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FREE GEOMETRIC THEORY
FOR
FOR HIGHER-SPIN FIELDS

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A Flavio, Filippo e Sveva,
trottolini sorridenti
alla scoperta del mondo

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Rome, December 13, 2005 Dario Francia

Contents

Introduction

Quantum Field Theory was born as an attempt to merge in a consistent fashion the principles of Quantum Mechanics with the (local) structure of space-time, as described by Special Relativity. From this point of view, in order to look for a fundamental description of elementary particles, it is quite reasonable to classify the irreducible representations of the symmetry group of space-time, and try then to build consistent non-trivial theories involving generic elements taken from this classification, since there is no reason *in principle* to suggest that certain irreducible representations should be regarded as "privileged" with respect to others. In particular, in four space-time dimensions all these representations are classified according to their mass and spin, and this explains why the construction of a consistent theory involving arbitrary-spin representations of the Poincaré group was felt as a natural challenge since the very beginning of Relativistic Quantum Mechanics, starting with the works of Majorana [1] and Dirac [2].

In that phase the efforts were mainly devoted to the definition of consistent wave equations, and simultaneously to the clarification of the group theoretical structure lying behind the notion of relativistic invariance [3], [4], [5], [6], [7], while the construction of consistent Lagrangians, at least for the free dynamics, was one of the main theoretical problems left open.

In the fifties, phenomenological support to the interest in the subject came from the observation in accelerator experiments of resonances of spin greater than one, before it was recognised that the proper way to interpret them was as composite states of lower spin constituents. A second series of works stimulated by these motivations culminated in the Feynman rules of [8], [9] and [10] allowing for the description of particles of any spin, both massive and massless, in the framework of the S-matrix program. Slightly later, the

Lagrangian problem was analysed again in [11], [12] and [13], and was finally solved for the massive free theory in [14] and [15]. A peculiar feature of the resulting Lagrangians is that the requirements to be met in order for the fields to define irreducible representations of the Poincaré group are there partly imposed as algebraic constraints on the tensors and partly recovered on-shell as dynamical conditions. This dichotomy will survive almost all the subsequent evolution of the subject, and it will be exactly on the possibility to surmount it that this Thesis will be focused.

In this first period of the higher-spin history (roughly starting in 1932 with [1] and ending in 1974 with the discovery of the massive Lagrangians in [14] and [15]) the first difficulties emerged with the interactions, involving once again the role of the constraints, together with non-causal propagations and other complications, all showing up in the analysis of interactions defined by means of simple vertices (like the ones defined by the minimal coupling) [4], [16], [17]. These results showed that, besides the lack of phenomenological evidence for higher-spin elementary constituents, there could also be some *theoretical* difficulties in extending to these systems the simpler properties of their lower-spin counterparts.

Novel motivations came when Fronsdal first observed in [18] that the massless limit of the Lagrangians of [14] and [15] leads to an abelian gauge theory for bosons of any spin. That paper, together with the related one dealing with fermions [19], is a milestone for higher-spin theory. First, it fixed what has been considered since then the form of the free theory for higher-spin gauge fields, supported in this respect by other independent results [20] [21], and, perhaps most important, because it led to evaluate the possibility that Yang-Mills theories and Gravity could be regarded as special members of a wider class of systems, whose dynamical properties and quantum behaviour could have been understood in a unifying framework. This possibility, in a sense, is still today the main motivation for the interest in higher spins.

From the point of view of the present work, it is to be noted that in the Fronsdal equation for a spin-s boson

$$\mathcal{F}_{\mu_1 \ldots \mu_s} \equiv \Box\, \varphi_{\mu_1 \,\ldots\, \mu_s} - (\partial_{\mu_1}\, \partial^{\alpha}\, \varphi_{\alpha\, \mu_2 \,\ldots\, \mu_s} + \text{symm.}) + (\partial_{\mu_1}\, \partial_{\mu_2}\, \varphi^{\alpha}{}_{\alpha\, \mu_3 \,\ldots\, \mu_s} + \text{symm.}) = 0\,, \tag{1}$$

the gauge field $\varphi_{\mu_1 \dots \mu_s}$ is a rank-s tensor subject to the condition that its double trace be *identically zero*

$$\varphi^{\alpha\beta}{}_{\alpha\beta\mu_5\dots\mu_s} \equiv 0 \,, \tag{2}$$

while the gauge parameter defining the transformation

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{\mu_1} \Lambda_{\mu_2 \dots \mu_s} + \text{symm.} \,, \tag{3}$$

under which eq. (1) is invariant, is a rank-$(s-1)$, *traceless* tensor:

$$\Lambda^{\alpha}{}_{\alpha\mu_3\dots\mu_{s-1}} \equiv 0 \,. \tag{4}$$

Similar conditions are met for fermions, described by symmetric, rank-n spinor-tensors, whose *triple* γ*-trace* is forced to vanish

$$\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\psi_{\mu\nu\rho\alpha_4\dots\alpha_n} \equiv 0 \,, \tag{5}$$

and where the gauge parameter defining the transformation law

$$\delta \psi_{\alpha_1 \dots \alpha_n} = \partial_{\alpha_1} \epsilon_{\alpha_2 \dots \alpha_n} + \text{symm.} \,, \tag{6}$$

is a symmetric, rank-$(n-1)$ spinor-tensor, whose γ*-trace* is assumed to vanish

$$\gamma^{\mu}\epsilon_{\mu\alpha_2\dots\alpha_{n-1}} \equiv 0 \,. \tag{7}$$

These constraints, that in the Fronsdal theory are a legacy of analogous conditions on the massive fields of [14] and [15], represent an unusual difference with respect to the way lower-spin gauge theories are usually described, and in this work I will explore the possibility of avoiding both of them in the free theory. Here I would only like to anticipate one of the conclusions that I will present.

If in the Fronsdal theory one couples a source $\mathcal{J}$ to the field φ in the "minimal" way, i.e. adding to the Lagrangian a term[1] $\varphi \cdot \mathcal{J}$, it is clear that this source should reflect the symmetries of the field. In particular, in this case gauge invariance *does not* require the usual condition that the current be conserved, but only the weaker constraint that *the traceless part of the divergence* of the current should be zero. On the other hand, this

[1] I indicate here with a dot "·" the contraction of indices.

is not the natural condition one would get from Noether's theorem, assuming that an invariant Lagrangian for the "matter" fields determining $\mathcal{J}$ could be given. In Chapter 3 it will be shown that it is possible to describe the same classical dynamics of the Fronsdal equations without any constraints on the field φ. To do that in a *local* theory, a couple of auxiliary fields are to be introduced, and it is exactly the equation of motion of one of these auxiliary fields that provides in this framework the condition for the source $\mathcal{J}$ to be *fully divergence free*, on shell. Hence, the formulation I will present allows to reconsider the theory of higher-spin currents (see for example [22]), that in this framework are indeed properly conserved.

In the years between the end of the sixties and the beginning of the eighties the possibility that higher-spin gauge theories could provide a clue to a deeper insight into the open problems of Quantum Field Theory and Quantum Gravity, was confronted with serious conceptual difficulties, that influenced drastically the research in the area.

In 1967 Coleman and Mandula [23] showed that under the hypotheses of Axiomatic Quantum Field Theory, together with reasonable assumptions on the spectrum of the theory itself, that in particular was bound to involve only a *finite* number of particles having mass $m < M$, for any fixed, positive value of M, the symmetry algebra of any Field Theory giving rise to a non-trivial S-matrix could only be *at most* given by the direct sum of a the Poincaré algebra[2] and an *internal* algebra, where "internal" means that the generators of this algebra commute with those of space-time symmetries. This theorem, together with its supersymmetric extension [24], ruled out the possibility of conventional couplings involving a finite number of gauge-fields of spin $s \geq 3$, which would be associated to forbidden symmetry generators of spin $s \geq 2$, with their lower-spin counterparts.

Moreover, also at the classical level, the construction of a Lagrangian in which the coupling among (higher-spin) gauge fields determines only a proper *deformation* of the gauge symmetry, and not its complete breaking, as needed in order to avoid the production of spurious states once the interaction is switched on, was found to be in general quite problematic.

A paradigmatic example is given by the so-called Aragone-Deser problem [25], [26]

[2]The extension to a conformal algebra is possible, according to the Coleman-Mandula theorem, at the algebraic level. In a conformal theory, on the other hand, there is no clear definition of an "S-matrix".

(see also [27]), according to which the mechanism that led to the discovery of Supergravity [28], [29] cannot be adapted to a spin $s = \frac{5}{2}$ particle . Briefly, to give a simplified description of the difference between these two settings, one can observe that in the latter case the variation of the Rarita-Schwinger equation for a spin-$\frac{3}{2}$ particle coupled to Gravity is proportional to the Ricci tensor, and thus vanishes if one can assume that the gravitational field satisfies its equations of motion in absence of matter. In the former case on the other hand, the same variation contains a term proportional to the Riemann tensor, and as a result the equation for the "hypergravitino" is not invariant even if the Ricci tensor vanishes. More generally, no modification of the equations of motion and of the transformation laws for the particles involved was found so that the gauge symmetry could be recovered (and generalised) at the end.

These "no-go theorems", two of the most important and influent among a wide set of negative results regarding the difficulties met in constructing consistent interactions for higher-spin gauge fields, called for a critical revision of the assumptions on which they were based, a process that in a sense is still going on nowadays, and that started to give the first results in the mid-eighties, when for the first time cubic vertices for self-interacting higher-spin fields (in a flat background, and *not* in interaction with Gravity) were exhibited in [30], [31], [32] and [33] (for recent work about cubic vertices in a flat background and general dimensions see [34], [35] and references therein). An interesting feature of these results was the realisation of a previous conjecture about higher-spin interactions (see for instance [36]), namely that a consistent system of interacting fields of spin greater than 2 would require the cooperation of *infinitely* many spin at the same time. This point, in particular, marked a relevant departure from the hypotheses of the Coleman-Mandula theorem.

After these positive signals, the real breakthrough was the discovery, by Fradkin and Vasiliev in [37] and [38], of a cubic vertex for higher-spin gauge fields *in interaction with the gravitational field*, followed in subsequent years by the construction of consistent equations of motion for systems of interacting higher-spin gauge fields, and by a detailed study of the properties of the related higher-spin algebras ([39], [40], [41], [42], [43], [44], [45], [46]; for a more recent review see [47], while interesting, related contributions are due Sezgin and Sundell [48], [49], [50] [51], who very recently obtained the important result of finding an *explicit*, non-trivial solution to the Vasiliev equations [52]). One main feature of the Vasiliev construction, that allows to forego the restrictions implied by the aforementioned no-go results, is the presence of a cosmological constant λ, that forces the background to be non flat, and allows the introduction of cubic couplings depending on inverse powers of λ (so that the flat limit $\lambda \to 0$ is singular) capable of eliminating the remainders in the Aragone-Deser result, at least to first order in the curvature expansion. Another salient aspect is the fact that the higher-spin algebra, on which the field equations are based, does not admit a consistent truncation to a finite subset of fields with spin greater than 2, thus providing a rationale to the previous results and conjectures about the need to consider a denumerable quantity of interacting fields simultaneously. In the Vasiliev equations indeed, consistency requires that infinitely many fields of increasing spin simultaneously cooperate to determine the dynamics.

This situation is somewhat reminiscent what happens in String Theory, where the spectrum of the string naturally involves infinitely many *massive* Regge trajectories, whose masses and spins are related, for open strings, by the formula

$$m_i^2\,(s)\;\sim\;\frac{1}{\alpha\,'}\,(s\;-\;s_0\,(i))\,, \tag{8}$$

where $\frac{1}{\alpha'}$ is the string tension, and "i" indicates the i-th Regge trajectory. It has been long conjectured that this massive phase could be the result of some mechanism of spontaneous breaking of the symmetry present in some more symmetric phase, where all states are massless and the theory should be described in terms of an higher-spin gauge field theory. This kind of speculation is indeed at the root of most of the contemporary interest for this subject, along with the basic observation that the presence of conformally-flat, but not flat, backgrounds, naturally calls for the investigation about the possibility that higher-spin theories could have non-trivial holographic duals at weak coupling, and stimulates research in the framework of the (A)dS-CFT correspondence on the properties of higher-spin currents [22], [53], [50], [54], [55], [56], [57], [58].

Aside from these nice features, the Vasiliev formulation presents a problem in the difficulty to identify an action principle for the interacting equations of motion[3] while the

[3] In a less recent formulation of these equations, the higher-spin algebra was realised by means of *spinor* oscillators in the van der Waerden formalism, and the basic fields of Vasiliev construction were expressed in

free limit form of the action is fixed by consistency with the Fronsdal formulation. This difficulty is the main obstacle to a full understanding of the properties of this interacting system, and one consistent step forward would be to understand at least if such difficulties are simply of technical origin or rather lie more deeply inside the structure of the theory.

The results that I shall describe in this Thesis cannot give so far an answer to this question, but my hope is that they will provide some insight in the right direction.

As already mentioned, the Vasiliev theory is defined in such a way that its free limit gives the Fronsdal equations (1). These equations are defined in terms of a constrained tensor and enjoy a constrained gauge symmetry, according to eqs. (2) and (4). Both features appear a bit unusual for a gauge theory, at least having in mind the basic examples of the Maxwell and (linearised) Einstein cases.

From a general point of view, given that the free fields are bound to satisfy certain conditions along classical trajectories, it could look as a matter of taste which ones of these conditions should be considered as intrinsic features, and which as dynamical relations to be produced by the equations of motion. For the higher-spin gauge fields, following Fronsdal, it has long been assumed that the double-tracelessness should be imposed as an algebraic constraint.

Our proposal is that this assumption is not necessary, and indeed it hides certain features of the free theory, that could prove to be relevant in a more interesting framework than the free theory itself. In order to clarify the roots of our idea, the first Part of the Thesis is devoted to tracing the origin of the constraints in the Fronsdal formulation, starting from the Fierz equations, where trace conditions had to be imposed on the massive fields.

In the first Chapter the Fierz-Pauli theory for the propagation of massive higher-spin fields is described [3], [4]. The conditions to be satisfied on-shell by the massive fields are of two types, since in order to describe an irreducible representation of the Poincaré group,

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terms of products of these oscillators, whose symmetry properties implied that the traces in their products were automatically removed, This determined, for technical reasons, that the system was automatically on-shell. More recently, an alternative formulation involving *vector* oscillators has been proposed and this allowed for the possibility to maintain the traces, avoiding that the system fall automatically on the equations of motion, but still an action principle is not available. See anyway [59], [60] for progress in this direction.

a symmetric tensor must be both *traceless* and *divergenceless*. The last requirement was found to be in contrast with the introduction of the coupling with an electromagnetic field via the minimal substitution, and in order to surmount such difficulties Fierz and Pauli proposed a Lagrangian approach in which the condition on the divergence had to be produced as a consequence of the variational principle. The trace constraint, on the other hand, that was harmless from the Fierz-Pauli point of view, was left as an algebraic property of the tensors, and as such was considered by Singh and Hagen [14], [15] when they solved the problem left open by Fierz and Pauli, obtaining explicitly Lagrangians for massive free equations of motion for any spin.

The Singh-Hagen Lagrangians, as suggested by Fierz and Pauli, involve a set of auxiliary fields, and in Chapter two it is described how the decoupling of almost all of these fields in the massless limit, studied by Fronsdal and Fang-Fronsdal [18] [19], takes place, and how the dynamical field happens to be described by a *doubly traceless* tensor, as a result of a field redefinition involving the only auxiliary field left over after the mass is removed. From this point of view, the tracelessness of the gauge parameter appear as a consistency condition given the choice of a doubly traceless field. Specific observations on this point are also presented in Section 4.2.3. The Fronsdal theory is then shown in Section 2.3 to propagate the correct number of degrees of freedom, by means of a standard calculation in light-cone coordinates, and to conclude the first Part, I collect some observations and add some more comments on the role of the constraints, in order to make a bridge with the second Part of the Thesis, where our results about the possibility of "covariantising" those constraints are given.

The starting point of our method [61] is the Fronsdal Lagrangian[4] for equation (1), and its fermionic analogue, eq. (2.37), here regarded as involving *unconstrained* fields, transforming according to (3), for the bosonic case and (6) for the fermionc one, but with an *unconstrained* gauge parameter. Under these conditions the Fronsdal Lagrangian cannot, of course, be invariant (and a more specific analysis of the contribution of the double trace of the field and of the trace of the gauge parameter is proposed). I show that

[4]To be precise, the Fronsdal equations are not directly Lagrangian equations, and an intermediate trace procedure on the Lagrangian equations is needed in order to end up with (1), as explained in Section 2.1.

by means of a compensator field α, transforming according to

$$\delta\,\alpha_{\,\mu_1\ldots\mu_{s-3}} \;=\; \Lambda^{\rho}{}_{\,\rho\,\mu_1\ldots\mu_{s-3}}\,, \tag{9}$$

it is possible to modify the original Lagrangian in order that in its variation *all terms involving the trace of the gauge parameter disappear*. More importantly, the remainder is proportional to the *gauge invariant combination*

$$\varphi^{\alpha\,\beta}{}_{\alpha\,\beta\,\mu_5\,\ldots\,\mu_s} \;-\; 4\,\partial\cdot\alpha_{\,\mu_5\ldots\mu_s} \;-\; (\partial_{\mu_5}\,\alpha^{\,\rho}{}_{\rho\mu_6\ldots\mu_s} \;+\; \text{symm.})\,, \tag{10}$$

in such a way that, by completing the Lagrangian with this last constraint enforced by a Lagrange multiplier β, and assigning to β the gauge transformation

$$\delta\,\beta_{\,\mu_1\ldots\mu_{s-3}} \;=\; \partial\cdot\partial\cdot\Lambda_{\,\mu_1\ldots\mu_{s-3}}\,, \tag{11}$$

one ends up with the *fully gauge invariant* Lagrangian (3.14), whose equations of motion are shown in Section 3.1.2 to propagate the correct number of polarisations, together with some novel properties about the coupling of the dynamical field φ with an external current, already discussed in this Introduction. The whole procedure is repeated step by step in Section 3.2, thus showing that, at the price of introducing a couple of auxiliary fields[5], it is possible to completely "covariantise" the constraints, deducing all the conditions needed for a consistent description of the dynamics from the dynamics itself together with the gauge symmetry of the theory.

In Chapter 4 I shall face with the issue of eliminating the auxiliary fields, without introducing back the conditions (2) and (4) on the traces. The end result [62], [63], similar from the methodological point of view to an old proposal by Fronsdal [12] and Chang [?], is a *non-local* theory, as the elimination of the compensator α unavoidably leads to the appearing of inverse powers of the d'Alembertian operator. For example, for spin three, the non-local equations are

$$\mathcal{F}_{\mu_1\mu_2\mu_3} \;-\; \frac{1}{3\,\Box}\,\left(\partial_{\mu_1}\,\partial_{\mu_2}\,\mathcal{F}'_{\mu_3} \,+\, \partial_{\mu_2}\,\partial_{\mu_3}\,\mathcal{F}'_{\mu_1} \;+\; \partial_{\mu_3}\,\partial_{\mu_1}\,\mathcal{F}'_{\mu_2}\right) \;=\; 0\,, \tag{12}$$

where the operator[6] $\mathcal{F}$ is defined in eq. (1). The dynamical content of this theory is anyway still equivalent to the Fronsdal one, first because it can be shown that *all non-localities*

[5] The role of β is further commented upon in Section 4.2.5.

[6] Here and in the following the right-hand side of the Fronsdal equation wil be loosely referred to as "the Fronsdal operator".

are actually pure gauge artifacts, being possible to eliminate them by means of a partial gauge fixing involving the trace of the gauge parameter. Moreover the double trace of the gauge field, that would present itself for spin $s \geq 4$ and that after the elimination of β can be no more recognised as carrying only pure gauge degrees of freedom, is shown to vanish *on-shell*.

The most appealing feature of the non-local version of the unconstrained formulation is that *it is directly related to the geometry* of these higher-spin gauge fields, as it allows to assign a dynamical meaning to the curvatures defined in [21] by de Wit and Freedman, that cannot be given a role in the Fronsdal formulation, as explained in Section 4.2.3. More precisely, by construction these curvatures involve as many derivatives as the spin of the basic field, and for this reason it is possible to deduce from them the correct dynamics only by means of a two-step procedure: first, acting with inverse powers of the d'Alembertian operator one restores the physical dimensions needed for a Relativistic wave equation, and then it must be recognised that the non-localities that remain after one has taken the proper number of traces (together with a divergence, if the spin is odd), in order to end with only s indices left, can be consistently eliminated performing a partial gauge-fixing involving the trace of the gauge parameter. The final result has the suggestive features of being a direct generalisation of the geometric equations known for the cases of spin one and two. For example, if $\mathcal{R}_{\mu_1\mu_2\mu_3,\,\rho_1\rho_2\rho_3}$ is the curvature for the spin three case, it is possible to show that the non-local equation (12) is indeed equivalent to

$$\frac{1}{\Box}\,\eta^{\,\rho_1\rho_2}\partial\cdot\,\mathcal{R}_{\,\rho_1\rho_2,\,\mu_1\mu_2\mu_3}\;=\;0\,, \tag{13}$$

where $\eta^{\,\rho_1\rho_2}$ is the flat Minkowski metric, a direct generalisation of the spin one case, where the vanishing of the divergence of the field strength gives the equation of motion for the vector potential. In a similar fashion for spin 4, the first non-trivial even-spin case, the non-local geometric equations are of the form

$$\frac{1}{\Box}\,\eta^{\,\rho_1\rho_2}\eta^{\,\rho_3\rho_4}\mathcal{R}_{\,\rho_1\rho_2\rho_3\rho_4,\,\mu_1\mu_2\mu_3\mu_4}\;=\;0\,, \tag{14}$$

a clear generalisation of the linearised Einstein equation, that in terms of the Riemann tensor $\mathcal{R}_{\,\rho_1\rho_2,\,\mu_1\mu_2}$ reads

$$\eta^{\,\rho_1\rho_2}\,\mathcal{R}_{\,\rho_1\rho_2,\,\mu_1\mu_2}\;=\;0\,. \tag{15}$$

It should also be appreciated by these examples that the spin one and spin two cases qualify as *the only* cases in which the geometric equations are local.

The possible relations between higher-spin field theory and String Theory, being among the main motivations behind the interest in the subject nowadays, are tested in a simplified setting in the last Chapter, by comparing the free equations of Open String Field Theory in the tensionless limit, with the unconstrained, local equations for higher spin presented in Chapter 3. In that limit, as was known for long time, the structure of the String equations for the leading Regge trajectory of the bosonic string, simplifies to that of a system of three fields, called "the triplet", subject to no constraints and displaying an unconstrained gauge symmetry. A comparison with our result was thus compelling, and in Section 5.1 I will show that indeed a consistent truncations of propagating modes leads to the local compensator equations (3.30), (3.67), for spin-s bosons and spin-$(n+\frac{1}{2})$ for fermions, respectively [63]. This result can be generalised to the mixed-symmetry case [64] with some technical complications but with no major conceptual difficulties, suggesting that the unconstrained formulation could simplify the search for a direct relationship between higher-spin systems and String theory, although this relationship is anyway still to be uncovered explicitly.

It is an interesting exercise to generalise the results about the unconstrained formulation on (A)dS space-times [63], [64], and in the very last paragraph it is briefly shown that the compensator equations (3.30), (3.67) can also be formulated on these backgrounds. This can be seen as another step towards the direction of establishing a link with the Vasiliev equations, given the crucial role that the choice of background plays in that framework, whereas the most suggestive hints that the removal of the constraints can tell us something relevant about the Vasiliev theory, is contained in a couple of recent works [59] [60], where it is discussed how the removal of trace constraints allows, for the first time, to take a look at those systems *off-shell*, if not still from the point of view of a true action principle.

Notation and Conventions

In this Thesis all results about higher-spin fields will be discussed in the tensor formalism. Hence, in order to deal with tensor indices in an efficient way, it is useful to simplify the notation that is used. Since "efficient" can have different meanings depending on circumstances, I have not been able to find a single option which could fit well enough in all situations, and at the end I chose three different possibilities throughout the work.

Type I: explicit notation

In places where I found it necessary to stress the pedagogical aspects of the discussion, I chose to display all tensor indices explicitly in the formulas. This happens tipically in the discussion of low-spin examples ($s \leq 2$).

Type II: symmetric notation

This is the notation most widely used in the Thesis, as well as in the works [62] [63] and [61], and is mainly motivated by the fact that our analysis of higher-spin gauge theories is restricted[7] to the class of *symmetric* tensors (or spinor-tensors).

The basic idea is that whenever a collection of indices are meant to be totally symmetrized, they are all left implicit.

For instance, a rank-s symmetric tensor $\varphi_{\mu_1 \ldots \mu_s}$ will be indicated simply with φ. As a less trivial example, the symmetric gradient of a rank-3 symmetric tensor, consisting of

[7] Of course, this is only a restriction in dimension $D > 4$, where mixed-symmetry representations of the extended Lorentz group are really distinct representations. In $D = 4$ every representation of the Lorentz group can be discussed by means of symmetric tensors.

four terms, will be denoted simply by $\partial \varphi$:

$$\partial_{\mu_1} \varphi_{\mu_2\mu_3\mu_4} + \partial_{\mu_2} \varphi_{\mu_1\mu_3\mu_4} + \partial_{\mu_3} \varphi_{\mu_1\mu_2\mu_4} + \partial_{\mu_4} \varphi_{\mu_1\mu_2\mu_3} \equiv \partial \varphi , \qquad (16)$$

where it is to be noted that in our convention there is no normalisation factor implicit, so that if the previous combination should be meant as having "strength one", the factor of $\frac{1}{4}$ should appear *explicitly* in the formulas, e.g.

$$\frac{1}{4}\{\partial_{\mu_1} \varphi_{\mu_2\mu_3\mu_4} + \partial_{\mu_2} \varphi_{\mu_1\mu_3\mu_4} + \partial_{\mu_3} \varphi_{\mu_1\mu_2\mu_4} + \partial_{\mu_4} \varphi_{\mu_1\mu_2\mu_3}\} \equiv \frac{1}{4} \partial \varphi . \qquad (17)$$

Lorentz traces will be denoted by "primes" or by numbers in square brackets, typically primes for single or double traces, and numbers in brackets for multiple traces; e.g.

$$\begin{aligned} &\eta^{\mu_1\mu_2} \varphi_{\mu_1\mu_2\ldots\mu_s} \equiv \varphi' \\ &\eta^{\mu_1\mu_2} \eta^{\mu_3\mu_4} \eta^{\mu_5\mu_6} \varphi_{\mu_1\mu_2\ldots\mu_s} \equiv \varphi^{[3]} . \end{aligned} \qquad (18)$$

In particular, the symbol "η" will denote the Minkowski metric tensor, whose indices are totally symmetrized with the rest of the monomial, and are never meant to be contracted. For instance, if φ is a rank-2 symmetric tensor, the symmetric product with the Minkowski metric, $\eta \varphi$, is a shortcut for the expression

$$\eta \varphi \equiv \eta_{\mu\nu} \varphi_{\rho\sigma} + \eta_{\mu\rho} \varphi_{\nu\sigma} + \eta_{\mu\sigma} \varphi_{\rho\nu} + \eta_{\rho\nu} \varphi_{\mu\sigma} + \eta_{\sigma\nu} \varphi_{\rho\mu} + \eta_{\rho\sigma} \varphi_{\mu\nu} . \qquad (19)$$

Divergences are indicated with the symbol "$\partial \cdot$", while multiple gradients are denoted by powers of a single gradient, as in the following examples:

$$\begin{aligned} &\partial^{\mu_1} \varphi_{\mu_1\mu_2\ldots\mu_s} \equiv \partial \cdot \varphi , \\ &\partial_{\mu_1} \partial_{\mu_1} \varphi_{\mu_3\mu_4\ldots\mu_{s+2}} \equiv \partial^2 \varphi . \end{aligned} \qquad (20)$$

Hence, differently from the notation used in many of the works that I am going to refer to in the first Part of the Thesis, let us note that in this notation the d'Alembertian operator will be always indicated with a $\Box$, and *never* with a ∂^2. It is also to be noted, as an unusual feature of this notation, that the "symmetric product" of two gradients *is not* the square of the gradient itself:

$$\partial \partial \neq \partial^2 ! \qquad (21)$$

One can convince himself of this peculiarity by simply looking at a specific case. For example, let us consider the symmetric gradient of a rank-2 tensor

$$\partial_{\mu_1} \varphi_{\mu_2\mu_3} + \partial_{\mu_2} \varphi_{\mu_1\mu_3} + \partial_{\mu_3} \varphi_{\mu_1\mu_2} \equiv \partial \varphi \, , \tag{22}$$

that defines a symmetric, rank-3 tensor. Now, if one takes the gradient of a rank-3 tensor the result is indicated in (16); on the other hand, if the rank-3 tensor is itself a gradient, as in (22), then in the final result a double-counting occurs, according to

$$\begin{aligned}
&\partial_{\mu_1}\{\partial_{\mu_2} \varphi_{\mu_3\mu_4} + \partial_{\mu_3} \varphi_{\mu_2\mu_4} + \partial_{\mu_4} \varphi_{\mu_2\mu_3}\} \\
&\partial_{\mu_2}\{\partial_{\mu_1} \varphi_{\mu_3\mu_4} + \partial_{\mu_3} \varphi_{\mu_1\mu_4} + \partial_{\mu_4} \varphi_{\mu_1\mu_3}\} \\
&\partial_{\mu_3}\{\partial_{\mu_1} \varphi_{\mu_2\mu_4} + \partial_{\mu_2} \varphi_{\mu_1\mu_4} + \partial_{\mu_4} \varphi_{\mu_1\mu_2}\} \\
&\partial_{\mu_4}\{\partial_{\mu_1} \varphi_{\mu_2\mu_3} + \partial_{\mu_2} \varphi_{\mu_1\mu_3} + \partial_{\mu_3} \varphi_{\mu_1\mu_2}\} \, ,
\end{aligned} \tag{23}$$

so that at the end, according to the second of (20) the correct relation is actually

$$\partial \, \partial = 2 \, \partial^{\,2} \, . \tag{24}$$

This is just one of the simplest examples of a series of rules allowing to manipulate quite efficiently symmetric tensors in this implicit notation . Here I give the list of these rules, that can be checked in simple examples, or completely proved paying some attention to combinatorics:

$$\begin{aligned}
\partial^{\,p} \, \partial^{\,q} \ &= \ \binom{p+q}{p} \, \partial^{\,p+q} \, , \\
\partial \cdot (\partial^{\,p} \, \varphi) \ &= \ \Box \, \partial^{\,p-1} \, \varphi \ + \ \partial^{\,p} \, \partial \cdot \varphi \, , \\
(\partial^{\,p} \, \varphi)^{\,\prime} \ &= \Box \, \partial^{\,p-2} \, \varphi \ + 2 \, \partial^{\,p-1} \, \partial \cdot \varphi \ + \partial^{\,p} \, \varphi^{\,\prime} \, , \\
\partial \cdot \eta^{\,k} \ &= \ \partial \, \eta^{\,k-1} \, , \\
\left(\eta^{k} \, \varphi\right)^{\prime} \ &= \ [\, D \ + \ 2\,(s+k-1)\,] \; \eta^{\,k-1} \, \varphi \ + \ \eta^{k} \, \varphi^{\,\prime} \, .
\end{aligned} \tag{25}$$

Type III: mixed-symmetric notation

This kind of notation is useful whenever a tensor happens to be symmetric with respect to *subgroups* of its indices, as is the case for the connections of de Wit and Freedman defined in Section 4.2.2. In this case a fully implicit notation could result in ambiguities, and in order to extend the benefits of the “symmetric notation” to this “mixed-symmetric” situation I found it preferable to display *explicitly* the presence of groups of symmetric indices by choosing the same letter for each index within a symmetric group.

To this end, indicate simply with a "μ" each index of the first set, with a "ρ" each index of the second set and so on. The only rule to be added is that complete symmetrization is meant to be understood only for indices belonging to *the same* set. Thus for instance, a rank-4 tensor $\varphi_{\mu\nu;\,\rho\sigma}$, symmetric in μ, ν, and *separately* in ρ, σ, will be denoted by $\varphi_{\mu\mu;\,\rho\rho}$, while if the tensor has rank $s + t$, being separately symmetric in the $s-$group and in the $t-$group, the synthetic notation $\varphi_{\mu_s;\,\rho_t}$ will be used. More generally, a rank-$(s_1 + s_2 + \cdots + s_n)$ tensor, separately symmetric in the first s_1 indices, in the second s_2 indices and so on, will be denoted by $\varphi_{\mu^{(1)}_{s_1},\mu^{(2)}_{s_2},\ldots\mu^{(n)}_{s_n}}$. All rules (25) of the symmetric calculus apply also in this mixed case, separately for each set of symmetric indices.

Some examples may help to clarify this option. Consider a rank-4, bi-symmetric tensor $\varphi_{\mu\mu;\,\rho\rho}$, and calculate first a double gradient, with respect to one index of the μ-group and one index of the ρ-group; the result is simply

$$\partial_\mu \, \partial_\rho \, \varphi_{\mu\mu;\,\rho\rho} \,, \tag{26}$$

and in particular no combinatoric factors appear in this case. Now, let us take first a divergence with respect to a ρ index; the result is simply

$$\partial_\mu \, \Box \, \varphi_{\mu\mu;\,\rho\rho} \, + \, \partial_\mu \, \partial_\rho \, \partial \cdot \varphi_{\mu\mu;\,\rho} \,, \tag{27}$$

while no modification involving the μ indices takes place. Let us then calculate a trace of (26) with respect to the ρ group; one can indicate this trace with a "prime" with a suffix, which could look somewhat awkward. Anyway in our cases it will always be clear from the context which kind of operation is being performed, and a simpler notation can be used. Here the important point is the effect of this trace,

$$(\partial_\mu \, \partial_\rho \, \varphi_{\mu\mu;\,\rho\rho})'^{\rho} \; = \; 2 \, \partial_\mu \, \partial \cdot \varphi_{\mu\mu;\,\rho} \, + \, \partial_\mu \, \partial_\rho \, \varphi'^{\rho}_{\mu\mu} \,, \tag{28}$$

which is understood to operate only on the ρ indices, while leaving again the μ indices untouched.

♋

I shall work mainly in flat, Minkowski space-time, with mostly positive metric. Wherever not otherwise specified the dimension of spacetime is left arbitrary. As long as the conditions to select an irreducible representation of the Poincaré group are not met, it should be more correct to speak about the "rank" of a tensor, rather than of its "spin". Nonetheless, I will often be cavalier with this point, and wherever there will be no risk of misunderstandings, I shall use the word "spin" both in a correct and a loose sense.

As usual, a Bibliography closes this work. In addition, at the end of each Part I have added shorter lists of specific references, that are only meant as a guide for the reader, and include the works that have been of more direct inspiration for the topics discussed in this Thesis.

Part I

Free theory I: constrained fields

Chapter 1

Constrained massive higher-spin fields

It is a recurrent situation in the history of Higher Spins that a consistent formulation of the free dynamics is found to run into some sort of difficulties as soon as one tries to introduce interactions in the theory. As pointed out among others by Weinberg in [65] these difficulties go from non-causality to loss of constraints, appearance of negative norm states and so on. In particular, it was to surmount obstacles of this kind that Fierz and Pauli in 1939 [4], after a previous work of Fierz on the free equations [3], were led to suggest a Lagrangian formulation for massive higher - spin fields, involving algebraic constraints on the physical field along with the introduction of auxiliary fields.

Their paper is usually regarded as marking the very beginning of the history for higher-spin field theory, and is of particular relevance in the framework of this Thesis, as I will try to show in what follows. Indeed, the classical theory for massless higher-spin gauge fields, due to Fronsdal and Fang-Fronsdal [18] [19], is based on constraints that are at the end inherited from the Fierz-Pauli picture, via the Singh-Hagen Lagrangians, and the main point of my work is to present a proposal to avoid the introduction of these constraints, and still reach a consistent, albeit fully unconstrained formulation. In this sense, the origin of the constraints in the massless case can be understood looking at the massive theory that Fronsdal chose as his starting point, and it is my goal in the next Sections to illustrate the main features of this theory.

1.1 Statement of the problem in the Fierz-Pauli work

In [3], after the works of Majorana [1] and Dirac [2], M. Fierz gave a field theoretical formulation for free massive higher-spin fields. In his work a free, spin-s massive boson is described by a totally symmetric rank-s tensor φ subjected to the conditions

$$(\Box + m^2)\,\varphi = 0\,, \tag{1.1}$$

$$\partial \cdot \varphi = 0\,, \tag{1.2}$$

$$\varphi' = 0\,. \tag{1.3}$$

It is possible to recognise that, whereas (1.1) is simply the expected wave equation for a free relativistic boson, (1.2) and (1.3) represent the conditions in order for the tensor to describe an *irreducible* representation of the Poincaré group. As long as one stays *on-shell*, and no interactions are introduced, one can simply regard the *whole* system given by (1.1), (1.2) and (1.3) as the "equation of motion" for a spin-s boson.

Problems arise if one tries and introduce interaction with an electromagnetic field via the "minimal" substitution $p_\mu \longrightarrow p_\mu - ieA_\mu \equiv \Pi_\mu$. In this case indeed the commutator $[\Pi^2, \Pi_\mu] \neq 0$ and for this reason the equations corresponding to (1.1) and (1.2),

$$(\Pi^2 + m^2)\,\varphi = 0\,, \tag{1.4}$$

$$\Pi \cdot \varphi = 0\,, \tag{1.5}$$

are no more compatible. Similar complications were also found in the work of Dirac, and these can be probably regarded as the very first hints that the general problem of higher-spin interactions displays complications invisible from the lower-spin point of view.

It was in order to surmount these difficulties that Fierz and Pauli proposed a Lagrangian approach. Their strategy can be better summarised by their own words[1] : "We shall not attack the problem of deriving such additional terms to make the equations compatible directly but solve it by an artifice. This consists in *introducing auxiliary tensors of lower rank than the original ones* [...] and deriving all equations from a variation principle without having to introduce extra conditions. By suitably choosing the numerical coefficients in the Lagrange function it will follow from the field equations [...] that in

[1] Cfr. ref. [4], pag. 213.

absence of an external field the auxiliary quantities vanish and the additional conditions [(1.2)] are satisfied automatically" (emphasis in the preceding quotation is mine). Moreover[2] "[...] it is important that a one-to-one correspondence should be possible between the states [...] with the external states and without. This is equivalent to saying that the number of conditions which the field and auxiliary variables [...] must satisfy *at a definite time* is not diminished by the presence of an external field[3]".

