Jump from Parallel to Sequential Proofs : On Polarities and Sequentiality in Linear Logic.

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Introduction

> *"If you knew time as well as I do - said the Hatter - you wouldn't talk about wasting it. [...] Now, if you only kept on good terms with him, he'd do almost anything you liked with the clock."*
> (Lewis Carroll, *Alice's adventures in Wonderland*.)

The topic of the present thesis is the relation between *parallelism* and *sequentiality* in proofs. More precisely, placing ourselves in the setting of Linear Logic, we study the connection between two of its most crucial discoveries: *proof nets* and *polarities*.

Let us try to explain better the point. Informally, a proof of a formula A could be represented as a chain of inference rules ending with an inference which concludes A. Such a representation does not seem to properly capture the intrinsic nature of a proof, since it makes distinction between proofs which are morally the *same*: consider for example, two proofs of the same formula differing only for the order of application of the rules.

Following such intuition, the nature of a proof appears to be somehow *independent from the temporal succession of its rules*, resembling much more a *parallel* process.

Linear Logic provides mathematical substance to this claim, introducing the notion of *proof net*.

Proof nets. In 1987, in the seminal paper [Gir87], Girard introduces Linear Logic (briefly LL) from a fine analysis of intuitionistic and classical logic; such a refinement provides a logical status to the structural rules of sequent calculus (thanks to the introduction of the exponential connectives, ! and ?) and splits the usual propositional connectives in two classes (the additives $\&, \oplus, \top, \mathbf{0}$, and the multiplicatives $\otimes, \parr, \perp, \mathbf{1}$).

The deep insight on the standard connectives operated by LL allows to represent proofs as *graphs* called *proof nets*, whose nodes correspond to linear logic rules and which satisfy some specific geometrical properties (called *correctness criteria*); such a discovery brought to the fore the *geometrical* nature of proofs.

The main characteristics of proof nets is to be *modular, parallel* objects: in a proof net, there is not any direct reference to the sequential succession of steps which brought to its construction.

As a consequence, a proof net turns out to be a canonical representative of a class of proofs equivalent modulo permutations of rules: *sequentialization*, one of the key results in the theory of proof nets, allows to recover a proof from a proof net, by proving that among the nodes of the proof net one can be chosen as the last rule of a proof; by iterated application of this property, called *splitting*, one can build up a proof from a proof net.

The discover of proof nets becomes even more interesting if we consider them in the light of the so-called *Curry-Howard isomorphism*, relating computer science with proof-theory. This isomorphism associates programs with proofs, and execution of programs with a procedure of transformation of proofs called *cut-elimination*, enlightening in this way the logical meaning of computation and the operational nature of proofs.

Cut-elimination in the setting of proof nets becomes a local, modular rewriting of graphs; due to the Curry Howard isomorphism, this provides then a parallel, geometrical account of computation.

Nevertheless, we must remark that outside multiplicative linear logic (briefly MLL), the beautiful theory of proof nets becomes less elegant and in (some cases) quite complicated: for instance, the search of a proper syntax extending proof nets to the the additives has been an open problem for a long time, only recently solved by Hughes and Van Glaabeek in [HVG03].

To sum up, proof nets allows to eliminate those *naive* aspects of sequentiality which are not *naturally* inherent to the structure of proofs.

However there exists also another side of sequentiality, more intrinsic than the simple ordered succession of rules, which has been disclosed by the discover inside linear logic of *polarities*.

Polarities. LL, tampering the structural rules with the exponentials connectives, allows for the first time to talk of *linear negation*, that is "negation without structural rules".

In the light of the Curry-Howard isomorphism, linear negation gets a clear operational meaning, as *a change of viewpoint*: in computer science, a program is executed in a given environment; this process can be either analyzed from the point of view of the program or from the one of the environment, and negation is here the switch between these two positions.

It is worth mentioning that the above intuition contributed in a remarkable way to the birth of *game semantics*, which interprets computation as a game between two players, the Proponent (the program) and the Opponent (the environment) (see [AMJ00], [HO00]).

The refinement of usual negation inside LL was the starting point of a deep analysis of the logical notion of *duality*, which eventually brought to the discover of *polarities*.

Multiplicative and additive connectives of LL splits into two dual families:

- *Positives (or synchronous)*: $\otimes, \oplus, \mathbf{1}, \mathbf{0}$;
- *Negatives (or asynchronous)*: $⅋, \&, \bot, \top$.

A formula is *positive* (resp. *negative*) if its outermost connective is positive (resp. negative).

In [And92], Andreoli proved that any proof of linear logic can be transformed modulo permutations of rules into a proof which satisfies (bottom-up) the following discipline:

i) negative formulas, if any, are decomposed immediately;

ii) otherwise, one chooses a positive formula, and keep decomposing it up to its negative subformulas;

such proofs are called *focusing*.

The alternation of positive and negative steps provides then a canonical way to construct a proof, yielding an intrinsic, *not trivial* notion of *time* in proofs, as pointed out by Girard in [Gir99].

Using *synthetic connectives* (that is considering cluster of connectives of the same polarity as a single connective), in [Gir00] Girard introduced a calculus for focusing proofs in multiplicative-additive Linear Logic (briefly $MALL$) called *hypersequentialized calculus*, with only two kind of logical rules (the positive and the negative), one strictly alternating with the other.

Looking at the hypersequentialized calculus (briefly HS) through the lens of interaction, a positive rule appears as the act of posing a question to the Opponent (the environment) by the Proponent (the program) , and a negative rule as the reception of an answer from the Opponent; if we apply linear negation, we switch the point of view, turning questions into answers and answers into questions.

The nature of a proof in HS then seems to be a *dialogue*, a strict alternation of questions and answers; this startling discover opened the way to game models of linear logic (see [Lau03] , [Lau04]) and to *ludics* [Gir01], a reconstruction of multiplicative-additive Linear Logic based only on the notion of interaction.

Nevertheless, the hypersequentialized approach has its limitations: mainly, it forces to leave the proof nets syntax, in spite of the simplicity and elegance of its multiplicative part.

The mismatch. Clearly there is a mismatch regarding the nature of proofs between proof nets and hypersequentialized calculus: while proof nets are *timeless*, parallel objects, in the hypersequentialized proofs there is an explicit marking of time, which makes them *sequential* in a strong sense.

The point is well captured in the following quoting of Girard, from [Gir99]:

> "We are perhaps explaining a *sequential* logic, and there might as well be a *parallel* logic -without temporality-; [...] I think

> that the ghost of an alternative parallel logic might vanish if we succeed to depart from the game intuition, in which a *strict* alternation of moves is so important."

The aim of the present thesis is to try to reconcile this mismatch in $MALL$, by proposing a notion of proof net for HS, recovering in this way polarities in a parallel setting; such proof nets will be called *J-proof nets*. In J-proof nets positive and negative rules are still alternating, but not *strictly*, that is, the set of rules following a given rule is *partially* ordered; as it is standard in the theory of proof nets, any J-proof net can be sequentialized into an hypersequentialized proof. In other words, time is still present, but while in hypersequentialized proofs it is *explicit*, in J-proof nets is *implicit*.

The ideas underlying J-proof nets come from recent development of ludics, namely the *ludics nets* (or *L-nets*) of Faggian and Maurel (see [FM05]).

Ludics nets. Game models of sequential computation interpret a program as a strategy in a game; from a geometrical point of view, usually such strategies appear as trees, (for instance, innocent Hyland-Ong strategies, see [HO00]), and composition between strategies yields a linearly ordered set of moves.

In the area of game semantics, several proposals are emerging (see among others [HS02, AM99, Mel04]) in order to capture more parallel forms of computation; in these approaches, strategies are no more trees, but more generally graphs, so that the order between the actions is not completely specified: the composition between graph strategies yields a partially ordered set of moves.

L-nets are a generalization of ludics designs (which correspond to Hyland-Ong strategies, see [FH02]) developed from the observation that a design is a merging of two kinds of orders between actions, as pointed out by Faggian in [Fag02]:

- the *causal* (or *spatial*) order, representing the causal dependencies between actions;

- the *sequential* (or *temporal*) order, representing how independent actions are scheduled.

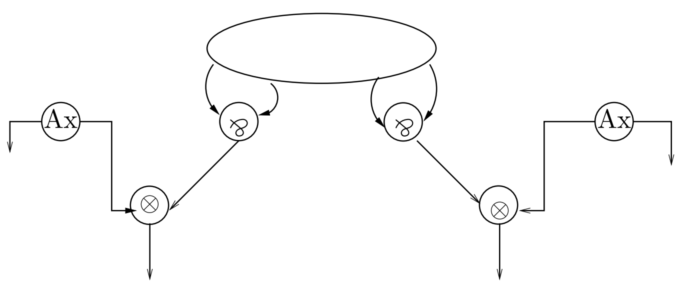
L-nets are built from usual designs by gradually relaxing the sequential order between actions, in such a way to have still enough information to compute; the main benefit of this approach is to provide an homogeneous space of strategies with different levels of sequentiality, within which one can move by adding or relaxing sequential order; such a space has as extremes from one side L-nets of minimal sequentiality, from the other designs (see [CF05]); by the way, designs are a special case of L-nets, as trees are a special case of graphs.

Jumps: sequentializing *à la carte.* It is well known that a design in ludics correspond to an abstract, untyped hypersequentialized proof. What we would like to achieve is a notion of proof net which corresponds to the one of L-net, as an hypersequentialized proof corresponds to a design. In order to accomplish this task, we must find a counterpart for both the causal and the sequential ordering in the syntax of proof nets. Concerning the causal ordering, if we restrict to $MALL$, we have already an answer in the *subformula* relation induced by the structure of the links in a proof net. The information provided by axiom links, instead, is closer to sequential ordering; however, to properly characterize sequential order in proof nets, we have to resort to the notion of *jump*.

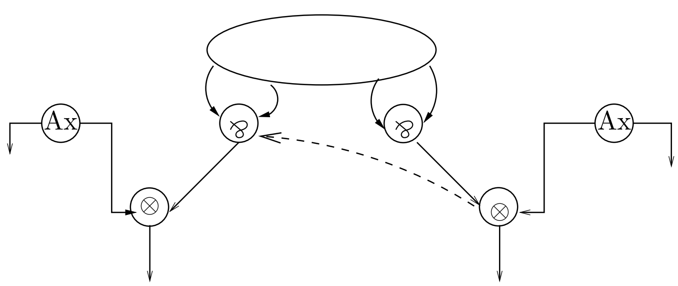
The idea of using edges to represent sequentiality constraints has been widely used into the study of correctness criteria for proof nets: in [Gir91] and [Gir96], Girard, as a part of the correctness criterion for proof nets, introduces *jumps*: if a link n is a $\parr$, &, $\perp$ or $\forall$ link, a jump is an untyped edge between n and another link m, which represent a sequential ordering between n and m; n precedes m (bottom-up) in every sequentialization.

J.Y. Girard, in several occasions, suggested that it could be possible to retrieve a sequent calculus proof from a proof net just by fixing some temporal information on the proof net, using jumps.

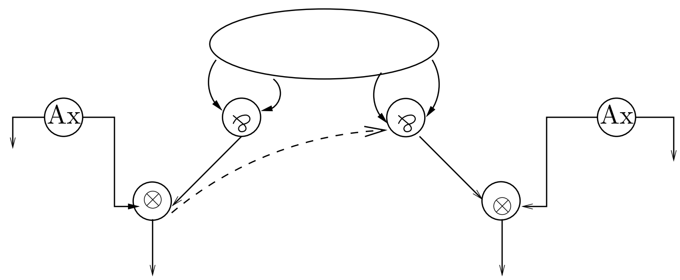
Let us try to make this point clearer with an example; consider the sketch of proof net below:



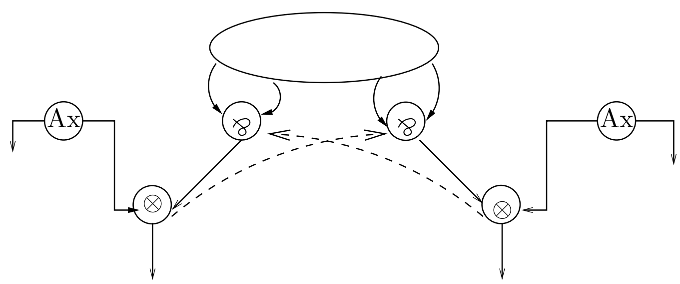
We remark that such a configuration is forbidden in HS: one must decide which one of the two $\otimes$ is the last rule of the proof. To retrieve the proof then we draw a jump between the leftmost (negative) $\parr$ and the rightmost (positive) $\otimes$, meaning that the corresponding $\parr$ rule must precede (bottom-up) the corresponding $\otimes$ rule in the sequentialization;



Now, the sequent calculus proof π induced from this proof net will have as last rule the leftmost $\otimes$, followed respectively by the leftmost $\parr$, the rightmost $\otimes$ and the rightmost $\parr$, so it respects the focusing discipline. We remark that, instead of fixing the order in the way above, we could as well draw a jump between the the rightmost (negative) $\parr$ and the leftmost (positive) $\otimes$, as below, obtaining a different focusing proof π':



Furthermore, once fixed an order between links using jumps, some other choices becomes unavailable: namely one cannot draw both the jumps above at the same time, without creating a *cycle*, which would prevent to get an order.



Once that all possible jumps have been chosen, one directly retrieve in this way a sequent calculus proof.

J-proof nets. Now, our method of work should be clear: we will introduce J-proof nets as proof nets with jumps for the hypersequentialized calculus, and we will gradually remove or add sequentiality between positive and negative links, in the form of jumps, in order to get more parallel or more sequential proofs, in the style of L-nets.

At any time, the information provided by jumps always makes possible to retrieve a fully sequentialized J-proof net, that is, an hypersequentialized proof: this key property of J-proof nets , stated in our main technical result, called *arborisation lemma*, provides a way to insert jumps in a J-proof net, up to a maximum.

In this way, as in L-nets, we get an homogenous space of J-proof nets with different degrees of sequentiality, having as extremes from one side J-proof nets of minimal sequentiality, from the other J-proof nets of maximal sequentiality, which directly correspond to HS proofs.

Content of the thesis. The thesis is divided into three chapters: in the first chapter we present the arborisation lemma, while in the remaining two chapters we introduce J-proof nets respectively for the multiplicative and the multiplicative-additive fragment of HS.

Chapter 1: The main contribution of this chapter is the proof of the arborisation lemma. In subsection 1.1 we recall some preliminary notions of graph theory; then, in subsection 1.2, we introduce a class of directed acyclic graphs, called *polarized graphs*, which are a sort of abstract proof nets, and we present the arborisation lemma as a general property of these graphs. Arborisation lemma states that, by inserting edges, is it possible to transform a polarized graph into a tree, preserving a particular geometrical condition on the graph, called *switching acyclicity* (which corresponds to the correctness criterion on proof nets). Actually, we provide two different formulations of the lemma: a stronger one, which constitutes the key of sequentialization in J-proof nets, and a weaker one, which will allow to provide a simple, alternative proof of the sequentialization theorem for the usual multiplicative proof nets of linear logic.

Chapter 2: In this chapter we introduce J-proof nets for the multiplicative fragment of the hypersequentialized calculus (briefly MHS). We start by presenting in subsection 2.1.1 the usual MLL sequent calculus in a slightly different way, by using synthetic connectives; then we retrieve the hypersequentialized calculus (in its multiplicative part) by imposing on proofs the constraint of *alternation*. In section 2.2 we define multiplicative J-proof nets, and in subsection 2.2.4 we prove the sequentialization theorem, using the strong arborisation lemma; then in subsection 2.2.5 we study cut-reduction on J-proof nets. In section 2.3 we isolate a mathematical structure, called *pointed set*, and we show that it describes what is invariant in a J-proof net under cut-reduction; pointed sets so define a model of cut-reduction in J-proof nets, called *pointed semantics*. Pointed semantics is an extension of usual relational semantics, developed in collaboration with Pierre Boudes and Damiano Mazza, allowing to semantically characterize jumps. In subsection 2.3.2 we prove that pointed semantics is *injective* with respect to J-proof nets, that is two J-proof nets with the same interpretation are sintactically equivalent. In section 2.4 we shift the focus on usual multiplicative proof nets, and we use the weak arborisation lemma to give an alternative proof of sequentialization theorem in this setting: in subsection 2.4.2 we prove how two standard results in the theory of proof nets, namely the splitting $\otimes$ lemma and the splitting $\parr$ lemma, are both consequences of the arborisation lemma. Finally in subsection 2.4.3 we study the relation between jumps and another standard notion in proof nets, the one of *empire*. Most of the results of this chapter are in [DGF06] (joint work with Claudia Faggian).

Chapter 3: In this chapter we extend J-proof nets to the additive fragment of hypersequentialized calculus. First, in section 3.1, we present the full hypersequentialized calculus; then in section 3.2 we introduce additive J-proof nets. In section 3.3 we extend sequentialization to include additives; the relevance of this result is clear, if one consider that the problem of sequentializing in presence of additives is one of the most difficult in the framework of proof nets. In order to properly take into account the superposition effects implicit in the structure of additives, we must resort to the notion of *slice* and *sharing equivalence*, defined in subsection 3.3.2. In section 3.4 we will study cut reduction, always using slices as main tools. In section 3.5, we extend pointed semantics to additives, proving that the injectivity result of the previous chapter is preserved. Using injectivity of pointed semantics, in section 3.6, we will prove that the correctness criterion is stable under cut reduction, (a similar strategy was used also by Laurent and Tortora de Falco in [LTdF04]). Finally, in section 3.7, in the style of [CF], we isolate two classes of J-proof nets, the ones with minimal (resp. maximal) sequentiality, and we provide some indications to recover within J-proof nets some of the usual syntaxes for additive proof nets; namely, additive boxes (see [Gir87]) , multiboxes (see [TdF03b], and sliced polarized proof nets (see [LTdF04]).

Contents

Chapter 1

The arborisation Lemma

In this chapter, after recalling some basic notions of graph theory, we introduce a class of directed acyclic graphs that we call **polarized graphs**. In this setting we state and prove the **arborisation lemma**, which will turn out to be our principal tool in chapter 2.

1.1 Preliminaries on graphs

A **directed graph** G is an ordered pair (V, E), where V is a finite set whose elements are called **nodes**, and E is a set of ordered pairs of nodes called **edges**.

We will denote nodes by small initial Latin letters $a, b, c, \ldots$ and edges by small final Latin letters $\ldots, x, y, z$.

To denote that there is an edge from a node a to a node b, we will write $a \to b$; we say that an edge x from a to b is **emergent** from a and **incident** on b; b is called the **target** of x and a is called the **source**.

The **in-degree** (resp. **out-degree**) of a node is the number of its incident (resp. emergent) edges; two edges are **coincident** when they have the same target.

Given a directed graph G a **path** (resp. **directed path**) r from a node b to a node c is a sequence $\langle a_1, \ldots, a_n \rangle$ of nodes such that $b = a_1$, $c = a_n$, and for each a_i, a_{i+1}, there is an edge x either from a_i to a_{i+1} , either from a_{i+1} to a_i (resp. from a_i to a_{i+1}); in this case, x is said to be *used* by r; given a path $r = \langle a_1, \ldots, a_n \rangle$ we will say that r **leaves** a_1 and **enters** in a_n.

We will denote a directed path from a to b with $a \xrightarrow{+} b$

A graph G is **connected** if for any pair of nodes a, b of G there exists a path from a to b.

A **cycle** (resp. **directed cycle**) is a path (resp. directed path) $\langle a_1, \ldots, a_n \rangle$ such that $a_1 = a_n$.

A **directed acyclic graph** (d.a.g.) is a directed graph without directed cycles.

When drawing a d.a.g we will represent edges oriented up-down so that we may speak of moving downwardly or upwardly in the graph; in the same spirit we will say that a node is above or below another node.

Given a d.a.g. G and a node a of G, we will call **cone** of a in G (denoted $C_G(a)$) the set of all the nodes hereditary above a; the set $\{C_G(a); a \text{ is a node of } G\}$ of the cones of a d.a.g. G is strictly ordered by inclusion.

We recall that we can represent a strict partial order as a d.a.g., where we have an edge $b \to a$ whenever $a <_1 b$ (i.e. $a < b$, and there is no c such that $a < c$ and $c < b$.) Conversely (the transitive closure of) a d.a.g. G induces a strict partial order $\prec_G$ on the nodes of G.

We call predecessor of a node c, a node which immediately precedes c in $\prec_G$.

An edge $a \to b$ is transitive if there is a node c such that $a \xrightarrow{+} c$ and $c \to b$.

We call **skeleton** of a directed graph G, denoted $Sk(G)$, the minimal graph whose transitive closure is the same as that of G.

A **root** of a d.a.g. is a node with no predecessors.

A **forest** (resp. **tree**) is a d.a.g. G such that, given a node a and a root b of G, there exists at most one (resp. exactly one) directed path from a to b.

A strict order on a set is **arborescent** when each element has a unique predecessor; it is straightforward that if the order $\prec_G$ associated to a directed graph G is arborescent, the skeleton of G is a forest.

A graph G is **bipartite** when there is a partition of the set V of nodes into two subsets V_1 and V_2, such that every edge of G connects an element of V_1 with an element of V_2.

A **graph with pairs** is a couple $(G, App(G))$ where G is a directed graph and $App(G)$ is a set of n-tuples of coincident edges of G.

Given a graph with pairs $(G, App(G))$, we call **switching edge** an edge x belonging to a n-tuple of $App(G)$. A **switching path** in $(G, App(G))$, is a path which uses at most one switching edge for each n-tuple of $App(G)$; a **switching cycle** is a path which is a cycle; G is **switching acyclic** when it does not contain switching cycles.

1.2 Polarized graphs

Definition 1 (Polarised graph) *A* ***polarized graph*** *G is a directed acyclic graph such that there is a partition of its nodes into three sets (I, N, P) which satisfy the following constraints:*

- *all the nodes in I (called* ***initials****) have in-degree 0 and out-degree 2;*
- *all the nodes in N have out-degree $n \leq 1$ and in-degree $n \geq 1$;*

- *all the nodes in P have in-degree $n \geq 1$;*
- *the graph $G \setminus I$ which contains only the nodes in N and P is bipartite.*

*We say that the nodes in N have **negative** polarity, and that the nodes in P have **positives** polarity.*

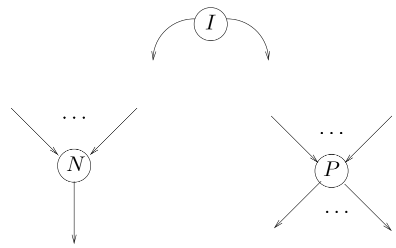


Figure 1.1: Nodes of a polarized graph

Definition 2 (Balanced polarised graph) *A polarized graph is **balanced** when for each initial node a, its immediate predecessors have different polarity.*

We turn a polarized graph G into a graph with pairs $(G, App(G))$, by taking as $App(G)$ the n-tuples of the edges incident on the same negative node. A polarized graph G is switching acyclic if and only if the corresponding graph with pairs $(G, App(G))$ is switching acyclic.

Definition 3 (Saturated polarised graph) *A switching acyclic polarized graph G is **saturated** iff for every negative node a and for every positive node b of G, adding an edge from b to a creates a switching cycle or doesn't increase the order $\prec_R$.*

Lemma 4 (Strong Arborisation Lemma) *Let G be a balanced polarized graph with at most one negative root. Then, if G is saturated, the order $\prec_G$ is arborescent.*

Proof.

Let us prove that if $\prec_G$ is not arborescent then G is not saturated, that is there exists a negative node c and a positive node b s.t. adding an edge between b and c doesn't create switching cycles and makes the order increase.

If $\prec_G$ is not arborescent, then in $\prec_G$ there exists a node a with two immediate predecessors b and c (they are incomparable). Observe that b and c are immediately below a in $Sk(G)$ and also in G.

If a is an initial node, since G is balanced then necessarily b and c are respectively a positive node and a negative node; we add an edge between b and c, this doesn't create cycles and the order increases.

Otherwise, a is a positive node, and b and c are two negative nodes; we distinguish two cases:

1. either b or c is a root in G. Let assume that b is a root; then c cannot be a root (by hypothesis), and there is a positive node c' which immediately precedes c. If we add an edge between b and c', this doesn't create cycles and the order increases (see fig 1.2).

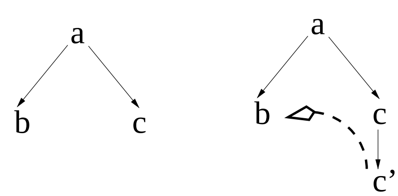


Figure 1.2:

2. Neither b or c are roots in G. Each of them has an immediate positive predecessor, respectively b' and c'. Suppose that adding an edge from b' to c creates a cycle: we show that adding an edge from c' to b cannot create a cycle.

   If adding to G the edge $b' \to c$ creates a cycle, this means that there is in G a switching path $r = \langle c, c' .... b \rangle$; if adding the edge $c' \to b$ creates a cycle then there is a switching path $r' = \langle b, b' ... c \rangle$.

   Assume that r and r' are disjoint: we exhibit a switching cycle in R $\langle c, c' ... b, b' ... c \rangle$ by concatenation of r and r'.This contradicts the fact that G is switching acyclic (see fig 1.3).

   Assume that r and r' are not disjoint. Let d be the first node of r' (starting from b) where r and r' meets. Observe that d must be negative (otherwise there would be a cycle). Each of r, r' uses one of the edges incident on d (hence the paths meet also in the node below d). From the fact that d is the first node of r' (starting from b) where r and r' meet it follows that: (i) r' enters in d using one of its incident edges; (ii) each of r and r' must use a different incident edge of d. Then we distinguish two cases (see 1.4):

   - r enters d using one its incident edges; we build a switching cycle taking the sub path $\langle b, ..., d \rangle$ of r' and the sub path $\langle d, ..., b \rangle$ of r.

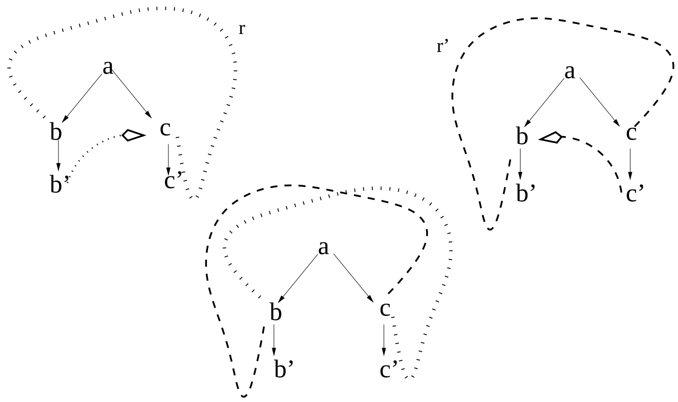


Figure 1.3:

- r enters d using its emergent edge; then we build a switching cycle composing the sub path of r $\langle c, ..., d\rangle$, the reversed sub path of r' $\langle d, ..., b\rangle$ and the path $\langle b, a, c\rangle$.

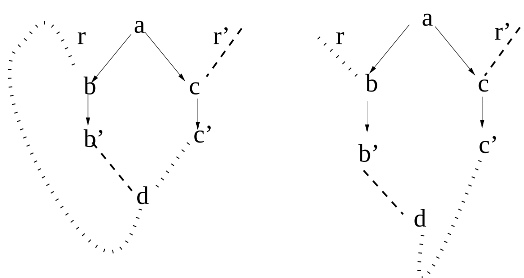


Figure 1.4:

□

The relation between arborescent order and saturation stated in the previous lemma actually holds only in the restricted case of balanced polarized graphs; it is easy to build a counterexample for the general case, using polarized graphs composed only of initial nodes and positive nodes.

However, even for not-balanced polarized graphs, the following property holds:

Lemma 5 (Weak Arborisation Lemma) *Let G be switching acyclic polarized graph with at most one negative root, and $C_G(b), C_G(c)$ be the cones of two negative nodes of G. Then if G is saturated, either $C_G(b) \cap C_G(c) = \emptyset$, either one among $C_G(b), C_G(c)$ is strictly included into the other.*

PROOF.

Assume that $C_G(b) \cap C_G(c) \neq \emptyset$, $c \notin C_G(b)$ and $b \notin C_G(c)$; now consider a node $a \in C_G(b) \cap C_G(c)$.

Every node in $C_G(b) \cap C_G(c)$ is hereditary above both b and c, so there is a a directed path r' (resp. r'') from a to b (resp. from a to c).

Let us assume that b, c are not roots of G (otherwise, at most one of them can be a root, so we reason as in the proof of lemma 4 and we find that G is not saturated: contradiction), so they are respectively immediately above two positive nodes b', c', such that $b' \notin C_G(c)$ (resp. $c' \notin C_G(b)$).

Since G is saturated , there is a switching path $\langle c, c', \ldots, b \rangle$ connecting c with b (otherwise we could add an edge from b' to c, and G would not be saturated); now this path cannot intersect r'', (otherwise there would be a cycle), and if it meets a node d of r', it follows r' from d to b: we call this path p'. In the same way we can build a switching path p'' $\langle b, b', \ldots, c \rangle$ from b to c.

