. Now $\widetilde{I}\backslash(\widetilde{\psi}^{-1}(Y)\cap\widetilde{I})$ is dense in $\widetilde{I}$ since it is open and not empty. Indeed, if it were empty then $\widetilde{I}\subseteq\widetilde{\psi}^{-1}(Y)$, so

$$\widetilde{\psi}(\widetilde{I})\subseteq Y.$$

That is impossible since Y has dimension 4 and $\widetilde{\psi}(\widetilde{I})$ has dimension 6 (because $\widetilde{I}$ is contained in the open set where $\widetilde{\psi}$ is finite). Since the map $\pi_{|\widetilde{I}} : \widetilde{I}\to I$ is dominant, the image of the dense open set $\widetilde{I}\backslash(\widetilde{\psi}^{-1}(Y)\cap\widetilde{I})$ is dense. But

$$\begin{aligned} \pi(\widetilde{I}\backslash(\widetilde{\psi}^{-1}(Y)\cap\widetilde{I})) &= \pi(\pi^{-1}(I\backslash\mathrm{Ind}(\psi))\backslash(\widetilde{\psi}^{-1}(Y)\cap\pi^{-1}(I\backslash\mathrm{Ind}(\psi)))) \\ &\subseteq \pi(\pi^{-1}(I)\backslash(\widetilde{\psi}^{-1}(Y)\cap\pi^{-1}(I))) = T \end{aligned}$$

then T is dense. □

Thus $\exists(l,l')\in T\cap(I\backslash\mathrm{Ind}(\psi))$, and since π is an isomorphism outside $\mathrm{Ind}(\psi)$,

$$\exists! u\in\widetilde{F\times F} \text{ such that } \pi(u)=(l,l'). \tag{3.4.10}$$

Claim. The point u is not in $\widetilde{\psi}^{-1}(Y)$, i.e.

$$u\notin\widetilde{\psi}^{-1}(Y). \tag{3.4.11}$$

Proof of the Claim. Since $(l,l')\in T$, by definition of T and by uniqueness of u, we have

$$u\in\pi^{-1}(I)\backslash(\widetilde{\psi}^{-1}(Y)\cap\pi^{-1}(I)).$$

In particular (3.4.11) holds. □

Now we want to check that

$$I \not\subseteq W, \tag{3.4.12}$$

where $W \subseteq F \times F$ is the closed subset defined in (3.4.8). Since $(l, l') \in I$, it is enough to prove the following.

CLAIM. The point (l, l') is not in W.

PROOF OF THE CLAIM. If $(l, l') \in W$, by definition of W there is some

$$u'' \in \widetilde{\psi}^{-1}(Y) \tag{3.4.13}$$

such that $(l, l') = \pi(u'')$. Again by uniqueness in (3.4.10), $u = u''$. Now (3.4.13) contradicts (3.4.11). □

We get that the open subset $I \backslash (W \cap I)$ of I is not empty by (3.4.12). If we apply Lemma 3.1.2 to diagram (3.4.9) we get

$$\mathrm{Ind}(\rho_{|F \times F \backslash W}) \subseteq \mathrm{Ind}(\psi_{|F \times F \backslash W}) \Rightarrow I \backslash (W \cap I) \subseteq \mathrm{Ind}(\psi) \backslash (W \cap \mathrm{Ind}(\psi)).$$

In particular, we have a dense subset of I contained in $\mathrm{Ind}(\psi)$, then $I \subseteq \mathrm{Ind}(\psi)$. Hence by (3.4.7) we get $I = \mathrm{Ind}(\psi)$. This contradicts the assumption $\mathrm{Ind}(\psi) \subsetneq I$ and we are done. □

Qual è ‘l geomètra che tutto s’affige
per misurar lo cerchio, e non ritrova,
pensando, quel principio ond’elli indige,
Dante Alighieri, La Divina Commedia, Paradiso XXXIII, v.133-135

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