Hence, the first part of the Fierz-Pauli program was to show how to write a Lagrangian for a *traceless* symmetric tensor, such to give (1.1) and (1.2) as equations of motion. The main result of [4] is that actually, at the free level, many possibilities are available, and the question of which possibility was to be judged as preferable remained open until the works of Singh and Hagen [14], [15] in 1974.

It is not my intention here to give an exhaustive review of [4]; in particular the detailed discussion of the spin-2 case, both at the massive and massless level, as long as with or without an external field, would go beyond the scope of this work. Rather, I will focus on the proposal for the construction of a free Lagrangian for bosons, where all conceptual ingredients needed for the forthcoming discussion are introduced.

1.2 The Fierz-Pauli proposal

To understand the idea proposed by Fierz and Pauli for the construction of a Lagrangian for the Fierz equations, it is useful to take a look at the two simplest examples of spin 1 and 2, that I will illustrate displaying the indices[4]. In the spin-1 case we can collect the two conditions (1.1) and (1.2) in the single equation

$$(\Box + m^2)\,\varphi_\mu \,-\, \partial_\mu \partial \cdot \varphi \,=\, 0\,, \tag{1.6}$$

from which in particular (1.2) follows simply by calculating the divergence. For spin 2, bearing in mind that each term of the equation should share the symmetry properties

[2]Ibid. pag. 214.

[3]This last requirement is meant to avoid the occurrence of singularities when the external field is slowly removed. Fierz and Pauli gave an example of this phenomenon in [4], showing how an "improper" interaction of a field with spin 1 with a scalar gives rise to the creation of new particles whose solutions are singular when the interaction is switched off.

[4]In this Section the discussion will be limited to the case of dimension $D = 4$.

of the field $\varphi_{\mu\nu}$, i. e. any contribution must be symmetric and traceless, one can try a similar combination,

$$(\Box + m^2)\,\varphi_{\mu\nu} - \alpha\left(\frac{1}{2}\,\partial_\mu\,\partial\cdot\varphi_\nu + \frac{1}{2}\,\partial_\nu\,\partial\cdot\varphi_\mu - \frac{1}{4}\,\eta_{\mu\nu}\,\partial\cdot\partial\cdot\varphi\right) = 0\,. \tag{1.7}$$

It is possible to verify that, if $\alpha = 2$, the divergence of the field $\varphi_{\mu\nu}$ vanishes *provided* the subsidiary condition $\partial\cdot\partial\cdot\varphi = 0$ is fulfilled. The idea is then to couple $\partial\cdot\partial\cdot\varphi$ to an auxiliary, *scalar* field C in the Lagrangian, such that the resulting equations of motion be precisely $\partial\cdot\partial\cdot\varphi = 0$, $C = 0$ and an equation equivalent to (1.7). This is the simplest way to suggest why there is no need to introduce an auxiliary field of spin $s - 1$, a peculiarity that will be still true for the general case of spin s. Hence, having in mind the requirements of locality, and second-order derivatives for the free action, Fierz and Pauli proposed the Lagrangian

$$\begin{aligned}\mathcal{L} =& \,\partial_\mu\,\varphi_{\rho\sigma}\,\partial^\mu\varphi^{\rho\sigma} - m^2\,\varphi^{\mu\nu}\,\varphi_{\mu\nu} + p\,\partial\cdot\varphi_\mu\,\partial\cdot\varphi^\mu \\ &+ q\,m^2 C^2 + r\,\partial_\mu C\,\partial^\mu C + C\,\partial\cdot\partial\cdot\varphi\,.\end{aligned} \tag{1.8}$$

whose equations of motion are

$$\begin{aligned}(\Box + m^2)\,\varphi_{\mu\nu} + \frac{p}{2}\,(\partial_\mu\,\partial\cdot\varphi_\nu + \partial_\nu\,\partial\cdot\varphi_\mu - \frac{1}{2}\,\eta_{\mu\nu}\,\partial\cdot\partial\cdot\varphi) \\ - \frac{1}{2}\,(\partial_\mu\,\partial_\nu - \frac{1}{4}\,\eta_{\mu\nu}\,\Box)\,C = 0\,,\end{aligned} \tag{1.9}$$

$$(r\,\Box - q\,m^2)\,C - \frac{1}{2}\,\partial\cdot\partial\cdot\varphi = 0\,. \tag{1.10}$$

Taking the double divergence of the first equation, one gets a linear differential system in $\partial\cdot\partial\cdot\varphi$ and C

$$\begin{pmatrix}(1 + \frac{3}{4}p)\Box + m^2 & -\frac{3}{8}\Box^2 \\ -\frac{1}{2} & r\Box - qm^2\end{pmatrix}\begin{pmatrix}\partial\cdot\partial\cdot\varphi \\ C\end{pmatrix} = 0\,, \tag{1.11}$$

and the aim is to fix the coefficients such that only the null solution is allowed. To this end, the determinant of the system must be algebraic and different from zero, and this can be accomplished by several choices for the triple p, q, r. The Fierz-Pauli choice, $p = -2$, $r = -\frac{3}{8}$, $q = \frac{3}{4}$, leading to a value for the determinant equal to $-\frac{3}{2}\,m^4$, is the right one in order to satisfy the further condition that the equation for φ be equivalent to the Fierz system (1.1) and (1.2):

$$(\Box + m^2)\,\varphi_{\mu\nu} - (\partial_\mu\,\partial\cdot\varphi_\nu + \partial_\nu\,\partial\cdot\varphi_\mu) = 0\,. \tag{1.12}$$

Again, that (1.12) satisfies this requirement can be seen simply by evaluating its divergence and using the condition $\partial \cdot \partial \cdot \varphi = 0$. To generalise this result to the case of arbitrary spin the idea is to introduce more and more auxiliary fields, and fix the coefficients in the Lagrangian so that no auxiliary field propagates, while the divergence of the physical field is found to vanish on-shell. More concretely, recall that a traceless, symmetric tensor of rank s corresponds to the representation $\mathcal{D}(\frac{s}{2}, \frac{s}{2})$ of the Lorentz group. This representation is reducible, and contains all lower spin values down to zero, according to $\mathcal{D}(\frac{s}{2}, \frac{s}{2}) = \sum_{j=0}^{s} \mathcal{D}(j)$. This means that starting from a Lagrangian involving the dynamical rank-s tensor, together with a number of auxiliary tensors of lower ranks, one must at the end get rid of *all* spurious particle introduced, and to this end it is necessary to have an adequate number of coefficients to adjust.

To translate this observation at a quantitative level, let us consider the building blocks of the Lagrangian. In general, given a rank-n tensor $\varphi^{(n)}$, the possible term in a local Lagrangian with derivatives up to two and quadratic couplings are of the form, in implicit notation

$$\begin{aligned} \mathcal{L}^{(n)} \sim \; & a\,(\partial_\mu \varphi^{(n)})^2 \,+\, b\,m^2\,\varphi^{(n)\,2} \,+\, c\,(\partial \cdot \varphi^{(n)})^2 \\ & + d\,\varphi^{(n-1)}\,\partial \cdot \varphi^{(n)} \,+\, e\,\varphi^{(n-2)}\,\partial \cdot \partial \cdot \varphi^{(n)}\,. \end{aligned} \tag{1.13}$$

Moreover, to calculate the number of arbitrary coefficients that will appear in $\mathcal{L}$ let us keep in mind the following observations:

- from the previous considerations we can infer that no auxiliary fields of rank $s-1$ are needed;
- the first two terms involving $\varphi^{(s)}$ are both normalised with a factor of $\frac{1}{2}$;
- in principle, for a given rank $0 \leq n \leq s-2$ one can introduce a number t_n of auxiliary fields.

On the basis of these considerations it is possible to evaluate the number of spurious "particles" introduced for a given spin, as well as the number of coefficients correspondingly present in the Lagrangian. One would then fix the minimum number of auxiliary fields by requiring that the second number be greater than or equal to the first.

For example, in the case $s = 3$ there is one rank-3 tensor, and in principle t_1 rank-1 auxiliary vectors and t_0 auxiliary scalars. Correspondingly, one must get rid of one spin-2 particle, $t_1 + 1$ vectors and $t_1 + t_0 + 1$ scalars. To compute the number of coefficients observe that from the terms involving $\varphi^{(3)}$ we get one coefficient from the self-coupling of its divergence and t_1 coefficients from the couplings with the vectors. The vectors bring also $3t_1$ coefficients from their kinetic terms and $t_1 t_0$ coefficients from the couplings with the scalars. Finally, the scalars contribute $2t_0$ coefficients from their kinetic terms. The "minimal" condition to impose on t_0, t_1 is then

$$2\,t_1 \,+\, t_0 \,+3 \,\leq\, 4\,t_1 \,+\, 2\,t_0 \,+\, t_1\,t_0 \,+\, 1\,, \tag{1.14}$$

which is clearly fulfilled if $t_1 \,=\, 1 \,=\, t_0$.

More generally it is possible to see that, if one considers only a single auxiliary tensor for any rank from $s-2$ down to zero, then the number of coefficients available is *less* than the number of particles to be removed, as soon as $s > 7$. To see this[5], observe that in such a case the number of spurious particles is equal to $\frac{s(s+1)}{2}$, while the number of coefficients is $3 + 5(s-2)$. Apparently, with fewer coefficients to adjust than particles to remove it would not be possible to determine all necessary conditions, and for this reason Fierz and Pauli were led to the conclusion that in general it is not possible to build a Lagrangian having the desired properties and involving *only one* auxiliary tensor for any rank $n \leq s-2$. On the other hand, adding more tensors the number of coefficients rapidly exceeds the number of unwanted states, and the conclusion was that, for a general spin, the kind of Lagrangian to be chosen in order to describe the Fierz dynamics of a spin-s particle is not determined, not to mention the fact that anyway, given that there are enough coefficients, it is still to be proved that these can be fixed in the general case to give the desired equations of motion.

At the end, the problem of finding a Lagrangian for these systems remained open, and several attempts were made to uncover the systematics behind the idea of Fierz and Pauli.

[5]Actually Fierz and Pauli proposed a Lagrangian in which the mass term of any tensor was always normalised to m^2. This implies a different counting of the number of coefficients with respect to the one proposed here, leading to the conclusion that starting from $s \,=\, 5$ one needs *two* auxiliary scalars, rather than one. Anyway, the qualitative conclusion that one auxiliary tensor for any rank seem to be *definitely* not enough, is the same.

One of this attempts, originally proposed by Fronsdal [12], and subsequently elaborated upon by Chang [?], is interesting for this work because it involves the use of *non-local* projectors, which are close in spirit to the method we found to describe the *geometry* of higher-spin gauge fields (see Chapter 4). The main idea is to collect the three conditions (1.1), (1.2) and (1.3) in *one* single equation involving suitable *differential projectors*, defined recursively starting from the vector case. The basic observation is that the rank-1 tensor φ_μ defines a basis for a reducible representation of the Poincaré group; namely, using Chang notation,

$$\varphi_\mu \,=\, \varphi_\mu^{(0)} \,+\, \varphi_\mu^{(1)}\,, \tag{1.15}$$

where $\varphi_\mu^{(0)}$ can be identified as the gradient of a scalar, while $\varphi_\mu^{(1)}$ satisfies the subsidiary condition $\partial \cdot \varphi^{(1)} \,=\, 0$. Starting from this decomposition Fronsdal defined the following, non-local projection operators,

$$\begin{aligned} \Theta^{(0)}_{\mu\nu} \,&=\, \frac{1}{\Box}\,\partial_\mu\,\partial_\nu\,, \\ \Theta^{(1)}_{\mu\nu} \,&=\, \eta_{\mu\nu} \,-\, \frac{1}{\Box}\,\partial_\mu\,\partial_\nu\,, \end{aligned} \tag{1.16}$$

such that, schematically

$$\varphi_\mu^{(0,1)} \,=\, \Theta^{(0,1)}_{\mu\nu}\,\varphi^\nu\,. \tag{1.17}$$

It is possible to construct spin-s projectors by taking the traceless part of the product of s operators $\Theta^{(1)}$ according to

$$\Theta^{(s)}_{\mu_1\ldots\mu_s;\,\nu_1\ldots\nu_s} \,=\, \left\{\Theta^{(1)}_{\mu_1\nu_1}\ldots\Theta^{(1)}_{\mu_s\nu_s}\right\}^T\,, \tag{1.18}$$

where the superscript T means "traceless part", and symmetrization among the μ_i indices, as well as among the ν_j indices is understood. Once it is verified that these projectors satisfy the correct properties, it is possible to define the equations of motion for a spin-s field simply as

$$\Box\,[\Theta^{(s)}\,\varphi]_{\mu_1\ldots\mu_s} \,+\, m^2\,\varphi_{\mu_1\ldots\mu_s} \,=\, 0\,, \tag{1.19}$$

while completely analogous constructions are possible for fermions. Two facts are worth stressing at this point:

- the equations are *non-local* for any spin $s \geq 1$;

- correspondingly, non local Lagrangians for these equations can be constructed, involving only the dynamical tensor $\varphi_{\mu_1 \dots \mu_s}$. *It is the need to turn these Lagrangians into local ones that justifies in this framework the introduction of the auxiliary fields* [**?**]. Although with reversed logic, the spirit behind this proposal is very close to the one that led us from the local, unconstrained formulation of massless higher-spin theory, involving auxiliary fields, to the non-local, "irreducible" version, in which the auxiliary fields are removed, and the necessary conditions for the propagation of pure-s degrees of freedom are partly recovered on-shell (see Chapters 3 and 4).

This proposal anyway did not lead at the end to a closed algorithm, and only for spin 2, 3 and 4 (along with spin $\frac{3}{2}$, $\frac{5}{2}$ and $\frac{7}{2}$ on the side of fermions) explicit Lagrangians were given in [**?**]. The best solution was finally found by Singh and Hagen [14] [15], who could give the closed form of the Lagrangian for any spin, showing unexpectedly that a *single* auxiliary tensor for any rank from $s-2$ down to zero *is* indeed enough, if one chooses carefully the coefficients.

The point that I would like to stress here about the Fierz-Pauli program, is that they introduced a *distinction* between the two types of conditions (1.2) and (1.3). The second one, that was not in principle in contrast with the introduction of an external field, was left as an algebraic constraint on the physical field, and for reasons of consistency became a feature of the auxiliary fields as well. The differential condition (1.2) on the other hand, that resulted to be problematic for the introduction of a minimal coupling, was "promoted" to be a dynamical condition, i.e. a condition produced by the variation of the Lagrangian, and in this sense it became a true part of the equations of motion. This point is relevant for the present discussion, because this distinction "propagated" itself to the massless Fronsdal theory, generating a description of free higher-spin gauge fields involving constrained tensors and constrained gauge parameters (see Sections 2.1 and 2.2). I will discuss in Section 2.3 which kind of reasons could suggest that such constraints in a gauge theory may not be completely desirable. Here I only wish to anticipate that the way we chose to forego those constraints in the massless case, that will be explained in Chapter 3, is much in the *spirit* of Fierz-Pauli, as quoted in the citation at page 28: we shall introduce auxiliary tensors, and derive all equations from a variational principle without having to introduce extra conditions, simply extending this philosophy also to the trace constraints.

1.3 The Singh-Hagen solution

In order to look for a solution to the Fierz Pauli problem, Singh and Hagen defined an algorithm generalising in a systematic way what we have seen in the previous Section for the construction of the spin-2 Lagrangian.

From the point of view of this work, what is really important is that such a solution *exists*, and that it involves a specific structure for the couplings among the various fields introduced. As will be described in the next Section, the massless limit of these Lagrangians, studied by Fronsdal in [18] and by Fang-Fronsdal in [66] for the case of fermions, displays the peculiar simplification that all auxiliary fields effectively decouple, in a sense to be specified, so that the resulting theory involves at the end only *a single* massless field. Moreover, the massless theory displays an abelian gauge symmetry, with gauge parameters constrained, as a heritage of the Fierz-Pauli constraints on the fundamental fields. It is in order to remove these constraints, and consequently to enlarge the gauge symmetry, that I shall introduce auxiliary fields at the massless level as well, as discussed in Section 3.

It will then suffice for our discussion to explain the logic behind the Singh-Hagen solution, limiting ourselves to the case of spin 3. The general proof of the existence of the Lagrangians and of the structure of the coefficients can be found in the original works [14] [15], or in the review part of the Thesis [67] (in Italian).

Let us then consider a rank-3 symmetric and traceless tensor φ, and let us recall once again that the arrival point of the whole procedure is to deduce from a Lagrangian the Fierz conditions (1.1) and (1.2), that I report here for simplicity

$$\begin{aligned} &(\Box + m^2)\,\varphi \,=\, 0\,, \\ &\partial \cdot \varphi \,=\, 0\,. \end{aligned} \tag{1.20}$$

The idea is to define sufficient conditions for these relations to be satisfied, and see what they imply for the possible auxiliary fields to be introduced. Since we assume that the coupling be at most quadratic and the order of derivatives less than or equal to two, what comes out is that the conditions to be imposed involve only a restricted number of fields at the same time, and can be solved recursively.

To begin with, let us note that, whatever the nature of the couplings with the auxiliary fields, we can assume for the Lagrangian the very general form

$$\mathcal{L} \,=\, \frac{1}{2}\,\partial_\mu\,\varphi\,\partial^\mu\,\varphi \,-\, \frac{1}{2}\,m^2\,\varphi^2 \,+\, \frac{1}{2}a\,(\partial\cdot\varphi)^2 \,+\, \Delta^{(0)}_{aux}\,, \tag{1.21}$$

where $\Delta^{(0)}_{aux}$ contains all unknown needed terms involving the auxiliary fields. The first step is to use (1.21) to fix the coefficient a. To this end let us calculate the equation of motion for φ, bearing in mind that the variation is with respect to a symmetric *traceless* field,

$$(\Box \,+\, m^2)\,\varphi \,+\, \frac{a}{3}\,(\partial\,\partial\cdot\varphi \,-\, \frac{1}{3}\,\eta\,\partial\cdot\partial\cdot\varphi) \,=\, \delta^{(0)}_{aux}\,; \tag{1.22}$$

where $\delta^{(0)}_{aux}$ is the remainder related to the variation w.r.t. φ of all terms involving auxiliary fields. Now, *suppose* that it is possible to fix all conditions so that on-shell all auxiliary fields, as well as $\partial\cdot\partial\cdot\varphi$, vanish. The equation reduces in this way to

$$(\Box \,+\, m^2)\,\varphi \,+\, \frac{a}{3}\,\partial\,\partial\cdot\varphi \,=\, 0\,. \tag{1.23}$$

Taking the divergence of this equation we can deduce the condition $\partial\cdot\varphi \,=\, 0$, provided $a \,=\, -\,3$. Now one can ask which kind of auxiliary field can force the double divergence of φ to vanish. The simplest choice, also by analogy with what we saw for the spin-2 case, is to introduce an *auxiliary vector*, ψ_1, and to couple it to the field φ in the only way allowed by the general conditions imposed on the system[6]. One can specify in this way a piece of $\Delta^{(0)}_{aux}$, obtaining

$$\begin{aligned}\mathcal{L} \,=&\, \frac{1}{2}\,\partial_\mu\,\varphi\,\partial^\mu\,\varphi \,-\, \frac{1}{2}\,m^2\,\varphi^2 \,-\, \frac{3}{2}\,(\partial\cdot\varphi)^2 \,+\, d_1\,\psi_1\,\partial\cdot\partial\cdot\varphi \\ &+\, \frac{1}{2}a_1\partial_\mu\,\psi_1\,\partial^\mu\,\psi_1 \,-\, \frac{1}{2}\,b_1\,m^2\,\psi_1^2 \,+\, \frac{1}{2}c_1\,(\partial\cdot\psi_1)^2 \,+\, \Delta^{(1)}_{aux}\,.\end{aligned} \tag{1.24}$$

[6] In the framework of the Fierz-Pauli, Singh-Hagen theory there is no reason *in principle* to discard the possibility of introducing also an auxiliary field of rank $s-1$. At the end such an auxiliary field is not necessary, and so for obvious reasons of "minimality" it is not introduced. Moreover, by dimensional arguments, its couplings with the rank-s and rank-$s-2$ tensors should be proportional to the mass m, and so it would decouple from them in the massless limit, together with all auxiliary tensors of rank $n \leq s-3$ (see Chapter 2), becoming completely irrelevant for this last case.

The equations of motion for φ and ψ_1 are

$$
\begin{aligned}
&(\Box + m^2)\,\varphi \,-\, (\partial\,\partial\cdot\varphi \,-\, \frac{1}{3}\,\eta\,\partial\cdot\partial\cdot\varphi) \,=\, \frac{d_1}{3}\{\partial^2\,\psi_1 \,-\, \frac{1}{3}\,\eta\,(\partial\,\partial\cdot\psi_1 \,+\, \frac{1}{2}\,\Box\,\psi_1)\}\,, \\
&(a_1\,\Box \,+\, b_1\,m^2)\,\psi_1 \,+\, c_1\,\partial\,\partial\cdot\psi_1 \,=\, d_1\,\partial\cdot\partial\cdot\varphi \,+\, \delta^{(1)}_{\psi_1}\,,
\end{aligned}
\tag{1.25}
$$

where $\delta^{(1)}_{\psi_1}$ involves all possible auxiliary fields coupled to ψ_1, while because of the constraint on the number of derivatives and on the order of monomials in $\mathcal{L}$, φ can only couple to a vector. If one takes divergences of the first of (1.25), in order to transform all φ's in $\partial\cdot\partial\cdot\varphi$'s, it is anyway manifest that the system cannot determine uniquely the values of $\partial\cdot\partial\cdot\varphi$ and ψ_1 because of the presence of the *scalar* contributions in $\partial\cdot\partial\cdot\partial\cdot\varphi$ and $\partial\cdot\psi_1$. Now, assuming that there is a consistent way to make these two quantities vanish, along with all contributions in $\delta^{(1)}_{\psi_1}$, eqs (1.25) reduce to the simpler form

$$
\begin{aligned}
&(\Box + m^2)\,\varphi \,-\, (\partial\,\partial\cdot\varphi \,-\, \frac{1}{3}\,\eta\,\partial\cdot\partial\cdot\varphi) \,=\, \frac{d_1}{3}\{\partial^2\,\psi_1 \,-\, \frac{1}{6}\,\eta\,\Box\,\psi_1)\}\,, \\
&(a_1\,\Box \,+\, b_1\,m^2)\,\psi_1 \,=\, d_1\,\partial\cdot\partial\cdot\varphi\,,
\end{aligned}
\tag{1.26}
$$

so that taking the double divergence of the first equation allows one to write a system for $\partial\cdot\partial\cdot\varphi$ and ψ_1:

$$
\begin{pmatrix} -\frac{2}{3}\Box + m^2 & -\frac{5}{18}d_1\Box^2 \\ -d_1 & a_1\Box + b_1 m^2 \end{pmatrix} \begin{pmatrix} \partial\cdot\partial\cdot\varphi \\ \psi_1 \end{pmatrix} \,=\, 0\,. \tag{1.27}
$$

The determinant of this matrix is algebraic and different form zero as long as the following relations among the coefficient are satisfied

$$
\begin{aligned}
a_1 \,&=\, -\,\frac{5}{12}\,d_1^2\,, \\
a_1 \,&=\frac{2}{3}\,b_1\,,
\end{aligned}
\tag{1.28}
$$

and what is left to prove is the consistency of the assumption that all scalar contributions can be included so that they vanish on shell. To this end let us write the complete Lagrangian as

$$
\begin{aligned}
\mathcal{L} \,=&\, \frac{1}{2}\,\partial_\mu\,\varphi\,\partial^\mu\,\varphi \,-\, \frac{1}{2}\,m^2\,\varphi^2 \,-\, \frac{3}{2}\,(\partial\cdot\varphi)^2 \,+\, d_1\,\psi_1\,\partial\cdot\partial\cdot\varphi \\
&+\frac{1}{3}b_1\partial_\mu\,\psi_1\,\partial^\mu\,\psi_1 \,-\, \frac{1}{2}\,b_1\,m^2\,\psi_1^2 \,+\, \frac{1}{2}c_1\,(\partial\cdot\psi_1)^2 \\
&+\frac{1}{2}\,a_0\,\partial_\mu\,\psi_0\,\partial^\mu\,\psi_0 \,-\, \frac{1}{2}\,b_0\,m^2\,\psi_0^2 \,+\, m\,e_0\,\psi_0\,\partial\cdot\psi_1\,,
\end{aligned}
\tag{1.29}
$$

where I have added contributions depending on an auxiliary scalar field, and I have taken into account the second of (1.28). The equations of motion are

$$
\begin{aligned}
&(\Box + m^2)\,\varphi - (\partial\,\partial\cdot\varphi - \frac{1}{3}\,\eta\,\partial\cdot\partial\cdot\varphi) = \frac{d_1}{3}\{\partial^2\,\psi_1 - \frac{1}{3}\,\eta\,(\partial\,\partial\cdot\psi_1 + \frac{1}{2}\,\Box\,\psi_1)\}\,,\\
&b_1(\frac{2}{3}\Box + m^2)\,\psi_1 + c_1\,\partial\,\partial\cdot\psi_1 = d_1\,\partial\cdot\partial\cdot\varphi - m\,e_0\,\partial\,\psi_0\,,\\
&(a_0\,\Box + b_0\,m^2)\,\psi_0 = m\,e_0\,\partial\cdot\psi_1\,.
\end{aligned}
\tag{1.30}
$$

The point here is to deduce form this equations a system involving the three scalar quantities introduced, namely $\partial\cdot\partial\cdot\partial\cdot\varphi$, $\partial\cdot\psi_1$ and ψ_0, and to adjust the coefficients in a way that they all vanish on-shell, proving in this way the consistency of all previous assumptions. Hence, calculating the triple divergence of the first equation, and the divergence of the second, one is left with the system

$$
\begin{pmatrix}
\Box - m^2 & -\frac{1}{2}d_1\Box^2 & 0\\
-d_1 & (\frac{2}{3}b_1 + c_1)\Box + b_1 m^2 & m e_0\Box\\
0 & -m e_0 & a_0\Box + b_0 m^2
\end{pmatrix}
\begin{pmatrix}
\partial\cdot\partial\cdot\partial\cdot\varphi\\
\partial\cdot\psi_1\\
\psi_0
\end{pmatrix}
= 0\,.
\tag{1.31}
$$

In this system there are six coefficients to adjust and it is conceivable that we can select them such that the determinant of the matrix be algebraic and different from zero. In particular, d_1 and b_1 are not independent because of (1.28), and moreover, following [14], one can normalise the kinetic term for φ such that in (1.28) $a_1 = -d_1$, and this in turn implies $d_1 = \frac{12}{5}$ and $b_1 = -\frac{18}{5}$. Similarly, imposing $a_0 = -e_0$ in the kinetic term for ψ_0, it is possible to determine the solution of the system, finding the values[7] $a_0 = \frac{18}{25}$, $b_0 = \frac{72}{25}$, $c_1 = -\frac{12}{25}$.

It is a non trivial result that this procedure can be generalised for every spin. The difficulty in principle is that the number of arbitrary coefficients one introduces, working with *a single* auxiliary field for any rank $n \leq s-2$, grows more slowly than the order of the polynomial in powers of $\Box$ that is necessary to reduce to its order-zero term to guarantee the vanishing of all unwanted states in the spectrum. This is just another way to look at the "counting problem" noted by Fierz and Pauli, that led them to conclude

[7]This choice of the coefficients in $\mathcal{L}$ is slightly different from that of [14]. To compare directly the results the following identifications are needed, where I indicated with a subscript "sh" the coefficient in [14]: $d_1 \equiv c_{sh}$, $b_1 \equiv -c_{sh}a_{2\,sh}$, $c_1 \equiv c_{sh}b_{2\,sh}$, $a_0 \equiv -c_{sh}c_{2\,sh}$, $b_0 \equiv -c_{sh}c_{2\,sh}a_{3\,sh}$, $e_0 \equiv c_{sh}c_{2\,sh}$.

that such a "minimal" set of auxiliary fields could not be sufficient[8]. From this point of view, the Singh-Hagen solution is remarkable, and I report here the general form of the Lagrangians, both for bosons and fermions, since these Lagrangians will be the starting point of Fronsdal's work on the massless theory.

For a massive, spin-s boson, following here the notation of [14], the Lagrangian for the Fierz-Pauli system (1.1), (1.2) is,

$$
\begin{aligned}
\mathcal{L} \;=\; & \frac{1}{2}(\partial_\mu \varphi^{(s)})^2 - \frac{m^2}{2}(\varphi^{(s)})^2 - \frac{s}{2}(\partial \cdot \varphi^{(s)})^2 \\
& + c \Big\{ \varphi^{(s-2)}\, \partial \cdot \partial \cdot \varphi^{(s)} - \frac{1}{2}(\partial_\mu \varphi^{(s-2)})^2 + a_2 \frac{m^2}{2}(\varphi^{(s-2)})^2 \\
& + \frac{1}{2} b_2 (\partial \cdot \varphi^{(s-2)})^2 - \sum_{q=3}^{s} \Big(\prod_{k=2}^{q-1} c_k\Big) \Big[\frac{1}{2}(\partial_\mu \varphi^{(s-q)})^2 - a_q \frac{m^2}{2}(\varphi^{(s-q)})^2 \\
& - \frac{1}{2} b_q (\partial \cdot \varphi^{(s-q)})^2 - m\, \varphi^{(s-q)}\, \partial \cdot \varphi^{(s-q+1)} \Big] \Big\} ,
\end{aligned}
\tag{1.32}
$$

where in particular the auxiliary field of rank $s-q$ has been indicated by $\varphi^{(s-q)}$, and the coefficients are

$$
\begin{aligned}
c &= \frac{s\,(s-1)^2}{2s-1} , \\
a_q &= \frac{q\,(2s-q+1)(s-q+2)}{2\,(2s-q+3)(s-q+1)} , \\
b_q &= -\frac{(s-q)^2}{2s-2q+1} , \\
c_q &= \frac{(q-1)\,(s-q)^2\,(s-q+2)(2s-q+2)}{2\,(s-q+1)(2s-2q+1)(2s-2q+3)} .
\end{aligned}
\tag{1.33}
$$

Let us note that in the final form of the Lagrangian (1.32) the couplings are only between "nearest-rank" tensors, and only for the dynamical field $\varphi^{(s)}$ a coupling with a *double* divergence survives. This implies that, for dimensional reasons, all these couplings (with the exception of $\varphi^{(s-2)}\, \partial \cdot \partial \cdot \varphi^{(s)}$) are proportional to the mass m, the only dimensionful parameter available in the free theory, and then in the massless limit almost all auxiliary fields will decouple from the physical one, as we will see in Section 2.

For the case of fermions the general pattern is the same, with the modifications that,

[8] Moreover, Fronsdal (see [12] and appendix of [18]) has proved that the number l of symmetric, traceless tensors needed to complete the Fierz-Pauli program for a spin-s boson is bounded from below: $l \geq s$, and so the Singh-Hagen solution is really "minimal".

if $\psi^{(n)}$ is the rank-n symmetric and γ-traceless[9] spinor-tensor carrying the degrees of freedom of a spin $s = n + \frac{1}{2}$ particle, the corresponding "minimal" set of auxiliary fields now comprises one symmetric and γ-traceless spinor-tensor of rank $n-1$, $\psi^{(n-1)}$, and *two* symmetric and γ-traceless spinor-tensors of rank $n-\lambda$, $\psi^{(n-\lambda)}$ and $\chi^{(n-\lambda)}$, for each $\lambda = 2, 3, \ldots n$. The Lagrangian is in this case

$$
\begin{aligned}
\mathcal{L} \,=\, & \frac{1}{2}\bar{\psi}^{(n)}\,(i\,\gamma\cdot\partial\,-\,m)\,\psi^{(n)}\,+\,c\Big\{\bar{\psi}^{(n-1)}\,\partial\cdot\psi^{(n)} \\
& +\,\frac{1}{2}\,\bar{\psi}^{(n-1)}(i\,\gamma\cdot\partial\,+\,a_1\,m\,)\,\psi^{(n-1)})\,+\,c_1\,\Big[\bar{\psi}^{(n-2)}\,\partial\cdot\psi^{(n-1)} \\
& +\,\frac{1}{2}\,\bar{\psi}^{(n-2)}(i\,\gamma\cdot\partial\,-\,d_2\,m\,)\,\psi^{(n-2)})\,-\,b_2\,m\,\bar{\psi}^{(n-2)}\,\chi^{(n-2)} \\
& -\,\frac{1}{2}\,b_2\,\bar{\chi}^{(n-2)}(i\,\gamma\cdot\partial\,+\,a_2\,m)\,\chi^{(n-2)})\Big]\Big\}\,+\,c\,c_1\,\sum_{q=3}^{n}(-)^q\,(\prod_{j=2}^{q-1}c_j\,b_j) \\
& \Big\{\bar{\psi}^{(n-q)}\,\partial\cdot\chi^{(n-q+1)}\,+\,\frac{1}{2}\bar{\psi}^{(n-q)}(i\,\gamma\cdot\partial\,-\,d_q\,m\,)\psi^{(n-q)} \\
& -\,b_q\,m\,\bar{\psi}^{(n-q)}\,\chi^{(n-q)}\,-\,\frac{1}{2}\,b_q\,\bar{\chi}^{(n-q)}(i\,\gamma\cdot\partial\,+\,a_q\,m)\,\chi^{(n-q)})\Big\}\,,
\end{aligned}
\tag{1.34}
$$

while the coefficients are

$$
\begin{aligned}
c \,&=\, \frac{2\,n^2}{2\,n\,+\,1}\,, \\
a_q \,&=\, \frac{n\,+\,1}{n\,-\,q\,+\,1}\,=\,d_q\,, \\
b_q \,&=\, -\,\frac{(q\,-\,1)\,(2\,n\,-\,q\,+\,3)}{2\,n\,-\,2\,q\,+\,3}\,, \\
c_q \,&=\, \frac{2\,(n\,-\,q\,+\,2)^2}{2\,n\,-\,2\,q\,+\,3}
\end{aligned}
\tag{1.35}
$$

♋

In order to understand *why* at the end the Singh-Hagen solution works, and why the mismatch between coefficients and spurious states does not cause any crucial trouble, one could take a closer look at the form of the determinant for the systems that there are to be solved, and see the algebraic reasons implying the peculiar relations among

[9] In explicit notation, $\gamma^{\mu_1}\,\psi_{\mu_1\ldots\mu_n}\,\equiv\,0$. Given the symmetry of these tensors, and the definition of the Clifford algebra, $\{\gamma^{\mu},\,\gamma^{\nu}\}\,=\,2\eta^{\mu\nu}$, it is easy to verify that γ-tracelessness implies Lorentz-tracelessness.

the coefficients needed to fix a number of conditions greater than the number of arbitrary parameters introduced. Another possibility to develop a picture of the mechanism at work is suggested in the papers [68], [69], [70], [71] where it is shown how to relate the *massless* theory in $D = 4{+}1$ to the massive one in $D = 4$. The first thing to note is that the number of propagating degrees of freedom in the two cases is the same (see Section 2.3). Moreover, in the Lagrangian description of the former, as will be explained in the next Chapter, no auxiliary fields are involved and a gauge symmetry is at work. Compactifying the massless theory from $D = 4 + 1$ to $D = 4$, and exploiting this gauge symmetry, it is then possible to show that only a single set of traceless, symmetric tensors of rank $n = 0, 1, \ldots, s-2, s$ survives, while all tensors of rank $n \leq s-2$ vanish because of the equations of motion, reconstructing in this way, modulo field redefinitions, the Singh-Hagen set.

Chapter 2

Constrained massless higher-spin fields

In this Chapter the works of Fronsdal and Fang-Fronsdal [18] [19] on the massless limit of the Singh-Hagen theory are briefly reviewed. These papers are important also for historical reasons, since they marked the beginning of research in higher-spin gauge field theory, and mostly because they have defined a sort of paradigm to which in a way or another all higher-spin theories are expected to conform, when their free massless limit is performed.

In this framework, the propagation of free massless bosons and fermions is described by means of tensors that are unavoidably constrained, since they are defined as suitable combinations of Singh-Hagen fields. Moreover, other contemporary, independent approaches led to the same tensor structure for the description of higher-spin gauge fields, without relying on any constrained massive ancestor [20], [21]. Consequently, the gauge freedom that the reduced Lagrangians and equations of motion display in these theories is realised by means of constrained gauge parameters and at the end of the Chapter, after showing the consistency of the dynamics described by these tensors, I will make some comments on the role of these constraints and on the possible benefits that could come from their elimination.

