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Minimal surfaces derived from the Costa-Hoffman-Meeks examples

Filippo Morabito

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Thesis Advisor
Laurent Hauswirth
Univ. Paris Est

Supervisor
Massimiliano Pontecorvo
Univ. Roma Tre

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Introduction

This thesis is mainly devoted to the construction of new families of examples of minimal surfaces derived from the family of Costa-Hoffman-Meeks surface.

The study of minimal surfaces in $\mathbb{R}^3$ started with Lagrange in 1762. He studied the problem of determining a graph over an open set W in $\mathbb{R}^2$, with the smallest area among all the surfaces that assume given values on the boundary of W.

In 1776, Meusnier supplied a geometric interpretation of the minimal graph equation: the mean curvature H of a minimal graph vanishes. Nowadays it has become customary to use the term minimal surface for any surface satisfying $H = 0$, notwithstanding the fact that such surfaces often do not provide a minimum for the area.

In all the questions I dealed with in this work, one minimal surface plays the key role. It is the Costa-Hoffmann-Meeks surface, the most famous minimal surface. The discovery of the Costa surface was responsible for the rekindling of interest in minimal surfaces in 1982. In that year C. Costa showed in [2, 3] the existence of a complete (i.e., it has no boundary) minimal surface of finite topology. It has genus 1 and three ends. In [14] D. Hoffman and W. H. Meeks III showed the embeddedness of this surface (i.e. it does not intersect itself). Until that moment the only other known embeddable complete minimal surfaces in $\mathbb{R}^3$ were the plane, the catenoid and the helicoid. They were discovered over two hundred years ago, and it was conjectured that these one were the only embedded complete minimal surfaces. Later D. Hoffman and W. H. Meeks III in [15, 16] generalized the work of C. Costa showing the existence of a family of complete embedded minimal surfaces with three ends and genus $k \geqslant 1$. We denote by M_k the surface of genus k. It is known as Costa-Hoffman-Meeks of genus k.

An important property of the minimal surfaces is the non degeneracy. The non degeneracy is defined in terms of the space of the Jacobi functions on the surface, that is the functions which belong to the kernel of the Jacobi operator. This operator is defined as the linearized of the mean curvature operator.

J. Pérez and A. Ros in [41] showed that the set of the non degenerate properly embedded minimal surfaces with finite total curvature and fixed topology in $\mathbb{R}^3$, has a structure of finite dimensional real-analytic manifold. As application they showed that for M_k with $2 \leqslant k \leqslant 37$, there exists a family of minimal surfaces with three horizontal ends that are obtained by infinitesimal deformations by M_k. This result is based on a work ([36]) of S. Nayatani which assures the non degeneracy of the Costa-Hoffman-Meeks surface only if its genus takes the values given above. In his work S. Nayatani computed the dimension of the kernel and the index (i. e. the number of the negative eigenvalues) of the Jacobi operator about M_k but only if $2 \leqslant k \leqslant 37$. He showed that the dimension of the kernel equals 4. The result is true also if $k = 1$ (see [37]). From that it follows the non degeneracy of M_k. In chapter 1 I show that it is possible to extend the result of S. Nayatani for bigger values of k. To be more precise I have shown that for $k \geqslant 38$ the dimension of the kernel and the index of the Jacobi operator about M_k, are respectively equal to 4 and $2k+3$. That allows us to state that the surface M_k is non degenerate also for $k \geqslant 38$.

The non degeneracy of the surface M_k is one of the essential ingredients of the proof due to L. Hauswirth and F. Pacard ([11]) of the existence of a new family of examples of minimal surfaces. Thanks to the result described in 1, their construction extends automatically to higher values of k. The same result is used in the other sections of the thesis. Without it the constructions that I will describe briefly in the following, would hold only for $k \leqslant 37$.

The last two chapters of the thesis are devoted to the construction of new families of examples of minimal surfaces by a gluing procedure (in the same spirit as [11]) which involves the surface M_k.

D. Hoffman and W. Meeks made in [17] a systematic study of sequence of complete minimal surfaces of increasing genus. In particular they proved that the limit of a sequence of Costa-Hoffman-Meeks surfaces tends to the union of a catenoid and a plane which intersects the catenoid through its waist. They also showed that if these surfaces are appropriately scaled and positioned, then the areas of high curvature tends to a classical periodic minimal surface, the Scherk fifth surface.

N. Kapouleas tried to give an answer to a question which naturally arises naturally from the work mentioned above. It is the possibility to desingularize the intersection of minimal surfaces replacing neighbourhoods of intersections with Scherk singly periodic surfaces. In [21] he presented a construction which answers positively the question above in highly symmetric cases. His construction allows to show the existence of minimal surfaces with finite total curvature, having at least three ends. Unfortunately their genus takes arbitrarly high values: it cannot be prescribed because it must be compatible with the symmetries. This technique cannot be used to construct low genus examples.

In [20] Kapouleas used the technique of the desingularization to construct constant mean curvature surfaces with arbitrarily high genus and without symmetries. Following this work M. Traizet in [48] obtained minimal surfaces with arbitrarily high genus by desingularization of the intersections of a finite number of vertical planes.

N. Kapouleas in [21] announced the preparation of a work concerning a more general theorem of desingularization valable in a general three-dimensional Riemannian manifold and which did not require any hypothesis about the symmetry and would have allowed the construction of minimal surfaces of arbitrary genus. This result was never published.

In last years the study of minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$, where $\mathbb{H}^2$ denotes the hyperbolic plane, and $\mathbb{S}^2 \times \mathbb{R}$ became more and more active. Various examples of minimal surfaces in these product manifolds have been constructed, one part of them being inspired by by known surfaces in $\mathbb{R}^3$. In chapter 2, I have shown the existence in the space $\mathbb{H}^2 \times \mathbb{R}$, of a family of minimal examples inspired to M_k, with $k \geqslant 1$. This result can be considered to be as a particular case of the announced general desingularization theorem of N. Kapouleas. The statement of the main theorem is

Theorem 1. *For all $k \geqslant 1$ there exists in $\mathbb{H}^2 \times \mathbb{R}$ a minimal surface of genus k, finite total extrinsic curvature with three horizontal ends: two catenoidal type ends and a middle planar end.*

I glue the image by a homothety of parameter ε^2, with ε sufficiently small, of a compact part of M_k along its three boundary curves to two minimal graphs that are respectively asymptotic to an upper half catenoid and a lower half catenoid defined in $\mathbb{H}^2 \times \mathbb{R}$ and to a minimal graph asymptotic to $\mathbb{H}^2 \times \{0\}$.

The chapter 3 is devoted to the construction of examples of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$. The results contained in this chapter generalize various previous results. In particular it is possible to prescribe an arbitrary value of the genus of the surface in the quotient. The results contained in this chapter (obtained in collaboration with Laurent Hauswirth and M. Magdalena Rodríguez Pérez) have been shown by the gluing procedure of surfaces.

The Scherk's second surface is one the most famous minimal surface. It is a properly embedded minimal surface in $\mathbb{R}^3$ invariant by one translation T we can suppose along the x_2-axis, and can be seen as the desingularization of two perpendicular planes P_1 and P_2 containing the x_2-axis. Also we suppose P_1, P_2 are symmetric with respect to the planes $\{x_1 = 0\}$ and $\{x_3 = 0\}$. By changing the angle between P_1, P_2 we obtain a 1-parameter family of properly embedded singly periodic minimal surfaces, we will refer to as Scherk surfaces. In the quotient $\mathbb{R}^3/T$ by its shortest translation T, each Scherk surface has genus zero and four ends asymptotic to flat annuli contained in $P_1/T, P_2/T$. Such ends are called Scherk-type ends.

F. Martin and V. Ramos Batista [27] have recently constructed a properly embedded singly periodic minimal example which has genus one and six Scherk-type ends in the quotient $\mathbb{R}^3/T$, called Scherk-Costa surface, based on Costa surface (from now on, T will denote a translation in the x_2-direction). Roughly speaking, they have removed each end of Costa surface (asymptotic to a catenoid or a plane) and replace it by two Scherk-type ends. We have obtain surfaces in the same spirit as Martin and Ramos Batista's one, but with a completely different method. We construct properly embedded singly periodic minimal surfaces with genus $k \geqslant 1$ and six Scherk-type ends in the quotient $\mathbb{R}^3/T$, by gluing (in an analytic way) a compact piece of M_k to two halves of a Scherk surface at the top and bottom catenoidal ends, and one periodic flat horizontal annulus with a disk removed at the middle planar end.

Theorem 2. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ invariant by T whose quotient in $\mathbb{R}^3/T$ has genus k and six Scherk-type ends.*

V. Ramos Batista [42] constructed a singly periodic Costa minimal surface with two catenoidal ends and two Scherk-type middle end, which has genus one in the quotient $\mathbb{R}^3/T$. This example is not embedded outside a slab in $\mathbb{R}^3/T$ which contains the topology of the surface. We observe that the surface we obtain by gluing a compact piece of M_1 (Costa surface) at its middle planar end to a periodic horizontal flat annulus with a disk removed has the same properties of Ramos Batista's one.

In 1988, H. Karcher [22, 23] defined a family of properly embedded doubly periodic minimal surfaces, called *toroidal halfplane layers*, which has genus one and four horizontal Scherk-type ends in the quotient. In 1989, Meeks and Rosenberg [31] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used an approach of minimax type to obtain the existence of a family of properly embedded doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. These Karcher's and Meeks and Rosenberg's surfaces have been generalized in [43], constructing a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called KMR examples (sometimes, they are also referred in the literature as toroidal halfplane layers). Such examples have been classified by Pérez, Rodríguez and Traizet [40] as the only properly embedded doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. Each $M_{\sigma,\alpha,\beta}$ is invariant by a horizontal translation T (by the period vector at its ends) and a non horizontal one. We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$, which has genus zero, infinitely many horizontal Scherk-type ends, and two limit ends.

In 1992, F.S. Wei [49] added a handle to a KMR example $M_{\sigma,0,0}$ in a periodic way, obtaining a properly embedded doubly periodic minimal surface invariant under reflection in three orthogonal planes, which has genus two and four horizontal Scherk-type ends in the quotient. Some years later, W. Rossman, E.C. Thayer and M. Wolgemuth [45] added a different type of handle to a KMR example $M_{\sigma,0,0}$, also in a periodic way, obtaining a different minimal surface with the same properties as Wei's one. They also added two handles to a KMR example, getting doubly periodic examples of genus three in the quotient. L. Mazet and M. Traizet [29] have recently added N handles to a KMR example $M_{\sigma,0,0}$, obtaining a genus N properly embedded minimal surface in $\mathbb{R}^3/T$ with an infinite number of horizontal Scherk-type ends and two limit ends. They have also constructed a properly embedded minimal surface in $\mathbb{R}^3/T$ with infinite genus, adding handles in a quasi-periodic way to a KMR example.

L. Hauswirth and F. Pacard [11] have obtained higher genus Riemann minimal examples in $\mathbb{R}^3$, by gluing two halves of a Riemann minimal example with the intersection of a conveniently chosen CHM surface with a slab. We follow their ideas to generalize Mazet and Traizet's examples by constructing properly embedded singly periodic minimal examples whose quotient to $\mathbb{R}^3/T$ has arbitrary finite genus, infinitely many horizontal Scherk-type ends and two limit ends. More precisely, we glue a compact piece of a slightly deformed CHM example M_k with bent catenoidal ends, to two halves of a KMR example $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ and a periodic horizontal flat annulus with a disk removed.

Theorem 3. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exist two 1-parameter families $\mathcal{K}_1, \mathcal{K}_2$ of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and two limit ends. The surfaces in $\mathcal{K}_1$ have a vertical plane of symmetry orthogonal to the x_1-axis, and the surfaces in $\mathcal{K}_2$ have a vertical plane of symmetry orthogonal to the x_2-axis.*

L. Mazet, M. Traizet and M. Rodriguez [28] have recently constructed saddle towers with infinitely many ends: they are non-periodic properly embedded minimal surfaces in $\mathbb{R}^3/T$ with infinitely many ends and one limit end. In the present paper, we construct (non-periodic) properly embedded minimal surfaces in $\mathbb{R}^3/T$ with arbitrary finite genus $k \geqslant 0$, infinitely many ends and one limit end. With this aim, we glue half a Scherk example with half a KMR example, in the case $k = 0$; and, when $k \geqslant 1$, we glue a compact piece of the CHM example M_k to half a Scherk surface (at the bottom catenoidal end of M_k), a periodic horizontal flat annulus with a disk removed (at the middle planar end) and half a KMR example (at the top catenoidal end).

Theorem 4. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exist a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus $k \geqslant 0$, infinitely many horizontal Scherk-type ends and one limit end.*

Introduction

Cette thèse porte sur la construction de nouvelles familles d'exemples de surfaces minimales derivées de la famille de Costa-Hoffman-Meeks.

L'étude des surfaces minimales dans $\mathbb{R}^3$ commença avec Lagrange en 1762. Il étudia le problème de la détermination d'un graphe sur un ensemble ouvert W de $\mathbb{R}^2$, d'aire la plus petite parmi toutes les surfaces qui prennent les mêmes valeurs au bord de W.

En 1776, Meusnier donna une interpretation géometrique de l'équation des graphes minimaux: leur courbure moyenne H est nulle. Aujourd'hui il est habituel d'utiliser l'expression surface minimale pour toute surface à courbure moyenne nulle, malgré le fait que souvent telles surfaces ne constituent pas un minimum de l'aire.

Dans toute question que j'ai abordée dans cette thèse, une surface minimale joue le rôle clé. Il s'agit de la surface de Costa-Hoffmann-Meeks, la plus fameuse parmi les surfaces minimales. La découverte de la surface de Costa a ravivé l'intérêt pour les surfaces minimales en 1982. En cette année C. Costa a montré dans [2, 3] l'existence d'une surface minimale complète (c'est-à-dire, sans bord) avec topologie finie. Elle est de genre 1 et a trois bouts. Dans [14] D. Hoffman et W. H. Meeks III ont montré que la surface est aussi plongée (c'est-à-dire elle n'a pas d'autointersections). En cette époque-la les seules autres surfaces minimales connues complètes et plongées dans $\mathbb{R}^3$, étaient le plan, le caténoïde et l'hélicoïde. Elles avaient été découvertes plus de deux cent ans plus tôt, et l'on conjecturait qu'elles fussent les uniques surfaces minimales complètes et plongées. Plus tard D. Hoffman et W. H. Meeks III dans [15] et [16] ont généralisé le travail de C. Costa en démontrant l'existence d'une famille de surfaces minimales complètes plongées avec trois bouts et genre $k \geqslant 1$. Nous posons M_k la surface de genre k de cette famille. Elle est connue sous le nom de surface de Costa-Hoffman-Meeks de genre k.

Une propriété rémarquable de certaines surfaces minimales est la non dégénérescence. La non dégénérescence est définie en termes de l'espace des fonctions de Jacobi de la surface, c'est-à-dire les fonctions du noyau de l'operateur de Jacobi. Cet operateur est le linéarisé de l'operateur courbure moyenne.

J. Pérez et A. Ros in [41] ont montré que l'ensemble des surfaces minimales non dégénérées, proprement plongées dans $\mathbb{R}^3$, de courbure totale finie et topologie fixée, a une structure de variété analytique réelle de dimension finie. Comme conséquence de cela, ils ont démontré qu'il existe une famille de surfaces minimales avec trois bouts horizontaux obtenues par déformations infinitésimales de M_k si $2 \leqslant k \leqslant 37$. Ce résultat est basé sur un travail ([36]) de S. Nayatani qui assure que la surface de Costa-Hoffman-Meeks est non dégénérée seulement si le genre prend les valeurs données ci-dessus. Dans son article S. Nayatani a calculé la dimension du noyau et l'indice (c'est-à-dire le nombre des valeurs propres negatives) de l'operateur de Jacobi de M_k seulement si $2 \leqslant k \leqslant 37$. Il a montré que la dimension du noyau est égale à 4. Le même résultat est valable si $k = 1$ (voir [37]). Cela implique que M_k est non dégénérée. Dans le chapitre 1 j'ai démontré que le résultat de S. Nayatani est valable pour valeurs superieures de k. Plus précisement j'ai démontré que pour $k \geqslant 38$ la dimension du noyau et l'indice de l'operateur de Jacobi de M_k, valent respectivement 4 et $2k + 3$. Cela nous permet de conclure que la surface M_k est non dégénérée aussi pour $k \geqslant 38$.

La non dégénérescence de la surface M_k est l'un des ingredients essentiels de la preuve dûe à L. Hauswirth et F. Pacard ([11]) de l'existence d'une nouvelle famille d'exemples de surfaces minimales. Grâce au résultat décrit dans le chapitre 1, leur construction s'étend automatiquement aux valeurs de k plus élevées. Le même résultat est utilisé dans les autres sections de la thèse. Sans ceci, les constructions brièvement décrites ici, ne seraient valables que pour $k \leqslant 37$.

Les deux derniers chapitres de la thèse sont consacrés à la construction de nouvelles familles d'exemples de surfaces minimales à l'aide d'une procédure de collage de surfaces déjà connues (parmi lesquelles y figure la surface M_k) du même style que dans [11].

D. Hoffman et W. Meeks ont présenté dans [17] un étude systématique des suites de surfaces minimales complètes de genre croissant. En particulier ils ont démontré que la limite d'une suite de surfaces de Costa-Hoffman-Meeks tend à l'union d'un caténoïde et d'un plan qui croise le caténoïde là où la section est le cercle de rayon le plus petit possible. Ils ont aussi montré que, si ces surfaces sont rescalées et disposées de façon appropriée, alors les morceaux de surfaces où la courbure est élevée tendent à une surface minimale périodique classique, la cinquième surface de Scherk.

N. Kapouleas a essayé de répondre à une question naturellement issue du travail cité ci-dessus. Il s'agit de démontrer la possibilité de désingulariser l'intersection de deux surfaces minimales en remplaçant des voisinages des intersections par des surfaces de Scherk simplement périodiques. Dans [21] il a presenté une construction qui repond affirmativement à la question ci-dessus dans les cas hautement symétriques. En fait sa construction permet de montrer l'éxistence de surfaces minimales de courbure totale finie et au moins trois bouts. Malheureusement le genre des surfaces ainsi construites prend des valeurs arbitrairement élevées: la valeur du genre ne peut pas être prescrite parce que elle doit être compatible avec les symétries. Donc l'on peut pas faire recours à cette technique à fin de construire des exemples de genre petit.

Dans [20], Kapouleas a utilisé la technique de la désingularisation pour construire des surfaces à courbure moyenne constante de genre arbitrairement grand et sans symétries.

En s'inspirant de cet article, M. Traizet dans [48] a construit des surfaces minimales simplement périodiques de genre arbitrairement grand par désingularisation des intersections d'un nombre fini de plans verticaux.

N. Kapouleas dans [21] a annoncé la rédaction d'un autre article dans lequel il aurait démontré un théorème de désingularisation plus générale valable dans une variété Riemannienne tridimensionelle quelconque qui ne requerrait aucune hypothèse sur la symétrie des surfaces et qui lui aurait permis de construire des surfaces minimales de genre arbitraire. Ce travail n'a jamais été publié.

Lors des dernières années, l'étude des surfaces minimales dans les variétés produit $\mathbb{H}^2 \times \mathbb{R}$, où $\mathbb{H}^2$ est le plan hyperbolique, et $\mathbb{S}^2 \times \mathbb{R}$ est devenu de plus en plus vif. Le développement de la théorie des surfaces minimales dans ces espaces a commencé avec [44] par H. Rosenberg et a continué par [33] et [32] par W. H. Meeks et H. Rosenberg. Dans [39] B. Nelli et H. Rosenberg ont démontré l'existence dans $\mathbb{H}^2 \times \mathbb{R}$ d'une riche famille d'exemples contenante les hélicoïdes, les caténoïdes et, en résolvant des problèmes de Plateau, des exemples de genre plus élevé inspirées de la théorie des surfaces minimales dans $\mathbb{R}^3$. Dans [10] L. Hauswirth a construit et classifié les surfaces minimales feuilletées par courbes horizontales de courbure constante dans $M \times \mathbb{R}$, où M est $\mathbb{H}^2, \mathbb{R}^2$ ou bien $\mathbb{S}^2$. D'autres exemples de surfaces minimales de genre 0 dans ces variétés produit ont été décrites par R. Sa Earp et E. Toubiana dans [46].

Dans le chapitre 2, j'ai démontré l'existence, dans $\mathbb{H}^2 \times \mathbb{R}$, d'une famille de surfaces minimales plongées inspirée de M_k, pour tout $k \geqslant 1$. Ce résultat peut être censé un cas particulier du théorème générale de désingularisation annoncé par N. Kapouleas. L'énoncé du théorème est le suivant.

Théorème 1. *Pour tout $k \geqslant 1$ il existe dans $\mathbb{H}^2 \times \mathbb{R}$ une surface minimale de genre k, courbure totale estrinseque finie avec trois bouts horizontaux: deux bouts de type caténoïdal et un bout plan au milieu.*

J'ai collé l'image par une homothétie de paramètre ε^2, avec ε suffisamment petit, d'un morceau compact de M_k le long de ses trois courbes de bord à deux graphes minimaux, qui sont, respectivement, asympotiques à la moitié superieure et à la moitié inferieure d'un caténoïde défini dans $\mathbb{H}^2 \times \mathbb{R}$, et à un graphe minimal asymptotique à $\mathbb{H}^2 \times \{0\}$.