The rest of the proof is the same as the proof of lemma 4 ; p' and p'' either do not meet on any node either they do; in any way, by composing them we get a cycle.

$\square$

Remark 6 *It easily follows from the lemma 5 that given a saturated polarized graph G, in the set $\{C_G(a); a \text{ is a node of } G \wedge a \text{ is negative}\}$ the strict order provided by inclusion is arborescent.*

Chapter 2

J-proof nets: multiplicatives

In this chapter we introduce and study J-proof nets for the multiplicative fragment of the hypersequentialized calculus. In section 2.1 we first present MLL grammar and sequent calculus, then MHS; in section 2.2 we define J-proof nets, and we prove sequentialization using the strong arborisation lemma. In section 2.3 we introduce *pointed sets* semantics in order to study the injectivity of pointed sets with respect to J-proof nets. Finally, in section 2.4, we present the usual MLL proof nets, and we provide an alternative proof of the sequentialization theorem using the weak arborisation lemma.

2.1 MLL and focusing proofs

The scope of this section is to present the language and the calculus of multiplicative linear logic. The calculus we present here is slightly different from the usual one and is based on the notion of *synthetic connective* introduced by Girard in [Gir00]. We first present in subsection 2.1.1 a variant of usual MLL grammar and calculus, where formulas are clustered modulo the usual associativity isomorphisms of linear logic; then in subsection 2.1.3 we introduce the *multiplicative hypersequentialized calculus* in order to restrict the scope to *focusing proofs*.

2.1.1 MLL

Definition 7 *Let $\mathcal{V} = \{X,\ Y,\ Z, \ldots\}$ be a countable set of propositional variables; the formulas of MLL are defined in the following way :*

- ***Atoms**: $X, Y, Z, \ldots$ and $X^\perp, Y^\perp, Z^\perp, \ldots$ are formulas of MLL*
- ***synchronous formulas**: given $A_1, \ldots, A_n$ formulas where $A_{i \in \{1,\ldots,n\}}$ is an atom or an asynchronous formula, then $\otimes(A_1, \ldots, A_n)$ is a formula;*

- ***asynchronous formulas***: *given $A_1, \ldots, A_n$ formulas where $A_{i\in\{1,\ldots,n\}}$ is an atom or a synchronous formula, then $\parr(A_1, \ldots, A_n)$ is a formula;*

Negation is defined as follows:

$$(\otimes(A_1, \ldots, A_n))^\perp = \parr(A_1^\perp, \ldots, A_n^\perp)$$

$$(\parr(A_1, \ldots, A_n))^\perp = \otimes(A_1^\perp, \ldots, A_n^\perp)$$

Note. We underline the following facts:

- By $\otimes(A_1, \ldots, A_n)$ we indicate the connective which represent all possible combinations of the formulas $A_{i\in\{1,\ldots,n\}}$ modulo the associativity of the usual $\otimes$ connective of LL; we denote the unary case of $\otimes(A_1, \ldots, A_n)$ as $\downarrow A$

- By $\parr(A_1, \ldots, A_n)$ we indicate the connective which represent all possible combinations of the formulas $A_{i\in\{1,\ldots,n\}}$ modulo the associativity of the usual $\parr$ connective of LL; we denote the unary case of $\parr(A_1, \ldots, A_n)$ as $\uparrow A$

The calculus has the following shape (where the capital Greek letters $\Gamma, \Delta, \ldots$ denote multisets of formulas) :

$$\frac{}{\vdash A, A^\perp}\,ax \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma_1,\ A_1 \qquad \ldots \qquad \vdash \Gamma_n, A_n}{\vdash \Gamma_1, \ldots, \Gamma_n, \otimes(A_1, \ldots, A_n)}\,(+) \qquad \frac{\vdash \Gamma, A_1, \ldots, A_n}{\vdash \Gamma, \parr(A_1, \ldots A_n)}\,(-)$$

MLL can be eventually enriched with the following rule, called **Mix** rule:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\,mix$$

2.1.2 Towards the hypersequentialized calculus

Let us consider the following two MLL proofs (that we denote respectively π_1, π_2) of the same sequent :

$$
\begin{prooftree}
\AxiomC{$\vdash A^{\perp}, A$}
\AxiomC{$\vdash B^{\perp}, B$}
\AxiomC{$\vdash C^{\perp}, C$}
\AxiomC{$\vdash D^{\perp}, D$}
\RightLabel{$\otimes$}
\QuaternaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ A,\ B,\ C,\ D$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \parr(A,B),\ C,\ D$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \parr(A,B),\ \parr(C,D)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \parr(A,B),\ \otimes(\parr(C,D), F^{\perp}),\ F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash E, E^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \otimes(\parr(A,B), E^{\perp}),\ \otimes(\parr(C,D), F^{\perp}),\ F,\ E$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$\vdash A^{\perp}, A$}
\AxiomC{$\vdash B^{\perp}, B$}
\AxiomC{$\vdash C^{\perp}, C$}
\AxiomC{$\vdash D^{\perp}, D$}
\RightLabel{$\otimes$}
\QuaternaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ A,\ B,\ C,\ D$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ A,\ B,\ \parr(C,D)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ A,\ B,\ \otimes(\parr(C,D), F^{\perp}),\ F$}
\RightLabel{$\parr$}
\UnaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \parr(A,B),\ \otimes(\parr(C,D), F^{\perp}),\ F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash E, E^{\perp}$}
\RightLabel{$\otimes$}
\BinaryInfC{$\vdash \otimes(A^{\perp}, B^{\perp}, C^{\perp}, D^{\perp}),\ \otimes(\parr(A,B), E^{\perp}),\ \otimes(\parr(C,D), F^{\perp}),\ F,\ E$}
\end{prooftree}
$$

The attentive reader has surely noticed the difference between π_1 and π_2; while in π_1 there can be two consecutive $\parr$ or $\otimes$ rules, in π_2 $\parr$ and $\otimes$ rules are **alternating**. If we want to restrict the proofs of MLL just to the alternating ones, we have to impose some constraints; in this way we will obtain the hypersequentialized calculus, (better, its multiplicative fragment), restricting the calculus to *focusing* proofs.

2.1.3 MHS

The formulas of the multiplicative hypersequentialized calculus (MHS) are obtained by the following restrictions on MLL formulas:

$$
\begin{array}{rcccl}
N & ::= & X^{\perp} & \mid & \parr(P, \ldots, P) \\
P & ::= & X & \mid & \otimes(N, \ldots, N)
\end{array}
$$

From now on, we will call the formulas in N **negatives** and the formulas in P **positives**.

The calculus is the following:

$$
\frac{}{\vdash P, P^{\perp}}\, ax \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\, cut
$$

$$
\frac{\vdash \Gamma_1,\ N_1 \qquad \ldots \qquad \vdash \Gamma_n, N_n}{\vdash \Gamma_1, \ldots, \Gamma_n, \otimes(N_1, \ldots, N_n)}\, (+) \qquad\qquad \frac{\vdash \Gamma, P_1, \ldots, P_n}{\vdash \Gamma, \parr(P_1, \ldots P_n)}\, (-)
$$

$$
\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\, mix
$$

where all contexts $\Gamma, \Delta, \ldots$ only contain P formulas.

Remark 8 *In this calculus the unary* $\parr$ *denoted by* $\uparrow$ *(resp. the unary* $\otimes$ *denoted by* $\downarrow$*) plays the role of a negative (resp. positive) polarity inverter (as in [Gir01] This polarity inverter is usually called a* ***negative*** *(resp.* ***positive****)* ***shift****.*

2.2 J-proof nets for MHS

The focus of this section is to provide a geometrical representation of proofs in MHS. We start by defining in subsection 2.2.1 a class of typed graphs called *J-proof structures*; then in subsection 2.2.2 we isolate some geometrical properties which allow to characterize all the J-proof structures with a *logical* meaning, that is *J-proof nets*; in subsection 2.2.4 we prove that every J-proof net can be sequentialized into a proof of MHS. Finally in subsection 2.2.5, we study cut reduction on J-proof nets.

2.2.1 J-proof structures

Definition 9 (Proof structure) *A* **MHS** ***proof structure*** *(briefly* ***proof structure****) is a directed acyclic graph with pending edges (that is some edges have a source but no target) whose edges are typed by formulas of* MHS *and whose nodes (also called* ***links****) are labelled by one of the symbols* $ax, cut, +, -$.

The edges incident on a link are called ***premises*** *and the edges emergent from a link are called* ***conclusions****; an edge which has no target is called a* ***conclusion*** *of the proof structure and its source is called a* ***terminal link****.*

The label of a link imposes some constraints on both the number and the types of its premises and conclusions:

- *the* ax*-link has two conclusions labeled by dual formulas, but no premises;*
- *the* cut*-link has two premises labeled by dual formulas but no conclusions;*
- *the* ***negative link*** *(or* $-$ *link) has* n *premises and one conclusion. If the* i*-th premise is labeled by the formula* P_i *for* $i \in \{1, \ldots, n\}$ *then the conclusion is labeled by* $\parr(P_1, \ldots, P_n)$*;*
- *the* ***positive link*** *(or* $+$ *link) has* n *premises and one conclusion. If the* i*-th premise is labeled by the formula* N_i *for* $i \in \{1, \ldots, n\}$ *then the conclusion is labeled by* $\otimes(N_1, \ldots, N_n)$*.*

Moreover, we ask that in a proof structure there is at most one negative conclusion.

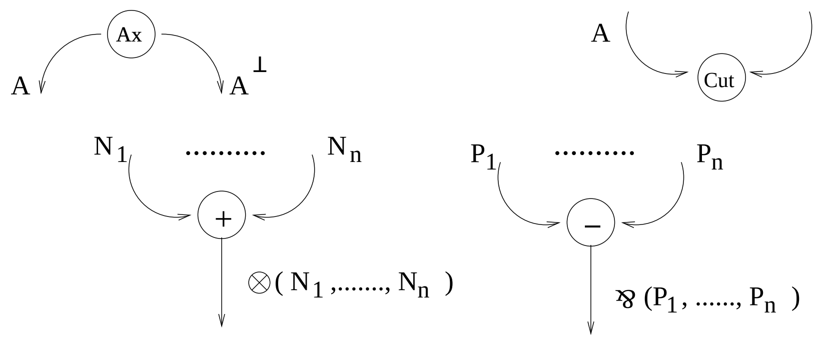


Figure 2.1: MHS links

Given a sequent calculus proof π of MHS (or MHS + Mix), we can associate to it a proof structure π^*, by induction on the height of π:

if $n = 1$, the last rule of π is an axiom with conclusions $P, P^\perp$; then π^* is an axiom link with conclusions $P, P^\perp$.

Otherwise:

- If the last rule r of π is a $-$ rule, having as premise the subproof π', then π^* is obtained by adding to π'^* the link corresponding to r.
- If the last rule of r is a $+$ rule or a *cut* rule having as premises the subproofs $\pi_1, \ldots, \pi_n$, then π^* is obtained by connecting $\pi_1^*, \ldots, \pi_2^*$ by means of the link corresponding to r.
- If the last rule of r is a Mix rule with premises the subproofs π_1 and π_2, then π^* is obtained by taking the union of π_1^* and π_2^*.

Definition 10 *A proof structure R is **sequentializable** iff there exists a proof π such that $\pi^* = R$.*

Now we refine our definition of proof structure, in order to take into account *jumps*:

Definition 11 (J-proof structure) *A **J-proof structure** (or **proof structure with jumps**) is a proof structure allowing untyped edges between negative and positive links called **jumps**, which are additional untyped premises of a negative link. Given a J-proof structure R, and a negative (resp. positive) link b (resp. a) we say that b **jumps on** a iff there is a jump between a and b.*

2.2.2 Correctness criterion

We can associate with a J-proof structure R the structure of a graph with pairs $(R, App(R))$, by taking as elements of $App(R)$ the n-tuples of the

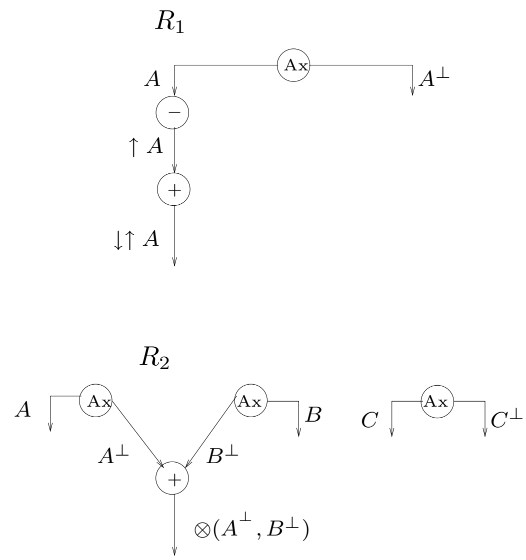


Figure 2.2: Two not sequentializable J-proof structures.

premises of a negative link; a J-proof structure R then is switching acyclic if and only if the graph with pairs $(R, App(R))$ is switching acyclic.

Unlike MLL + Mix, the switching acyclic condition is not enough to characterize all sequentializable proof structures: consider for example the proof structures in fig. 2.2, which are clearly not sequentializable in $MHS+$ Mix. The reason why they are not sequentializable is that in a proof of MHS a $-$rule or a Mix rule can be applied only after a $+$ rule has been applied.

In order to avoid this incongruousness, we have to impose one more condition in the correctness criterion, called **positivity condition**, similar to the homonymous condition on L-nets (see [FM05].

Definition 12 (Positivity condition) *A J-proof structure R satisfy the* ***positivity*** *condition if and only if*

1. *For every $-$ link b, such that a premise of b is a conclusion of an ax link a, there exists a positive link c (called* ***justifier*** *of b) and a path $\langle b, a, \ldots, c\rangle$ from b to c which crosses only cut and ax links;*

2. *if R is composed by more than one connected component, then each component contains at least one positive link.*

Using positivity condition we can interdict J-proof structures as the ones in fig. 2.2.

Definition 13 (J-proof net) *A J-proof structure (resp. proof structure) R is called a **J-proof-net** (resp. **proof net**) iff is switching acyclic, and it satisfies the positivity condition.*

The following theorem states that the purely geometrical condition of being a proof net characterizes exactly all the proof structures with a logical meaning, that is proof structures which come from a sequent calculus proof.

Theorem 14 (sequentialization) *A proof structure R is sequentializable if and only if is a proof net.*

The right to left direction is trivial; in section 2.2.4 we provide a simple proof of the left to right one.

Remark 15 *It is obvious that, if we do not consider cut-links, J-proof structures are balanced polarized graphs (whose edges are labelled by MHS formulas) where the ax links are I nodes, $+$ links are P nodes and $-$ links are N nodes .*

Note. By now, we will only consider J-proof nets without cut links, since in this case is simpler to prove sequentialization; we will speak about sequentialization with cut in section 2.2.5.

2.2.3 J-proof-nets and sequent calculus

In the next section we will induce a sequentialization of a proof net by adding jumps. Let us start with an example.

Consider the proof-net in fig. 2.3:

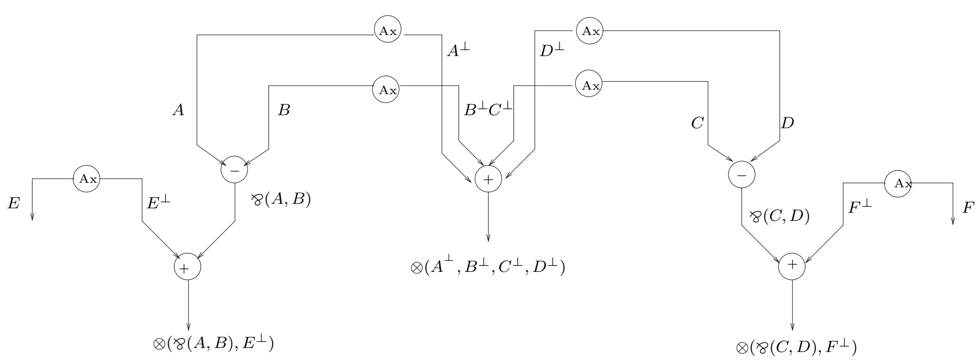


Figure 2.3:

We make the leftmost $-$ link jump on the middle $+$ link, and the rightmost $-$ link jump on the leftmost $+$ link, obtaining the J-proof net in fig 2.4;

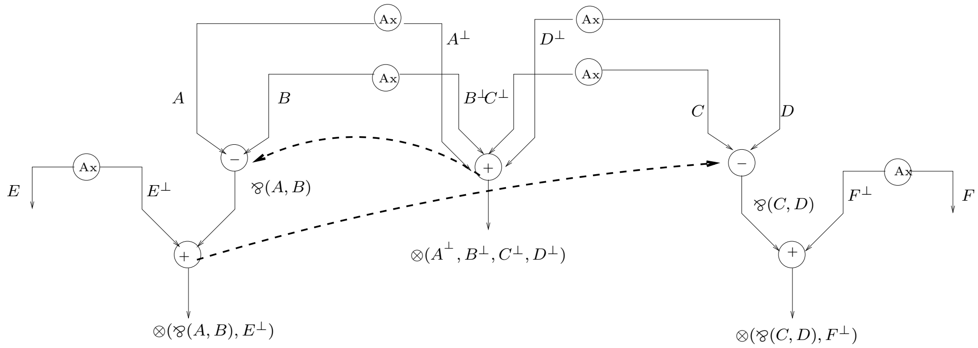


Figure 2.4:

Now we consider the partial order induced by the proof-net as a directed graph; the order is arborescent, so the skeleton of the graph is the tree in fig. 2.5.

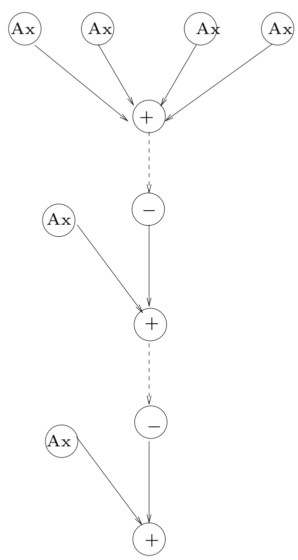


Figure 2.5:

Such a tree directly correspond to the following sequent calculus proof:

$$\begin{prooftree}
\AxiomC{$\vdash A^\perp, A$}
\AxiomC{$\vdash B^\perp, B$}
\AxiomC{$\vdash C^\perp, C$}
\AxiomC{$\vdash D^\perp, D$}
\RightLabel{$+$}
\QuaternaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ A,\ B,\ C,\ D$}
\RightLabel{$-$}
\UnaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ C,\ D,\ \parr(A,B)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash E, E^\perp$}
\RightLabel{$+$}
\BinaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ C,\ D,\ \otimes(\parr(A,B),E^\perp),\ E$}
\RightLabel{$-$}
\UnaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ \parr(C,D),\ \otimes(\parr(A,B),E^\perp),\ E$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\RightLabel{$+$}
\BinaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ \otimes(\parr(C,D),F^\perp),\ \otimes(\parr(A,B),E^\perp),\ F,\ E$}
\end{prooftree}$$

We remark that we could as well make the links jump as in fig 2.6

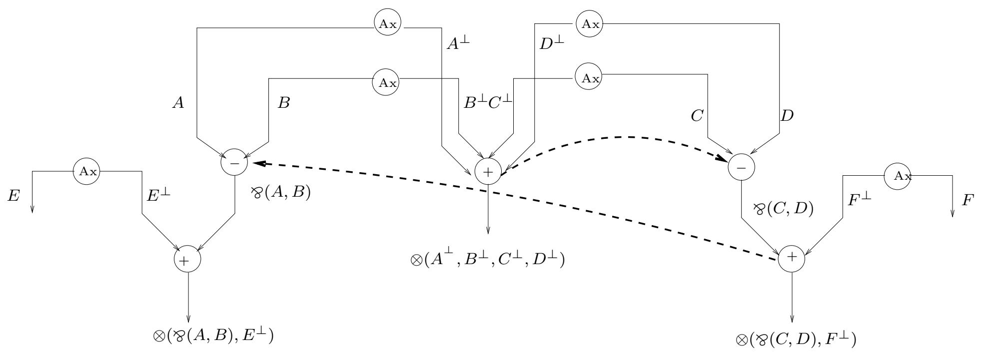


Figure 2.6:

retrieving the following, *different* sequent calculus proof:

$$\begin{prooftree}
\AxiomC{$\vdash A^\perp, A$}
\AxiomC{$\vdash B^\perp, B$}
\AxiomC{$\vdash C^\perp, C$}
\AxiomC{$\vdash D^\perp, D$}
\RightLabel{$+$}
\QuaternaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ A,\ B,\ C,\ D$}
\RightLabel{$-$}
\UnaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ A,\ B,\ \parr(C,D)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash F, F^\perp$}
\RightLabel{$+$}
\BinaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ A,\ B,\ \otimes(\parr(C,D),F^\perp),\ F$}
\RightLabel{$-$}
\UnaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ \parr(A,B),\ \otimes(\parr(C,D),F^\perp),\ F$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash E, E^\perp$}
\RightLabel{$+$}
\BinaryInfC{$\vdash \otimes(A^\perp, B^\perp, C^\perp, D^\perp),\ \otimes(\parr(A,B),E^\perp),\ \otimes(\parr(C,D),F^\perp),\ F,\ E$}
\end{prooftree}$$

To sequentialize a proof net, we will then consider the order associated to a proof net as a directed acyclic graph, and add to it enough jumps, to make the order arborescent, and hence proof-like.

Let us show that if the order on the nodes of a J-proof net is arborescent, it corresponds to a sequent calculus derivation.

We first have to prove the following lemma:

Lemma 16 *If R is a J-proof net with more than one link and without terminal negative links, then all the conclusions of R are positive.*

PROOF. Suppose there is a negative conclusion of R which is the conclusion of an ax link a. If R is composed by a single connected component, there must exist a link b a premise of which is the other (positive) conclusion of a, so b must be a negative link. But then by point 1) of the positivity condition on b, there must exists also a positive link d such that there is a path $\langle b, a, \ldots, d\rangle$, but this is impossible, since a is a terminal link : contradiction. If R is composed by more than one connected component, then by point 2) of the positivity condition, the connected component containing a must contain also a positive link; this means that a cannot be the only link in its connected component. Then there must exist a link b a premise of which is the other (positive) conclusion of a, and b must be a negative link; by point 1) of the positivity condition on b we find a contradiction, as above. □

Now we are in the position to prove the following proposition:

Proposition 17 (A forest is a sequent calculus proof) *Let R be a J-proof net of conclusions $A_1, \ldots, A_n$ and such that $Sk(R)$ is a forest.*

We can associate to R a sequent calculus proof π^R of conclusion $\vdash A_1, \ldots, A_n$ in MHS + Mix.

Moreover, if $Sk(R)$ is a tree where each negative node has exactly one incident edge, π^R is a sequent calculus proof in MHS (without Mix).

PROOF.

First, we observe that given a J-proof net R:

- $Sk(R)$ is obtained from R by removing the edges which are transitive;
- Only an edge incident on a negative link can be transitive.

Then we reason by induction on the number of nodes in $Sk(R)$:

1. $n = 1$. The only node in R is an Axiom link with conclusions $P,\ P^{\perp}$, to which we associate $\dfrac{}{\vdash P,\ P^{\perp}}$.
2. $n > 1$; if R has a terminal negative link c of type $\wp(P_1, \ldots, P_n)$, then $Sk(R)$ is a tree with c as root. Let $Sk(R)'$ be the forest obtained by erasing the root c; to this forest corresponds a subnet R' of R with conclusion $\Gamma, P_1, \ldots, P_n$. By induction we associate a proof $\pi^{R'}$ of conclusion $\Gamma, P_1, \ldots, P_n$ to R'. π^R is

$$\dfrac{\dfrac{\pi^{R'}}{\vdash \Gamma, P_1, \ldots, P_n}}{\vdash\ \Gamma, \wp(P_1, \ldots, P_n)}$$

whose last rule is a $-$ rule on $P_1, \ldots P_n$;

If R has no terminal negative links, by lemma 16 we can suppose that all the conclusions of R are positive. Now we reason by cases, depending if $Sk(R)$ is a tree or a forest:

- *Sk*(R) is a tree with root c of conclusion $\otimes(N_1, \ldots, N_n)$: by erasing c we obtain n trees *Sk*$(R_1), \ldots,$ *Sk*(R_n). To each tree corresponds a different subnet R_i, of R (because *Sk*(R) is obtained just by erasing transitive edges) of conclusion Γ_i, N_i; by induction we associate a proof π^{R_i} to each R_i.
  π^R is
  $$\dfrac{\dfrac{\pi^{R_1}}{\vdash \Gamma_1, N_1} \quad \cdots \quad \dfrac{\pi^{R_n}}{\vdash \Gamma_n, N_n}}{\vdash \Gamma_1, \ldots, \Gamma_n,\ \otimes(N_1, \ldots, N_n)}$$
  whose last rule is a $+$ rule on $N_1, \ldots, N_n$.
- *Sk*(R) is a forest and to each tree corresponds a different subnet of R; we apply to them the induction hypothesis, obtaining n proofs , which we compose by using a Mix rule.

□

2.2.4 Sequentialization

Definition 18 (Saturated J-proof net) *A J-proof net R is* ***saturated*** *if for every negative link a and for every positive link b, making a jump on b creates a switching cycle or does not increase the order $\prec_R$.*

Given a J-proof net R, a saturation $Sat(R)$ of R is a saturated J-proof net obtained from R by adding jumps.

Our sequentialization argument is as follows:

- Any J-proof net can be saturated.
- The order associated to a saturated J-proof net is arborescent.
- If the order $\prec_R$ associated to a J-proof net R is arborescent, we can associate to R a proof π^R in the sequent calculus.

Lemma 19 (Arborisation) *Let R be a J-proof net. If R is saturated then $\prec_R$ is arborescent. Any J-proof net can be saturated.*

Proof. It is easy to check that a J-proof net is a balanced polarized graph; then the proof follows from the strong arborisation lemma. □

Now we deal with three standard results one usually has on proof nets: we give an immediate proof of the usual splitting Lemma, we prove that the sequentialization we have defined is correct w.r.t. Definition 10 and we get rid of the Mix rule.

The novelty here is the argument: when adding jumps, we gradually transform the skeleton of a graph into a tree. We observe that some properties are invariant under the transformation we consider: adding jumps and removing transitive edges. Our argument is always reduced to simple observations on the final tree (the skeleton of $Sat(R)$), and on the fact that each elementary graph transformation preserves some properties of the nodes.

Splitting

We observe that given a d.a.g., adding edges, or deleting transitive edges, preserves connectedness. The following properties are all immediate consequences of this remark.

Lemma 20 *(i) Two nodes in a d.a.g. G are connected iff they are connected in the skeleton of G.*

(ii) Given a J-proof net R, if two nodes are connected in R, then they are connected in $Sat(R)$.

(iii) If R is connected as a graph so are $Sat(R)$ and $Sk(Sat(R))$.

The above lemma allows us to give a simple proof of a standard result, the Splitting Lemma, which we state below.

Definition 21 (Splitting) *Let G be a d.a.g., c a root, and $b_1, \ldots, b_n$ the nodes which are immediately above c. We say that the root c is* ***splitting*** *for G if, when removing c, any two of the nodes b_i, b_k become not connected.*

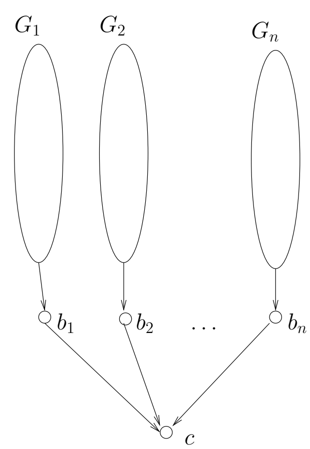


Figure 2.7: An example of splitting node

Remark 22 *It is immediate that if R is a J-proof net without negative conclusions, and c is splitting, the removal of c splits R into n disjoint components $R_1, \ldots, R_n$, and each component is a J-proof net.*

Lemma 23 (Splitting positive lemma) *Let R be a J-proof net without negative conclusions, and $Sat(R)$ a saturation such that $Sk(Sat(R))$ is a tree; the minimal link c of $Sat(R)$ (i.e. the root of $Sk(Sat(R))$) is splitting for R.*

Proof. Observe that c is obviously splitting in the skeleton of $Sat(R)$, because c is the root of a tree. Hence it is splitting in $Sat(R)$, as a consequence of Lemma 20, (i). Similarly, c must be splitting in R, as a consequence of Lemma 20, (ii). $\square$

Sequentialization Is Correct

Proposition 24 *Let R be a proof-net of conclusion Γ. For any saturation $Sat(R)$ of R, if $\pi = \pi^{Sat(R)}$ then $(\pi)^* = R$. Any proof net is sequentializable.*

Proof.

The proof is by induction on the number of links of R.