2.1 Constrained massless bosons

Let us consider the Lagrangian (1.32) in the simple case of spin 2, setting $m = 0$,

$$\mathcal{L} \,=\, \frac{1}{2}(\partial_\mu \varphi^{(2)})^2 \,-\, (\partial \cdot \varphi^{(2)})^2 + \frac{2}{3}\varphi^{(0)}\,\partial \cdot \partial \cdot \varphi^{(2)} \,-\, \frac{1}{3}(\partial_\mu \varphi^{(0)})^2\,, \tag{2.1}$$

whose equations of motion are

$$\begin{aligned} &\Box\,\varphi^{(2)} \,-\, (\partial\,\partial \cdot \varphi^{(2)} \,-\, \frac{1}{2}\,\eta\,\partial \cdot \partial \cdot \varphi^{(2)}) \,-\, \frac{2}{3}\,(\partial^2\,\varphi^{(0)} \,-\, \frac{1}{4}\,\eta\,\Box\,\varphi^{(0)}) \,=\, 0\,, \\ &\Box\,\varphi^{(0)} \,+\, \partial \cdot \partial \cdot \varphi^{(2)} \,=\, 0\,. \end{aligned} \tag{2.2}$$

Let us look for solutions in which no spin-2 degrees of freedom propagate. This can be achieved demanding that the field $\varphi^{(2)}$ be the (symmetric, traceless) gradient of a vector:

$$\hat{\varphi}^{(2)} \,=\, \partial\,\Lambda \,-\, \frac{1}{2}\,\eta\,\partial \cdot \Lambda\,; \tag{2.3}$$

in order for the system to be consistent, the scalar $\varphi^{(0)}$ must satisfy

$$\hat{\varphi}^{(0)} \,=\, -\,\frac{3}{2}\,\partial \cdot \Lambda\,. \tag{2.4}$$

A solution to the equations (2.2) of the form (2.3), (2.4) defines a *pure gauge* field; of course adding to a general solution of (2.2) a pure gauge does not modify the physical content of the solution itself. It is possible to compactify the results by combining the fields $\varphi^{(2)}$ and $\varphi^{(0)}$ together, to form a symmetric, *traceful* field

$$h \,\equiv\, \varphi^{(2)} \,-\, \frac{1}{3}\,\eta\,\varphi^{(0)}\,, \tag{2.5}$$

such that, under the gauge transformation defined by (2.3) and (2.4) it transforms as

$$\delta\,h \,=\, \partial\Lambda\,. \tag{2.6}$$

In terms of the field h eq. (2.2) becomes

$$\begin{aligned} &\Box\,h \,-\, \partial\,\partial \cdot h \,+\, \partial^2\,h' \,-\, \frac{1}{2}\,(\Box\,h' \,-\, \partial \cdot \partial \cdot h) \,=\, 0\,, \\ &\Box\,h' \,=\, \partial \cdot \partial \cdot h\,, \end{aligned} \tag{2.7}$$

which, combined together, give the linearised Einstein equation

$$\mathcal{R} \,\equiv\, \Box\,h \,-\, \partial\,\partial \cdot h \,+\, \partial^2\,h' \,=\, 0\,. \tag{2.8}$$

The same substitutions in (2.1) give the Lagrangian in terms of the field h. It is possible to verify that the result can be expressed in terms of the linearised Ricci tensor $\mathcal{R}$ in the usual form

$$\mathcal{L} \,=\, \frac{1}{2}\, h\, \left(\mathcal{R} \,-\, \frac{1}{2}\,\eta\,\mathcal{R}'\right)\,. \tag{2.9}$$

In this sense what we have found is just that the Singh-Hagen system for spin 2 is consistent in the massless limit with the known results of linearised gravity. As Fronsdal showed in [18], the same properties generalise to symmetric tensors of arbitrary rank, and therefore an entire class of (abelian) gauge systems can be defined along these lines.

First, let us stress once again that in the massless limit the auxiliary fields $\varphi^{(n)}$ with $n \leq s-3$ decouple, and one is left with a Lagrangian where the dynamical field $\varphi^{(s)}$ is coupled only to the $\varphi^{(s-2)}$ tensor

$$\begin{aligned}\mathcal{L} \,=& \frac{1}{2}(\partial_\mu \varphi^{(s)})^2 \,-\, \frac{s}{2}\,(\partial \cdot \varphi^{(s)})^2 \,-\, \frac{s\,(s\,-\,1)^2\,(s\,-\,2)^2}{2\,(2\,s\,-\,1)^2}\,(\partial \cdot \varphi^{(s-2)})^2 \\ &+\, \frac{s\,(s\,-\,1)^2}{2\,s\,-\,1}\,\{\varphi^{(s-2)}\,\partial \cdot \partial \cdot \varphi^{(s)} \,-\, \frac{1}{2}(\partial_\mu \varphi^{(s-2)})^2\}\,,\end{aligned} \tag{2.10}$$

while the left-over kinetic operators for the lower-rank auxiliary fields will be ignored in the following. The corresponding equations of motion are

$$\begin{aligned}\Box\,\varphi^{(s)} \,-\, \{\partial\,\partial \cdot \varphi^{(s)}\}_T \,&=\, \frac{2\,(s\,-\,1)}{2\,s\,-\,1}\,\{\partial^2\,\varphi^{(s-2)}\}_T\,, \\ \Box\,\varphi^{(s-2)} \,+\, \frac{(s\,-\,2)}{2\,s\,-\,1}\,\{\partial\,\partial \cdot \varphi^{(s-2)}\}_T \,&=\, -\,\partial \cdot \partial \cdot \varphi^{(s)}\,,\end{aligned} \tag{2.11}$$

where the subscript "T" indicates that only the traceless part of the expressions in brackets contributes; explicitly:

$$\begin{aligned}\{\partial\,\partial \cdot \varphi^{(s)}\}_T \,&=\partial\,\partial \cdot \varphi^{(s)} \,-\, \frac{1}{s}\,\eta\,\partial \cdot \partial \cdot \varphi^{(s)}\,, \\ \{\partial^2\,\varphi^{(s-2)}\}_T \,&=\partial^2\,\varphi^{(s-2)} \,-\, \frac{1}{2\,s}\eta\,\left(\Box\,\varphi^{(s-2)} \,+\, 2\,\partial\,\partial \cdot\, \varphi^{(s-2)}\right) \\ &\quad+\, \frac{1}{s\,(s\,-\,1)}\,\eta^2\,\partial \cdot \partial \cdot \varphi^{(s-2)}\,.\end{aligned} \tag{2.12}$$

Again, let us look for pure gauge solution, in the form of a symmetric, traceless gradient of a rank-$(s-1)$ tensor Λ, that in dimension $D = 4$ takes the form

$$\hat{\varphi}^{(s)} \,=\, (\partial\,\Lambda)_T \,\equiv\, \partial\,\Lambda \,-\, \frac{1}{s}\,\eta\,\partial \cdot \Lambda\,, \tag{2.13}$$

where Λ is symmetric *and traceless*[1]. From the equation for $\hat{\varphi}^{(s)}$ one can recognise that

$$\hat{\varphi}^{(s-2)} = -\frac{2s-1}{s}\,\partial\cdot\Lambda\,. \tag{2.14}$$

This kind of gauge symmetry suggests to define a new tensor from a combination of $\varphi^{(s)}$ and $\varphi^{(s-2)}$, in such a way that its gauge transformation be as simple as possible. Indeed, the combination

$$\phi^{(s)} \equiv \varphi^{(s)} - \frac{1}{2s-1}\,\eta\,\varphi^{(s-2)}\,, \tag{2.15}$$

defines a symmetric, *doubly traceless* tensor, transforming as

$$\delta\,\phi^{(s)} = \partial\,\Lambda\,, \tag{2.16}$$

a simple generalisation of the spin 1 and spin 2 cases.

It is possible to express the Singh-Hagen tensors in terms of the new one:

$$\begin{aligned} \varphi^{(s)} &= \phi^{(s)} - \frac{1}{2s}\,\eta\,\phi^{(s)\,\prime}\,, \\ \varphi^{(s-2)} &= -\frac{2s-1}{2s}\,\eta\,\phi^{(s)\,\prime}\,, \end{aligned} \tag{2.17}$$

and substituting (2.17) in (2.10) one finds at the end the Fronsdal Lagrangian for a massless, spin-s boson

$$\begin{aligned} \mathcal{L} = &\frac{1}{2}\,(\partial_\mu\phi)^2 - \frac{s}{2}\,(\partial\cdot\phi)^2 - \binom{s}{2}\,\phi'\,\partial\cdot\partial\cdot\phi \\ &- \frac{1}{2}\binom{s}{2}\,(\partial_\mu\,\phi')^2 - \frac{3}{4}\binom{s}{3}\,(\partial\cdot\phi')^2\,, \end{aligned} \tag{2.18}$$

whose equations of motion are

$$\Box\,\phi - \partial\,\partial\cdot\,\phi + \partial^2\,\phi' - \frac{1}{2}\,\eta\,\left(2\,\Box\,\phi' - 2\,\partial\cdot\partial\cdot\phi + \partial\,\partial\cdot\phi'\right) = 0\,. \tag{2.19}$$

These are just the equations one would get from a suitable combination[2] of (2.11) after the substitutions (2.17).

---

[1]At this level one could also try with a *traceful* tensor Λ, taking in (2.13) the traceless part of $\partial\Lambda$. For example, for spin 4 one could take $\hat{\varphi}^{(4)} = \partial\,\Lambda - \frac{1}{8}\,\eta\,(2\,\partial\cdot\Lambda + \partial\,\Lambda') + \frac{1}{12}\,\eta^2\,\partial\cdot\Lambda'$. Correspondingly one finds $\partial\cdot\partial\cdot\hat{\varphi}^{(4)} = \frac{7}{4}\,\Box\,\partial\cdot\Lambda + \frac{1}{2}\,\partial\,\partial\cdot\partial\cdot\Lambda - \frac{3}{8}\,\Box\,\partial\,\Lambda' - \frac{1}{3}\,\partial^2\,\partial\cdot\Lambda' - \frac{1}{4}\,\eta\,\partial\cdot\partial\cdot\partial\cdot\Lambda - \frac{1}{6}\,\eta\,\Box\,\partial\cdot\Lambda'$, and it can be shown that in this case $\hat{\varphi}^{(2)} = a\,\partial\cdot\Lambda + b\,\partial\,\Lambda' + c\,\partial\cdot\Lambda'$ cannot be fixed consistently with (2.11).

[2]The combination is $\Box\,\varphi^{(s)} - \{\partial\,\partial\cdot\varphi^{(s)}\}_T - \frac{2\,(s-1)}{2\,s-1}\,\{\partial^2\,\varphi^{(s-2)}\}_T + \frac{s-1}{s}\,\eta\,\{\Box\,\varphi^{(s-2)} + \frac{(s-2)}{2\,s-1}\,\{\partial\,\partial\cdot\varphi^{(s-2)}\}_T - -\partial\cdot\partial\cdot\varphi^{(s)}\} = 0\,.$

In order to stress the formal analogy with the case of linearised gravity, let us define the *Fronsdal operator*

$$\mathcal{F} \equiv \Box \phi - \partial \partial \cdot \phi + \partial^2 \phi' , \tag{2.20}$$

such that the Lagrangian can be written as

$$\mathcal{L} = \frac{1}{2} \phi \left(\mathcal{F} - \frac{1}{2} \eta \mathcal{F}' \right) , \tag{2.21}$$

and correspondingly the equations of motion are[3]

$$\mathcal{F} - \frac{1}{2} \eta \mathcal{F}' = 0 . \tag{2.22}$$

Taking the trace of this last relation one can show that $\mathcal{F}' = 0$, that in turn in eq. (2.22) implies the *Fronsdal equation*

$$\mathcal{F} = 0 , \tag{2.23}$$

where the formal analogy among (2.21) and (2.23) on the one side, for a general spin-s free boson, and (2.9) and (2.8) describing the spin-2 massless field should be appreciated . This analogy can be used backwards, to deduce the Lagrangian (2.21) starting from the Fronsdal equation (2.23). Indeed, in the case of linearised gravity, starting from the equation

$$\mathcal{R} = 0 , \tag{2.24}$$

the possibility to find a gauge invariant Lagrangian for this dynamics is related to the fact that the tensor $\mathcal{R}$ satisfies the *Bianchi identity*

$$\partial \cdot \mathcal{R} - \frac{1}{2} \partial \mathcal{R}' \equiv 0 . \tag{2.25}$$

Hence, the gauge transformation of the Lagrangian (2.9) vanishes:

$$\begin{aligned} \delta \mathcal{L} &= \frac{1}{2} \delta h \left(\mathcal{R} - \frac{1}{2} \eta \mathcal{R}' \right) \\ &\sim h \left(\partial \cdot \mathcal{R} - \frac{1}{2} \partial \mathcal{R}' \right) \equiv 0 . \end{aligned} \tag{2.26}$$

Similarly, the Fronsdal operator $\mathcal{F}$ satisfies the Bianchi-like identity

$$\partial \cdot \mathcal{F} - \frac{1}{2} \partial \mathcal{F}' = - \frac{3}{2} \partial^3 \phi'' , \tag{2.27}$$

[3] It can be verified that $\varphi \frac{\delta}{\delta \varphi} (\mathcal{F} - \frac{1}{2} \eta \mathcal{F}') = \mathcal{F} - \frac{1}{2} \eta \mathcal{F}'$, so that $\delta \mathcal{L} = \delta \varphi (\mathcal{F} - \frac{1}{2} \eta \mathcal{F}')$.

where in particular the contribution in the double trace of the field vanishes by construction in this framework. Hence this term, that will be important in the formulation of the *unconstrained* theory in Part II, here cannot contribute, and this implies that the gauge transformation of the Lagrangian (2.21),

$$\delta\,\mathcal{L}\,\sim\,\Lambda\,\left(\partial\cdot\mathcal{F}\,-\,\frac{1}{2}\,\partial\,\mathcal{F}\,'\,-\,\frac{1}{2}\eta\,\partial\cdot\mathcal{F}\,'\right)\,, \tag{2.28}$$

vanishes, both because of the Bianchi identity 2.27 *and* because of the tracelessness of the gauge parameter Λ.

Formulas (2.21), (2.23) and (2.27), along with the constraints $\phi''\,=\,0$ and $\Lambda'\,=\,0$ are the basic relations for the Fronsdal theory of massless bosons. In the following Section I shall describe the same results for fermions, to pass in Section 2.3 to the evaluation of the degrees of freedom propagating in the massless theory, confining for brevity my attention to the boson case. In order to conclude this Part and introduce the following one I will then make some additional observations on the role of the constraints in the Fronsdal theory.

2.2 Constrained massless fermions

The discussion of the massless limit for fermions follows step by step the bosonic case, with no real conceptual differences. For this reason the presentation of the fermionic case will be somewhat abridged. The main reference for this Section is [19]. First, one can observe that, because of the structure of the couplings in the Lagrangian (1.34), when $m = 0$ all spinor-tensors of rank $l \leq n - 3$ decouple, and can be ignored in the rest of the Section. The interesting part of the Lagrangian reduces to

$$\begin{aligned}\mathcal{L}\,=\,&\frac{i}{2}\bar{\psi}^{\,(n)}\,\,\partial\!\!\!/\psi^{\,(n)}\,+\,\frac{2\,n^{\,2}}{2\,n\,+\,1}\Big\{\bar{\psi}^{\,(n-1)}\,\partial\cdot\psi^{\,(n)}\\&+\,\frac{i}{2}\bar{\psi}^{\,(n-1)}\,\,\partial\!\!\!/\psi^{\,(n-1)}\,-\,\frac{2\,(n\,+\,1)^{\,2}}{2\,n\,+\,1}\,\Big[\bar{\psi}^{\,(n-2)}\,\partial\cdot\psi^{\,(n-1)}\\&+\,\frac{i}{2}\bar{\psi}^{\,(n-2)}\,\,\partial\!\!\!/\psi^{\,(n-2)}\Big]\Big\}\end{aligned} \tag{2.29}$$

whose equations of motion are

$$\begin{aligned} i\,(\not\partial\,\psi^{(n)})_T &= \frac{2\,n^2}{2\,n\,+\,1}\,(\partial\,\psi^{(n-1)})_T \\ i\,(\not\partial\,\psi^{(n-1)})_T &= -\,\frac{2\,(n\,+\,1)^{2\,n+1}}{2\,n\,+\,1}\,(\partial\,\psi^{(n-2)})_T\,-\,\partial\cdot\psi^{(n)}\,, \\ i\,(\not\partial\,\psi^{(n-2)})_T &= -\,\partial\cdot\psi^{(n-1)}\,, \end{aligned} \tag{2.30}$$

where the subscript "T" here indicates the γ-traceless part of the corresponding expression[4].

These equations are invariant under the following gauge transformations

$$\begin{aligned} \delta\,\psi^{(n)} &= \partial\,\Lambda\,-\,\frac{1}{2\,(n\,+\,1)}\,\gamma\,\not\partial\,\partial\,\Lambda\,-\,\frac{1}{n\,+\,1}\,\eta\,\partial\cdot\Lambda\,, \\ \delta\,\psi^{(n-1)} &= i\,\frac{2\,n\,+\,1}{2\,(n\,+\,1)}\,\{\not\partial\,\Lambda\,-\,\frac{1}{n}\,\gamma\,\partial\cdot\Lambda\}\,, \\ \delta\,\psi^{(n-2)} &= \frac{(n\,-\,1)(2\,n\,+\,1)}{2\,n\,(n\,+\,1)}\,\partial\cdot\Lambda\,, \end{aligned} \tag{2.31}$$

where Λ is a rank-$(n-1)$ spinor-tensor, symmetric and γ-*traceless*. It is possible to combine the three spinor-tensors into a single, *triply γ-traceless field*

$$\hat{\psi}\,=\,\psi^{(n)}\,-\,\frac{i}{2\,n\,+\,1}\,\gamma\,\psi^{(n-1)}\,-\,\frac{2\,(n\,+\,1)}{(n\,-\,1)\,(2\,n\,+\,1)}\,\eta\,\psi^{(n-2)}\,, \tag{2.32}$$

where the coefficients have been chosen such that the gauge transformation acquires the particularly simple form

$$\delta\,\hat{\psi}\,=\,\partial\,\Lambda\,. \tag{2.33}$$

It is useful to define *the Fang-Fronsdal operator*

$$\mathcal{S}\,\equiv\,\not\partial\,\hat{\psi}\,-\,\partial\,\hat{\not\psi}\,, \tag{2.34}$$

in terms of which the Lagrangian (2.29) takes the simple form

$$\frac{1}{i}\,\mathcal{L}\,=\,\frac{1}{2}\,\bar{\psi}\left\{\mathcal{S}\,-\,\frac{1}{2}\,\gamma\,\not\mathcal{S}\,-\,\frac{1}{2}\,\eta\,\mathcal{S}\,'\right\}\,-\,\frac{1}{2}\,\left\{\bar{\mathcal{S}}\,-\,\frac{1}{2}\,\bar{\not\mathcal{S}}\,\gamma\,-\,\frac{1}{2}\,\eta\,\bar{\mathcal{S}}\,'\right\}\,\psi\,. \tag{2.35}$$

The corresponding equations of motion are

$$\mathcal{S}\,-\,\frac{1}{2}\,\gamma\,\not\mathcal{S}\,-\,\frac{1}{2}\,\eta\,\mathcal{S}\,'\,=\,0\,, \tag{2.36}$$

[4]The γ-trace of $\not\partial\psi$ is not zero, even if $\not\psi = 0$. Indeed, in general $\gamma^\alpha\,\not\partial\psi_{\alpha\mu_{n-1}}\,=\,2\,\partial\cdot\psi_{\mu_{n-1}}\,-\,\not\partial\,\not\psi_{m_{n-1}}$.

and it is simple to show, in analogy with the bosonic case, that they imply *the Fang-Fronsdal equation*

$$\mathcal{S} \,=\, 0\,. \tag{2.37}$$

The gauge invariance of this Lagrangian rests on the γ-traceless nature of the gauge parameter Λ, *and* on the following Bianchi identity for the Fronsdal operator $\mathcal{S}$:

$$\partial \cdot \mathcal{S} \,-\, \frac{1}{2}\,\partial\,\mathcal{S}\,' \,-\, \frac{1}{2}\,\not{\partial}\,\not{\mathcal{S}} \,=\, i\,\partial^{\,2}\,\not{\psi}\,'\,, \tag{2.38}$$

where the r.h.s. vanishes in the Fang-Fronsdal case, while it will have to be taken into account in the unconstrained formulation.

2.3 Degrees of freedom and analysis of the constraints

Here I evaluate the number of degrees of freedom propagating in the Fronsdal theory, restricting my attention for brevity to the bosonic case. It turns out is that the synergic effects of dynamics, gauge freedom and constraints determine that only the correct number of polarisations expected for an irrep of the Poincaré group actually propagates, e.g. only the two proper states of helicity $\pm s$, in $D = 4$. The calculation is performed in light-cone coordinates. This Section closes with some observations on the role of the constraints, that are intended as a bridge to the next Part of the Thesis, where I will introduce and explain the main proposal of this work, namely that the same dynamics analysed in this Chapter can be described without the need of any *a priori* constraints.

Consider the Fronsdal equation for a symmetric tensor in $D-$dimensions

$$\Box\,\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \partial^2\,\varphi\,' \,=0\,, \tag{2.39}$$

and choose the de Donder gauge

$$\partial\cdot\varphi \,-\, \frac{1}{2}\,\partial\,\varphi\,' \,=\, 0\,. \tag{2.40}$$

The de Donder condition is indeed a gauge: suppose to start with a field configuration ϕ that does not satisfy the de Donder constraint, so that $\partial\cdot\phi - \frac{1}{2}\partial\,\phi' \neq 0\,$, and perform the gauge transformation $\varphi = \phi + \partial\Lambda\,$. Substituting and demanding that the result be equal to 0 we find the condition for the gauge parameter that is needed in order to reach the de Donder gauge:

$$\partial \cdot \phi + \Box \Lambda + \partial \partial \cdot \Lambda - \frac{1}{2} \partial \left(\phi' + 2 \, \partial \cdot \Lambda + \partial \Lambda' \right) = 0 \, . \tag{2.41}$$

Recalling that in the Fronsdal case the gauge parameter is traceless, this is just a naive equation with a source term,

$$\Box \Lambda = - \partial \cdot \phi + \frac{1}{2} \partial \phi' \, , \tag{2.42}$$

a condition that can always be satisfied.

In this gauge the field equation (2.39) reduces to

$$\Box \, \varphi = 0 \tag{2.43}$$

and we are allowed to study the number of independent polarisations in any reference frame, as the representation does not depend on such a choice. It is useful to choose a plane wave propagating along the $D-$axis, whose form in light-cone coordinates, writing the indices explicitly, is

$$\varphi_{\mu_1 \ldots \mu_s} \; = \; \epsilon_{\mu_1 \ldots \mu_s} \, e^{ip \cdot x} \; = \; \epsilon_{\mu_1 \ldots \mu_s} \, e^{ip_- x^-} \; = \; \epsilon_{\mu_1 \ldots \mu_s} \, e^{ip^+ x^-} \, , \tag{2.44}$$

where ϵ is a symmetric, doubly traceless, constant tensor.

If the tensors entering the de Donder condition carry n lower indices along the minus direction, then:

$$\partial \cdot \, \varphi = \; ip^+ \, \epsilon_{+(-)^n \mu_1 \ldots \mu_{s-n-1}} \, e^{ip^+ \, x^-} \, , \tag{2.45}$$

$$\partial \, \varphi' \; = \; ip^+ \, (2n \epsilon_{+(-)^n \mu_1 \ldots \mu_{s-n-1}} \; - \; n \, \epsilon_{ii(-)^{n-1} \mu_1 \ldots \mu_{s-n-1}}) \, e^{ip^+ \, x^-} \, , \tag{2.46}$$

so that the gauge condition implies

$$(n-1) \epsilon_{+(-)^n \mu_1 \ldots \mu_{s-n-1}} \; = \frac{n}{2} \epsilon_{ii(-)^{n-1} \mu_1 \ldots \mu_{s-n-1}} \, , \tag{2.47}$$

where the μ_i can be " $+$ " or "transverse" indices.

Let us now exploit the residual gauge transformations. Suppose the field φ carries k lower components along the minus direction, and perform a gauge transformation with a parameter Λ of the form

$$\Lambda_{\mu_1 \ldots \mu_{s-1}} \; = \; \lambda_{\mu_1 \ldots \mu_{s-1}} \, e^{ip_- x^-} \, , \tag{2.48}$$

with λ a constant, traceless, symmetric tensor. As a result

$$\varphi_{(-)^k\mu_1\ldots\mu_{s-k}} \rightarrow \varphi_{(-)^k\mu_1\ldots\mu_{s-k}} + ikp^+\Lambda_{(-)^{k-1}\mu_1\ldots\mu_{s-k}}\,. \tag{2.49}$$

It is clear that if $k = 0$ the gauge field is not modified by this transformation, while all components carrying at least one " $-$ " lower component can be eliminated by this gauge choice.

Now consider again eq. (2.47); if $n = 0$

$$\epsilon_{+\mu_1\ldots\mu_{s-1}} = 0\,, \tag{2.50}$$

so that all components with at least one $+$ lower index are set to zero, and the only components left are the "transverse" ones $\epsilon_{i_1\ldots i_s}$, $i_k \in \{1, 2, \ldots D-2\}$. If we take $n = 1$ in (2.47) we have a list of constraints on these components of the form:

$$\epsilon_{ii\mu_1\ldots\mu_{s-2}} = 0\,. \tag{2.51}$$

In other words, we are left with a symmetric, rank-s tensor in $(D-2)$-dimensions, whose independent components are in general $C^s_{D+s-3} \equiv \binom{D+s-3}{s}$; on these components we can impose as many constraints as the number of components of a symmetric, rank-$(s-2)$ tensor in $(D-2)$-dimensions, which means C^{s-2}_{D+s-5} constraints. At the end, the number I of independent components of a Fronsdal, spin-s boson in dimension D are

$$I \,=\, C^s_{D+s-3} \,-\, C^{s-2}_{D+s-5} \tag{2.52}$$

In particular, in $D = 4$ they are

$$\binom{s+1}{s} - \binom{s-1}{s-2} = s + 1 - (s-1) = 2\,,$$

and it is possible to see, going to complex coordinates, that they correspond to the two states of helicity $\pm s$. To this end, define

$$z = \frac{x + iy}{\sqrt{2}}\,, \tag{2.53}$$

$$\bar{z} = \frac{x - iy}{\sqrt{2}}\,, \tag{2.54}$$

and observe that the only surviving components in $\varphi_{\iota_1\ldots\iota_s}$ when we go to complex coordinates are the two with only z or only $\bar{z}$ indices: $\varphi_{z\ldots z}$, $\varphi_{\bar{z}\ldots\bar{z}}$.

In $D = 5$ they are

$$\binom{s+2}{s} - \binom{s}{s-2} = \frac{(s+2)(s+1)}{2} - \frac{s(s-1)}{2} = 2s+1\,,$$

that is, the same number of independent components for a *massive* spin-s tensor in 4 dimensions.

In $D = 6$ we have

$$I = \binom{s+3}{s} - \binom{s+1}{s-2} = \frac{(s+3)(s+2)(s+1) - (s+1)s(s-1)}{6} = (s+1)^2\,.$$

♋

Since the main point of this Thesis is to look for possible formulations of higher-spin theories in which no constraints are involved, I would like to conclude this Part collecting some observations from the previous Sections and making some brief comments on the role of the constraints in the Fronsdal formulation.

As we saw in Sections 2.1 and 2.2, from the historical point of view the constraints are simply a heritage of the Fierz-Pauli conditions, via the Singh-Hagen choice for the auxiliary fields to be introduced in the Lagrangian. In this respect, in order to understand the nature of these conditions and to look for a possible way out, one should begin by criticising the massive theory, to understand whether the tracelessness condition on the fields is really unavoidable, a point which had been not touched upon by the aforementioned authors, mainly because the trace constraints seemed to cause no difficulties in the coupling with external fields. At this level, I shall limit myself to observing that the conditions imposed by Fierz and Pauli are meant to guarantee that only physical polarisations propagate, and I shall assume that this is their role in the massless case as well. From this point of view, all that is required from a possible unconstrained theory is to propagate only physical polarisations, while avoiding the introduction of ghosts.

Anyway, it should be noted that other authors reached the same conclusions as Fronsdal about the need for conditions on the traces of the fields and of the gauge parameters, without any reference to a pre-existing massive theory. Notably Curtright [20], and deWit and Freedman [21], tried to construct the more general theory for free fields of any spin based on the assumption that the gauge transformation be of the form (2.16) for bosons and (2.33) for fermions. What they found was that the gauge parameters should be traceless, for otherwise it would not be possible to write a gauge invariant Lagrangian, local and with order of derivatives up to two[5]. The double trace of the gauge field for bosons, and the triple γ-trace for fermions, were then assumed to vanish for reasons related to the possibility of deducing the Lagrangian from a Bianchi identity in [21], or to avoid the introduction of independent, decoupled degrees of freedom in the theory in [20]. Moreover, in [20] it is stated that it would not be possible to relate bosonic and fermionic Lagrangian by a supersymmetric transformation, if the conditions $\varphi'' = 0$ and $\not\psi' = 0$ were not fulfilled.

Still, it should be appreciated that these constraints represent an unusual novelty with respect to what we know from the lower-spin cases. Briefly, for the spin-1 and spin-2 cases we know that the dynamics together with the gauge freedom suffice to provide all necessary and sufficient conditions for the theory to be consistent, and afford a geometrical description. For higher-spins it seems that dynamics and gauge freedom are not enough, and one is forced to help from the outside, imposing extra conditions on fields and parameters. It is absolutely conceivable at this point that there is no way out, and that what one obtains for the lower spin cases is just due to peculiar simplifications taking place only in those specific situations. Nonetheless, at least to satisfy a naive curiosity, one could ask himself whether it is really impossible to provide the necessary conditions in a different way. Moreover, whereas in the frame-like formulation of Vasiliev the trace constraints are realised as algebraic conditions in the tangent space, in a possible metric-like formulation of interacting higher-spin theory they would represent strong dynamical conditions, and it could be helpful not to be forced to impose them from the very beginning.

The end result is that *it is* indeed possible to discard the constraints in a consistent way.

In [72], [73] and [74], it was shown how to construct a fully unconstrained formulation, both for the massless and the massive theories, by making extensive use of BRST tech-

[5] Indeed, the motivations for the constraints were a bit different in [20] and [21]. The second paper will be analysed in some detail in Section 4.2.2, while here I will limit myself to some general comments.

niques. A complication of this elegant approach is the need of a number of auxiliary fields and gauge parameters that increases linearly with the spin.

The second Part of this Thesis will deal with an alternative possibility to eliminate all constraints, in which only a minimum number of auxiliary fields are needed, and no extra gauge parameters have to be introduced. The removal of the auxiliary fields leads unavoidably to a *non-local* formulation, and indeed this last possibility is the *only one* to allow for a geometrical description of these gauge theories, at least at the free level.

The final hope, supported so far by concrete hints [59], [60] is that this step will prove necessary also to arrive at an off-shell formulation for the Vasiliev theory for systems of interacting higher-spin fields.

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Part II

Free theory II: unconstrained fields

Chapter 3

Free local Lagrangians for unconstrained fields

As discussed in the first Part of this work, the propagation of free higher-spin gauge fields relies on the choice of suitable constraints on the double trace of the gauge field and on the trace of the gauge parameter for bosons, on the triple γ-trace of the field, and on the γ-trace of the gauge parameter for fermions. Supported by the motivations exposed at the end of the previous Chapter, I would like to explore the possibility of attaining a more general formulation of the same dynamics, in which no constraints are required *a priori*, and nonetheless the physical properties of the theory remain unchanged. In order to achieve this goal it is necessary to introduce a couple of auxiliary fields, whose roles are different and will be clarified in the following. This choice leads to an *unconstrained, local theory* describing the free propagation of individual higher-spin fields, since the additional fields introduced to make the removal of the constraints consistent can be shown not to propagate any physical degrees of freedom. In the following chapter I will then show that actually the auxiliary fields can be consistently removed without introducing any conditions on the physical field; the price to pay is that the final theory, when formulated in terms of a single physical field, contains *non-local* contributions which, albeit apparently unfriendly, can consistently be removed by a partial gauge fixing involving the trace of the gauge parameter. This non-local, irreducible formulation, while more difficult to use, has the virtue of unveiling the *geometry* underlying these gauge theories, as it allows to encode all the relevant informations about the dynamics and the gauge structure of the theory in tensors which can be considered, in a sense to be clarified later, as proper generalisations of the Maxwell and Einstein curvatures known from the lower-spin cases.

I start with the description of the bosonic case, first constructing the local Lagrangians, and discussing the meaning of the corresponding equations of motion. I will then repeat the arguments for fermions. In this Part the dimension of space-time is left arbitrary.

3.1 Bosons

3.1.1 Lagrangians

The starting point is the Fronsdal Lagrangian (2.21)

$$\mathcal{L}_0 \,=\, \frac{1}{2}\,\varphi \left(\,\mathcal{F}\, - \,\frac{1}{2}\,\eta\,\mathcal{F}\,' \right)\,. \tag{3.1}$$

As shown in Section 2.1, this is gauge invariant under $\delta\,\varphi \,=\, \partial\,\Lambda$ if and only if *two* conditions hold: the field φ is doubly-traceless, $\varphi'' \,\equiv\, 0$, *and* the gauge parameter is traceless, $\Lambda' \,\equiv\, 0$. Here I take a different attitude, and consider the Lagrangian (3.1) *without any constraint.* In this respect it should be noted that (3.1), when expanded in terms of the basic field φ, is *not* identical to the expression (2.18) for the Fronsdal Lagrangian, as it also contains the additional term

$$-\,\frac{3}{2}\binom{s}{4}\varphi''\,\partial\cdot\partial\cdot\varphi\,'\,, \tag{3.2}$$

which of course is not present in the Fronsdal theory, that involves a doubly-traceless field.

The unconstrained variation of (3.1) is

$$\begin{aligned} \delta\,\mathcal{L}_0 \,=&\, +\,\Lambda\,'\binom{s}{3}\left\{\frac{3}{4}\,\partial\cdot\mathcal{F}\,' - \,\frac{3}{2}\,\partial\cdot\partial\cdot\partial\cdot\varphi \,+\, \frac{9}{4}\Box\,\partial\cdot\varphi\,'\right\} \\ &-\;9\binom{s}{4}\,\partial\cdot\partial\cdot\varphi\,'\,\partial\cdot\Lambda\,' \,+\, \frac{15}{2}\binom{s}{5}\partial\cdot\partial\cdot\partial\cdot\varphi\,'\,\Lambda'' - 3\binom{s}{4}\,\varphi''\,\partial\cdot\partial\cdot\partial\cdot\Lambda\,. \end{aligned} \tag{3.3}$$

As expected, this variation comprises two types of contributions: a first one due to the presence of a non zero trace for Λ, and a second one proportional to the double trace of φ, times the triple divergence of the gauge parameter. For spin $s \,=\, 3$ this last contribution is not present, and the total variation reduces to

$$\delta\,\mathcal{L}_0^{(3)} \,=\, \frac{3}{2}\,\Lambda\,'\,\partial\cdot\mathcal{F}\,'\,. \tag{3.4}$$

In order to compensate this contribution, one can introduce a rank-$(s-2)$ auxiliary field α, characterised by the gauge transformation

$$\delta\,\alpha \,=\Lambda^{\,\prime}\,, \tag{3.5}$$

and add to (3.1) the contribution

$$\mathcal{L}_1^{(3)} \;=\; -\,\frac{3}{2}\,\alpha\,\partial\cdot\mathcal{F}^{\,\prime}\,, \tag{3.6}$$

so that

$$\delta\left\{\mathcal{L}_0^{(3)} \,+\, \mathcal{L}_1^{(3)}\right\} \;=\; -\,\frac{9}{2}\,\alpha\,\Box^{\,2}\,\Lambda^{\,\prime}\,. \tag{3.7}$$

It is then possible to complete the kinetic term for α, thus obtaining a fully gauge invariant Lagrangian having the form

$$\mathcal{L}^{(3)} \;=\; \frac{1}{2}\,\varphi\left(\,\mathcal{F}\,-\,\frac{1}{2}\,\eta\,\mathcal{F}^{\,\prime}\right)\,-\,\frac{3}{2}\,\alpha\,\partial\cdot\mathcal{F}^{\,\prime}\,+\,\frac{9}{4}\,\alpha\,\Box^2\,\alpha\,. \tag{3.8}$$

What is suggested at this stage is that the possibility of making the Fronsdal Lagrangian *fully* gauge invariant simply by introducing terms in the compensator α can probably work only in the $s = 3$ case, because of the contributions in $\partial\cdot\partial\cdot\partial\cdot\Lambda$ appearing in $\delta\,\mathcal{L}_0$ from spin $s = 4$ onwards. It is less evident that terms in the compensator α can provide the cancellation of *any* contribution in $\Lambda^{\,\prime}$ in (3.3); if this were possible, one could try and see whether the remaining terms can be dealt with in a sufficiently simple way.

The first step in this direction is simply the generalisation of (3.6) suitable to account for the cancellation of each term involving $\Lambda^{\,\prime}$ in (3.3). To this end define

$$\begin{aligned}\mathcal{L}_1 \;=&\; -\,\alpha\binom{s}{3}\left\{\frac{3}{4}\,\partial\cdot\mathcal{F}^{\,\prime}\,-\,\frac{3}{2}\,\partial\cdot\partial\cdot\partial\cdot\varphi\,+\,\frac{9}{4}\,\Box\,\partial\cdot\varphi^{\,\prime}\right\}\\ &+\,9\binom{s}{4}\,\partial\cdot\alpha\,\partial\cdot\partial\cdot\varphi^{\,\prime}\,-\,\frac{15}{2}\binom{s}{5}\alpha^{\prime}\,\partial\cdot\partial\cdot\partial\cdot\varphi^{\,\prime}\,,\end{aligned} \tag{3.9}$$

and add it to $\mathcal{L}_0$; the combination $\mathcal{L}_0\,+\,\mathcal{L}_1$ is such that under variation only the term in

φ'' survives of (3.3), but new terms are generated:

$$\begin{aligned}
\delta\{\mathcal{L}_0 + \mathcal{L}_1\} = & -3\binom{s}{4}\varphi''\,\partial\cdot\partial\cdot\partial\cdot\Lambda - \frac{9}{2}\binom{s}{3}\alpha\,\Box^2\,\Lambda' \\
& + 54\binom{s}{4}\,\partial\cdot\alpha\,\Box\,\partial\cdot\Lambda' - 90\binom{s}{5}\,\partial\cdot\partial\cdot\alpha\,\partial\cdot\partial\cdot\Lambda' \\
& - \frac{45}{2}\binom{s}{5}\partial\cdot\partial\cdot\alpha\,\Box\,\Lambda'' + 45\binom{s}{6}\,\partial\cdot\partial\cdot\partial\cdot\,\alpha\,\partial\cdot\Lambda'' \\
& + 12\binom{s}{4}\partial\cdot\alpha\,\partial\cdot\partial\cdot\partial\cdot\Lambda - \frac{45}{2}\binom{s}{5}\alpha'\,\Box\,\partial\cdot\partial\cdot\Lambda' \\
& - 15\binom{s}{5}\alpha'\partial\cdot\partial\cdot\partial\cdot\partial\cdot\Lambda + 45\binom{s}{6}\partial\cdot\alpha'\,\partial\cdot\partial\cdot\partial\cdot\Lambda'\,.
\end{aligned} \tag{3.10}$$

It is important at this stage to recognise that, even in this general case, one can cancel all terms containing the trace of Λ by completing the kinetic operator for α, adding to $\mathcal{L}_0 + \mathcal{L}_1$ the terms

$$\begin{aligned}
\mathcal{L}_2 = & \frac{9}{4}\binom{s}{3}\alpha\,\Box^2\,\alpha - 27\binom{s}{4}\,\partial\cdot\alpha\,\Box\,\partial\cdot\alpha + 45\binom{s}{5}(\partial\cdot\partial\cdot\alpha)^2 \\
& + \frac{45}{2}\binom{s}{5}\partial\cdot\partial\cdot\alpha\,\Box\,\alpha' - 45\binom{s}{6}\,\partial\cdot\partial\cdot\partial\cdot\,\alpha\,\partial\cdot\alpha'\,.
\end{aligned} \tag{3.11}$$

In this way one obtains a "quasi gauge-invariant" Lagrangian, $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$, whose variation only involves *divergences* of the gauge parameter:

$$\begin{aligned}
\delta\{\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2\} & = -3\binom{s}{4}\,\{\varphi'' - 4\partial\cdot\alpha\}\,\partial\cdot\partial\cdot\partial\cdot\Lambda - 15\binom{s}{5}\,\alpha'\,\partial\cdot\partial\cdot\partial\cdot\partial\cdot\Lambda \\
& = -3\binom{s}{4}\,\{\varphi'' - 4\partial\cdot\alpha - \partial\alpha'\}\,\partial\cdot\partial\cdot\partial\cdot\Lambda\,.
\end{aligned} \tag{3.12}$$

The last expression displays the key point of the idea: first, it should be recognised that the expression in brackets, $\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha'$, is *identically gauge invariant*, since $\delta\,\varphi'' = 4\,\partial\cdot\Lambda' + \partial\,\Lambda''$; this means that differently from the Fronsdal formulation, where the double trace of φ is drastically ruled out from the dynamics by imposing the condition that it be identically zero, here we can obtain an equivalent "physical" condition adding to the Lagrangian the constraint

$$\mathcal{L}_3 \sim \beta\,\left(\varphi'' - 4\,\partial\cdot\alpha - \partial\alpha'\right)\,, \tag{3.13}$$

such that the equation of motion for the Lagrange multiplier β simply gives the condition that φ'' be *pure gauge*. Moreover, by comparison with (3.12) it is simple to realise that, by allowing the multiplier β to participate in the gauge transformation of φ and α according to $\delta \beta = \partial \cdot \partial \cdot \partial \cdot \Lambda$, we can reach a *fully* gauge invariant Lagrangian by simply adding the constraint (3.13), with a suitable normalisation, to the combination $\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$.