Le chapitre 3 est consacré à la construction d'exemples de surfaces minimales simplement périodiques proprement plongées dans $\mathbb{R}^3$. Les résultats presentés dans ce chapitre (obtenus en collaboration avec Laurent Hauswirth et M. Magdalena Rodríguez Pérez) généralisent plusieurs anciennes constructions. L'une des nouveautés c'est la possibilité de produire des surface minimales périodiques dont le quotient a une valeur arbitraire du genre. Tous les résultats que je vais décrire brièvement ont été démontrés à l'aide de la méthode du collage de surfaces déjà introduite ci-dessus.

Après le plan et l'helïcoide, la première surface minimale simplement périodique a été découverte par Scherk [47] en 1835. Cette surface, connue sous le nom de *deuxième surface de Scherk*, est proprement plongée dans $\mathbb{R}^3$ et invariante par une translation T, qui nous allons supposer le long de l'axe x_2, et qui peut être vue comme la désingularisation de deux plans orthogonaux, notés P_1 et P_2, dont l'intersection est l'axe x_2. En outre, nous supposons que P_1, P_2 soient symétriques par rapport aux plans $\{x_1 = 0\}$ et $\{x_3 = 0\}$. Par la variation de l'angle parmi P_1, P_2 l'on obtient une famille à un paramétre de surfaces minimales proprement plongées et simplement périodiques, que nous nommons surfaces de Scherk. Dans le quotient $\mathbb{R}^3/T$, où T est la translation la plus petite, toute surface de Scherk a genre égal à zero et quatre bouts plans asymptotiques à des anneaux plats contenus dans $P_1/T, P_2/T$. Tels bouts sont nommés bouts de type Scherk.

F. Martin et V. Ramos Batista [27] ont récemment construit une surface minimale proprement plongée, simplement périodique, de genre égal à un et six bout de type Scherk dans le quotient $\mathbb{R}^3/T$, appelée surface de Scherk-Costa, basée sur la surface de Costa (dorénavant nous noterons T une translation dans la direction x_2). En gros, ils ont oté tout bout de la surface de Costa (asymptotique à un caténoïde ou à un plan) et l'ont remplacé par un bout de type Scherk. Dans ce chapitre nous avons obtenu des surfaces du même type que M. Traizet et V. Ramos Batista, mais à l'aide d'une méthode tout à fait différente. Nous avons montré l'existence de surfaces minimales proprement plongées, simplement périodiques, de genre $k \geqslant 1$ avec six bouts de type Scherk dans le quotient $\mathbb{R}^3/T$, par collage d'un morceau compact de M_k, des deux moitiés d'une surface de Scherk et d'un anneau plat horizontal P/T privé d'un disque. Voici l'énoncé du théorème.

Théorème 2. *Soit T la translation le long de l'axe x_2. Pour tout $k \geqslant 1$, il existe une famille à un paramétre de surfaces minimales simplement périodiques, proprement plongées dans $\mathbb{R}^3$, invariantes par T, dont le quotient dans $\mathbb{R}^3/T$ a genre égal k et six bout de type Scherk.*

H. Karcher dans [22, 23] et W.H. Meeks and H. Rosenberg dans [31] ont montré l'existence de deux différentes familles de surfaces minimales plongées doublement périodiques de genre égal à un et quatre bouts de type Scherk dans le quotient. Ces familles ont été généralisées par M.M. Rodríguez dans [43], en construisant une famille d'exemples à 3 paramétres, notée $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$. Ces exemples sont nommés exemples KMR (ils apparaissent dans la literature aussi sous le nom de "toroidal halfplane layers"). Ils ont été classifiés par J. Pérez, M. M. Rodríguez et M. Traizet dans [40]. Il s'agit des uniques surfaces minimales doublement périodiques, proprement plongées, de genre un et un nombre fini de bouts parallèles de type Scherk dans le quotient. Toute surface $M_{\sigma,\alpha,\beta}$ est invariante par translations soit horizontales, notées T, (il s'agit du vecteur période au bouts) soit non horizontales. Soit $\widetilde{M_{\sigma,\alpha,\beta}}$ le relèvement de $M_{\sigma,\alpha,\beta}$ à $\mathbb{R}^3/T$: il s'agit d'une famille de surfaces de genre zero, un nombre infini de bouts horizontaux de type Scherk et deux bouts limites.

En 1992, F.S. Wei [49] a rajouté une anse à un exemple KMR de type $M_{\sigma,0,0}$ de façon périodique, en obtenant une surface minimale proprement plongée doublement périodique invariante par réflections dans trois plans orthogonaux, de genre 2 et avec quatre bouts horizontaux de type Scherk dans le quotient. Quelques années plus tard, W. Rossman, E.C. Thayer et M. Wolgemuth [45] ont rajouté une anse de type différente à un exemple KMR de type $M_{\sigma,0,0}$, en obtenant d'autres surfaces minimales avec les mêmes propriétés. En outre ils ont rajouté deux anses à un exemple KMR, en montrant l'existence d'exemples doublement périodiques de genre 3 dans le quotient. Récemment, L. Mazet and M. Traizet [29] ont rajouté N anses à un exemple KMR de type $M_{\sigma,0,0}$, en obtenant une surface minimale proprement plongée de genre N dans $\mathbb{R}^3/T$ avec un nombre infini de bouts horizontaux de type Scherk et deux bouts limite. De plus ils ont construit une surface minimale proprement plongée dans $\mathbb{R}^3/T$ de genre infini, en rajoutant des anses de façon quasi périodique à un exemple KMR.

L. Hauswirth et F. Pacard [11] ont construit des exemples de type Riemann de genre élevé dans $\mathbb{R}^3$, par collage de deux moitié d'une surface de Riemann avec un morceau compact d'une surface CHM choisie de façon appropriée. Nous avons suivi leurs idées à fin da généraliser les exemples de Mazet et Traizet en construant des surfaces minimales proprement plongées, simplement périodiques, dont le quotient dans $\mathbb{R}^3/T$ a genre fini arbitraire, un nombre infini de bouts horizontaux de type Scherk et deux bouts limites. Plus précisément nous avons collé un morceau compact de la surface M_k (avec les bouts caténoïdaux penchés), les deux moitiés d'un exemple KMR (soit $M_{\sigma,\alpha,0}$ soit $M_{\sigma,0,\beta}$) et un anneau plat horizontal périodique privé d'un disque. Ci-dessous l'énoncé du théorème.

Théorème 3. *Soit T une translation le long de l'axe x_2. Pour tout $k \geqslant 1$, il existe deux familles à un paramétre $\mathcal{K}_1, \mathcal{K}_2$ de surfaces minimales proprement plongées dans $\mathbb{R}^3$ simplement périodiques dont le quotient dans $\mathbb{R}^3/T$ a genre égal à k, un nombre infini de bouts horizontaux de type Scherk et deux bouts limites. Les surfaces dans $\mathcal{K}_1$ ont un plan vertical de symétrie orthogonal à l'axe x_1, et les surfaces dans $\mathcal{K}_2$ ont un plan vertical de symétrie orthogonal à l'axe x_2.*

L. Mazet, M. Traizet and M. M. Rodríguez [28] ont récemment construit des "saddle towers": ce sont des surfaces minimales non-périodiques proprement plongées dans $\mathbb{R}^3/T$ avec un nombre infini de bouts et un bout limite. Le dernier résultat présenté dans le chapitre 3 est la construction de surfaces minimales (non-périodiques) proprement plongées dans $\mathbb{R}^3/T$, de genre arbitraire et fini $k \geqslant 0$, avec un nombre infini de bouts et un bout limite. La preuve de ce résultat est basée, dans le cas $k = 0$, sur le collage d'une moitié de la surface de Scherk et une moitié d'un exemple KMR; et dans le cas $k \geqslant 1$, sur le collage d'un morceau compact de la surface M_k avec une moitié de la surface de Scherk, un anneau plat horizontal périodique (privé d'un disque) et une moitié d'un exemple KMR.

Théorème 4. *Soit T une translation le long de l'axe x_2. Pour tout $k \geqslant 0$, il existe une famille à un paramétre de surfaces minimales proprement plongées dans $\mathbb{R}^3$, simplement périodiques, dont le quotient dans $\mathbb{R}^3/T$ a genre égal à k, un nombre infini de bouts horizontaux de type Scherk et un bout limite.*

Introduzione

Questa tesi verte sulla construzione di nuove famiglie di esempi di superfici minime derivate dalla famiglia delle superfici di Costa-Hoffman-Meeks.

Lo studio delle superfici minime in $\mathbb{R}^3$ ha avuto inizio con Lagrange nel 1762. Egli studiò il problema della determinazione di un grafico su un insieme aperto W di $\mathbb{R}^2$ di area minima tra tutte le superfici che assumono gli stessi valori sul bordo di W.

Nel 1776, Meusnier diede una interpretazione geometrica dell'equazione dei grafici minimi: la curvatura media H di un grafico minimo è nulla. Oggi si continua a utilizzare l'espressione superfice minima per ogni superficie a curvatura media nulla, nonostante il fatto che spesso tali superfici non minimizzino l'area.

In tutte le questioni di cui mi sono occupato nella tesi, c'è una superficie minima che gioca un ruolo chiave. Si tratta della superficie di Costa-Hoffmann-Meeks di genere $k \geqslant 1$, la più famosa tra tutte le superfici minime. La scoperta della superficie di Costa risvegliò l'interesse per le superfici minime nel 1982. In tale anno C. Costa dimostrò in [2, 3] l'esistenza di una superficie minima completa con topologia finita, di genere pari a 1 e con tre terminazioni. In [14] D. Hoffman e W. H. Meeks III dimostrarono che la superficie è anche embedded. Fino a quel momento le sole altre superfici minime conosciute che fossero complete e embedded in $\mathbb{R}^3$, erano il piano, il catenoide e l'elicoide. Esse erano state scoperte più di 200 anni prima, e si congetturava che si trattasse delle uniche superfici minime complete e embedded. Più tardi D. Hoffman e W. H. Meeks III in [15, 16] generalizzarono il lavoro di C. Costa dimostrando l'esistenza d'una famiglia di superfici minime complete embedded con tre terminazioni e genere $k \geqslant 1$. Indichiamo con M_k la superficie di genere k appartenente a questa famiglia. Essa è nota sotto il nome di superficie di Costa-Hoffman-Meeks di genere k.

Una proprietà importante di alcune superfici minime è la non degenericità. Tale proprietà è definita in termini dello spazio di funzioni di Jacobi della superficie, cioè le funzioni del nucleo dell'operatore di Jacobi. Tale operatore è il linearizzato dell'operatore curvatura media.

J. Pérez e A. Ros in [41] hanno mostrato che l'insieme delle superfici minime non degenere, propriamente embedded in $\mathbb{R}^3$, di curvatura totale finita e topologia fissata, ha una struttura di varietà analitica reale di dimensione finita. Come consequenza di ciò, essi ricavarono che esiste una famiglia di superfici minime con tre terminazioni orizzontali ottenute mediante infinitesime deformazioni di M_k se $2 \leqslant k \leqslant 37$. Questo risultato si basa su un lavoro ([36]) di S. Nayatani che assicura che la superficie di Costa-Hoffman-Meeks è non degenere solo se il genere assume i valori dati sopra. Nel suo articolo S. Nayatani calcolò la dimensione del nucleo e l'indice (cioé il numero degli autovalori negativi) dell'operatore di Jacobi di M_k solo nel caso $2 \leqslant k \leqslant 37$. Egli dimostrò che la dimensione del nucleo è uguale a 4. Lo stesso risultato vale anche per $k = 1$ (vedere [37]). Ciò garantisce che M_k è non degenere. Nel capitolo 1 ho dimostrato che il risultato di S. Nayatani è valido per valori superiori di k. Più precisamente ho dimostrato che per $k \geqslant 38$ la dimensione del nucleo e l'indice dell'operatore di Jacobi di M_k, valgono rispettivamente 4 e $2k+3$. Ciò mi permette di concludere che la superficie M_k è non degenere anche per $k \geqslant 38$.

La non degenericità della superficie M_k è uno degli ingredienti fondamentali della dimostrazione trovata da L. Hauswirth e F. Pacard ([11]) dell'esistenza di una nuova famiglia d'esempi di superfici minime. Grazie al risultato descritto nel capitolo 1, la loro costruzione si estende automaticamente ai valori di k più grandi. Lo stesso risultato è utilizzato negli altri capitoli della tesi. Senza di esso i resultati ivi esposti e che descriverò brevemente qui, sarebbero validi solo per $k \leqslant 37$.

Gli ultimi due capitoli della tesi sono dedicati alla costruzione di nuove superfici minime mediante il metodo dell'incollamento di parti di superfici minime già note (tra le quali M_k) nello stesso stile di [11].

D. Hoffman e W. Meeks presentarono in [17] uno studio sistematico delle successioni di superfici minime complete di genere crescente. In particolare essi hanno dimostrato che il limite di una successione di superfici di Costa-Hoffman-Meeks di genere crescente tende all'unione di un catenoide e di un piano che interseca il catenoide nel cerchio di diametro più piccolo possibile. Hanno anche dimostrato che, se queste superfici sono riscalate e disposte in modo appropriato, allora le parti di superfici dove la curvatura è più elevata tendono a un'altra superficie minima semplicemente periodica classica, la quinta superficie di Scherk.

N. Kapouleas cercò di rispondere ad una questione messa in evidenza dal lavoro citato sopra. Si tratta di dimostrare la possibilità di desingolarizzare le intersezioni di due superfici minime sostituendo degli intorni delle intersezioni con delle superfici di Scherk semplicemente periodiche. In [21] egli presentò una costruzione che permette di rispondere affermativamente alla questione nei casi in cui le superfici minime che si intersecano hanno molte proprietà di simmetria. Tale costruzione permette di mostrare l'esistenza di superfici minime di curvatura totale finita e almeno tre terminazioni. Sfortunatamente il genere delle superfici così costruite assume valori arbitrariamente elevati: esso non può essere prescritto poichè deve essere compatibile con le simmetrie. Quindi non si può fare ricorso a questa tecnica per ottenere degli esempi di genere basso.

Lo stesso autore in [20] ha utilizzato la tecnica della desingolarizzazione per ottenere delle superfici a curvatura media constante di genere arbitrariamente elevato e senza simmetrie.

Seguendo questo articolo M. Traizet ha ottenuto superfici minime di genere arbitrariamente alto mediante la desingolarizzazione delle intersezioni di un numero finito di piani verticali.

N. Kapouleas annunciò in [21] la redazione di un altro articolo in cui avrebbe dimostrato un teorema di desingolarizzazione più generale valido in una varietà Riemanniana tridimensionale qualunque e che non avrebbe richiesto alcuna ipotesi sulla simmetria delle superfici e che gli avrebbe permesso di ottenere delle superfici minime di genere arbitrario. Questo risultato non è mai stato pubblicato.

Gli ultimi anni hanno visto diventare sempre più vivo lo studio delle superfici minime nelle varietà prodotto $\mathbb{H}^2 \times \mathbb{R}$, dove $\mathbb{H}^2$ denota il piano iperbolico, e $\mathbb{S}^2 \times \mathbb{R}$. In queste varietà sono stati scoperti vari esempi di superfici minime ispirandosi a quelle conosciute in $\mathbb{R}^3$. Nel capitolo 2 ho dimostrato l'esistenza, in $\mathbb{H}^2 \times \mathbb{R}$, di una famiglia di superfici minime embedded ispirata a M_k, per ogni $k \geqslant 1$. Questo risultato può essere considerato come un caso particolare del teorema generale di desingolarizzazione annunciato da N. Kapouleas. L'enunciato del teorema è il seguente.

Teorema 1. *Per ogni $k \geqslant 1$ esiste in $\mathbb{H}^2 \times \mathbb{R}$ una superficie minima di genere k, con curvatura estrinseca totale finita con tre terminazioni orizzontali: due di tipo catenoidale e una di tipo planare.*

Questa superficie è stata ottenuta mediante incollamento dell'immagine mediante una omotetia di parametro ε^2, con ε sufficentemente piccolo, di una parte compatta di M_k lungo le sue tre curve di bordo a due grafici minimi, che sono asintotici, rispettivamente, alla metà superiore e alla metà inferiore di un catenoide definito in $\mathbb{H}^2 \times \mathbb{R}$ e a un grafico minimo asintotico a $\mathbb{H}^2 \times \{0\}$.

Il capitolo 3 è dedicato alla costruzione di superfici minime semplicemente periodiche propriamente embedded in $\mathbb{R}^3$. I risultati presentati in questo capitolo (ottenuti in collaborazione con Laurent Hauswirth e M. Magdalena Rodríguez Pérez) generalizzano varie precedenti costruzioni e sono stati dimostrati mediante la tecnica dell'incollamento. Una delle novità rispetto al passato è la possibilità di produrre superfici minime periodiche il cui quoziente ha genere arbitrario.

La seconda superficie di Scherk è una delle superfici minime più famose. E' propriamente embedded in $\mathbb{R}^3$, invariante rispetto alla traslazione T, che supporremo essere lungo l'asse x_2, e può essere considerata come la desingolarizzazione di due piani ortogonali, che denotiamo con P_1 e P_2, la cui l'intersezione è l'asse x_2. Inoltre supporremo che P_1, P_2 siano simmetrici rispetto ai piani $\{x_1 = 0\}$ e $\{x_3 = 0\}$. Mediante la variazione dell'angolo tra P_1, P_2 si ottiene una famiglia a un parametro di superfici minime propriamente embedded e semplicemente periodiche, che chiamiamo superfici di Scherk. Nel quoziente $\mathbb{R}^3/T$, ogni superficie di Scherk ha genere pari a zero e quattro terminazioni planari asintotiche a dei cilindri piatti contenuti in $P_1/T, P_2/T$. Tali terminazioni sono dette di tipo Scherk.

Noi abbiamo costruito delle superfici semplicemente periodiche di genere $k \geqslant 1$ con 6 terminazioni di tipo Scherk nel quoziente $\mathbb{R}^3/T$, mediante incollamento di una parte compatta di M_k, due metà di una superficie di Scherk e un cilindro piatto orizzontale periodico privato di un disco. L'enunciato del teorema è il seguente.

F. Martin and V. Ramos Batista [27] hanno recentemente construito una superficie minima propriamente embedded e semplicemente periodica di genere 1 e sei terminazioni di tipo Scherk nel quoziente $\mathbb{R}^3/T$, detta superficie di Scherk-Costa, (d'ora in poi T denota la traslazione nella direzione x_2). Semplificando, essi hanno rimosso ogni terminazione dalla superficie di Costa rimpiazzandola da due terminazioni di tipo Scherk. Nel capitolo 3 noi otteniamo superfici con caratteristiche analoghe a quella di M. Traizet e V. Ramos Batista, ma mediante un metodo completamente diverso. Si tratta di superfici propriamente embedded semplicemente periodiche di genere $k \geqslant 1$ e con 6 terminazioni di tipo Scherk nel quoziente $\mathbb{R}^3/T$, incollando (in modo analitico) una parte compatta di M_k a due metà di una superficie di Scherk e un annulus piatto orizzontale periodico P/T, da cui abbiamo rimosso un disco.

Teorema 2. *Sia T la traslazione lungo l'asse x_2. Per ogni $k \geqslant 1$, esiste una famiglia a un parametro di superfici minime semplicemente periodiche, propriamente embedded in $\mathbb{R}^3$, invarianti rispetto a T, e il cui quoziente in $\mathbb{R}^3/T$ ha genere pari a k e sei terminazioni di tipo Scherk.*

V. Ramos Batista [42] ha costruito una superficie minima di tipo Costa ma semplicemente periodica con due terminazioni di tipo catenoidale e due terminazioni di tipo Scherk, di genere 1 nel quoziente $\mathbb{R}^3/T$. Si tratta di un esempio non embedded all'esterno di uno slab in $\mathbb{R}^3/T$ contenente la topologia della superficie. Osserviamo che la superficie da noi ottenuta incollando una parte compatta di M_1 (la superficie di Costa) con un cilindro piatto orizzontale privato di un disco, ha le stesse proprietà di quella descritta da Ramos Batista.

Nel 1988, H. Karcher [22, 23] ha definito una famiglia di superfici minime propriamente embedded e doppiamente periodiche, chiamate *toroidal halfplane layers*, di genere 1 e con quattro terminazioni di tipo Scherk nel quoziente. Nel 1989, Meeks and Rosenberg [31] hanno sviluppato la teoria generale per le superfici minime doppiamente periodiche con topologia finita nel quoziente, e usarono un approccio di tipo minimax per dimostrare l'esistenza di una famiglia di superfici propriamente embedded, doppiamente periodiche, di genere 1 e con 4 terminazioni orizzontali di tipo Scherk nel quoziente. Queste famiglie sono state generalizzate da M.M. Rodríguez in [43], che ha costruito una famiglia a 3 parametri, denotata $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ e nota sotto il nome di famiglia di esempi KMR (una denominazione alternativa presente in letteratura è toroidal halfplane layers). Tali esempi sono stati classificati da J. Pérez, M. M. Rodríguez e M. Traizet in [40] come le uniche superfici minime doppiamente periodiche, propriamente embedded, di genere uno e un numero finito di terminazioni parallele di tipo Scherk nel quoziente. Ogni superficie $M_{\sigma,\alpha,\beta}$ è invariante per traslazioni sia orizzontali, denotate con T, (si tratta del vettore periodo in corrispondenza delle terminazioni) sia non orizzontali. Sia $\widetilde{M}_{\sigma,\alpha,\beta}$ il sollevamento di $M_{\sigma,\alpha,\beta}$ a $\mathbb{R}^3/T$: si tratta di una famiglia di superfici di genere zero, un numero infinito di terminazioni orizzontali di tipo Scherk e due terminazioni limite.