1. $n = 1$: then R consists of a single Axiom link of conclusions $P, P^\perp$, and π is the corresponding Axiom rule $\dfrac{}{\vdash P,\ P^\perp}$;
2. $n > 1$: if R has a terminal negative link c then $Sk(Sat(R))$ is a tree with c as root; observe that the last rule r of π is the rule which correspond to the root c. Assume that c is a $-$ link of conclusion $\parr(P_1, \ldots, P_n)$. We call R_0^J the saturated J-proof net of conclusion $\Gamma, P_1, \ldots, P_n$ obtained erasing c from $Sat(R)$; the forest obtained erasing c from $Sk(Sat(R))$ it clearly equal to $Sk(R_0^J))$, so by proposition 17 we associate to R_0^J a proof π_0. We call R_0 the subnet of conclusion $\Gamma, P_1, \ldots P_n$, obtained by removing c from R. Now, $R_0^J = Sat(R_0)$, so by induction hypothesis $\pi_0^* = R_0$. By applying the $-$ rule r of conclusion $\vdash \Gamma, \parr(P_1, \ldots, P_n)$ to the proof $\dfrac{\pi_0}{\vdash \Gamma, P_1, \ldots, P_n}$, we get a proof which is equal to π and such that that $R = \pi^*$.

   Otherwise by lemma 16, all conclusions of R are positive; we reason by cases, depending if $Sk(Sat(R))$ is a tree or a forest:

   - $Sk(Sat(R))$ is a tree with a $+$ link c of conclusion $\otimes(N_1, \ldots, N_n)$ as root; observe that the last rule r of π is the rule which correspond to the root c. By erasing c from $Sat(R)$ we get $R^J{}_1, \ldots, R_n^J$ saturated J-proof nets of conclusions respectively $\Gamma_1, N_1 \ldots \Gamma_n, N_n$. Erasing the root c in $Sk(Sat(R))$ we get n trees, such that each tree is the skeleton $Sk(R_i^J)$ of an $R^J{}_i$; let us call π_i the proof associated to each R_i^J by proposition 17.

     By the splitting lemma, c is splitting in R; let $R_1, \ldots, R_n$ be the n sub nets of conclusions respectively $\Gamma_1, N_1 \ldots \Gamma_n, N_n$, obtained by removing c from R. Now for each R_i^J, $R_i^J = Sat(R_i)$ so by induction hypothesis $\pi_i^* = R_i$; by applying the $+$ rule r of conclusion $\vdash \Gamma_1, \dots \Gamma_n, \otimes(N_1, \dots N_n)$ to the proofs $\dfrac{\pi_i}{\vdash \Gamma_i, N_i}$, we get a proof which is equal to π and such that that $R = \pi^*$.

- Otherwise, $Sk(Sat(R))$ is a forest, and by lemma 20 each tree corresponds to a different connected component of $Sat(R)$, and to a different sub-net of R; we conclude by applying induction hypothesis on them, followed by a sequence of Mix rules.

□

Connectedness

We now deal with a more peculiar notion of connectedness, to get rid of the Mix rule, as is standard in the theory of proof-nets.

Definition 25 (Correction graph) *Given a J-proof net R (resp. its skeleton $Sk(R)$), a* ***switching*** *s is the choice of an incident edge for every negative link of R (resp. $Sk(R)$); a* ***correction graph*** *$s(R)$ (resp. $s(Sk(R))$) is the graph obtained by erasing the edges of R (resp. of $Sk(R)$) not chosen by s.*

Definition 26 (s-connected) *A J-proof net R is s-connected if given a switching of R, its correction graph is connected.*

Remark 27 *We only need to check a single switching. The condition that a proof structure has not switching cycles is equivalent to the condition that* all *correction graphs are acyclic.*

A simple graph argument shows that assuming that all correction graphs are acyclic, if for a switching s the correction graph $s(R)$ is connected, then for all other switching s', $s'(R)$ is connected.

Proposition 28 *If R is s-connected, then the skeleton of $Sat(R)$ is a tree which only branches on positive nodes (i.e., each negative link has a unique successor).*

Proof. First we observe that:

- any switching of R is a switching of $Sat(R)$, producing the same correction graph. Hence if R is s-connected, $Sat(R)$ is s-connected.
- Given a J-proof net R, any switching of its skeleton is also a switching of R, because the skeleton is obtained by erasing the edges which are transitive. A transitive edge can be premise only of a negative node.

As a consequence, any switching of $Sk(Sat(R))$ induces a correction graph which is a correction graph also for $Sat(R)$ and hence is connected (so $Sk(Sat(R))$ must be a tree). Moreover, we observe that there is only one possible switching. In fact, since $Sk(Sat(R))$ is a tree, we cannot erase any edge and still obtain a graph which is connected; so each negative link has a unique successor.

□

From Proposition 17, it follows that

Proposition 29 *If R is s-connected, and $Sat(R)$ a saturation, we can associate to it a proof $\pi^{Sat(R)}$ which does not use the Mix rule.*

Partial sequentialization and Desequentialization

Our approach is well suited for *partially* introducing or removing sequentiality, by adding (deleting) a number of jumps.

Actually, it would be straightforward to associate to a sequent calculus proof π a saturated J-proof net. In this way, to π we could associate either a maximal sequential or a maximal parallel J-proof net.

Given a J-proof net R, let us indicate with $Jump(R)$ (resp. $DeJump(R)$) a J-proof net resulting from (non deterministically) introducing (resp. eliminating) a number of jumps in such a way that every time the order associated increases (decreases).

The following proposition applies to J-proof nets of *any* degree of sequentiality.

Proposition 30 (Partial sequentialization/desequentialization.) *Let R, R' be J-proof nets:*

- *if $R' = Jump(R)$ then there exists an $R'' = DeJump(R')$ such that $R'' = R$;*
- *if $R' = Dejump(R)$ then there exists an $R'' = Jump(R')$ such that $R'' = R$.*

Proof. Immediate, since we can reverse any step... □

2.2.5 Cut

Sequentialize with cuts

We have already observed that J-proof nets are balanced polarized graphs only if there are not *cut*-links. In order to extend our proof of sequentialization in presence of *cut*-links, using the strong arborisation lemma, we have

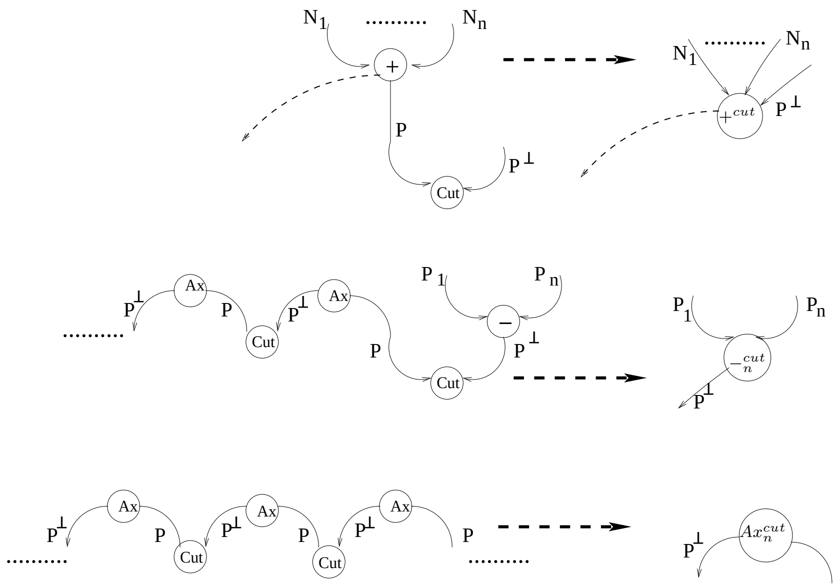


Figure 2.8: Turning R into R^+.

to establish a correspondence between J-proof nets with cuts and balanced polarized graphs.

Hence we define a procedure to turn a J-proof net R with *cut*-links into a balanced polarized graph R^{pol}, following these three steps on the links of R, depicted in fig. 2.8 :

1. if c is a *cut* link of R, whose premises are typed by $P, P^{\perp}$ and such that the link whose conclusion is P is a $+$ link b, we substitute b and c with a single positive node b' in R^{pol}, labeled by $+^{cut}$;

2. if c is a *cut* link of R whose premises are typed by $P, P^{\perp}$ and such that the link whose conclusion is P is an ax link a and the link whose conclusion is $P^{\perp}$ is a $-$ link b, we substitute a and b with a single negative node b' in R^{pol}, labeled by $-^{cut}_{1}$; if b' is still just above a *cut* link whose positive premise is the conclusion of an ax-link, we iterate the procedure until we get a $-^{cut}_{n}$ node in R^{pol};

3. if c is a *cut* link of R whose premises are typed by $P, P^{\perp}$ and such that the link whose conclusion is P is an axiom link a and the link whose conclusion is $P^{\perp}$ is an axiom link b, we substitute a and b with a single initial node b' of R^{pol}, labeled by ax^{cut}_{1}; if b' is still just above a cut link whose positive premise is the conclusion of an ax-link, we iterate the procedure until we get a ax^{cut}_{n} node in R^{pol}.

Now R^{pol} is a balanced polarized graph, so we can apply the arborisation lemma; when we get a saturated graph $Sat(R^{pol})$, it is easy to check that the graph obtained by reverting each $+^{cut}$, ax_n^{cut} and $-_n^{cut}$ node into the former links of R is a saturated J-proof net $Sat(R)$.

We can now prove the extension in presence of *cut* links of proposition 17; before doing that we must prove the extension of lemma 16 in presence of *cut* links.

Lemma 31 *If R is a J-proof net with more than one link such that R^{pol} has no negative roots, then all the conclusions of R are positive.*

PROOF.

Suppose there is a negative conclusion of R which is the conclusion of an ax link a. If R is composed by a single connected component, there must exist a link b a premise of which is the positive conclusion of a, and b is either a negative link, either a *cut*-link. If b is a negative link by point 1) of the positivity condition on b, there must exists also a positive link d such that there is a path $\langle b, a, \ldots, d\rangle$, but this is impossible, since a is a terminal link : contradiction. Now suppose b is a cut link: the other premise of b is negative and is the conclusion of a link c which is either a negative link, either the conclusion of an ax link. Now if c is a negative link, by construction of R^{pol} the link corresponding to c in R^{pol} is a negative root $-_1^{cut}$, contradiction. Otherwise c is an axiom link: then we iterate the procedure, until we find a contradiction. If R is composed by more than one connected component we just adapt the the proof of lemma 16, reasoning as above.

□

Proposition 32 (A forest is a sequent calculus proof) *Let R be a J-proof net (possibly with cut-links) of conclusions $A_1, \ldots, A_n$ and such that $Sk(R^{pol})$ is a forest.*

We can associate to R a sequent calculus proof π^R of conclusion $\vdash A_1, \ldots, A_n$ in MHS + Mix.

PROOF. The proof is by induction on the number of nodes in $Sk(R^{pol})$; using lemma 31, and having as reference the graph R^{pol}, the only difference with respect to the proof of 17 is when the root c of $Sk(R^{pol})$ is a $+^{cut}$, an ax_n^{cut} or a $-_n^{cut}$ node. Then there are three possibilities:

1. the root c of $Sk(R^{pol})$ is an ax_n^{cut} node whose edges are labelled by the formulas $P, P^\perp$; then R is composed by $n+1$ ax links of conclusions $P, P^\perp$, connected together by n *cut*-links: π^R is the proof obtained by applying to $n+1$ axiom rules of conclusion $P, P^\perp$ n consecutive *cut* rules.

2. the root c of $Sk(R^{pol})$ is a $-_n^{cut}$ node; by erasing c we obtain one forest $Sk(R_0^{pol})$. To this forest correspond a subnet R_0 of R with conclusion $\Gamma, P_1, \ldots, P_n$; by induction we associate a proof π^{R_0} to $Sk(R_0^{pol})$. π^R is

$$\dfrac{\dfrac{\pi_0}{\vdash P, P^\perp} \qquad \dfrac{\dfrac{\pi^{R_0}}{\vdash \Gamma, P_1, \ldots, P_n}}{\vdash \Gamma, \parr(P_1, \ldots, P_n)}(-)}{\vdash \Gamma, P^\perp}(cut)$$

, with $P^\perp = \parr(P_1, \ldots, P_n)$ and where π_0 is the proof obtained by applying to n axiom rules of conclusion $P, P^\perp$ $n-1$ consecutive *cut* rules.

3. the root c of $Sk(R^{pol})$ is a $+^{cut}$-node: by erasing c we obtain $n+1$ trees $Sk(R_0^{pol}), Sk(R_1^{pol}), \ldots, Sk(R_n^{pol})$. To each tree among $Sk(R_1^{pol}), \ldots, Sk(R_n^{pol})$ correspond a subnet R_i of R with conclusions Γ_i, N_i, for $i \in \{1, \ldots, n\}$; to $Sk(R_0^{pol})$ correspond a subnet R_0 of R with conclusion $\Delta, P^\perp$, where $P = \otimes(N_1, \ldots, N_n)$. By induction we get $n+1$ proofs $\pi^{R_0}, \pi^{R_1}, \ldots, \pi^{R_n}$. π^R is

$$\dfrac{\dfrac{\pi^{R_0}}{\vdash \Delta, P^\perp} \qquad \dfrac{\dfrac{\pi^{R_1}}{\vdash \Gamma_1, N_1} \quad \cdots \quad \dfrac{\pi^{R_n}}{\vdash \Gamma_n, N_n}}{\vdash \Gamma_1, \ldots, \Gamma_n,\ \otimes(N_1, \ldots, N_n)}}{\vdash \Gamma_1, \ldots, \Gamma_n, \Delta}(cut)$$

whose last rule is a *cut* rule on $P, P^\perp$.

$\square$

All results in subsection 2.2.4, can be straightforwardly generalized in presence of *cut* links.

When we will represent a J-proof net R with cuts, in order to remind the order associated to R^{pol}, we will revert the orientation of the positive premise of a cut link, as in figure 2.9,2.10.

Cut elimination

A proof structure without cut-links is called **cut-free**. **Cut reduction rules** are graph rewriting rules which locally modify a J-proof structure R, obtaining a J-proof structure R' with the same conclusions.

There are two kinds of cut-elimination steps, the $+/-$ step and the *ax* step, depicted in Fig. 2.9 and Fig. 2.10; we denote by $R \rightsquigarrow R'$ the relation "R reduces to R'".

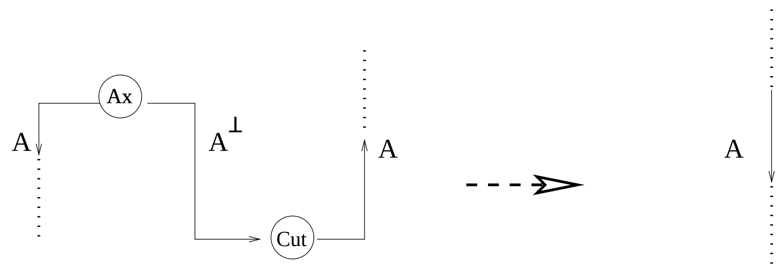


Figure 2.9: *ax* cut reduction.

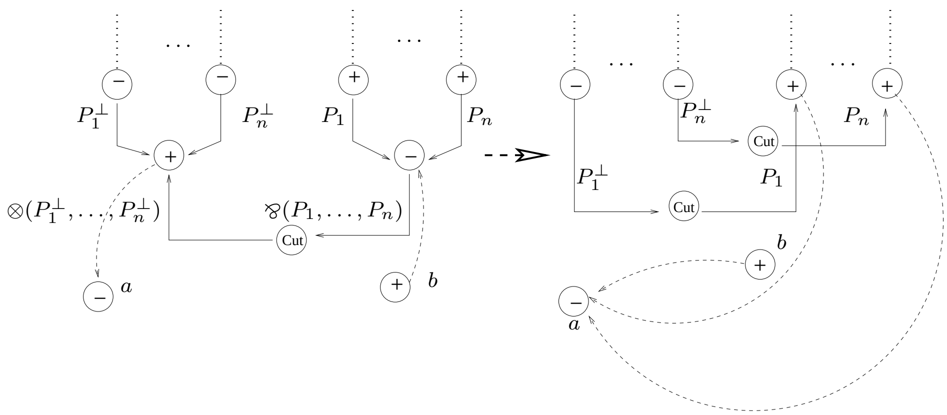


Figure 2.10: $+/-$ cut reduction.

With respect to the rewriting rules $+/-$ and ax, reduction enjoys the following properties:

Theorem 33 (Preservation of correctness) *Given a J-proof structure R, if R is a J-proof net and $R \rightsquigarrow R'$, then R' is a J-proof net.*

Proof. Checking the preservation of switching acyclicity is a straightforward generalization of the proof given by Girard in [Gir87] for MLL; we only have to verify that the jumps added in the $+/-$ step do not introduce cycles. Consider fig 2.10; if $R \rightsquigarrow R'$ with a step $+/-$ and $b \to a$ is a jump added in the step, suppose $b \to a$ creates a cycle: then there is a switching path r in R' from a to b which does not use any switching edge of a. If r does not cross any of the *cut* links generated by the $+/-$ step then r belongs to R too, and then we have cycle also in R. Otherwise, let c be the first *cut* link generated by the the $+/-$ step that we meet following r from a to b, and d the node which precedes c in r; obviously the conclusion of d is a premise of c. Then consider the subpath r' of r from a to d in R'; it is a path of R too, so looking at fig. 2.10 it is easy to conclude that there is a cycle in R too, contradiction.

Now we prove the preservation of point 1) of the positivity condition: in the case of an ax step, the result is obvious. Concerning the $+/-$ step, consider a J-proof net R with a *cut* link c between a positive link a and a negative link b, such that a is the justifier of a negative link d, and R reduces to R' by reducing c: we prove that d has still a justifier in R'. By reducing c, either d becomes connected in R' with a positive link b' whose conclusion is a premise of b in R, and then b' becomes the justifier of d in R', either d becomes connected in R' with an ax link a conclusion of which is a premise of b in R; then by positivity condition on b in R, b has a justifier b' in R, which becomes the justifier of d in R'.

To prove the preservation of point 2) of positivity condition, let us consider a J-proof net R with a *cut* link c between a positive link a and a negative link b such that by reducing c with a $+/-$ step we get a J-proof structure R' with more connected components than R: we prove that each connected component of R' contains at least one positive link.

If the premises of b are conclusions of positive links, then the result is immediate; if a premise of b is the conclusion of an axiom link d, then by positivity condition on R, b has a justifier b' which is connected with d in R; then by reducing c, b' will be in the same connected component as d in R'. The case of the ax step is obvious. □

Theorem 34 (Strong normalization) *For every J-proof net R, there is no infinite sequences of reductions $R \rightsquigarrow R_1 \rightsquigarrow R_2 \ldots \rightsquigarrow R_n \ldots$*

Proof. By the fact that at each step the number of links decreases, and that we never reach a **deadlock** (that is a *cut*-link whose premises are conclusions of the same ax-link) during reduction, by theorem 108 (see [Gir87]). □

Theorem 35 (Confluence) *For every J-proof net R_1, R_2 and R_3, such that $R_1 \rightsquigarrow R_2$ and $R_1 \rightsquigarrow R_3$, there is a J-proof net R_4, s.t. $R_2 \rightsquigarrow R_4$ and $R_3 \rightsquigarrow R_4$.*

Proof.

It easily follows from confluence of usual multiplicative proof nets (see [Gir87]) and from the simple observation that if $R \rightsquigarrow R'$, the displacing of a jump after a $+/-$ step does not influence the other jumps of R'.

□

Jumps and η-expansion

It is well known that the *Curry-Howard isomorphism* relates the β-reduction of the λ-calculus to the cut reduction in the proof nets; the η-expansion corresponds to a rewriting rule of proof nets too, i.e. to the reduction of complex axioms in simpler ones. Let us define the η expansion of an *ax* link as depicted in fig 2.11.

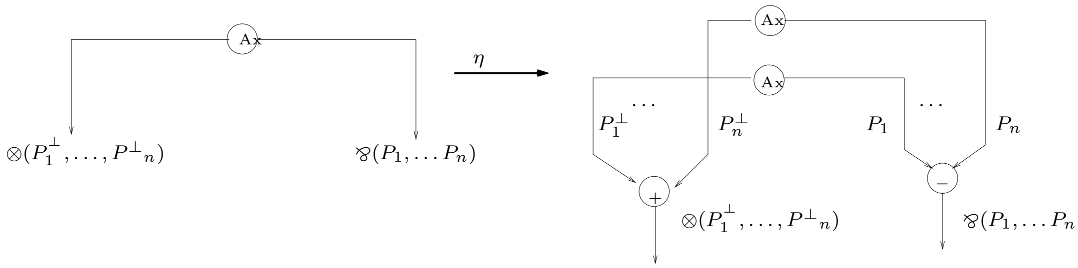


Figure 2.11: η expansion.

From a computational point of view, we should expect that in a J-proof net the result of the reduction of cut against an *ax* link , and against its η expansion was the same.

This is not the case, as we can see in figure 2.12, 2.13.

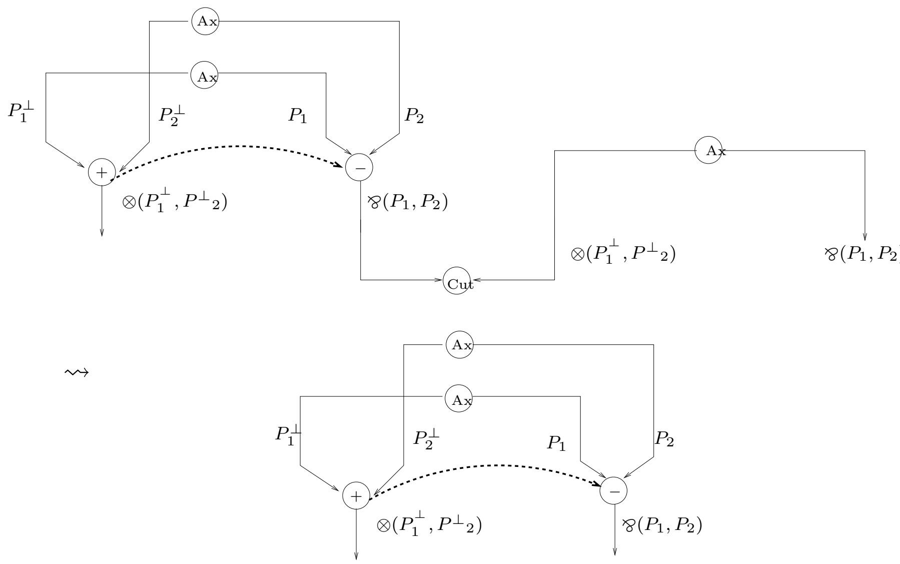


Figure 2.12:

In order to avoid this incongruousness, we must modify the positivity condition in the following way (as shown in fig 2.14), and consequently the η rewriting step:

Definition 36 (Extended positivity condition) *A J-proof structure R satisfy the* ***extended positivity*** *condition if and only if*

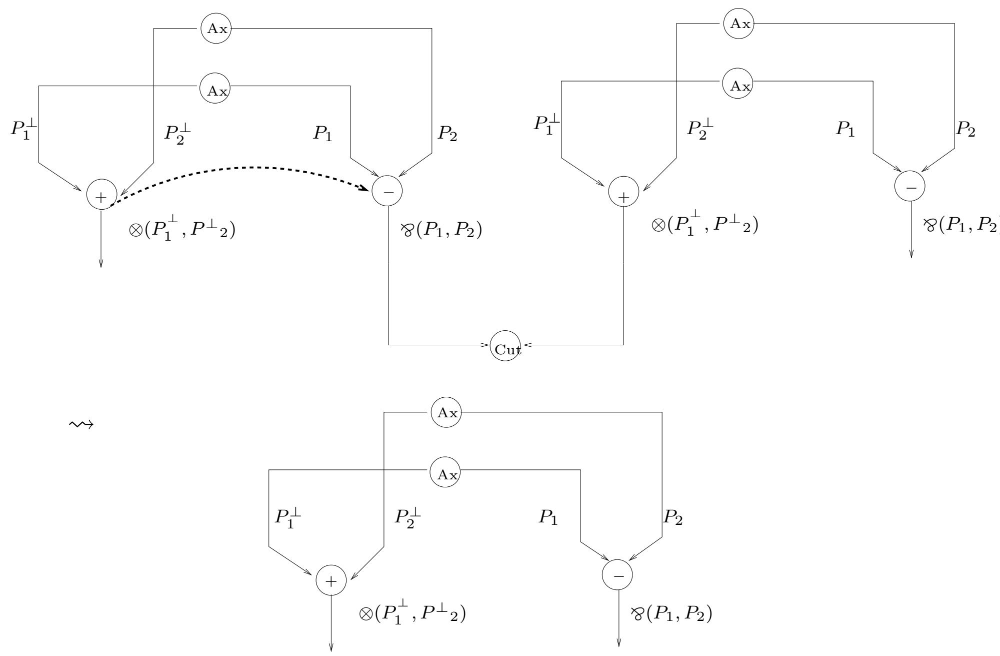


Figure 2.13:

1. *For every* − *link* b*, such that a premise of* b *is a conclusion of an ax link* a*, there exists a positive link* c *(called* ***justifier*** *of* b*) and a path* $\langle b, a, \dots, c\rangle$ *from* b *to* c *which crosses only cut and ax links; moreover,* **b** ***jumps on*** **c** *in* R*.*

2. *if* R *is composed by more than one connected component, then each component contains at least one positive link.*

It is easy to verify that the extended positivity condition is stable under cut reduction.

2.3 A denotational semantics for J-proof nets

In this section we provide a denotational semantics of J-proof nets, which is a variation of standard relational semantics based on the notion of **pointed set**. The aim is to refine the relational model, in order to be able to semantically characterize sequential order, in our case *jumps* (which usually are not captured by relational semantics) ; actually, our approach is inspired by [Bou04], where a step is made in the direction of developping a unified framework for both *static* (sets, coherence spaces, etc) and *dynamic* (games) denotational semantics.

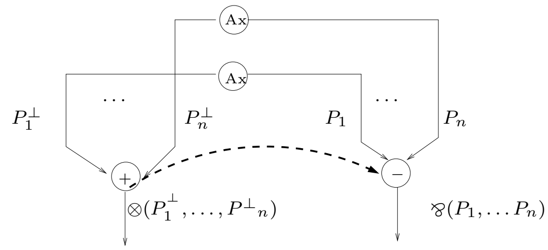


Figure 2.14:

By now we will denote sets by $\mathsf{A}, \mathsf{B}, \mathsf{C}, \ldots$ and elements of a set by $\mathsf{a}, \mathsf{b}, \mathsf{c}, \ldots$.

2.3.1 Pointed sets.

A **pointed set** A^* is given by a set $\mathsf{A} \cup \{0_{\mathsf{A}^*}\}$ where 0_{A^*} is a distinguished object which does not belong to A; this object is called the **point** of A^*

The **product** $\mathsf{A_1}^* \circledast \ldots \circledast \mathsf{A_n}^*$ of n pointed sets $\mathsf{A_1}^*, \ldots, \mathsf{A_n}^*$ is the pointed set $\mathsf{A}^* \times \ldots \times \mathsf{A_n}^* \ \cup \{0_{\mathsf{A}^* \circledast \mathsf{A_n}^*}\}$ whose elements are the elements of the cartesian product $\mathsf{A_1}^* \times \ldots \times \mathsf{A_n}^*$ (resp. the set of singletons of elements of $\mathsf{A_1}^*$ if $n = 1$) together with a distinguished *fresh* object $0_{\mathsf{A}^* \circledast \ldots \circledast \mathsf{A_n}^*}$ which does not belong to $\mathsf{A}^* \times \ldots \times \mathsf{A_n}^*$.

For simplicity's sake we will often refer to the point 0_{A^*} of a pointed set A^* simply as 0.

The formulas of MHS are interpreted in the following way:

- an atomic formula X is interpreted by a pointed set X^*;
- a positive formula $\otimes(P_1, \ldots, P_n)$ (resp. a negative formula $⅋(N_1, \ldots, N_n)$) is interpreted by $\mathsf{P}_1^* \circledast \ldots \circledast \mathsf{P}_n^*$ (resp. $\mathsf{N}_1^* \circledast \ldots \circledast \mathsf{N}_n^*$);

Given a J-proof structure R, we define the interpretation of R in pointed sets semantics, and we denote it by $[\![R]\!]$; in case R has no conclusions, we let $[\![R]\!]$ be undefined. Otherwise, let x_1 of type $C_1, \ldots, x_n$ of type C_n be the conclusions of R; $[\![R]\!]$ is a subset of $\mathsf{C}_1^* \circledast \cdots \circledast \mathsf{C}_n^*$, which we define using the notion of **experiment**. The experiments have been introduced by Girard in [Gir87], and extensively studied in [TdF00] by Tortora de Falco.