We arrive in this way at the complete Lagrangian for an *unconstrained* spin-s boson, which is of the form

$$\begin{aligned}
\mathcal{L} = & \frac{1}{2}\varphi\left(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'\right) - \binom{s}{3}\alpha\left\{\frac{3}{4}\partial\cdot\mathcal{F}' - \frac{3}{2}\partial\cdot\partial\cdot\partial\cdot\varphi + \frac{9}{4}\Box\,\partial\cdot\varphi'\right\} \\
& + 9\binom{s}{4}\partial\cdot\alpha\,\partial\cdot\partial\cdot\varphi' - \frac{15}{2}\binom{s}{5}\alpha'\partial\cdot\partial\cdot\partial\cdot\varphi' + \frac{9}{4}\binom{s}{3}\alpha\,\Box^2\,\alpha \\
& - 27\binom{s}{4}\partial\cdot\alpha\,\Box\,\partial\cdot\alpha + 45\binom{s}{5}(\partial\cdot\partial\cdot\alpha)^2 + \frac{45}{2}\binom{s}{5}\partial\cdot\partial\cdot\alpha\,\Box\,\alpha' \\
& - 45\binom{s}{6}\partial\cdot\partial\cdot\partial\cdot\alpha\,\partial\cdot\alpha' + 3\binom{s}{4}\beta\left(\varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha'\right) ,
\end{aligned} \tag{3.14}$$

and is gauge invariant under the *unconstrained* transformations

$$\begin{aligned}
\delta\,\varphi &= \partial\,\Lambda\,, \\
\delta\,\alpha &= \Lambda'\,, \\
\delta\,\beta &= \partial\cdot\partial\cdot\partial\cdot\Lambda\,.
\end{aligned} \tag{3.15}$$

In order to stress the peculiarities of this unconstrained formulation, let me recall that in the Fronsdal formulation the double trace of the physical field cannot play any role, neither at the classical nor at the quantum level. In this unconstrained theory on the other hand φ'' is recognised to carry only pure gauge degrees of freedom which, although not directly related to physical informations, could still reveal to be important for the geometrical properties of the theory and for its quantum behaviour, as is known to be the case for the simplest example of spin one, both at the abelian and at the non-abelian level. Actually in the following Chapter will be clarified that, at least at the free level, *only via the unconstrained formulation it is possible to recover the geometry of free higher-spin fields.* In addition, there are hints that only the removal of the constraints could allow for the formulation of an action principle leading to the Vasiliev equations, i.e. for the only known consistent interacting systems of higher-spin fields [59] [60]. Moreover, the Lagrangian (3.14) displays a symmetry under transformations involving the trace of Λ that the Fronsdal theory simply does not possess, and finally the coupling with currents in this setting displays a substantial difference with respect to the Fronsdal case, as will be discussed in the next Section.

The condition that the double trace of the fundamental field be pure gauge, however, is not enough to guarantee that the physical content of (3.14) be classically equivalent, as it should, to the Fronsdal one. In order to analyse this point it is necessary to study the equations of motion implied by (3.14). This will be done in the next Section.

3.1.2 Equations of motion and their interpretation

The equations of motion following from the Lagrangian (3.14) for the fields φ, β and α are

$$\begin{aligned}\varphi: \quad & \mathcal{F} - 3\,\partial^{3}\,\alpha - \frac{1}{2}\,\eta\,(\mathcal{F}' - \frac{1}{2}\,\partial^{2}\,\varphi'' - 3\,\Box\,\partial\,\alpha - 4\,\partial^{2}\,\partial\cdot\alpha - \frac{3}{2}\,\partial^{3}\,\alpha') \\ & + \eta^{2}\,(\beta + \frac{1}{2}\,\partial\,\partial\cdot\partial\cdot\alpha + \Box\,\partial\cdot\alpha - \frac{1}{2}\,\partial\cdot\partial\cdot\varphi') = 0\,,\end{aligned} \tag{3.16}$$

$$\beta: \quad \varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha' = 0\,, \tag{3.17}$$

$$\begin{aligned}\alpha: \quad & 6\,\Box^{2}\,\alpha + 18\,\Box\,\partial\,\partial\cdot\,\alpha + 12\,\partial^{2}\,\partial\cdot\partial\cdot\alpha + 3\,\Box\,\partial^{2}\,\alpha' + 3\,\partial^{3}\,\partial\cdot\alpha' \\ & - 3\,\partial\,\partial\cdot\partial\cdot\varphi' - \partial\cdot\mathcal{F}' + 2\,\partial\cdot\partial\cdot\partial\cdot\varphi - 3\,\Box\,\partial\cdot\varphi' + 4\,\partial\,\beta \\ & + \eta\,(3\,\Box\,\partial\cdot\partial\cdot\alpha + \partial\,\partial\cdot\partial\cdot\partial\cdot\alpha - \partial\cdot\partial\cdot\partial\cdot\varphi' + 2\,\partial\cdot\beta) = 0\,.\end{aligned} \tag{3.18}$$

First, I would like to show that the dynamics for φ can be reduced to the Fronsdal one. To this end let me start by writing the system in the more compact form

$$\begin{aligned}& \mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{B} + \eta^{2}\,\mathcal{C} = 0\,, \\ & \varphi'' - 4\,\partial\cdot\alpha - \partial\,\alpha' = 0\,, \\ & \mathcal{G}_{\varphi,\beta}(\alpha) = 0\,,\end{aligned} \tag{3.19}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{G}_{\varphi,\beta}(\alpha)$ are defined by comparison with (3.16) and (3.18). Let us then notice that, because of (3.17), i.e. when β is on-shell, the first of (3.19) reduces to

$$\mathcal{A} - \frac{1}{2}\,\eta\,\mathcal{A}' + \eta^{2}\,\mathcal{C} = 0\,, \tag{3.20}$$

and that, under the same condition, the double trace of $\mathcal{A}$ vanishes *identically*: $\mathcal{A}'' \equiv 0$. One can take successive traces of (3.20); the first trace gives

$$-\frac{D + 2\,(s-3)}{2}\,\mathcal{A}' + [D + 2\,(s-3)]\,\eta\,\mathcal{C} + \eta^{2}\,\mathcal{C}' = 0\,, \tag{3.21}$$

while tracing a second time one gets

$$[D + 2\,(s-2)]\,[D + 2\,(s-4)]\mathcal{C} + \{[D + 2\,(s-3)] + [D + 2\,(s-5)]\}\,\eta\,\mathcal{C}' + \eta^{2}\,\mathcal{C}'' = 0\,. \tag{3.22}$$

Defining the coefficients

$$\rho_k \equiv D + 2(s - k), \tag{3.23}$$

and tracing again until the k-th step, one finds

$$\left(\eta^2\, \mathcal{C}\right)^{[k]} = \eta^2\, \mathcal{C}^{[k]} + \sum_{i=1}^{k} \rho_{2i+1}\, \eta\, \mathcal{C}^{[k-1]} + \sum_{i \le j=2}^{k} \rho_{2i-1}\, \rho_{2j}\, \mathcal{C}^{[k-2]} = 0\,. \tag{3.24}$$

The crucial point is that these relations cannot become trivial identities, since the traces of $\mathcal{C}$ are of the form $\mathcal{C}^{[k]} = \beta^{[k]} - \partial \cdot \partial \cdot \partial \cdot \alpha^{[k-1]}$, and thus each term in (3.24) is never identically zero, while the coefficients ρ_k are strictly positive. This means that if we define

$$p \equiv \text{integer part of} \left\{ \frac{s - 4}{2} \right\} \tag{3.25}$$

the $(p+1)$-th and $(p+2)$-th traces are *saturating*, in the sense that they can only act on the η's, and cannot generate further traces of $\mathcal{C}$. Explicitly one finds, for the p-th trace:

$$\eta^2\, \mathcal{C}^{[p]} + \sum_{i=1}^{p} \rho_{2i+1}\, \eta\, \mathcal{C}^{[p-1]} + \sum_{i \le j=2}^{p} \rho_{2i-1}\, \rho_{2j}\, \mathcal{C}^{[p-2]} = 0\,; \tag{3.26}$$

for the $(p+1)$-th trace:

$$\sum_{i=1}^{p+1} \rho_{2i+1}\, \eta\, \mathcal{C}^{[p]} + \sum_{i \le j=2}^{p} \rho_{2i-1}\, \rho_{2j}\, \mathcal{C}^{[p-1]} = 0\,, \tag{3.27}$$

and, for the $(p+2)$-th trace:

$$\sum_{i \le j=2}^{p+2} \rho_{2i-1}\, \rho_{2j}\, \mathcal{C}^{[p]} = 0\,. \tag{3.28}$$

This last equation, together with the positivity of the coefficients, implies

$$\mathcal{C}^{[p]} = 0\,, \tag{3.29}$$

and inserting this condition in (3.27) and then in (3.26) we get first $\mathcal{C}^{[p-1]} = 0$, and subsequently $\mathcal{C}^{[p-2]} = 0$. Iterating backwards it is possible to convince oneself that, *on-shell, all traces of* $\mathcal{C}$*, including* $\mathcal{C}$ *itself, vanish.* This result in (3.21) implies that $\mathcal{A}' = 0$ on-shell [1], and the equation of motion for φ finally reduces to

[1]The case $D = 2$, $s = 2$ is a known exception. In this case $\rho_3 = 0$, and one gets no information about $\mathcal{A}'$.

$$\mathcal{F} \,=\, 3\,\partial^{\,3}\,\alpha\,. \tag{3.30}$$

This last equation is very important in our discussion, and will appear again in the following Chapters. Actually, it will be the link between the local unconstrained formulation of this Section and the non-local geometry of the higher-spin gauge fields. Moreover, it will also provide the connection between their geometric equations and the propagation of modes in the tensionless limit of String Field Theory. Given the central role that this equation plays in the unconstrained formulation, it is reasonable that the possibility of extending and improving the comprehension of the dynamics of higher-spin gauge fields, by generalising the point of view from the Fronsdal theory to the unconstrained formulation, will pass through a better understanding of the role of this equation, and of its possible deformations, in the framework of an interacting theory. For the time being, it will suffice to observe that by fixing the gauge to $\alpha \,=\, 0$, making use of $\Lambda\,'$, one ends with the system

$$\begin{aligned} \mathcal{F} \,&=\, 0\,, \\ \varphi'' \,&=\, 0\,, \end{aligned} \tag{3.31}$$

which describes exactly the Fronsdal dynamics, with a doubly-traceless field φ, and an equation of motion which is gauge invariant under $\delta\,\varphi \,=\, \partial\,\Lambda$, with Λ now a traceless gauge parameter. In particular, we can repeat the calculation of Section 2.3 to find that only physical polarisations are propagating, thus showing that the unconstrained Lagrangian (3.14) is dynamically equivalent to the Fronsdal theory.

Turning to the equation of motion of β (3.17), it is quite clear that its meaning is to reduce the degrees of freedom in φ'' to be pure gauge, as observed in the previous Section. One could ask whether the presence of the Lagrange multiplier is really necessary, or whether a more economical formulation is still possible. I will come back to this point in Section 4.2.5

Finally, we should investigate the meaning of the equation for α. Actually, this equation has not played any role in the reduction to the Fronsdal form, and so it is expected that at the level of the free theory it should not introduce any new information. In fact, at the free level, eq. (3.18) can be shown to be *a consequence* of (3.16) and (3.17). To see this, take the divergence of the equation for φ in the form (3.20), where we have used the equation for β:

$$\partial \cdot \left\{ \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' + \eta^2 \mathcal{C} \right\} = 0 \,, \tag{3.32}$$

and expand it in the form

$$\partial \cdot \mathcal{A} - \frac{1}{2} \partial \mathcal{A}' + \eta \, (\partial \, \mathcal{C} - \frac{1}{2} \partial \cdot \mathcal{A}' + \frac{1}{2} \eta \, \partial \cdot \mathcal{C}) = 0 \,. \tag{3.33}$$

In this expression, the part which is not proportional to η vanishes identically,

$$\partial \cdot \mathcal{A} - \frac{1}{2} \partial \mathcal{A}' = \partial \cdot \mathcal{F} - \frac{1}{2} \partial \mathcal{F}' + 6 \, \partial^3 \, \partial \cdot \alpha + 6 \, \partial^4 \, \alpha' = 0, \tag{3.34}$$

making use of the Bianchi identity (2.27) and again of the equation for β; in other words, in the unconstrained formulation it is $\mathcal{A} = \mathcal{F} - 3 \, \partial^3 \, \alpha$, rather than $\mathcal{F}$, that verifies the "true" Bianchi identity. For the remaining part, a direct calculation shows that

$$\begin{aligned} \partial \, \mathcal{C} - \frac{1}{2} \partial \cdot \mathcal{A}' = \frac{1}{4} \{ & 6 \, \Box^2 \, \alpha + 18 \, \Box \, \partial \, \partial \cdot \alpha + 12 \, \partial^2 \, \partial \cdot \partial \cdot \alpha + 3 \, \Box \, \partial^2 \, \alpha' + 3 \, \partial^3 \, \partial \cdot \alpha' \\ & - 3 \, \partial \, \partial \cdot \partial \cdot \varphi' - \partial \cdot \mathcal{F}' + 2 \, \partial \cdot \partial \cdot \partial \cdot \varphi - 3 \, \Box \, \partial \cdot \varphi' + 4 \, \partial \, \beta \} \,, \end{aligned} \tag{3.35}$$

and

$$\partial \cdot \mathcal{C} = + \frac{3}{2} \Box \, \partial \cdot \partial \cdot \alpha + \frac{1}{2} \partial \, \partial \cdot \partial \cdot \partial \cdot \alpha - \frac{1}{2} \partial \cdot \partial \cdot \partial \cdot \varphi' + \partial \cdot \beta \,, \tag{3.36}$$

so that, at the end

$$\partial \cdot \left\{ \mathcal{A} - \frac{1}{2} \eta \mathcal{A}' + \eta^2 \mathcal{C} \right\} = \frac{\eta}{4} \, \mathcal{G}_{\varphi,\beta}(\alpha) \,. \tag{3.37}$$

In other words, as a consequence of the equations of motion for φ and β we find that the combination $\frac{\eta}{4} \, \mathcal{G}_{\varphi,\beta}(\alpha) = 0$. An argument of the same kind as the one used to show that all traces of the $\mathcal{C}$ part of the equation for φ vanish can be used here to conclude that, because of (3.37), *all traces of* $\mathcal{G}_{\varphi,\beta}(\alpha)$*, including* $\mathcal{G}_{\varphi,\beta}(\alpha)$ *itself are zero.* That is to say, the equation of motion for α is *implied* by the equations for φ and β.

The role of the compensator α acquires a neat meaning in the presence of an external source. Let us suppose to introduce the coupling with a source $\mathcal{J}$ in the usual manner, by adding to the Lagrangian (3.14) a term $\varphi \cdot \mathcal{J}$; the equations of motion become in this case:

$$\begin{aligned} & \mathcal{A} - \frac{1}{2} \eta \, \mathcal{B} + \eta^2 \, \mathcal{C} = \mathcal{J} \,, \\ & \varphi'' - 4 \, \partial \cdot \alpha - \partial \, \alpha' = 0 \,, \\ & \mathcal{G}_{\varphi,\beta}(\alpha) = 0 \,. \end{aligned} \tag{3.38}$$

Now, taking the divergence of the first equation, and using (3.37) one finds

$$\frac{\eta}{4}\, \mathcal{G}_{\varphi,\beta}(\alpha) \;=\; \partial \cdot \mathcal{J}\,, \tag{3.39}$$

and therefore, because of the equation for α, *the source must be divergenceless, on-shell*, as expected from Noether's theorem, if $\mathcal{J}$ is to be linked to the gauge symmetry of the free theory. This result is relevant both because it shows that the properties of the unconstrained system are consistent with the physical expectations for the most elementary kind of interactions conceivable in a Field Theory, and because it marks a first "physical" difference with the Fronsdal scheme. In this last case, in fact, the equations with a source are of the kind

$$\mathcal{F} \;-\; \frac{1}{2}\,\eta\, \mathcal{F}' \;=\; \mathcal{J}\,, \tag{3.40}$$

and consequently the divergence results to be

$$-\frac{1}{2}\,\eta\, \partial \cdot \mathcal{F}' \;=\; \partial \cdot \mathcal{J}\,. \tag{3.41}$$

The direct consequence of this equation is that, while for $s = 1$ and $s = 2$ the usual condition $\partial \cdot \mathcal{J} = 0$ is found, from spin $s = 3$ onwards consistency only requires the weaker assumption that *the traceless part of the divergence* vanish. This assumption, although sufficient to guarantee that only physical polarisations contribute in the exchange of quanta between sources [18], does not completely fit the condition for a current to be a Noether's one.

3.2 Fermions

A nice feature of the procedure proposed in the previous Section to transform the Fronsdal Lagrangian for bosons in a fully gauge invariant one, is that it can be reproduced step by step also in the case of half-integer spin. It is then possible to obtain a fully gauge invariant Lagrangian for unconstrained fermions, at the price of introducing two additional fields providing the necessary conditions for the triple γ-trace of the basic field ψ, which cannot carry physical degrees of freedom, and for the γ-trace of the gauge parameter ϵ, which should be used to partially gauge fix the unconstrained equations to the Fronsdal form. Similarly, the interpretation of the equations of motion follows very closely what we have seen for bosons, and for this reason I will not repeat here observations and comments that can be easily borrowed from the bosonic case.

3.2.1 Lagrangians

The starting point is now the Fang-Fronsdal Lagrangian (2.35)

$$\frac{1}{i}\,\mathcal{L} \,=\, \frac{1}{2}\,\bar{\psi}\left\{\mathcal{S} \,-\, \frac{1}{2}\gamma\,\not{\mathcal{S}} \,-\, \frac{1}{2}\eta\,\mathcal{S}'\right\} \,-\, \frac{1}{2}\left\{\bar{\mathcal{S}} \,-\, \frac{1}{2}\bar{\not{\mathcal{S}}}\gamma \,-\, \frac{1}{2}\eta\,\bar{\mathcal{S}}'\right\}\psi\,, \tag{3.42}$$

where, as for bosons, it must be observed that this expression, when expanded in terms of the fundamental fields, it is not identical to the usual Fang-Fronsdal Lagrangian because of the term

$$-\frac{3}{4}\binom{n}{3}\left(\partial\cdot\bar{\psi}'\,\not{\psi}' \,-\, \bar{\not{\psi}}'\,\partial\cdot\psi'\right)\,, \tag{3.43}$$

which would vanish under the usual constraint on the triple γ-trace of ψ. Here I keep $\not{\psi}' \neq 0$ and evaluate the variation of (3.42) under $\delta\,\psi \,=\, \partial\,\epsilon$, $\delta\,\bar{\psi} \,=\, \bar{\epsilon}$ with unconstrained parameters:

$$\begin{aligned}
\delta\,\mathcal{L}_0 = &+ 2\,i\binom{n}{2}\bar{\not{\epsilon}}\,\partial\cdot\partial\cdot\psi \,-\, 2i\binom{n}{2}\bar{\not{\epsilon}}\;\not{\partial}\,\partial\cdot\not{\psi} \,-\, i\binom{n}{2}\bar{\not{\epsilon}}\,\Box\,\psi' \\
&+ \frac{9i}{2}\binom{n}{3}\partial\cdot\bar{\not{\epsilon}}\,\partial\cdot\psi' \,+\, \frac{3i}{4}\binom{n}{3}\bar{\epsilon}'\;\not{\partial}\,\partial\cdot\psi' \,-\, 3\,i\binom{n}{3}\bar{\epsilon}'\,\partial\cdot\partial\cdot\not{\psi} \\
&- \frac{3i}{4}\binom{n}{3}\bar{\epsilon}'\,\Box\;\not{\psi}' \,+\, 3i\binom{n}{4}\partial\cdot\bar{\epsilon}'\,\partial\cdot\not{\psi}' \,-\, 3\,i\binom{n}{4}\bar{\not{\epsilon}}'\,\partial\cdot\partial\cdot\psi' \\
&- \frac{3i}{2}\binom{n}{3}\partial\cdot\partial\cdot\bar{\epsilon}\;\not{\psi}' \,+\, h.c.\,.
\end{aligned} \tag{3.44}$$

In this expression one can again distinguish the terms proportional to single or multiple traces of the parameter from the contribution proportional to the double divergence of the parameter itself, present from spin $s \,=\, \frac{7}{2}$ onwards. To clarify once more the basic idea it is useful to take a look at $\delta\mathcal{L}_0$ in the particularly simple case of spin $s \,=\, \frac{5}{2}$, the first non-trivial possibility after the known cases described by the Dirac and Rarita-Schwinger theories. In this case the variation reduces to

$$\delta\mathcal{L}_0^{(\frac{5}{2})} \,=\, i\,\bar{\not{\epsilon}}\,\partial\cdot\,\not{\mathcal{S}} \,+\, h.c.\,, \tag{3.45}$$

that can be compensated adding to $\delta\mathcal{L}_0^{(\frac{5}{2})}$ the counterterm

$$\mathcal{L}_1^{(\frac{5}{2})} \,=\, -\,i\,\bar{\xi}\,\partial\cdot\,\not{\mathcal{S}} \,+\, h.c.\,, \tag{3.46}$$

where the field ξ is a rank-$\frac{3}{2}$ spinor-tensor, transforming as $\delta\,\xi\ = \not\epsilon$. The variation of $\mathcal{L}_0\ +\ \mathcal{L}_1$ does not vanish,

$$\delta\mathcal{L}_0^{(\frac{5}{2})}\ +\ \delta\mathcal{L}_1^{(\frac{5}{2})}\ =\ 2\,i\,\bar{\xi}\,\Box\ \not\partial\ \not\epsilon + h.c.\,, \tag{3.47}$$

but can be compensated completing the kinetic operator for ξ with the contribution $\mathcal{L}_2\ =\ -\,2\,i\,\bar{\xi}\,\Box\ \not\partial\,\xi$. The end result is the fully gauge-invariant Lagrangian

$$\mathcal{L}^{(\frac{5}{2})} = \frac{1}{2}\,\bar{\psi}\{\mathcal{S}\ -\ \frac{1}{2}\gamma\ \not\mathcal{S}\ -\ \frac{1}{2}\eta\,\mathcal{S}\,'\}\ -\ i\,\bar{\xi}\,\partial\cdot\ \not\mathcal{S}\ -\ \frac{i}{2}\,\bar{\xi}\,\Box\ \not\partial\,\xi\ + h.c.\,. \tag{3.48}$$

For spin $s\ \geq\ \frac{7}{2}$, where this program cannot work so simply, we can still compensate all terms involving γ-traces of the parameter in (3.44) by new couplings between the gauge field and the compensator ξ of the form

$$\begin{aligned}\mathcal{L}_1 =& -\ \frac{3i}{4}\binom{n}{3}\,\bar{\not\xi}\ \not\partial\,\partial\cdot\psi'\ +\ 3i\binom{n}{3}\,\bar{\not\xi}\,\partial\cdot\partial\cdot\not\psi\ +\ \frac{3i}{4}\binom{n}{3}\,\bar{\not\xi}\,\Box\ \not\psi\,'\\ &-\ 3i\binom{n}{4}\,\partial\cdot\bar{\not\xi}\,\partial\cdot\not\psi\,'\ -\ 2i\binom{n}{2}\,\bar{\xi}\,\partial\cdot\partial\cdot\psi\ +\ 2i\binom{n}{2}\,\bar{\xi}\ \not\partial\,\partial\cdot\not\psi\\ &+\ i\binom{n}{2}\,\bar{\xi}\,\Box\,\psi'\ -\ \frac{9i}{2}\binom{n}{3}\ \partial\cdot\bar{\xi}\,\partial\cdot\psi'\ +\ 3i\binom{n}{4}\,\bar{\xi}'\,\partial\cdot\partial\cdot\psi'\ +\ h.c.\,.\end{aligned} \tag{3.49}$$

Apart from the terms proportional to divergences of the parameter, the variation of $\mathcal{L}_0\ +\ \mathcal{L}_1$ generates new terms in $\not\epsilon$. The relevant result at this level is that all these new terms can be compensated by the variation of a suitable kinetic operator for ξ, of the form

$$\begin{aligned}\mathcal{L}_2\ =& -\ \frac{15\,i}{2}\binom{n}{3}\,\bar{\not\xi}\,\Box\,\partial\cdot\xi\ -\ i\binom{n}{2}\,\bar{\xi}\,\Box\ \not\partial\,\xi\ +\ 3\,i\binom{n}{3}\ \partial\cdot\bar{\xi}\ \not\partial\,\partial\cdot\xi\\ &+\ 18\,i\binom{n}{4}\,\partial\cdot\bar{\not\xi}\,\partial\cdot\partial\cdot\xi\ +\ 6\,i\binom{n}{4}\ \partial\cdot\bar{\not\xi}\,\Box\,\xi'\\ &-\ 15\,i\binom{n}{5}\,\partial\cdot\partial\cdot\bar{\not\xi}\ \partial\cdot\xi'\ +\ h.c.\,.\end{aligned} \tag{3.50}$$

Here again we arrive at a "quasi gauge-invariant" Lagrangian, $\mathcal{L}_0\ +\ \mathcal{L}_1\ +\ \mathcal{L}_2$, whose variation only involves multiple divergences of the parameter:

$$\begin{aligned}\delta\{\mathcal{L}_0\ +\ \mathcal{L}_1\ +\ \mathcal{L}_2\}\ =& -\ i\,\frac{3}{2}\binom{n}{3}\,\{\partial\cdot\partial\cdot\bar{\epsilon}\,(\not\psi'\ -\ 2\,\partial\cdot\xi -\ \not\partial\ \not\xi)\ -\ (\bar{\not\psi}'\ -\ 2\,\partial\cdot\bar{\xi}\ +\ \bar{\not\xi}\ \not\partial)\,\partial\cdot\partial\cdot\epsilon\}\\ &+\ 6\,i\binom{n}{4}\{\bar{\xi}'\,\partial\cdot\partial\cdot\partial\cdot\epsilon\ -\ \partial\cdot\partial\cdot\partial\cdot\bar{\epsilon}\,\xi'\}\\ =& -\ i\,\frac{3}{2}\binom{n}{3}\,\{\partial\cdot\partial\cdot\bar{\epsilon}\,(\not\psi\,'\ -\ 2\,\partial\cdot\xi -\ \not\partial\ \not\xi\ -\ \partial\,\xi').\\ &-\ (\bar{\not\psi}'\ -\ 2\,\partial\cdot\bar{\xi}\ +\ \bar{\not\xi}\ \not\partial\ -\ \partial\,\bar{\xi}')\,\partial\cdot\partial\cdot\epsilon\}\,.\end{aligned} \tag{3.51}$$

The key feature of this variation is that, similarly to the case of bosons, it is related to the *gauge invariant* constraint $\not\psi\,' - 2\,\partial\cdot\xi - \not\partial\,\not\xi - \partial\xi' = 0$, and to its hermitian conjugate. This suggests to compensate the last terms by introducing in the Lagrangians such constraints, via a Lagrange multiplier λ that transforms according to $\delta\,\lambda = \partial\cdot\partial\cdot\epsilon$. The final result is the complete Lagrangian for an *unconstrained* spin-$(n+1/2)$ spinor-tensor ψ:

$$
\begin{aligned}
\mathcal{L} = & \ \frac{1}{2}\,\bar{\psi}\left(\mathcal{S} - \frac{1}{2}\gamma\,\not\mathcal{S} - \frac{1}{2}\eta\,\mathcal{S}'\right) \\
& - \frac{3i}{4}\binom{n}{3}\,\bar{\not\xi}\,\not\partial\,\partial\cdot\psi' + 3i\binom{n}{3}\,\bar{\not\xi}\,\partial\cdot\partial\cdot\not\psi + \frac{3i}{4}\binom{n}{3}\,\bar{\not\xi}\,\Box\,\not\psi' \\
& - 3i\binom{n}{4}\,\partial\cdot\bar{\not\xi}\,\partial\cdot\not\psi' - 2i\binom{n}{2}\,\bar{\xi}\,\partial\cdot\partial\cdot\psi + 2i\binom{n}{2}\,\bar{\xi}\,\not\partial\,\partial\cdot\not\psi \\
& + i\binom{n}{2}\,\bar{\xi}\,\Box\,\psi' - \frac{9i}{2}\binom{n}{3}\,\partial\cdot\bar{\xi}\,\partial\cdot\psi' + 3i\binom{n}{4}\,\bar{\xi}'\,\partial\cdot\partial\cdot\psi' \\
& - \frac{15i}{2}\binom{n}{3}\,\bar{\not\xi}\,\Box\,\partial\cdot\xi - i\binom{n}{2}\,\bar{\xi}\,\Box\,\not\partial\,\xi + 3i\binom{n}{3}\,\partial\cdot\bar{\xi}\,\not\partial\,\partial\cdot\xi \\
& + 18i\binom{n}{4}\,\partial\cdot\bar{\not\xi}\,\partial\cdot\partial\cdot\xi + 6i\binom{n}{4}\,\partial\cdot\bar{\not\xi}\,\Box\,\xi' - 15i\binom{n}{5}\,\partial\cdot\partial\cdot\bar{\not\xi}\,\partial\cdot\xi' \\
& + \frac{3i}{2}\binom{n}{3}\,\bar{\lambda}\,\left(\not\psi\,' - 2\,\partial\cdot\xi - \not\partial\,\not\xi - \partial\xi'\right) + h.c.\,,
\end{aligned}
\tag{3.52}
$$

invariant under the gauge transformations

$$
\begin{aligned}
\delta\,\psi &= \partial\,\epsilon\,, \\
\delta\,\xi &= \not\epsilon\,, \\
\delta\,\lambda &= \partial\cdot\partial\cdot\epsilon\,.
\end{aligned}
\tag{3.53}
$$

with an *unconstrained* gauge parameter.

3.2.2 Equations of motion and their interpretation

The equations of of motion for the fields $\bar{\psi}$, $\bar{\lambda}$ and $\bar{\xi}$ are

$$
\begin{aligned}
\bar{\psi}\,: \quad & -i\,\mathcal{S} + 2\,\partial^2 \xi - \frac{1}{2}\eta\,(-i\,\mathcal{S}' + \frac{1}{2}\partial\,\not{\psi}' - \frac{1}{2}\partial\,\not{\partial}\,\not{\xi} + 2\Box\xi + 3\,\partial\partial\cdot\,\xi + \partial^2\,\xi') \\
& - \frac{1}{2}\gamma\,(-i\,\not{\mathcal{S}} + 2\,\partial^2\,\not{\xi} + 2\,\partial\,\not{\partial}\xi) + \frac{1}{4}\gamma\,\eta\,(\partial\cdot\psi' - \Box\,\not{\xi} - \partial\,\partial\cdot\,\not{\xi} - 2\,\lambda) = 0\,,
\end{aligned}
\tag{3.54}
$$

$$
\bar{\lambda}\,: \quad \not{\psi}\,' - 2\,\partial\cdot\xi - \not{\partial}\,\not{\xi} - \partial\xi' = 0\,, \tag{3.55}
$$

$$
\begin{aligned}
\bar{\xi}\,: \quad & \Box\,\psi' + 2\,\not{\partial}\,\partial\cdot\not{\psi} - 2\,\partial\cdot\partial\cdot\psi + \frac{3}{2}\partial\,\partial\cdot\,\psi' - 2\Box\,\not{\partial}\xi - \frac{5}{2}\Box\,\partial\,\not{\xi} - 2\,\not{\partial}\,\partial\,\partial\cdot\xi \\
& - 3\,\partial^2\,\partial\cdot\,\not{\xi} - \partial\lambda + \eta\,(\frac{1}{2}\partial\cdot\partial\cdot\psi' - \Box\,\partial\cdot\,\not{\xi} - \frac{1}{2}\partial\,\partial\cdot\partial\cdot\,\not{\xi} - \partial\cdot\lambda) \\
& + \gamma\,(-\frac{1}{4}\,\not{\partial}\,\partial\cdot\psi' + \partial\cdot\partial\cdot\,\not{\psi} + \frac{1}{4}\Box\,\not{\psi}' + \frac{1}{4}\partial\partial\cdot\,\not{\psi}' \\
& - \frac{5}{2}\Box\,\partial\cdot\xi - \frac{3}{2}\partial\partial\cdot\partial\cdot\xi - \frac{1}{2}\Box\,\partial\xi' - \frac{1}{2}\partial^2\partial\cdot\xi' - \frac{1}{2}\,\not{\partial}\lambda) = 0\,.
\end{aligned}
\tag{3.56}
$$

I would now like to discuss the reduction to the Fang-Fronsdal equations (2.37) together with the role of the field equation for the compensator.

To begin with, let us rewrite the system in compact notation as

$$
\begin{aligned}
& \mathcal{W} - \frac{1}{2}\gamma\,\mathcal{X} - \frac{1}{2}\eta\,\mathcal{Y} + \frac{1}{4}\eta\,\gamma\,\mathcal{Z} = 0\,, \\
& \not{\psi}\,' - 2\,\partial\cdot\xi - \not{\partial}\,\not{\xi} - \partial\xi' = 0\,, \\
& \mathcal{H}_{\psi,\lambda}(\xi) = 0\,,
\end{aligned}
\tag{3.57}
$$

where the meaning of the various operators can be deduced by comparison with (3.54) and (3.56), and where it can be recognised that, after substituting (3.55) in (3.54), this last equation reduces to

$$
\mathcal{W} - \frac{1}{2}\gamma\,\not{\mathcal{W}} - \frac{1}{2}\eta\,\mathcal{W}' + \frac{1}{4}\eta\,\gamma\,\mathcal{Z} = 0\,; \tag{3.58}
$$

moreover, under the same conditions one can see that $\not{\mathcal{W}}' = 0$. We can now evaluate successive γ-traces of (3.58). To compute the first γ-trace we need the following three formulas:

$$
\begin{aligned}
\gamma\cdot(\gamma\,T) &= [D\,+\,2\,m\,]\,T\,-\,\gamma\,\not{T}\,,\\
\gamma\cdot(\eta\,T) &= \gamma\,T\,+\,\eta\,\not{T}\,,\\
\gamma\cdot(\eta\,\gamma\,T) &= \eta\,T\,[D\,+\,2\,(m\,+\,1)]\,-\,\eta\,\gamma\,\not{T}\,.
\end{aligned}
\tag{3.59}
$$

that hold for any rank-m, symmetric, spinor-tensor T. Using these results we can evaluate the γ-trace of (3.58) to find

$$
-\,\frac{1}{2}\,\rho_2\,\mathcal{W}\,+\,\frac{1}{4}\,\rho_2\,\eta\,\mathcal{Z}\,-\,\frac{1}{4}\,\eta\,\gamma\,\not{\mathcal{Z}}\,=\,0\,,
\tag{3.60}
$$

where the coefficients ρ_i are defined as in (3.23): $\rho_i\,=\,[D\,+\,2\,(n\,-\,i)]$.
Letting p denote the integer part of $\frac{n-3}{2}$, we can take at most p traces of $\mathcal{Z}$, together with possibly one further γ-trace, if n is even. Taking one trace of (3.60) we can eliminate $\mathcal{W}$, finding

$$
\frac{1}{4}\,\rho_2\,\rho_3\,\mathcal{Z}\,+\,\frac{1}{4}\rho_3\,\eta\,\mathcal{Z}'\,-\,\frac{1}{4}\,\rho_3\,\gamma\,\not{\mathcal{Z}}\,-\,\frac{1}{4}\,\eta\,\gamma\,\not{\mathcal{Z}}'\,=\,0\,,
\tag{3.61}
$$

and in general it is possible to show by induction that the p-th trace of (3.60) gives, for n even,

$$
\rho_{p+1}\,(\sum_{i=1}^{p}\rho_{2i+1})\,\mathcal{Z}^{[p-1]}\,+\,\rho_{p+2}\,\eta\,\mathcal{Z}^{[p]}\,-\,(\sum_{i=1}^{p}\rho_{2i+1})\,\gamma\,\not{\mathcal{Z}}^{[p-1]}\,-\,\eta\,\gamma\,\not{\mathcal{Z}}^{[p]}\,=\,0\,.
\tag{3.62}
$$

Consequently, the $(p+1)$-th trace reduces to

$$
\rho_{p+2}\,(\sum_{i=1}^{p+1}\rho_{2i+1})\,\mathcal{Z}^{[p]}\,-\,\rho_{2p+3}\,\gamma\,\not{\mathcal{Z}}^{[p]}\,=\,0\,,
\tag{3.63}
$$

and the γ-trace of this last relation gives at the end

$$
\gamma\cdot\left(\frac{1}{4}\,\eta\,\gamma\,\mathcal{Z}\right)^{[p+1]}\,=\,\left\{\rho_{p+2}\,(\sum_{i=1}^{p+1}\rho_{2i+1})\,-\,D\,\rho_{2p+3}\right\}\not{\mathcal{Z}}^{[p]}\,=\,0\,.
\tag{3.64}
$$

One can see that in this relation the coefficient of $\not{\mathcal{Z}}^{[p]}$ is always positive. To this end it suffices to explicitate the first term in the sum: $\rho_{p+2}\,\rho_{2p+3}\,=\,[D+2(n-p-2)]\rho_{2p+3}$, and recognise that the whole coefficient reduces to $2(n\,-\,p\,-\,2)\rho_{2p+3}\,+\,\rho_{p+2}\,(\sum_{i=1}^{p}\rho_{2i+1})$, which is manifestly positive. Moreover, the γ-traces of $\mathcal{Z}$, evaluated when the equation for λ is satisfied, are of the form

$$
\not{\mathcal{Z}}^{[k]}\,=\,2\,\partial\cdot\partial\cdot\xi^{[k]}\,-\,2\,\not{\chi}^{[k]}\,,
\tag{3.65}
$$

and hence do not vanish identically. As a consequence of the equation for ψ and λ, one finds in this way

$$\not{\mathcal{Z}}^{[p]} = 0 . \tag{3.66}$$

It is then possible to proceed backwards to conclude that, *on-shell, all γ-traces of $\mathcal{Z}$, including $\mathcal{Z}$ itself, vanish.* This implies that the γ-trace of $\mathcal{W}$ vanishes as well, and so the equation of motion for ψ reduces to

$$\mathcal{S} + 2\, i\, \partial^2 \xi = 0 . \tag{3.67}$$

In complete analogy with the bosonic case, this equation can be partially gauge fixed to the Fang-Fronsdal form using the γ-trace of the parameter to fix the value of ξ to zero. For n odd the argument is almost identical, with the only difference that there is no need to evaluate a further γ-trace after the saturating $(p+1)$-th trace.