Nel 1992, F.S. Wei in [49] ha aggiunto un manico a un esempio KMR di tipo $M_{\sigma,0,0}$ in modo periodico, ottenendo una superficie minima propriamente embedded doppiamente periodica invariante per riflessioni rispetto a tre piani ortogonali, di genere 2 e con 4 terminazioni di tipo Scherk nel quoziente. Alcuni anni più tardi, W. Rossman, E.C. Thayer e M. Wolgemuth in [45] hanno aggiunto un diverso tipo di manico all'esempio $M_{\sigma,0,0}$, in modo periodico, ottenendo una superficie differente ma con proprietà simili a quella di Wei. Inoltre essi hanno aggiunto due manici a un esempio KMR, ottenendo superfici doppiamente periodiche di genere 3 nel quoziente. L. Mazet e M. Traizet [29] hanno recentemente aggiunto N manici a un esempio KMR di tipo $M_{\sigma,0,0}$, ottenendo una superficie di genere N, propriamente embedded in $\mathbb{R}^3/T$ con un numero infinito di terminazioni orizzontali di tipo Scherk e due terminazioni limite. Essi hanno anche costruito una superficie minima propriamente embedded in $\mathbb{R}^3/T$ di genere infinito, aggiungendo dei manici in modo quasi periodico a un esempio KMR.

L. Hauswirth e F. Pacard in [11] hanno costruito superfici minime di tipo Riemann ma di genere non nullo in $\mathbb{R}^3$, incollando le due metà di un esempio di tipo Riemann con l'intersezione di M_k con terminazioni catenoidali inclinate e uno slab. Noi abbiamo seguito le loro idee per generalizzare gli esempi di Mazet e Traizet, ottenendo nuove superfici minime semplicemente periodiche, propriamente embedded, il cui quoziente in $\mathbb{R}^3/T$ ha genere arbitrario finito, un numero infinito di terminazioni orizzontali di tipo Scherk e due terminazioni limite. Più precisamente abbiamo incollato una parte compatta della superficie M_k, con le terminazioni di tipo catenoidale inclinate, le due metà di un esempio KMR ($M_{\sigma,\alpha,0}$ o $M_{\sigma,0,\beta}$) e un cilindro piatto orizzontale periodico privato di un disco. Segue l'enunciato del teorema.

Teorema 3. *Sia T una traslazione lungo l'asse x_2. Per ogni $k \geqslant 1$, esistono due famiglie a un parametro $\mathcal{K}_1, \mathcal{K}_2$ di superfici minime propriamente embedded in $\mathbb{R}^3$ semplicemente periodiche il cui quoziente in $\mathbb{R}^3/T$ ha genere pari a k, un numero infinito di terminazioni orizzontali di tipo Scherk e due terminazioni limite. Le superfici in $\mathcal{K}_1$ hanno un piano verticale di simmetria ortogonale all'asse x_1, e le superfici in $\mathcal{K}_2$ hanno un piano verticale di simmetria ortogonale all'asse x_2.*

L. Mazet, M. Traizet and M. Rodriguez in [28] hanno recentemente construito delle "saddle towers": si tratta di superfici non periodiche propriamente embedded in $\mathbb{R}^3/T$ con infinite terminazioni e una terminazione limite. L'ultimo risultato presentato nel capitolo 3 è la costruzione di superfici minime, propriamente embedded in $\mathbb{R}^3/T$, di genere arbitrario e finito $k \geqslant 0$, con un numero infinito di terminazioni di tipo Scherk e una terminazione limite. La dimostrazione di ciò é basata, nel caso $k = 0$, sull'incollamento di una metà della superficie di Scherk e una metà di un esempio KMR; e, nel caso $k \geqslant 1$, sull'incollamento di una parte compatta della superficie M_k con una metà della superficie di Scherk, un cilindro piatto orizzontale periodico privato di un disco e una metà di un esempio KMR.

Teorema 4. *Sia T la traslazione lungo l'asse x_2. Per ogni $k \geqslant 0$, esiste una famiglia a un parametro di superfici minime propriamente embedded in $\mathbb{R}^3$, semplicemente periodiche, il cui quoziente in $\mathbb{R}^3/T$ ha genere pari a k, un numero infinito di terminazioni orizzontali di tipo Scherk e una terminazione limite.*

Chapter 1

Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces

1.1 Introduction

In the years 80's and 90's the study of the index of minimal surfaces in Euclidean space has been quite active. D. Fisher-Colbrie in [7], R. Gulliver and H. B. Lawson in [9] proved independently that a complete minimal surface M in $\mathbb{R}^3$ with Gauss map G has finite index if and only if it has finite total curvature. D. Fisher-Colbrie also observed that if M has finite total curvature its index coincides with the index of an operator $L_{\bar{G}}$ (that is the number of its negative eigenvalues) associated to the extended Gauss map $\bar{G}$ of $\bar{M}$, the compactification of M. Moreover $N(\bar{G})$, the null space of $L_{\bar{G}}$, if restricted to M consists of the bounded solutions of the Jacobi equation. The nullity, $\mathrm{Nul}(\bar{G})$, that is the dimension of $N(\bar{G})$, and the index are invariants of $\bar{G}$ because they are independent of the choice of the conformal metric on $\bar{M}$.

The computation of the index and of the nullity of the Gauss map of the Costa surface and of the Costa-Hoffman-Meeks surface of genus $g = 2, \ldots, 37$ appeared respectively in the works [37] and [36] of S. Nayatani. The aim of this work is to extend his results to the case where $g \geqslant 38$.

In [37] he studied the index and the nullity of the operator L_G associated to an arbitrary holomorphic map $G : \Sigma \to S^2$, where Σ is a compact Riemann surface. He considered a deformation $G_t : \Sigma \to S^2$, $t \in (0, +\infty)$, with $G_1 = G$ (see equation (1.2)) and gave lower and upper bounds for the index of G_t, $\mathrm{Ind}(G_t)$, and its nullity, $\mathrm{Nul}(G_t)$, for t near to 0 and $+\infty$ and $t = 1$. The computation of the index and the nullity in the case of the Costa surface is based on the fact that the Gauss map of this surface is a deformation for a particular value of t of the map G defined by $\pi \circ G = 1/\wp'$, that is its stereographic projection is equal to the inverse of the derivative of the Weierstrass $\wp$-function for an unit square lattice. S. Nayatani computed $\mathrm{Ind}(G_t)$ and $\mathrm{Nul}(G_t)$ for $t \in (0, +\infty)$, where G is the map defined above. So the result concerning the Costa surface follows as a simple consequence from that. He obtained that for this surface the index and the nullity are equal respectively to 5 and 4.

In [36] S. Nayatani extended the last result treating the case of the Costa-Hoffman-Meeks surface of genus g but only for $2 \leqslant g \leqslant 37$. He obtained that the index is equal to $2g+3$ and the nullity is equal to 4. Here we will show that these results continue to hold also for $g \geqslant 38$.

J. Pérez and A.Ros in [41] call a minimal surface non degenerate if the bounded Jacobi functions about the surface are induced by the isometries of the ambient space. As consequence of the works [36] and [37], the Costa-Hoffman-Meeks surface was known to be non degenerate with respect to this definition, but only for $1 \leqslant g \leqslant 37$.

The result of S. Nayatani about the nullity of the Gauss map of the Costa-Hoffman-Meeks surface is essential for the construction due to L. Hauswirth and F. Pacard [11] of a family of minimal surfaces with two limit ends asymptotic to half Riemann minimal surfaces and of genus g with $1 \leqslant g \leqslant 37$. Their construction is based on a gluing procedure which involves the Costa-Hoffman-Meeks surface of genus g and two half Riemann minimal surfaces. In particular the authors needed show the existence of a family of minimal surfaces close to the Costa-Hoffman-Meeks surface, invariant under the action of the symmetry with respect to the vertical plane $x_2 = 0$, having one horizontal end asymptotic to the plane $x_3 = 0$ and having the upper and the lower end asymptotic (up to translation) respectively to the upper and the lower end of the standard catenoid whose axis of revolution is directed by the vector $\sin\theta\, e_1 + \cos\theta\, e_3$, $\theta \leqslant \theta_0$ with θ_0 sufficiently small. That was obtained by Schauder fixed point theorem and using the fact that the nullity of the Gauss map of the surface is equal to 4. In [11] the authors refer to this last result as a non degeneracy property of the Costa-Hoffman-Meeks surface. It is necessary to remark that here the choice of working with symmetric deformations of the surface with respect to the plane $x_2 = 0$, has a key role. Because of the restriction on the value of the genus which affects the result of S. Nayatani, it was not possible to prove the existence of this family of minimal surfaces for higher values of the genus.

So one of the consequences of our work is the proof of the non degeneracy of the Costa-Hoffman-Meeks surface for $g \geqslant 1$ in the sense of the definition given in [41] and also, only in a symmetric setting, in [11]. So we can state that the family of examples constructed by L. Hauswirth and F. Pacard exists for all the values of the genus.

The author wishes to thank S. Nayatani for having provided the background computations

on which are based the results about the Costa-Hoffman-Meeks surfaces contained in [36].

1.2 Preliminaries

Let M be a complete oriented minimal surface in $\mathbb{R}^3$. The Jacobi operator of M is

$$L = -\Delta + 2K$$

where Δ is the Laplace-Beltrami operator and K is the Gauss curvature. Moreover we suppose that M has finite total curvature. Then M is conformally equivalent to a compact Riemann surface with finitely many punctures and the Gauss map $G : M \to S^2$ extends to the compactified surface holomorphically. So in the following we will pay attention to a generic compact Riemann surface, denoted by Σ and $G : \Sigma \to S^2$ a not constant holomorphic map, where S^2 is the unit sphere in $\mathbb{R}^3$ endowed with the complex structure induced by the stereographic projection from the north pole (denoted by π). We fix a conformal metric ds^2 on Σ and consider the operator $L_G = -\Delta + |dG|^2$, acting on functions on Σ.

We denote by $N(G)$ the kernel of L_G. We define $\mathrm{Nul}(G)$, the nullity of G, as the dimension of $N(G)$. Since $L(G) = \{a \cdot G \,|\, a \in \mathbb{R}^3\}$ is a three dimensional subspace of $N(G)$, then $\mathrm{Nul}(G) \geqslant 3$. We denote the index of G, that is the number of negative eigenvalues of L_G, by $\mathrm{Ind}(G)$. The index and the nullity are invariants of the map G: they are independent of the metric on the surface Σ. So we can consider on Σ the metric induced by G from S^2.

N. Ejiri and M. Kotani in [5] and S. Montiel and A. Ros in [33] proved that a non linear element of $N(G)$ is expressed as the support function of a complete branched minimal surface with planar ends whose extended Gauss map is G. In the following we will review briefly some results contained in [33] used by S. Nayatani in [37].

We will use some definitions and concepts of algebraic geometry. They are recalled in subsection 1.6.1.

Let γ be the meromorphic function defined by $\pi \circ G$. Let p_j and r_i be respectively the poles and the branch points of γ. We denote by $P(G) = \sum_{i=1}^{\nu} n_j p_j$, $S(G) = \sum_{i=1}^{\mu} m_i r_i$ respectively the polar and ramification divisor of γ. Here n_j, m_i denote, respectively, the multiplicity of the pole p_j and the multiplicity with which γ takes its value at r_i. We define on the surface Σ the divisor

$$D(G) = S(G) - 2P(G)$$

and introduce the vector space $\bar{H}(G)$ (see [33], theorem 4)

$$\bar{H}(G) = \Big\{ \omega \in H^{0,1}(k_\Sigma + D(G)) \,|\, Res_{r_i}\omega = 0, 1 \leqslant i \leqslant \mu,$$

$$Re \int_\alpha (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega = 0,\ \forall \alpha \in H_1(\Sigma, \mathbb{Z}) \Big\},$$

where k_Σ is a canonical divisor of Σ and $H_1(\Sigma, \mathbb{Z})$ is the first group of homology of Σ. Suppose that the divisor D has an expression of the form $\sum n_j v_j - \sum m_i u_i$, with $n_j, m_i \in \mathbb{N}$. An element of $H^{0,1}(D)$ can be expressed as $f dz$, where f is a meromorphic function on Σ with poles of order not bigger than n_j at v_i and zeroes of order not smaller than m_i at u_i. Equivalently, if $g dz$, where g is a meromorphic function, is the differential form associated with the divisor D, the product fg must be holomorphic.

For $\omega \in \bar{H}(G)$, let $X(\omega) : \Sigma \setminus \{r_1, \ldots, r_\mu\} \to \mathbb{R}^3$ be the conformal immersion defined by

$$X(\omega)(p) = Re \int^p (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega.$$

Then $X(\omega) \cdot G$, the support function of $X(\omega)$, extends over the ramification points $r_1, \ldots, r_\mu$ smoothly and thus gives an element of $N(G)$. Conversely, every element of $N(G)$ is obtained in this way. In fact the map

$$\begin{array}{rcl} i : \bar{H}(G) & \to & N(G)/L(G) \\ \omega & \to & [X(\omega) \cdot G] \end{array} \tag{1.1}$$

is an isomorphism. This result, used in association with the Weierstrass representation formula, gives a description of the space $N(G)$. To obtain the dimension of $N(G)$ it is sufficient to compute the dimension of $\bar{H}(G)$. Since the dimension of $L(G)$ is equal to 3, then $\mathrm{Nul}(G) = 3 + \dim \bar{H}(G)$.

We denote by A_t a one parameter family $(0 < t < +\infty)$ of conformal diffeomorphisms of the sphere S^2 defined by

$$\pi \circ A_t \circ \pi^{-1} w = tw, \quad w \in \mathbb{C} \cup \{\infty\}.$$

We define for $0 < t < \infty$

$$G_t = A_t \circ G. \tag{1.2}$$

S. Nayatani in [37] gave lower and upper bounds for the index and, applying the method recalled above, for the nullity of G_t, $t \in (0, \infty)$, a deformation of an arbitrary holomorphic map $G : \Sigma \to S^2$, where Σ is a compact Riemann surface. In the same work, choosing appropriately the map G and the surface Σ, he computed the index and the nullity for the Gauss map of the Costa surface. In fact the extended Gauss map of this surface is a deformation of G for a particular value of t. We describe briefly the principal steps to get this result.

Firstly it is necessary to study the vector space $\bar{H}(G_t)$. A differential $\omega \in H^{0,1}(k_\Sigma + D(G))$ with null residue at the ramification points, is an element of $\bar{H}(G_t)$ if and only if the pair $(t\gamma, \omega)$ defines a branched minimal surface by the Weierstrass representation. If one sets $\gamma = 1/\wp'$ then there exist only two values of t, denoted by $t' < t''$, for which the condition above is verified and moreover $\dim H(G_t) = 1$. In other words, thanks to the characterization of the non linear elements of $N(G_t)$ by the isomorphism described by (1.7), if $t = t', t''$, $\mathrm{Nul}(G_t) = 4$. As for the index, if $t = t', t''$ then $\mathrm{Ind}(G_t) = 5$. Since $G_{t''}$ is the extended Gauss map of the Costa surface, one can state:

Theorem 1.2.1. *Let $\bar{G}$ be the extended Gauss map of the Costa surface. Then*

$$\mathrm{Nul}(\bar{G}) = 4, \quad \mathrm{Ind}(\bar{G}) = 5.$$

The same author in [36] treated the more difficult case of the Costa-Hoffman-Meeks surfaces of genus $2 \leqslant g \leqslant 37$ by a slightly different method. That is the subject of next section.

1.3 The case of the Costa-Hoffman-Meeks surface of genus smaller than 38

In this section we expose some of the background details at the base of section 3 of the work [36]. S. Nayatani provided them to us in [38].

We denote by M_g the Costa-Hoffman-Meeks surface of genus g. Let Σ_g be the compact Riemann surface

$$\Sigma_g = \{(z, w) \in (\mathbb{C} \cup \{\infty\})^2 \,|\, w^{g+1} = z^g(z^2 - 1)\} \tag{1.3}$$

and let $Q_0 = (0,0), P_+ = (1,0), P_- = (-1,0), P_\infty = (\infty,\infty)$. It is known that $M_g = \Sigma_g \setminus \{P_+, P_-, P_\infty\}$.

The following result describes the properties of symmetry of M_g and Σ_g.

Lemma 1.3.1. *([14]) Consider the conformal mappings of $(\mathbb{C} \cup \{\infty\})^2$:*

$$\kappa(z, w) = (\bar{z}, \bar{w}) \quad \lambda(z, w) = (-z, \rho w), \tag{1.4}$$

where $\rho = e^{\frac{i\pi g}{g+1}}$. The map κ is of order 2 and λ is of order $2g + 2$. The group generated by κ and λ is the dihedral group D_{2g+2}. This group of conformal diffeomorphisms leaves

M_g invariant, fixes both Q_0 and P_∞ and extend to Σ_g. Also κ fixes the points $P_\pm$ while λ interchanges them.

We set $\gamma(w) = w$. Let $G : \Sigma_g \to S^2$ be the holomorphic map defined by

$$\pi \circ G(z, w) = \gamma(w). \tag{1.5}$$

We denote by r_i, $i = 1, \dots, \mu$, the ramification points of γ and by $R(G)$ the ramification divisor $\sum_{i=1}^{\mu} r_i$. Theorem 5 of [33] shows that the space $N(G)/L(G)$, that we have introduced in previous section, is also isomorphic to a space of meromorphic quadratic differentials. This alternative description of $N(G)/L(G)$ that we present in the following, was adopted by S. Nayatani in [36]. We start defining the vector spaces $\hat{H}(G)$ and $H(G)$.

$$\hat{H}(G) = \left\{ \sigma \in H^{0,2}(2k_\Sigma + R(G)) \,|\, Res_{r_i} \frac{\sigma}{d\gamma} = 0, i = 1, \dots, \mu \right\}, \tag{1.6}$$

$$H(G) = \left\{ \sigma \in \hat{H}(G) \,|\, Re \int_\alpha (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \frac{\sigma}{d\gamma} = 0, \, \forall \alpha \in H_1(\Sigma, \mathbb{Z}) \right\},$$

where k_Σ is a canonical divisor of Σ. We remark that the elements of $H^{0,2}(2k_\Sigma + R(G))$ are quadratic differentials (see subsection 1.6.1). Since hereafter we will work only with quadratic differentials, we can set $H^0(\cdot) = H^{0,2}(\cdot)$ to simplify the notation. If we suppose that the divisor $2k_\Sigma + R(G)$ has an expression of the form $\sum n_j v_j - \sum m_i u_i$, with $n_j, m_i \in \mathbb{N}$, an element of $H^0(2k_\Sigma + R(G))$ can be expressed as $f(dz)^2$, where f is a meromorphic function on Σ with poles of order not bigger than n_j at v_i and zeroes of order not smaller than m_i at u_i. Equivalently, if $g(dz)^2$, where g is a meromorphic function, is the differential form associated with the divisor $2k_\Sigma + R(G)$, the product fg must be holomorphic.

For $\sigma \in H(G)$, let $X(\sigma) : \Sigma \setminus \{r_1, \dots, r_\mu\} \to \mathbb{R}^3$ be the conformal immersion defined by

$$X(\sigma)(p) = Re \int^p (1 - \gamma^2, i(1 + \gamma^2), 2\gamma) \frac{\sigma}{d\gamma}.$$

Then $X(\sigma) \cdot G$, the support function of $X(\sigma)$, extends over the ramification points $r_1, \dots, r_\mu$ smoothly and thus gives an element of $N(G)$. Conversely, every element of $N(G)$ is obtained in this way. In fact the map

$$\begin{array}{rcl} i : H(G) & \to & N(G)/L(G) \\ \sigma & \to & [X(\sigma) \cdot G] \end{array} \tag{1.7}$$

is an isomorphism. So to obtain the dimension of $N(G)$ it is sufficient to compute the dimension of $H(G)$. We recall that the dimension of $L(G)$ is equal to 3, so $\mathrm{Nul}(G) =$

$3 + \dim H(G)$.

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is a deformation in the sense of the definition (1.2) of the map G, we need to study the space $H(G_t)$. From (1.6) and (1.2) it is clear that $\hat{H}(G) = \hat{H}(G_t)$ and

$$H(G_t) = \left\{ \sigma \in \hat{H}(G_t) \mid Re \int_\alpha (1 - t^2\gamma^2, i(1 + t^2\gamma^2), 2t\gamma)\frac{\sigma}{d\gamma} = 0, \forall \alpha \in H_1(\Sigma_g, \mathbb{Z}) \right\}.$$

Long computations ([38], see subsection 1.6.2 for some details) show that a basis of the differentials of the form $\sigma/d\gamma$, where $\sigma \in \hat{H}(G) = \hat{H}(G_t)$, and whose residue at the ramification points of $\gamma(w) = w$ is zero, is formed by

$$\omega_k^{(1)} = \frac{z^{k-1}}{w^k}\frac{dz}{w}, \quad \text{with} \quad k = 0, \ldots, g-1,$$

$$\omega_k^{(2)} = \frac{((k-2)z^2 - kA^2)}{(z^2 - A^2)^2}\left(\frac{z}{w}\right)^{k-1}\frac{dz}{w}, \quad \text{with} \quad k = 0, \ldots, g,$$

$$\omega_k^{(3)} = \frac{((k-2)z^2 - kA^2)}{w(z^2 - A^2)^2}\left(\frac{z}{w}\right)^{k-1}\frac{dz}{w}, \quad \text{with} \quad k = 0, \ldots, g-1,$$

where $A = \sqrt{\frac{g}{g+2}}$.