Definition 37 (Experiments) *Let R be a J-proof structure and e an application associating with every edge a of type A of R an element of A^*; e is an **experiment** of R when the following conditions hold:*

- *if x, y are the conclusions of an ax link then $e(x_1) = e(x_2)$;*

- *if x, y are premises of a cut link with premises x and y, then $e(x) = e(y)$;*
- *if x of type $\parr(A_1, \ldots, A_n)$ (resp. $\otimes(A_1, \ldots, A_n)$) is the conclusion of a negative (resp. positive) link with premises x_1 of type $A_1, \ldots, x_n$ of type A_n and there exist an $i \in \{1, \ldots, n\}$ such that $e(x_i) \neq 0_{\mathsf{A}_i^*}$, then if $e(x_1) = \mathsf{a}_1, \ldots e(x_n) = \mathsf{a}_n$, $e(x) =< \mathsf{a}_1, \ldots, \mathsf{a}_n >$; otherwise either $e(x) =< 0_{\mathsf{A}_1^*}, \ldots, 0_{\mathsf{A}_n^*} >$ either $e(x) = 0_{\mathsf{A}_1^* \circledast \ldots \circledast \mathsf{A}_n^*}$;*
- *if a is a positive link of conclusion x of type A and b is a negative link of conclusion y of type B, and b jumps on a, then if $e(x) \neq 0_{\mathsf{A}^*}$ then $e(y) \neq 0_{\mathsf{B}^*}$.*

If the conclusions of R are the edges $x_1, \ldots, x_n$ of type respectively $A_1, \ldots, A_n$ and e is an experiment of R such that $\forall i \in \{1, \ldots, n\}$ $e(x_i) = a_i$ then we shall say that $< \mathsf{a}_1, \ldots, \mathsf{a}_n >$ is the **conclusion** or the **result** of the experiment e of R, and we will denote it by $|e|$. The set of the results of all experiments on R is the interpretation $[\![R]\!]$ of R.

In the following proposition we prove that pointed sets semantics is stable under cut reduction:

Proposition 38 *If R is a J-proof net, and $R \rightsquigarrow R'$, then $[\![R]\!] = [\![R']\!]$.*

Proof.

If $R \rightsquigarrow R'$ with an ax step, the result is trivial. Having as reference fig. 2.10 let us suppose that $R \rightsquigarrow R'$ with a $+/-$ step reducing a cut in R between a $+$ link a of conclusion x of type $\otimes(N_1, \ldots, N_n)$, and a $-$ link b of conclusion y of type $\parr(P_1, \ldots, P_n)$; we denote the edges of type $P_1, \ldots, P_n$ (resp. $N_1, \ldots, N_n$) by $y_1, \ldots, y_n$ (resp. $x_1, \ldots, x_n$).

Suppose that b jumps on a positive link c of typed conclusion z and that a negative link d of conclusion w jumps on a.

We must show that for every experiments e on R', there is an experiment e' of R with the same result, and *vice versa*.

The delicate part is the one dealing with jumps, the rest of the proof being the same as the one given in [Gir87]; the cases to check are the followings:

If e is an experiment of R' such that $e(w) = 0$, then $e(z) = 0$, $e(y_1) = 0, \ldots, e(y_n) = 0$ we can build an experiment e' of R with the same values on the same edges by assigning $e(x) = 0$ and $e(y) = 0$; if in R' $e(w) \neq 0$, $e(z) = 0$ $e(y_1) = 0, \ldots, e(y_n) = 0$, we can build an experiment e' of R with the same values on the same edges by assigning $e(y) =< 0_{\mathsf{P}_1^*}, \ldots, 0_{\mathsf{P}_n^*} >$ and $e(x) =< 0_{\mathsf{N}_1^*}, \ldots, 0_{\mathsf{N}_n^*} >$.

□

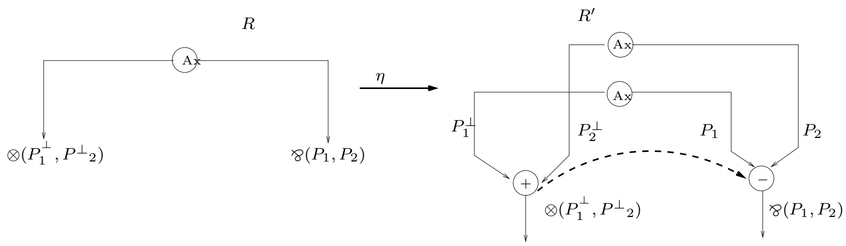


Figure 2.15:

Remark 39 (Pointed sets and η-expansion) *We observe that pointed set semantics is not stable under the η-expansion rewriting step; let us consider the result of the experiments on the J-proof structure R in fig. 2.15 and on its η-expansion R'; for the experiment e of R' with result $< 0_{\mathsf{P}_1^{\perp *} \circledast \mathsf{P}_n^{\perp *}}, < 0_{\mathsf{P}_1^*}, 0_{\mathsf{P}_2^*} >>$ there is no corresponding experiment with the same result in R.*

2.3.2 Injectivity

Semantic injectivity has been studied in the setting of linear logic mainly by Tortora (see [TdF00] and Pagani in [Pag06]; however, it is a traditional question in the denotational semantics of λ-calculus; *Statman theorem*, for example, states that the relational model is injective for the simply typed λ-calculus ([Sta83]).

We remark also that the notion of semantic injectivity is deeply related with the one of *syntactical separability*, stated in the *Böhm theorem* for pure λ-calculus ([Boh68]): if t, t' are two closed λ-terms, such that t is not $\beta\eta$ equivalent to t', then there are $u_1, \ldots, u_n$ λ-terms such that $tu_1 \ldots u_n \rightarrow_\beta 1$ and $t'u_1 \ldots u_n \rightarrow_\beta 0$; that is, t and t' compute two different functions on the λ-terms, and $u_1, \ldots, u_n$ are arguments on which t and t' give different values.

Furthermore, syntactical separability is also one of the main properties of designs in ludics ([Gir01]) and of L-nets too (see [FM05]).

For an extensive analysis of the relation between syntactical separability and semantic injectivity in the framework of linear logic, we refer to [Pag06].

In this subsection we study the injectivity of pointed set semantics with respect to J-proof nets. Given any two cut-free J-proof nets R, R', we say that R and R' are **syntactically equivalent** when $R = R'$; we consider this equality up to *transitive jumps* (a transitive jump of a J-proof net R is a jump which is a transitive edge in R); we say that R and R' are **semantically equivalent** when $[\![R]\!] = [\![R']\!]$.

We will prove that pointed sets semantics is **injective** with respect to J-proof nets, that is for any two J-proof nets R, R' , if $[\![R]\!] = [\![R']\!]$ then

$R = R'$.

The proof follows the lines of the proof of injectivity of relational semantics with respect to MLL proof nets provided in [TdF03a], with some details more to take into account jumps.

Definition 40 (Relational result) *Let R be a J-proof structure and $|e|$ the result of an experiment on R; $|e|$ is **relational** it does not contain any occurrence of* 0.

The set of relational results of experiments on a J-proof structure R is called the **relational part** of $[\![R]\!]$; we will denote it by $[\![R]\!]^{Rel}$.

Remark 41 *Given two J-proof structure R, R', if $[\![R]\!] = [\![R']\!]$ then $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$.*

Remark 42 *Let R be a J-proof structure and e, e' be two experiments of R. If $|e| = |e'|$, then $e = e'$; in other words an experiment is completely determined by its result.*

Definition 43 (Injective result) *Let R be a J-proof structure and $|e|$ be a relational result of an experiment on R; $|e|$ is **injective** when in $|e|$ does not occur two times a same element of a pointed set X^* interpreting an atomic formula.*

Given a J-proof net R, we denote by R^- the proof net obtained by erasing all the jumps of R.

Lemma 44 *Let R_0 be a J-proof net without jumps; then for all J-proof nets R, such that $R^- = R_0$, given an element γ of $[\![R]\!]$ there exists a unique experiment e_0 of R_0 such that $|e_0| = \gamma$.*

Proof. The proof follows from the observation that for any J-proof net R, $[\![R]\!] \subseteq [\![R^-]\!]$, and from remark 42.

□

Given a J-proof net R we denote its η expansion by R^η.

Lemma 45 *Let R be a J-proof net and R' be an η expanded proof net without jumps with the same conclusions, such that $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$. Then $R^{\eta-} = R'$.*

Proof.

Since for any J-proof net R, $[\![R]\!]^{Rel} = [\![R^\eta]\!]^{Rel}$, the proof is a consequence of injectivity of relational semantics for (η expanded) proof nets given by Tortora de Falco in [TdF03a]; the proof uses the fact that an injective result (which always exists) in the interpretation of a proof net allows to completely determine the proof net modulo the η-expansion of the axioms.

□

Lemma 46 *Given a J-proof structure R, a positive link a with conclusion x and a negative link b with conclusion y, b jumps on a (eventually with a transitive jump) iff for all experiments e of R, $e(x) \neq 0 \Rightarrow e(y) \neq 0$.*

Proof. The proof is an easy consequence of definition of experiment. □

Theorem 47 (Injectivity) *Let R and R' be two cut-free J-proof nets with the same conclusions. If $[\![R]\!] = [\![R']\!]$ then $R = R'$.*

Proof.

$[\![R]\!] = [\![R']\!]$, so $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$.

Since $[\![R^{\eta -}]\!]^{Rel} = [\![R'^{\eta -}]\!]^{Rel}$, by lemma 45, $R^{\eta\ -} = R'^{\eta\ -}$.

Now, by remark 42 , given an element γ of $[\![R]\!]$ (resp. $[\![R']\!]$) there exists a unique experiment e of $R^{\eta\ -}$ (resp. of $R'^{\eta\ -}$ such that $|e| = \gamma$.

Starting from $R^{\eta\ -}$ (resp. $R'^{\eta\ -}$), we build a proof net R_1 (resp. R_2), eventually with non atomic axioms, in the following way: for any configuration of links as in fig 2.16, we check that for all elements $\gamma_1, \ldots, \gamma_n$ of $[\![R]\!]$ the unique experiment e_i of $R^{\eta\ -}$ (resp. $R^{\eta\ -}$) induced by γ_i assigns the same values to the edges x, y; if it is the case we substitute in $R^{\eta\ -}$ (resp. $R'^{\eta\ -}$) the configuration of fig 2.16 with an axiom link with conclusions x, y; otherwise we leave it as it is. Now $R_1 = R^-$ and $R_2 = R'^-$; since $[\![R]\!] = [\![R']\!]$ and $R^{\eta\ -} = R'^{\eta\ -}$, $R^- = R'^-$.

Now, by lemma 44 , given an element γ of $[\![R]\!]$ (resp. $[\![R']\!]$) there exists a unique experiment e of R^- (resp. of R'^-) such that $|e| = \gamma$.

Now, we build from R^- (resp. R'^-) a J-proof net R^J (resp. R'^J) in the following way; for any positive link a of typed conclusion x and for any negative link b of typed conclusion y, we check that for every element γ of $[\![R]\!]$, given the unique experiment e of R^- (resp. R'^-) induced by γ, $e(x) \neq 0 \Rightarrow e(y) \neq 0$; if it is the case we make b jump on a in R^- (resp. in R'^-). By lemma 46, $R^J = R$, and $R' = R'^J$; since $[\![R]\!] = [\![R']\!]$ and $R^- = R'^-$, $R = R'$.

□

The above result of injectivity allows also to semantically recognize if a given J-proof net R' is obtained from another J-proof net R by adding jumps on R; it is enough to check that $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$, so that $R^- = R'^-$, and that $[\![R']\!]^{Rel} \subseteq [\![R]\!]^{Rel}$, so that all the jumps of R are jumps of R'; if we add the remaining jumps of R' to R then we retrieve R'.

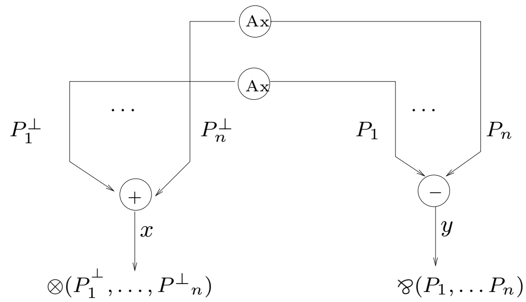


Figure 2.16:

2.4 J-proof nets and MLL

In this section we extend our proof of sequentialization with jumps to proof nets of MLL, by showing that both the splitting $\parr$ and splitting $\otimes$ lemmas (see [Gir87] and [Dan90]), which are two of the standard results used to prove sequentialization, are consequences of the weak arborisation lemma.

2.4.1 MLL proof nets

An **MLL proof structure** is a proof structure in the sense of definition 9, whose edges are labelled by MLL formulas (following the grammar we introduced in subsection 2.1), and whose typing respects the following constraints (see fig 2.17)

- the *ax*-link has two conclusions labeled by dual formulas, but no premises;
- the *cut*-link has two premises labeled by dual formulas but no conclusions;
- the $\parr$ link has n premises and one conclusion. If the i-th premise is labeled by the formula A_i for $i \in \{1, \ldots, n\}$ then the conclusion is labeled by $\parr(A_1, \ldots, A_n)$;
- the $\otimes$ link has n premises and one conclusion. If the i-th premise is labeled by the formula A_i for $i \in \{1, \ldots, n\}$ then the conclusion is labeled by $\otimes(A_1, \ldots, A_n)$.

As in the case of MHS, we can associate to an MLL proof π a **sequentializable** proof structure π^* by induction on the height of π.

We call $\mathbf{J^{MLL}}$**-structures** the generalization to MLL proof structures of J-proof structures. Exactly as we did in subsection 2.2.2, we can associate to a J^{MLL}-structure the structure of a graph with pairs $(R, App(R))$; then

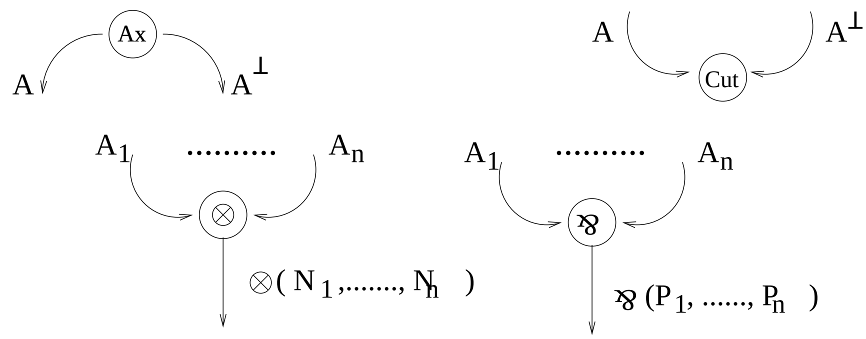


Figure 2.17: MLL links

we will call $\mathbf{J^{MLL}}$**-net** a switching acyclic J^{MLL}-structure. The notion of saturation, correction graph and s-connectedness, are directly retrieved from the ones of subsections 2.2.4, 2.2.4.

Remark 48 *Since in this section we are mainly interested in sequentialization, from now on, for sake of simplicity, we will consider only cut-free structures; we will also assume w.l.o.g., that all our proof structures have at most one terminal* $\parr$ *link (otherwise, we put together all* $\parr$ *conclusions by substituting all terminal* $\parr$*-links with a single one).*

2.4.2 Arborisation lemma and splitting lemmas

Lemma 49 *Let R be a saturated J^{MLL}-net R and a a $\parr$-link of R. If a conclusion of a node $b \in C_R(a)$ is a premise of a link $c \notin C_R(a)$, then c is a* $\parr$*-link.*

Proof.

Suppose c is a $\otimes$-link; then by saturation, making a jump on c would create a cycle: but then there is a switching path r from a to c which does not use any switching edge of a. Since $b \xrightarrow{+} a$ and $b \to c$ it is straightforward that the existence of r would induce a switching cycle in R, contradiction.

$\square$

Definition 50 (Splitting $\parr$**-link)** *Given a J^{MLL}-net R and a* $\parr$*-link a, we say that a is* ***splitting*** *for R if there exist two subgraphs G_1, G_2 of R, such that G_1 does not contain the conclusion of a, which is contained by G_2, and the only edge of R connecting a node in G_1 with a node in G_2 is the conclusion of a.*

Lemma 51 (Splitting $\parr$**-lemma)** *Given a saturated J^{MLL}-net R with at least one* $\parr$*-link, there exists a splitting* $\parr$*-link.*

Proof.

Let us consider a $⅋$-link a such that $C_R(a)$ is maximal with respect to inclusion among all the cones of the $⅋$-links in R; we prove that a is splitting in R.

Suppose that a conclusion of a link in $C_R(a)$ is the premise of a link (different from a) which does not belong to $C_R(a)$ so it must be a premise of another $⅋$-link b by lemma 49; now, $C_R(a) \cap C_R(b) \neq \emptyset$, and $b \notin C_R(a)$ so by the weak arborisation lemma $C_R(a) \subset C_R(b)$, contradicting the maximality of $C_R(a)$.

We observe also that if a $⅋$-link c different from a jumps on a link d which belongs to $C_R(a)$, then $c \in C_R(a)$; otherwise (that is if $c \notin C_R(a)$), $C_R(a) \cap emp_R(c) \neq \emptyset$ and $c \notin C_R(a)$ and again by the weak arborisation lemma $C_R(a) \subset C_R(c)$, contradicting the maximality of $C_R(a)$.

So, each conclusion of a link in $C_R(a)$ is a premise of a link in $C_R(a)$, or a premise of a, or a conclusion of R.

Now, if we consider the subgraph G of R which corresponds to $C_R(a)$, by the above observations, all the paths connecting a node in G with a node in $R \setminus G$ must use the conclusion of a; but then a is splitting for R.

□

Lemma 52 (Splitting $\otimes$ lemma) *Given a saturated J^{MLL}-net R which has only terminal $\otimes$ links, there exists at least one splitting $\otimes$ link.*

Proof.

The proof is an adaptation of a similar proof in [CF]; we reason by induction on the number n of $⅋$-links in R. If $n = 0$, then it is easy to check that all terminal $\otimes$ links must be splitting. If $n > 0$, we consider a $⅋$-link a such that $C_R(a)$ is maximal with respect to inclusion among all the cones of the $⅋$-links in R; the conclusion of a must be the premise of a terminal $\otimes$ link b, which is not above any other $⅋$-link of R, (otherwise $C_R(a)$ should not be maximal).

Now, reasoning as in the proof of lemma 51 we can conclude that any path from a node in $C_R(a)$ to any other node in $R \setminus C_R(a)$ must pass trough the conclusion of a and go up trough b ; but then if we disconnect the conclusion of a from b, and we modify consequently the type of the conclusion of b, we get two disjoint J^{MLL}-nets R_1, R_2, respectively containing a, b. By induction hypothesis on R_2, R_2 has a splitting $\otimes$ link c. Now either $c \neq b$, either $c = b$; in any case the splitting $\otimes$ link of R_2 is splitting also for R.

□

Remark 53 *Given an MLL proof net R and a saturated J^{MLL}-net $Sat(R)$, all splitting $⅋$-links and splitting $\otimes$-links of $Sat(R)$ are splitting also for R*

(as a consequence of lemma 20); since any MLL proof net can be saturated, lemma 52 and lemma 51 provides also a proof of the existence of both a splitting $\parr$ link and a splitting $\otimes$ link for MLL proof nets.

Theorem 54 *An MLL proof net is sequentializable.*

PROOF.

By induction on the number n of links of the proof net (for simplicity, we consider only the case where R is s-connected):

- if $n = 1$ then R is composed by a single axiom link, trivial;
- if $n > 1$ then we consider two sub-cases:
  - if R contains one terminal $\parr$ link, then we erase it getting a proof structure R'; R' is a proof net (erasing a $\parr$ link cannot create cycles); by induction hypothesis on R', we get a proof π' such that $\pi'^* = R'$; we add to π' a proper $\parr$ rule to get a proof π such that $\pi^* = R$;
  - Otherwise, R contains only terminal $\otimes$ links, so by the splitting $\otimes$ lemma and remark 53 there exist a splitting $\otimes$ link n in R: we erase it, getting n graphs $R_1, \ldots R_n$ which must be proof nets (otherwise there would be a cycle in R); by induction hypothesis on them we get n proofs $\pi_1, \ldots \pi_n$ such that $\pi_i^* = R_i$; we add to them a proper $\otimes$ rule to get a proof $\pi = R^*$.

□

2.4.3 Jumps and geography of subnets

Our object of study in this section will be the notion of **empire**, a class of subnets which has been introduced by Girard in [Gir87] (and further studied by Bellin and Van De Wiele in [BVDW95]), to prove sequentialization.

For simplicity's sake, in this subsection we will make the assumption that all J^{MLL}-nets we consider are s-connected.

A **sub-structure** of a J^{MLL}- net R is a subgraph R' of R which is a J-proof structure and such that for any link a of R which belongs to R', R' contains also all the premises of a in R.

A **sub-net** of a J^{MLL}-net R is a sub-structure which is a J^{MLL}-net.

Given a correction graph $s(R)$ of a J^{MLL}-net R, a path r $\langle a, .., b\rangle$ from a link a with typed conclusion x to a link b is said to **go up** from a, when it does not use neither x neither any untyped edge emergent from a; otherwise r is said to **go down**.

In the following definition 55, we will modify the standard definition of empires, in order to take into account jumps.

Definition 55 (Empire) *Let x be a typed conclusion of a link a in a J^{MLL}-net R: the* ***empire*** *of x in R (denoted $emp_R(x)$) is the smallest substructure of R closed under the following conditions:*

- *a belong to $emp_R(x)$;*
- *if b is a link of R connected with a with a path that goes up from x in all correction graphs of R, then $b \in emp_R(x)$.*

We call ***border*** *of $emp_R(x)$ the set of links $a_1, \ldots, a_n$ such that $a_i \in emp_R(x)$ and its conclusions either are conclusions of R either are premises of a link b which does not belong to $emp_R(x)$.*

Remark 56 *Of course for any typed edge x, $emp_R(x)$ is a sub-net of R. It is easy to check that if b is a link in the border of $emp_R(x)$, and one of its conclusion is premise of a link c such that c does not belongs to $emp_R(x)$, then c must be a $\parr$-link.*

Lemma 57 *Let R be a J^{MLL}-net and b a $\parr$-link with typed conclusion x: given the J^{MLL}-structure R' obtained by making b jump on another link a, then R' is a J^{MLL}-net iff $a \in emp_R(x)$.*

Proof.

We first prove the right to left direction: if $a \in emp_R(x)$ this means that for every correction graph of R (which is a correction graph of R' too), there is a path going up from x to a; if in R' there were a cycle, this means that in some correction graph of R' there would be a path from b to a which doesn't uses any switching edge of b, so it's also in a correction graph of R, but then we have a cycle in some correction graph of R, contradiction. To prove the other direction, we simply observe that if a does not belong to $emp_R(x)$ this means that there is at least one correction graph in R such that there is a path from b to a which goes down; but then if we make b jump on a we get a cycle.

□

Definition 58 (Kingdoms) *Let x be a typed edge of a proof net R; the* kingdom *of x in R (denoted $k_R(x)$) is the smallest sub-net of R having x as conclusion.*

Proposition 59 *Given a J^{MLL}-net R for any typed edge x, $k_R(x) \subseteq emp_R(x)$.*

Proof. Let us suppose that there is a link c which belongs to $k_R(x)$ and does not belong to $emp_R(x)$; so for some switching s in $s(R)$ there is a path r which goes down from x to c. But then if we consider the graph $s(k_R(x))$ (which is the correction graph obtained by restricting the switching s to $k_R(x)$) is not connected, and so $k_R(x)$ is not a subnet, contradiction. □

Definition 60 *Let R be a J^{MLL}-net and a a link of R with typed conclusion x; we denote by $C_R(x)$ the smallest sub-structure which contains only a and the links in $C_R(a)$.*

Remark 61 *In a J^{MLL}-net R given a typed edge x, by definition of sub-structure, $C_R(x) \subseteq emp_R(x)$ and $C_R(x) \subseteq k_R(x)$.*

The following proposition will allow us to characterize saturated J^{MLL}-nets by the shape of the empires of their $\parr$-links:

Proposition 62 *A J^{MLL}-net R is saturated, iff for any $\parr$ link a of typed conclusion x, $C_R(x) = k_R(x) = emp_R(x)$.*

Proof.

To prove the left to right direction, let us assume R saturated, and suppose $emp_R(x) \neq C_R(x)$; obviously $C_R(x) \subset emp_R(x)$. Now consider a link b of $emp_R(x)$ which isn't in $C_R(x)$: if there is not such an element, then $emp_R(x) = C_R(x)$; otherwise we make a jump on b , and by lemma 57 this doesn't create cycles so R is not saturated. Since $C_R(x) \subseteq k_R(x) \subseteq emp_R(x)$ and $emp_R(x) = C_R(x)$ it follows that $C_R(x) = k_R(x) = emp_R(x)$

To prove the other direction, if one makes a jump on a link which is in $emp_R(x)$ then it is transitive by definition of $C_R(x)$; if one makes a jump on a link which is outside $emp_R(x)$ then it creates a cycle by lemma 57.

$\square$

The following proposition is a standard property of empires in MLL proof nets, which, in the case of saturated J^{MLL}-nets becomes a simple consequence of the weak arborisation lemma.

Proposition 63 (Nesting of empires) *Given a saturated J^{MLL}-net R and two edges x, y resp. typed conclusions of two $\parr$ links a, b, either $emp_R(x)$ and $emp_R(y)$ are disjoint, either one is strictly included into the other.*

Proof. By lemma 62, $emp_R(x) = C_R(x)$, and $emp_R(y) = C_R(y)$; since R is a saturated polarized graph, the rest of the proof easily follows from the weak arborisation lemma. $\square$

Chapter 3

J-proof nets: additives

In this chapter we introduce and study J-proof nets for the hypersequentialized calculus. In section 3.1, as we did in chapter 1, we will present first $MALL$ grammar and sequent calculus, then HS; in section 3.2 we define J-proof nets for HS and in section 3.3 we prove the sequentialization theorem, while in section 3.4 we study cut-reduction on J-proof nets, keeping aside for the moment the question of the preservation under reduction of the correctness criterion. In section 3.5 we extend pointed set semantics to include additives, and we prove that the injectivity result of the previous chapter still holds in the additive setting; in the following section 3.6 we will use this result to prove that the correctness criterion is stable under reduction. Finally in the last section 3.7, we provide a classification of J-proof nets with respect to their degree of sequentiality, and we study the correspondence between them and some of the usual syntaxes of additive proof nets.

3.1 Hypersequentialized calculus

The scope of this section is to present the multiplicative-additive hypersequentialized calculus. As in section 2.1 we first present in subsection 3.1.1 a variant of usual $MALL$ grammar and calculus, where formulas are clustered modulo the usual associativity and distributivity isomorphisms of linear logic; then in subsection 3.1.2 we introduce the *hypersequentialized calculus* in order to restrict the scope to *focusing proofs*.

3.1.1 $MALL$

Definition 64 *Let $\mathcal{V} = \{X,\ Y,\ Z,\ldots\}$ be a countable set of propositional variables; the formulas of clustered $MALL$ are defined in the following way:*

- Atoms*: $X, Y, Z, \ldots$ and $X^{\perp}, Y^{\perp}, Z^{\perp}, \ldots$ are formulas of $MALL$*

- synchronous formulas*: let $\mathcal{N} = \{I, J, \ldots, K\}$ be a family of index sets, and $A_{i \in I}$ a set of atoms or asynchronous formulas indexed by some $I \in \mathcal{N}$; then $\oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(A_i))$ is a formula;*

- asynchronous formulas*: let $\mathcal{N} = \{I, J, \ldots, K\}$ be a family of index sets, and $A_{i \in I}$ a set of atoms or synchronous formulas indexed by some $I \in \mathcal{N}$; then $\&_{I \in \mathcal{N}}(\parr_{i \in I}(A_i))$ is a formula.*

Negation is defined as follows:

$$(\oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(A_i)))^{\perp} = \&_{I \in \mathcal{N}}(\parr_{i \in I}(A_i^{\perp}))$$

$$(\&_{I \in \mathcal{N}}(\parr_{i \in I}(A_i)))^{\perp} = \oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(A_i^{\perp}))$$

Note We remark the following facts:

- By $\&_{I \in \mathcal{N}}(\parr_{i \in I}(A_i))$ we indicate the connective which represent all possible combinations of the formulas $A_{i \in I \in \mathcal{N}}$ modulo the associativity and distributivity properties of usual $\parr$ and $\&$ connectives of LL; in case $\mathcal{N}$ is a singleton, we shall use the abbreviation $(\parr_{i \in I}(A_i))$; we denote the unary case of $(\parr_{i \in I}(A_i))$ as $\uparrow A_i$

- By $\oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(N_i))$ we indicate the connective which represent all possible combinations of the formulas $A_{i \in I \in \mathcal{N}}$ modulo the associativity and distributivity properties of the usual $\otimes$ and $\oplus$ connectives of LL; in case $\mathcal{N}$ is a singleton, we shall use the abbreviation $(\otimes_{i \in I}(A_i))$; we denote the unary case of $(\otimes_{i \in I}(A_i))$ as $\downarrow A_i$.

The calculus is the following:

$$\frac{}{\vdash A, A^{\perp}}\, ax \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\, cut$$

$$\frac{\vdash \Gamma_1,\ A_1 \qquad \ldots \qquad \vdash \Gamma_n, A_n}{\vdash \Gamma_1, \ldots, \Gamma_n, \oplus_{I \in \mathcal{N}}(\otimes_{i \in I} A_i)}\, (+, I) \qquad \frac{\vdash \Gamma, A_1^1 \ldots, A_{k_1}^1 \qquad \ldots \qquad \vdash \Gamma, A_1^n \ldots, A_{k_n}^n}{\vdash \Gamma, \&_{J \in \mathcal{N}}(\parr_{j \in J} A_i)}\, (-, \mathcal{N})$$

$MALL$ can be eventually enriched with the Mix rule:

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\, mix$$

3.1.2 From $MALL$ to HS

.