Also for the analysis of the role of ξ one finds complete analogy with the bosonic case. In particular, at the free level the equation for ξ is actually a consequence of the equations for ψ and λ, since it is possible to show that

$$\partial \cdot \left\{ \mathcal{W} - \frac{1}{2}\gamma\, \not{\mathcal{W}} - \frac{1}{2}\eta\, \mathcal{W}' + \frac{1}{4}\eta\,\gamma\, \mathcal{Z} \right\} = \frac{1}{2}\gamma\, \mathcal{H}_{\psi,\lambda}(\xi) , \tag{3.68}$$

and so that $\frac{1}{2}\gamma\, \mathcal{H}_{\psi,\lambda}(\xi) = 0$, when ψ and λ are on shell. It then follows that all traces of the last expression vanish:

$$(\gamma\, \mathcal{H})^{[k]} = k\, \not{\mathcal{H}}^{[k-1]} + \frac{1}{2}\gamma\, \mathcal{H}^{[k]} = 0 . \tag{3.69}$$

An iterative argument close in spirit to the one used to derive (3.67) from (3.54) leads to conclude that, because of (3.68), *all traces of $\mathcal{H}$, including $\mathcal{H}$ itself, vanish*, and hence (3.56) is implied, in the free case, by (3.54) and (3.55).

To see explicitly how (3.68) emerges, one can write the divergence of (3.58) as

$$\begin{aligned} \partial \cdot \mathcal{W} - \frac{1}{2}\, \not{\partial}\, \not{\mathcal{W}} - \frac{1}{2}\partial\, \mathcal{W}' - \frac{1}{2}\gamma \left\{ \partial \cdot \not{\mathcal{W}} - \frac{1}{2}\partial\, \mathcal{Z} \right\} & \\ - \frac{1}{2}\eta \left\{ \partial \cdot \mathcal{W}' - \frac{1}{2}\, \not{\partial}\, \mathcal{Z} - \frac{1}{2}\gamma\, \partial \cdot \mathcal{Z} \right\} &= 0 , \end{aligned} \tag{3.70}$$

and compare with the equation for $\bar{\xi}$, written in the compact form

$$\mathcal{H}_{\psi,\lambda}(\xi) = \mathcal{A} + \eta\, \mathcal{B} + \gamma\, \mathcal{C} = 0 , \tag{3.71}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are defined here by comparison with (3.56). The first contribution in (3.70) vanishes because of the proper Bianchi identity satisfied by the operator $\mathcal{W}$ via the constraint (3.55):

$$\begin{aligned}
\partial \cdot \mathcal{W} - \frac{1}{2}\,\not\partial\,\not{\mathcal{W}} - \frac{1}{2}\partial\,\mathcal{W}' &= -i\,(\partial \cdot \mathcal{S} - \frac{1}{2}\partial\,\mathcal{S}' - \frac{1}{2}\,\not\partial\,\not{\mathcal{S}}) \\
&\quad - \partial^2\,(2\,\partial \cdot \xi + \not\partial\,\not\xi + \partial\,\xi') \\
&= \partial^2(\not\psi' - 2\,\partial \cdot \xi - \not\partial\,\not\xi - \partial\,\xi') = 0\,.
\end{aligned} \tag{3.72}$$

The proof of (3.68) can be completed looking explicitly at the remaining terms, divided for the sake of clarity in three groups, and comparing group by group with (3.56):

$$\begin{aligned}
-\frac{1}{2}\gamma\left\{\partial\cdot \not{\mathcal{W}} - \frac{1}{2}\,\partial\,\not{\mathcal{Z}}\right\} &= \frac{1}{2}\gamma\{\Box\,\psi' + 2\,\not\partial\,\partial\cdot\not\psi - 2\,\partial\cdot\partial\cdot\psi + \frac{3}{2}\partial\partial\cdot\psi' \\
&\quad 2\,\Box\,\not\partial\xi - \frac{5}{2}\Box\,\partial\,\not\xi - 2\,\not\partial\,\partial\partial\cdot\xi - 3\,\partial^2\partial\cdot\not\xi - \partial\lambda\}\,, \\
\frac{1}{4}\,\eta\,\gamma\,\partial\cdot\mathcal{Z} &= \frac{1}{2}\gamma\,\{\eta\,(\frac{1}{2}\,\partial\cdot\partial\cdot\psi' - \Box\,\partial\cdot\not\xi - \frac{1}{2}\,\partial\,\partial\cdot\partial\cdot\not\xi - \partial\cdot\lambda)\}\,, \\
-\frac{1}{2}\,\eta\,\{\partial\cdot\mathcal{W}' - \frac{1}{2}\,\not\partial\,\mathcal{Z}\} &= \frac{1}{2}\gamma\,\{\gamma\,(-\frac{1}{4}\,\not\partial\partial\cdot\psi' + \partial\cdot\partial\cdot\not\psi + \frac{1}{4}\Box\,\not\psi' + \frac{1}{4}\,\partial\partial\cdot\not\psi' \\
&\quad -\frac{5}{2}\Box\,\partial\cdot\xi - \frac{3}{2}\partial\partial\cdot\partial\cdot\xi - \frac{1}{2}\Box\,\partial\,\xi' - \frac{1}{2}\partial^2\partial\cdot\xi' - \frac{1}{2}\,\not\partial\lambda)\}\,,
\end{aligned} \tag{3.73}$$

where I used again the equation for $\bar{\lambda}$, along with the relation $\gamma\,\gamma = 2\,\eta$, to be understood as an identity of the symmetric calculus. As for the bosonic case, the role of the equation for the compensator acquires some relevance in the presence of an external current coupled to the physical field in the "minimal" way, adding in the Lagrangian the term $\psi\cdot\mathcal{J}$. Differently from the Fang-Fronsdal case, and in complete analogy with the results of Section 3.1, it can be recognised in this situation that, on-shell, the current $\mathcal{J}$ *must* be divergenceless, by consistency with (3.56).

Chapter 4

Unconstrained formulation and higher-spin geometry

In the previous Chapter I have shown that it is possible to formulate the free theory of higher-spin bosons and fermions without any constraint on the gauge fields or on the gauge parameters, at the price of introducing in the Lagrangians two auxiliary fields providing the necessary conditions for a consistent dynamics. Here I face with the issue of solving the same problem *without* auxiliary fields, i.e. I show how to arrive for these gauge theories at a formulation more resembling the lower-spin counterparts of spin less or equal than two.

Section 4.1 is devoted to show how to eliminate the auxiliary fields and still get fully gauge-invariant unconstrained equations of motion for arbitrary spin bosons. The main point here is that the theory becomes *non-local*, because of the presence of inverse powers of the D'Alembertian operator that enter the game when the auxiliary fields α and ξ are eliminated from the bosonic and fermionic equations respectively.

This main departure from the Fronsdal form is compensated by the enlargement of the gauge group; it can be shown, as also expected by consistency with the local compensator formulation, that the trace of the gauge parameter can be used to eliminate any non-localities in the equations of motion, while the double trace of the gauge field can be shown to vanish on-shell, thus showing the full equivalence of the non-local formulation with the Fronsdal theory. This non-local theory admits an independent Lagrangian formulation, as the corresponding Lagrangians can be obtained by integration of suitable, non-local Einstein-like tensors, verifying the correct Bianchi identity.

As a natural step forward, one could ask whether these gauge theories can be given a "geometrical" formulation. By "geometrical" I mean a formulation in which all the information on the dynamics and on the gauge structure of the theory can be deduced by suitable, gauge-covariant tensors or spinor-tensors, generalising in some convincing way the Maxwell and Einstein curvatures for spin 1 and 2. These generalised "curvatures", along with a whole set of connections, were actually constructed in 1980 by B. de Wit and D. Freedman in [21], but it is only in the unconstrained setting that they can underlie the dynamics, and in this sense it is only by discarding the constraints that it is possible to give the higher-spin gauge theory a truly geometrical formulation. The construction of de Wit and Freedman is presented in Section 4.2.2, while in Section 4.2.4 the link with the unconstrained, non-local formulation is discussed.

4.1 Unconstrained formulation and non-local equations

The main results of the previous Chapter are in the two systems

$$\begin{aligned} \mathcal{F} &= 3\,\partial^3\,\alpha\,, \\ \varphi'' &= 4\,\partial\cdot\alpha\,+\,\partial\,\alpha'\,, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \mathcal{S} + 2\,i\,\partial^2\,\xi \,&=\, 0\,, \\ \not\psi\,' &= 2\,\partial\cdot\xi - \not\partial\,\not\xi\, - \partial\,\xi'\,, \end{aligned} \tag{4.2}$$

describing respectively the dynamics of unconstrained bosons and fermions derived from the Lagrangians (3.14) and (3.52). I would like to show that it is possible to eliminate from these equations the auxiliary fields, and still arrive at an unconstrained dynamics for the physical fields. This dynamics is in principle non-local, but it is possible to reduce it to the local Fronsdal one performing a partial gauge fixing involving the trace of the gauge parameter. To elucidate the main ideas in a relatively simple setting, Let me begin with the description of the cases of spin 3 and 4, to pass then to the general bosonic case and to fermions.

4.1.1 Spin 3

The idea is to look for combinations of the $\mathcal{F}$ operator with its traces and divergences, such that the resulting expression vanishes identically, if $\mathcal{F} \,=\, 3\,\partial^{\,3}\,\alpha$ is satisifed. For spin 3 the compensator α is a scalar, and this allows for three possible constructions verifying the same equations as $\mathcal{F}$:

$$\begin{aligned} \frac{1}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F} \,&=\, 3\,\partial^{\,3}\,\alpha\,, \\ \frac{1}{3}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{\,\prime} \,&=\, 3\,\partial^{\,3}\,\alpha\,, \\ \frac{\partial^{\,3}}{\Box^2}\,\partial\cdot\mathcal{F}^{\,\prime} \,&=\, 3\,\partial^{\,3}\,\alpha\,. \end{aligned} \tag{4.3}$$

Correspondingly, three possible fully gauge invariant unconstrained equations are available:

$$\begin{aligned} \mathcal{F} \,-\, \frac{1}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F} \,&=\, 0, \\ \mathcal{F} \,-\, \frac{1}{3}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{\,\prime} \,&=\, 0 \\ \mathcal{F} \,-\, \frac{\partial^{\,3}}{\Box^2}\,\partial\cdot\mathcal{F}^{\,\prime} \,&=\, 0. \end{aligned} \tag{4.4}$$

It is simple to recognise, considering the explicit form of $\mathcal{F}$ in terms of φ, that the first two equations in (4.4) are actually *the same* equation, while the third is a consequence of the preceding two, for example calculating the trace of $\mathcal{F}$ in the second equation and then substituting back. This last form is anyway quite interesting, being of the kind

$$\mathcal{F} \,=\, 3\,\partial^3\,\mathcal{H}(\varphi)\,, \tag{4.5}$$

where $\mathcal{H}(\varphi) \,=\, \frac{\partial^3}{3\,\Box^2}\,\partial\cdot\mathcal{F}^{\,\prime}$ is such that $\delta\,\mathcal{H}(\varphi) \,=\, \Lambda^{\,\prime}$. We have then shown two facts: first, it is possible to eliminate α from the equation with compensator to obtain a fully gauge invariant equation in terms of the single field φ, without introducing any constraints. This form of the equation is non-local, as anticipated in the introduction of this Chapter, but it can be cast in the useful form (4.5), in which it is manifest that, making use only of the trace of the gauge parameter, *all non localities can be removed*, while the equation itself reaches the Fronsdal form $\mathcal{F} \,=\, 0$. In other words, it is possible to substitute the compensator of the local theory with a non-local construction involving only φ, behaving as the compensator itself under gauge transformations. Once the form $\mathcal{F} \,=\, 0$ is reached one still has gauge freedom on φ with a traceless gauge parameter, and so it is possible to reproduce the calculation of the propagating degrees of freedom, exactly as in the Fronsdal case discussed in Section 2.3.

In order to find Lagrangians for these non-local equations, the first step is to recognise that the building block of the Einstein-like tensors must be the full kinetic operator $F^{(3)} \equiv \mathcal{F} - \frac{1}{3}\frac{\partial^2}{\Box}\mathcal{F}'$, or alternatively $\hat{F}^{(3)} \equiv \mathcal{F} - \frac{\partial^3}{\Box^2}\partial\cdot\mathcal{F}'$, just as the Fronsdal operator $\mathcal{F}$ was the elementary input for the construction of the Einstein tensor $\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'$ for the constrained case. In that case, as shown in Section 2.1, the gauge invariance of the Lagrangian was assured by the Bianchi identity together with the tracelessness of the gauge parameter. In this situation, with Λ traceful, we must combine $F^{(3)}$ (or $\hat{F}^{(3)}$) with its trace in such a way that the resulting tensor be *identically divergenceless*. In this way, if $\mathcal{G}^{(3)}$ is the resulting tensor, the Lagrangian obtained by integrating $\sim\ \varphi\,\mathcal{G}^{(3)}$ will automatically be gauge invariant under $\delta\,\varphi\ =\ \partial\,\Lambda$.

It is possible to verify that the right combination is given by

$$\mathcal{G}^{(3)}\ =\ F^{(3)}\ -\ \frac{1}{4}\,\eta\,F^{(3)\,\prime}\,, \tag{4.6}$$

and the resulting, non-local Lagrangian is

$$\begin{aligned}\mathcal{L}^{(3)} =& -\frac{1}{2}\,(\partial\,\varphi)^{\,2}\,+\,\frac{3}{2}\,(\partial\cdot\varphi)^{\,2}\,+\,\varphi'\,\partial\cdot\partial\cdot\varphi\,-\,\frac{1}{2}\,(\partial\cdot\varphi')^{\,2}\\ &+\,\frac{1}{2}(\partial\,\varphi')^{\,2}\,+\,\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box}\,\partial\cdot\partial\cdot\varphi\,+\,\partial\cdot\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box}\,\partial\cdot\varphi'\,.\end{aligned} \tag{4.7}$$

Equivalently, choosing $\hat{F}^{(3)}$ one finds that the tensor $\hat{\mathcal{G}}^{(3)}$,

$$\hat{\mathcal{G}}^{(3)}\ =\ \hat{F}^{(3)}\ -\ \frac{1}{2}\,\eta\,\hat{F}^{(3)\,\prime}\,, \tag{4.8}$$

is identically divergenceless, while the corresponding Lagrangian $\mathcal{L}\ =\ \frac{1}{2}\varphi\,(\hat{F}^{(3)} - \frac{1}{2}\,\eta\,\hat{F}^{(3)\,\prime})$ is in this case:

$$\begin{aligned}\hat{\mathcal{L}}^{(3)}\ =& -\frac{1}{2}\,(\partial\,\varphi)^{\,2}\,+\,\frac{3}{2}\,(\partial\cdot\varphi)^{\,2}\,+\,3\,\varphi'\,\partial\cdot\partial\cdot\varphi\,-\,\partial\cdot\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box^2}\,\partial\cdot\partial\cdot\partial\cdot\varphi\\ &+\,3\,\partial\cdot\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box}\,\partial\cdot\varphi'\,+\,\frac{3}{2}\,(\partial\,\varphi')^{\,2}\,-\,\frac{3}{2}(\partial\cdot\varphi')^{\,2}\,.\end{aligned} \tag{4.9}$$

As for Fronsdal, the equations of motion (4.4) are not directly Lagrangian equations. Rather, as it is quite explicit by looking for example at (4.6), to recover them one should take a suitable combination of the Lagrangian equation with its trace

4.1.2 Spin 4

In passing from spin 3 to spin 4, two novelties emerge: first, the compensator α is no longer a scalar, and so the evaluation of traces and divergences of the basic equation $\mathcal{F} \,=\, 3\,\partial^3\,\alpha$ will no more produce in general equivalent results; it will be necessary to combine both kinds of constructions in order to get a fully gauge invariant equation. Secondly, the double trace of the gauge field appears, and this implies the need to remove the anomalous factors in $\partial^3\,\varphi''$ in the construction of identically divergenceless Einstein tensors. Moreover, given that by eliminating the auxiliary fields α and β one looses memory of the constraint $\varphi'' \,-\, 4\,\partial\cdot\alpha \,-\, \partial\,\alpha' = 0$, it is to be clarified whether and how the introduction of spurious degrees of freedom is avoided.

Once we have explained how to forego these difficulties in this case, all conceptual obstacles will be clarified, and we will then pass to face the technicalities involved in the discussion of the general case of spin s.

Let us start from the compensator equation, and let us compute its trace and divergence:

$$\begin{aligned} \mathcal{F}' \quad &= 3\,\Box\,\partial\,\alpha \,+\, 6\,\partial^2\,\partial\cdot\,\alpha \\ \partial\cdot\mathcal{F} &= 3\,\Box\,\partial^2\,\alpha \,+\, 3\,\partial^3\,\partial\cdot\,\alpha\,. \end{aligned} \tag{4.10}$$

In order to obtain something proportional to $\partial^3\,\alpha$, let us act on the first equation with $\frac{\partial^2}{\Box}$ and on the second with $\frac{\partial}{\Box}$:

$$\begin{aligned} \frac{\partial^2}{\Box}\,\mathcal{F}' \quad &= 9\,\partial^3\,\alpha \,+\, 36\,\frac{\partial^4}{\Box}\,\partial\cdot\,\alpha \\ \frac{\partial}{\Box}\,\partial\cdot\mathcal{F} &= 9\,\partial^3\,\alpha \,+\, 12\,\frac{\partial^4}{\Box}\,\partial\cdot\,\alpha \end{aligned} \tag{4.11}$$

The idea is to combine linearly the two left-hand sides of (4.11) with $\mathcal{F}$ in order to get something identically zero when $\mathcal{F} \,=\, 3\,\partial^3\,\alpha$. The right combination is

$$\mathcal{F} \,+\, \frac{1}{6}\,\frac{\partial^2}{\Box}\,\mathcal{F}' \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F} \,=\, 0\,. \tag{4.12}$$

This shows how to eliminate the compensator α to obtain an unconstrained equation. The next issue is to prove that a partial gauge-fixing with the trace of Λ removes the non-local terms, gauge-fixing the equation to the Fronsdal form. To this end let us write (4.12) in the form

$$\mathcal{F} - \frac{1}{2}\frac{\partial}{\Box}\left\{\partial\cdot\mathcal{F} - \frac{1}{6}\,\partial\,\mathcal{F}'\right\} = 0\,, \tag{4.13}$$

and use the Bianchi identity (2.27) $\partial\cdot\mathcal{F} - \frac{1}{2}\,\partial\,\mathcal{F}' = -\frac{3}{2}\,\partial^3\,\varphi''$, to turn it into

$$\mathcal{F} + 3\,\frac{\partial^4}{\Box}\,\varphi'' - \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{F}' = 0\,. \tag{4.14}$$

Now one can trace this last relation and solve for $\mathcal{F}'$ finding

$$\mathcal{F}' = \frac{\partial}{\Box}\,\partial\cdot\mathcal{F}' + \frac{1}{2}\,\frac{\partial^2}{\Box}\,\mathcal{F}'' - \frac{9}{2}\,\partial^2\,\varphi''\,; \tag{4.15}$$

substituting back in (4.12) the end result is the desired form

$$\mathcal{F} = 3\,\partial^3\,\mathcal{H}(\varphi)\,, \tag{4.16}$$

where in this case

$$\mathcal{H}(\varphi) = \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}' + \frac{1}{12}\,\frac{\partial}{\Box^2}\,\mathcal{F}'' - \frac{\partial}{\Box}\,\varphi''\,, \tag{4.17}$$

which can be directly verified to transform into the trace of Λ.

By gauge-fixing $\mathcal{H}(\varphi)$ to zero we can thus end with a Fronsdal-like equation of motion, in which nonetheless the double trace of φ is in principle different from zero, and so it is not clear whether the gauge freedom left, with a traceless gauge parameter, suffices to remove all unphysical polarisations. On the other hand, the Bianchi identity (2.27) implies that, when $\mathcal{F} = 0$, $\partial^3\,\varphi'' = 0$ as well, and consequently in momentum space, taking for example a light-like momentum p_+ it is possible to convince oneself that $\varphi'' = 0$. At this point nothing forbids to choose as a gauge condition the usual de Donder form, with a field constrained as in [18]. The non-local equation thus propagates correctly spin-4 polarisations, despite the fact that a *traceless* gauge parameter, left over after setting $\mathcal{H} = 0$, is in principle accompanied by an *unconstrained* gauge field.

Also for spin 4 one can look for a non-local Lagrangian from which equation (4.12) could be derived. As for spin 3, the building block for the Einstein-tensor is the kinetic operator defined by (4.12):

$$F^{(4)} \equiv \mathcal{F} + \frac{1}{6}\,\frac{\partial^2}{\Box}\,\mathcal{F}' - \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}\,. \tag{4.18}$$

This tensor satisfies the Bianchi identity

$$\partial\cdot F^{(4)} - \frac{1}{4}\,\partial\,F^{(4)\,\prime} = 0\,, \tag{4.19}$$

but, differently from the spin 3 case, one cannot use directly the combination

$$F^{(4)} \,-\, \frac{1}{4}\,\eta\, F^{(4)\,\prime}\;, \tag{4.20}$$

as an Einstein tensor, because this combination is not identically divergenceless. In particular, $\partial \cdot F^{(4)\,\prime} \neq 0$, and so one must add to (4.20) a term to compensate the remainder. The right combination,

$$\mathcal{G}^{(4)} \,=\, F^{(4)} \,-\, \frac{1}{4}\,\eta\, F^{(4)\,\prime} \,+\, \frac{1}{8}\,\eta^2\, F^{(4)\,\prime\prime}\,, \tag{4.21}$$

is such that

$$\partial \cdot \mathcal{G}^{(4)} \,\equiv\, 0\,. \tag{4.22}$$

The corresponding Lagrangian results

$$\begin{aligned}
\mathcal{L}^{(4)} \,=\, \frac{1}{2}\,\varphi\,\mathcal{G}^{(4)} =& \frac{1}{2}\,\varphi\,\Box\,\varphi \,+\, 2\,(\partial\cdot\varphi)^2 \,+\, 2\,\varphi'\,\partial\cdot\partial\cdot\varphi \,-\, \varphi'\,\Box\,\varphi' \\
&-\, 2\,(\partial\cdot\varphi')^2 \,-\, 2\,\varphi''\,\partial\cdot\partial\cdot\varphi' \,+\, \frac{1}{2}\,\varphi''\,\Box\,\varphi'' \\
&+\, 2\,\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box}\,\partial\cdot\partial\cdot\varphi \,+\, 4\,\partial\cdot\varphi'\,\frac{1}{\Box}\,\partial\cdot\partial\cdot\partial\cdot\varphi \\
&+\, \varphi''\,\frac{1}{\Box}\,\partial\cdot\partial\cdot\partial\cdot\partial\cdot\varphi \,+\, \partial\cdot\partial\cdot\varphi'\,\frac{1}{\Box}\,\partial\cdot\partial\cdot\varphi'\,.
\end{aligned} \tag{4.23}$$

4.1.3 General bosonic case

For bosons of any spin s one can reproduce the scheme described for spin 4: assuming that the compensator equation is satisfied, I show that it is possible to find a linear combination of constructs built up from $\mathcal{F}$ that is identically zero, without introducing constraints of any kind. These equations can be reduced to the form $\mathcal{F} \,=\, 3\,\partial^{\,3}\mathcal{H}(\varphi)$, with $\delta\,\mathcal{H}(\varphi) \,=\, \Lambda'$, and once the Fronsdal gauge is reached the double trace of the field is shown not to participate in the dynamics. I then display the Einstein tensors for these kinetic operators, and indicate how to deduce non-local Lagrangians, to be compared with the local ones along the lines of the last part of the previous Section.

To see how to find kinetic operators for a general bosonic field φ by eliminating α from the compensator equation, consider again the kinetic operator (4.12) we found for $s = 4$, now with φ a tensor of rank s, and correspondingly α a tensor of rank $s - 3$:

$$\mathcal{F} \,+\, \frac{1}{6}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}' \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F} \,=\, +\,5\,\frac{\partial^{\,5}}{\Box}\,\alpha'\,. \tag{4.24}$$

As expected, a remainder appears that is proportional to the trace of the compensator, not present in the $s = 4$ case, where the compensator α was a rank-1 tensor. The idea is to start with (4.24), and to combine it with its trace and divergence in order to eliminate the piece in α'. New terms proportional to higher traces of α are expected to appear, and the goal is to define the iteration in such a way that, when all indices in α are saturated and there is no room to accommodate for more traces, the last iteration gives an equation for the field φ only.

To this end, let us change slightly the notation from the one adopted in the previous two Sections, and define

$$\begin{aligned} \mathcal{F}^{(1)} &\equiv \mathcal{F} , \\ \mathcal{F}^{(2)} &\equiv \mathcal{F}^{(1)} + \frac{1}{6} \frac{\partial^2}{\Box} \mathcal{F}^{(1)\prime} - \frac{1}{2} \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(1)} , \end{aligned} \tag{4.25}$$

and search for coefficients a_3 and b_3 such that

$$\mathcal{F}^{(3)} \equiv \mathcal{F}^{(2)} + a_3 \frac{\partial^2}{\Box} \mathcal{F}^{(2)\prime} + b_3 \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(2)} \sim \alpha'' ; \tag{4.26}$$

in this way, for $s = 5, 6$ we could use $\mathcal{F}^{(3)} = 0$ as our equation of motion. Given the two expressions

$$\begin{aligned} \frac{\partial^2}{\Box} \mathcal{F}^{(2)\prime} &= 50 \frac{\partial^5}{\Box} \alpha' + 150 \frac{\partial^6}{\Box^2} \partial \cdot \alpha' + 105 \frac{\partial^7}{\Box^2} \alpha'' , \\ \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(2)} &= 25 \frac{\partial^5}{\Box} \alpha' + 30 \frac{\partial^6}{\Box^2} \partial \cdot \alpha' , \end{aligned} \tag{4.27}$$

it is easy to find $a_3 = \frac{1}{15}$ and $b_3 = -\frac{1}{3}$, and then

$$\mathcal{F}^{(3)} = \mathcal{F}^{(2)} + \frac{1}{15} \frac{\partial^2}{\Box} \mathcal{F}^{(2)\prime} - \frac{1}{3} \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(2)} = 7 \frac{\partial^7}{\Box^2} \alpha'' . \tag{4.28}$$

In general, by induction, it is possible to define the following sequence of kinetic operators:

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)} \frac{\partial^2}{\Box} \mathcal{F}^{(n)\prime} - \frac{1}{n+1} \frac{\partial}{\Box} \partial \cdot \mathcal{F}^{(n)} , \tag{4.29}$$

with the crucial property

$$\mathcal{F}^{(n)} = (2n+1) \frac{\partial^{2n+1}}{\Box^{n-1}} \alpha^{[n-1]} , \tag{4.30}$$

which makes clear that for spin s, if p denotes the integer part of $\{\frac{s-3}{2}\}$, the non-local kinetic operator $\mathcal{F}^{(p+2)}$ satisfies the "irreducible", gauge-invariant equation

$$\mathcal{F}^{(p+2)} = 0 . \tag{4.31}$$

To summarise, the gauge-invariant non-local equations we propose, depending only on the physical field, are [1]

$$\mathcal{F} \,=\, \Box\,\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \partial^{\,2}\,\varphi' \,=\, 0\,, \qquad\qquad s \,=\, 1,2\,;$$

$$\mathcal{F}^{(2)} \,=\, \mathcal{F} \,+\, \frac{1}{6}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}' \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F} \,=\, 0\,, \qquad\qquad s \,=\, 3,4\,;$$

$$\mathcal{F}^{(3)} \,=\, \mathcal{F}^{(2)} \,+\, \frac{1}{15}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(2)\,\prime} \,-\, \frac{1}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(2)} \,=\, 0\,, \qquad\qquad s \,=\, 5,6\,;$$

$$\ldots \qquad\qquad \ldots$$

$$\mathcal{F}^{(n+1)} = \mathcal{F}^{(n)} + \frac{1}{2n^2+3n+1}\,\frac{\partial^{\,2}}{\Box}\mathcal{F}^{(n)\,\prime} - \frac{1}{n+1}\,\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n)} = 0\,, \qquad\qquad s = 2n+1, 2n+2\,.$$

The next issues are to show that these equations can be gauge-fixed to the Fronsdal form, without propagation of degrees of freedom arising from φ'', and that identically divergenceless Einstein tensors are available, allowing for the construction of corresponding non-local Lagrangians.

For both purposes the following generalisations of the Bianchi identity for the operators $\mathcal{F}^{(n)}$, that can be proved by induction, are quite useful:

$$\partial\cdot\mathcal{F}^{(n)} \,-\, \frac{1}{2n}\,\partial\,\mathcal{F}^{(n)\,\prime} \,=\, -\,\left(1+\frac{1}{2n}\right)\,\frac{\partial^{\,2n+1}}{\Box^{\,n-1}}\,\varphi^{[n+1]}\,. \tag{4.32}$$

Let us then consider the general non-local equation

$$\mathcal{F}^{(n)} + \frac{1}{(n+1)(2n+1)}\,\frac{\partial^{2}}{\Box}\mathcal{F}^{(n)\,\prime} - \frac{1}{n+1}\,\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n)} = 0 \tag{4.33}$$

and let us write it in the form

$$\mathcal{F}^{(n)} \,=\, \frac{1}{n+1}\,\frac{\partial}{\Box}\,\{\partial\cdot\mathcal{F}^{(n)} \,-\, \frac{1}{2n}\,\partial\,\mathcal{F}^{(n)\,\prime}\} \,+\, \frac{1}{n(2n+1)}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(n)\,\prime}\,. \tag{4.34}$$

Using the identity (4.32) it becomes

$$\mathcal{F}^{(n)} \,=\, \frac{1}{n(2n+1)}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{F}^{(n)\,\prime} \,-\, \frac{2n+1}{n}\,\frac{\partial^{(2n+2)}}{\Box^{n}}\,\varphi^{[n+1]}\,. \tag{4.35}$$

[1] Note that in the $s=3$ case the operators $\frac{\partial^2}{\Box}\,\mathcal{F}'$, and $\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}$ are in fact *the same* operator, and so the proposed equation of motion coincides with the first two expressions in (4.4).

The crucial point is that taking the trace of this relation we can express $\mathcal{F}^{(n)\,\prime}$ as the gradient of something else. Explicitly:

$$\begin{aligned}\mathcal{F}^{(n)\,\prime} \;=\; \frac{1}{n(2n+1)-1}\,\frac{\partial}{\Box}\{2\,\partial\cdot\mathcal{F}^{(n)\,\prime} \,+\, \frac{1}{2}\partial\,\mathcal{F}^{(n)\,\prime\prime} \,-\, \frac{(2n+1)^2}{2n}\,\frac{\partial^{2n-1}}{\Box^{n-2}}\,[\varphi^{[n+1]} \,+\\ \frac{2}{2n+1}\,\frac{\partial}{\Box}\,\partial\cdot\varphi^{[n+1]} \,+\, \frac{2}{(2n+1)(2n+2)}\,\frac{\partial^2}{\Box}\,\varphi^{[n+2]}]\}\,;\end{aligned} \tag{4.36}$$

this relation can be substituted back in (4.35) and then leads to the equation

$$\mathcal{F}^{(n)} \;=\; 3\,\partial^3\,\mathcal{H}_n(\varphi)\,, \tag{4.37}$$

where the form of $\mathcal{H}_n(\varphi)$, that I report for completeness, is

$$\begin{aligned}\mathcal{H}_n(\varphi) \;=\; \frac{1}{4n^4+4n^3-n^2-n}\,\frac{1}{\Box}\{2\,\partial\cdot\mathcal{F}^{(n)\,\prime} \,+\, \frac{1}{2}\partial\,\mathcal{F}^{(n)\,\prime\prime} \,-\, \frac{(2n+1)^2}{2n}\,\frac{\partial^{2n-1}}{\Box^{n-2}}\,[\varphi^{[n+1]}\\ +\,\frac{2}{2n+1}\,\frac{\partial}{\Box}\,\partial\cdot\varphi^{[n+1]} \,+\, \frac{1}{(2n+1)(n+1)}\,\frac{\partial^2}{\Box}\,\varphi^{[n+2]}] \,-\, \frac{3}{2n^2(n+1)}\,\frac{\partial^{2n-1}}{\Box^{n}}\,\varphi^{[n+1]}\}\,.\end{aligned} \tag{4.38}$$

Now, we can repeat this construction writing $\mathcal{F}^{(n)}$ according to

$$\mathcal{F}^{(n)} \;=\; \mathcal{F}^{(n-1)} \,+\, \frac{1}{(n)(2n-1)}\,\frac{\partial^2}{\Box}\,\mathcal{F}^{(n-1)\,\prime} \,-\, \frac{1}{n}\frac{\partial}{\Box}\partial\cdot\mathcal{F}^{(n-1)}\,, \tag{4.39}$$

using again the "Bianchi" identity to get rid of the term in $\partial\cdot\mathcal{F}^{(n-1)}$, and taking the trace of the resulting formula to find for $\mathcal{F}^{(n-1)\,\prime}$ an expression as a pure gradient. These steps allow to reformulate (4.37) as

$$\mathcal{F}^{(n-1)} \;=\; 3\,\partial^3\,\{\mathcal{H}_{(n-1)}(\varphi) \,+\, \mathcal{H}_n(\varphi)\}\,. \tag{4.40}$$

We can thus iterate this pattern; at the end the desired relation emerges

$$\mathcal{F} \;=\; 3\,\partial^3\,\mathcal{H}(\varphi)\,, \tag{4.41}$$

so that one can reach the Fronsdal form $\mathcal{F} = 0$ using Λ' to eliminate $\mathcal{H}$.

At this point one can observe again that, as we saw for the spin-4 case, because of the Bianchi identity (2.27) the double trace of the field vanishes if $\mathcal{F} = 0$ is satisfied, allowing for the choice of the usual de Donder gauge under the same conditions of Section 2.3. This means in particular that, at the level of the equations of motion, there is no real inconsistency in the Fronsdal theory if the double trace of the gauge field is retained, even in the presence of a traceless gauge parameter, given that the condition $\varphi'' = 0$ can be consistently reached on-shell. It is at the level of Lagrangians that the presence of a double trace of φ actually changes the theory, as I will discuss in Section 4.2.3.

To conclude this Section, note that the generalised Bianchi identities (4.32) suffice to define for all spin-s fields fully gauge invariant analogues of the Einstein tensor,

$$\mathcal{G}^{(n)} \;=\; \sum_{p \leq n} \frac{(-1)^p}{2^p \; p! \, \binom{n}{p}} \; \eta^p \; \mathcal{F}^{(n)\,[p]} \tag{4.42}$$

that, for n large enough, have vanishing divergence like their spin-2 counterpart. From $\mathcal{G}^{(n)}$, one can then construct generalized Lagrangians that are fully gauge invariant without any restrictions on the gauge fields or on the gauge parameters.

4.1.4 Spin $\frac{5}{2}$

As for the local compensator theory, also for the non-local formulation there is specular symmetry between bosons and fermions. It is then possible to put the same ideas at work, and find non-local Lagrangians and equations of motion in terms of the single field ψ, describing the same dynamics of the Fang-Fronsdal case. In particular, all non-localities can be shown to be pure gauge, and can eliminated exploiting the extra gauge freedom allowed by the presence of a non-zero γ-trace of the gauge parameter. Because of this strong similarity, the treatment of fermions will be more sketchy, and only the main logic and the basic results will be discussed.

Let me start with the two simplest cases of spin $\frac{5}{2}$ and $\frac{7}{2}$, and then let me pass to the general case.