Now we put attention to the space $H(G_t)$. We recall that we are interested in the computation of its dimension. By the definition of $H(G_t)$, a differential $\sigma \in \hat{H}(G_t)$ belongs to $H(G_t)$ if and only if $\forall \alpha \in H_1(\Sigma_g, \mathbb{Z})$ the differential form $\omega = \frac{\sigma}{d\gamma} = \frac{\sigma}{dw}$ satisfies

$$\int_\alpha \omega = t^2 \overline{\int_\alpha \gamma^2(w)\omega}, \tag{1.8}$$

$$Re \int_\alpha \gamma(w)\omega = 0. \tag{1.9}$$

If these two conditions are satisfied then (γ, w) are the Weierstrass data of a branched minimal surface. Of course, it is sufficient to impose that these equations are satisfied when α varies between the elements of a basis of $H_1(\Sigma_g, \mathbb{Z})$. The convenient basis of $H_1(\Sigma_g, \mathbb{Z})$ is constructed as follows. Let $\beta(s) = \frac{1}{2} + e^{i2\pi s}$, $0 \leqslant s \leqslant 1$. Let $\tilde{\beta}(s) = (\beta(s), w(\beta(s)))$ be a lift of β to Σ_g such that, for example, $\tilde{\beta}(0) = (\frac{3}{2}, w(0))$, with $w(0) \in \mathbb{R}$. As stated in lemma 1.3.1 the group of conformal diffeomorphisms of Σ_g is isomorphic to the dihedral group D_{2g+2}. The collection $\{\lambda^l \circ \tilde{\beta}, l = 0, \ldots, 2g-1\}$, where λ is the generator of D_{2g+2} of order $2g+2$, is a basis of $H_1(\Sigma_g, \mathbb{Z})$ (see [14]).

Now we must impose (1.8) and (1.9) for $\alpha = \lambda^l \circ \tilde{\beta}$, with $l = 0, \dots, 2g-1$. To do that we collapse β to the unit interval. In other terms we deform continously β in such a way the limit curve is the union of two line segments lying on the real line. We set

$$\omega = \sum_0^{g-1} c_k^{(1)} \omega_k^{(1)} + \sum_0^{g} c_k^{(2)} \omega_k^{(2)} + \sum_0^{g-1} c_k^{(3)} \omega_k^{(3)},$$

where $c_k^{(i)} \in \mathbb{C}$.

Taking into account these assumptions, it is possible to show that the equation (1.8), if the genus g is 2, is equivalent to the following system of four equations (see subsection 1.6.3)

$$\begin{cases} f_0 = -t^2 \bar{h}_0 \\ f_1 = 0 \\ p_1 = -t^2 \bar{q}_1 \\ p_2 = -t^2 \bar{q}_0. \end{cases} \tag{1.10}$$

If $g \geqslant 3$ there are the following additional $2g-4$ equations to consider

$$\begin{cases} f_k = -t^2 \bar{q}_{g-k+2} \\ p_{g-k+2} = -t^2 \bar{h}_k \end{cases} \tag{1.11}$$

where $k = 2, \dots, g-1$ and

$$f_0 = \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin\left(\frac{\pi}{g+1}\right) K_0,$$

$$f_k = \left(-c_k^{(1)} + \frac{(g+2)(g+2+k)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{(k+1)\pi}{g+1}\right) K_k, \quad k = 1, \dots, g-1,$$

$$h_0 = \frac{(g+2)^2}{2(g+1)} c_0^{(3)} \sin\left(\frac{-\pi}{g+1}\right) J_0,$$

$$h_k = \left(c_k^{(1)} + \frac{(g+2)(g+2-k)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{(k-1)\pi}{g+1}\right) J_k, \quad k = 2, \dots, g-1,$$

$$p_k = -\frac{(g+2)k}{2(g+1)} c_k^{(2)} \sin\left(\frac{k\pi}{g+1}\right) I_k, \quad k = 1, \dots, g,$$

$$q_k = \frac{(g+2)(2g+4-k)}{2(g+1)} c_k^{(2)} \sin\left(\frac{(k-2)\pi}{g+1}\right) L_k, \quad k = 0, 1, 3, \dots, g,$$

and

$$I_m = \frac{g+1}{m} \frac{\Gamma\left(1+\frac{m}{2(g+1)}\right)\Gamma\left(1-\frac{m}{g+1}\right)}{\Gamma\left(1-\frac{m}{2(g+1)}\right)},$$

$$J_m = \frac{g+1}{g-m+2} \frac{\Gamma\left(\frac{1}{2}+\frac{m-1}{2(g+1)}\right)\Gamma\left(1-\frac{m-1}{g+1}\right)}{\Gamma\left(\frac{1}{2}-\frac{m-1}{2(g+1)}\right)},$$

$$K_m = J_{m+2},$$

$$L_m = \frac{m-2}{2g-m+4} I_{m-2}.$$

The equation (1.9) if the genus g is 2, is equivalent to the following system of two equations (see subsection 1.6.3)

$$\begin{cases} d_1 = 0 \\ e_2 = \bar{e}_0. \end{cases} \tag{1.12}$$

If $g \geqslant 3$ there are the following additional $g-2$ equations to consider

$$d_k = \bar{e}_{g-k+2} \tag{1.13}$$

where $k = 2, \dots, g-1$, and

$$d_k = \left(c_k^{(1)} - \frac{k(g+2)}{2(g+1)} c_k^{(3)}\right) \sin\left(\frac{k\pi}{g+1}\right) I_k, \quad k = 1, \dots, g-1,$$

$$e_k = \frac{(g+2)(g+2-k)}{2(g+1)} c_k^{(2)} \sin\left(\frac{(k-1)\pi}{g+1}\right) J_k, \quad k = 0, 2, \dots, g.$$

We are looking for the values of t such that the previous systems have non trivial solutions in terms of $c_i^{(j)}$. Only for these special values of t it holds $\dim H(G_t) > 0$ or equivalently $\mathrm{Nul}(G_t) > 3$.

We start with the analysis of the system (1.10). This system admits non trivial solutions if and only if t takes three values denoted by t_1, t_2, t_3. Obviously they are functions of g.

If we set $s = \frac{1}{g+1}$ then we can write

$$t_1 = \sqrt{\frac{K_0}{J_0}} = \frac{\sqrt{1-s^2}}{2} \sqrt{\frac{\Gamma(1-s)}{\Gamma(1+s)} \frac{\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(1+\frac{s}{2}\right)}},$$

$$t_2 = \sqrt{\frac{I_1}{(2g+3)L_1}} = \sqrt{\frac{\Gamma\left(1-s\right)}{\Gamma\left(1+s\right)}\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}},$$

$$t_3 = \sqrt{\frac{I_2 J_0}{g L_0 K_0}} = \frac{2}{1-s}\sqrt{\left(\frac{\Gamma(1+s)}{\Gamma(1-s)}\right)^3}\sqrt{\frac{\Gamma(1-2s)}{\Gamma(1+2s)}\frac{\Gamma(3/2-s/2)}{\Gamma(1/2+s/2)}}.$$

We recall that if $g \geqslant 3$ there are other equations to consider. They are

$$\begin{cases} f_k = -t^2 \bar{q}_{g-k+2} \\ p_{g-k+2} = -t^2 \bar{h}_k \\ d_k = \bar{e}_{g-k+2} \end{cases}$$

where $k = 2, \ldots, g-1$. Thanks to the particular structure of the equations, it is possible to study separately for each set of three equations the existence of solutions. Each set of three equations admits non trivial solutions if and only if the following matrix has determinant equal to zero

$$\begin{pmatrix} -K_k & (g+2+k)K_k & (g+2+k)t^2 L_{g-k+2} \\ t^2 J_k & (g+2-k)t^2 J_k & (g+2-k)I_{g-k+2} \\ I_k & -kI_k & -kJ_{g-k+2} \end{pmatrix}.$$

After the change of variable $l = g-k+1$ so that $2 \leqslant l \leqslant g-1$, it is possible to show that the determinant is

$$-(g+2)(at^4 + bt^2 + c), \tag{1.14}$$

with

$$\begin{aligned} a &= (2g-l+3)I_{g-l+1}J_{g-l+1}L_{l+1} \\ b &= -2(g-l+1)J_{l+1}J_{g-l+1}K_{g-l+1} \\ c &= (l+1)I_{g-l+1}I_{l+1}K_{g-l+1}. \end{aligned}$$

We are interested in finding the positive values of t such that

$$at^4 + bt^2 + c = 0. \tag{1.15}$$

To simplify the notation we introduce the following three functions

$$F(v) = \left(\frac{\Gamma(\frac{1}{2}+\frac{v}{2})}{\Gamma(\frac{1}{2}-\frac{v}{2})}\right)^2 \frac{\Gamma(1-v)}{\Gamma(1+v)},$$

$$I(v) = \left(\frac{\Gamma(1-\frac{v}{2})}{\Gamma(1+\frac{v}{2})}\right)^2 \frac{\Gamma(1+v)}{\Gamma(1-v)},$$

$$L(v) = \left(\frac{\Gamma(1+\frac{v}{2})}{\Gamma(1-\frac{v}{2})}\right)^2 \frac{\Gamma(1-v)}{\Gamma(1+v)} = \frac{1}{I(v)}.$$

The discriminant $b^2 - 4ac$ of the equation (1.15), seen like an equation of degree two in the variable t^2, is negative if and only if $X = b^2/4ac < 1$. It is possible to show that

$$X = \frac{l^2}{l^2-1} F^2 \left(\frac{l}{g+1}\right) I \left(\frac{l-1}{g+1}\right) I \left(\frac{l+1}{g+1}\right). \tag{1.16}$$

S. Nayatani showed that if $2 \leqslant g \leqslant 37$, then $X < 1$ and as consequence the equation (1.15) has not any solution since its discriminant is negative. Then $\dim H(G_t) > 0$ only for $t = t_1, t_2, t_3$. Summarizing we can state (see [36] for other details):

Theorem 1.3.2. *If* $2 \leqslant g \leqslant 37$ *and* $t \in (0, +\infty)$, *then*

$$\mathrm{Nul}(G_t) = \begin{cases} 4 & \text{if} \quad t = t_1, t_2 \\ 5 & \text{if} \quad t = t_3 \\ 3 & \text{elsewhere.} \end{cases}$$

Since the extended Gauss map of the Costa-Hoffman-Meeks surfaces is exactly G_{t_2}, it is possible to state that the null space of the Jacobi operator of M_g has dimension equal to 4 for $2 \leqslant g \leqslant 37$.

Other values of t for which $\mathrm{Nul}(G_t) > 3$ are admitted only if $g \geqslant 38$. In [36] S. Nayatani conjectured these values were bigger than t_3. The proof of the conjecture and its consequences will be showed in sections 1.4 and 1.5.

1.4 The case $g \geqslant 38$

S.Nayatani proved that X is a decreasing function in the variables l,

$$x = \frac{l}{g+1}, y = \frac{l+1}{g+1}, z = \frac{l-1}{g+1}$$

with $2 \leqslant l \leqslant g-1$. We recall that we have set $s = \frac{1}{g+1}$. We know that for $l = 2$ and $g = 37$ the discriminant of the equation (1.15) is negative. For these values of l and g the variables x, y, z, s are respectively equal to $x_{max} = 2s_{max}$, $y_{max} = 3s_{max}$, $z_{max} = s_{max} = 1/38$. Then we will study the solutions of (1.15) for $i \in [0, i_{max}]$ (we call admissible values the values in these intervals) where i denotes x, y, z, s, because for bigger values of the variables the discriminant continues to be negative and so the equation (1.15) does not admit solutions.

All the solutions of (1.15), that we denote by $t_{\pm}(l,g)$, satisfy $t_{\pm}^2(l,g) = T_1 \pm T_2$, with

$$T_1 = \frac{l}{l-1} F\left(x\right) I\left(z\right) \tag{1.17}$$

and

$$T_2 = \sqrt{\left(\frac{l}{l-1}\right)^2 F^2\left(x\right) I^2\left(z\right) - \frac{l+1}{l-1} L\left(y\right) I\left(z\right).} \tag{1.18}$$

We will prove that, for all the values of l and g, such that $0 \leqslant \frac{l}{g+1} \leqslant x_{max} = \frac{2}{38}$, with $2 \leqslant l \leqslant g-1$ and $g \geqslant 38$, such that T_2 is a real number, it holds

$$t_3^2\left(s\right) < t_-^2(l,g). \tag{1.19}$$

We need study the behaviour of the functions F, I, L, F^2, I^2 that appear in (1.17) and (1.18). This aim is pursued by the use of zero order series of these functions.
The Mac-Laurin series of the functions $F(x), G(z), L(y), F^2(x), I^2(z)$ for admissible values of x, y, z are

$$F(x) = 1 + R_F(d_1x)x, \quad I(z) = 1 + R_I(d_2z)z, \quad L(y) = 1 + R_L(d_3y)y, \tag{1.20}$$

$$F^2(x) = 1 + R_{F^2}(c_1x)x, \quad I^2(x) = 1 + R_{I^2}(c_2x)x,$$

where $c_i, d_i \in (0,1)$. So we can write

$$F(x)I(z) = 1 + R_{FI}(x,z),\ F^2(x)I^2(z) = 1 + R_{F^2I^2}(x,z),\ L(y)I(z) = 1 + R_{LI}(y,z),$$

with

$$\begin{aligned} R_{FI}(x,z) &= R_F(d_1x)x + R_I(d_2z)z + R_F(d_1x)R_I(d_2z)xz, \\ R_{F^2I^2}(x,z) &= R_{F^2}(c_1x)x + R_{I^2}(c_2z)z + R_{F^2}(c_1x)R_{I^2}(c_2z)xz, \\ R_{LI}(y,z) &= R_L(d_3y)y + R_I(d_2z)z + R_I(d_2z)R_L(d_3y)zy. \end{aligned}$$

In the following $\psi(x)$ denotes the digamma function. It is related to $\Gamma(x)$, the gamma function, by

$$\psi(x) = \frac{d}{dx}\left(\ln \Gamma(x)\right).$$

For the properties of these special functions we will refer to [1].

The following proposition gives useful properties of the functions just introduced.

Proposition 1.4.1. *If $x \in [0, x_{max}]$, $z \in [0, z_{max}]$, and $y \in [0, y_{max}]$, the following assertions hold:*

1. $R_F(x) < 0$
2. $R_I(z) \leqslant 0$
3. $R_L(y) \geqslant 0$
4. $(R_F)'_x(x) > 0$
5. $min(R_I)'_z(z) = -0.095\cdots$
6. $R_{FI}(x,z) \geqslant Cx$ *with* $C = -4\ln 2$
7. $R_{LI}(y,z) \geqslant 0$
8. $R_{I^2}(z) \leqslant 0$
9. $W(x) = R_{F^2}(x) < 0$
10. $W'_x(x) > 0$, *so* $R_{F^2}(x)$ *is an increasing function*
11. $W''_{xx}(x) < 0$
12. $W'''_{xxx}(x) > 0$
13. *If we set* $Y(x) = x\,W(x)$, *then* $Y'_x(x) < 0$
14. $Y''_{xx}(x) > 0$
15. $Y'''_{xxx}(x) < 0$.

Proof.

1. $R_F(x) = F'_x(x) = F(x)\Psi_F(x)$, where

$$\Psi_F(x) = -\psi(1-x) - \psi(1+x) + \psi\left(\frac{1}{2} - \frac{x}{2}\right) + \psi\left(\frac{1}{2} + \frac{x}{2}\right).$$

We observe that

$$\Psi_F(x) = 2\sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right)x^{2k}.$$

Since $\Psi_F(0) = 2\psi\left(\frac{1}{2}\right) - 2\psi(1) = -4\ln 2$, $\psi^{(2k)}(1) < 0$ and $\psi^{(2k)}\left(\frac{1}{2}\right) = (2^{2k+1} - 1)\psi^{(2k)}(1) < 0$, if $k \geqslant 1$ (see formulas 6.4.2 and 6.4.4 of [1]), we can conclude that $\Psi_F(x) < 0$ and it is a decreasing function. Since $F(x) > 0$ then $R_F(x) < 0$ and $F(x)$ is a decreasing function.

2. $R_I(z) = I'_z(z) = I(z)\Psi_I(z)$, where

$$\Psi_I(z) = \psi(1-z) + \psi(1+z) - \psi\left(1 - \frac{z}{2}\right) - \psi\left(1 + \frac{z}{2}\right).$$

We observe that

$$\Psi_I(z) = 2\sum_{k=1}^{\infty} \frac{1}{(2k)!}\psi^{(2k)}(1)\left(1 - \frac{1}{2^{2k}}\right) z^{2k}.$$

Since $\psi^{(2k)}(1) < 0$ for $k \geqslant 1$ then $\Psi_I(z) \leqslant 0$ and it is a decreasing function. Since $I(z) > 0$ then $R_I(z) \leqslant 0$.

3. $R_L(y) = L'_y(y) = L(y)\Psi_L(y)$, where $\Psi_L(y) = -\Psi_I(y)$. Then $\Psi_L(y) \geqslant 0$ and it is an increasing function. Since $L(y) = 1/I(y) > 0$, then $R_L(y) \geqslant 0$.

4. The derivative of R_F is $F''_{xx}(x) = F(x)(\Psi_F^2(x) + (\Psi_F)'_x(x))$. Since $\Psi_F(x) < 0$ and it is a decreasing function, $\Psi_F^2(x) > 0$ and increasing. It holds $\Psi_F^2(x) \geqslant \Psi_F^2(0) = 16\ln^2 2$.

$$(\Psi_F)'_x(x) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right) x^{2k-1}.$$

All the coefficients of the series are negative (see the point 1) so $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. In particular $(\Psi_F)'_x(x) \geqslant (\Psi_F)'_x(x_{max}) = -0.19\cdots$. Since $F(x) > 0$ and it is a decreasing function we can conclude that

$$F''_{xx}(x) \geqslant F(x_{max})(\Psi_F^2(0) + (\Psi_F)'_x(x_{max})) = 6.4\cdots.$$

5. The derivative of R_I is $I''_{zz}(z) = I(z)(\Psi_I^2(z) + (\Psi_I)'_z(z))$. Since $\Psi_I(z) \leqslant 0$ and it is a decreasing function (see the point 2), $\Psi_I^2(z) \geqslant 0$ and increasing. It holds $\Psi_I^2(z) \leqslant \Psi_I^2(z_{max}) = 1.5\cdots \cdot 10^{-6}$.

$$(\Psi_I)'_z(z) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\psi^{(2k)}(1)\left(1 - \frac{1}{2^{2k}}\right) z^{2k-1}.$$

All the coefficients of the series are negative so $(\Psi_I)'_z(z) \leqslant 0$ and it is a decreasing function. In particular $(\Psi_I)'_z(z) \geqslant (\Psi_I)'_z(z_{max}) = -0.095\cdots$. Since $I(z) > 0$ and it is a decreasing function we can conclude that

$$I''_{zz} \geqslant I(z_{max})(\Psi_I^2(0) + (\Psi_I)'_z(z_{max})) = -0.095\cdots.$$

6. Since $R_F < 0$ and $R_I \leqslant 0$, it holds that

$$R_{FI}(x,z) \geqslant R_F(d_1x)x + R_I(d_2z)z,$$

where $d_i \in (0,1)$. The point 4 implies that R_F is an increasing function and we have computed the positive minimum (that we denote by m) value of its derivative. Thanks to the point 5 we have $m > |n|$, where n denotes the negative minimum value of the derivative of R_I. Now we observe that

$$R_F(d_1x)x + R_I(d_2z)z \geqslant (R_F(0) + mx)x + (R_I(0) + nz)z \geqslant$$

$$R_F(0)x + R_I(0)z + (m+n)x^2 \geqslant R_F(0)x + R_I(0)z = Cx.$$

To obtain this chain of inequalities we used the fact that $m + n > 0$ and $x \geqslant z$. Then $R_{FI} \geqslant Cx$.

7. We recall that $R_{LI}(y,z) = L(y)I(z) - 1$, $L(t) = 1/I(t)$ and

$$y = \frac{l+1}{g+1} > \frac{l-1}{g+1} = z.$$

We want to prove that $L(y)I(z) - 1 \geqslant 0$ or equivalently $L(y) \geqslant 1/I(z)$. But thanks to the point 3, we have

$$L(y) \geqslant L(z) = \frac{1}{I(z)}.$$

8. $R_{I^2}(z) = (I^2)'_z(z) = 2I^2(z)\Psi_I(z)$. From the proof of the point 2, $\Psi_I(z) \leqslant 0$ and it is a decreasing function. Since $2I^2(z) > 0$, then also $R_{I^2}(z) \leqslant 0$.

9. $W(x) = (F^2)'_x(x) = 2F^2(x)\Psi_F(x)$. In the point 1 we have observed that $\Psi_F(x)$ is a negative and decreasing function. Since $2F^2(x) > 0$, then also $W(x)$ is a negative function.

10. $W'_x(x) = F^2\left(4\Psi_F^2(x) + 2(\Psi_F)'_x(x)\right)$. Since $\Psi_F(x) < 0$ and it is a decreasing function, $\Psi_F^2(x)$ is a positive and increasing function. In the proof of the point 4 we observed that $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. Since $2(\Psi_F)'_x(x_{max}) = -0.38\cdots$ and $4\Psi_F^2(x) \geqslant 4\Psi_F^2(0) = 64\ln^2 2 = 30.74\cdots$, we can conclude that $W'_x(x) > 0$.