As we did for the multiplicative fragment, we restrict the grammar and calculus of $MALL$ in order to make proofs *alternating*, retrieving in this way the hypersequentialized calculus. The formulas of HS are obtained by the following restrictions on $MALL$ formulas:

$$\begin{array}{lcl} N & ::= & X^{\perp} \quad | \quad \&_{I\in\mathcal{N}}(\parr_{i\in I}(P_i)) \\ P & ::= & X \quad | \quad \oplus_{I\in\mathcal{N}}(\otimes_{i\in I}(N_i)) \end{array}$$

From now on, we will call the formulas in N *negatives* and the formulas in P *positives*.

The rules for proving sequents are the following:

$$\frac{}{\vdash X, X^{\perp}}\,ax \qquad\qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, A^{\perp}}{\vdash \Gamma, \Delta}\,cut$$

$$\frac{\vdash \Gamma_1,\ N_1 \qquad \dots \qquad \vdash \Gamma_n, N_n}{\vdash \Gamma_1, \dots, \Gamma_n, \oplus_{I\in\mathcal{N}}(\otimes_{i\in I} N_i)}\,(+,I) \qquad \frac{\vdash \Gamma, P_1^1 \dots, P_{k_1}^1 \qquad \dots \qquad \vdash \Gamma, P_1^n \dots, P_{k_n}^n}{\vdash \Gamma, \&_{J\in\mathcal{N}}(\parr_{j\in J} P_i)}\,(-,\mathcal{N})$$

$$\frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}\,mix$$

where $\Gamma, \Delta, \dots$ only contain positive formulas.

In the additive fragment, we restrict ourself to axioms introducing just atomic formulas; the reason behind this choice will become clearer in section 3.2.

Decomposing the additives. Before presenting J-proof nets, we want to give a first intuition about two fundamentals notions which come out dealing with additives: the notion of *slice* and the one of *superposition*.

An additive proof containing some & rules can be thought of as a superposition of multiplicative proofs, called *slices*.

Let us look for example at the following sequent proof π of
$\vdash \oplus(\otimes(X_1^{\perp}, X_2^{\perp}), \otimes(X_3^{\perp}, X_4^{\perp})),\ \&(\parr(X_1, X_2), \parr(X_3, X_4))$:

$$\frac{\dfrac{\dfrac{}{\vdash X_1, X_1^{\perp}}\,ax \qquad \dfrac{}{\vdash X_2, X_2^{\perp}}\,ax}{\vdash \oplus(\otimes(X_1^{\perp}, X_2^{\perp}), \otimes(X_3^{\perp}, X_4^{\perp})), X_1, X_2}\,(+,\{1,2\}) \qquad \dfrac{\dfrac{}{\vdash X_3, X_3^{\perp}}\,ax \qquad \dfrac{}{\vdash X_4, X_4^{\perp}}\,ax}{\vdash \oplus(\otimes(X_1^{\perp}, X_2^{\perp}), \otimes(X_3^{\perp}, X_4^{\perp})), X_3, X_4}\,(+,\{3,4\})}{\vdash \oplus(\otimes(X_1^{\perp}, X_2^{\perp}), \otimes(X_3^{\perp}, X_4^{\perp})), \&(\parr(X_1, X_2), \parr(X_3, X_4))}\,(-,\{\{1,2\},\{3,4\}\})$$

now, we choose a branch of each $(-,\mathcal{N})$ rule, in this case just one; by erasing the right branch we get the following derivation s_1:

$$\dfrac{\dfrac{\dfrac{}{\vdash X_1, X_1^\perp}\,ax \qquad \dfrac{}{\vdash X_2, X_2^\perp}\,ax}{\vdash \oplus(\otimes(X_1^\perp, X_2^\perp), \otimes(X_3^\perp, X_4^\perp)), X_1, X_2}\,(+,\{1,2\})}{\vdash \oplus(\otimes(X_1^\perp, X_2^\perp), \otimes(X_3^\perp, X_4^\perp)), \&(⅋(X_1, X_2), ⅋(X_3, X_4))}\,(-,\{\{1,2\},\{3,4\}\})$$

and by erasing the left branch, we get the following derivation s_2:

$$\dfrac{\dfrac{\dfrac{}{\vdash X_3, X_3^\perp}\,ax \qquad \dfrac{}{\vdash X_4, X_4^\perp}\,ax}{\vdash \oplus(\otimes(X_1^\perp, X_2^\perp), \otimes(X_3^\perp, X_4^\perp)), X_3, X_4}\,(+,\{3,4\})}{\vdash \oplus(\otimes(X_1^\perp, X_2^\perp), \otimes(X_3^\perp, X_4^\perp)), \&(⅋(X_1, X_2), ⅋(X_3, X_4))}\,(-,\{\{1,2\},\{3,4\}\})$$

In both s_1 and s_2 the $(-, \mathcal{N})$-rule is unary (as in MHS).

Following this idea, a $(-, \mathcal{N})$-rule is a set (superposition) of unary rules having the same active formulas. If we consider a sequent calculus derivation in HS, and for each $(-, \mathcal{N})$-rule we select one of the premises, we obtain a derivation where all $(-, \mathcal{N})$-rules are unary. This is called a *slice*.

Actually, the notion of slice is old as Linear Logic itself: it appears for the first time in the seminal paper [Gir87]; it has been used by Laurent and Tortora de Falco for studying normalization on polarized proof nets (see [LTdF04]) and is a key notion of ludics and L-nets. Furthermore, as Pagani pointed out in his PhD thesis, (see [Pag06]) slices correspond to the basic objects in the syntax of Hughes and Van Glabbeek additive proof nets, namely *linkings* (see [HVG03]).

The main point when one deals with additive proof nets is to properly reconstruct the structure of the multiplicative proofs of which an additive proof is composed, and to correctly superpose them; such a task is usually fulfilled by boxes (as in [Gir87], [LTdF04], [TdF03b]) or by boolean weights (as in [Gir96], [Lau99], [Mai07]), which provide enough "synchronization points" to glue slices together. In additive J-proof nets this role will be played by *jumps*.

3.2 J-proof nets

In this section we present J-proof nets for HS, using a syntax which is directly inspired from L-nets.

Firstly, in subsection 3.2.1, we modify the definition of J-proof structure given in the previous chapter, in order to take into account the "additive contraction" effect induced by the $(-, \mathcal{N})$-rule. While in others syntaxes (as [Lau99]) this is done by introducing an explicit "additive contraction" link, we adopt the convention of incorporate contraction in the links, by enriching them with a structure of *ports* and by defining the conclusion of a proof structure no more as a pending edge but as a *link*.

In subsection 3.2.2 we study the relation between J-proof structures and HS, by defining *sequentializable* J-proof structures; then, in subsection 3.2.3, we introduce the correctness criterion and we define J-proof nets.

3.2.1 J-proof structures

Definition 65 (Graph with ports) *We call **graph with ports** a directed graph where for each node b the edges entering on b are partitioned into subsets called* ports *; given a node b we denote its ports by b^j, b^k, b^l.*

Definition 66 (Pre-proof structure) *A **pre-proof structure** is a directed acyclic graph with ports whose edges are possibly typed by formulas of HS and whose nodes (also called **links**) are labelled by one of the symbols $ax, cut, +_{I \in \mathcal{N}}, -_{I \in \mathcal{N}}$ (we call such links **logical links**) or by a formula of HS (we call such links **conclusion links**).*

*The edges incident on a link are called **premises** and the edges emergent from a link are called **conclusions**; the label of a link imposes some constraints on its ports and the number and the types of its incident edges and emergent edges:*

- *an ax-link has two conclusions labeled by dual atomic formulas, but no premises.*
- *a cut-link has no conclusions, and two ports (called left and right), one containing $n > 1$ premises all typed by a formula A and the other containing $k > 1$ premises all typed by $A^{\perp}$;*
- *a $-_{I \in \mathcal{N}}$ link b (also called **negative** link) has:*
  - *one port b^i for each $i \in I$; each port b^i contains $n > 1$ premises which are typed by the same formula P_i for $i \in I$;*
  - *one port b^*, which contains only untyped edges (called **jumps**);*
  - *exactly one conclusion, typed by a formula N.*

  If the premises in b^i are typed by a formula P_i, then the conclusion is typed by $\&_{I \in \mathcal{N}}(\parr_{i \in I}(P_i))$.
- *a $+_{I \in \mathcal{N}}$ link b (also called **positive** link) has:*
  - *one port b^i for each $i \in I$; each port b^i contains $n > 1$ premises which are typed by the same formula N_i for $i \in I$;*
  - *$n > 1$ conclusions, exactly one among them typed by a formula P.*

  If the premises in b^i are typed by a formula N_i, then the conclusion is typed by $\oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(N_i))$.

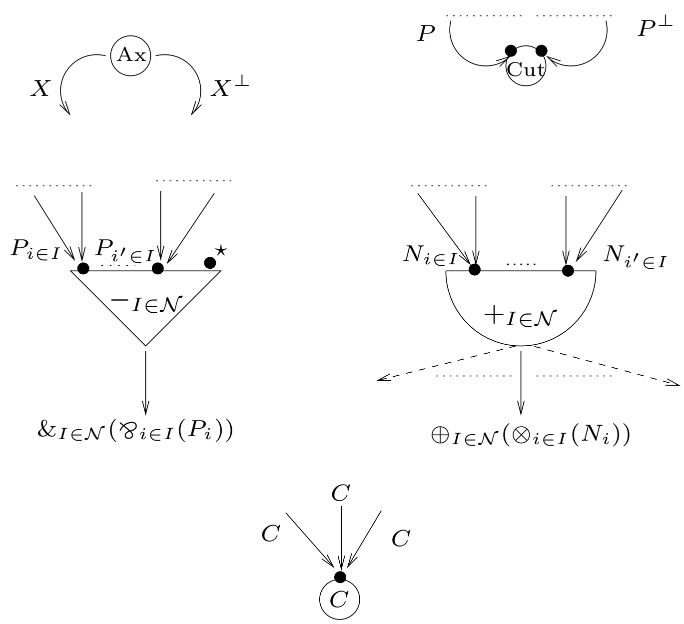


Figure 3.1: Links of a pre-proof structure

- *a conclusion link with label A has no conclusions and one port containing $n > 1$ edges all typed by the formula A.*

A link whose conclusion belongs to a conclusion link is called a ***terminal link****; the types $C_1, \ldots, C_n$ of the conclusions links are called* ***conclusions*** *of the pre-proof structure*

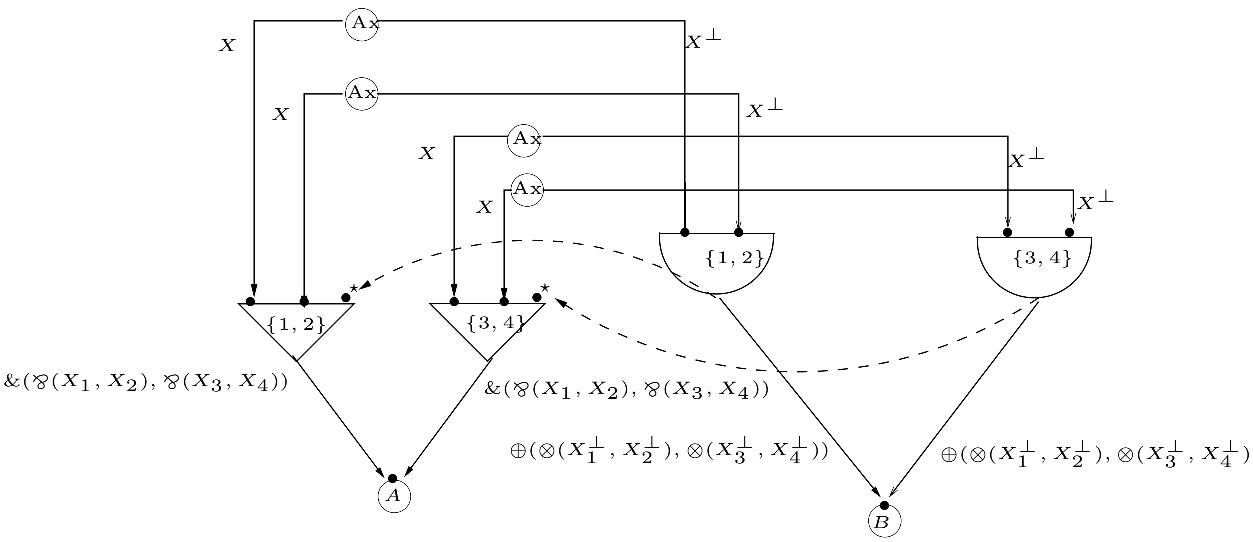
The constraints and the links of definition 66 are synthetically represented in fig. 3.1; we denote ports by black spots, and we distinguish positive and negative links by their shape; using this graphical convention, sometimes when drawing a preproof structure we will label a positive (resp. negative) link simply with $I \in \mathcal{N}$ instead of $+_{I\in\mathcal{N}}$ (resp. $-_{I\in\mathcal{N}}$).

To properly take into account the structures of additives, that is to retrieve slices, we must refine our definition of proof structure, as follows:

Definition 67 (Sibling links and negative rule) *Given a pre-proof structure , two links a, b are* ***sibling*** *iff the typed conclusion of a and the typed conclusion of b belong to the same port of a link c.*

An ***negative rule*** *W is a maximal set $\{w_1, \ldots, w_n\}$ of negative sibling links ; the premises (resp. conclusion) of a negative rule are the premises (resp. conclusion) of its elements.*

An ***additive pair*** *is a pair of negative links belonging to the same negative rule.*



$$A = \&(⅋(X_1, X_2), ⅋(X_3, X_4))$$

$$B = \oplus(\otimes(X_1^\perp, X_2^\perp), \otimes(X_3^\perp, X_4^\perp))$$

Figure 3.2: example of J-proof structure

Definition 68 (View) *We call **view** of a link a (denoted $a^\downarrow$) the set of links $\{b : a \xrightarrow{+} b\} \cup \{a\}$.*

Definition 69 (J-proof structure) *A pre-proof structure R is a **J-proof structure** iff it satisfies the followings:*

Positivity: *see definition 36.*

Additives: *if two links a, b of R belong to the same negative rule, then the label of a is $-^{J \in \mathcal{N}}$ and the label of b is $-^{J' \in \mathcal{N}}$ and $J \neq J'$;*

Views: *given a link a , $a^\downarrow$ doesn't contain any two elements of the same negative rule;*

Contraction: *given two non negative sibling links a, b, there exists an additive pair w_1, w_2 s.t. $a \xrightarrow{+} w_1$ and $b \xrightarrow{+} w_2$.*

a negative rule containing terminal links is called a terminal negative rule*; in addition to the above conditions, a J-proof structure must have at most one terminal negative rule.*

Let us give some explanations on the conditions of definition 69:

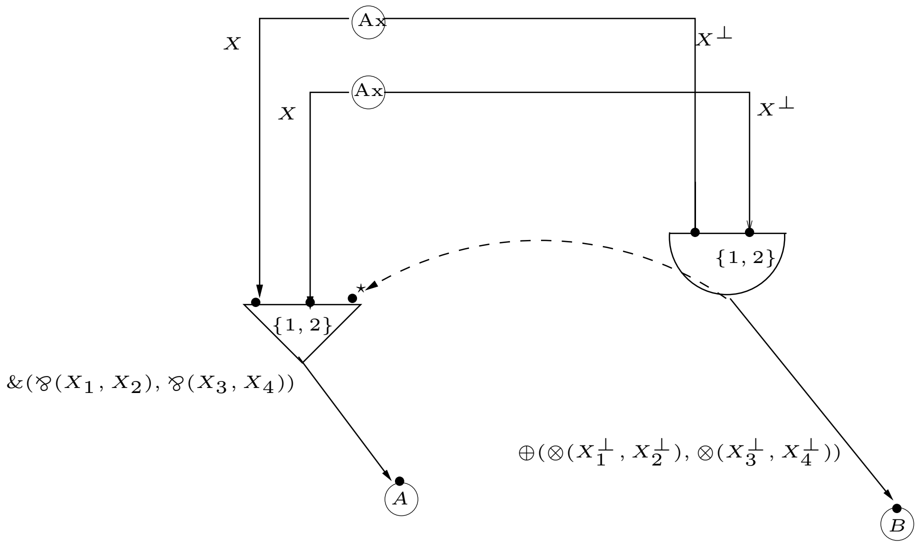


Figure 3.3: a slice of the J-proof structure in fig. 3.2

- Positivity has the same function of the analogous condition on multiplicative J-proof nets; we chose to present it as a constraint on proof structures to make clearer the relation between J-proof structures and L-nets; as a matter of fact, each condition of definition 69 corresponds to a condition in the definition of L-nets (see [CF05]).

- The Additives condition allow to recognize the different components of which a $(-, \mathcal{N})$-rule is composed, and to work independently with each of them.

- The Views condition assures that there cannot be *conflicts* between components of the same negative rule; to give an intuition on the meaning of this condition in the usual syntax of additive proof nets, it is enough to consider the constraint of disjointness between the different component of an additive box.

- The Contraction condition assures that superposition is not "wild"; each time two links are contracted, there is always *at least* one negative rule which justifies the superposition.

Now we are in the position to define the notion of slice:

Definition 70 (Slice) *A **slice** is a J-proof structure where all negative rules are singleton; a slice of a J-proof structure R is a subgraph S of R which is a slice.*

We remark that in this setting jumps has a more preeminent role with respect to the multiplicative case; they not only graduate sequentiality, but allow also to keep track of the additive structure, as it appears clearly from fig. 3.2.

3.2.2 Sequent calculus and J-proof structures

As we did in the multiplicative case, we isolate the J-proof structures which correspond to HS proof, introducing the notion of *sequentializable* J-proof structure.

Given a J-proof structure R and a terminal link b of R we define **the removal of** b in the following way:

- if b is a terminal link of type $+_{I\in\mathcal{N}}$ (resp. $-_{I\in\mathcal{N}}$) of R , the removal of b is the substitution in R of b with one conclusion link for each port b^i of b for $i \in I$.

- if b is a cut link, the removal of b is the substitution in R of b with one conclusion link for each port of b.

Definition 71 (Scope) *Let R be a J-proof structure and $W = \{w_1, \dots, w_n\}$ an negative rule of R: we call* ***scope*** *of an element w_i of W (denoted $\mathcal{R}_i$) the graph obtained from R by erasing all w_j and all the links of R above w_j for $j \neq i$;*

The following definitions are adapted from [Lau99].

Definition 72 (sequentialization of a proof structure) *We define the relation "L* ***sequentializes*** *R* ***in*** *ε", where R is a J-proof structure, L is a terminal link or a terminal negative rule of R and ε is a set of J-proof structures , in the following way, depending from L:*

- *If L is an axiom link, and is the only link of R, then L sequentializes R into $\emptyset$;*

- *if L is a cut link, and it is possible to split the graph obtained by removing L into two J-proof structures R_1, R_2, then L sequentializes R into $\{R_1, R_2\}$;*

- *if L is a positive link with n ports, and it is it is possible to split the graph obtained by removing L into n J-proof structures $R_1, \dots, R_n$, then L sequentializes R into $\{R_1, \dots, R_n\}$ J-proof structures;*

- *if L is a terminal negative rule $W = \{w_1, \dots w_n\}$ of conclusions $\&_{I\in\mathcal{N}}(\bindnasrepma_{i\in I}(P_i))$ such that for each $I \in \mathcal{N}$ there is an element w_j of W, we consider for each w_j the scope $\mathcal{R}_j$ of w_j; if $\mathcal{R}_j$ is a J-proof structure, then L sequentializes R into $\{\mathcal{R}'_1, \dots, \mathcal{R}'_n\}$ J-proof structures, where $\mathcal{R}'_j$ is the J-proof structure obtained by removing w_j in $\mathcal{R}_j$.*

Definition 73 (sequentializable J-proof structure) *A J-proof structure R is **sequentializable** iff*

- *R has a terminal negative rule, which sequentializes R into a set of sequentializable J-proof structures;*
- *R has no terminal negative rule and*
  - *R is composed by a single connected component, and one of its link sequentializes R into a set of sequentializable J-proof structures or into the empty set;*
  - *R is composed by more than one connected component and each component is a sequentializable J-proof structure.*

Proposition 74 *If a J-proof structure R is sequentializable, we can associate to it a proof π of HS.*

PROOF.
the proof is an easy induction on the number of logical links of R:

1. $n = 1$: the only node in R is an Axiom link with conclusions $X,\ X^{\perp}$, to which we associate $\dfrac{}{\vdash P,\ P^{\perp}}$;
2. $n > 1$: suppose R contains one terminal negative rule $W = \{w_1, \dots w_n\}$ of conclusion $\&_{J\in\mathcal{N}}(\parr_{j\in J}(P_i))$; then by definition of sequentializable J-proof structure, W sequentializes R into $R_1, \dots, R_n$ J-proof structures of conclusions respectively $\Gamma, P_1^1, \dots P_{k_1}^1, \dots \Gamma, P_1^n \dots P_{k_n}^n$; to each R_j by induction hypothesis we can associate a proof π_i of conclusion $\vdash \Gamma, P_1^i, \dots P_{k_i}^i$. We obtain π by applying a $(-, \mathcal{N})$ rule of conclusion $\vdash \Gamma, \&_{J\in\mathcal{N}}(\parr_{j\in J}(P_J))$ to all $\pi_1, \dots \pi_n$.

   Otherwise R has no terminal negative rule; suppose R is composed by a single connected component; since it is sequentializable there exists a link L which sequentializes R. Then we reason by cases:

   - L is cut link whose premises are typed by $P, P^{\perp}$; then L sequentializes R into two proof structures R_1, R_2 of conclusions respectively Γ, P and $\Delta, P^{\perp}$; by induction hypothesis we associate to R_1 (resp. R_2) a proof π_1 of conclusion $\vdash \Gamma, P$ (resp. π_2 of conclusion $\vdash \Delta, P^{\perp}$). We obtain π by applying to π_1, π_2 a cut rule of conclusion $\vdash \Gamma, \Delta$;
   - L is a positive link $+_{I\in\mathcal{N}}$ with conclusion $\oplus_{I\in\mathcal{N}}(\otimes_{i\in I}(N_i))$; we recall that each port of L correspond to an $i \in I$. L sequentializes R into $R_1, \dots, R_n$ J-proof structures of conclusions respectively $\Gamma_1, N_1 \dots \Gamma_n, N_n$; to $R_1, \dots, R_n$ we can associate by induction hypothesis n proof $\pi_1, \dots, \pi_n$ of conclusions respectively

$\vdash \Gamma_1, N_1, \ldots \vdash \Gamma_n, N_n$. We obtain π by applying a $(+, I)$ rule of conclusion $\vdash \Gamma_1, \Gamma_n, \oplus_{J\in\mathcal{N}}(\otimes_{j\in J}(N_j))$ to $\pi_1, \ldots, \pi_n$.

Otherwise, R is composed by more than one connected component, and each of them is a sequentializable J-proof structure; we conclude by applying induction hypothesis on them, followed by a sequence of Mix rules.

□

Note Contrarily as we did in the previous chapter, we do not define the "desequentialization" π^* of an HS proof π, because we want to establish a correspondence between HS proofs and J-proof structures of *any* degree of sequentiality (not only the most parallel ones); in section 3.7, we will show how to associate with a proof a J-proof net of *minimal* sequentiality.

3.2.3 Correctness criterion

As we recalled in the previous chapter a correctness criterion must allow to characterize in an intrinsic, purely geometrical way all sequentializable J-proof structures, that is the ones which correspond to sequent calculus proofs. The correctness criterion for J-proof structures in the additive case is composed by two conditions:

- a *qualitative* one, called **cycles** condition, (due to Curien and Faggian , see [CF05]), which is a reformulation in our setting of Hughes and van Glabbeek's toggling condition (see [HVG03]);

- a *quantitative* one, called **totality** condition, which assure that in a J-proof net there are enough slices to retrieve a sequent calculus proof.

One of the differences between J-proof nets and L-nets is the totality condition, which is not required for proving the correctness of L-nets (since they are *partial objects*, in the sense of ludics); cycles condition instead is identical to the homonymous condition on L-nets.

As we did for the multiplicative case, we can associate with a J-proof structure the structure of a graph with pairs $(R, App(R))$, by taking as elements of $App(R)$ the n-tuples of the premises of a negative link; due to the presence of additives, we have to modify our notion of switching path in the following way:

Definition 75 (Switching path and cycle) *Given a J-proof structure, a* ***switching path*** *is a path which never uses two different premises of the same negative rule (called* switching edges*); a* ***switching cycle*** *is a switching path which is a cycle.*

Definition 76 (Cycles-correct J-proof-structure) *A J-proof structure R is **cycles-correct** if and only if, given a non empty union C of switching cycles of R, there is a negative rule $W \in R$ not intersecting C and a pair $w_1, w_2 \in W$ such that for some links $c_1, c_2 \in C$, $c_1 \xrightarrow{+} w_1$ and $c_2 \xrightarrow{+} w_2$; in this case we say that the additive pair $w_1, w_2 \in W$ **breaks** C.*

Remark 77 *The above condition deals with cycles which crosses* different *slices of the same J-proof structure; it is well known, from [Gir96], that the switching acyclicity of the single slices of a proof structure does not imply the sequentiability of the whole proof net.*

A &**-resolution** of a J-proof structure R, is the graph obtained by choosing for each negative rule W of conclusion $\&_{J\in\mathcal{N}}(\parr_{j\in J}(P_j))$ an $I \in \mathcal{N}$ and erasing each component w of W which is not labelled by I, together with all the links hereditary above w.

Definition 78 (Total J-proof structure) *A J-proof structure R is **total** iff each &-resolution of R yields a unique slice with the same conclusions of R.*

Definition 79 (J-proof net) *A **J-proof net** is a J-proof structure which is total and cycles-correct.*

3.3 Sequentialization

In this section we extend the technique of sequentialization used in the previous chapter to additive J-proof nets; that is, we prove that gradually adding jumps to a J-proof net, we retrieve a sequent calculus proof. In subsection 3.3.1 we show that, if the order associated with a J-proof net R is arborescent, then R is sequentializable. In order to sequentialize an additive J-proof net by adding jumps, we must take into account the effect of duplication of the context induced by a $(-, \mathcal{N})$-rule; to properly deal with it, when adding a jump to a J-proof net R, we will add it *separately* to each slice, and then we will superpose the slices so obtained. In subsection 3.3.2, we define precisely the operation of superposition of slices; then in subsection 3.3.3 we define the notion of *bundle of jumps*, which allow to add a jump in all the slices of a J-proof net at the same time. Finally, in subsection 3.3.4, we will prove that we can make arborescent the order of any J-proof-net by adding bundles of jumps.

Note. By now, we will only consider J-proof nets without cut links; we will speak about the question of sequentialization with cut-links in section 3.4.

3.3.1 Arborescence and sequent calculus

Due to the introduction of ports and conclusion links, we have to modify the notion of order associated with a J-proof net and the one of skeleton.

A J-proof net R is a directed acyclic graph (d.a.g.); we define the order $\prec_R$ associated with R as the strict partial order induced by R as a d.a.g. restricted to the logical links of R (that is without conclusion links).

The *skeleton* of a J-proof net R (denoted as always $Sk(R)$) is the directed graph with ports obtained from R by erasing all the edges which are transitive and all conclusion links.

Since $Sk(R)$ is obtained from R just by erasing transitive edges, the order associated to $Sk(R)$ as a d.a.g and the order $\prec R$ associated with R are equal; so if the order $\prec_R$ is arborescent, the skeleton of R is a forest.

Now we prove that if the order associated with a J-proof net R is arborescent, then R is sequentializable.

We first state the following lemma:

Lemma 80 *If R is a J-proof net with more than one logical link and without terminal negative links, then all the conclusions of R are positive.*

PROOF. It follows from the positivity condition (the proof is the same of lemma 16). □

We modify definition 21 in order to adapt it to graphs with ports:

Definition 81 (Splitting node) *Let G be a d.a.g. with ports and c a node with n ports, which is a root of G; let us call $b_i^1, \dots, b_i^l$ the nodes of G which are sources of an edge belonging to the port c^i of c for $i \in \{1, \dots, n\}$. We say that c is **splitting** for G if erasing c, any two of the nodes $b_j^{l'}, b_k^{l''}$ become not connected. I.e. by erasing c, the graph splits into n components, one for each port of c .*

Proposition 82 *Let R be a J-proof net such that $\prec_R$ is arborescent. Then R is sequentializable.*

PROOF.

The proof is by induction on the number of logical links of R:

$n = 1$: in this case, R is composed by just an axiom link, and is trivially sequentializable;

$n = k + 1$: suppose R has a terminal negative rule W, whose elements w_i are minimal in $\prec_R$; then we consider for each i the graph $\mathcal{R}'_i$ (which is obtained by deleting w_i from its scope $\mathcal{R}_i$). Due to totality, each $\mathcal{R}'_i$ is obviously a J-proof net whose order associated is arborescent, so by induction hypothesis is sequentializable; then R is sequentializable.