For spin $\frac{5}{2}$, starting from the compensator equation $\mathcal{S} \;=\; -\,2\,i\,\partial^{\,2}\,\xi$, it is possible to find three non-local constructs solving the same equation:

$$\begin{aligned} \frac{\partial^2}{\Box}\,\mathcal{S}\,' \quad &= \;-\,2\,i\,\partial^{\,2}\,\xi\,, \\ \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \;&=\; -\,2\,i\,\partial^{\,2}\,\xi\,, \\ \frac{1}{2}\;\not{\partial}\frac{\partial}{\Box}\;\not{\mathcal{S}} \;&=\; -\,2\,i\,\partial^{\,2}\,\xi\,, \end{aligned} \tag{4.43}$$

and hence three possible non-local equations involving only ψ are available, along with suitable linear combinations of the listed operators:

$$
\begin{aligned}
\mathcal{S} \,-\, \frac{\partial^2}{\Box}\,\mathcal{S}' \qquad &= \, 0\,, \\
\mathcal{S} \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \, &= \, 0\,, \\
\mathcal{S} \,-\, \frac{1}{2}\,\not{\partial}\frac{\partial}{\Box}\,\not{\mathcal{S}} \, &= \, 0\,.
\end{aligned}
\tag{4.44}
$$

In particular the first form is of the kind

$$
\mathcal{S} \,=\, -\,2\,i\,\partial^2\,\mathcal{K}(\psi)\,, \tag{4.45}
$$

with $\mathcal{K} \equiv \frac{-1}{2i}\,\frac{\mathcal{S}'}{\Box}$ such that

$$
\delta\,\mathcal{K} \,=\, \not{\epsilon}\,. \tag{4.46}
$$

In this form it is manifest that the non local term can be algebraically gauge-fixed to zero using the γ-trace of the parameter. To build up a Lagrangian for these equations we must look for a suitable Einstein tensor, and this requires the knowledge of the correct Bianchi identity in this unconstrained case. We can search for inspiration by looking at the "anomalous" Bianchi identity satisfied by $\mathcal{S}$ when the triple γ-trace of ψ is not identically zero,

$$
\partial\cdot\mathcal{S} \,-\, \frac{1}{2}\,\partial\,\mathcal{S}' \,-\, \frac{1}{2}\not{\partial}\,\not{\mathcal{S}} \,=\, i\,\partial^{\,2}\,\not{\psi}'\;, \tag{4.47}
$$

where the right-hand side is not present for $s \leq \frac{3}{2}$. It is conceivable that the "correct" kinetic operator, let us call it $\mathcal{S}^{(2)}$, satisfy an identity of the kind

$$
\partial\cdot\mathcal{S}^{(2)} \,+\, a_2\,\partial\,\mathcal{S}^{(2)\,\prime} \,+\, b_2\,\not{\partial}\,\not{\mathcal{S}}^{(2)} \,=\, 0. \tag{4.48}
$$

The idea is then to look for a combination of the operators in (4.44), such to allow for (4.48) with coefficients a_2 and b_2 to be determined. What comes out is that the "right" kinetic operator to this end is

$$
\mathcal{S}^{(2)} \,=\, \mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S}\,, \tag{4.49}
$$

obtained in particular by the combination of the two gauge-invariant expressions in (4.44)

$$
\mathcal{S}^{(2)} \,=\, \frac{4}{3}\,(\mathcal{S} \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S}) \,-\, \frac{1}{3}\,(\mathcal{S} \,-\, \frac{\partial^2}{\Box}\,\mathcal{S}')\,, \tag{4.50}
$$

and that the corresponding Bianchi identity is, fixing the coefficients:

$$
\partial\cdot\mathcal{S}^{(2)} \,-\, \frac{1}{4}\,\partial\,\mathcal{S}^{(2)\,\prime} \,-\, \frac{1}{4}\,\not{\partial}\,\not{\mathcal{S}}^{(2)} \,=\, 0. \tag{4.51}
$$

Equivalently to (4.44), it is then possible to choose as an equation of motion the relation

$$\mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,. \tag{4.52}$$

We can also define the Einstein tensor

$$\mathcal{G}^{(\frac{5}{2})} \,=\, \mathcal{S}^{(2)} \,-\, \frac{1}{4}\,\eta\,\mathcal{S}^{(2)\,\prime} \,-\, \frac{1}{4}\,\gamma\,\not{\mathcal{S}}^{(2)}\,, \tag{4.53}$$

whose divergence is identically zero:

$$\partial\cdot\mathcal{G}^{(\frac{5}{2})} \,=\, \partial\cdot\mathcal{S}^{(2)} \,-\, \frac{1}{4}\,\partial\,\mathcal{S}^{(2)\,\prime} \,-\, \frac{1}{4}\,\not{\partial}\,\not{\mathcal{S}}^{(2)} \,-\, \frac{1}{4}\,\gamma\partial\cdot\,\not{\mathcal{S}}^{(2)} \,=\, 0\,. \tag{4.54}$$

In particular the last term is proportional to the γ-trace of the Bianchi identity, and so it is separately vanishing. It is then possible to build the non-local Lagrangian

$$\mathcal{L} \,\sim\, \bar{\psi}\,\mathcal{G}^{(2)} \,+\, h.c.\,. \tag{4.55}$$

4.1.5 Spin $\frac{7}{2}$

As expected from the corresponding case of spin 4, for spin $\frac{7}{2}$ the kinetic operator is *in form* identical to (4.49), the one used eventually for $s \,=\, \frac{5}{2}$, with of course a different number of terms contributing in the expansion in ψ. The non-local, gauge invariant, unconstrained equation of motion is then

$$\mathcal{S}^{(2)} \,=\, \mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,. \tag{4.56}$$

One can reduce this equation to the form $\mathcal{S} \,=\, -\,2\,i\,\partial^2\,\mathcal{K}(\psi)$ again using the Bianchi identity, first, writing (4.56) as

$$\mathcal{S} \,=\, \frac{2}{3}\,\frac{\partial}{\Box}\,\left\{\partial\cdot\mathcal{S} \,-\, \frac{1}{2}\,\partial\,\mathcal{S}\,' \,-\, \frac{1}{2}\not{\partial}\,\not{\mathcal{S}}\right\} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,+\, \frac{1}{3}\,\frac{\partial}{\Box}\,\not{\partial}\,\not{\mathcal{S}}\,, \tag{4.57}$$

and then using (4.47) to put the preceding relation in the form

$$\mathcal{S} \,=\, \partial^2\,\{\frac{2i}{3}\,\partial\,\not{\psi}\,' \,-\, \frac{1}{6\Box}\,\mathcal{S}\,'\} \,+\, \frac{1}{3}\,\frac{\partial}{\Box}\,\not{\partial}\,\not{\mathcal{S}}\,. \tag{4.58}$$

Taking the γ-trace of this last relation it is possible to express $\not{\mathcal{S}}$ as the gradient of something,

$$\not{\mathcal{S}} \,=\, 3\,i\,\partial^2\,\not{\psi}\,' + \frac{1}{6\Box}\,\partial\,\not{\partial}\,\mathcal{S}\,' \,+\, \frac{1}{2\Box}\,\partial^2\,\mathcal{S}\,' \,+\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\,\not{\mathcal{S}}\,, \tag{4.59}$$

and inserting this last formula in the previous equation one finds at the end the "compensator-like" form, with

$$\mathcal{K}(\psi) \;=\; -\left\{\frac{1}{3}\partial\;\psi\!\!\!/\,' \,+\, \frac{1}{6i\Box}\,\partial\;\partial\!\!\!/\;\psi\!\!\!/\,' \,+\, \frac{1}{18i}\,\frac{\partial\!\!\!/}{\Box^2}\,\mathcal{S}\,' \,+\, \frac{1}{12i\Box^2}\,\partial\;\mathcal{S}\!\!\!/\,' \,+\, \frac{4}{9i\Box^2}\,\partial\cdot\;\mathcal{S}\!\!\!/\right\}\,. \quad (4.60)$$

We can build an Einstein tensor starting from the kinetic operator $\mathcal{S}^{(2)}$. Differently from the $s = \frac{5}{2}$ case, here the combination $\mathcal{S}^{(2)} - \frac{1}{4}\eta\,\mathcal{S}^{(2)\,\prime} - \frac{1}{4}\gamma\;\mathcal{S}\!\!\!/^{(2)}$ is not identically divergenceless, because of terms in $\partial\cdot\mathcal{S}\,'$, that must be cancelled with the insertion of other terms. It is possible to verify directly that the combination

$$\mathcal{G}^{(\frac{7}{2})} \;=\; \mathcal{S}^{(2)} \,-\, \frac{1}{4}\eta\,\mathcal{S}^{(2)\,\prime} \,-\, \frac{1}{4}\gamma\;\mathcal{S}\!\!\!/^{(2)} \,+\, \frac{1}{8}\eta\gamma\;\mathcal{S}\!\!\!/^{(2)\,\prime}\,, \quad (4.61)$$

is identically divergenceless, and finally one can get a non-local Lagrangian, as in (4.55), from which equation (4.56) can be retrieved, although not directly.

4.1.6 General fermionic case

The scheme to face the general case should now be clear. One first looks for a gauge invariant kinetic operator built from the Fronsdal operator, and such that it vanishes because of the compensator equation $\mathcal{S} = -2i\,\partial^2\,\xi$, but not involving the compensator itself, or any other field in the resulting equation. To obtain this result, I follow once more an iterative procedure and define recursively the operators

$$\mathcal{S}^{(n+1)} \;=\; \mathcal{S}^{(n)} \,+\, \frac{1}{n(2n+1)}\,\frac{\partial^{\,2}}{\Box}\,\mathcal{S}^{(n)\,\prime} \,-\, \frac{2}{2n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S}^{(n)} \quad (4.62)$$

where $\mathcal{S}^{(1)} \equiv \mathcal{S}$, such that, as a consequence of the compensator equation

$$\mathcal{S}^{(n)} \;=\; -\,2\,i\,n\,\frac{\partial^{\,2n}}{\Box^{\,n-1}}\,\xi^{\,[n-1]}\,. \quad (4.63)$$

At the end that the following unconstrained equations for fermions of any spin can be defined:

$$\begin{aligned}
&\mathcal{S} \;=\; i\,(\partial\!\!\!/\,\psi \,-\, \partial\;\psi\!\!\!/) \;=\; 0\,, && s = \frac{1}{2},\,\frac{3}{2}\,;\\
&\mathcal{S}^{(2)} \;=\; \mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \;=\; 0\,, && s = \frac{5}{2},\,\frac{7}{2}\,;\\
&\ldots && \ldots\\
&\mathcal{S}^{(n+1)} \;=\; \mathcal{S}^{(n)} + \frac{1}{n(2n+1)}\frac{\partial^2}{\Box}\mathcal{S}^{(n)\prime} - \frac{2}{2n+1}\frac{\partial}{\Box}\partial\cdot\mathcal{S}^{(n)} \;=\; 0, && s = 2n+\frac{1}{2},\,2n+\frac{3}{2}\,.
\end{aligned} \quad (4.64)$$

To reduce these equations to the compensator-like form $\mathcal{S} = -2i\,\partial^2\,\mathcal{K}(\psi)$, and also to find the Einstein tensor needed to evaluate the Lagrangians, the following generalised Bianchi identities are very useful:

$$\partial \cdot \mathcal{S}^{(n)} \;-\; \frac{1}{2n}\,\partial\,\mathcal{S}^{(n)\,\prime} \;-\; \frac{1}{2n}\,\not{\partial}\,\not{\mathcal{S}}^{(n)} \;=\; i\,\frac{\partial^{2n}}{\Box^{n-1}}\,\not{\psi}^{[n]}\,. \tag{4.65}$$

It is then possible to show that the tensors

$$\mathcal{G}^{(n)} \;=\; \mathcal{S}^{(n)} \;+\; \sum_{0<p\leq n} \frac{(-1)^p}{2^p\,p!\,\binom{n}{p}}\,\eta^{p-1}\left[\,\eta\,\mathcal{S}^{(n)\,[p]} \;+\; \gamma\,\not{\mathcal{S}}^{(n)\,[p-1]}\,\right], \tag{4.66}$$

are identically divergenceless if n is large enough, and thus allow for the construction of non-local Lagrangians for the unconstrained theory. They also provide a crucial tool to show that the non-local equations can be partially gauge-fixed to the Fang-Fronsdal form. To this end the strategy is again the same as that used for bosons. First, write eq. (4.64) as

$$\mathcal{S}^{(n)} \,+\, a_n\,\frac{\partial}{\Box}\,\{\partial\cdot\mathcal{S}^{(n)} \;-\; \frac{1}{2n}\,\partial\,\mathcal{S}^{(n)\,\prime} \;-\; \frac{1}{2n}\;\not{\partial}\,\not{\mathcal{S}}^{(n)}\} \,+\, b_n\,\frac{\partial^2}{\Box}\,\mathcal{S}^{(n)\,\prime} \,+\, c_n\,\frac{\partial}{\Box}\;\not{\partial}\,\not{\mathcal{S}} \,=\, 0\,. \tag{4.67}$$

then use (4.65) to cast it in the form

$$\mathcal{S}^{(n)} \,+\, a_n\,\frac{\partial}{\Box}\,\{i\,\frac{\partial^{2n}}{\Box^{n-1}}\,\not{\psi}^{[n]}\} \,+\, b_n\,\frac{\partial^2}{\Box}\,\mathcal{S}^{(n)\,\prime} \,+\, c_n\,\frac{\partial}{\Box}\;\not{\partial}\,\not{\mathcal{S}} \,=\, 0\,. \tag{4.68}$$

Now compute the γ-trace of the resulting formula, in order to express $\not{\mathcal{S}}$ as the gradient of something. This allows one to rewrite (4.68) as

$$\mathcal{S}^{(n)} \;=\; -\,2\,i\,\partial^2\,\mathcal{K}^{(n)}\,. \tag{4.69}$$

Now substitute for $\mathcal{S}^{(n)}$ its definition in terms of $\mathcal{S}^{(n-1)}$, eq. (4.62), and iterate the procedure, getting in this way

$$\mathcal{S}^{(n-1)} \;=\; -\,2\,i\,\partial^2\,\{\mathcal{K}^{(n-1)} \,+\, \mathcal{K}^{(n)}\}\,. \tag{4.70}$$

This algorithm can be repeated for each n, until one finds the desired expression

$$\mathcal{S} \;=\; -\,2\,i\,\partial^2\,\mathcal{K}(\psi)\,, \tag{4.71}$$

in which all non-localities are in the pure-gauge term $\mathcal{K}$, and can then be eliminated using the trace of the gauge parameter, reducing in this way the non-local dynamics to the local Fronsdal form.

4.1.7 Additional remarks

Before discussing the geometry of these gauge fields I would like to make two observations on the non-local formulation. The first is, in some sense, a matter of notation. As we limit ourselves to totally symmetric representations, we can exploit this property of our tensors in order to simplify the formulas, as it has been done so far by making use of the "symmetric notation".

An alternative possibility is to introduce new commuting vectors χ_μ; it is then possible to encode the symmetry properties of a tensor $\varphi_{\mu_1\cdots\mu_s}(x)$ in the definition of a new function,

$$\hat{\Phi}(x,\chi) \,=\, \frac{1}{s!}\,\chi^{\mu_1}\cdots\chi^{\mu_s}\;\varphi_{\mu_1\cdots\mu_s} \tag{4.72}$$

where all indices of the tensor φ are contracted with the vectors χ. To reproduce the "elementary" operations of trace and divergence on the tensor φ, when encoded in $\hat{\Phi}(x,\chi)$ one has to use derivatives with respect to the χ variables, $\partial_{\chi_\mu}\,\equiv\,\frac{\partial}{\partial\chi_\mu}$. To be specific

$$\frac{\partial}{\partial\chi_\alpha}\,\hat{\Phi}(x,\chi) \,=\, \frac{1}{(s-1)!}\,\chi^{\mu_1}\cdots\chi^{\mu_{s-1}}\;\varphi_{\mu_1\cdots\mu_{s-1}\alpha}\,, \tag{4.73}$$

and so, defining

$$\begin{aligned} \frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial\chi_\alpha} &\equiv \partial_\chi\cdot\partial\,,\\ \frac{\partial}{\partial\chi^\alpha}\frac{\partial}{\partial\chi_\alpha} &\equiv \partial_\chi\cdot\partial_\chi\,, \end{aligned} \tag{4.74}$$

one can get at the end

$$\begin{aligned} \partial_\chi\cdot\partial\,\hat{\Phi}(x,\chi) &\,=\, \frac{1}{(s-1)!}\,\chi^{\mu_1}\cdots\chi^{\mu_{s-1}}\,\partial\cdot\varphi_{\mu_1\cdots\mu_{s-1}}\,,\\ \partial_\chi\cdot\partial_\chi\,\hat{\Phi}(x,\chi) &\,=\, \frac{1}{(s-2)!}\,\chi^{\mu_1}\cdots\chi^{\mu_{s-2}}\,\varphi'_{\mu_1\cdots\mu_{s-2}}\,. \end{aligned} \tag{4.75}$$

In this notation the Fronsdal operator can be written as

$$\hat{\mathcal{F}}(\hat{\Phi}) = \left[\,\Box\,-\,\chi\cdot\partial\,\partial\cdot\partial_\chi\,+\,\frac{1}{s\,(s\,-\,1)}\,(\chi\cdot\partial)^2\,\partial_\chi\cdot\partial_\chi\right]\hat{\Phi}\,, \tag{4.76}$$

while the kinetic operator at the n-th step can be written in the compact form

$$\prod_{k=0}^{n-1}\left[\,1\,+\,\frac{1}{(k+1)(2k+1)}\,\frac{(\chi\cdot\partial)^2}{\Box}\,\partial_\chi\cdot\partial_\chi\,-\,\frac{1}{k+1}\,\frac{\chi\cdot\partial}{\Box}\,\partial_\chi\cdot\partial\right]\hat{\mathcal{F}}(\hat{\Phi})\,=\,0\,, \tag{4.77}$$

where, as we have seen, for spin s a fully gauge invariant operator is first obtained after $\left[\frac{s+1}{2}\right]$ iterations. Expanding this expression and combining it with its trace it is possible give an alternative proof of the basic fact that the non-local field equations can all be reduced to the form

$$\mathcal{F} = \partial^{3} \mathcal{H} . \tag{4.78}$$

The second observation concerns the relationship between local and non-local formulations.

I first presented the traditional description of the dynamics for higher spin fields, in the final form due to Singh and Hagen for the massive case, and to Fronsdal and Fang-Fronsdal for the massless limit. This description is local, Lagrangian and constrained, as explained in Sections 1 and 2.

Then I argued that it was conceivable at least to discuss critically the role of the constraints, and see whether they were really unavoidable. I then showed, at the massless level, that this is not the case, and that it is possible to obtain an alternative formulation in which no constraints are present, while the same no inconsistencies are introduced. This is accomplished by modifying the Lagrangians by the introduction of a couple of auxiliary fields, *independently from the spin.* It is in this respect, to start with, that this proposal differs from the BRST approach of [72], [73] and [74], in which an unconstrained formulation is given (also for the massive case), by means of the introduction of more and more fields and gauge parameters, as the spin increases.

Finally, I faced with the issue of eliminating all auxiliary fields without fixing the gauge and/or constraining the physical fields, and in these last Sections I showed that this task can actually be completed, at the price of introducing *non localities* both in the Lagrangians and in the equations of motion. This non-local formulation is obtained starting with the simplest version of the local equations of motion in the compensator form, eliminating the auxiliary fields in a systematic way. In this sense the non-local theory is presented as a consequence of the local one. The heritage of the role of the compensator fields is in the pure gauge nature of the non-local part of the equations of motion, which as a matter of fact is their true substitute.

Here I would like to observe that the non-local formulation could be presented independently of the local compensator one. Still confining the attention for simplicity to bosons, we could start from the Fronsdal equation, noting that its unconstrained variation is

$$\delta \mathcal{F} \,=\, 3\,\partial^3\,\Lambda^{\,\prime}\,. \tag{4.79}$$

Then we could look for constructs built from $\mathcal{F}$ and transforming in the same way, and combine them with $\mathcal{F}$ in order to reach a fully gauge invariant equation. The non-localities appear at this level unavoidably, because to build constructions transforming as $\Lambda^{\prime}$ one must take traces and divergences of $\mathcal{F}$, and so one must eliminate all spurious d'Alembertian operators produced in this way. Once we have the sequence of kinetic operators, we can proceed as in Section 4.1.3 and build Einstein tensors and non-local Lagrangians. Finally, the proof that for these systems $\varphi^{\prime\prime} \,=\, 0$ on-shell is completely independent on any previous knowledge of the local theory, and in particular of the role of the Lagrange multiplier β.

In fact, from the chronological point of view, we found the non-local theory in [62] and [63], *before* the local compensator form was found for any spin in [61].

Here I chose to present things not abiding to the historical order, mainly because I judge that there is a *logical* dependence of the non-local version from the local one, since the first can be *explained* in terms of the second, as I tried to describe.

Nonetheless the non-local formulation still plays a peculiar role, often anticipated in the previous Sections: it allows to describe the unconstrained, free dynamics for these gauge fields in a *geometrical* fashion, closely resembling the description of the basic cases of spin 1 and 2.

In the next chapter this peculiar feature of the non-local theory will be described in some detail.

4.2 Higher-spin geometry

4.2.1 Suggestions from lower-spins

Let us begin by recalling how a geometrical description is obtained for the cases of spin 1 and 2. In the Maxwell case, the fundamental field is the vector A_μ, subject to the gauge transformation $\delta A_\mu = \partial_\mu \Lambda$. In this simple case, the basic field plays also the role of a *connection*, in the sense that taking its curl yields the gauge-invariant quantity

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{4.80}$$

From the tensor $F_{\mu\nu}$, that we call a *curvature* or a *field strenght*, one can obtain the dynamics of the field A_μ; actually taking a divergence of F and putting it equal to zero gives

$$\partial \cdot F^{\,\mu} \,=\, \Box A^{\,\mu} \,-\, \partial^{\,\mu}\, \partial \cdot A \,=\, 0\,, \tag{4.81}$$

that is, the Fronsdal equation for $s = 1$.

For the free Einstein theory, the basic field is a rank-two symmetric tensor $h_{\mu\nu}$, subject to the transformation law $\delta h_{\mu\nu} = \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$. In this case the fundamental field is not yet the connection, and an intermediate step is necessary before defining a gauge-invariant curvature: one has to take a "suitable" combination of first derivatives of the field $h_{\mu\nu}$, where "suitable" means such that the gauge transformation of the new object should be as simple as possible. Defining thus the linear (Christoffel) connection

$$\Gamma^{\,\alpha}_{\,\mu\nu} = \partial_\nu\, h^{\,\alpha}_{\,\mu} \,+\, \partial_\mu\, h^{\,\alpha}_{\,\nu} \,-\, \partial^{\,\alpha} h_{\,\mu\nu} \tag{4.82}$$

as is well known, one finds

$$\delta\, \Gamma^{\,\alpha}_{\,\mu\nu} \,=\, 2\, \partial_\mu\, \partial_\nu\, \Lambda^{\,\alpha}\,, \tag{4.83}$$

and so, taking the curl of this quantity finally yields a gauge-invariant curvature,

$$\mathcal{R}^{\,\alpha}_{\,\beta\mu\nu} \,=\partial_\nu\, \Gamma^{\,\alpha}_{\,\beta\mu} \,-\, \partial_\mu\, \Gamma^{\,\alpha}_{\,\beta\nu}\,, \tag{4.84}$$

In this even-spin case, the equation of motion is deduced from the curvature taking a trace and setting it to zero:

$$\mathcal{R}^{\,\alpha}_{\,\beta\alpha\nu} \,\equiv\, \mathcal{R}_{\,\mu\nu} \,=\, \Box\, h_{\,\mu\nu} \,-\, \partial_\mu\, \partial \cdot h_{\,\nu} \,-\, \partial_\nu\, \partial \cdot h_{\,\mu} \,+\, \partial_\mu \partial_\nu\, h' \,=\, 0\,. \tag{4.85}$$

where one can recognise the full Fronsdal form of the equation of motion. I have recalled these very basic facts just to suggest a possible method to look for generalisations of the concepts of connections and curvatures for gauge fields of spin greater than two. This task was systematically developed by B. de Wit and D. Freedman in [21], and will be reviewed in the next Section.

4.2.2 Hierarchy of connections: de Wit and Freedman

Consider a symmetric rank-s tensor φ subject to the gauge transformation $\delta\,\varphi \;=\; \partial\,\Lambda$, with Λ a symmetric, rank-$(s-1)$ tensor. In this Section I would like to explain how to construct proper generalisations of the concepts of "connection" and "curvature" for these tensors. To this end, *I will assume no constraints a priori*, taking this as an opportunity to discuss again from a technical viewpoint their role in the Fronsdal-de Wit-Freedman formulation.

To begin with, let us consider the $s\;=\;3$ case and, for the sake of clarity, restore for a while the indices in the notation. In analogy with the lower spin cases, take a linear combination of first derivatives of the basic field $\varphi_{\alpha\beta\gamma}$, such that the gauge transformation of the resulting object be as simple as possible; to this end define the quantity

$$\Gamma^{(1)}_{\rho,\alpha\beta\gamma} \;=\; \partial_\rho\,\varphi_{\alpha\beta\gamma} - (\partial_\alpha\varphi_{\rho\beta\gamma} + \partial_\beta\varphi_{\rho\alpha\gamma} + \partial_\gamma\varphi_{\alpha\beta\rho}) \;, \tag{4.86}$$

such that

$$\delta\Gamma^{(1)}_{\rho,\alpha\beta\gamma} \;=\; -\,2(\partial_\alpha\partial_\beta\Lambda_{\rho\gamma} + \partial_\alpha\partial_\gamma\Lambda_{\rho\beta} + \partial_\beta\partial_\gamma\Lambda_{\rho\alpha}) \;. \tag{4.87}$$

This is the best one can do with a combination of *first* derivatives, but one can simplify the gauge transformation further by taking a combination of *second* derivatives of φ, in the form

$$\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma} = \partial_\rho\Gamma^{(1)}_{\sigma,\alpha\beta\gamma} - \frac{1}{2}(\partial_\alpha\Gamma^{(1)}_{\sigma,\rho\beta\gamma} + \dots)\,. \tag{4.88}$$

The variation of $\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma}$ is indeed

$$\delta\,\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma} = 3\,\partial_\alpha\partial_\beta\partial_\gamma\Lambda_{\rho\sigma}\,, \tag{4.89}$$

which represents an evident simplification of the previous formula, and resembles the analogous forms of the lower spin cases. In this sense, for spin 3 this $\Gamma^{(2)}$ is what one would properly call a "connection".

The symmetry properties of these connections imply a distinction between two sets of indices, and for this reason I judge to be preferable to use in this framework the "mixed-symmetric" notation described at page 21, according to which, just to recall, all indices in a symmetric group are denoted by the same letter. In this fashion one can write explicitly $\Gamma^{(2)}_{\rho\sigma,\alpha\beta\gamma}$, which will be now denoted by $\Gamma^{(2)}_{\rho\rho,\,\alpha\alpha\alpha}$, as

$$\Gamma^{(2)}_{\rho\rho,\,\alpha\alpha\alpha} \;=\; \partial^2_\rho\,\varphi_{\,\alpha\alpha\alpha} \;-\; \frac{1}{2}\partial_\alpha\,\partial_\rho\,\varphi_{\,\rho\alpha\alpha} \;+\; \partial^2_{\alpha\alpha}\,\varphi_{\,\rho\rho\alpha}\,. \tag{4.90}$$

It is quite simple in this form to evaluate the gauge transformation of $\Gamma^{(2)}$:

$$\begin{aligned}\delta\,\Gamma^{(2)}_{\rho\rho,\,\alpha\alpha\alpha} &= \partial^2_\rho\,\partial_\alpha\,\Lambda_{\,\alpha\alpha} \;-\; \frac{1}{2}\partial_\alpha\,\partial_\rho\,(\partial_\rho\,\Lambda_{\,\alpha\alpha} \;+\; \partial_\alpha\,\Lambda_{\,\rho\alpha}) \;+\; \partial^2_{\alpha\alpha}\,(\partial_\rho\,\Lambda_{\,\rho\alpha} \;+\; \partial_\alpha\,\Lambda_{\,\rho\rho})\,, \\ &= \partial^2_\rho\,\partial_\alpha\,\Lambda_{\,\alpha\alpha} \;-\partial_\alpha\,\partial^2_\rho\,\Lambda_{\,\alpha\alpha} \;-\; \partial^2_\alpha\,\partial_\rho\,\Lambda_{\,\rho\alpha} \;+\; \partial^2_\alpha\,\partial_\rho\,\Lambda_{\,\rho\alpha} \;+3\,\partial^3_\alpha\,\Lambda_{\,\rho\rho} \\ &= 3\,\partial^3_\alpha\,\Lambda_{\,\rho\rho}\,.\end{aligned} \tag{4.91}$$

One can then take a sort of "symmetric curl" of $\Gamma^{(2)}$, ending with a gauge invariant "curvature" for the spin 3 field:

$$\Gamma^{(3)}_{\rho\sigma\tau,\,\alpha\beta\gamma} \equiv \mathcal{R}^{(3)}_{\rho\sigma\tau,\,\alpha\beta\gamma} \;=\; \partial_\rho\Gamma^{(2)}_{\sigma\tau,\,\alpha\beta\gamma} - \frac{1}{3}(\partial_\alpha\Gamma^{(2)}_{\sigma\tau,\,\rho\beta\gamma} + \dots)\ , \tag{4.92}$$

such that

$$\delta\mathcal{R}^{(3)}_{\rho\sigma\tau,\,\alpha\beta\gamma} = 0\ . \tag{4.93}$$

In our new, "mixed-symmetric" notation, it is possible to give a quite compact, explicit expression for $\mathcal{R}^{(3)}$:

$$\mathcal{R}^{(3)}_{\rho\rho\rho,\,\alpha\alpha\alpha} \;=\; \partial^3_\rho\,\varphi_{\,\alpha\alpha\alpha} \;-\; \frac{1}{3}\partial^2_\rho\,\partial_\alpha\,\varphi_{\,\rho\alpha\alpha} \;+\frac{1}{3}\partial^2_\alpha\,\partial_\rho\,\varphi_{\,\rho\rho\alpha} \;-\; \partial^3_\alpha\,\varphi_{\,\rho\rho\rho}\,, \tag{4.94}$$

which is easily verified to be gauge-invariant.

One can now proceed to the case of general tensors, following a strategy which should be clear from the previous examples: in each $\Gamma^{(m)}_{\rho_1\,\dots\,\rho_m,\,\alpha_1\,\dots\,\alpha_s} \equiv \Gamma^{(m)}_{\rho_m,\,\alpha_s}$, the two groups of indices *are not equivalent*. The α's are "field indices", even if they are mixed in the combination with the indices of derivatives, while the ρ's are "special" indices, related to the introduction of m derivatives at the m-th step. The combination should be chosen so that in the gauge variation of $\Gamma^{(m)}$ *all* special indices end in the gauge parameter $\Lambda_{\,\mu_1\,\dots\,\mu_{s-1}}$. If this can be realised, then in the gauge transformation of $\Gamma^{(s-1)}$ all indices of the gauge parameter will be the special ones, and then $\Gamma^{(s)}$ (along with any $\Gamma^{(s+k)}$ for all $k > 0$) will be a gauge invariant quantity, simply because there will be no more room in Λ to accomodate a number of special indices greater than or equal to s.

Explicitly, in this "mixed-symmetric" notation, for any field of spin s, after m iterations, one can define

$$\Gamma^{(m)}_{\rho_m,\,\alpha_s} = \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}}\, \partial_\rho^{m-k} \partial_\alpha^k \varphi_{\,\rho_k,\,\alpha_{s-k}}\ , \tag{4.95}$$

It is simple to show, by an inductive argument, that the gauge transformation of $\Gamma^{(m)}$ is

$$\delta\, \Gamma^{(m)}_{\rho_m,\,\alpha_s} = (-1)^m\, (m+1)\, \partial_\alpha^{m+1} \Lambda_{\,\rho_m,\,\alpha_{s-m-1}}\ , \tag{4.96}$$

so that, as anticipated, all m indices of the first set are within the gauge parameter. Hence,

$$\Gamma^{(s-1)}_{\rho_{s-1},\,\alpha_s} = \sum_{k=0}^{s-1} \frac{(-1)^k}{\binom{s-1}{k}}\, \partial_\rho^{s-k-1} \partial_\alpha^k \varphi_{\,\rho_k,\,\alpha_{s-k}}\ , \tag{4.97}$$

is the proper analogue of the Christoffel connection for a spin-s gauge field, since its gauge transformation contains a single term. Moreover, *all* Γ's with $m \geq s$ are gauge invariant, and in particular

$$\Gamma^{(s)}_{\rho_s,\,\alpha_s} \equiv \mathcal{R}^{(s)}_{\rho_s,\,\alpha_s} = \sum_{k=0}^{s} \frac{(-1)^k}{\binom{s}{k}}\, \partial_\rho^{s-k} \partial_\alpha^k \varphi_{\,\rho_k,\,\alpha_{s-k}}\ , \tag{4.98}$$

is the proper analogue of the Riemann curvature tensor[2]. This generalised curvature is totally symmetric under the interchange of any pair of indices within each of the two sets. In addition, as shown in [21],

$$\mathcal{R}_{\rho_s,\,\alpha_s} = (-1)^s\, \mathcal{R}_{\alpha_s,\,\rho_s}\ , \tag{4.99}$$

[2]In order to clean up (4.97) and (4.98) from indices, it is possible to choose a notation in which the "∂" symbol is associated with ρ-derivatives, while a "∇" is chosen to indicate α-derivatives. Bearing in mind that symmetrization is only among indices belonging to separate sets, one could then write the de Wit-Freedman symbols in the form

$$\Gamma^{(m)} = \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}}\, \partial^{m-k}\, \nabla^k\, \varphi\,,$$

with gauge transformations $\delta\, \Gamma^{(m)} = (-1)^m\, (m+1)\, \nabla^{m+1}\, \Lambda$. In particular the gauge-invariant curvature in this notation becomes

$$\mathcal{R}^{(s)} = \sum_{k=0}^{s} \frac{(-1)^k}{\binom{s}{k}}\, \partial^{s-k}\, \nabla^k\, \phi\,.$$

and a generalised cyclic identity holds.

There is another, perhaps more obvious way, to generate a gauge invariant quantity from a connection $\Gamma^{(s-1)}$ that transforms as in (4.96), a curl with respect to any of its α indices. However, the choice (4.98) has the virtue of simplicity, since it results automatically in a tensor with two totally symmetric sets of indices. For instance, the symmetric curvature of [21] for the spin-2 case is a linear combination of ordinary Riemann tensors, while its trace in the first symmetric set of indices is proportional to the ordinary Ricci tensor.

4.2.3 Back to Fronsdal

Given that, as we have seen in the previous Section, we have at our disposal fully gauge-invariant quantities generalising the curvatures of Maxwell and Einstein to the case of higher-rank tensors, it is interesting to discuss whether (and how) they can play a role in the dynamics of these tensors.

The first obstacle to a direct use of the $\mathcal{R}_{\rho_s,\alpha_s}$ was, for the authors of [21], the number of derivatives. As "standard quantum field theory requires linear second-order wave equations for bosons", to cite them literally[3] it is then not possible to make direct use of the curvatures just defined, which contain a number of derivatives equal to the spin of the basic field, even if the $\Gamma^{(s)}$'s are the only *fully* gauge invariant objects in the whole hierarchy of connections (for $m \leq s$), according to (4.96). From the same equation, on the other hand, one can also note that, taking a trace of a generic element of the hierarchy in a couple of *special* indices, the resulting gauge transformation will always be proportional to the *trace* of the gauge parameter:

$$\delta\,\Gamma^{(m)\,\prime} \propto \nabla^{m+1}\Lambda^{\prime}\ . \tag{4.100}$$

Hence, if one declares *a priori* the trace of Λ to be identically zero, then a whole set of gauge invariant quantities becomes available, and we can try and pick from this set the most suitable one for the description of the dynamics. Given the "constraint" on the number of derivatives, at this level one is naturally led to choose $\Gamma^{(2)}$, the only element of

[3] [21], pag. 360.

the set of second order in derivatives, and for these reasons in [21]

$$\Gamma^{(2)\,\prime} \,=\, 0\,. \tag{4.101}$$

was selected as an equation of motion for higher rank tensors. Computing explicitly this trace it is found that

$$\Gamma^{(2)\,\prime} \,=\, \Box\,\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \partial^2\,\varphi\,'\,, \tag{4.102}$$

and this is the reason why in [21] the choice of a traceless gauge parameter for the Fronsdal equations was justified.

As we have observed, the equations (4.101) are not Lagrangian equations, exactly for the same reason why $\mathcal{R}_{\mu\nu} \,=\, 0$ is not the equation of motion descending from the linearised Einstein-Hilbert action for gravity. Instead, from the Ricci tensor one can construct the Einstein tensor $\mathcal{G}_{\mu\nu} \,=\, \mathcal{R}_{\mu\nu} \,-\, \frac{1}{2}\,\eta_{\mu\nu}\,\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar, which is identically divergenceless because of the Bianchi identity and so allows for the construction of a Lagrangian of the form $\mathcal{L} \,=\, \frac{1}{2}\,h^{\,\mu\nu}\,\mathcal{G}_{\mu\nu}$. In the same spirit, one can try and construct, starting from the Fronsdal operator, a generalised Einstein tensor

$$\mathcal{G} \equiv \mathcal{F} \,-\, \frac{1}{2}\,\eta\,\mathcal{F}\,'\,. \tag{4.103}$$

The main point to justify the constraint $\varphi'' \,=\, 0$ in [21] is to observe that the “would-be” Bianchi identity

$$\partial\cdot\mathcal{F} \,-\, \frac{1}{2}\partial\mathcal{F}\,' \,=\, -\,\frac{3}{2}\partial^{\,3}\,\varphi'' \tag{4.104}$$

is in fact “anomalous, if $\varphi'' \,\neq\, 0$. This means that if one tries to define a Lagrangian of the usual form, $\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,\mathcal{G}$, then under gauge transformations one gets

$$\delta\,\mathcal{L} \,=\, -\,\frac{1}{2}\,\Lambda\,\{\partial\cdot\mathcal{F} \,-\, \frac{1}{2}\partial\mathcal{F}'\} \,-\, \frac{1}{2}\,\Lambda\,'\partial\cdot\,\mathcal{F}\,'\,, \tag{4.105}$$

which is *not* zero, even with a traceless gauge parameter. Hence, the only way to use (4.103) to construct a Lagrangian leading to the equations (4.101), although in an indirect way (i.e. via a trace procedure), is recognised in the possibility to declare *a priori* the gauge field φ to be doubly traceless: $\varphi'' \,\equiv\, 0$. Alternatively, if the double trace is retained, the Lagrangian cannot be determined by the Einstein tensor (4.103), and terms involving higher traces of φ, being gauge invariant[4] could not be excluded a priori. The most general form for a gauge-invariant Lagrangian would be in this case

$$\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,\{\mathcal{F} \,-\, \frac{1}{2}\,\eta\,(\mathcal{F}' \,-\, \partial^2\,\varphi'')\} \,+\, (\text{higher-traces terms})\,. \tag{4.106}$$

In this last case, on the other hand, this generalised Lagrangian would give rise to a set of *independent* equations involving the Fronsdal operator together with the traces of the field. This implies[5] "that in order to derive [the Fronsdal equation] as the universal wave equation from an action principle requires the constraint [of double-tracelessness] on the fields". *A latere*, the authors of [21] also observed that the constraint $\varphi'' \,=\, 0$ is conceivable, both because it represents the strongest gauge-invariant constraint possible, and for a matter of irreducibilty, as φ'' carries in general lower-spin components which could give rise to propagating ghosts.