11. The explicit expression of W''_{xx} is

$$W''_{xx} = \frac{1}{2}F^2(x)\left(16\Psi_F^3(x) + 24\Psi_F(x)(\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x)\right).$$

In the proof of the point 1 we observed that $\Psi_F(x)$ is a negative and decreasing function. So $16\Psi_F^3(x) \leqslant 16\Psi_F^3(0) = -1024\ln^3 2 = -341.\cdots$. Thanks to the proof of the point 10 we know that $(\Psi_F)'_x(x) \leqslant 0$ and it is a decreasing function. In particular $0 \geqslant (\Psi_F)'_x(x) \geqslant (\Psi_F)'_x(x_{max}) = -0.19\cdots$. We can conclude that

$$24\Psi_F(x)(\Psi_F)'_x(x) \leqslant 24(\Psi_F)'_x(x_{max})\Psi_F(x_{max}) = 12.\cdots .$$

As for the last summand, it is negative. In fact

$$(\Psi_F)''_{xx}(x) = 2\sum_{k=1}^{\infty} \frac{1}{(2k-2)!}\left(\frac{1}{2^{2k}}\psi^{(2k)}\left(\frac{1}{2}\right) - \psi^{(2k)}(1)\right)x^{2k-2}.$$

Since all the coefficients of the series are negative, we get

$$4(\Psi_F)''_{xx}(x) \leqslant 4(\Psi_F)''_{xx}(0) = -12\zeta(3) = -14.4\cdots ,$$

where $\zeta(\cdot)$ denotes the Riemann zeta function. We can conclude that

$$16\Psi_F^3(x) + 24\Psi_F(x)(\Psi_F)'_x(x) + 4(\Psi_F)''_{xx}(x) \leqslant$$
$$\leqslant 16\Psi_F^3(0) + 24\Psi_F(x_{max})(\Psi_F)'_x(x_{max}) + 4(\Psi_F)''_{xx}(0) = -342.7\cdots .$$

That assures $W''_{xx} < 0$.

12. The explicit expression of W'''_{xxx} is

$$W'''_{xxx} = \frac{1}{4}F^2(x)\left(64\Psi_F^4 + 192\Psi_F^2(\Psi_F)'_x + 48((\Psi_F)'_x)^2 + \right.$$
$$\left. +64\Psi_F(\Psi_F)''_{xx} + 8(\Psi_F)'''_{xxx}\right).$$

We start observing that, since Ψ_F is a negative decreasing function,

$$64\Psi_F^4(x) \geqslant 64\Psi_F^4(0) = 64(4\ln 2)^4 = 3782.\cdots .$$

Since $(\Psi_F)'_x(x)$ is a not positive and decreasing function (point 10), then $192\Psi_F^2(\Psi_F)'_x$ enjoys the same property. In particular

$$192\Psi_F^2(\Psi_F)'_x \geqslant 192\Psi_F^2(x_{max})(\Psi_F)'_x(x_{max}) = -282.\cdots .$$

From the previous observations it follows that $64\Psi_F(\Psi_F)''_{xx} \geqslant 0$, $48((\Psi_F)'_x)^2 \geqslant 0$ and they are increasing functions.

As for the last summand which appears in the expression of W'''_{xxx}, we observe that

$$(\Psi_F)'''_{xxx} = 2\sum_{k=1}^{\infty} \frac{1}{(2k-1)!}\left(\frac{1}{2^{(2k+2)}}\psi^{(2k+2)}\left(\frac{1}{2}\right) - \psi^{(2k+2)}(1)\right)x^{2k-1}.$$

It is a not positive and decreasing function. So we can write

$$8(\Psi_F)'''_{xxx}(x) \geqslant 8(\Psi_F)'''_{xxx}(x_{max}) = -19.9\cdots .$$

We can conclude $W'''_{xxx}(x) > 0$. Furthermore from our observations it follows that

$$W'''_{xxx}(x) \leqslant \big(16\Psi_F^4(x_{max}) + 24((\Psi_F)'_x)^2(x_{max})$$

$$+16\Psi_F(x_{max})(\Psi_F)''_{xx}(x_{max})\big) < C_W$$

with $C_W = 1125$.

13. It holds that $Y'_x(x) = W(x) + x\,W'_x(x)$. From the points 9, 10 and 11 we know that $W(x)$ is a negative increasing function and $W'_x(x)$ is positive and decreasing for $x \in [0, x_{max}]$. So we can write $W(x) \leqslant W(x_{max}) = -4.1\cdots$ and $W'_x(x) \leqslant W'_x(0) = 64\ln^2 2 = 30.7\cdots$. Then $Y'_x(x) \leqslant W(x_{max}) + x_{max}W'_x(0) < 0$.

14. It holds that $Y''_{xx}(x) = 2W'_x(x) + x\,W''_{xx}(x)$. From the points 10, 11 and 12 we know that $W'_x(x)$ is a positive decreasing function and $W''_{xx}(x)$ is negative and increasing. So we can write $W'_x(x) \geqslant W'_x(x_{max}) = 22.\cdots$. and $W''_{xx}(x) \geqslant W''_{xx}(0) = -64\ln^3 4 - 6\zeta(3) = -177.\cdots$. Then $Y'_x(x) \geqslant 2W'_x(x_{max}) + x_{max}W''_{xx}(0) > 0$.

15. It holds that $Y'''_{xxx}(x) = 3W''_{xx}(x) + x\,W'''_{xxx}(x)$. From the points 11 and 12 we know that $W''_{xx}(x)$ is a negative increasing function and $0 < W'''_{xxx}(x) < C_W$. Then $Y'''_{xxx}(x) \leqslant 3W''_{xx}(x_{max}) + x_{max}C_W < 0$.

□

Proposition 1.4.2. *For all the values of l, x, y, z for which $T_2(l, x, y, z)$ is real, it holds that*

$$T_2(l, x, y, z) \leqslant \frac{1 + Cl^2x}{l-1},$$

where $C = -4\ln 2$.

Proof. The epression of T_2 is given by (1.18). We rewrite it in the following way

$$T_2 = \frac{1}{l-1}\sqrt{l^2F^2\,(x)\,I^2\,(z) - (l^2-1)L(y)I(z)}.$$

We start studying the case of T_2 not zero. If $1 + \bar{R}(x, y, z, l)$ is the Mac-Laurin series of the function under the square root then we can write

$$T_2 = \frac{1}{l-1}\sqrt{1 + \bar{R}(x, y, z, l)},$$

where $\bar{R}(x,y,z,l) = l^2(R_{F^2}(c_1x)x(1+R_{I^2}(c_2z)z)+R_{I^2}(c_2z)z)-(l^2-1)R_{LI}(y,z)$, and $c_1, c_2 \in (0,1)$. Thanks to the points 7,8,9 and 10 of proposition 1.4.1, we know that $R_{LI}(y,z) \geqslant 0$, $R_{I^2}(x) \leqslant 0$ and that $R_{F^2}(x)$ is a negative increasing function, so $R_{F^2}(c_1x) \leqslant R_{F^2}(x)$. We can conclude that, if we set

$$R(x,z,l) = l^2 R_{F^2}(x)x(1+R_{I^2}(c_2z)z),$$

$$\bar{R}(x,y,z,l) \leqslant l^2 R_{F^2}(c_1x)x(1+R_{I^2}(c_2z)z) \leqslant R(x,z,l),$$

then

$$T_2 = \frac{1}{l-1}\sqrt{1+\bar{R}(x,y,z,l)} \leqslant \frac{1}{l-1}\sqrt{1+R(x,z,l)}.$$

We know that

$$\sqrt{1+f(x)} = \sqrt{1+f(0)} + \frac{f'_t(t)}{2\sqrt{1+f(t)}}|_{t=cx}x,$$

where $c \in (0,1)$. If we apply this result to the function $f(x) = R(x,z,l)$, we get

$$T_2 \leqslant \frac{\sqrt{1+R(x,z,l)}}{l-1} = \frac{1}{l-1}\left(\sqrt{1+R(0,z,l)} + \frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}}|_{t=cx}x\right),$$

where $c \in (0,1)$. We observe that $R(0,z,l) = 0$. Then

$$T_2 \leqslant \frac{1}{l-1}\left(1 + \frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}}|_{t=cx}x\right).$$

The proof will be completed after having proved the following result. □

Proposition 1.4.3. *Under the same hypotheses of proposition 1.4.2*

$$\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \leqslant Cl^2,$$

where $C = -4\ln 2$.

Proof. We set $H(z,l) = l^2(1+R_{I^2}(c_2z)z) \leqslant l^2$ and $Y(t) = R_{F^2}(t)t$. From the expression of $R(t,z,l) = H(z,l)Y(t)$, it follows that $R'_t(t,z,l) = H(z,l)Y'_t(t)$. Furthermore we can write

$$\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} = \frac{H(z,l)Y'_t(t)}{2\sqrt{1+H(z,l)Y(t)}}.$$

We know from proposition 1.4.1 that $Y(t) \leqslant 0$ and $Y'_t(t) < 0$, then $R'_t(t,z,l) = H(z,l)Y'_t(t) \geqslant l^2 Y'_t(t)$, and

$$-\frac{1}{2\sqrt{1+R(t,z,l)}} \geqslant -\frac{1}{2\sqrt{1+l^2Y(t)}}.$$

We can conclude that

$$-\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \geqslant -\frac{l^2Y'_t(t)}{2\sqrt{1+l^2Y(t)}}.$$

We shall show that the function on the left side is increasing with respect to the variable t. The derivative with respect to the variable t of this function is

$$D(t,l) = -\frac{l^2}{2}\frac{Y''_{tt}\sqrt{1+l^2Y} - l^2\frac{(Y'_t)^2}{2\sqrt{1+l^2Y}}}{1+l^2Y}.$$

We want to study the sign of $D(t,l)$. We start observing that $1+l^2Y \geqslant 1+\bar{R} > 0$. So it is sufficient to prove that the quantity

$$E(t,l) = 2Y''_{tt}(1+l^2Y) - l^2(Y'_t)^2$$

is always not positive. It holds that

$$Y'_t(t) = R_{F^2}(t) + t(R_{F^2})'_t(t)$$

and

$$Y''_{tt}(t) = 2(R_{F^2})'_t(t) + t(R_{F^2})''_{tt}(t).$$

Then $Y(0) = 0$, $Y'_t(0) = R_{F^2}(0) = 2C$ and $Y''_{tt}(0) = 2(R_{F^2})'_t(0) = 8\Psi_F(0)^2 = 8C^2$. Furthermore we observe that $l \geqslant 2$. So

$$E(0,l) = 16C^2 - 4l^2C^2 \leqslant 0$$

and the equality holds if $l = 2$. The next step is to show that $E'_t(t,l) \leqslant 0$. It is possible to find the following relation

$$E'_t(t,l) = Y'''_{ttt}(1+l^2Y)$$

Observing that $1+l^2Y > 0$ and $Y'''_{ttt} < 0$ (see the point 15 of proposition 1.4.1), we can conclude that $D(t,l) \geqslant 0$ (the equality holding if $t=0$). We have showed that

$$-\frac{l^2Y'_t(t)}{2\sqrt{1+l^2Y(t)}}$$

is a non decreasing function. It gets the minimum for $t=0$ and its value is $-Cl^2$. Then

$$-\frac{R'_t(t,z,l)}{2\sqrt{1+R(t,z,l)}} \geqslant -Cl^2,$$

and the proof of proposition 1.4.3 is completed. □

To achieve the proof of proposition 1.4.2, we need to show that the statement continues to hold also for values of l, x, y, z for which $T_2 = 0$. To get this aim it is sufficient to observe that we can extend the result obtained under the hypothesis $T_2 > 0$ for continuity.

As for the first summand which appears in the expression of t_-^2, that is T_1, the following result holds.

Proposition 1.4.4. *For all the admissible values of x, z, it holds that*

$$T_1 \geqslant \frac{l}{l-1}(1 + Cx)$$

where $C = -4\ln 2$.

Proof. We recall that

$$T_1 = \frac{l}{l-1}F(x)I(z) = \frac{l}{l-1}(1 + R_{FI}(x, z)).$$

Thanks to the point 6 of proposition 1.4.1 we have $R_{FI}(x, z) \geqslant Cx$. Then the result is immediate. □

The following result gives the estimate of t_-^2.

Proposition 1.4.5. *For all the values of x, y, z for which $t_-^2 \in \mathbb{R}$, it holds*

$$t_-^2 \geqslant 1 - Clx,$$

where $C = -4\ln 2$.

We recall that $t_-^2 = T_1 - T_2$. Thanks to propositions 1.4.2 and 1.4.4 we get

$$t_-^2 \geqslant \frac{l}{l-1}(1 + Cx) + \frac{1}{l-1}(-1 - Cl^2x) =$$

$$1 + \left(\frac{Cl}{l-1} - \frac{Cl^2}{l-1}\right)x = 1 + \left(\frac{-Cl}{l-1}(l-1)\right)x = 1 - Clx.$$

□

Now we turn our attention to the function t_3. We recall that $s_{max} = \frac{1}{38}$.

Proposition 1.4.6. *For $s \in [0, s_{max}]$*

$$t_3^2(s) \leqslant 1 + \frac{7}{2}s.$$

Proof. We recall that

$$t_3^2(s) = T(s) = \frac{4}{(1-s)^2}\left(\frac{\Gamma(1+s)}{\Gamma(1-s)}\right)^3 \frac{\Gamma(1-2s)}{\Gamma(1+2s)}\left(\frac{\Gamma(3/2-s/2)}{\Gamma(1/2+s/2)}\right)^2.$$

It holds that

$$T_s'(s) = \frac{1}{(1-s)}T(s)B(s),$$

where

$$B(s) = 2 + (1-s)\left(-2\psi(1-2s) - 2\psi(1+2s) + 3\psi(1-s) + 3\psi(1+s) - \right.$$
$$\left. -\psi\left(\frac{3}{2}-\frac{s}{2}\right) - \psi\left(\frac{1}{2}+\frac{s}{2}\right)\right).$$

To complete the proof we need the following result.

Proposition 1.4.7. *If* $s \in [0, s_{max}]$ *then* $1 < B(s) < 3$.

Proof. We observe that for $s \in [0, s_{max}]$

$$0 < \psi\left(\frac{3}{2}-\frac{s}{2}\right) < \psi\left(\frac{3}{2}\right) = 0.036\cdots, \quad \frac{3}{2} < -\psi\left(\frac{1}{2}+\frac{s}{2}\right) < -\psi\left(\frac{1}{2}\right) < 2.$$

We can conclude that

$$1 < -\psi\left(\frac{1}{2}+\frac{s}{2}\right) - \psi\left(\frac{3}{2}-\frac{s}{2}\right) < 2.$$

Furthermore

$$\psi(1-s) + \psi(1+s) = 2\sum_{k\geqslant 0}\frac{\psi^{(2k)}(1)}{(2k)!}s^{2k},$$

from which it follows that

$$D(s) = -2\psi(1-2s) - 2\psi(1+2s) + 3\psi(1-s) + 3\psi(1+s) = 2\sum_{k\geqslant 0}\frac{\psi^{(2k)}(1)}{(2k)!}s^{2k}(3-2^{2k+1}).$$

If $k \geqslant 1$ then $3 - 2^{2k+1} < 0$ and $\psi^{(2k)}(1) < 0$ (see formula 6.4.2 of [1]) then

$$2\psi(1) = -2\gamma_{EM} = D(0) \leqslant D(s) \leqslant D(s_{max}) = -1.146\cdots,$$

where $\gamma_{EM} = 0.577\cdots$ is the Euler-Mascheroni constant. So

$$1 < B(s) \leqslant 2 + (1-s)(2 + D(s_{max})) < 4 + D(s_{max}) < 3.$$

□

Since $B(s) > 0$ then $T(s)$ is an increasing function and we can deduce that

$$T'(s) = \frac{1}{1-s} T(s) B(s) \leqslant \frac{3}{1-s_{max}} T(s_{max}) < 7/2.$$

The Mac-Laurin series of order zero of $T(s)$ is $1 + T'_s(cs)s$, where $c \in (0,1)$. So it is immediate to conclude that

$$T(s) \leqslant 1 + \frac{7}{2}s.$$

□

The following proposition shows that the eventual solutions $t_+(l,g) \geqslant t_-(l,g)$ of the equation (1.15) are always bigger than t_3.

Proposition 1.4.8. *$t_3(\frac{l}{g+1}) < t_-(l,g)$ for $g \geqslant 1$ and $2 \leqslant l \leqslant g-1$ such that $t_-(l,g) \in \mathbb{R}$.*

Proof. From our observations, it is sufficient to show that $t_3^2(s) < t_-^2(l,g)$ holds for $g \geqslant 38$. Propositions 1.4.5 and 1.4.6 assure that

$$t_-^2 \geqslant 1 - Clx,$$

$$t_3^2(s) \leqslant 1 + \frac{7}{2}s.$$

We recall that $x = ls$ and $2 \leqslant l \leqslant g-1$. Then the result is obvious. □

1.5 The index and the nullity of the Costa-Hoffman-Meeks surfaces

We start recalling some results described in previous sections. We denoted by G_t, $t \in (0,+\infty)$, a deformation of the map G defined by (1.5). Thanks to theorem 1.3.2 $\mathrm{Nul}(G_t) > 3$ only if t assumes special values. If $2 \leqslant g \leqslant 37$ these values are t_1, t_2, t_3. If $g \geqslant 38$ there are additional values. They are the positive solutions of the equation (1.15). We denoted them by $t_\pm(l,g)$, where $2 \leqslant l \leqslant g-1$, and for definition $t_+ \geqslant t_-$. In previous section we have proved that the inequality $t_3(s) < t_-(l,g)$ holds. S. Nayatani showed in [36] that $t_3 > t_2$ for $g \geqslant 2$. We can conclude that no one of the $t_\pm$ can be equal to t_2. As consequence $\mathrm{Nul}(G_{t_2})$ continues to be equal to 4 also for $g \geqslant 38$, because $\dim H(G_{t_2})$ is equal to 1 for all $g \geqslant 2$.

We recall that M_g denotes the Costa-Hoffman-Meeks surface of genus g. Since the extended Gauss map of M_g is exactly G_{t_2}, and taking into account the result of S. Nayatani about the Costa surface (theorem 1.2.1) showed in [37] we have proved the following result.

Theorem 1.5.1. *The null space of the Jacobi operator of M_g has dimension equal to* 4 *for all* $g \geqslant 1$.

Using the definition of non degeneracy given in [41], we can also rephrase this result giving the following statement.

Corollary 1.5.2. *The surface M_g is non degenerate for all $g \geqslant 1$.*

Now we turn our attention to the results relative to the index of the map G_t. We recall that Σ_g denotes the compactification of M_g. S. Nayatani proved in [36] the following result.

Theorem 1.5.3. *Let $G : \Sigma_g \to S^2$ be the holomorphic map defined by (1.5). If $2 \leqslant g \leqslant 37$, then*

$$\mathrm{Ind}(G_t) = \begin{cases} 2g+3 & if\ t \leqslant t_1,\ t_2 \leqslant t < t_3,\ t > t_3, \\ 2g+4 & if\ t_1 < t < t_2, \\ 2g+2 & if\ t = t_3. \end{cases}$$

For $t = t_1, t_2, t_3$ we have $\mathrm{Nul}(G_t) > 3$, that is the kernel of L_{G_t} contains at least one non linear element. The eigenvalue associated to this function is zero. The proof of theorem 1.5.3 is based on the analysis of the behaviour of these null eigenvalues under a variation of the value of t. Let's suppose that $t \neq t_1, t_2, t_3$ but remaining in a neighbourhood of one of these values. For example we choose t_1. Then the eigenvalue E that before the variation was associated to a non linear element of $N(G_{t_1})$, is not more equal to zero. To compute the index, it was necessary to understand which is the sign assumed by E, respectively for $t > t_1$ and $t < t_1$. Similar considerations are applicable to the eigenvalues associated with t_2 and t_3. See [36] for the details.

If $g \geqslant 38$, we have just proved that the other values for which $\mathrm{Nul}(G_t) > 3$ are bigger than t_3. The presence of these additional values $t_\pm$ does not influence the value of $\mathrm{Ind}(G_t)$ if $t \leqslant t_3$. In other terms theorem 1.5.3 continues to hold for $g \geqslant 38$ if we consider $0 < t \leqslant t_3$. Taking into account also the result of S. Nayatani about the Costa surface ($g = 1$) showed in [37], we can give the following statement

Theorem 1.5.4. *For all $g \geqslant 1$ the index of the Gauss map of M_g is equal to $2g+3$.*

1.6 Appendix

This section contains some additional details of the computations made by S. Nayatani.

1.6.1 Divisors and Riemann-Roch theorem

Here we introduce some definitions and concepts of the algebraic geometry. See for example [4].

Let Σ_g be a compact Riemann surface of genus g. A divisor on Σ_g is a finite formal sum of integer multiples of points of Σ_g,

$$D = \sum_{x \in \Sigma_g} n_x x, \quad n_x \in \mathbb{Z}, n_x = 0 \quad \text{for almost all} \quad x.$$

The set of the divisors on Σ_g is denoted by $\mathrm{Div}(\Sigma_g)$. The degree of a divisor is the integer $\deg(D) = \sum n_x$.