Otherwise, by lemma 80, all the conclusions of R are positive; we reason by cases, depending if R is composed by one or several connected component:

- if R is composed by a single connected component, there is a positive link c which is terminal and minimal in $\prec_R$. Obviously c is splitting in $Sk(R)$, (because in $Sk(R)$, c is the root of a tree); since passing from R to $Sk(R)$ we only erase transitive edge, it easy to check that the removal of c from R splits R into $R_1, \dots, R_n$ J-proof nets (one for each port of c) ,whose order associated is arborescent, so by induction hypothesis they are sequentializable; then R is sequentializable.
- If R is composed by more than one connected component then $Sk(R)$ is a forest, and each tree of the forest corresponds to a J-proof net whose order is arborescent (so sequentializable by induction hypothesis). Then R is sequentializable.

□

3.3.2 Superposition

We define the operation of superposition of J-proof structures, using the notion of **sharing equivalence**, which has been introduced by Laurent and Tortora de Falco in [LTdF04], and refined by Pagani in [Pag06]; the operation we define is analogous to the union of *chronicles* in ludics and L-nets (see [CF]).

Let $R_1, \dots R_n$ be J-proof structures; Two typed edges x, y of $R_1, \dots, R_n$, premises respectively of two nodes b, b', are **similar** when

- b and b' are both conclusion nodes;
- b and b' are labelled by $+_{I \in \mathcal{N}}$ (resp. $-_{I \in \mathcal{N}}$) and x, y both belong to the i-th port of b, b' for $i \in I$.

Given a J-proof structure R we say that a is a **sublink** of b when a typed premise of b is a conclusion of a; a link a is an **hereditary sublink** of b when there exist a sequence of link $a_1, \dots, a_n$ such that a_i is a sublink of a_{i+1} and $a = a_1,\ b = a_n$.

Given two nodes a, b in a J-proof structure R, we say that a **is a sublink of** b **due to an edge** x , if x is both a typed conclusion of a and a premise of b; we denote it by $a \xrightarrow{x} b$.

Definition 83 (Sharing equivalence) *Given $R_1, \dots, R_n$ J-proof structures with the same conclusions $C_1, \dots C_n$, a* ***sharing equivalence*** *is an equivalence relation $\equiv$ on the links of $R_1, \dots, R_n$ such that for any link a, a', b :*

identity *if a, a' belong to the same R_i, then $a \equiv a'$ iff $a = a'$;*

bottom *if a, a' are conclusion links, then $a \equiv a'$ iff a, a' have the same label among $C_1, \dots, C_n$;*

bottom-up *if $b \xrightarrow{x} a$, and $a \equiv a'$, then for every link b' such that $b' \xrightarrow{x'} a'$, $b \equiv b'$ iff*

- *b and b' have the same label;*
- *x and x' are similar;*
- *for all edge $b \to c$ there exist an edge $b' \to c'$ such that $c \equiv c'$ (and* vice versa*).*

up-bottom *if $a \xrightarrow{x} b$, and $a \equiv a'$ then there exist a link b' such that $a' \xrightarrow{x'} b'$ and $b \equiv b'$.*

If $a \equiv a'$, we say that $\mathbf{a}, \mathbf{a'}$ ***are superimposed by*** *$\equiv$. We denote by $[a]$ the equivalence class of a link a w.r.t. $\equiv$.*

Proposition 84 *Let $R_1, \dots, R_n$ be J-proof structures with the same conclusions, $\equiv$ be the sharing equivalence on $R_1, \dots, R_n$, and a, a' be two nodes in $R_1, \dots, R_n$. If $a \equiv a'$ then the types of the conclusions of a (resp. the type of a is a is a conclusion link) and the types of the conclusions of a' (resp. the type of a' if a' is a conclusion link) are equal.*

Proof. We prove the proposition by induction on the number of links below a. If a is a conclusion link, then the proposition is a consequence of condition **bottom**. Otherwise, there exists a node b such that $a \xrightarrow{x} b$ and since $a \equiv a'$, by condition **up-bottom** there exist a node b' such that $a' \xrightarrow{x'} b'$ and $b \equiv b'$; by induction hypothesis, the types of the conclusions of b and b' are the same. Hence, by condition **bottom-up** and by definition of similar edges, x and x' have the same type. Moreover, if a, a' are axioms it is clear that the other conclusions than x, x' are of same type too. $\square$

We can generalize $\equiv$ to the edges; if x, x' are two edges of $R_1, \dots, R_n$, we say that $x \equiv x'$ iff x is a typed edge (resp. a jump) conclusion of a node a and premise of a node b, x' is a typed edge (resp. a jump) conclusion of a node a' and premise of a node b' and $a \equiv a'$, $b \equiv b'$; we denote by $[x]$ the equivalence class of an edge x w.r.t. $\equiv$ By proposition 84 if $x \equiv x'$ then x and x' either have the same type either they are both jumps.

Fact 85 *Let $R_1, \ldots, R_n$ be n J-proof structures with the same conclusions, and let $\equiv$ denote the sharing equivalence on $R_1, \ldots, R_n$ extended to the edges of $R_1, \ldots, R_n$. If x is an edge conclusion (resp. premise) of a link a, then all the edges in $[x]$ are conclusion (resp. premise) of links in $[a]$.*

Definition 86 (Superposition) *Let $R_1, \ldots, R_n$ be a set of J-proof structures with the same conclusions, and let $\equiv$ denote the sharing equivalence on $(R_1, \ldots, R_n)$ extended to the edges of $(R_1, \ldots, R_n)$. The* ***superposition*** *of $(R_1, \ldots, R_n)$, denoted by $\between\ (R_1, \ldots, R_n)$, is the pre-proof structure whose links (resp. edges) are the equivalence classes w.r.t. $\equiv$ of the links (resp. edges) of $R_1, \ldots, R_n$.*

In particular if a is a link of $R_1, \ldots, R_n$, then:

1. *in case a is an axiom with conclusions x, y, then $[a]$ is an axiom of $\between\ (R_1, \ldots, R_n)$ with conclusions $[x], [y]$;*
2. *in case a is a $+_{I \in \mathcal{N}}$ link, then $[a]$ is a $+_{I \in \mathcal{N}}$ link of $\between\ (R_1, \ldots, R_n)$ such that:*
   - *for each typed edge x premise of a which belongs to a port a^i, for $i \in I$, $[x]$ is a premise of $[a]$ which belongs to the port $[a]^i$ for $i \in I$;*
   - *for each edge y conclusion of a, then $[y]$ is a conclusion of $[a]$.*
3. *in case a is a $-_{I \in \mathcal{N}}$ link, then $[a]$ is a $+_{I \in \mathcal{N}}$ link of $\between\ (R_1, \ldots, R_n)$ such that:*
   - *for each typed edge x premise of a which belongs to a port a^i, for $i \in I$, $[x]$ is a premise of $[a]$ which belongs to the port $[a]^i$ for $i \in I$;*
   - *for each jump x' which belongs to the port a^*, $[x']$ is a premise of $[a]$ which belongs to the port $[a]^*$;*
   - *if y is the conclusion of a, then $[y]$ is a conclusion of $[a]$.*
4. *if a is a conclusion link of type A, $[a]$ is a conclusion link of $\between\ (R_1, \ldots, R_n)$ of type A, such that for each premise x of a, $[x]$ is a premise of $[a]$*

Given a set $R_1, \ldots, R_n$ of J-proof structures with the same conclusions and the sharing equivalence $\equiv$, we say that R_j **shares** a link b of R_k, if and only if there is a link b' of R_j such that $b \equiv b'$.

Proposition 87 *Let $R_1, \ldots, R_n$ be J-proof structures with the same conclusions; then $R = \between\ (R_1, \ldots, R_n)$ is a J-proof structure iff R satisfies condition* ***Contraction*** *of definition 69.*

Proof. Condition **positivity** and **views** are easily verified. Let us show the preservation of condition **additives**. Suppose that $[w], [w']$ are two negative sibling links in R. Then there exist in a slice R_i a $w \in [w]$ (resp. in a slice R_k a $w' \in [w']$) such that $w \stackrel{x}{\to} a$ in R_i (resp. $w' \stackrel{x'}{\to} a'$ in R_k) and $a \equiv a'$, x, x' are similar, but w and w' are not sharing equivalent. Since w, w' are negative links, they have just one conclusion: but then in order to be not sharing equivalent they must have different labels.

$\square$

Remark 88 *If $S_1, \ldots, S_n$ are the slices induced by the &-resolutions of a J-proof structure (resp. J-proof net) R, then $\between\ (S_1, \ldots, S_n) = R$*

Note. It should be clear now the reason of our choice of considering only η-expanded axioms: the presence of non atomic axioms would unnecessarily complicate the definition of superposition of J-proof structures.

3.3.3 Bundle of jumps

Given a negative rule $W : \{w_1, \ldots, w_n\}$ of a J-proof structure R, we say that a link c *depends from* W if for some $w_i \in W$, $w_i \in c^{\downarrow}$.

Definition 89 (Bundle of jumps) *Given a J-proof net R, adding a **bundle of jumps** in R between a positive link a and a negative link b sums up to:*

1. *taking the set of all the slices $S_1, \ldots, S_n$ induced by the &-resolution of R;*
2. *if b depends from some additive pair $W_1 \ldots W_n$ in R , we consider all the slices containing some elements $w_1 \ldots w_n$ of $W_1, \ldots W_n$;*
3. *for any slice S_i containing a and some components $w_j, \ldots, w_k$ of $W_1, \ldots, W_n$ we add a jump in S_i between a and $w_j, \ldots, w_k$; if S_i contains b too, we add also a jump between a and b: in this way we get a slice S'_i;*
4. *we take the superposition $\between\ (S'_1, \ldots, S'_n)$ of all $S'_1, \ldots, S'_n$.*

Proposition 90 *Let R be a J-proof net, a a positive link and b a negative link depending from $W_1 \ldots W_n$, and R' the pre-proof structure obtained by adding a bundle of jumps between a and b; then R' is a total J-proof structure.*

Proof.

By proposition 87 we just have to check the preservation of **contraction**.

To do that, we just have to prove that for any two positive sibling links c_1, c_2 in R' which became siblings due to the bundle of jumps there exists an additive pair w_1, w_2 in R' such that $c_1 \xrightarrow{+} w_1$ and $c_2 \xrightarrow{+} w_2$.

Let us consider two positive links a', a'' respectively belonging to two slices S_j, S_k of R such that $[a'], [a'']$ are siblings in R': if $[a'], [a'']$ were siblings in R, we are done (by contraction condition on R). Otherwise it is easy to check that $a' = a''$ in R; then in the slice S'_j obtained from S_j (resp. in the slice S'_k obtained from S_k) by definition of bundle of jumps $a' \xrightarrow{+} w'$ (resp. $a'' \xrightarrow{+} w''$) where w', w'' form an additive pair in R ,and then $[w'], [w'']$ form an additive pair in R'.

Totality is trivially preserved.

□

Given a cycles-correct J-proof structure, one add a **correct jump** when one add a jump in such a way to get a J-proof structure R' which is still a J-proof net; given a J-proof net R, one add a **correct bundle of jumps**, when one add a bundle of jumps in such a way to get a J-proof structure R' which is still a J-proof net.

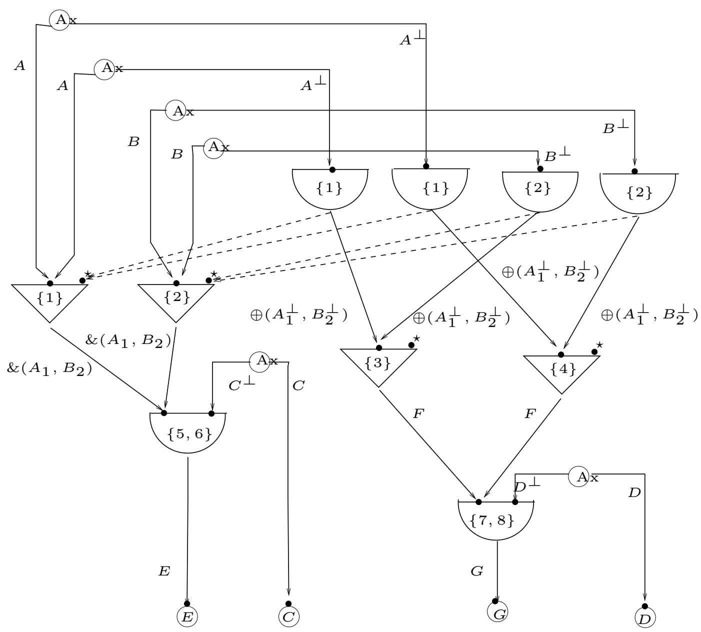
Definition 91 (Strong switching path) *Given a negative link w belonging to a negative rule W of a J-proof structure, a **strong switching path** $\langle w, \dots, a\rangle$ from w to a node b is a switching path which does not use any switching edge of W.*

Remark 92 *Let a be a positive link and b a negative link depending from some additive pairs $W_1, \dots, W_n$ of a J-proof net R; if there isn't any strong switching path from b to a in R, then there isn't any strong switching path from $w_i \in W_i$ to a in R.*

Proposition 93 *Let R be a J-proof net, a a positive link and b a negative link of R; if there isn't any strong switching path from b to a in R, then we can add a correct bundle of jumps between a and b in R.*

PROOF. By the above remark, every jump added by the bundle of jumps is correct, so no new switching cycles are created; but then cycles condition is preserved. □

An example of sequentialization Now let us consider the J-proof net R in fig 3.4; let us add a bundle of jumps between the leftmost terminal positive link and the rightmost negative rule. To add the bundle of jumps , we consider separately each of the four slices S_1, S_2, S_3, S_4 of R and we add in each slice the jump induced by the bundle of jumps as in fig. 3.5; then we superpose the slices to obtain the J-proof net R' in fig. 3.6.



$$F = \&((\oplus(A_1^{\perp}, B_2^{\perp}))_3, (\oplus(A_1^{\perp}, B_2^{\perp})_4)$$
$$G = \otimes(\&((\oplus(A_1^{\perp}, B_2^{\perp}))_3, (\oplus(A_1^{\perp}, B_2^{\perp})_4)_7, D_8^{\perp})$$
$$E = \otimes((\&(A_1, B_2))_5, C_6^{\perp})$$

Figure 3.4:

If we consider the skeleton $Sk(R')$ of R' in fig. 3.7 it directly corresponds to the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{(+,\{1\})\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax}{\vdash A, \oplus(A_1^{\perp}, B_2^{\perp})}\;\dfrac{\dfrac{}{\vdash B, B^{\perp}}\,ax}{\vdash B, \oplus(A_1^{\perp}, B_2^{\perp})}(+,\{2\})}{\vdash \&(A_1, B_2), \oplus(A_1^{\perp}, B_2^{\perp})}(-,\mathcal{N}) \quad \dfrac{}{\vdash C, C^{\perp}}\,ax}{\vdash \otimes((\&(A_1, B_2))_5, C_6^{\perp}), \oplus(A_1^{\perp}, B_2^{\perp}), C}(+,K) \quad \dfrac{\dfrac{(+,\{1\})\dfrac{\dfrac{}{\vdash A, A^{\perp}}\,ax}{\vdash A, \oplus(A_1^{\perp}, B_2^{\perp})}\;\dfrac{\dfrac{}{\vdash B, B^{\perp}}\,ax}{\vdash B, \oplus(A_1^{\perp}, B_2^{\perp})}(+,\{2\})}{\vdash \&(A_1, B_2), \oplus(A_1^{\perp}, B_2^{\perp})}(-,\mathcal{N}) \quad \dfrac{}{\vdash C, C^{\perp}}\,ax}{\vdash \otimes((\&(A_1, B_2))_5, C_6^{\perp}), \oplus(A_1^{\perp}, B_2^{\perp}), C}(+,K)}{\vdash \otimes((\&(A_1, B_2))_5, C_6^{\perp}), \&((\oplus(A_1^{\perp}, B_2^{\perp})_3, (\oplus(A_1^{\perp}, B_2^{\perp})_4, C}(-,\mathcal{M}) \quad \dfrac{}{\vdash D, D^{\perp}}\,ax}{\vdash \otimes((\&(A_1, B_2))_5, C_6^{\perp}), \otimes((\&(\oplus(A_1^{\perp}, B_2^{\perp}))_3, (\oplus(A_1^{\perp}, B_2^{\perp}))_4)_7, D_8^{\perp}), C, D.}(+,L)$$

where $\mathcal{N} = \{\{1\}, \{2\}\}$, $\mathcal{M} = \{\{3\}, \{4\}\}$, $L = \{7, 8\}$, $K = \{5, 6\}$.

We could as well add a bundle of jumps in R from the rightmost terminal positive link to the leftmost negative rule, obtaining the J-proof net R'' in fig 3.8.

If we consider the skeleton $Sk(R'')$ of R'' in fig. 3.9 it directly corresponds to the following proof:

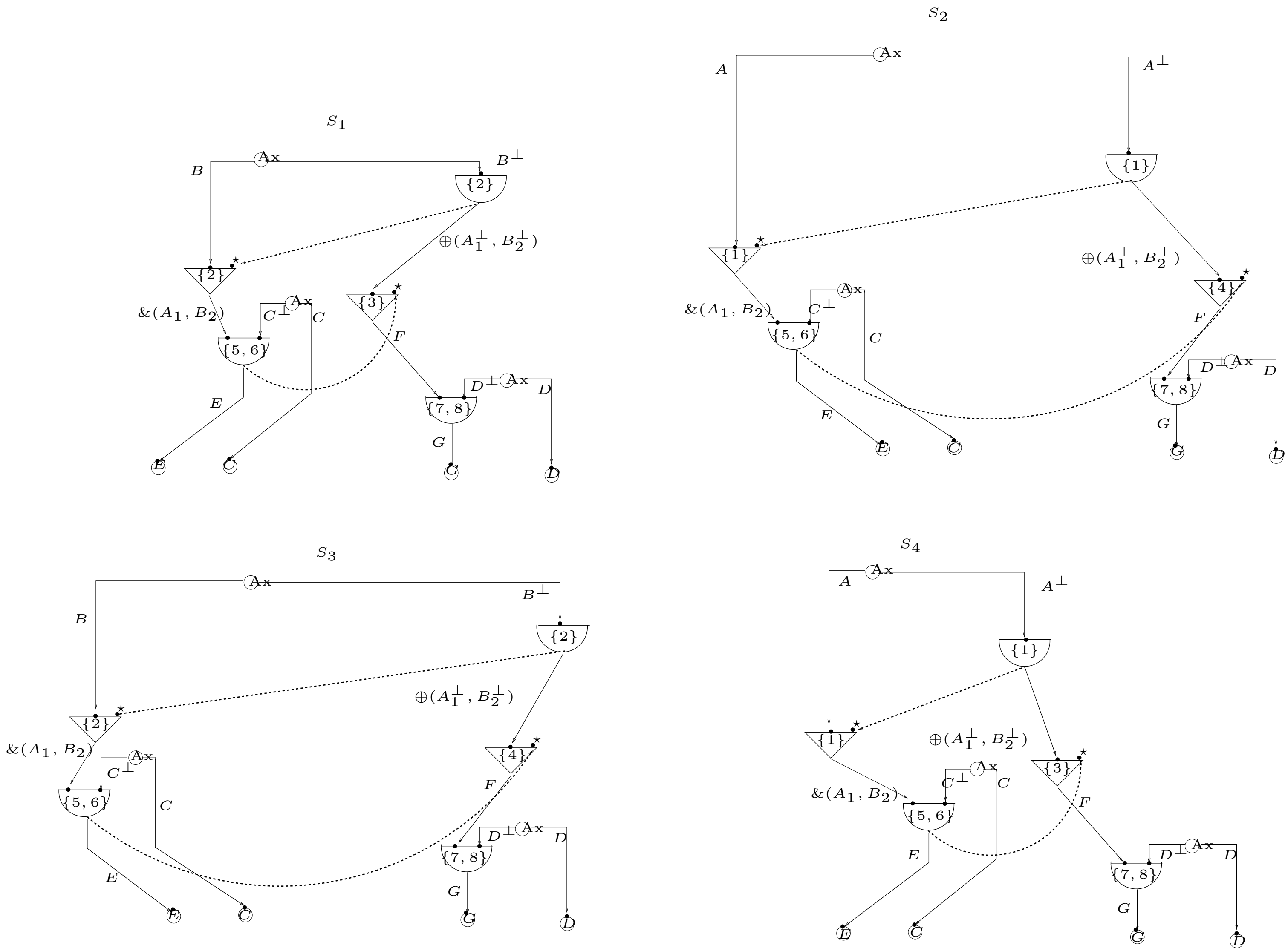
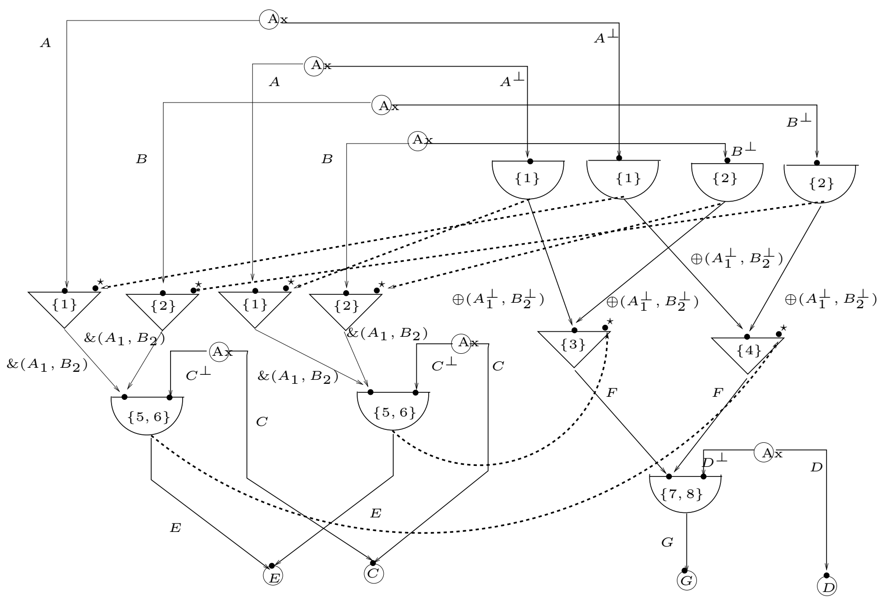


Figure 3.5: adding a bundle of jumps on R



$$F = \&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp)_4)$$

$$G = \otimes(\&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp)_4)_7, D_8^\perp)$$

$$E = \otimes((\&(A_1, B_2))_5, C_6^\perp)$$

Figure 3.6: The superposition of S_1, S_2, S_3, S_4

$$
\begin{prooftree}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash A, A^\perp$}
\LeftLabel{$(+,\{1\})$}
\UnaryInfC{$\vdash A, \oplus(A_1^\perp, B_2^\perp)$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash A, A^\perp$}
\RightLabel{$(+,\{1\})$}
\UnaryInfC{$\vdash A, \oplus(A_1^\perp, B_2^\perp)$}
\LeftLabel{$(-,\mathcal{M})$}
\BinaryInfC{$\vdash \&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4), A$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash D, D^\perp$}
\LeftLabel{$(+,L)$}
\BinaryInfC{$\vdash \otimes(\&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4)_7, D_8^\perp), A, D$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash B, B^\perp$}
\LeftLabel{$(+,\{2\})$}
\UnaryInfC{$\vdash B, \oplus(A_1^\perp, B_2^\perp)$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash B, B^\perp$}
\RightLabel{$(+,\{2\})$}
\UnaryInfC{$\vdash B, \oplus(A_1^\perp, B_2^\perp)$}
\LeftLabel{$(-,\mathcal{M})$}
\BinaryInfC{$\vdash \&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4), B$}
\AxiomC{}
\LeftLabel{ax}
\UnaryInfC{$\vdash D, D^\perp$}
\LeftLabel{$(+,L)$}
\BinaryInfC{$\vdash \otimes(\&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4)_7, D_8^\perp), B, D$}
\LeftLabel{$(-,\mathcal{N})$}
\BinaryInfC{$\vdash \&(A_1, B_2), \otimes(\&((\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4)_7, D_8^\perp), D$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\vdash C, C^\perp$}
\LeftLabel{$(+,K)$}
\BinaryInfC{$\vdash \otimes((\&(A_1, B_2))_5, C_6^\perp), \otimes((\&(\oplus(A_1^\perp, B_2^\perp))_3, (\oplus(A_1^\perp, B_2^\perp))_4)_7, D_8^\perp), C, D.$}
\end{prooftree}
$$

where $\mathcal{N} = \{\{1\}, \{2\}\}$, $\mathcal{M} = \{\{3\}, \{4\}\}$, $L = \{7, 8\}$, $K = \{5, 6\}$.

3.3.4 Arborisation

Definition 94 (Saturated J-proof net) *A J-proof net R is* saturated *if for every negative link a and for every positive link b, it is not possible to add any correct bundle of jumps between a and b such that the order increases.*

Given a J-proof net R, a saturation $Sat(R)$ of R is a saturated J-proof net obtained from R by adding bundles of jumps.

As before, our sequentialization argument is the following:

- If the order $\prec_R$ associated with a J-proof net R is arborescent, we can associate with R a proof π of HS.

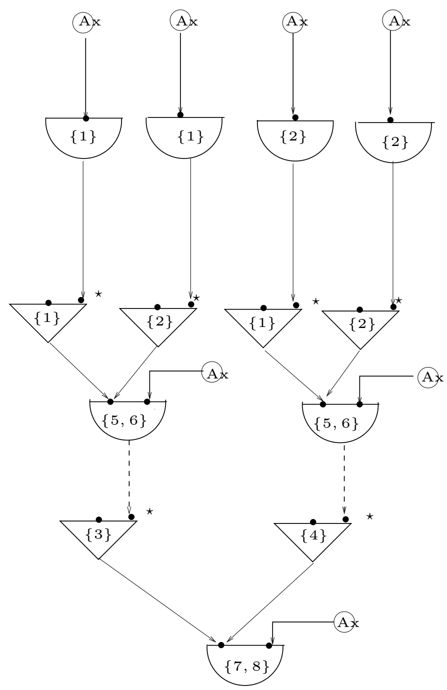


Figure 3.7: The skeleton of R'

- The order associated to a saturated J-proof net is arborescent.
- Any J-proof net can be saturated.

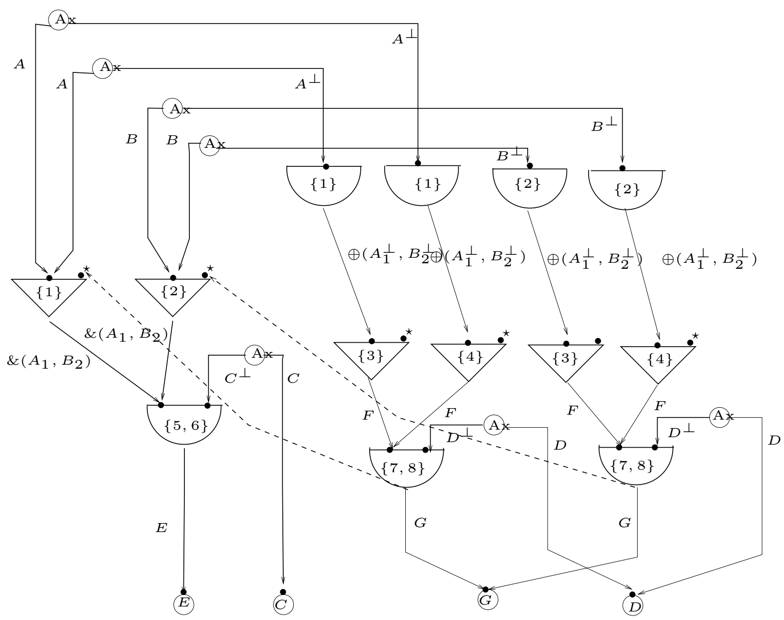
Lemma 95 *Let R be a cycles-correct J-proof structure, and C a union of switching cycles of R; then there exists an additive pair $w_1, w_2 \in W$ in R which breaks C and positive node $c \in C$ s.t.*

1. $\neg(c \xrightarrow{+} w_1)$ *and* $\neg(c \xrightarrow{+} w_2)$*;*
2. *c belongs to a cycle $C' \in C$ which **sees** W (a cycle C'* sees *W iff there exists a node $d \in C'$ which is hereditary above w_1 or w_2).*

PROOF.

The proof is by induction on the number of cycles in C:

1: By the correctness criterion there exists in R an additive pair $w_1, w_2 \in W$ which breaks C. Let's suppose by absurd that every link of C is above w_1 or w_2; then we can partition the nodes of C in two sets, $A = \{a : a \xrightarrow{+} w_1\}$ and $B = \{b : b \xrightarrow{+} w_2\}$, disjoint by condition views of the definition of J-proof structure. Given any two elements $a \in A$ and $b \in B$, there exists a path $r : \langle a \dots b \rangle$ connecting them.