These were the reasons why the curvatures were not used to describe the dynamics of higher-rank tensors, and the constrained formulation of Fronsdal was judged to be the only possible answer. In the next Section I will recall the arguments used in support of the unconstrained formulation to justify the use of the deWit-Freedman curvatures to produce the dynamics. This geometrical dynamics will be straightforwardly related to the non-local one defined by the sequence of kinetic operators (4.29).

4.2.4 Geometric theory for free higher-spin fields: bosons

To summarise, the geometric program could not be completed, in the sense that the curvatures could not be used to describe the dynamics, because of the following difficulties:

1. for spin greater than 2 the curvatures are higher-derivative operators. It is then not clear how to use them to derive the expected relativistic, second-order differential equations for free fields;

2. a second-order, gauge-invariant operator can be picked up from the hierarchy of connections provided $\Lambda' = 0$. Setting $\Gamma^{(2)}{}'$ to zero one gets an equation that results

[4]The double trace of φ indeed transforms according to

$$\delta\,\varphi'' \,=\, 4\,\partial\cdot\,\Lambda' \,+\, \partial\,\Lambda''\,,$$

and so is gauge invariant if (and only if) $\Lambda' \,\equiv\, 0$.

[5][21] pag. 362.

to be exactly the Fronsdal equation, which was known to describe correctly the dynamics of free gauge fields;

3. to deduce the Fronsdal equation from a gauge-invariant Lagrangian it is necessary to impose $\varphi'' = 0$. Moreover, without this constraint the Fronsdal equation could not be regarded as "the universal wave equation [derived] from an action principle", as other terms would be possible in the Lagrangian, giving rise to independent equations for φ, as explained in [21]. Finally, the double trace "describes lower spin components of the field. If not eliminated, the theory would not contain pure spin s, and would very likely have negative metric ghosts".

To see how all these difficulties can be surmounted in the non-local formulation, let us start again with the spin 3 case, and consider the form (4.94) of the curvature:

$$\mathcal{R}^{(3)}_{\rho\rho\rho\,;\alpha\alpha\alpha} = \partial_\rho^3\,\varphi_{\alpha\alpha\alpha} - \frac{1}{3}\partial_\rho^2\,\partial_\alpha\,\varphi_{\rho\alpha\alpha} + \frac{1}{3}\partial_\alpha^2\,\partial_\rho\,\varphi_{\rho\rho\alpha} - \partial_\alpha^3\,\varphi_{\rho\rho\rho}\,. \tag{4.107}$$

The examples of electromagnetism and linearised gravity, briefly recalled in Section 4.2.1, would suggest to use this curvature by taking traces and divergences in order to define the equation of motion for the spin 3 field. Adding to these basic observations the discussion of the non-local kinetic operators of Section 4.1.3 one can see a possible way out to the problem of higher derivatives: in that Section we saw that it is possible to add to a second-order kinetic operator terms of higher order in the derivatives, whose dimensions can be at least formally restored acting upon those terms with suitable inverse powers of the d'Alembertian operator $\Box$. If we try and exploit this possibility, the reasonable choice to generalise the Maxwell equation for spin one, $\partial \cdot F_\mu = 0$, and the linearised Einstein equation for spin two, $\mathcal{R}^\alpha_{\beta\alpha\nu} = 0$, is to take one trace and one divergence of $\mathcal{R}^{(3)}$, and divide the result by one power of $\Box$.

I then define the geometrical equation for spin 3 as

$$\frac{1}{\Box}\,\partial \cdot \mathcal{R}^{(3)\,\prime} = 0\,. \tag{4.108}$$

This equation is gauge-invariant with an unconstrained parameter, but nonetheless can be easily connected to the Fronsdal equation; it is enough to display the explicit form of (4.108) to recognise that

$$\frac{1}{\Box}\,\partial \cdot \mathcal{R}^{(3)\,\prime} = \Box\,\varphi - \partial\,\partial \cdot \varphi + \frac{1}{3}\,\partial^2\,\varphi' + \frac{2}{3}\,\frac{\partial^2}{\Box}\,\partial \cdot \partial \cdot \varphi - \frac{\partial^3}{\Box}\,\partial \cdot \varphi' \equiv \mathcal{F}^{(2)}\,, \tag{4.109}$$

i.e. *the geometrical equation of motion is exactly the non-local equation described in Section 4.1.1*, that in particular has been shown to be gauge-equivalent to the Fronsdal one, by means of a partial gauge-fixing involving Λ'.

For spin 3, of course, no difficulties associated to the double trace can arise. It is then useful to take a look at the spin 4 case, to discuss whether and how the geometrical equation can avoid the difficulties listed at the beginning of this Section.

For spin 4 in the "mixed-symmetric" notation the curvature has the form

$$\mathcal{R}^{(4)}_{\rho\rho\rho\rho\,;\alpha\alpha\alpha\alpha} \,=\, \partial_\rho^4\,\varphi_{\alpha\alpha\alpha\alpha} \,-\, \frac{1}{4}\,\partial_\rho^3\,\partial_\alpha\,\varphi_{\rho\alpha\alpha\alpha} \,+\, \frac{1}{6}\,\partial_\rho^2\,\partial_\alpha^2\,\varphi_{\rho\rho\alpha\alpha} \,-\, \frac{1}{4}\,\partial_\rho\partial_\alpha^3\varphi_{\rho\rho\rho\alpha} \,+\, \partial_\alpha^4\,\varphi_{\rho\rho\rho\rho}\,. \tag{4.110}$$

By analogy with what we have just seen for spin 3, one can consider as the more straightforward generalisation of the linearised Einstein equation the formula:

$$\frac{1}{\Box}\,\mathcal{R}^{(4)\,\prime\prime}_{\alpha\alpha\alpha\alpha} \,=\, 0\,. \tag{4.111}$$

As for spin 3, the key point is to recognise, by explicit calculation, that

$$\frac{1}{\Box}\,\mathcal{R}^{(4)\,\prime\prime} \,\equiv\, \mathcal{F}^{(2)}\,, \tag{4.112}$$

and that, as a consequence, *the Fronsdal equation for* $s = 4$ *can be regarded as the result of a partial gauge-fixing on the geometric equation, performed by fixing the trace of* Λ *in such a way that all non-localities be removed.*

We have already discussed the role of φ'' in this equation in Section 4.1.2, showing that it does not propagate any degrees of freedom. Hence, in particular, no negative metric ghosts can appear in the theory. It is still to be clarified the possibility to derive (4.111) from a Lagrangian, avoiding the difficulties listed in point 3 at the beginning of this Section. We already know that a non-local Lagrangian for the unconstrained theory can be derived by the generalised Einstein tensor defined in (4.21), that I report here for clarity in the case of spin four:

$$\mathcal{G}^{(4)} \,=\, F^{(4)} \,-\, \frac{1}{4}\,\eta\,F^{(4)\,\prime} \,+\, \frac{1}{8}\,\eta^2\,F^{(4)\,\prime\prime}\,. \tag{4.113}$$

What is still to be clarified is the possibility that the Lagrangian written in absence of constraints gives rise to *more* independent equation for φ, according to the observation reported in point 3 from [21].

To analyse this point, recall that the Lagrangian is given by the expression

$$\mathcal{L}^{(4)} \;=\; \frac{1}{2}\,\varphi\,\mathcal{G}^{(4)}\,, \tag{4.114}$$

where $\mathcal{G}^{(4)}$ in terms of the fundamental field is given by

$$\begin{aligned}\mathcal{G}^{(4)} \;=& \Box\,\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \frac{1}{3}\,\partial^2\,\varphi' \,+\, \frac{2}{3}\,\frac{\partial^2}{\Box}\,\partial\cdot\partial\cdot\varphi \,-\, \frac{\partial^3}{\Box}\,\partial\cdot\varphi' \,+\, \frac{\partial^4}{\Box}\,\varphi'' \\ &-\,\frac{1}{3}\,\eta\,\{\Box\,\varphi' \,-\, \partial\cdot\,\partial\cdot\varphi \,-\, \partial\,\partial\cdot\varphi' \,+\, \partial^2\,\varphi'' \,+\, \frac{\partial}{\Box}\,\partial\cdot\partial\cdot\partial\cdot\varphi \,-\, \frac{\partial^2}{\Box}\,\partial\cdot\partial\cdot\varphi'\} \\ &+\,\frac{1}{3}\,\eta^2\,\{\Box\,\varphi'' \,-\, 2\,\partial\cdot\,\partial\cdot\varphi' \,+\, \frac{1}{\Box}\,\partial\cdot\partial\cdot\partial\cdot\partial\cdot\varphi\}\,.\end{aligned} \tag{4.115}$$

The form of $\mathcal{G}^{(4)}$ allows one to verify that $\varphi\,\frac{\delta\,\mathcal{G}}{\delta\,\varphi} \;=\; \mathcal{G}$, and this implies that $\delta\,\mathcal{L} \;=\; \delta\,\varphi\,\mathcal{G}$; the Lagrangian equation of motion is then

$$\mathcal{G}^{(4)} \;=\; F^{(4)} \;-\; \frac{1}{4}\,\eta\,F^{(4)\,\prime} \;+\; \frac{1}{8}\,\eta^2\,F^{(4)\,\prime\prime} \;=\; 0. \tag{4.116}$$

Now, taking twice the trace of this relation gives

$$\frac{D}{4}\,(\frac{D}{2}\,-1)\,F^{(4)\,\prime\prime} \;=0\,, \tag{4.117}$$

which implies, in the first trace of (4.116), $F^{(4)\,\prime} \;=\; 0$, and then finally

$$F^{(4)} \;=\; 0\,. \tag{4.118}$$

The end result is that the unconstrained Lagrangian for $s = 4$, involving the double trace of the field φ, yields *one independent equation:* $F^{(4)} \;=\; 0$, given that the coefficients multiplying η and η^2 are just traces of the fundamental kinetic operator. From this basic equation, as we already know, we can deduce the Fronsdal equation by gauging to zero all non-localities.

These two examples suggest how to proceed for the general case of spin s. The first point is to consider the de Wit-Freedman curvature (4.98)

$$\mathcal{R}^{(s)}_{\rho_s,\,\alpha_s} \;=\; \sum_{k=0}^{s}\frac{(-1)^k}{\binom{s}{k}}\;\partial_\rho^{s-k}\partial_\alpha^k\varphi_{\,\rho_k,\,\alpha_{s-k}}\,, \tag{4.119}$$

taking enough traces, together with one divergence if the spin is odd, to saturate all indices belonging to one symmetric set; then act with an inverse power of the d'Alembertian operator suitable to restore the dimensions of the relativistic wave equation. Setting the result to zero yields the geometric equations

$$\frac{1}{\Box^{n-1}}\, \mathcal{R}^{[n]}{}_{\mu_1 \cdots \mu_{2n}} \;=\; 0 \qquad\qquad s \;=\; 2\,n\,, \tag{4.120}$$

$$\frac{1}{\Box^{n-1}}\, \partial \cdot \mathcal{R}^{[n-1]}{}_{\mu_1 \cdots \mu_{2n-1}} \;=\; 0 \qquad\qquad s \;=\; 2\,n \;-\; 1\,. \tag{4.121}$$

It is possible to show that the resulting operators coincide with the kinetic operators defined in (4.29)[6]; in particular, the geometric equations (4.120), (4.121) are both equivalent to the non-local form

$$\mathcal{F}^{(n)} \;=\; 0\,, \tag{4.122}$$

that can always be reduced to the Fronsdal one, as already discussed, and in which no propagation of degrees of freedom due to the double trace is present, as shown in Section 4.1.3. Lagrangians for these unconstrained operators can be obtained by means of the identically divergenceless tensors (4.42)

$$\mathcal{G}^{(n)} \;=\; \sum_{p \leq n} \frac{(-1)^p}{2^p\, p!\, \binom{n}{p}}\; \eta^p\, \mathcal{F}^{(n)\,[p]}\,, \tag{4.123}$$

giving rise to equations of motion of the form $\mathcal{G}^{(n)} \;=\; 0$, where the only independent relations can be shown to reduce to eq. (4.122).

Before illustrating the geometry of free fermions, it seems convenient to pause and discuss in more detail the relation between local and non-local formulations. This will be the object of the next Section, where emphasis will be mainly on the spin-4 case.

4.2.5 Comments on the relationship between local Lagrangians and non-local geometry

According to the scheme proposed in this Thesis, it is possible to derive a non-local version for the free theory of higher-spin fields, substantially by eliminating the compensator α from the equation $\mathcal{F} \;=\; 3\,\partial^{\,3}\,\alpha$ deduced in Section 3.1.2 . This non-local theory has in turn the feature of providing a geometric description of higher spin gauge fields. In this Section I would like to discuss some features of this relationship, trying to elucidate the steps by which the local Lagrangian (3.14) can be reduced to a non-local form.

[6] See the Remark at the end of Section 4.2.6 for a discussion of this point.

To begin with, let us observe that the form (4.41) of the non-local equations, $\mathcal{F} = 3\,\partial^3\,\mathcal{H}(\varphi)$, is clearly the non-local analogue of the compensator equation (3.30), $\mathcal{F} = 3\,\partial^3\,\alpha$. This means that the relationship we are looking for *is not*, at least not directly, between local theory and geometry. Rather, it appears as a possibility of linking the local theory to *a consequence* of the geometric equations.

For instance, in the spin-3 case the local Lagrangian (3.8), that I report here for simplicity:

$$\mathcal{L}^{(3)} \,=\, \frac{1}{2}\,\varphi\left(\,\mathcal{F}\,-\,\frac{1}{2}\eta\,\mathcal{F}'\right)\,-\,\frac{3}{2}\,\alpha\,\partial\cdot\mathcal{F}'\,+\,\frac{9}{4}\,\alpha\,\Box^2\,\alpha\,, \tag{4.124}$$

can be shown to reduce to the non-local Lagrangian (4.9)

$$\begin{aligned}\hat{\mathcal{L}}^{(3)} \,=&\, -\,\frac{1}{2}\,(\partial\,\varphi)^{\,2}\,+\,\frac{3}{2}\,(\partial\cdot\varphi)^{\,2}\,+\,3\,\varphi'\,\partial\cdot\partial\cdot\varphi\,-\,\partial\cdot\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box^2}\,\partial\cdot\partial\cdot\partial\cdot\varphi\\ &+\,3\,\partial\cdot\partial\cdot\partial\cdot\varphi\,\frac{1}{\Box}\,\partial\cdot\varphi'\,+\,\frac{3}{2}\,(\partial\,\varphi')^{\,2}\,-\,\frac{3}{2}(\partial\cdot\varphi')^{\,2}\,,\end{aligned} \tag{4.125}$$

simply substituting for α in (4.124) its on-shell value

$$\alpha \,=\, \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}'\,, \tag{4.126}$$

according to equation (3.18), that in the spin-3 case becomes

$$6\,\Box^2\,\alpha \,=\, 2\,\partial\cdot\,\mathcal{F}'\,, \tag{4.127}$$

The Lagrangian (4.125) on the other hand, is the one built from the Einstein tensor obtained using $\mathcal{F}\,-\,3\,\partial^3\,\mathcal{H}(\varphi)$ as a starting point, and therefore it is not the one giving the geometric equations of motion (4.108). From this point of view, it is still an open question whether a local theory exists that is *directly* related to the geometric equations (4.120) and (4.121)

This observation generalises to the case when the double trace of the field enters the game, that is from $s\,=\,4$ onwards. Limiting myself to the relatively simple example of spin 4, I would then like to compare the local Lagrangian (3.14) with the non-local one built out of the equation

$$\mathcal{D} \,=\, \mathcal{F}\,-\,3\,\partial^3\,\mathcal{H}(\varphi)\,=\,0\,. \tag{4.128}$$

To this end, let us recover the explicit form of $\mathcal{H}$, given in (4.17)

$$\mathcal{H}(\varphi) \,=\, \frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}'\,-\,\frac{3}{4}\,\frac{\partial}{\Box}\,\varphi''\,, \tag{4.129}$$

and let us use it to write $\mathcal{D}$ as

$$\mathcal{D} \,=\, \mathcal{F} \,-\, \frac{\partial^3}{\Box^2}\,\partial\cdot\mathcal{F}' \,-\, 9\,\frac{\partial^4}{\Box}\,\varphi'' \,. \tag{4.130}$$

Let me look for the Einstein tensor for $\mathcal{D}$. The naive starting point

$$\mathcal{E}_0 \,\equiv\, \mathcal{D} \,-\, \frac{1}{2}\,\eta\,\mathcal{D}' \,, \tag{4.131}$$

does not give a divergenceless tensor, since

$$\partial\cdot\{\mathcal{D} \,-\, \frac{1}{2}\,\eta\,\mathcal{D}'\} \,=\, -\,2\,\frac{\partial^3}{\Box^2}\,\{\Box\mathcal{F}'' \,-\, \partial\cdot\partial\cdot\mathcal{F}'\} \,-\, \frac{3}{2}\,\eta\,\frac{\partial}{\Box}\,\{\Box\mathcal{F}'' \,-\, \partial\cdot\partial\cdot\mathcal{F}'\} \,. \tag{4.132}$$

The form of the divergence of $\mathcal{E}_0$ implies that, in order to obtain a divergenceless tensor, one must not only add a correction term proportional to η^2, as was the case for the non-local Einstein tensors defined in (4.42) and related to the geometric equations, but also a correction term in the coefficient of η. Given that

$$\mathcal{D}'' \,=\, \frac{4}{\Box}\,\{\Box\mathcal{F}'' \,-\, \partial\cdot\partial\cdot\mathcal{F}'\} \,, \tag{4.133}$$

one can find at the end the divergenceless tensor

$$\mathcal{E} \,\equiv\, \mathcal{D} \,-\, \frac{1}{2}\,\eta\,\left(\mathcal{D}' \,-\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{D}''\right) \,+\, \frac{5}{24}\,\eta^2\,\mathcal{D}'' \,. \tag{4.134}$$

From this tensor it is possible to derive the non-local Lagrangian $\mathcal{L} \,=\, \frac{1}{2}\,\varphi\,\mathcal{E}$; moreover, the explicit expression of $\mathcal{E}$ in terms of φ,

$$\begin{aligned}\mathcal{E} \,=\,& \Box\varphi \,-\, \partial\,\partial\cdot\varphi \,+\, \partial^2\,\varphi\prime \,-\, 3\,\frac{\partial^3}{\Box}\,\partial\cdot\,\varphi' \,+\, 2\,\frac{\partial^3}{\Box^2}\,\partial\cdot\partial\cdot\partial\cdot\,\varphi \,-\, 4\,\frac{\partial^4}{\Box^2}\,\partial\cdot\partial\cdot\,\varphi' \,+\, 5\,\frac{\partial^4}{\Box}\,\varphi'' \\ &-\, \frac{1}{2}\,\eta\,\{2\,\Box\varphi' \,-\, 2\,\partial\cdot\partial\cdot\varphi \,-\, 2\,\partial\,\partial\cdot\varphi' \,+\, \frac{10}{3}\,\partial^2\,\varphi'' \,+\, 2\,\frac{\partial}{\Box}\,\partial\cdot\partial\cdot\partial\cdot\,\varphi \,-\, \frac{8}{3}\,\frac{\partial^2}{\Box}\,\partial\cdot\partial\cdot\,\varphi' \\ &+\, \frac{4}{3}\,\frac{\partial^2}{\Box^2}\,\partial\cdot\partial\cdot\partial\cdot\partial\cdot\,\varphi\} \,+\, \frac{5}{3}\,\eta^2\,\{\Box\,\varphi'' \,-\, 2\,\partial\cdot\partial\cdot\varphi' \,+\, \partial\cdot\partial\cdot\partial\cdot\partial\cdot\varphi\} \,,\end{aligned} \tag{4.135}$$

makes it possible to verify that $\varphi\,\frac{\delta\mathcal{E}}{\delta\varphi} \,=\, \mathcal{E}$, and so the Lagrangian equation of motion is simply

$$\mathcal{D} \,-\, \frac{1}{2}\,\eta\,\left(\mathcal{D}' \,-\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{D}''\right) \,+\, \frac{5}{24}\,\eta^2\,\mathcal{D}'' \,=\, 0\,; \tag{4.136}$$

this in turn implies the relations

$$\begin{aligned}\mathcal{D}'' \,&=\, 0\,, \\ \mathcal{D}' \,&=\, 0\,, \\ \mathcal{D} \,&=\, 0\,.\end{aligned} \tag{4.137}$$

These results are to be compared with the outcomes of the elimination of the auxiliary fields α and β from the local Lagrangian (3.14) evaluated for $s = 4$,

$$\begin{aligned} \mathcal{L}_4 = & \frac{1}{2}\varphi\left(\mathcal{F} - \frac{1}{2}\eta\mathcal{F}'\right) - 6\,\alpha\left\{\frac{1}{2}\partial\cdot\mathcal{F}' - \partial\cdot\partial\cdot\partial\cdot\varphi + \frac{3}{2}\Box\,\partial\cdot\varphi'\right\} \\ & + 9\,\partial\cdot\alpha\,\partial\cdot\partial\cdot\varphi' + 9\,\alpha\,\Box^2\,\alpha - 27\,\partial\cdot\alpha\,\Box\,\partial\cdot\alpha \\ & + 3\,\beta\,\left(\varphi'' - 4\,\partial\cdot\alpha\right)\,. \end{aligned} \tag{4.138}$$

In the non-local theory there is no information *a priori* on the double trace of the field, hence what I will first try is to eliminate β from this Lagrangian, to then calculate the equations of motion for φ and α. In order to eliminate β, one can recall that the equation for φ

$$\begin{aligned} \mathcal{F} - 3\,\partial^3\,\alpha - \frac{1}{2}\eta\,(\mathcal{F}' - \frac{1}{2}\partial^2\,\varphi'' - 3\,\Box\,\partial\,\alpha - 4\,\partial^2\,\partial\cdot\alpha) & \\ + \eta^2\,(\beta + \Box\,\partial\cdot\alpha - \frac{1}{2}\partial\cdot\partial\cdot\varphi') = 0\,, & \end{aligned} \tag{4.139}$$

implies, in particular, that

$$\beta = \frac{1}{2}\partial\cdot\partial\cdot\varphi' - \Box\,\partial\cdot\alpha\,; \tag{4.140}$$

substituting in $\mathcal{L}_4$ and calculating the variation with respect to φ and α one finds the equations of motion of the "reduced" local theory

$$\begin{aligned} \varphi_r : \quad & \mathcal{F} - 3\,\partial^3\,\alpha - \frac{1}{2}\eta\,(\mathcal{F}' - \partial^2\,\varphi'' - 3\,\Box\,\partial\,\alpha - 2\,\partial^2\,\partial\cdot\alpha) = 0\,, \\ \alpha_r : \quad & 3\,\Box^2\,\alpha + 5\,\Box\,\partial\,\partial\cdot\,\alpha - \partial\cdot\mathcal{F}' + \Box\,\partial\,\varphi'' = 0\,. \end{aligned} \tag{4.141}$$

The first one, in particular, can be cast in the form

$$\mathcal{A} - \frac{1}{2}\eta\left\{\mathcal{A}' - \frac{1}{3}\frac{\partial^2}{\Box}\mathcal{A}''\right\} = 0\,, \tag{4.142}$$

where $\mathcal{A} \equiv \mathcal{F} - 3\,\partial^3\,\alpha$, resembling in form eq. (4.136), save for the missing term in $\eta^2\,\mathcal{A}''$. The consequences of this last equation are anyway the same we found for the $\mathcal{D}$ operator:

$$\begin{aligned} \mathcal{A}'' & = 0\,, \\ \mathcal{A}' & = 0\,, \\ \mathcal{A} & = 0\,. \end{aligned} \tag{4.143}$$

To show how to relate the compensator to the non-local construct $\mathcal{H}(\varphi)$, one must solve for α in terms of φ. To this end write the equation for α in the form

$$3\,\Box^2\,\alpha = \partial\cdot\mathcal{F}' - \partial\,\Box\{\varphi'' + 5\,\partial\cdot\alpha\}\,, \tag{4.144}$$

and substitute for $\Box\,\partial\cdot\alpha$ its value according to the equation $\mathcal{A}''\,=\,0$:

$$\Box\,\partial\cdot\alpha\,=\,\frac{1}{4}\,\Box\,\varphi''\,. \tag{4.145}$$

In this fashion, acting on both sides of (4.144) with the singular operator $\frac{1}{3\,\Box^2}$, one arrives at the identification of the compensator α with $\mathcal{H}(\varphi)$:

$$\alpha\,=\,\frac{1}{3\,\Box^2}\,\partial\cdot\mathcal{F}'\,-\,\frac{3}{4\,\Box}\,\partial\,\varphi''\,. \tag{4.146}$$

To complete the reduction to the non-local form it is necessary to discuss the behaviour of the double trace of φ. We know that in the local theory, once β is on-shell, for any configuration φ the double trace carries only pure gauge degrees of freedom, whereas in the non-local framework we showed that φ'' vanishes on-shell. Here we obtain a similar result; indeed, by evaluating

$$\partial\cdot\mathcal{A}\,-\,\frac{1}{2}\partial\,\mathcal{A}'\,=\,-\,\frac{3}{2}\,\partial^{\,3}\,(\varphi''\,-\,4\,\partial\cdot\alpha)\,, \tag{4.147}$$

it is possible to recognise that, when the system is on-shell, $\partial^{\,3}\,(\varphi''\,-\,4\,\partial\cdot\alpha)\,=\,0$, and then, going to momentum space, $\varphi''\,=\,4\,\partial\cdot\alpha$.

This last observation leads me to a further comment on the role of the Lagrange multiplier β.

In Section 3.1.1 we found that the cancellation of all terms involving the trace of the gauge parameter in the variation of the Lagrangian, obtained by the introduction of contributions in the compensator α, leaves at the end the triple divergence of Λ, times the gauge invariant expression $\varphi''-4\,\partial\cdot\alpha-\partial\,\alpha'$. This suggested the introduction of the gauge dependent Lagrange multiplier β in (3.13) in order to compensate this last contribution. Eq. (4.140) on the other hand indicates that it is possible to construct an expression with the same gauge transformation, only involving the fields φ and α, and so one could ask whether the introduction of the independent field β is really necessary. From this respect, eq. (4.147) indicates that, from the point of view of the *free* theory, we could probably avoid the introduction of a new field, since in any case all physical requirements are at the end fullfilled, at least on-shell, as was the case for the irreducible, non-local theory. It is on the side of the *interacting* theory, in my opinion, that the role of the Lagrange multiplier could reveal itself to be relevant. In this last case indeed, we could no more make use of the equations of motion to conclude that the double trace vanishes on shell, because in general it would be no more true that the equations of motion imply that $\mathcal{A}$ or its traces vanish, while the presence of the Lagrange multiplier would anyway ensure that the degrees of freedom of φ'' are not physical ones.

In this sense the free, non-local formulation of the theory can be seen as a partially on-shell version of the local one, and it is reasonable from this point of view that a piece of information is lost. To summarise, the two main manifestations of this phenomenon are in the appearance of non-local contributions, due to the substitution for α, and in the missing knowledge of the pure-gauge nature of φ'', which is only recovered on-shell.

4.2.6 Geometric theory for free higher-spin fields: fermions

To describe fermions in a geometrical fashion, it is possible to reproduce the construction of the hierarchy of connections, with the only modification that the fundamental field now also carries a spinor index. Hence here a symmetric rank-s spinor-tensor, in the "mixed-symmetric" notation, will be indicated

$$\psi^{\alpha}_{\mu_s} \,\equiv\, \psi^{\alpha}_{\mu_1 \ldots \mu_s}\,, \tag{4.148}$$

where α denotes the spinor index. This field is subject to the transformation law

$$\delta \psi^{\alpha}_{\mu_s} \,=\, \partial_\mu \, \epsilon^{\alpha}_{\mu_{s-1}}\,, \tag{4.149}$$

where $\epsilon^{\alpha}_{\mu_{s-1}}$ is a rank-$(s-1)$ spinor-tensor.

Although the construction of curvatures does not present any relevant novelties with respect to the bosonic case, nonetheless the analysys of fermion geometry diplays some peculiar features which is worth discussing.

We can start with the simplest case of spin $\frac{3}{2}$, where the first step in the construction of the system of connections

$$\Gamma^{(1)}_{\rho,\mu} \,=\, \partial_\rho \, \psi^{\alpha}_{\mu} \,-\, \partial_\mu \, \psi^{\alpha}_{\rho}\,, \tag{4.150}$$

yields immediately a gauge invariant quantity:

$$\delta \, \Gamma^{(1)}_{\rho,\mu} \,=\, \partial_\rho \, \partial_\mu \, \epsilon^{\alpha} \,-\, \partial_\mu \, \partial_\rho \, \epsilon^{\alpha} \,=\, 0\,. \tag{4.151}$$

For spin $\frac{5}{2}$ the transformation law of the $\Gamma^{(1)}_{\rho,\mu\mu}$, defined as

$$\Gamma^{(1)}_{\rho,\mu\mu} \;=\; \partial_\rho \, \psi^\alpha_{\mu\mu} \;-\; \partial_\mu \, \psi^\alpha_{\rho\mu} \, , \tag{4.152}$$

is

$$\delta \, \Gamma^{(1)}_{\rho,\mu\mu} \;=\; -2 \, \partial^2_\mu \, \epsilon^\alpha_\rho \, , \tag{4.153}$$

and so the next step gives a gauge invariant spinor-tensor:

$$\Gamma^{(2)}_{\rho\rho,\mu\mu} \;=\; \partial^2_\rho \, \psi^\alpha_{\mu\mu} \;-\; \frac{1}{2} \, \partial_\rho \, \partial_\mu \, \psi^\alpha_{\rho\mu} \;+\; \partial^2_\mu \, \psi^\alpha_{\rho\rho} \, . \tag{4.154}$$

It is clear that the construction follows exactly the same steps we saw for bosons; we can define for a general rank-s spinor-tensor the quantities

$$\Gamma^{(m)}_{\rho_m \, \mu_s} \;=\; \sum_{k=0}^{m} \frac{(-1)^k}{\binom{m}{k}} \; \partial^{m-k}_\rho \partial^k_\mu \, \psi^\alpha_{\rho_k \, \mu_{s-k}} \, , \tag{4.155}$$

having the gauge transformations

$$\delta \, \Gamma^{(m)}_{\rho_m \, \mu_s} \;=\; (-1)^m \, (m+1) \, \partial^{m+1}_\mu \, \epsilon^\alpha_{\rho_m \, , \, \mu_{s-m-1}} \, . \tag{4.156}$$

A fully gauge invariant tensor is reached first at the s-th step, and it is this object that I will call a "curvature" for fermionic gauge fields:

$$\mathcal{R}^{(s)}_{\rho_s \, , \, \mu_s} \;=\; \sum_{k=0}^{s} \frac{(-1)^k}{\binom{s}{k}} \; \partial^{s-k}_\rho \partial^k_\mu \, \psi^\alpha_{\rho_k \, \mu_{s-k}} \, . \tag{4.157}$$

Difficulties of the same kind of those we faced for bosons present themselves also for fermions. In particular, the need to define the equation of motion by means of a *first order*, relativistic differential operator, may lead to conclude that the curvatures are not the proper objects to describe the dynamics. On the other hand, as it is manifest from (4.156), the γ-*trace* of each connection, with respect to one of the special indices ρ, is gauge invariant if we assume that the gauge parameter is γ-traceless: $\not{\epsilon} \equiv 0$. Under this assumption we can dispose of a bunch of tensors to describe -in this restricted sense- a gauge-invariant dynamics, and the natural choice is to select the γ-trace of the first-order connection $\Gamma^{(1)}$. The deWit-Freedman equation of motion for gauge fermions of any spin is then

$$\not{\Gamma}^{(1)} \;= \not{\partial} \, \psi \;-\; \partial \, \not{\psi} \;=\; 0 \, , \tag{4.158}$$

that is just the Fang-Fronsdal equation (2.37).

The triple γ-trace of the field, $\not{\psi}'$, makes the construction of a Lagrangian difficult, as the violation of the Bianchi identity given by (4.47)

$$\partial \cdot \mathcal{S} \, - \, \frac{1}{2} \, \partial \, \mathcal{S}' \, - \, \frac{1}{2} \not{\partial} \, \not{\mathcal{S}} \, = \, i \, \partial^{\, 2} \, \not{\psi}' \, , \tag{4.159}$$

implies that the Einstein tensor

$$\mathcal{G} \, = \, \mathcal{S} \, - \, \frac{1}{2} \left(\eta \, \mathcal{S}' \, + \, \gamma \, \not{\mathcal{S}} \right) \, , \tag{4.160}$$

would give rise to a gauge-dependent Lagrangian even in the presence of a γ-traceless gauge parameter:

$$\delta \bar{\psi} \, \mathcal{L} \, \sim \, \bar{\epsilon} \, \partial \cdot \mathcal{G} \, \neq \, 0 \, . \tag{4.161}$$

Because of this motivation, and to avoid the risk that lower-spin components in $\not{\psi}'$ could manifest themselves as ghosts, in the deWit-Freedman formulation the triple γ-trace of ψ is constrained to be zero, and so at the end the Fronsdal formulation is fully recovered.

From the point of view proposed in this work, a possible way out to surmount the higher-derivative difficulty is to restore the physical dimensions of the differential operators defined by the curvatures, acting with suitable inverse power of the $\Box$ operator. This procedure can be given a meaning if, in the end, the non-localities introduced can be consistently removed from the theory, as was the case for the fermionic kinetic operators defined in Section 4.1.6. Moreover, we would expect that the geometric equations so defined bear a direct relation with the "kinetic" ones defined in that Section, as was the case for bosons. As we shall see, some more comments will be needed in order to clarify the relationship between these different constructions.

Before going ahead in the construction of the non-local, geometric equations, it is useful to make some preliminary comments.

First, we want our differential operators to carry an odd number of derivatives so that, after acting with the appropriate inverse power of $\Box$, only an operator with the dimension of a first derivative is left.

Second, while for bosons the only ways to saturate indices are provided by traces or divergences, in the fermionic case we can saturate indices also with γ-matrices. Of the many possibilities one has at his disposal in order to define gauge invariant equations of motion starting from curvatures, I will call *primary* the one which involves the maximum number of traces and γ-traces of $\mathcal{R}$, aside from a single divergence in the case of even rank, to ensure that the total number of derivatives at the numerator is odd. In this way, only the minimum number of inverse powers of $\Box$ is needed to restore dimensions, and the operator defined in this manner has the lower possible order of singularity.

Third, given that *the form* of the curvatures is identical for bosons and fermions, we can always start with the bosonic geometrical equations (4.120) and (4.121), interpret the fields as carrying also a spinor index, and act on them with the operator $\frac{\not\partial}{\Box}$. I will call *secondary* the equations generated in this way. These equations are related to a formal link that can be established between the fundamental Fronsdal and Fang-Fronsdal operators, for spin respectively s and $s+\frac{1}{2}$. Namely, if we interpret the field in the Fronsdal $\mathcal{F}$ operator for a spin-s boson as a fermion field of spin $s+\frac{1}{2}$, than it is possible to prove the following identity:

$$\mathcal{S} \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\not\partial\,\not\mathcal{S} \,=\, i\,\frac{\not\partial}{\Box}\,\mathcal{F}\,, \tag{4.162}$$

whose counterparts for the recursive kinetic operators of Sections 4.1.3 and 4.1.6 are of the form

$$\mathcal{S}^{(n)} \,-\, \frac{1}{2n}\,\frac{\partial}{\Box}\,\not\partial\,\not\mathcal{S}^{(n)} \,=\, i\,\frac{\not\partial}{\Box}\,\mathcal{F}^{(n)}\,. \tag{4.163}$$

Primary and secondary equations, arising directly from curvatures, are manifestly *geometrical* equations. The key point is that these two types of non-local equations exhibit the same order of pole *only if the rank of ψ is even.* Differently, if the rank is odd, the primary equations are *less* singular of the secondary ones. So, it is only in the first case that an actual ambiguity can arise in the choice of the equation. I will add comments on this point later in this Section.

Finally, I shall call *kinetic* the non-local equations for unconstrained fermions defined in Section 4.1.6, that I report here for clarity:

$$\mathcal{S}^{(n+1)} \,=\, \mathcal{S}^{(n)} + \frac{1}{n(2n+1)}\,\frac{\partial^2}{\Box}\,\mathcal{S}^{(n)\prime} \,-\, \frac{2}{2n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S}^{(n)} \,=\, 0, \tag{4.164}$$

fro spin $s=2n+\frac{1}{2}$ and $s=2n+\frac{3}{2}$.