Let $\mathbb{C}(\Sigma_g)$ be the field of the meromorphic functions on Σ_g and let $\mathbb{C}(\Sigma_g)^*$ be its multiplicative group of nonzero elements. Every $f \in \mathbb{C}(\Sigma_g)^*$ has a divisor

$$\mathrm{div}(f) = \sum \nu_x(f) x,$$

where $\nu_x(f)$ denotes the order of f at x.

Let ω be a nonzero meromorphic differential n-form on Σ_g. Then ω has a local representation $\omega_x = f_x(z)(dz)^n$ about each point x of Σ_g, where z is the local coordinate about x and $f_x(z) \in \mathbb{C}(\Sigma_g)^*$. So we can define in a natural way $\nu_x(\omega) = \nu_0(f_x)$ and also associate a divisor with a differential form:

$$\mathrm{div}(\omega) = \sum \nu_x(\omega) x.$$

A canonical divisor on Σ_g is a divisor of the form $\mathrm{div}(\omega)$ where ω is a nonzero meromorphic differential form.

Let $D \in \mathrm{div}(\Sigma_g)$. We denote by $H^{0,n}(D)$ the vector space of the meromorphic differential n-forms ω such that

$$\mathrm{div}(\omega) + D \geqslant 0.$$

In other terms, if $D = \mathrm{div}(\eta)$, with η differential form with local representation $\eta_x = g_x(z)(dz)^n$, then the elements of $H^{0,n}(D)$ are the differential forms ω having a local representation $\omega_x = f_x(z)(dz)^n$ with $f_x \in \mathbb{C}(\Sigma_g)$ vanishing to high enough order to make the product $f \cdot g$ holomorphic. We set $\dim H^{0,n}(D) = \ell(D)$.

We are ready to state the following result.

Theorem 1.6.1 (Riemann-Roch). *Let Σ_g be a compact Riemann surface of genus g. Let k_{Σ_g} be a canonical divisor on Σ. Then for any divisor $D \in \mathrm{Div}(\Sigma_g)$,*

$$\ell(D) = \deg(D) - g + 1 + \ell(k_{\Sigma_g} - D).$$

The next result gives information about the canonical divisor and a simpler version of Riemann-Roch theorem for divisors of large enough order.

Corollary 1.6.2. *Let* $\Sigma_g, g, D, k_{\Sigma_g}$ *as above.*

- $\deg(k_{\Sigma_g}) = 2g - 2$,
- *If* $\deg(D) > 2g - 2$ *then* $\ell(k_{\Sigma_g} - D) = 0$. *Equivalently* $\ell(D) = \deg(D) - g + 1$.

1.6.2 The determination of a basis of differential forms with null residue at the ramification points

The ramification points (or branch points) of $\gamma(w) = w$ are the zeroes of

$$\frac{dw}{dz} = \frac{g+2}{g+1}\frac{z^{g-1}(z^2 - A^2)}{w^g} = \frac{g+2}{g+1}\frac{z^{g-1}(z^2 - A^2)}{(z^g(z^2-1))^{\frac{g}{g+1}}},$$

with $A = \sqrt{\frac{g}{g+2}}$, where g denotes the genus, the pole of γ and the origin of $\mathbb{C}^2$. That is $Q_0 = (0,0)$, $P_\infty = (\infty, \infty)$, $P_m = (A, B_m)$ and $S_m = (-A, C_m)$ for $m = 0, \ldots, g$, where B_m, C_m denote, respectively, the m-th complex value of $\sqrt[g+1]{A^g(A^2-1)}$ and $\sqrt[g+1]{(-A)^g(A^2-1)}$. We have set $P_\pm = (\pm 1, 0)$. We recall that

$$\hat{H}(G) = \left\{\sigma \in H^0(2k_{\Sigma_g} + R(G)) \,|\, Res_{r_i}\frac{\sigma}{dw} = 0, i = 1, \ldots, \mu\right\}, \tag{1.21}$$

where k_{Σ_g} is a canonical divisor of Σ_g and $R(G) = \sum_1^\mu r_i$ is the ramification divisor of G. In our case it is given by $R(G) = Q_0 + P_\infty + \sum_{m=0}^g (P_m + S_m)$. Furthermore it holds $\hat{H}(G) = \hat{H}(G_t)$.

As for the canonical divisor k_{Σ_g}, we consider $k_{\Sigma_g} = (g-1)P_+ + (g-1)P_-$. We observe that $\deg(k_{\Sigma_g}) = 2g - 2$ like stated by corollary 1.6.2.

To study the space $\hat{H}(G_t)$ we need understand which are the elements of the space $H^0(2k_{\Sigma_g} + R(G))$. Taking into account the definitions of k_{Σ_g} and $R(G)$, then $2k_{\Sigma_g} + R(G) = 2(g-1)P_+ + 2(g-1)P_- + Q_0 + P_\infty + \sum_{m=0}^g P_m + \sum_{m=0}^g S_m$. Between the quadratic differentials σ that are in $H^0(2k_{\Sigma_g} + R(G))$, we consider ones having one of the following forms:

$$z^k w^j \left(\frac{dz}{w}\right)^2, \tag{1.22}$$

$$z^k w^j \frac{1}{z \pm A}\left(\frac{dz}{w}\right)^2. \tag{1.23}$$

In fact from the definition of H^0, it follows that the quadratic differentials to consider can have a pole of order 0 (differentials of type (1.22)) or of order 1 (differentials of type (1.23)) at P_m and S_m for $k = 0, \ldots, g$. We will determine separately which are the differential forms of type (1.22) and (1.23) belonging to $H^0(2k_{\Sigma_g}+R(G))$. To select the differential forms of type (1.23) it is convenient to introduce an auxiliar divisor.

$$D = Q_0 + (g+2)P_\infty + 2(g-1)P_+ + 2(g-1)P_-.$$

Actually to determine the differential forms of type (1.23) which belong to $H^0(2k_{\Sigma_g}+R(G))$ is equivalent to look for the differential forms of type (1.22) which are in $H^0(D)$. We observe that the elements of the vector space $H^0(D)$ after the multiplication by the factor $z \pm A$ are elements of $H^0(2k_{\Sigma_g}+R(G))$. It is necessary to remark that to obtain a basis of $H^0(2k_{\Sigma_g}+R(G))$, we will not take into account the differentials of $H^0(2k_{\Sigma_g}+R(G))$ that can be constructed from an element of $H^0(D)$ as described above. Otherwise the number of the founded differential forms would exceed the dimension of $H^0(2k_{\Sigma_g}+R(G))$, that we can compute as follows. We observe that $\deg(2k_{\Sigma_g}+R(G)) = 6g$. Then thanks to corollary of Riemann-Roch theorem 1.6.2 we conclude that $\dim H^0(2k_{\Sigma_g}+R(G)) = 5g+1$. So the basis we are looking for counts $5g+1$ elements. From the observations made above we can deduce that among the forms of type (1.22), we will consider ones which satisfy the following conditions

$$\begin{cases} k(g+1)+jg \geqslant -1, \\ j \geqslant -2(g-1), \\ -k(g+1)-j(g+2) \geqslant -1. \end{cases}$$

These relations assure that a differential form ω of type $z^k w^j \left(\frac{dz}{w}\right)^2$, satisfies $\operatorname{div}(\omega) + 2k_{\Sigma_g} + R(G) \geqslant 0$. These differentials can be classified in three families. Each family is characterized by particular values of l and k. That is

1. $j = -g+1, \dots, 0, 1$ and $k = -j$,
2. $j = 2-2g, \dots, -g$ and $k = -j$,
3. $j = 2-2g, \dots, -g$ and $k = -j-1$.

As for the forms of type (1.23) we shall consider only the ones which satisfy

$$\begin{cases} k(g+1)+jg \geqslant -1, \\ j \geqslant -2(g-1), \\ -k(g+1)-j(g+2) \geqslant -(g+2). \end{cases}$$

These relations assure that a differential form of type $z^k w^j \frac{1}{z\pm A}\left(\frac{dz}{w}\right)^2$, satisfies $\operatorname{div}(\omega)+D \geqslant 0$. We obtain that $j = -g+1, \dots, 0, 1$ and $k = -j+1$.

Since we are looking for a basis of a vector space we can replace each couple of differentials $\frac{f}{z-A}\left(\frac{dz}{w}\right)^2$, $\frac{f}{z+A}\left(\frac{dz}{w}\right)^2$ by an appropriate linear combination. We observe that

$$\frac{1}{z-A} \pm \frac{1}{z+A} = \begin{cases} \eta_1 = \frac{z}{z^2-A^2} \\ \eta_2 = \frac{1}{z^2-A^2}. \end{cases}$$

So in the following we will work with the forms $f\eta_1\left(\frac{dz}{w}\right)^2$ and $f\eta_2\left(\frac{dz}{w}\right)^2$, where $f = z^k w^j$ as described above.

The $5g+1$ quadratic differentials we have found forms a basis of $\hat{H}(G_t)$. The last step is to divide each elements of this basis by dw. After simple algebraic manipulations, we obtain the following $5g+1$ differential 1-forms:

$$\begin{array}{lll}
\frac{w^k}{z^{k-1}}\frac{dz}{(z^2-A^2)^2} & \text{for} & k=-1,0,\ldots,g-1, \\
\frac{w^k}{z^k}\frac{dz}{(z^2-A^2)^2} & \text{for} & k=-1,0,\ldots,g-1, \\
\frac{w^k}{z^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=-1,0,\ldots,g-1, \\
\frac{z^k}{w^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=1,\ldots,g-1, \\
\frac{z^{k-1}}{w^{k+1}}\frac{dz}{(z^2-A^2)} & \text{for} & k=1,\ldots,g-1.
\end{array} \tag{1.24}$$

Now it is necessary to select the 1-forms having residue equal to zero at the points Q_0, P_m and S_m with $m=0,\ldots,g$. Thanks to the properties of symmetry of the surface it is sufficient to verify the null residue condition at the points Q_0, $P_1 = (A, e^{\frac{2\pi i}{g+1}}\sqrt[g+1]{A^g(A^2-1)})$. In fact from the coordinates of the points P_m and S_m, we can deduce that for each $Q \in \{P_m, S_m, k=0,\ldots,g\}$ there exists $n \in \{0,\ldots,2g+1\}$ such that $Q = \lambda^n(P_1)$, where λ is the conformal diffeomorphism described in lemma 1.3.1. So we can state that the residue of an arbitrary form ω at the point Q is related to the residue at P_1 by

$$\mathrm{Res}_Q\omega = \mathrm{Res}_{P_1}(\lambda^{n-1})^*\omega.$$

Applying this result to the differential forms of the list (1.24) and using the the definition (1.4) of λ, it is easy to obtain that $\mathrm{Res}_Q\omega$ is equal to $\mathrm{Res}_{P_1}\omega$ times a power of $\pm\rho$. So if $\mathrm{Res}_{P_1}\omega = 0$ then $\mathrm{Res}_Q\omega = 0$.

Thanks to algebraic manipulations inspired by the simpler cases where $g = 2, 3$, it is possible to find $3g$ linear independent differential forms satisfying the null residue condition. They constitute the wanted basis.

$$\omega_k^{(1)} = \frac{z^{k-1}}{w^k}\frac{dz}{w} \quad \text{for} \quad k=1,\ldots,g-1,$$

$$\omega_k^{(2)} = \frac{z^{k-1}((k-2)z^2-kA^2)}{w^k(z^2-A^2)^2}dz \quad \text{for} \quad k=0,\ldots,g,$$

$$\omega_k^{(3)} = \frac{z^{k-1}((k-2)z^2-kA^2)}{w^{k+1}(z^2-A^2)^2}dz \quad \text{for} \quad k=0,\ldots,g-1.$$

1.6.3 The equations equivalent to the condition of existence of a branched minimal surface.

Let ω_1 and ω_2 two meromorphic differential forms on Σ_g. We write $\omega_1 \sim \omega_2$ if there exists a meromorphic function f on Σ_g such that $\omega_2 = \omega_1 + df$. It is possible to prove that:

$$\omega_k^{(2)} \sim -\frac{k(g+2)}{2(g+1)} \frac{z^{k-1}}{w^k} dz \quad \text{for} \quad k = 0, \dots, g,$$

$$\omega_k^{(3)} \sim -\frac{(g+2)(g+k+2)}{2(g+1)} \frac{z^{k-1}}{w^{k+1}} dz \quad \text{for} \quad k = 0, \dots, g-1.$$

Using these relations we get:

$$\int_{\tilde{\beta}} \omega_k^{(1)} = -2i \sin \frac{(k+1)\pi}{g+1} K_k, \quad \int_{\tilde{\beta}} \omega_k^{(2)} = -\frac{(g+2)k}{2(g+1)} 2i \sin \frac{k\pi}{g+1} I_k,$$

$$\int_{\tilde{\beta}} \omega_k^{(3)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k+1)\pi}{g+1} K_k,$$

$$\int_{\tilde{\beta}} \gamma\omega_k^{(1)} = 2i \sin \frac{k\pi}{g+1} I_k, \quad \int_{\tilde{\beta}} \gamma\omega_k^{(2)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k-1)\pi}{g+1} J_k,$$

$$\int_{\tilde{\beta}} \gamma\omega_k^{(3)} = -\frac{(g+2)k}{2(g+1)} 2i \sin \frac{k\pi}{g+1} I_k,$$

$$\int_{\tilde{\beta}} \gamma^2\omega_k^{(1)} = 2i \sin \frac{(k-1)\pi}{g+1} J_k, \quad \int_{\tilde{\beta}} \gamma^2\omega_k^{(2)} = \frac{(g+2)(2g+4-k)}{2(g+1)} 2i \sin \frac{(k-2)\pi}{g+1} L_k,$$

$$\int_{\tilde{\beta}} \gamma^2\omega_k^{(3)} = \frac{(g+2)(g+2-k)}{2(g+1)} 2i \sin \frac{(k-1)\pi}{g+1} J_k.$$

We recall that we must impose that $\omega = \sum_0^{g-1} c_k^{(1)} \omega_k^{(1)} + \sum_0^g c_k^{(2)} \omega_k^{(2)} + \sum_0^{g-1} c_k^{(3)} \omega_k^{(3)}$, where $c_k^{(i)} \in \mathbb{C}$, satisfies

$$\int_\alpha \omega = t^2 \overline{\int_\alpha \gamma^2(w)\omega}, \quad Re \int_\alpha \gamma(w)\omega = 0$$

for $\alpha = \lambda^l \circ \tilde{\beta}$ for $l = 0, \dots, 2g-1$. Now it is convient to introduce some additional notation.

Let

$$\mathcal{L} = \begin{bmatrix} \mathcal{R}_\theta & 0 \\ 0 & 1 \end{bmatrix} \tag{1.25}$$

where $\mathcal{R}_\theta$ is the rotation in the plane by $\theta = g\pi/(g+1)$.
If we denote $\Phi(\omega) = (1-\gamma^2, i(1+\gamma^2), 2\gamma)\omega$, then it is possible to prove

$$\int_{\lambda^l \circ \tilde{\beta}} \Phi(\omega) = \int_{\tilde{\beta}} \lambda^* \Phi(\omega).$$

Since we want to apply this last relation to the differential form ω, it is convenient to remark that:

$$\lambda^*\Phi(\omega_k^{(1)}) = (-1)^k \rho^{-k} \mathcal{L}\Phi(\omega_k^{(1)}),$$
$$\lambda^*\Phi(\omega_k^{(2)}) = (-1)^k \rho^{-k+1} \mathcal{L}\Phi(\omega_k^{(2)}),$$
$$\lambda^*\Phi(\omega_k^{(3)}) = (-1)^k \rho^{-k} \mathcal{L}\Phi(\omega_k^{(3)}),$$

where $\rho = e^{i\frac{g\pi}{g+1}}$. Then the equations

$$\mathrm{Re}\int_{\lambda^l \circ \tilde{\beta}} (1 - t^2\gamma^2, i(1 + t^2\gamma^2))\omega = 0, \quad \text{for} \quad l = 0, \ldots, 2g-1,$$

are equivalent to:

$$Im\left[\sum_{k=0}^{g-1}\{(-1)^k\rho^{-k}\}^l f_k + \sum_{k=1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l p_k\right] =$$

$$t^2 Im\left[\sum_{k=0,k\neq 1}^{g-1}\{(-1)^k\rho^{-k}\}^l h_k + \sum_{k=0,k\neq 2}^{g}\{(-1)^k\rho^{-(k-1)}\}^l q_k\right]$$

$$Re\left[\sum_{k=0}^{g-1}\{(-1)^k\rho^{-k}\}^l f_k + \sum_{k=1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l p_k\right] =$$

$$-t^2 Re\left[\sum_{k=0,k\neq 1}^{g-1}\{(-1)^k\rho^{-k}\}^l h_k + \sum_{k=0,k\neq 2}^{g}\{(-1)^k\rho^{-(k-1)}\}^l q_k\right],$$

$l = 0, \ldots, 2g-1$. These last equations can be arranged as in the systems (1.10) and (1.11). The equations

$$\mathrm{Re}\int_{\lambda^l \circ \tilde{\beta}} 2t\gamma\omega = 0, \quad \text{for} \quad l = 0, \ldots, 2g-1,$$

are equivalent to:

$$Im\left[\sum_{k=1}^{g-1}\{(-1)^k\rho^{-k}\}^l d_k + \sum_{k=0,k\neq 1}^{g}\{(-1)^k\rho^{-(k-1)}\}^l e_k\right] = 0,$$

$l = 0, \ldots, 2g-1$. These last equations can be arranged as in the systems (1.12) and (1.13).

Chapter 2

A Costa-Hoffman-Meeks type surface in $\mathbb{H}^2 \times \mathbb{R}$

2.1 Introduction

In the last years the study of the minimal surfaces in the product spaces $M \times \mathbb{R}$ with $M = \mathbb{H}^2, \mathbb{S}^2$ has been becoming more and more active. The development of the theory of the minimal surfaces in these spaces started with [44] by H. Rosenberg and continued with [33] and [32] by W. H. Meeks and H. Rosenberg. In [39] B. Nelli and H. Rosenberg showed the existence in $\mathbb{H}^2 \times \mathbb{R}$ of a rich family of examples including helicoids, catenoids and, solving Plateau problems, of higher topological type examples inspired by the theory of minimal surfaces in $\mathbb{R}^3$. In [10] L. Hauswirth constructed and classified the minimal surfaces foliated by horizontal constant curvature curves in $M \times \mathbb{R}$, where M is $\mathbb{H}^2, \mathbb{R}^2$ or $\mathbb{S}^2$. Other examples of minimal surfaces of genus 0 in these product manifolds are described by R. Sa Earp and E. Toubiana in [46].

C. Costa in [2, 3] and D. Hoffman and W.H. Meeks in [14], [15] and [16] described in $\mathbb{R}^3$ a minimal surface of genus $k \geqslant 1$, finite total curvature with two ends asymptotic to the two ends of a catenoid and a middle end asymptotic to a plane. We shall denote the Costa-Hoffman-Meeks surface of genus $k \geqslant 1$ by M_k.

The aim of this work is to show the existence in the space $\mathbb{H}^2 \times \mathbb{R}$ of a family of surfaces inspired to M_k. We shall prove the following result

Theorem 2.1.1. *For all $k \geqslant 1$ there exists in $\mathbb{H}^2 \times \mathbb{R}$ a minimal surface of genus k, finite total extrinsic curvature with three horizontal ends: two catenoidal type ends and a middle planar end.*

We shall observe that it is more convenient to construct a minimal surface enjoying the same properties mentioned in the statement of theorem in the riemannian manifold $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$ where $g_{hyp} = \frac{dx_1^2+dx_2^2}{(1-x_1^2-x_2^2)^2} + dx_3^2$. It is usually denoted by $\mathbb{M}^2(-4) \times \mathbb{R}$, to point out that the sectional curvature of $\mathbb{D}^2 \times \{0\}$ endowed with the metric $\frac{dx_1^2+dx_2^2}{(1-x_1^2-x_2^2)^2}$ equals -4. We observe that $\mathbb{H}^2 = \mathbb{M}^2(-1)$. Once constructed this surface it is easy to obtain by a diffeomorphism the wanted minimal surface in $\mathbb{H}^2 \times \mathbb{R}$.

The main result is proved by a gluing procedure (see for example [11]) usually adopted to construct in $\mathbb{R}^3$ new examples starting from known minimal surfaces. We consider a scaled version of a compact part of a Costa-Hoffman-Meeks type surface, such that it can be contained in a cylindrical neighbourhood of $\{0,0\} \times \mathbb{R} \subset \mathbb{M}^2(-4) \times \mathbb{R}$ of sufficiently small radius. Actually it's possible to prove that, in the same set, the mean curvature of such a surface with respect the metric g_{hyp}, up to an infinitesimal term, equals the euclidean one. We glue the surface described above along its three boundary curves to two minimal graphs that are respectively asymptotic to an upper half catenoid and a lower half catenoid defined in $\mathbb{M}^2(-4) \times \mathbb{R}$ and to a minimal graph about $\mathbb{M}^2(-4) \times \{0\}$ which goes to zero in a neighbourhood of $\partial_\infty \mathbb{M}^2(-4) \times \{0\}$. The existence of these surfaces is proven in sections 2.5 and 2.7.