$F = \&((\oplus(A_1^{\perp}, B_2^{\perp}))_3, (\oplus(A_1^{\perp}, B_2^{\perp})_4)$

$G = \otimes(\&((\oplus(A_1^{\perp}, B_2^{\perp}))_3, (\oplus(A_1^{\perp}, B_2^{\perp})_4)_7, D_8^{\perp})$

$E = \otimes((\&(A_1, B_2))_5, C_6^{\perp})$

Figure 3.8: The J-proof net R''

We consider the first edge of r starting from a which connects a node d of A with a node d' of B; either is an incident edge $d \rightarrow d'$, and then $d \xrightarrow{+} w_1$ and $d \xrightarrow{+} w_2$, or is an emergent edge $d \leftarrow d'$, and then $d' \xrightarrow{+} w_1$ and $d' \xrightarrow{+} w_2$; in any case we contradict the condition views of definition 69, so there exists some link c s.t. $\neg(c \xrightarrow{+} w_i)$. Furthermore there has to be at least one positive link which enjoys the property, otherwise C would not be switching; C obviously sees W.

$n+1$: By the correctness criterion there exists in R an additive pair $w_1, w_2 \in W$ which breaks C. If there is a node c belonging to some cycle $C' \in C$ which sees W and s.t. c is not hereditary above W, we have done. Otherwise, we can partition the cycles of C in three groups: C_1 (the cycles with all elements above w_1), C_2 (the cycles with all elements above w_2) and C_0 (the cycles with no element above w_1 or w_2). Now by induction hypothesis on $C_1 \cup C_0$ there exists an additive pair $w_1', w_2' \in W'$ which breaks $C_1 \cup C_0$ and a positive link c' belonging to some $C' \in C_1 \cup C_0$ which sees W' , such that c' is not above W'; W' cannot belong to C_2, otherwise is above w_2, and then either there is

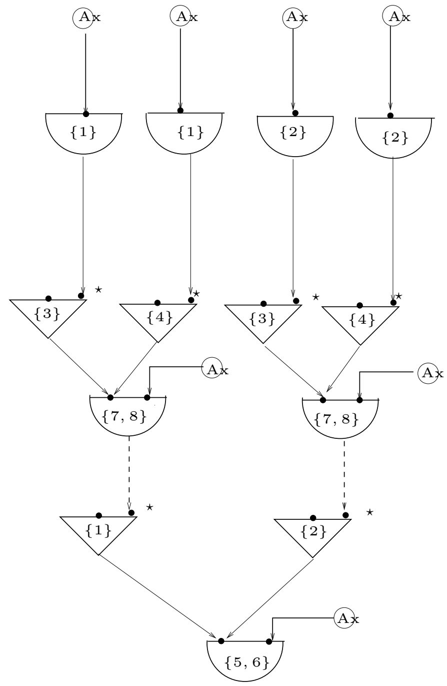


Figure 3.9:

some $c_1 \in C_1$ which is above w_1 and w_2, impossible, either there is some $c_0 \in C_0$ which is above w_2, impossible; so W' breaks C too, and we have done.

□

Lemma 96 *Let R be a cycles-correct J-proof structure; if R contains a switching cycle , then R is not saturated.*

Proof.

We consider the union C of all cycles of R (there is at least one).There exists, by lemma 95, an additive pair $W = \{w_1, w_2\}$ belonging to some negative rule not intersecting C, which breaks C and a positive link c, belonging to a cycle C' of C, which is not above any of w_1, w_2; by the fact that C' sees W, there exists a path r' from c to W, which contains only nodes of C' and nodes in a directed path from some $b \in C'$ to w_1 or w_2.

Let's suppose that W is a conclusion of R: in this case we add a bundle of jumps between c and w_1 (or w_2), this doesn't create cycles and increases the order.

If W isn't a conclusion of R, we show that there cannot be any strong switching path from a link in W to c. Let us suppose that there is a strong switching path $r : \langle w_1(w_2) \dots c\rangle$ in R; now if r and r' are disjoint by composing them we get a switching cycle intersecting W, contradicting the fact that W doesn't intersect any switching cycle of R.

If r and r' do intersect, let's take the first point d starting from $w_1(w_2)$ and going down on r where r meets C' (if r doesn't meet C', this means that r and r' intersect on the directed path from some node in C' to w_1 or w_2, and so we have a cycle). The only interesting case is if d is negative: by the fact that d is in a switching cycle where at least one node is above W, there exists a strong switching path r'' from d to W, so we compose the subpath of r from $w_1(W_2)$ to d with r'' and we get a switching cycle, contradiction.

So there isn't any strong switching path from w_i to c, then by proposition 93 we can add a correct bundle of jumps from c to w_i.

$\square$

Lemma 97 (Arborisation of J-proof nets) *Let R be a J-proof net. If R is saturated then $\prec_R$ is arborescent. Any J-proof net can be saturated.*

Proof.

If R contains some cycles, then we apply lemma 96 and we have done; so we can restrict ourselves to the case where R doesn't contain any switching cycle.

We prove that if $\prec_R$ is not arborescent, then there exists a negative link c and a positive link b s.t. we can add a bundle of correct jumps between b and c which makes the order increase.

The proof is just an adaption to J-proof nets of the proof of lemma 4:

if $\prec_R$ is not arborescent, then in $\prec_R$ there exists a link a with two immediate predecessors b and c (they are incomparable). Observe that b and c are immediately below a in $Sk(R)$ and also in R; observe also that b and c cannot belong to the same rule and b (resp. c) cannot be above any link in the same negative rule than c (resp. b), by condition views.

Either 1) a is an axiom link, either 2) is a positive link, and b and c are two negative links; we consider just the case 2), the first one being slightly simpler.

We have two possibilities:

1. either b or c is terminal in R. Let assume that b is terminal; then c cannot be terminal (by definition of J-proof structure), and there is a positive link c' which immediately precedes c. If we add a bundle of jumps between b and c', we preserve cycles condition and the order increases (see fig 3.10).

2. Neither b or c are terminal in R. Each of them has an immediate positive predecessor, respectively b' and c'.

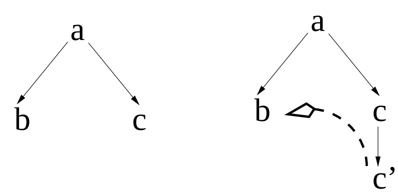


Figure 3.10:

Now we want to prove that either we can add a bundle of correct jumps from b' to c, either we can add a bundle of correct jumps from c' to b.

Let's suppose that we cannot add any bundle of correct jumps in R from b' to c; then by proposition 93 there is in R a strong switching path $r = \langle c, c' .... b\rangle$. If we cannot add a bundle of correct jumps from c' to b too, then there is a strong switching path $r' = \langle b, b' ... c\rangle$ in R.

Assume that r and r' are disjoint: we exhibit a switching cycle in R $\langle c, c' ... b, b' ... c\rangle$ by concatenation of r and r', contradicting the hypothesis that R has no switching cycles (see fig 3.11).

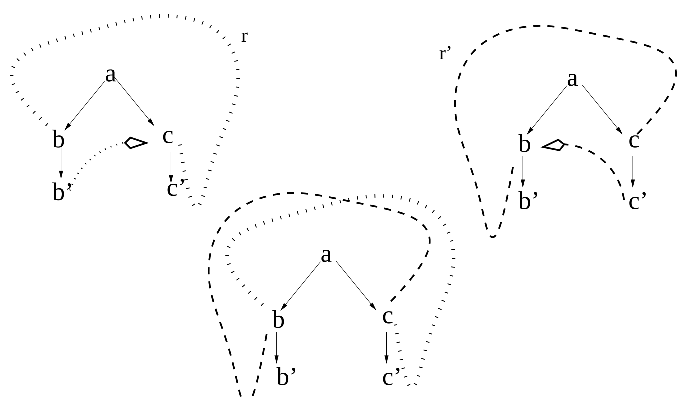


Figure 3.11:

If r and r' are not disjoint, we reason as in the proof of lemma 4, and and we still find a cycle .

$\square$

Theorem 98 (sequentialization) *Let R be a J-proof structure of conclusion $C_1, \ldots, C_n$. If R is a J-proof net then is sequentializable.*

PROOF.

Let us take a saturation $Sat(R)$ of R; we reason by induction on the number of logical links in R:

$n = 1$: in this case, R is composed by just an axiom link, and is trivially sequentializable;

$n = k + 1$: if $Sat(R)$ has a terminal negative rule W, (whose elements $w_1, \dots w_n$ are minimal in $\prec_{Sat(R)}$), then R too has a terminal negative rule W'; due to totality, it is straightforward that W' sequentializes R into $\{R_1, \dots R_n\}$ J-proof nets, which are sequentializable by induction hypothesis. Otherwise, by lemma 80 all conclusions of R are positive; we reason by cases, depending if $Sat(R)$ is composed by one or more than one connected component:

- if $Sat(R)$ is composed by a single connected component, there is a terminal positive link c with conclusion C_i in $Sat(R)$ which is minimal in $\prec_{Sat(R)}$ (and splitting in $Sk(R)$) whose removal splits $Sat(R)$ into n J-proof nets; but then also the removal of the terminal link c' with conclusion C_i in R splits R into n J-proof nets (otherwise, c would not be splitting in $Sat(R)$) so c' sequentializes R into $\{R_1, \dots, R_n\}$ J-proof nets which are sequentializable by induction hypothesis.
- if $Sat(R)$ is composed by more than one connected component, each component correspond to a subnet of R (so is sequentializable by induction hypothesis). Then R is sequentializable.

□

J-proof nets and Mix The proof of sequentialization provided above, could be easily adjusted in order to take out the Mix rule, just by properly extending the notion of correction graph and s-connectedness, as we did in chapter 2.

3.4 Cut

In this section we study J-proof structures with *cut*-links.

First in subsection 3.4.1 we deal with sequentialization in presence of cut links; then in subsection 3.4.2 we study cut-elimination on J-proof structures.

3.4.1 Cut and sequentialization

Unfortunately, we cannot straightforwardly extend our proof of sequentialization in presence of *cut*-links. The reason of the problem is in the operation of superposition of slices, which allows to define the bundle of jumps: superposing slices in presence of cut links is quite difficult. This is not a novelty: actually, in the sliced polarized proof nets of [LTdF04], a similar problem is present, which makes hard to conciliate the presence of cuts *inside* proof nets and sequentialization.

The way out is to consider only cut-free J-proof nets (for which we can prove sequenzialization), compose them using *cut*-links, and then reducing the J-proof net obtained until we reach the normal form (which is cut-free, so that we can deal with it again).

The central point of this argument is the preservation of the property of being sequentializable under cut reduction; we prove this result in section 3.6 by using the injectivity of pointed semantics with respect to J-proof nets, that we state in subsection 3.5.2; actually, this strategy is the same used by Laurent and Tortora de Falco for sliced polarized proof nets, using relational semantics.

3.4.2 Cut elimination

Definition 99 *Given two J-proof structure R_1, R_2 with conclusion respectively Γ, P and $\Delta, P^\perp$, the* ***composition*** *of R_1, R_2 is the J-proof structure obtained by :*

1. *erasing the conclusions links with label $P, P^\perp$ of R_1, R_2;*
2. *connecting the graphs so obtained with a cut-link with premises $P, P^\perp$.*

Remark 100 *If R_1, R_2 are sequentializable (i.e. J-proof nets) then their composition R is sequentializable (i.e. a J-proof net).*

Now we define cut elimination on J-proof structures. As in L-nets, reduction is defined on slices: so to reduce a J-proof structure R, we will decompose R in slices, perform reduction separately on each slice, until we reach a cut-free slice, and then superpose all the cut free slices.

We first begin by defining cut reduction on slices.

Cut elimination on slices In order to define cut elimination on slices, we have to extend our definition of slice to include the empty slices with conclusion $C_1, \dots, C_n$.

Cut reduction rules are graph rewriting rules which locally modify a slice S obtaining a slice S' with the same conclusions.

There are three kinds of cut-elimination steps (we denote by $S \rightsquigarrow S'$ the relation "S reduces to S'"), depicted in Fig. 3.12, Fig. 3.13 and fig. 3.14..

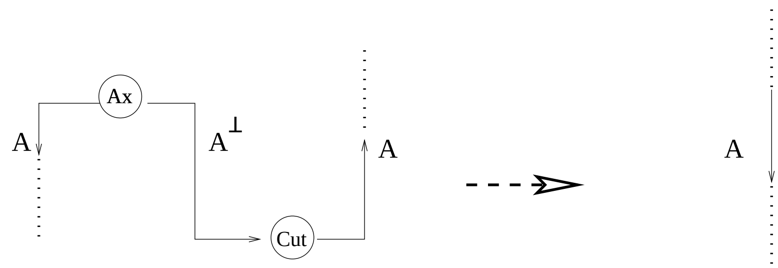


Figure 3.12: *ax* cut reduction.

Definition 101 (Correct slice) *A slice is **correct** iff it is switching acyclic.*

With respect to the rewriting rules $+_{I\in\mathcal{N}}/-_{I\in\mathcal{N}}$, $+_{K\in\mathcal{N}}/-_{J\in\mathcal{N}}$ and ax, reduction enjoys the following properties:

Theorem 102 (Preservation of correctness) *Given a slice S, if S is correct and $S \rightsquigarrow S'$, then S' is correct.*

Theorem 103 (Strong normalization) *For every correct slice S, there is no infinite sequences of reductions $S \rightsquigarrow S_1 \rightsquigarrow S_2 \ldots \rightsquigarrow S_n \ldots$*

Theorem 104 (Confluence) *For every correct slice S_1, S_2 and S_3, such that $S_1 \rightsquigarrow S_2$ and $S_1 \rightsquigarrow S_3$, there is a slice S_4, s.t. $S_2 \rightsquigarrow S_4$ and $S_3 \rightsquigarrow S_4$.*

The proofs of the above theorems are a straightforward generalization of the proofs of the analogous theorems of section 2.2.5

Cut elimination on J-proof structures In order to properly define reduction we must isolate the class of proof structures which have a good computational behaviour: we call them **weakly correct**.

Definition 105 *A total J-proof structure is **weakly correct** when all its slices are correct.*

Let R be a weakly correct J-proof structure and $\{S_1, \ldots, S_n\}$ be the set of the slices induced by the &-resolutions of R. If S'_i is the cut-free slice obtained by reducing S_i , we call **normal form of** R (denoted by $[R]$) the superposition $\between(S'_1, \ldots, S'_n)$ of $S'_1, \ldots, S'_n$.

Given a link a of a J-proof structure R, we say that a belongs to the **right (resp. left) branch** of a *cut*-link c iff there exists a link b, whose conclusion belongs to the right (resp. left) port of c, such that a is an hereditary sublink of b or $a = b$.

Given a J-proof structure R and a link a of R, we say that a is **hidden** if a is an hereditary sub-link of a *cut* link of R, we call it **visible** otherwise.

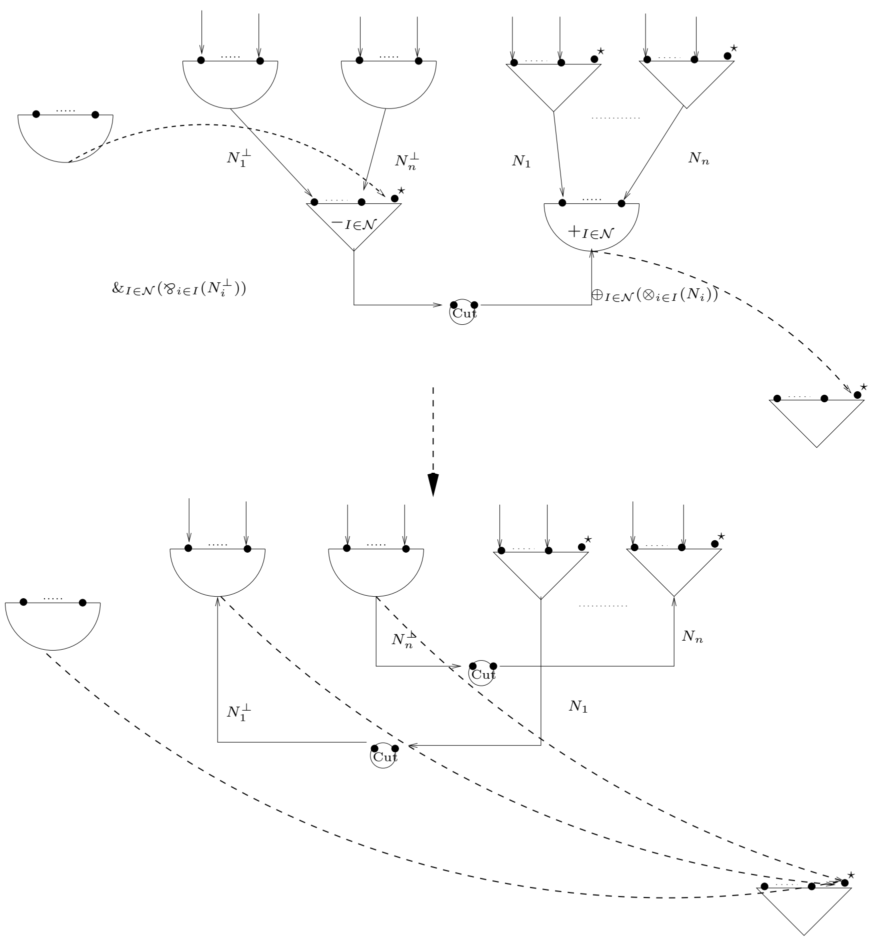


Figure 3.13: $+_{I\in\mathcal{N}}/-_{I\in\mathcal{N}}$ cut reduction.

A slice is **persistent** if it does not reduce itself to the empty slice.

If S is a persistent slice, and a is an hidden link in the right (resp. left) branch of a *cut*-link of S, the **opposite link** of a is the link b in the left (resp. right) branch of c such that in a slice S' obtained by reducing S, the conclusion of a and the conclusion of b becomes premises of the same cut link c'.

Remark 106 *Consider an hidden negative link a of a persistent slice S and its opposite link b; a and b are hereditary sublink of the same cut-link c. Now if there are two links a', b' in S such that $a' \xrightarrow{+} a$ and $b \xrightarrow{+} b'$ and a', b' are not hereditary sublink of c, it easy to verify, following the reduction steps and the definition of opposite link, that there is a slice S' obtained by reducing S such that in S' $a' \xrightarrow{+} b'$.*

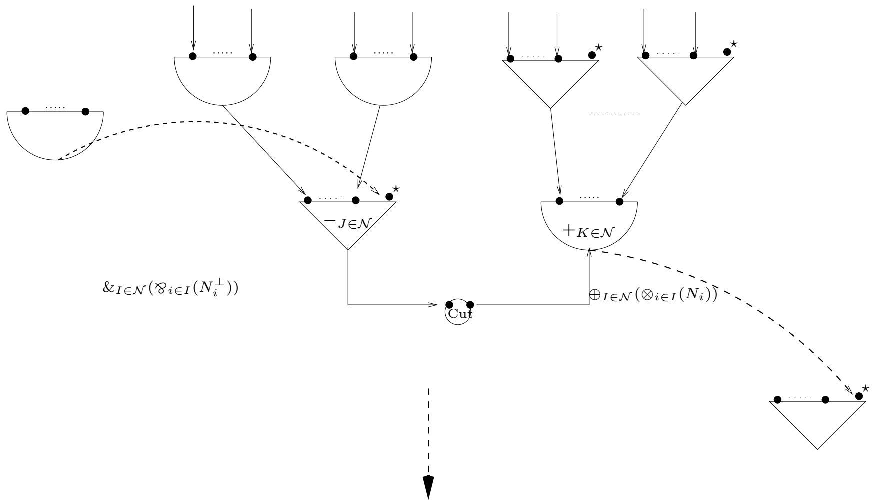


Figure 3.14: $+_{K\in\mathcal{N}}/-_{J\in\mathcal{N}}$ cut reduction (with $J \neq K$.

Proposition 107 *Given a weakly correct J-proof structure R and a normal form $[R] = \between (S'_1, \ldots, S'_n)$ of R, $[R]$ is a weakly correct J-proof structure.*

Proof.

To check that $[R]$ is a J-proof structure, by proposition 87, it is enough to check that the condition **contraction** of definition 69 is respected. First we observe that a weakly correct J-proof structure must contain least one persistent slice (by totality).

Now let us suppose that there is a positive link $a \in S'_j$ and a positive link $b \in S'_k$ such that $[a]$ and $[b]$ are siblings in $[R]$.

Then a, b belong to two different slices S_j, S_k of R, such that a, b are visible. Now we have the following cases:

1. there is in R a $-_{J\in\mathcal{N}}$-link w_j and a $-_{K\in\mathcal{N}}$-link w_k, with $J \neq K$ such that w_j, w_k are an additive pair and $a \xrightarrow{+} w_j$ in S_j and $b \xrightarrow{+} w_k$ in S_k.

   If w_j, w_k are visible in R, then $a \xrightarrow{+} w_j$ in S'_j and $b \xrightarrow{+} w_k$ in S'_k, and we have done.

   Otherwise, w_j, w_k are hidden; suppose they belong to the left branch of a *cut*-link c in R; then in the right branch of c there are two positive sibling links a', b' with label respectively $+_{J\in\mathcal{N}}$ and $+_{K\in\mathcal{N}}$, such that a' is the opposite link of w_j in S_j, and b' is the opposite link of w_k in S_k (because S_j, S_k are persistent).

   Since R is a J-proof structure, there exist in R two negative links w'_j, w'_k with $a' \xrightarrow{+} w'_j$ in S_j and $b' \xrightarrow{+} w'_k$ in S_k, such that w'_j, w'_k

form an additive pair in R and they are not hereditary sublinks of c; moreover, since R is weakly correct, $w'_j, w'_k \neq w_j, w_k$.

(a) Let us suppose that w'_j, w'_k are visible; since $a \xrightarrow{+} w_j$ and $a' \xrightarrow{+} w'_j$ in S_j, by remark 106 and theorem 104 it is easy to check that in S'_j $a \xrightarrow{+} w'_j$; similarly we can found that $b \xrightarrow{+} w'_k$ in S'_k, so we have done.

(b) If w'_j, w'_k are hidden, we search for their opposite positive links in S_j, S_k and we iterate the procedure on them, until we get a visible additive pair w''_j, w''_k of R, with $w''_j \in S_j, w''_k \in S_k$ (it must exists, by finiteness and switching acyclicity of S_j, S_k); by remark 106 and theorem 104 in S'_j $a \xrightarrow{+} w''_j$ and $b \xrightarrow{+} w''_k$ in S'_k, and we have done.

2. the view of a in S_j (resp. the view of b in S_k) does not contain a link w_j (resp. a link w_k) such that w_j, w_k form an additive pair in R; it is easy to check that in this case the view of a and the view of b contains the same links of R, so a, b are two different occurrence of the same link of R in S_j, S_k. If all links in the view of a (resp. b) are visible, then a and b must be sharing equivalent in S'_j, S'_k: contradiction. Otherwise, suppose that the view of a in S_j contains a hidden negative link d, hereditary sublink of a *cut*-link c of R; then the view of b in S_k contains d too. We have the following cases:

   (a) the opposite positive link d' of d in S_j is hereditary above a negative link w'_j and the opposite positive link d'' of d in S_k is hereditary above a negative link w'_k such that w'_j, w'_k forms an additive pair in R and they are not hereditary sublinks of c; if w'_j, w'_k are visible, then by remark 106 and theorem 104, $a \xrightarrow{+} w'_j$ in S'_j and $b \xrightarrow{+} w'_k$ in S'_k, and we are done; if w'_j, w'_k are hidden, then we reason as in point 1-(b) and we conclude.

   (b) the view of the opposite link d' of d in S_j (resp. the view of the opposite link d'' of d in S_k) does not contain a link w'_j (resp. a link w'_k) such that w_j, w_k form an additive pair in R; it is easy to check that in this case the view of d' and the view of d'' contains the same links of R, so d', d'' are two different occurrence of the same link of R in S_j, S_k. If all links in the view of d' (resp. d'') are visible, then a and b must be sharing equivalent in S'_j, S'_k: contradiction. Otherwise the view of d' in S_j contains a hidden negative link e of R, and the view of d'' in S_k contains e too; then we search for the positive opposite links of e in S_j,S_k and we iterate the procedure on them until by finiteness of R either we find a visible additive pair w''_j, w''_k of R, with $w''_j \in S_j, w''_k \in S_k$, (and then by remark 106 and theorem 104 $a \xrightarrow{+} w''_j$ in S'_j and $b \xrightarrow{+} w''_k$ in S'_k), either we find a contradiction.

The property of being weakly correct is trivially preserved, due to theorem 102.

□

Theorem 108 (Existence of a normal form) *Given a weakly correct J-proof structure R, there exists a weakly correct J-proof structure R' such that $R' = [R]$.*

Proof.
The proof is an easy consequence of theorem 103. □

Theorem 109 (Confluence) *If R, R', R'' are weakly correct J-proof structures, such that R', R'' are normal forms of R, then $R' = R''$.*

Proof. Trivial, from theorem 104. □

3.5 Pointed sets and injectivity

In this section we extend pointed sets semantics to additive J-proof nets.

In subsection 3.5.1 we define the interpretation of a J-proof structure and we prove that is stable under reduction; in subsection 3.5.2, we prove the injectivity of pointed semantics in presence of additives.

By $\mathsf{A_1}^* \uplus \ldots \uplus \mathsf{A_n}^*$ we denote the pointed set obtained by taking the disjoint union $\bigcup_{i \in \{1 \ldots n\}} (\{i\} \times \mathsf{A_i}^*)$ reunited with a distinguished element $0_{\mathsf{A_1}^* \uplus \ldots \uplus \mathsf{A_n}^*}$.

The formulas of HS are interpreted in the following way:

- an atomic formula X is interpreted by a pointed set X^*
- a positive formula $\oplus_{I \in \mathcal{N}}(\otimes_{i \in I}(N_i))$ is interpreted by $\uplus_{I \in \mathcal{N}}(\circledast_{i \in I}(\mathsf{P}^*_i))$;
- a negative formula $\&_{I \in \mathcal{N}}(\parr_{i \in I}(P_i))$ is interpreted by $\uplus_{I \in \mathcal{N}}(\circledast_{i \in I}(\mathsf{N}^*_i))$.

3.5.1 Experiments

Given a J-proof structure R of conclusions $C_1, \ldots, C_n$, we define the interpretation $[\![R]\!]$ of R as in the multiplicative case, that is as a subset of $\mathsf{C}^*_1 \circledast \cdots \circledast \mathsf{C}^*_n$, which we define extending the notion of **experiment**.

In defining the interpretation of R, given a pointed set $\mathsf{A} = \uplus_{I\in\mathcal{N}}(\circledast_{i\in I}(\mathsf{A}_i^*))$ which interprets a formula $\&_{I\in\mathcal{N}}(\parr_{i\in I}(A_i))$ (resp. a formula $\oplus_{I\in\mathcal{N}}(\otimes_{i\in I}(A_i))$) occurring in R, we will not make use of the point $0_{\uplus_{I\in\mathcal{N}}(\circledast_{i\in I}(\mathsf{A}_i^*))}$ of A; so in the following when we will refer to 0_{A} we will mean one of the $\langle I, 0_{\circledast_{i\in I}(\mathsf{A}_i^*)}\rangle$ (for $I \in \mathcal{N}$) which belongs to A.

Definition 110 (Experiments) *Let S be a slice and e an application associating with every edge a of type A of S an element of $\mathcal{A}^*$; e is an **experiment** of S when the following conditions hold:*

- *if x, y are the conclusions of an ax link then $e(x_1) = e(x_2)$.*
- *if x, y are premises of a cut link with premises x and y, then $e(x) = e(y)$.*
- *if x is the conclusion of a negative link $-_{I\in\mathcal{N}}$ with premises x_1 of type $P_1, \ldots, x_n$ of type P_n and there exist an $i \in \{1, \ldots, n\}$ such that $e(x_i) \neq 0_{\mathsf{P}_i^*}$, then if $e(x_1) = \mathsf{a}_1, \ldots e(x_n) = \mathsf{a}_n$, $e(x) =< I, < \mathsf{a}_1, \ldots, \mathsf{a}_n >>$; otherwise either $e(x) =< I, < 0_{\mathsf{P}_1^*}, \ldots, 0_{\mathsf{P}_n^*} >>$ either $e(x) =< I, 0_{\mathsf{P}_1^* \circledast \ldots \circledast \mathsf{P}_n^*} >$;*
- *if x is the conclusion of a positive link $+_{I\in\mathcal{N}}$ with premises x_1 of type $N_1, \ldots, x_n$ of type N_n and there exist an $i \in \{1, \ldots, n\}$ such that $e(x_i) \neq 0_{\mathsf{N}_i^*}$, then if $e(x_1) = \mathsf{a}_1, \ldots e(x_n) = \mathsf{a}_n$, $e(x) =< I, < \mathsf{a}_1, \ldots, \mathsf{a}_n >>$; otherwise either $e(x) =< I, < 0_{\mathsf{N}_1^*}, \ldots, 0_{\mathsf{N}_n^*} >>$ either $e(x) =< I, 0_{\mathsf{N}_1^* \circledast \ldots \circledast \mathsf{N}_n^*} >$.*
- *if a is a positive link of conclusion x of type A and b is a negative link of conclusion y of type B, and there is a jump between b and a, then if $e(x) \neq 0_{\mathsf{A}^*}$ then $e(y) \neq 0_{\mathsf{B}^*}$.*

If the conclusion links of S have premises $x_1, \ldots, x_n$ of type respectively $A_1, \ldots, A_n$ and e is an experiment of S such that $e(x_i) = \mathsf{a}_i$ then we shall say that $< \mathsf{a}_1, \ldots, \mathsf{a}_n >$ is the **conclusion** or the **result** of the experiment e of S, and we will denote it by $|e|$. The set of the results of all experiments on S is the interpretation $[\![S]\!]$ of S; in case S is the empty slice, then its interpretation is the empty set.