These three possibilities are expected to give rise to equations that must bear some definite relation to one another, at least due to gauge invariance. In order to proceed systematically in the analysis of these options, let us begin by studying the first lower-spin possibilities before going to the general case.

spin $\frac{3}{2}$

As we have seen, the curvature in this case is simply

$$\Gamma^{(1)}_{\rho,\mu} \;=\; \partial_\rho\, \psi^{\alpha}_{\mu} \;-\; \partial_\mu\, \psi^{\alpha}_{\rho}\,, \tag{4.165}$$

and correspondingly, the primary geometrical equation is just the kinetic one:

$$\not{\Gamma}^{(1)} \;=\not{\partial}\, \psi \;-\; \partial\, \not{\psi} \;= \mathcal{S} \;=\; 0\,. \tag{4.166}$$

The secondary equation is obtained taking the geometric equation for spin 1, which is just the Fronsdal (Maxwell) equation, to be interpreted as an equation for a fermion, and acting on it with $\frac{\not{\partial}}{\Box}$:

$$\frac{\not{\partial}}{\Box}\,(\partial \cdot \mathcal{R}) \;=\; \frac{\not{\partial}}{\Box}\,\mathcal{F}(\psi) \;=\; 0\,. \tag{4.167}$$

In terms of the $\mathcal{S}$ operators it looks

$$\mathcal{S} \;-\; \frac{1}{2}\,\frac{\partial}{\Box}\;\not{\partial}\;\not{\mathcal{S}} \;=\; 0\,, \tag{4.168}$$

or, equivalently, using the Bianchi identity (4.47):

$$\mathcal{S} \;-\; \frac{\partial}{\Box}\,\partial \cdot \mathcal{S} \;=\; 0\,. \tag{4.169}$$

As we can see, in this case the secondary equation *is not* manifestly equivalent to the primary-kinetic one. Anyway, it is simple to convince oneself that (4.168) implies $\not{\mathcal{S}} \;=\; 0$ (by evaluating $\not{\mathcal{S}}$ directly from the equation) and so at the end for spin $\frac{3}{2}$ all three equations are equivalent.

spin $\frac{5}{2}$

In this case, given the curvature

$$\mathcal{R}^{(2)}_{\rho\rho,\mu\mu} \;=\; \partial^2_\rho\, \psi^{\alpha}_{\mu\mu} \;-\; \frac{1}{2}\,\partial_\rho\, \partial_\mu\, \psi^{\alpha}_{\rho\mu} \;+\; \partial^2_\mu\, \psi^{\alpha}_{\rho\rho}\,, \tag{4.170}$$

one has two possibilities to saturate the ρ indices: taking one γ-trace and one divergence, or a simple trace. In both cases one has to act with $\frac{1}{\Box}$ to restore dimensions, and so in the second case it is necessary to act also with a $\not\partial$. The first case correspond to the primary equation, that looks like, in symmetric notation

$$\frac{1}{\Box}\,\partial\cdot\ \not{\mathcal{R}} \,=\, \mathcal{S} \,-\, \frac{1}{2}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,, \tag{4.171}$$

to be compared with the secondary equation

$$\frac{\not\partial}{\Box}\,\mathcal{R}\,' \,=\, \mathcal{S} \,+\, \frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,. \tag{4.172}$$

It is to be stressed that because of the Bianchi identity, it is always possible to give different forms to these equations. Here I display only the form useful for the mutual comparison.

On the other hand, the kinetic equation in this case has the form

$$\mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,, \tag{4.173}$$

and so no clear relation among the three possibilities arises, apart from the obvious fact that they all differ by gauge invariant quantities. At this level it is possible to observe that the kinetic equation can be seen as a combination of the two “geometric” equations, with coefficients respectively $\frac{2}{3}$ and $\frac{1}{3}$. In this sense, as expected because of gauge invariance, the kinetic equation *is* geometric; moreover, the requirement of having the least number of poles in the non-local part of the operator leaves anyway open an infinite number of possibilities, given by all linear combinations of primary and secondary equations.

It is the purpose of this Section to try and see whether, among the various options, it is possible to select one as “distinguishable” from the others, and what is the peculiarity, if any, of the kinetic operators defined in Section 4.1.6.

spin $\frac{7}{2}$

From the expression of the curvature

$$\mathcal{R}^{(3)\,\alpha}_{\rho\rho\rho\,;\mu\mu\mu} \,=\, \partial^3_\rho\,\psi^{\,\alpha}_{\mu\mu\mu} \,-\, \frac{1}{3}\partial^2_\rho\,\partial_\mu\,\psi^{\,\alpha}_{\rho\mu\mu} \,+\, \frac{1}{3}\partial^2_\mu\,\partial_\rho\,\psi^{\,\alpha}_{\rho\rho\mu} \,-\, \partial^3_\mu\,\psi^{\,\alpha}_{\rho\rho\rho}\,, \tag{4.174}$$

one can easily derive the primary equation:

$$\frac{1}{\Box}\ \not{\mathcal{R}}\,' \,=\, \mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^2}{\Box}\,\mathcal{S}\,' \,-\, \frac{2}{3}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,=\, 0\,, \tag{4.175}$$

that is identical, in this case, to the kinetic one.

On the other hand, as expected, the secondary equation is more singular:

$$\frac{\not\partial}{\Box}\left(\frac{1}{\Box}\,\partial\cdot\mathcal{R}'\right) \;=\; \mathcal{S} \,+\, \frac{1}{3}\,\frac{\partial^{2}}{\Box}\,\mathcal{S}' \,-\, \frac{\partial}{\Box}\,\partial\cdot\mathcal{S} \,+\, \frac{2}{3}\,\frac{\partial^{2}}{\Box^{2}}\,\partial\cdot\partial\cdot\mathcal{S} \,-\, \frac{\partial^{3}}{\Box^{2}}\,\partial\cdot\mathcal{S}'\,, \qquad (4.176)$$

and for this reason it is not to be considered among the simplest choices.

We are now in the position to discuss the general case, for which it is useful to further distinguish the possibilities of odd and even ranks for the field ψ.

spin $s \,=\, (2n+1) + \frac{1}{2}$

As anticipated, if the rank of ψ is odd, then the secondary equations will *always* be more singular with respect to the primary ones. In this case, in fact, the bosonic equation involves a divergence, which can be replaced with a γ-trace in the primary choice, reducing the number of derivatives at the numerator, and consequently the degree of singularity for the whole expression.

In order to establish contact with the kinetic equations defined via the operators $\mathcal{S}^{(n)}$, it is useful to compare their gauge transformations with those of the de Wit-Freedman connections:

$$\delta\,\mathcal{S}^{(n)} \;=\; -\,2\,i\,n\,\frac{\partial^{2n}}{\Box^{n-1}}\;\not\epsilon^{[n-1]}\,, \qquad\qquad \delta\,\Gamma^{(m)} \;=\; (-1)^{m}\,(m+1)\,\partial^{\,m+1}\,\epsilon\,. \qquad (4.177)$$

The crucial observation to derive from this comparison, is that the gauge transformations of the Γ's can be related to that of the $\mathcal{S}$'s *if and only if* $m \,=\, 2n \,-\, 1$.

Moreover, while gauge invariance does not imply equivalence in general, given that two gauge invariant quantities can differ by a third gauge invariant object, on the other hand, in this particular case, if *the same gauge transformation* is implemented by one $\mathcal{S}$ operator and one connection Γ, suitably traced and divided by a certain power of $\Box$, than we can infer that these two quantities are actually *the same* quantity. This is so because if the gauge transformation are equal, then the two tensors could only differ by gauge invariant quantities, but *by construction* the kinetic operators as well as the connections are built out of elementary constituents none of which is gauge invariant.

This justifies the following table, where the equalities are to be understood up to signs:

$$\delta\,\mathcal{S}^{(1)} \;=\; \delta\;\not{\mathcal{F}}^{(1)} \qquad \Longrightarrow \qquad \mathcal{S}^{(1)} \;=\not{\mathcal{F}}^{(1)}\,, \tag{4.178}$$

$$\delta\,\mathcal{S}^{(2)} \;=\; \delta\frac{1}{\Box}\;\not{\mathcal{F}}^{(3)\,\prime} \qquad \Longrightarrow \qquad \mathcal{S}^{(2)} \;=\; \frac{1}{\Box}\;\not{\mathcal{F}}^{(3)\,\prime}\,, \tag{4.179}$$

$$\delta\,\mathcal{S}^{(3)} \;=\; \delta\frac{1}{\Box^2}\;\not{\mathcal{F}}^{(5)\,\prime\prime} \qquad \Longrightarrow \qquad \mathcal{S}^{(3)} \;=\; \frac{1}{\Box^2}\;\not{\mathcal{F}}^{(5)\,\prime\prime}\,, \tag{4.180}$$

$$\ldots \qquad \Longrightarrow \qquad \ldots$$

$$\delta\,\mathcal{S}^{(n)} \;=\; \delta\frac{1}{\Box^{n-1}}\;\not{\mathcal{F}}^{(2n-1)\,[n-1]} \qquad \Longrightarrow \qquad \mathcal{S}^{(n)} \;=\; \frac{1}{\Box^{n-1}}\;\not{\mathcal{F}}^{(2n-1)\,[n-1]}\,. \tag{4.181}$$

Now, consider the minimum spin such that these operators are gauge invariant: for $s\;=\;\frac{3}{2}$ $\Gamma^{(1)}$ is the curvature, and the geometric equation of motion, $\not{\mathcal{F}}^{(1)}\;=\;0$, is just the one defined by the kinetic operator: $\mathcal{S}^{(1)}\;=\;0$.

For $s\;=\;\frac{7}{2}$ the curvature is $\Gamma^{(3)}$, and in this case primary and kinetic equations coincide, as shown by (4.179).

In general, for $s\;=\;(2n-1)+\frac{1}{2}$ the curvature is $\Gamma^{(2n-1)}$, and the primary geometric equation, defined as $\frac{1}{\Box^{n-1}}\;\not{\mathcal{F}}^{(2n-1)\,[n-1]}\;=\;0$, is equal to the kinetic equation of Section (4.1.6).

It remains to be discussed what happens when the rank of ψ is even, and there seems to be no direct relation between the kinetic operators and the curvatures.

spin $s\;=\;2n+\frac{1}{2}$

According to the point of view proposed in this Section, when the rank of ψ is even, say $2n$, the "natural" choice for a curvature is the first gauge invariant symbol of de Wit and Freedman:

$$\Gamma^{(2n)}_{\rho_{2n}\,,\,\mu_{2n}}\;\equiv\;\mathcal{R}^{(2n)}\,. \tag{4.182}$$

Correspondingly, the primary, geometric equations of motion are defined as

$$\frac{1}{\Box^n}\,\partial\cdot\;\not{\mathcal{R}}^{[n-1]}\;=\;0\,, \tag{4.183}$$

while the secondary equations, i.e. the ones mutuated by the bosonic case, are

$$\frac{\not{\partial}}{\Box^n}\,\mathcal{R}^{[n]}\;=\;0\,. \tag{4.184}$$

Yet, as I have observed, in this case these two types of equations actually display the same degree of singularity, and in this sense there is no manifest reason to prefer one or another of the two (and, to be honest, to call one "primary" and the other "secondary"). Moreover, as can be seen from the example of spin $\frac{5}{2}$, they do not happen to have same form in terms of the basic field, and so we have really at our disposal two different geometric equations, and in fact all their possible linear combinations. Finally, the kinetic equations of Section 4.1.6 now seems a third possibility, given that, from the table of the former subsection, it is not clear at all which kind of relationship could connect these operators to the geometric ones. For spin $\frac{5}{2}$ it was found that the kinetic equation of motion is in fact a linear combination of the two geometric possibilities. This result gives the right indication on where to search, and at the end it will result to be just a particular case of a general relation between the kinetic operators $\mathcal{S}^{(n+1)}$ and the curvatures $\mathcal{R}^{(2n)}$.

To see how this relation comes out, recall once more the recursive definition of the kinetic operators:

$$\mathcal{S}^{(n+1)} = \mathcal{S}^{(n)} + \frac{1}{n(2n+1)} \frac{\partial^2}{\Box} \mathcal{S}^{(n)\prime} - \frac{2}{2n+1} \frac{\partial}{\Box} \partial \cdot \mathcal{S}^{(n)} , \tag{4.185}$$

and of the deWit-Freedman symbols:

$$\Gamma^{(m)}_{\sigma \rho_{m-1}, \mu_s} = \partial_\sigma \Gamma^{(m-1)}_{\rho_{m-1}, \mu_s} - \frac{1}{m} \partial_\mu \Gamma^{(m-1)}_{\rho_{m-1}, \sigma \mu_{s-1}} . \tag{4.186}$$

If $m = 2n$, this last equality relates the $\Gamma^{(2n)}$ to the $\Gamma^{(2n-1)}$'s, and for this last ones we know how to relate them to the operators $\mathcal{S}^{(n)}$, according to (4.181):

$$\mathcal{S}^{(n)} = \frac{1}{\Box^{n-1}} \not{\Gamma}^{(2n-1)\,[n-1]} . \tag{4.187}$$

By means of (4.181) and (4.186) one can relate the connection $\Gamma^{(2n)}$ to the operator $\mathcal{S}^{(n)}$, and in this way one can try to determine whether a combination of traces and divergences of the $\Gamma^{(2n)}$ can reproduce, at least under certain circumstances, the $\mathcal{S}^{(n+1)}$ operator.

Explicitly, the following identities can be verified:

$$\frac{1}{\Box^n} \partial \cdot \not{\Gamma}^{(2n)\,[n-1]} = \mathcal{S}^{(n)} - \frac{1}{2n} \frac{\partial}{\Box} \partial \cdot \mathcal{S}^{(n)} , \tag{4.188}$$

$$\frac{\not{\partial}}{\Box^n} \partial \cdot \not{\Gamma}^{(2n)\,[n]} = \mathcal{S}^{(n)} - \frac{1}{2n} \frac{\partial}{\Box} \not{\partial} \not{\mathcal{S}}^{(n)} . \tag{4.189}$$

In the last equality we can substitute for $-\frac{1}{2n}\, \not\partial\, \not\mathcal{S}^{(n)}$ via the generalised Bianchi identities (4.65):

$$-\frac{1}{2n}\, \not\partial\, \not\mathcal{S}^{(n)} = -\partial\cdot\mathcal{S}^{(n)} + \frac{1}{2n}\, \partial\, \mathcal{S}^{(n)\,\prime} + i\, \frac{\partial^{2n}}{\Box^{n-1}}\, \not\psi^{[n]}\,, \tag{4.190}$$

then obtaining

$$\frac{\not\partial}{\Box^{n}}\, \partial\cdot\, \not\mathcal{F}^{(2n)\,[n]} = \mathcal{S}^{(n)} - \frac{\partial}{\Box}\, \partial\cdot\mathcal{S}^{(n)} + \frac{1}{n}\, \frac{\partial^{2}}{\Box}\, \mathcal{S}^{(n)\,\prime} + i\,(2n+1)\, \frac{\partial^{2n+1}}{\Box^{n}}\, \not\psi^{[n]}\,. \tag{4.191}$$

At this point, looking for a combination of (4.188) and (4.191) such to reproduce $\mathcal{S}^{(n+1)}$, *modulo* the term in $\not\psi^{[n]}$, it is possible to verify that the following equality holds:

$$\frac{2\,n}{2\,n\,+\,1}\left[\frac{1}{\Box^{n}}\, \partial\cdot\, \not\mathcal{F}^{(2n)\,[n-1]}\right] + \frac{1}{2\,n\,+\,1}\left[\frac{\not\partial}{\Box^{n}}\, \not\mathcal{F}^{(2n)\,[n]}\right] = \mathcal{S}^{(n+1)} + i\, \frac{\partial^{2n+1}}{\Box^{n}}\, \not\psi^{[n]}\,. \tag{4.192}$$

Now, consider all the cases not accounted for in the table at page 119;

- for $s = \frac{5}{2}$ the kinetic equation is $\mathcal{S}^{(2)} = 0$. The two geometrical equations are both possible, having the same degree of singularity, and in this case (4.192) implies

$$\frac{2}{3}\, \frac{1}{\Box}\, \partial\cdot\, \not\mathcal{F}^{(2)} + \frac{1}{3}\, \frac{\not\partial}{\Box}\, \not\mathcal{F}^{(2)\,\prime} = \mathcal{S}^{(2)}\,, \tag{4.193}$$

  as previuosly anticipated.

- for $s = \frac{9}{2}$ the kinetic equation is $\mathcal{S}^{(3)} = 0$. One can still combine the two geometric equations derived by $\Gamma^{(4)}$ as in (4.192), and find

$$\frac{4}{5}\left[\frac{1}{\Box^{2}}\, \partial\cdot\, \not\mathcal{F}^{(4)\,\prime}\right] + \frac{1}{5}\left[\frac{\not\partial}{\Box^{2}}\, \not\mathcal{F}^{(4)\,\prime\prime}\right] = \mathcal{S}^{(3)}\,, \tag{4.194}$$

  where in this case, as in the previous one, the fact that in these situations the term proportional to traces of $\not\psi$ is not present has been exploited.

- It is straightforward at this point to generalise the result. If $s = 2\,n + \frac{1}{2}$ then $\not\psi^{[n]}$ is not present, and (4.192) shows that the kinetic equation $\mathcal{S}^{(n+1)} = 0$ is just a combination of the two geometric possibilities allowed in this case, thus completing the proof that the unconstrained non-local formalism bears a direct geometrical meaning, at least for free fields.

Remarks

The last discussion leaves open at least two points.

First, if in the case of even-rank spinors we have an infinite number of possibilities to describe the dynamics in a geometric fashion, why should we choose the kinetic operators $\mathcal{S}^{(n)}$? Actually, I cannot exclude that other possibilities would reveal themselves to have some useful peculiar features. Anyway, on the one hand the kinetic option seems quite natural for consistency with the odd-rank case (in which, I recall, it coincides with the primary, geometric one), moreover, it is via the kinetic operator that we can give generalisations of the Bianchi identity, build identically divergenceless Einstein tensors, and then construct Lagrangians for the non-local equations. This last argument may therefore suggest that the choice of the $\mathcal{S}^{(n)}$ is the more convenient one.

Second, one could ask whether for bosons it is always true that the geometric equations coincide with the kinetic ones, or whether any ambiguities of the kind we have seen for fermions arise. Actually, by comparing the gauge transformations of the kinetic operators, given by

$$\delta\,\mathcal{F}^{(n)} \;=\; (2n+1)\,\frac{\partial^{\,2n+1}}{\Box^{\,n-1}}\,\Lambda^{[n]}\;, \tag{4.195}$$

with those of the connections,

$$\delta\,\Gamma^{(m)}_{\rho_m\,,\,\alpha_s} \;=\; (-1)^m\,(m+1)\,\partial_{\alpha}^{\,m+1}\Lambda_{\rho_m\,,\,\alpha_{s-m-1}}\;, \tag{4.196}$$

we can conclude that, possibly up to a sign

$$\mathcal{F}^{(n)} \;=\; \frac{1}{\Box^{\,n-1}}\,\Gamma^{(2n)[n]}\,. \tag{4.197}$$

It is then manifest that for *even* spin the kinetic operators define the same equation we derived from the corresponding curvature. The point is to discuss what happens for odd spins. To this end, as we did for fermions, consider the definition of the connection at the $(2n+1)$-th step:

$$\Gamma^{(2n+1)}_{\sigma\rho_{2n}\,,\,\mu_s} \;=\; \partial_\sigma\Gamma^{(2n)}_{\rho_{2n}\,,\,\mu_s} \;-\; \frac{1}{2n+1}\,\partial_\mu\Gamma^{(2n)}_{\rho_{2n}\,,\,\sigma\mu_{s-1}}\;, \tag{4.198}$$

and trace over all ρ-indices; then act with $\Box^{n-1}$ and express in this way the $\Gamma^{(2n+1)}$ in terms of the $\mathcal{F}^{(n)}$ operators:

$$\frac{1}{\Box^{n-1}}\,\Gamma^{(2n+1)\,[n]}_{\sigma,\,\mu_s} \;=\; \partial_\sigma\,\mathcal{F}^{(n)}_{\mu_s} \;-\; \frac{1}{2n+1}\,\partial_\mu\,\mathcal{F}^{(n)}_{\sigma\mu_{s-1}}\,. \tag{4.199}$$

The key difference with the fermionic case is that for bosons, to saturate the last special index σ, *we have only one possibility*, namely, to take a divergence. In this way, acting further on the result with $\frac{1}{\Box}$ we end with the equation

$$\frac{1}{\Box^n}\,\partial\cdot\Gamma^{(2n+1)\,[n]} \;=\; \mathcal{F}^{(n)} \;-\; \frac{1}{2n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)}\,. \tag{4.200}$$

To establish contact with the recursive definition of the kinetic operators,

$$\mathcal{F}^{(n+1)} \;=\; \mathcal{F}^{(n)} \;+\; \frac{1}{(n+1)(2n+1)}\,\frac{\partial^2}{\Box}\,\mathcal{F}^{(n)\,\prime} \;-\; \frac{1}{n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)}\;, \tag{4.201}$$

I again make use of the generalised Bianchi identity (4.32)

$$\partial\cdot\mathcal{F}^{(n)} \;-\; \frac{1}{2n}\,\partial\,\mathcal{F}^{(n)\,\prime} \;=\; -\,\left(1+\frac{1}{2n}\right)\,\frac{\partial^{\,2n+1}}{\Box^{\,n-1}}\,\varphi^{[n+1]}\,, \tag{4.202}$$

and transform (4.200) into

$$\frac{1}{\Box^{\,n}}\,\partial\cdot\Gamma^{(2n+1)\,[n]} \;=\; \mathcal{F}^{(n)} \;+\; \frac{1}{(n+1)(2n+1)}\,\frac{\partial^2}{\Box}\,\mathcal{F}^{(n)\,\prime} \;-\; \frac{1}{n+1}\,\frac{\partial}{\Box}\,\partial\cdot\mathcal{F}^{(n)} \;-\; \frac{\partial^{\,2n+2}}{\Box^{\,n}}\,\varphi^{\,[n+1]}\,. \tag{4.203}$$

It is then clear that, if $s = 2n+1$, the contribution in $\varphi^{\,[n+1]}$ does not appear, and the operator defining the geometric equation is exactly the kinetic operator recalled in (4.201).

Chapter 5

Some developments

The Higher-Spin problem represents a conceptual challenge for (Quantum) Field Theory. Indeed, its first principles allow for the presence of a whole tower of fields, classified by irreducible representations of the (local) symmetry group of space-time. For this reason, as already stressed in the Introduction, it is in a sense natural to investigate their possible role, without assuming any kind of "selection rule" introduced in order to assign to some restricted set ($s \leq 2$) a privileged status.

On the other hand, it should be honestly recognised that there is no *compelling* motivation in this framework calling for an improvement of the understanding of the subject. Indeed, the Field Theories that are commonly investigated as candidates to extend the Standard Model do not require higher-spin fields for their consistency. Moreover, there is no direct phenomenological input, up to now, suggesting that any help to solve presently open problems could come from a deeper insight in this area.

To simplify, one could say that it is not quite clear *why* higher spins seem to be "decoupled" from the physical world, but at the same time this "decoupling" apparently causes no real trouble, and at the end, one could simply take the attitude to ignore the problem.

The situation is different in String Theory, where higher-spin states are an intrinsic, inevitable part of the string spectrum, whose presence is required for the consistency of the construction, and for this reason gaining a deeper insight into the dynamics of these states is a real open problem of this area of Theoretical Physics. Hence, it is in a sense mandatory to look for some quantitative link between what we know of higher-spin fields and what can be deduced from String models, at least in simplified limits.

In this brief Chapter my aim is twofold. First, I would like to provide support for the role of the unconstrained formulation for free higher spin gauge fields, showing that it can be related to the tensionless limit of the free equations of String Field Theory.

These results are related to a particular structure normally referred as "the triplet", that has a long history in the literature. It first appeared in [75], for the case of spin 3, and shortly thereafter for arbitrary spin, even in non fully symmetric case, in [76]. It then served as a paradigmatic example of the BRST construction in [77], [78], [79], and more recently these systems, whose connection with the unconstrained higher-spin equations in the compensator form was presented in [63], while the (A)dS generalisation, together with the analysis of the mixed-symmetry case was discussed in [64], were studied together with their fermionic analogue, in connection with higher-spin dynamics in [80]. In Section 5.1 I shall briefly describe the setting, and define the *triplet* systems directly descending from the String equations for bosons, and guessed by analogy for fermions. I will then show in which sense these systems can be related to the unconstrained equations for higher-spin fields, in the compensator forms (3.30) and (3.67). Second, in Section 5.2, I will show how these systems can be consistently described in generic maximally symmetric backgrounds. This last issue, of course, is particularly relevant in the perspective of establishing a link with the Vasiliev theory, where the presence of a cosmological constant different form zero is a crucial ingredient in the construction of consistent equations of motion for interacting higher-spin systems. Here in particular I will limit myself to the description of the basic results, without entering the peculiarities related to the propagation of fields in conformally flat backgrounds, that would require a wider exposition (see [81], [82], [83]).

5.1 Unconstrained higher-spins and String Field theory

The free equations of String Field Theory can be written in the form [84], [85]

$$\mathcal{Q} \, |\Phi\rangle \; = \; 0 \, , \tag{5.1}$$

where $\mathcal{Q}$ is the BRST operator of the first-quantized string. These systems display the chain of gauge invariances

$$\delta|\Phi^{(n)}\rangle \;=\; \mathcal{Q}\,|\Phi^{(n+1)}\rangle\;, \tag{5.2}$$

with $\Phi^{(1)}$ the string gauge field and $\Phi^{(n)}$ $(n > 1)$ the corresponding chain of gauge parameters, and describe *massive* higher-spin states. In particular, it is relevant for the present discussion that neither (5.1) nor (5.2) involve trace conditions. Hence it is conceivable that in the tensionless limit $\alpha' \to \infty$, where α' is the string tension, the massless fields that emerge describe a dynamics more naturally related to the unconstrained one presented in Sections 3 and 4, rather than to the Fronsdal equations.

The complete analysis of the spectrum of these states would require to deal with mixed-symmetry tensors, which is beyond the scope of this Thesis[1], and has been discussed in [64]. Here it will suffice to note that for symmetric tensors (corresponding to states generated by the lowest string oscillators α_{-1} in the bosonic string) the limit $\alpha' \to \infty$ yields the equations

$$\begin{aligned} \Box\,\phi \,&=\, \partial\, C\,,\\ \partial\cdot\phi\,-\,\partial\, D\,&=\,C\,,\\ \Box\, D\,&=\,\partial\cdot C\,, \end{aligned} \tag{5.3}$$

where ϕ, C and D are symmetric tensors of rank respectively s, $s-1$ and $s-2$. These equations are invariant under the *unconstrained* gauge transformations

$$\begin{aligned} \delta\,\phi \,&=\, \partial\,\Lambda\,,\\ \delta\, C\,&=\,\Box\,\Lambda\,,\\ \delta\, D\,&=\,\partial\cdot\Lambda\,, \end{aligned} \tag{5.4}$$

and follow from the Lagrangians

$$\begin{aligned} \mathcal{L} = -\,\frac{1}{2}\,\partial_\mu\,\phi\,\partial^\mu\,\phi\,-\,\phi\,\partial\, C\,-\,s\, C\,\partial\, D\\ -\,\frac{s}{2}\, C^2\,+\,\binom{s}{2}\,\partial_\mu\, D\,\partial^\mu\, D\,. \end{aligned} \tag{5.5}$$

[1] The free theory for higher-spin gauge fields in this more general setting has been developed by many authors. In particular, in [86], non-local equations and Lagrangians that generalise the results presented in Section 4 are given, while in [87] those results are discussed in (A)dS background. Other related work can be found in [88] and [89]while in [90], [91], [92] and [93] the mixed symmetry case is discussed in a perspective closer to the Vasiliev's one.

In order to establish contact with the unconstrained equations discussed in Section 3.1 one can try and manipulate eq. (5.3) in order to reproduce the Fronsdal operator

$$\mathcal{F} \,=\, \Box\,\phi \,-\, \partial\,\partial\cdot\phi \,+\, \partial^{\,2}\,\phi^{\,\prime}\,. \tag{5.6}$$

In particular, one can use the second of (5.3) to evaluate the gradient of $\partial\cdot\phi$

$$\partial\,\partial\cdot\phi \,=\, 2\,\partial^{\,2}\,D \,+\, \partial\,C\,, \tag{5.7}$$

while acting with $\frac{1}{\Box}$ on the first and taking the square gradient of the trace of the resulting expression one gets

$$\partial^{\,2}\,\phi^{\,\prime} \,=\, 2\,\partial^{\,2}\,D \,+\, 3\,\frac{\partial^{\,3}}{\Box}\,C^{\,\prime}\,. \tag{5.8}$$

Putting together (5.7) and (5.8) with the first of (5.3) the final result is

$$\mathcal{F} \,=\, 3\,\frac{\partial^{\,3}}{\Box}\,C^{\,\prime}\,, \tag{5.9}$$

which is clearly of the form (3.30), with the obvious identification $\alpha \,\equiv\, \frac{1}{\Box}\,C^{\,\prime}$. Analogously, the relation $\phi \,=\, \frac{1}{\Box}\,\partial\,C$ implies

$$\phi^{\,\prime\prime} \,=\, 4\,\frac{\partial\cdot C^{\,\prime}}{\Box} \,+\, \partial\,\frac{C^{\,\prime\prime}}{\Box} \tag{5.10}$$

which shows the consistency of the triplet system (5.3) with the gauge invariant constraint (3.13) forced in the local theory by the introduction in the Lagrangian (3.14) of the Lagrange multiplier β. In this fashion it is manifest that the equations of String Field Theory, at least in this particular limit, bear a relationship with the unconstrained formalism for higher spins[2]

By analogy with the bosonic case, the following triplet system for fermions can be

[2] As discussed in [63], [64], the triplet system (5.3) propagates the polarisations of a spin-s field, together with all its traces, i.e. all lower spin $s-2$, $s-4$ and so on, down to one or zero according to whether s is odd or even. Actually, in order to reproduce the local compensator equation I have used the first equation in (5.3) to evaluate the trace of φ in terms of C. This actually amounts to the statement that the trace of φ be pure gauge, as C can be eliminated by a gauge choice, and in this sense what I have shown here is that the triplet system can be *consistently truncated* to the compensator equations (3.30). Analogous considerations are to be done for the case of fermions.

considered

$$
\begin{aligned}
\not\partial \psi &= \partial \chi \,, \\
\partial \cdot \psi - \partial \lambda &= \not\partial \chi \,, \\
\not\partial \lambda &= \partial \cdot \chi \,,
\end{aligned}
\tag{5.11}
$$

that link the unconstrained spin-$(s+1/2)$ spinor-tensor ψ, spin-$(s-1/2)$ spinor-tensor χ and spin-$(s-3/2)$ spinor-tensor λ, is invariant under the unconstrained gauge transformations

$$
\begin{aligned}
\delta \psi &= \partial \epsilon \,, \\
\delta \chi &= \not\partial \epsilon \,, \\
\delta \lambda &= \partial \cdot \epsilon \,,
\end{aligned}
\tag{5.12}
$$

and follow from the rather simple Lagrangian

$$
\begin{aligned}
\mathcal{L} = & \, i \bar{\psi} \not\partial \psi - i \bar{\psi} \partial \chi + i \partial \bar{\chi} \psi - i s \bar{\chi} \not\partial \chi \\
& + i s \bar{\chi} \partial \lambda - i s \partial \bar{\lambda} \chi - i s (s-1) \bar{\lambda} \not\partial \lambda \,.
\end{aligned}
\tag{5.13}
$$

Similarly to the bosonic triplet, also the system (5.11), that anyway does *not* descend directly from a String model[3], can be related quite simply to the local unconstrained equations discussed for fermions in Section 3.2.

To see explicitly this relationship let us take the γ- trace of the first of (5.11)

$$
2 \partial \cdot \psi - \not\partial \not\psi = \not\partial \chi + \partial \not\chi \,, \tag{5.14}
$$

and substitute for $\partial \cdot \psi$ the value it attains according to the second equation, to get

$$
\not\partial \not\psi = 2 \partial \lambda + \not\partial \chi - \partial \not\chi \,. \tag{5.15}
$$

Acting on this last relation with $\frac{\not\partial}{\Box}$ one gets a non-local expression for $\not\psi$ in term of χ,

$$
\not\psi = 2 \frac{\partial}{\Box} \partial \cdot \chi + \chi - \frac{\partial}{\Box} \not\partial \not\chi \,, \tag{5.16}
$$

where the third relation from (5.11) has also been used. This result allows to relate the Fang-Fronsdal operator $\mathcal{S} = \not\partial \psi - \partial \not\psi$ to the field χ according to the following equation

$$
\mathcal{S} = -2 \partial^2 \{ \frac{2}{\Box} \partial \cdot \chi - \frac{\not\partial}{\Box} \not\chi \} \,, \tag{5.17}
$$

[3]Indeed, totally symmetric spinor-tensor do not contribute to the dynamics of the superstring, being eliminated by the GSO projection.

which is identifiable with the compensator equation (3.67), $\mathcal{S} \, + \, 2\,\partial^2\,\xi \;=\; 0$, with the auxiliary field ξ here defined by the non-local expression

$$\xi \,\equiv\, \frac{2}{\Box}\,\partial\cdot\chi \,-\, \frac{\not{\partial}}{\Box}\,\not{\chi}\,, \tag{5.18}$$

and having the gauge transformation $\delta\,\xi \,=\, \not{\epsilon}$.

5.2 Compensator equations in (A)dS

Once the consistency of the free theory is verified, the very first step forward towards any non trivial generalisations is the study of the same systems in non-flat, maximally symmetric backgrounds.

This exercise is useful in general, since it allows to test the "reaction" of the free theory to the mildest deformation conceivable[4], and in particular is inescapable in the framework of higher-spin theories at present, given the central role of (A)dS background in the Vasiliev formulation. For this reason, in this Section I report the extension of the compensator equations to (A)dS spaces first presented in [63], and analysed in [64] as well.

From the technical point of view, in order to perform this extension only two additional inputs are needed: the commutator of two covariant derivatives on a vector, that in explicit notation reads

$$[\nabla_{\,\mu}, \nabla_{\,\nu}]\,V_{\rho} \;=\; \frac{1}{L^2}\,(g_{\,\nu\rho}\,V_{\mu} \;-\; g_{\,\mu\rho}\,V_{\nu})\;, \tag{5.19}$$

where L determines the AdS curvature (and the dS case is recovered formally continuing L to imaginary values), and the corresponding commutator on a spinor,

$$[\nabla_{\,\mu}, \nabla_{\,\nu}]\,\eta \;=\; -\,\frac{1}{2L^2}\,\gamma_{\,\mu\nu}\,\eta\,, \tag{5.20}$$

where $\gamma_{\,\mu\nu}$ is antisymmetric in μ and ν and equals the product $\gamma_{\,\mu}\gamma_{\,\nu}$ when μ and ν are different, since these determine all other cases.

Let us therefore begin by considering the bosonic case, noticing that the gauge transformations for the fields φ and α in such a curved background take naturally the form

$$\begin{aligned} \delta\,\varphi \;&=\; \nabla\Lambda\,, \\ \delta\,\alpha \;&=\; \Lambda'\;, \end{aligned} \tag{5.21}$$

[4] For the (A)dS extension of the Fronsdal theory see [36].

where ∇ denotes and (A)dS covariant derivative.

One can then proceed in various ways, for instance starting from the gauge variation of the (A)dS form of the Fronsdal operator

$$\begin{aligned} \delta\,\mathcal{F}_L &\equiv \delta \left\{ \mathcal{F} - \frac{1}{L^2}\,[(3 - \mathcal{D} - s)(2 - s) - s]\varphi - 2\,g\,\varphi\,' \right\} \\ &= 3\,(\nabla^{\,3}\Lambda\,') - \frac{4}{L^2}\,g\,\nabla\,\Lambda\,', \end{aligned} \tag{5.22}$$

so that

$$\mathcal{F}_L = \mathcal{F} - \frac{1}{L^2}\,\{[(3 - \mathcal{D} - s)(2 - s) - s]\,\varphi + 2\,g\,\varphi'\}\ , \tag{5.23}$$

and it is then simple to conclude that the compensator form of the higher-spin equations in (A)dS is

$$\begin{aligned} \mathcal{F} &= 3\,\nabla^{\,3}\alpha + \frac{1}{L^2}\{[(3 - \mathcal{D} - s)(2 - s) - s]\varphi + 2\,g\,\varphi'\} - \frac{4}{L^2}\,g\,\nabla\,\alpha\,, \\ \varphi'' &= 4\,\nabla\cdot\alpha + \nabla\alpha'\,. \end{aligned} \tag{5.24}$$

These are again nicely consistent, as can be seen making use of the Bianchi identity, that now becomes

$$\nabla\cdot\mathcal{F}_L - \frac{1}{2}\,\nabla\,\mathcal{F}_L' = -\frac{3}{2}\,\nabla^{\,3}\,\varphi'' + \frac{2}{L^2}\,g\,\nabla\,\varphi''\,, \tag{5.25}$$

and one can in fact verify that the first of (5.24) implies the second.

The fermionic compensator equations (3.67) can be also generalized to (A)dS backgrounds, where the gauge transformation for a spin-s fermion becomes in this case

$$\delta\,\psi = \nabla\,\epsilon + \frac{1}{2L}\,\gamma\,\epsilon\,. \tag{5.26}$$

For a spin-s fermion ($s = n + \frac{1}{2}$), the compensator equations in an (A)dS background are then

$$\begin{aligned} &(\not\nabla\,\psi - \nabla\,\not\psi) + \frac{1}{2L}\,[\mathcal{D} + 2\,(n - 2)]\,\psi + \frac{1}{2L}\,\gamma\,\not\psi \\ &= -\{\nabla\,\nabla\}\xi + \frac{1}{L}\,\gamma\,\nabla\,\xi + \frac{3}{2\,L^2}\,g\,\xi\,, \\ &\not\psi' = 2\,\nabla\cdot\xi + \not\nabla\,\not\xi + \nabla\xi' + \frac{1}{2L}\,[\mathcal{D} + 2\,(n - 2)]\,\not\xi - \frac{1}{2L}\,\gamma\,\xi'\,, \end{aligned} \tag{5.27}$$

and are invariant under

$$\begin{aligned} \delta\psi &= \nabla\,\epsilon\,, \\ \delta\xi &= \not\epsilon\,, \end{aligned} \tag{5.28}$$

with an unconstrained parameter ϵ.

Eqs. (5.27), like their flat-space counterparts (3.67) and (3.55), are again consistent, on account of the (A)dS deformation of the Bianchi identity (2.38),

$$\nabla \cdot \mathcal{S} - \frac{1}{2} \nabla \mathcal{S}' - \frac{1}{2} \not\nabla \not\mathcal{S} = \frac{i}{4L} \gamma S' + \frac{i}{4L} [(\mathcal{D} - 2) + 2(n-1)] \not\mathcal{S} + \frac{i}{2} \left[\{\nabla, \nabla\} - \frac{1}{L} \gamma \nabla - \frac{3}{2L^2} \right] \not\psi', \tag{5.29}$$

where now the Fang-Fronsdal operator $\mathcal{S}$ is also deformed and becomes

$$\mathcal{S} = i\,(\not\nabla \psi - \nabla \not\psi) + \frac{i}{2L} [\mathcal{D} + 2(n-2)] \psi + \frac{i}{2L} \gamma \not\psi . \tag{5.30}$$

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