2.2 Preliminaries

In this work we shall consider the unit disk model for $\mathbb{H}^2$. Let (x_1, x_2) denote the coordinates in the unit disk $\mathbb{D}^2$ and x_3 the coordinate in $\mathbb{R}$. Then the space $\mathbb{D}^2 \times \mathbb{R}$ is endowed with the metric

$$g_{\mathbb{H}^2 \times \mathbb{R}} = \frac{4(dx_1^2 + dx_2^2)}{(1 - x_1^2 - x_2^2)^2} + dx_3^2.$$

As mentioned in the introduction, one of the surfaces involved in the gluing procedure is a compact part of a scaled version of the Costa-Hoffman-Meeks surface. That is a minimal surface in $\mathbb{R}^3$ endowed with the euclidean metric g_0. To simplify as much as possible the proof of main theorem, it is convenient to consider a riemannian manifold endowed with a metric more similar to g_0 than the standard metric of $\mathbb{H}^2 \times \mathbb{R}$. The best choice is

$$g_{hyp} = \frac{dx_1^2 + dx_2^2}{(1 - x_1^2 - x_2^2)^2} + dx_3^2,$$

because $g_{hyp} \to g_0$ if $(x_1, x_2) \to (0, 0)$. This is the reason which induces us to give a proof of theorem 2.1.1 working in the riemannian manifold $\mathbb{M}^2(-4) \times \mathbb{R}$. Now we suppose having shown the existence of a minimal surface in this riemannian manifold. We need to show how it is possible to obtain a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$. Let $\bar{g}$ be the metric defined on $\mathbb{D}^2 \times \mathbb{R}$ by

$$\bar{g} = 4g_{hyp} = \frac{4(dx_1^2 + dx_2^2)}{(1 - x_1^2 - x_2^2)^2} + 4dx_3^2.$$

We consider the map $f : (\mathbb{D}^2 \times \mathbb{R}, g_{\mathbb{H}^2\times\mathbb{R}}) \to (\mathbb{D}^2 \times \mathbb{R}, \bar{g})$ defined by

$$(x_1, x_2, x_3) \to \left(x_1, x_2, \frac{x_3}{2}\right). \tag{2.1}$$

It is easy to see that it is an isometric embedding. That is the pull-back of the metric $\bar{g}$ by f equals $g_{\mathbb{H}^2\times\mathbb{R}}$. So if Σ is a minimal surface in $(\mathbb{D}^2 \times \mathbb{R}, \bar{g})$, then the image of Σ by f^{-1} is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$.

Now we turn our attention to the riemannian manifold $\mathbb{M}^2(-4) \times \mathbb{R}$, mentioned in the introduction. In the following we shall adopt the simplified notation $\mathbb{M}^2 \times \mathbb{R}$. We recall that the metric $\bar{g}$ has been defined as $4g_{hyp}$, being g_{hyp} the metric of $\mathbb{M}^2 \times \mathbb{R}$. As consequence the mean curvature of a surface Σ in $\mathbb{M}^2 \times \mathbb{R}$ equals the mean curvature of Σ in $(\mathbb{D}^2 \times \mathbb{R}, \bar{g})$ multiplied by 4. So if a surface is minimal in $\mathbb{M}^2 \times \mathbb{R}$, also it is minimal with respect to the metric $\bar{g}$.

We can conclude that if Σ is a minimal surface in $\mathbb{M}^2 \times \mathbb{R}$, then $f^{-1}(\Sigma)$ is a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$.

Remark 2.2.1. *To prove theorem 2.1.1 we shall need consider spaces of functions invariant under the actions of the isometries of* $\mathbb{R}^3$ *which let invariant the Costa-Hoffman-Meeks surface (the rotation about the vertical coordinate axis* x_3*, the reflection with respect the horizontal plane* $x_3 = 0$ *and the vetical plane* $x_2 = 0$*). These are isometries of* $\mathbb{M}^2 \times \mathbb{R}$ *as well. So we will continue using the same language as we are in* $\mathbb{R}^3$.

2.3 Minimal graphs in $\mathbb{M}^2 \times \mathbb{R}$

We denote by H_u the mean curvature of the graph of the function u over a domain in $\mathbb{D}^2$. Its expression is

$$2H_u = F\text{div}\left(\frac{\nabla u}{\sqrt{1 + F|\nabla u|^2}}\right), \tag{2.2}$$

where $F = (1 - x_1^2 - x_2^2)^2 = (1 - r^2)^2$ and div denotes the divergence in $\mathbb{R}^2$. For the details of the computation see subsection 2.12.3.

Let Σ_u be the graph of the function u. In this section we want to obtain the expression of the mean curvature of the surface Σ_{u+v} that is the graph of the function v over Σ_u and close to it. We shall show how it follows from (2.2) that the linearized mean curvature, that we denote with L_u, is given locally by:

$$L_u v := F\text{div}\left(\frac{\nabla v}{\sqrt{1 + F|\nabla u|^2}} - F\nabla u \frac{\nabla u \cdot \nabla v}{\sqrt{(1 + F|\nabla u|^2)^3}}\right). \tag{2.3}$$

Furthermore we shall give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is H_u. In the following we shall restrict our attention to two cases: the plane (in section 2.5), that is $u=0$, and (in section 2.7) a part of catenoid defined on the domain $\{(r,\theta) \in \mathbb{M}^2 \,|\, r \in [r_\varepsilon, 1]\}$, where $r_\varepsilon = \varepsilon/2$.

Here we shall show that:

$$2H_{u+v} = 2H_u + L_u v + F Q_u(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v), \tag{2.4}$$

where Q_u has bounded coefficients if $r \in [r_\varepsilon, 1]$ which satisfies

$$Q_u(0,0) = \nabla Q_u(0,0) = 0.$$

To show this, we start observing that:

$$\frac{1}{\sqrt{1+F|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+F|\nabla u|^2}} - F\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}} + Q_{u,1}(v). \tag{2.5}$$

$Q_{u,1}(v)$ has the following expression

$$\frac{-F|\nabla v|^2}{(1+F|\nabla(u+\bar{t}v)|^2)^{3/2}} + \frac{3F^2\left(\nabla u \cdot \nabla v + \bar{t}|\nabla v|^2\right)^2}{(1+F|\nabla(u+\bar{t}v)|^2)^{5/2}}, \tag{2.6}$$

with $\bar{t} \in (0,1)$, and it satisfies $Q_{u,1}(0) = \nabla Q_{u,1}(0) = 0$. To prove (2.5) it's sufficient to set

$$f(t) = \frac{1}{\sqrt{1+F|\nabla(u+tv)|^2}}$$

and to write down the Taylor's series of order one of this function and to evaluate it in $t=1$. That is $f(1) = f(0) + f'(0) + \frac{1}{2}f''(\bar{t})$, with $\bar{t} \in (0,1)$. We insert (2.5) in the expression that defines $2H_{u+v}$ to get

$$F\mathrm{div}\left(\frac{\nabla(u+v)}{\sqrt{1+F|\nabla u|^2}} - F\nabla(u+v)\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}} + \nabla(u+v)Q_{u,1}(v)\right) =$$

$$2H_u + F\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+F|\nabla u|^2}} - F\nabla u\frac{\nabla u \cdot \nabla v}{\sqrt{(1+F|\nabla u|^2)^3}}\right) + FQ_u(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v).$$

Since we assume that Σ_u is a minimal surface, we shall consider $H_u = 0$.

Remark 2.3.1. *The minimal surfaces in the families we shall construct in sections 2.5 and 2.7, have finite total extrinsic curvature. These minimal surfaces are graphs about the domain* $\{(r,\theta) \in \mathbb{M}^2 \,|\, r \in [r_\varepsilon, 1]\}$ *of functions of class* $\mathcal{C}^{2,\alpha}$. *The total extrinsic curvature of the graph* S *of a function* u *defined on* $\mathbb{M}^2$, *is the integral of the extrinsic curvature, that is*

$$\int_S K_{ext} dA = \int_S \frac{II}{I} dA, \tag{2.7}$$

where I, II *denote the determinants of the first and of the second differential form. It hold that* $II = b_{11}b_{22} - b_{12}^2$, $I = g_{11}g_{22} - g_{12}^2$, $dA = \sqrt{I}$. *For the expressions of the coefficients of the first and of the second differential form see subsection 2.12.3. Once their expressions replaced in* (2.7) *it is clear that, taking into account that* u *is a* $\mathcal{C}^{2,\alpha}$ *class function,* $\int_S K_{ext} dA$ *is bounded. This observation allows us to state that this property holds also for the surface obtained by a gluing procedure in section 2.11. In fact the total extrinsic curvature of this last surface equals the sum of the total extrinsic curvature of the surfaces glued together: that is a compact piece of a Costa-Hoffman-Meeks type surface and three minimal graphs about the domain described above. Because of the compactness, the contribution to the total curvature of the piece of the Costa-Hoffman-Meeks type example is bounded. Then the result follows immediately, taking into account the observation made above concerning the graph of* $\mathcal{C}^{2,\alpha}$ *class functions about* $\mathbb{M}^2$.

2.4 The mapping properties of the Laplace operator

Now we restrict our attention to the case of the minimal surfaces close to $\mathbb{M}^2 \times \{0\}$, that is the graph of the function $u = 0$. In this case we obtain immediately from (2.3) that $L_{u=0} = F\Delta_0$, where Δ_0 denotes the Laplacian in the flat metric g_0 of the unit disk $\mathbb{D}^2$.

In this section we shall study the mapping properties of Δ_0. In the following we shall use the polar coordinates (r,θ). In particular our aim is to solve in an unique way the problem:

$$\begin{cases} \Delta_0 w = f & \text{in } S^1 \times [r_0, 1] \\ w|_{r=r_0} = \varphi \end{cases}$$

with $r_0 \in (0,1)$, considering a convenient normed functions space for w, f and φ, so that the norm of w is bounded by the one of f.

Now we shall define the space of functions we shall work with.

Definition 2.4.1. *Given* $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, *and the closed interval* $I \subset [0,1]$, *we define*

$$\mathcal{C}^{\ell,\alpha}(S^1 \times I)$$

to be the space of functions $w := w(\theta, r)$ in $\mathcal{C}^{\ell,\alpha}_{loc}(S^1 \times I)$ for which the norm

$$\|w\|_{\mathcal{C}^{\ell,\alpha}(S^1\times I)}$$

is finite and which are invariant with respect to symmetry with respect to the $x_2 = 0$ plane, with respect to the rotation of an angle $\frac{2\pi}{k+1}$ about the vertical x_3 axis, with respect to the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

We recall that one of the surfaces involved in the gluing procedure we shall follow to prove the main theorem, is a surface derived by the Costa-Hoffman-Meeks surface. This surface, as explained in subsection 2.9.1 enjoys many properties of symmetry that we want to be inherited by the surface obtained by the gluing procedure. This is the reason for which we have chosen the function space described above.

Proposition 2.4.2. *Given $r_0 \in (0,1)$, there exists an operator*

$$\begin{array}{rccc} G_{r_0}: & \mathcal{C}^{0,\alpha}(S^1 \times [r_0,1]) & \longrightarrow & \mathcal{C}^{2,\alpha}(S^1 \times [r_0,1]) \\ & f & \longmapsto & w := G_{r_0}(f) \end{array}$$

satisfying the following statements

(i) $\Delta_0\, w = f$ *on* $S^1 \times [r_0, 1]$,

(ii) $w = 0$ *on* $S^1 \times \{r_0\}$ *and* $S^1 \times \{1\}$,

(iii) $||w||_{\mathcal{C}^{2,\alpha}(S^1\times[r_0,1])} \leqslant c\,||f||_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])}$, *for some constant $c > 0$ which does not depend on r_0.*

The proof of this result is contained in subsection 2.12.2.

2.5 A family of minimal surfaces close to $\mathbb{M}^2 \times \{0\}$

In this section we shall show the existence of minimal graphs over $\mathbb{D}^2 - B_{r_\varepsilon}$, having prescribed boundary and which are asymptotic to it. We recall that $r_\varepsilon = \varepsilon/2$. We shall reformulate the problem to use Schäuder fixed point theorem. We know already that the graph of a function v, denoted with Σ_v, is minimal, if and only if the function v is a solution of

$$F\left(\Delta_0\, v + Q_0\left(\sqrt{F}\nabla v, \sqrt{F}\nabla^2 v\right)\right) = 0. \tag{2.8}$$

This equation is a simplified version (since $u = 0$) of (2.4). The operator Q_0 has bounded coefficients for $r \in [r_\varepsilon, 1]$. Its expression is $\operatorname{div}(\nabla v\, Q_{0,1})$ where $Q_{u,1}$ is given by (2.6).

Now let's consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, collinear to $\cos(j(k+1)\theta)$, (for $k \geqslant 1$ fixed) with $j \geqslant 1$ and odd and such that

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\varepsilon^2. \tag{2.9}$$

We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\varepsilon,\varphi}(\cdot,\cdot),$$

where $\mathcal{H}$ is the operator of harmonic extension introduced in proposition 2.12.1. The particular choice of φ assures that its harmonic extension belongs to the functional space of definition 2.4.1.
In order to solve the equation (2.8), we look for v of the form $v = w_\varphi + w$ where $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ and $v = \varphi$ on $S^1 \times \{r_\varepsilon\}$. Using proposition 2.4.2, we can rephrase this problem as a fixed point problem

$$w = S(\varphi, w) \tag{2.10}$$

where the nonlinear mapping S which depends on ε and φ is defined by

$$S(\varphi, w) := -G_{r_\varepsilon}\left(Q_0\left(w_\varphi + w\right)\right),$$

where the operator G is defined in proposition 2.4.2. To prove the existence of a fixed point for (2.10) we need the following result that states that S is a contraction mapping:

Lemma 2.5.1. *There exist some constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^4 \tag{2.11}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|S(\varphi, v_2) - S(\varphi, v_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant \frac{1}{2}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}}$$

$$\|S(\varphi_2, v) - S(\varphi_1, v)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2\,\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v_1, v_2 \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ such that $\|v_i\|_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^4$ and for all boundary data $\varphi, \varphi_1, \varphi_2 \in \mathcal{C}^{2,\alpha}(S^1)$ satisfying (2.9).

Proof. We know from proposition 2.4.2 that $\|G_{r_\varepsilon}(f)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}}$, then

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}},$$

because w_φ is an harmonic function.
To find an estimate of the norm above we recall that $\|\varphi\|_{2,\alpha} \leqslant \kappa\varepsilon^2$ and thanks to proposition 2.12.1 we obtain

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^2.$$

Then
$$\|Q_0\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\|w\|^2_{2,\alpha} \leqslant c\|\varphi\|^2_{C^{2,\alpha}(S^1)} \leqslant c_\kappa \varepsilon^4.$$
Then we can conclude
$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^4.$$
As for the second estimate, we observe that
$$\|S(\varphi,v_2)-S(\varphi,v_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|Q_0\left(w_\varphi+v_2\right)-Q_0\left(w_\varphi+v_1\right)\|_{\mathcal{C}^{0,\alpha}}.$$
Thanks to the considerations made above it follows that
$$||Q_0\left(w_\varphi+v_2\right)-Q_0\left(w_\varphi+v_1\right)||_{\mathcal{C}^{0,\alpha}} \leqslant c||v_2-v_1||_{\mathcal{C}^{2,\alpha}}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant$$
$$\leqslant c_\kappa \varepsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}}.$$
Then
$$\|S(\varphi,v_2)-S(\varphi,v_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}}.$$
To show the third estimate we proceed as above:
$$\|S(\varphi_2,v)-S(\varphi_1,v)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|Q_0\left(w_{\varphi_2}+v\right)-Q_0\left(w_{\varphi_1}+v\right)\|_{\mathcal{C}^{0,\alpha}}$$
$$\leqslant c||w_{\varphi_2}-w_{\varphi_1}||_{\mathcal{C}^{2,\alpha}}\|v\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2||\varphi_2-\varphi_1||_{\mathcal{C}^{2,\alpha}(S^1)}.$$
□

Theorem 2.5.2. *Let be $B := \{w \in \mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1)) \,|\, ||w||_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^4\}$. Then the non-linear mapping S defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^4$ in $\mathcal{C}^{2,\alpha}(S^1\times[r_\varepsilon,1])$ into itself. This value follows from the estimate of the norm of $S(0)$. Consequently thanks to Schäuder fixed point theorem, S has an unique fixed point w in this ball. □

We have proved the existence of a minimal surface, denoted with $S_m(\varphi)$, which is close to $\mathbb{M}^2\times\{0\}$, and close to its boundary is the vertical graph over the annulus $B_{2r_\varepsilon}-B_{r_\varepsilon}$ of a function which can be expanded as
$$\bar{U}_m(r,\theta)=\mathcal{H}_{r_\varepsilon,\varphi}(r,\theta)+\bar{V}_m(r,\theta), \quad \text{with} \quad ||\bar{V}_m||_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2.$$
The function $\bar{V}_m$ depends non linearly on ε,φ. Furthermore as it is easy to proof thanks to the third estimate of lemma 2.5.1, it satisfies
$$\|\bar{V}_m(\varepsilon,\varphi)(r_\varepsilon\cdot,\cdot)-\bar{V}_m(\varepsilon,\varphi')(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2-B_1)} \leqslant c\varepsilon\|\varphi-\varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{2.12}$$

2.6 The catenoid in $\mathbb{M}^2 \times \mathbb{R}$

The catenoid in the space $\mathbb{M}^2 \times \mathbb{R}$ can be obtained by the revolution around the x_3 axis, $\{0,0\} \times \mathbb{R}$, of an appropriate curve γ (see [39]). We consider a vertical geodesic plane containing the origin of $\mathbb{M}^2$ and the curve γ. Let r be the euclidean distance between the point of γ at height t and the x_3 axis: we denote with $r = r(t)$ a parametrization of γ. The surface obtained by revolution of γ, is minimal with respect to the metric g_{hyp} if and only if $r = r(t)$ satisfies the following differential equation (see subsection 2.12.4):

$$r(t)\frac{\partial^2 r}{\partial t^2} - \left(\frac{\partial r}{\partial t}\right)^2 - (1 - r(t)^4) = 0. \tag{2.13}$$

A first integral for this equation is:

$$\left(\frac{\partial r}{\partial t}\right)^2 = Cr^2 - (1 + r^4) \tag{2.14}$$

with $C > 2$ and constant. By the resolution of equation $\left(\frac{\partial r}{\partial t}\right)^2 = 0$, it is easy to prove that the function $r(t)$ has a minimum value r_{min} given by:

$$r_{min} = \sqrt{\frac{C - \sqrt{C^2 - 4}}{2}} = \sqrt{\frac{C/2 + 1}{2}} - \sqrt{\frac{C/2 - 1}{2}} < 1.$$

Since we assume $C = \frac{1}{\varepsilon^4}$, we get

$$r_{min} = \sqrt{\frac{C/2 + 1}{2}} - \sqrt{\frac{C/2 - 1}{2}} = \sqrt{C}\left(1 + \frac{1}{C} - 1 + \frac{1}{C} + \mathcal{O}\left(\frac{1}{C^2}\right)\right) =$$

$$\frac{1}{\sqrt{C}} + \mathcal{O}\left(\frac{1}{C^{3/2}}\right) = \varepsilon^2 + \mathcal{O}(\varepsilon^6).$$

We denote with C_t and C_b, respectively, the part of the catenoid contained in $\mathbb{M}^2 \times \mathbb{R}^+$ and $\mathbb{M}^2 \times \mathbb{R}^-$.

We set

$$t_\varepsilon = -\varepsilon^2 \ln \varepsilon$$

We need find the parametrization of C_t and C_b as graphs on the horizontal plane respectively for $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$ and $t \in [-t_\varepsilon - \varepsilon^2 \ln 2, -t_\varepsilon + \varepsilon^2 \ln 2]$. We start finding the expression of $r(t)$ for t in the interval specified before. We denote it by $r_\varepsilon(t)$.

Lemma 2.6.1. *For $\varepsilon > 0$ small enough, we have*

$$r_\varepsilon(t) = \varepsilon^2 \cosh \frac{t}{\varepsilon^2} + \mathcal{O}(\varepsilon^6 e^{\frac{t}{\varepsilon^2}}) \text{ and } \partial_t r_\varepsilon(t) = \sinh \frac{t}{\varepsilon^2} + \mathcal{O}(\varepsilon^4 e^{\frac{t}{\varepsilon^2}})$$

for $t \in [0, t_\varepsilon + \varepsilon^2 \ln 2]$. Moreover if $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, we derive

$$r_\varepsilon(t) = \mathcal{O}(\varepsilon) \text{ and } \partial_t r_\varepsilon = \mathcal{O}\left(\varepsilon^{-1}\right).$$

Proof. We define the function $v(t)$ in such a way $r_\varepsilon(t) = r_\varepsilon(0)\cosh v(t)$, with $v(0) = 0$ and $r_\varepsilon(0)$ the minimum for $r_\varepsilon(t)$. It satisfies

$$Cr_\varepsilon^2(0) - (1 + r_\varepsilon^4(0)) = 0,$$

from which

$$1 = Cr_\varepsilon^2(0) - r_\varepsilon^4(0). \tag{2.15}$$

Plugging $r_\varepsilon(t)$ in (2.14) and using (2.15), we have

$$(\partial_t v)^2 = C - r_\varepsilon^2(0)(1 + \cosh^2 v(t))$$

and under the hypothesis

$$\frac{t}{\varepsilon^2} \leqslant v\left(t\right) \leqslant \frac{t}{\varepsilon^2} + 1$$

we obtain that $(\partial_t v)^2 = C + \mathcal{O}(\varepsilon^4 e^{\frac{t}{\varepsilon^2}})$ and then $v(t) = \sqrt{C}t + \mathcal{O}(\varepsilon^6 e^{\frac{t}{\varepsilon^2}})$. We remark a posteriori that $\frac{t}{\varepsilon^2} \leqslant v(t) \leqslant \frac{t}{\varepsilon^2} + 1$ holds for $t \in [0, t_\varepsilon + \varepsilon^2 \ln 2]$, $\varepsilon > 0$ small enough. Since $r_\varepsilon(0) = r_{min} = \varepsilon^2 + \mathcal{O}(\varepsilon^6)$, we get

$$r_\varepsilon(t) = r_\varepsilon(0)\cosh v(t) = \varepsilon^2 \cosh\left(\frac{t}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon^6 e^{\frac{t}{\varepsilon^2}}) \tag{2.16}$$

Now we assume that $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, then $r_\varepsilon(t) = \mathcal{O}(\varepsilon)$ and $\partial_t r_\varepsilon(t) = \sinh\left(\frac{t}{\varepsilon^2}\right) + \mathcal{O}(\varepsilon^4 e^{\frac{t}{\varepsilon^2}}) = \mathcal{O}(\varepsilon^{-1})$. □

Now we can prove a lemma that give us the parametrization of the part of catenoid whose height t belongs to a neighbourhood of t_ε.