Let R be a total J-proof structure and $\{S_1, \ldots, S_n\}$ the set of slices induced by the & resolutions of R; the interpretation $[\![R]\!]$ of R is the union of $[\![S_1]\!], \ldots, [\![S_n]\!]$.

Proposition 111 *If S is a correct slice, and $S \rightsquigarrow S'$, then $[\![S]\!] = [\![S']\!]$.*

Proof. Easily follows from the proof of proposition 38. □

Proposition 112 *If R, R' are weakly correct J-proof structures, such that $R' = [R]$, then $[\![R]\!] = [\![R']\!]$.*

Proof. The proof is a consequence of proposition 111. □

3.5.2 Injectivity

Definition 113 (Relational result) *Let S be a slice and $|e|$ the result of an experiment on S; $|e|$ is* ***relational*** *it does not contain any occurrence of* 0.

The set of relational results of experiments on a slice S is called the **relational part** of $[\![S]\!]$; we will denote it by $[\![S]\!]^{Rel}$.

Definition 114 (Injective result) *Let S be a slice and $|e|$ be a relational result of an experiment on S; $|e|$ is* ***injective*** *when in $|e|$ does not occur two times a same element of a pointed set X^* interpreting an atomic formula.*

Lemma 115 *Given a slice S, a positive link a with typed conclusion x and a negative link b with typed conclusion y, there is a jump (eventually transitive) between a and b iff for all experiments e of R, $e(x) \neq 0 \Rightarrow e(y) \neq 0$.*

Proof. The proof is an easy consequence of definition of experiment. □

Theorem 116 (Injectivity of slices) *Let S and S' be two cut-free correct slices with the same conclusions. If $[\![S]\!] = [\![S']\!]$ then $R = R'$.*

Proof. It easily follows from the proof of theorem 47. □

Given a slice S, we denote by S^- the slice obtained by erasing all the jumps of S.

Proposition 117 *If S is a correct, cut free slice and S' is a saturated, correct cut-free slice with the same conclusions as S, such that $[\![S]\!]^{Rel} = [\![S']\!]^{Rel}$ and $[\![S']\!] \subseteq [\![S]\!]$, then $S' = Sat(S)$.*

Proof. Let e be an injective experiment on S, which always exists. Since the result of e is in $[\![S]\!]^{Rel} = [\![S']\!]^{Rel}$, then there is an experiment e' on S', such that e and e' have the same result. Now, let c be a conclusion link of S, and c' be the correspondent of S'. Since c and c' have same type, it is simple to note that the values of e and e' on the correspondent premises of such links are equals. Hence by going from the conclusions $c_1, \ldots, c_n$ to the atomic edges, we can prove that S and S' are the same graph up to the axioms and jumps. Now since e' has the same values as e, e' is injective too, therefore the two slices have the same axioms, that is $S^- = S'^-$. Since $[\![S']\!] \subseteq [\![S]\!]$, using lemma 115 we can say that all the jumps of S are jumps of S'. In order to saturate S, we just add to S all the jumps of S' which are not jumps of S; in this way we obtain a slice $Sat(S) = S'$.

□

Given a J-proof structure R of conclusions Γ, a **&-assignment** of R is a function ϕ associating to any formula of type $\&_{I\in\mathcal{N}}(\parr_{i\in I}(P_i))$ occurring in Γ a $J \in \mathcal{N}$.

It is easy to check that if R is total, to any &-assignment ϕ it corresponds a unique slice S^ϕ of R, and to each slice induced by a & resolution of R correspond (at least one) &-assignment ϕ.

Let us consider an element $< J, \delta >$ of a pointed set $\uplus_{I\in\mathcal{N}}(\circledast_{i\in I}(\mathsf{P}_i^*))$, interpretation of a formula $\&_{I\in\mathcal{N}}(\parr_{i\in I}(P_i))$, and an element γ of the interpretation $[\![R]\!]$ of a total J-proof structure R with conclusions Γ; we say that γ is **compatible** with a &-assignment ϕ on Γ iff for any occurrence of $< J, \delta >$ in γ, on the corresponding occurrence of $\&_{I\in\mathcal{N}}(\parr_{i\in I}(P_i))$ in Γ, $\phi(\&_{I\in\mathcal{N}}(\parr_{i\in I}(P_i))) = J$.

Proposition 118 *Given a total J-proof structure R, an element γ of $[\![R]\!]$ is compatible with a &-assignment ϕ, iff γ is the result of an experiment on S^ϕ.*

Proof. Suppose γ is not a result of an experiment on S^ϕ; then it is a result of an experiment on another slice S' of R, which differ from S^ϕ for at least one component of a negative rule. But then is easy to observe that γ cannot be compatible with S'. $\square$

Proposition 119 *Given a total J-proof structure R and a &-assignment ϕ, $[\![S^\phi]\!] = \{\gamma \in [\![R]\!] | \gamma$ is compatible with $\phi\}$.*

Proof. Easy consequence of proposition 118. $\square$

Theorem 120 (Injectivity) *Let R and R' be two cut-free J-proof nets with the same conclusions Γ. If $[\![R]\!] = [\![R']\!]$ then $R = R'$.*

Proof.

Let us take the slice S^ϕ of R corresponding to the &-assignment ϕ of Γ, and suppose S^ϕ does not belong to R'. By proposition 119 $[\![S^\phi]\!] = \{\gamma \in [\![R]\!] | \gamma$ is compatible with $\phi\}$. Since $[\![R]\!] = [\![R']\!]$, $\{\gamma \in [\![R]\!] | \gamma$ is compatible with $\phi\} = \{\gamma \in [\![R']\!] | \gamma$ is compatible with $\phi\}$; then for the unique slice S'^ϕ of R' which corresponds to ϕ by proposition 119 $[\![S'^\phi]\!] = \{\gamma \in [\![R']\!] | \gamma$ is compatible with $\phi\}$, and $[\![S'^\phi]\!] = [\![S^\phi]\!]$; but then by theorem 116 $S'^\phi = S^\phi$ so S^ϕ belongs to R', contradiction.

$\square$

Proposition 121 *If R is a cut-free J-proof net and R' is a saturated cut-free J-proof net with the same conclusions as R, such that $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$ and $[\![R']\!] \subseteq [\![R]\!]$, then $R' = Sat(R)$.*

PROOF.

We prove that for every slice S of R there exist a slice S' of R' such that $S' = Sat(S)$.

Let us take the slice S^ϕ of R corresponding to the &-assignment ϕ of Γ, and suppose that for no slices S' of R', $S' = Sat(S^\phi)$. By proposition 119 $[\![S^\phi]\!] = \{\gamma \in [\![R]\!] | \gamma \text{ is compatible with } \phi\}$, and $[\![S^\phi]\!]^{Rel} = \{\gamma \in [\![R]\!]^{Rel} | \gamma$ is compatible with $\phi\}$.

Since $[\![R]\!]^{Rel} = [\![R']\!]^{Rel}$, then $\{\gamma \in [\![R]\!]^{Rel} | \gamma \text{ is compatible with } \phi\} = \{\gamma \in [\![R']\!]^{Rel} | \gamma \text{ is compatible with } \phi\}$ and since $[\![R]\!] \supseteq [\![R']\!]$, $\{\gamma \in [\![R]\!] | \gamma$ is compatible with $\phi\} \supseteq \{\gamma \in [\![R']\!] | \gamma \text{ is compatible with } \phi\}$; then for the unique slice S'^ϕ of R' which corresponds to ϕ by proposition 119 $[\![S'^\phi]\!] = \{\gamma \in [\![R']\!] | \gamma$ is compatible with $\phi\}$, and $[\![S'^\phi]\!]^{Rel} = \{\gamma \in [\![R']\!]^{Rel} | \gamma$ is compatible with $\phi\}$; so $[\![S'^\phi]\!]^{Rel} = [\![S^\phi]\!]^{Rel}$, and $[\![S'^\phi]\!] \subseteq [\![S^\phi]\!]$; but then by proposition 117 $S'^\phi = Sat(S^\phi)$, contradiction.

Similarly, we can prove that for every slice S' of R' there exist a slice S of R such that $S' = Sat(S)$; but then it is immediate that $R' = Sat(R)$.

$\square$

3.6 Correctness criterion is stable under reduction

In this section we solve the question, left opened since subsection 3.4.1, of the stability of correctness under cut-reduction. Our strategy is the following: first in subsection 3.6.1 we prove that pointed semantics is a model also for HS; then, in subsection 3.6.2 from this result and from injectivity of pointed semantics we prove that the normal form of a sequentializable J-proof net is still sequentializable.

3.6.1 Pointed set semantics and HS

We provide an interpretation $[\![\pi]\!]$ of an HS proof π in pointed sets:

if π is a proof of conclusions $\vdash \Gamma$, where Γ is a sequence of formulas $A_1, \ldots, A_n$, then $[\![\pi]\!]$ is a subset of $\mathsf{A}_1^* \circledast \ldots \circledast \mathsf{A}_n^*$, defined inductively in the following way:

- if π is the proof

$$\dfrac{}{\vdash X, X^\perp}\, ax$$

  then $[\![\pi]\!] = \{< \mathsf{a}, \mathsf{a} > | \mathsf{a} \in \mathsf{X}^*\}$.

- if π is the proof

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, A} \quad \dfrac{\pi_2}{\vdash \Delta, A^\perp}}{\vdash \Gamma, \Delta}\, cut$$

then $[\![\pi]\!] = \{< \gamma, \delta > | \exists \mathsf{a} < \gamma, \mathsf{a} > \in [\![\pi_1]\!] \text{and} < \delta, a > \in [\![\pi_2]\!]\}$.

- if π is the proof

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma} \quad \dfrac{\pi_2}{\vdash \Delta}}{\vdash \Gamma, \Delta}\, mix$$

then $[\![\pi]\!] = \{< \gamma, \delta > | \gamma \in [\![\pi_1]\!] \text{ and } \delta \in [\![\pi_2]\!]\}$.

- if π is the proof

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma_1,\ N_1} \quad \dots \quad \dfrac{\pi_n}{\vdash \Gamma_n, N_n}}{\vdash \Gamma_1, \dots, \Gamma_n, \oplus_{I \in \mathcal{N}}(\otimes_{i \in I} N_i)}\, (+, I)$$

then $[\![\pi]\!] = \{< \gamma_1, \dots, \gamma_n, < I, a >> | < \gamma_1, \mathsf{a}_1 > \in [\![\pi_1]\!], \dots, < \gamma_n, \mathsf{a}_n > \in [\![\pi_n]\!]\}$, where $a = 0_{\mathsf{N}_1^* \circledast \dots \circledast \mathsf{N}_n^*}$ or $a =< \mathsf{a}_1, \dots, \mathsf{a}_n >$; if $a = 0_{\mathsf{N}_1^* \circledast \dots \circledast \mathsf{N}_n}$ then for all i, $\mathsf{a}_i = 0_{\mathsf{N}_i^*}$.

- if π is the proof

$$\dfrac{\dfrac{\pi_1}{\vdash \Gamma, P_1^1 \dots, P_{k_1}^1} \quad \dots \quad \dfrac{\pi_n}{\vdash \Gamma, P_1^n \dots, P_{k_n}^n}}{\vdash \Gamma, \&_{J \in \mathcal{N}}(\invamp_{j \in J} P_i)}\, (-, \mathcal{N})$$

then $[\![\pi]\!] = \bigcup_{I \in \mathcal{N}} \{< \gamma, < I, a >> | < \gamma, \mathsf{a}_1^i, \dots, \mathsf{a}_{k_i}^i > \in [\![\pi_i]\!]\}$ where $a = 0_{\mathsf{P}_i^{1*} \circledast \dots \circledast \mathsf{P}_i^{k_i *}}$ or $a =< \mathsf{a}_i^1, \dots, \mathsf{a}_i^{k_i} >$; if $a = 0_{\mathsf{P}_i^{1*} \circledast \dots \circledast \mathsf{P}_i^{k_i *}}$ then

i) for all j, $\mathsf{a}_i^j = 0_{\mathsf{P}_i^{j*}}$ and

ii) $\gamma = 0_{\mathsf{C}_1^*}, \dots, 0_{\mathsf{C}_l^*}$ (if $\Gamma = C_1, \dots, C_l$). .

In order to prove that pointed sets are a semantics for HS proofs, we first prove that we can simulate cut-reduction on HS proofs using slices (for a precise definition of cut-elimination in HS we refer to [Gir07]).

To an HS proof π we can associate a set of slices $S(\pi)$ by induction on the height of π in the following way:

let r be the last rule of the HS proof π. We define the set of slices $\mathcal{S}(\pi)$ (with the same conclusions as π) by induction on π.

- If r is an axiom with conclusions $X, X^{\perp}$, then the unique slice of $\mathcal{S}(\pi)$ is an axiom link with conclusions $X, X^{\perp}$.
- If r is a a *cut* rule with premises the subproofs π_1 and π_2, then $\mathcal{S}(\pi)$ is obtained by connecting every slice of $\mathcal{S}(\pi_1)$ and every slice of $S(\pi_2)$ by means of a *cut*-link.
- if r is a Mix rule, with premises the subproofs π_1 and π_2, then $\mathcal{S}(\pi)$ is obtained by taking for every slice in $\mathcal{S}(\pi_1)$ and every slice in $\mathcal{S}(\pi_2)$ their disjoint union.
- If r is a $(+, I)$-rule with premises the subproofs $\pi_1, \ldots, \pi_n$, then $\mathcal{S}(\pi)$ is obtained by connecting every slice in $\mathcal{S}(\pi_j)$ with every slice of $\mathcal{S}(\pi_k)$ (with $j \neq k$) by means of a $+_{I \in \mathcal{N}}$-link.
- If r is a $(-, \mathcal{N})$-rule with premises the subproofs $\pi_1 \ldots \pi_n$ (one subproof π_i for each $I \in \mathcal{N}$), then $\mathcal{S}(\pi)$ is obtained by adding to every slice S of $\mathcal{S}(\pi_i)$:
  - a $-_{I \in \mathcal{N}}$-link b_i;
  - for all a positive terminal link a of S, add a jump between a and b_i in S;

  and then by taking the union of all this sets of slices.

Given a set of slice $\mathcal{S}$, we say that $\mathcal{S}$ **reduces** to the set of slices $\mathcal{S}'$ if $\mathcal{S}'$ is obtained from $\mathcal{S}$ by reducing some (even none) of the slices of $\mathcal{S}$.

Proposition 122 *Let π be an HS proof, and π' be a cut-free proof obtained from π by a cut-elimination step. Then $\mathcal{S}(\pi)$ reduces to $\mathcal{S}(\pi')$.*

Proof.
If π reduces to π' with a commutative step then it is clear that $\mathcal{S}(\pi) = \mathcal{S}(\pi')$; otherwise, the proof easily follows from the fact that to each slice S' of $\mathcal{S}(\pi')$ correspond a slice S of $\mathcal{S}(\pi)$ such that either $S' = S$ either S' is obtained from S by a reduction step. The slices of $\mathcal{S}(\pi)$ which are not (or not reduces to) slices of $\mathcal{S}(\pi')$, are all the slices which reduces in a step to the empty slice. □

Proposition 123 *Let π be an HS proof, and $\mathcal{S}(\pi) = S_1, \ldots, S_n$ the set of slices associated to π. Then $[\![\pi]\!] = \cup_{i \in \{1,\ldots,n\}}([\![S_i]\!])$.*

Proof. Easy induction on π. □

Proposition 124 *Let π be an HS proof and π' be a proof obtained by reducing a cut in π. Then $[\![\pi]\!] = [\![\pi']\!]$.*

PROOF. By proposition 123 $[\![\pi]\!] = \cup_{i\in\{1,\ldots,n\}}([\![S_i]\!])$ for $\mathcal{S}(\pi) = S_1, \ldots, S_n$, and $[\![\pi']\!] = \cup_{i\in\{1,\ldots,k\}}([\![S'_i]\!])$ for $\mathcal{S}(\pi') = S'_1, \ldots, S'_k$; then the proof follows from proposition 122 and theorem 111. □

3.6.2 Stability of the criterion

Proposition 125 *If R is a J-proof structure sequentializable into a proof π, then $[\![\pi]\!]^{Rel} = [\![R]\!]^{Rel}$ and $[\![\pi]\!] \subseteq [\![R]\!]$.*

PROOF. Trivial, from the fact that any element of $[\![\pi]\!]$ induces an experiment of R. □

Theorem 126 *Given a sequentializable J-proof structure R, $[R]$ is sequentializable.*

PROOF.

Since R is sequentializable, we can associate to it a proof π, and by proposition 125, $[\![\pi]\!] \subseteq [\![R]\!]$. We reduce π into a cut-free proof π_0; Since semantics is preserved by cut-elimination by proposition 124, $[\![\pi]\!] = [\![\pi_0]\!]$. Now consider the normal form R_0 of R; by proposition 112 $[\![R]\!] = [\![R_0]\!]$

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \big\downarrow & & \big\downarrow \\ \pi & \longrightarrow & \pi_0 \end{array}$$

□

If $\mathcal{S}(\pi_0) = S_1, \ldots, S_n$ it is immediate that the superposition $\between\,(S_1, \ldots, S_n)$ is a saturated J-proof net R', and obviously $[\![\pi_0]\!] = [\![R']\!]$. Since $[\![R']\!]^{Rel} = [\![\pi_0]\!]^{Rel} = [\![\pi]\!]^{Rel} = [\![R]\!]^{Rel} = [\![R_0]\!]^{Rel}$, and $[\![R']\!] = [\![\pi_0]\!] = [\![\pi]\!] \subseteq [\![R]\!] = [\![R_0]\!]$ by proposition 121 $R' = Sat(R_0)$; but then, R_0 is sequentializable into π_0.

3.7 J-proof nets and degrees of sequentiality

In this section we isolate some specific classes of J-proof nets, with respect to their degree of sequentiality.

In subsection 3.7.1, we define two subsets of J-proof nets, the ones with minimal sequentiality and the ones with maximal sequentiality, by providing inductive procedures for constructing them. Such procedures are based on the grammars for generating *parallel L-nets* and *L-forests* defined in [CF]. Then in the remaining two subsections we show how the notion of *box* can be retrieved using jumps, by relating J-proof nets with sliced polarized proof nets of [LTdF04] (in subsection 3.7.2) and with proof nets with additive boxes, both the standard ones of [Gir87] and the *multiboxes* of [TdF03b] (in subsection 3.7.3).

For simplicity's sake in this section we will deal only with cut-free J-proof nets.

3.7.1 Minimal and maximal sequentiality

Definition 127 *A **J-proof net of minimal sequentiality** is a J-proof net which is built inductively in the following way:*

- *An axiom link is a J-proof net of minimal sequentiality;*
- *If $R_1, \ldots, R_n$ are J-proof nets of minimal sequentiality of conclusions respectively $\Gamma_1, \ldots, \Gamma_n$, where the formulas in Γ_i are all positive, the the union of $R_1, \ldots, R_n$ is a J-proof net of minimal sequentiality.*
- *If $R_1 \ldots, R_n$ are J-proof nets of minimal sequentiality of conclusions respectively $\Gamma_1, N_1, \ldots \Gamma_n, N_n$ then the J-proof net of conclusions $\Gamma_1, \ldots, \Gamma_n, \oplus_{J\in\mathcal{N}}(\otimes_{j\in J}(N_j))$ obtained by erasing from each R_i the conclusion link of type N_i and then connecting all R_i together with a $+_{I\in\mathcal{N}}$-link of conclusion $\oplus_{J\in\mathcal{N}}(\otimes_{j\in J}(N_j))$, is a J-proof net of minimal sequentiality.*
- *If $R_1, \ldots, R_n$ are J-proof nets of minimal sequentiality of conclusions respectively $\Gamma, P^1_1, \ldots P^1_{k_1}, \ldots \Gamma, P^n_1 \ldots P^n_{k_n}$ then the J-proof net of conclusion $\Gamma, \&_{J\in\mathcal{N}}(\invamp_{j\in J}(P_j))$ obtained in the following way is a J-proof net of minimal sequentiality:*
  1. *for each R_i erase the conclusion links of type $P^i_1 \ldots P^i_{k_i}$ and add a $-_{I\in\mathcal{N}}$ link b_i of conclusion $\&_{J\in\mathcal{N}}(\invamp_{j\in J}(P_j))$ in such a way to get a J-proof net R'_i of conclusions $\Gamma, \&_{J\in\mathcal{N}}(\invamp_{j\in J}(P_j))$ for each $I \in \mathcal{N}$;*
  2. *for every R'_i and for every positive link a of R'_i such that there exist an $R'_{i'}$ with $i \neq i'$ which does not share a, add a jump in R'_i between a and b_i, obtaining a J-proof net R''_i;*
  3. *take the superposition of all $R''_1, \ldots, R''_n$.*

Definition 128 *A **J-proof net of maximal sequentiality** is a J-proof net which is built inductively in the following way:*

- *An axiom link is a J-proof net of maximal sequentiality;*
- *If $R_1, \ldots, R_n$ are J-proof nets of maximal sequentiality of conclusions respectively $\Gamma_1, \ldots, \Gamma_n$, where the formulas in Γ_i are all positive, then the union of $R_1, \ldots, R_n$ is a J-proof net of maximal sequentiality.*

- *If $R_1 \dots, R_n$ are J-proof nets of maximal sequentiality of conclusions respectively $\Gamma_1, N_1, \dots \Gamma_n, N_n$ then the J-proof net of conclusions $\Gamma_1, \dots, \Gamma_n, \oplus_{J\in\mathcal{N}}(\otimes_{j\in J}(N_j))$ obtained by erasing from each R_i the conclusion link of type N_i and then connecting all R_i together with a $+_{I\in\mathcal{N}}$-link of conclusion $\oplus_{J\in\mathcal{N}}(\otimes_{j\in J}(N_j))$, is a J-proof net of maximal sequentiality.*

- *If $R_1, \dots, R_n$ are J-proof nets of maximal sequentiality of conclusions respectively $\Gamma, P^1_1, \dots P^1_{k_1}, \dots \Gamma, P^n_1 \dots P^n_{k_n}$ then the J-proof net of conclusion $\Gamma, \&_{J\in\mathcal{N}}(\parr_{j\in J}(P_j))$ obtained in the following way is a J-proof net of maximal sequentiality:*

  1. *for each R_i erase the conclusion links of type $P^i_1 \dots P^i_{k_i}$ and add a $-_{I\in\mathcal{N}}$ link b_i of conclusion $\&_{J\in\mathcal{N}}(\parr_{j\in J}(P_j))$ in such a way to get a J-proof net R'_i of conclusions $\Gamma, \&_{J\in\mathcal{N}}(\parr_{j\in J}(P_j))$ for each $I \in \mathcal{N}$;*
  2. *for every R'_i and for every positive terminal link a of R'_i, add a jump in R'_i between a and b_i, obtaining a J-proof net R''_i;*
  3. *take the superposition of all $R''_1, \dots, R''_n$.*

Given a sequent calculus proof π, we could either associate a J-proof net of minimal sequentiality π^{*min}, either a J-proof net of maximal sequentiality π^{*max} by induction on the height of π in the obvious way.

3.7.2 J-proof nets and polarized boxes

In [LTdF04] Laurent and Tortora de Falco introduced a notion of proof net for the polarized fragment of linear logic, LL_{pol} (see [Lau02]) as set of slices glued together using exponential boxes; they called such a proof net *sliced polarized proof net*.

In this subsection we study the relation between sliced polarized proof nets without structural rules (that is in the fragment $MALL^{\uparrow\downarrow}_{pol}$) and J-proof nets.

We do not provide a direct translation of one syntax into the other; nevertheless, we define a condition on J-proof nets (the *polarized boxing condition*), and we show that this condition is analogous to the condition on boxes in sliced polarized proof nets; furthermore we prove that the J-proof nets which satisfy the polarized boxing condition are exactly the ones with maximal sequentiality.

Definition 129 (Polarised box) *Given a J-proof net R, we call* ***polarized box*** *of a negative rule $W = \{w_1, \dots, w_n\}$ the set of links hereditary above some $w_i \in W$ in R.*

Definition 130 *A J-proof net R satisfy the* ***polarized boxing condition*** *iff given two polarized boxes B_1, B_2 of R, either they are disjoint, either one of them is strictly included into the other.*

Proposition 131 *A J-proof net R satisfies the polarized boxing condition iff R is a J-proof net of maximal sequentiality.*

PROOF. The proof is an easy induction on R. □

Proposition 132 *Let R be a J-proof net which satisfies the boxing condition: then each polarized box B of R can be decomposed into a set of slices $\mathcal{S}(B)$.*

PROOF. By proposition 131, R is a J-proof net of maximal sequentiality, so by construction given a polarized box B of a negative rule $W = w_1, \ldots, w_n$ to each w_i corresponds a subnet R_i of R; we take as $\mathcal{S}(B)$ the set of slices of each R_i. □

Given a J-proof net R which satisfies the boxing condition, we define the **depth** of a node b in R as the maximal number of polarized boxes containing b. Given a node b which belongs to a polarized box B of R, the depth of b with respect to B is the maximal number of boxes included in B which contains b.

Proposition 133 *In an s-connected J-proof net R which satisfies the polarized boxing condition:*

- *there is at most one positive link at depth 0;*
- *for any slice in the set $\mathcal{S}(B)$ associated with a polarized box B of R there is at most one positive link at depth 0 with respect to B.*

PROOF. Using proposition 131 and 132, the proof is an easy induction on the construction of R. □

3.7.3 J-proof-net and additive boxes

The first solution proposed in [Gir87] to represent the &-rule in proof nets, was to deal with it explicitly, using a box called *additive box*; analyzing the interpretation of the &-rule in coherent semantics, Tortora de Falco in [TdF03b] refined the notion of additive box in the one of *multibox*, as the superposition of several additive boxes.

In this subsection, as in the previous one, we do not give a direct translation of proof nets with additive boxes (resp. multiboxes) into J-proof nets; we provide instead a condition on J-proof nets called *additive boxing* (resp. *multiboxing*) condition, characterizing a subclass of J-proof nets.

The additive boxing (resp. multiboxing) condition is analogous to the condition on boxes given in [Gir87] (resp. in [TdF03b]).

Definition 134 (Additive box) *Given a J-proof net R, we call* ***additive box*** *of a negative rule $W = \{w_1, \dots, w_n\}$ with $n \geq 2$ the set of links hereditary above some $w_i \in W$ in R; the i-th component of an additive box is the set of links hereditary above $w_i \in W$ in R.*

Proposition 135 *Let R be a J-proof net. Given two different components of an additive box B, they are disjoint.*

Proof. An easy consequence of the condition **views** of definition 69. □

Given a J-proof net R:

i) R satisfies the **additive boxing condition** if given two additive boxes B_1, B_2 of a J-proof net R, either they are disjoint, either one of them is strictly included into the other;

ii) R satisfies the **additive multiboxing condition** if given two additive boxes B_1, B_2, if they are not disjoint, they are equal. Given a maximal set $W_1, \dots, W_n$ with the same additive box B, we call B the **multibox** of $W_1, \dots, W_n$.

An example of J-proof net respecting the additive boxing (resp. the multiboxing) condition is the one depicted in fig.3.7 (resp. 3.4). It easy to build an example of J-proof net which does not satisfy neither the additive neither the multiboxing condition.

3.8 Final remarks

To conclude, let us spent some few words on some points which still need further investigation:

- in the last section, we gave some hints on how to recover some standard syntaxes for additive proof nets in the setting of J-proof nets; nevertheless, the relation between J-proof nets and the proof nets defined by Hughes and Van Glabbeek needs still to be clarified. In this spirit, it could be interesting to verify if our approach to sequentialization can still be applied in their setting;

- ludics taught us the intrinsic interest of considering *partial* objects in proof theory; discarding the constraint of totality from the correctness criterion for J-proof nets and introducing the *Daimon* rule of ludics, may enlighten interesting computational features;

- recent works by Faggian and Piccolo [FP07] exploit the relation between L-nets and linear π-calculus, a typed π-calculus introduced by Berger, Honda and Yoshida in [MBY03], enlightening the operational content of the additives as a kind of non-deterministic choice; the bridge is the correspondence between L-nets and event structures, a model of concurrency introduced by Nielsen, Plotkin and Winskel in

[MNW81]. Following this approach, the analysis could be extended to the relation between event structures and J-proof-nets, in order to give a proof theoretical characterization of terms in linear π-calculus; this should contribute to the general purpose of bringing together proof theory, game semantics and concurrency theory.

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