Lemma 2.6.2. *For $\varepsilon > 0$, small enough and $t \in [t_\varepsilon - \varepsilon^2 \ln 2, t_\varepsilon + \varepsilon^2 \ln 2]$, the surface C_t can be seen on the annulus $\left\{re^{i\theta}; \frac{r_\varepsilon}{2} \leqslant r \leqslant 2r_\varepsilon\right\}$ as the graph of the function $W_t(r,\theta)$ which satisfies*

$$W_t(r,\theta) = \varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^3), \tag{2.17}$$

Similarly if $t \in [-t_\varepsilon - \varepsilon^2 \ln 2, -t_\varepsilon + \varepsilon^2 \ln 2]$, the surface C_b can be seen on $\left\{(r,\theta); \frac{r_\varepsilon}{2} \leqslant r \leqslant 2r_\varepsilon\right\}$ as the graph of the function

$$W_b(r,\theta) = -\varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^3).$$

Proof. The first result follow easily from the hypothesis and the equation (2.16). The second one can be shown observing that C_b is the image of C_t by the reflection with respect to the $x_3 = 0$ plane. In other terms $W_b(r,\theta) = -W_t(r,\theta)$.

□

2.7 A family of minimal surfaces close to a catenoid on $S^1 \times [r_\varepsilon, 1]$

In this section we want to show the existence of minimal graphs over the parts of the surfaces C_t and C_b (described in previous section) defined on $S^1 \times [r_\varepsilon, 1] \subset \mathbb{M}^2$ and asymptotic to them. We know that the graph of the function $u + v$ is minimal, being u the function whose graph is the catenoid, if and only if v is a solution of the equation

$$H_{u+v} = 0 \tag{2.18}$$

whose expression is given by (2.4). The explicit expression of $L_u v$ is

$$F\left(\frac{1}{\sqrt{A}}\Delta_0 v + \partial_r\left(\frac{1}{\sqrt{A}}\right)\partial_r v - \frac{1}{A^{\frac{3}{2}}}\partial_r u\,\partial_r\left(F\partial_r u\right)\partial_r v - F\partial_r u\,\partial_r\left(\frac{1}{A^{\frac{3}{2}}}\partial_r u\,\partial_r v\right)\right), \tag{2.19}$$

where $F = (1-r^2)^2$,

$$A = 1 + F|\nabla u|^2 = \frac{(C-2)r^2}{Cr^2 - 1 - r^4}$$

and

$$\partial_r u = \pm\frac{1}{\sqrt{Cr^2 - 1 - r^4}},$$

as it is easy to obtain using (2.14). It's useful to observe that since we assume $C = \frac{1}{\varepsilon^4}$ and $r_\varepsilon = \varepsilon/2$, we have that, for $r \in [r_\varepsilon, 1]$, $A = 1 + \mathcal{O}(\varepsilon^2)$, $\partial_r u = \mathcal{O}(\varepsilon)$,

$$\partial_r A = \frac{(2C-4)(-r+r^5)}{(Cr^2-1-r^4)^2} = \mathcal{O}(\varepsilon)$$

and

$$\partial^2_{rr} u = -\frac{(2Cr - 4r^3)}{\sqrt{(Cr^2-1-r^4)^3}} = \mathcal{O}(1).$$

Taking into account these estimates, we can conclude it holds that the operator

$$\bar{L}_u v := \sqrt{A}\left(\partial_r\left(\frac{1}{\sqrt{A}}\right)\partial_r v - \frac{1}{A^{\frac{3}{2}}}\partial_r u\,\partial_r\left(F\partial_r u\right)\partial_r v - F\partial_r u\,\partial_r\left(\frac{1}{A^{\frac{3}{2}}}\partial_r u\,\partial_r v\right)\right) =$$

$$l_1\partial_r v + l_2\partial^2_{rr} v, \tag{2.20}$$

where $l_1, l_2 = \mathcal{O}(\varepsilon)$. Then we can write $\sqrt{A} L_u v = F\left(\Delta_0 v + \bar{L}_u v\right)$.

We remark that we have already studied the mapping properties of the operator Δ_0 in section 2.4.

Let Σ_u be the graph of the function u. Then the graph of a function v over Σ_u is minimal if and only if v is a solution of the following equation

$$\Delta_0 v + \bar{L}_u v + \sqrt{A} Q_u(v) = 0. \tag{2.21}$$

Thanks to the observations on the functions A and u_r, we can conclude that Q_u has bounded coefficients in $[r_\varepsilon, 1]$. Now we consider a function $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ which is even with respect to θ, collinear to $\cos(j(k+1)\theta)$ (for $k \geqslant 1$ fixed) and such that

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa \varepsilon^2. \tag{2.22}$$

We define

$$w_\varphi(\cdot,\cdot) := \mathcal{H}_{r_\varepsilon,\varphi}(\cdot,\cdot)$$

where the operator $\mathcal{H}$ has been introduced in proposition 2.12.1. In order to solve the equation (2.21), we look for v of the form $v = w_\varphi + w$ where $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ and $v = \varphi$ on $S^1 \times \{r_\varepsilon\}$. We can rephrase this problem as a fixed point problem, that is

$$w = S(\varphi, w) \tag{2.23}$$

where the nonlinear mapping S is defined by

$$S(\varphi, w) := -G_{r_\varepsilon}\left(\Delta_0 w_\varphi + \bar{L}_u(w_\varphi + w) + \sqrt{A} Q_u\left(w_\varphi + w\right)\right),$$

where the operator G is defined in proposition 2.4.2. To prove the existence of a solution for (2.23) we need the following result which states that S is a contraction mapping.

Lemma 2.7.1. *There exist some constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|S(\varphi, 0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^3 \tag{2.24}$$

and, for all $\varepsilon \in (0, \varepsilon_k)$

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant \frac{1}{2} \|w_2 - w_1\|_{\mathcal{C}^{2,\alpha}}$$

$$\|S(\varphi_2, w) - S(\varphi_1, w)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon \left\|\varphi_2 - \varphi_1\right\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $w_1, w_2 \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ such that $\|w_i\|_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa \varepsilon^4$ and for all boundary data $\varphi, \varphi_1, \varphi_2 \in C^{2,\alpha}(S^1)$ satisfying (2.22).

Proof. We know from the proposition 2.4.2 that $\|G_{r_\varepsilon}(f)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}}$. Then

$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|\bar{L}_u w_\varphi + \sqrt{A}Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant$$
$$c\left(\|\bar{L}_u w_\varphi\|_{\mathcal{C}^{0,\alpha}} + \|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}}\right).$$

Here we have used the fact that $A = 1 + \mathcal{O}(\varepsilon^2)$ and that w_φ is harmonic.

So we need to find the estimates of each summand. We recall that $\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\varepsilon$. Thanks to proposition 2.12.1 we get that

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant c_\kappa\varepsilon^2.$$

We use (2.20) for finding the estimate of $\bar{L}_u w_\varphi$. We obtain

$$\|\bar{L}_u w_\varphi\|_{\mathcal{C}^{0,\alpha}} \leqslant c\varepsilon\|w_\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^3.$$

The last term is estimated observing that

$$\|Q_u\left(w_\varphi\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\|w_\varphi\|^2_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^4.$$

Putting together all these estimates we get

$$\|S(\varphi,w_\varphi)\|_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa\varepsilon^3.$$

As for the second estimate, we observe that

$$S(\varphi,w_2) - S(\varphi,w_1) = -G_{r_\varepsilon}\left(\bar{L}_u(w_\varphi + w_2) + \sqrt{A}Q_u\left(w_\varphi + w_2\right)\right) +$$
$$G_{r_\varepsilon}\left(\bar{L}_u(w_\varphi + w_1) + \sqrt{A}Q_u\left(w_\varphi + w_1\right)\right)$$

and

$$\|S(\varphi,w_2) - S(\varphi,w_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant$$
$$c\|\bar{L}_u(w_\varphi + w_2) - \bar{L}_u(w_\varphi + w_1) + Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)\|_{\mathcal{C}^{0,\alpha}} =$$
$$= c\|\bar{L}_u(w_2 - w_1) + Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)\|_{\mathcal{C}^{0,\alpha}} \leqslant$$
$$\leqslant \|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}} + ||Q_u\left(w_\varphi + w_2\right) - Q_u\left(w_\varphi + w_1\right)||_{\mathcal{C}^{0,\alpha}}.$$

We observe that from the considerations above it follows that

$$\|\bar{L}_u(w_2 - w_1)\|_{\mathcal{C}^{0,\alpha}} \leqslant c\varepsilon||w_2 - w_1||_{\mathcal{C}^{2,\alpha}},$$

and

$$||Q_u(w_\varphi + w_2) - Q_u(w_\varphi + w_1)||_{\mathcal{C}^{0,\alpha}} \leqslant c||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}\|w_\varphi\|_{\mathcal{C}^{2,\alpha}}$$
$$\leqslant c_\kappa \varepsilon ||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}.$$

Then

$$\|S(\varphi, w_2) - S(\varphi, w_1)\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon ||w_2 - w_1||_{\mathcal{C}^{2,\alpha}}.$$

To show the third estimate we observe that

$$\|S(\varphi_2, w) - S(\varphi_1, w)\|_{\mathcal{C}^{2,\alpha}} \leqslant$$

$$\|\bar{L}_u(w_{\varphi_2} - w_{\varphi_1})\|_{\mathcal{C}^{0,\alpha}} + ||Q_u(w_{\varphi_2} + w) - Q_u(w_{\varphi_1} + w)||_{\mathcal{C}^{0,\alpha}} \leqslant$$
$$c\varepsilon\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|w\|_{\mathcal{C}^{2,\alpha}}\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant$$
$$c\varepsilon\|\varphi_2 - \varphi_1\|_{\mathcal{C}^{2,\alpha}(S^1)}.$$

□

Theorem 2.7.2. *Let be $B := \{w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1]) \,|\, ||w||_{2,\alpha} \leqslant 2c_\kappa \varepsilon^4\}$. Then the nonlinear mapping S defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping S is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^4$ in $\mathcal{C}^{2,\alpha}(S^1 \times [r_\varepsilon, 1])$ into itself. This value follows from the estimate of the norm of $S(\varphi, 0)$. Consequently thanks to Schäuder fixed point theorem, S has a unique fixed point w in this ball. □

We have proved the existence of a minimal surface $S_t(\varphi)$, which is close to the part of catenoid C_t introduced in section 2.6 and close to its boundary is a graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the function

$$\bar{U}_t(r, \theta) = \varepsilon^2 \ln \frac{2r}{\varepsilon^2} + \mathcal{H}_{r_\varepsilon, \varphi}(r, \theta) + \bar{V}_t(r, \theta),$$

with $\|\bar{V}_t\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^2$. The function $\bar{V}_t$ depends non linearly on ε, φ. Furthermore it satisfies

$$\|\bar{V}(\varepsilon, \varphi)(r_\varepsilon \cdot, \cdot) - \bar{V}(\varepsilon, \varphi')(r_\varepsilon \cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon\|\varphi - \varphi'\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{2.25}$$

This estimate follows from the lemma 2.7.1.

Now it is easy to show the existence of a minimal surface $S_b(\varphi)$, which is close to the part of catenoid denoted by C_b introduced in section 2.6 and close to its boundary is a graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$. We start observing that C_b can be obtained by reflection of C_t with respect to the $x_3 = 0$ plane. So we can define $S_b(\varphi)$ as the image of $S_t(\varphi)$ by the composition of a rotation by an angle $\frac{\pi}{k+1}$ about the x_3 axis and the reflection with respect the horizontal plane. This choice (in particular the apparently unnecessary rotation) is indispensable to assure to the surface we shall construct by the gluing procedure in section 2.11, to have the same properties of symmetry as the Costa-Hoffman-Meeks surface. See subsection 2.9.1 for more information.

It is clear that $S_b(\varphi)$, over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$, is the graph of the function

$$\bar{U}_b(r,\theta) = -\bar{U}_t\left(r, \theta - \frac{\pi}{k+1}\right).$$

2.8 The relation between the mean curvatures of a surface in $\mathbb{D}^2 \times \mathbb{R}$ with respect to two different metrics

In this section we want to express the mean curvature H_{hyp} of a surface in $\mathbb{D}^2 \times \mathbb{R}$ with respect the metric g_{hyp} in terms of the mean curvature H_e of the same surface with respect to the euclidean metric g_0.

We recall that, if x_1, x_2 denote the coordinates in $\mathbb{D}^2$ and x_3 the coordinate in $\mathbb{R}$, then

$$g_{hyp} = \frac{dx_1^2 + dx_2^2}{F} + dx_3^2, \quad \text{where} \quad F = \left(1 - x_1^2 - x_2^2\right)^2 = \left(1 - r^2\right)^2$$

and

$$g_0 = dx_1^2 + dx_2^2 + dx_3^2.$$

If N_{hyp} denotes the normal vector to a surface Σ with respect to the metric g_{hyp}, then its mean curvature with respect the same metric is given by

$$H_{hyp}(\Sigma) := -\frac{1}{2}\,\text{trace}\left(X \to -[\bar{\nabla}_X N_{hyp}]^T\right),$$

where $[\cdot]^T$ denotes the projection on the tangent bunble $T\Sigma$ and $\bar{\nabla}$ is the riemannian connection relative to g_{hyp}. The mean curvature of Σ with respect g_0, denoted by $H_e(\Sigma)$, is given by

$$H_e(\Sigma) := -\frac{1}{2}\,\text{trace}\left(X \to -[\nabla_X N_e]^T\right),$$

where N_e denotes the normal vector to Σ with respect to the metric g_0 and ∇ is the flat riemannian connection.

The Christoffel symbols, Γ_{ij}^k, associated to the metric g_{hyp} all vanish except

$$\Gamma_{11}^1 = \Gamma_{21}^2 = \Gamma_{12}^2 = -\Gamma_{22}^1 = \frac{2x_1}{\sqrt{F}},$$

$$\Gamma^1_{12} = \Gamma^2_{22} = \Gamma^1_{21} = -\Gamma^2_{11} = \frac{2x_2}{\sqrt{F}}.$$

Let $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$, $\partial_3 = \frac{\partial}{\partial x_3}$ be the elements of a basis of the tangent space. Now, if $X = \sum_i X^i \partial_i$ and $Y = \sum_j Y^j \partial_j$ are two tangent vector fields, the expression of the covariant derivative in $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$ is given by

$$\bar{\nabla}_X Y = \sum_k \left(X(Y^k) + \sum_{i,j} X^i Y^j \Gamma^k_{ij} \right) \partial_k.$$

It is clear that

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{k=1}^{2} \sum_{i,j} X^i Y^j \Gamma^k_{ij} \partial_k. \tag{2.26}$$

We suppose that $N_{hyp} = (N^1, N^2, N^3)$. From (2.26) we get the relation

$$\bar{\nabla}_X N_{hyp} = \nabla_X N_{hyp} + \sum_{k=1}^{2} \sum_{i,j=1}^{2} X^i N^j \Gamma^k_{ij} \partial_k. \tag{2.27}$$

We start evaluating the term $\nabla_X N_{hyp}$. We observe that the normal vector $N_e = (N_1, N_2, N_3)$ to Σ with respect the metric g_0 does not coincide with N_{hyp}. But it is clear that

$$N_{hyp} = (N^1, N^2, N^3) = (\sqrt{F} N_1, \sqrt{F} N_2, N_3).$$

We observe that

$$\nabla_X N_{hyp} = \sum_{k=1}^{3} X(N^k) \partial_k = X(\sqrt{F} N_1)\partial_1 + X(\sqrt{F} N_2)\partial_2 + X(N_3)\partial_3.$$

We can write $X(\sqrt{F} N_k) = (1 - r^2) X(N_k) - X(r^2) N_k$, for $k = 1, 2$. Since

$$X(N_k) = \sum_l X_l \partial_{x_l} N_k \quad \text{and} \quad X(r^2) = 2x_1 X_1 + 2x_2 X_2,$$

it holds that

$$X(\sqrt{F} N_k) = X(N_k) - (2x_1 X_1 + 2x_2 X_2) N_k - r^2 \left(\sum_{l=1}^{3} X_l \partial_{x_l} N_k \right),$$

for $k = 1, 2$. We can conclude that $\nabla_X N_{hyp} = \sum_k X(N^k)\partial_k$ is given by

$$\sum_{k=1}^{3} X(N_k)\partial_k - (2x_1 X_1 + 2x_2 X_2) \sum_{k=1}^{2} N_k \partial_k - r^2 \sum_{k=1}^{2} \left(\sum_{l=1}^{3} X_l \partial_{x_l} N_k \right) \partial_k.$$

Inserting this equality in (2.26) and observing that $\sum_{k=1}^{3} X(N_k)\partial_k = \nabla_X N_e$, we obtain

$$\nabla_X N_{hyp} = \nabla_X N_e - (2x_1X_1 + 2x_2X_2)\sum_{k=1}^{2} N_k\partial_k - r^2\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X_l\partial_{x_l}N_k\right)\partial_k.$$

Replacing this result into (2.27) we find the expression of $\bar{\nabla}_X N_{hyp}$ we shall consider to compute the trace. We will assume X to be a vector field tangent to Σ.

$$\nabla_X N_e - (2x_1X_1 + 2x_2X_2)\sum_{k=1}^{2} N_k\partial_k - r^2\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X_l\partial_{x_l}N_k\right)\partial_k + \sum_{k=1}^{2}\sum_{i,j=1}^{2} X^iN^j\Gamma_{ij}^k\partial_k. \tag{2.28}$$

We start studying the second summand. $(2x_1X_1 + 2x_2X_2)\sum_{k=1}^{2} N_k\partial_k$ is the vector whose components with respect the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} 2x_1N_1 & 2x_2N_1 & 0 \\ 2x_1N_2 & 2x_2N_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

So the trace of the mapping $\sum_{i=1}^{3} X_i\partial_i \to (2x_1X_1 + 2x_2X_2)\sum_{k=1}^{2} N_k\partial_k$ equals $2(x_1N_1 + x_2N_2)$.

The components of the vector

$$\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X_l\partial_{x_l}N_k\right)\partial_k$$

with respect the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} \partial_{x_1}N_1 & \partial_{x_2}N_1 & \partial_{x_3}N_1 \\ \partial_{x_1}N_2 & \partial_{x_2}N_2 & \partial_{x_3}N_2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}.$$

So the trace of the mapping

$$\sum_{i=1}^{3} X_i\partial_i \to r^2\sum_{k=1}^{2}\left(\sum_{l=1}^{3} X_l\partial_{x_l}N_k\right)\partial_k$$

equals $r^2\left(\partial_{x_1}N_1 + \partial_{x_2}N_2\right)$.

As for the last term of (2.28), we can state that $\sum_{k=1}^{2}\sum_{i,j} X^i N^j \Gamma_{ij}^k \partial_k$ is the vector whose components with respect the basis $(\partial_1, \partial_2, \partial_3)$ are given by

$$\begin{bmatrix} \frac{2x_1N^1}{\sqrt{F}} + \frac{2x_2N^2}{\sqrt{F}} & \frac{2x_2N^1}{\sqrt{F}} - \frac{2x_1N^2}{\sqrt{F}} & 0 \\ -\frac{2x_2N^1}{\sqrt{F}} + \frac{2x_1N^2}{\sqrt{F}} & \frac{2x_1N^1}{\sqrt{F}} + \frac{2x_2N^2}{\sqrt{F}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}. \tag{2.29}$$

Taking into account the equalities $N^1 = \sqrt{F} N_1$ and $N^2 = \sqrt{F} N_2$ it easy to conclude that the trace of the mapping

$$\sum_i X_i \partial_i \to \sum_{k=1}^{2} \sum_{i,j} X^i N^j \Gamma_{ij}^k \partial_k$$

equals $4\,(x_1N_1 + x_2N_2)$. From the definition of the mean curvatures it is easy to obtain the following relation.

$$H_{hyp}(\Sigma) = H_e(\Sigma) - (x_1N_1 + x_2N_2) + \frac{r^2}{2}\left(\partial_{x_1}N_1 + \partial_{x_2}N_2\right). \tag{2.30}$$

We have proved the following result

Proposition 2.8.1. *Let S be a surface in $\mathbb{D}^2 \times \mathbb{R}$ endowed with the metric g_{hyp}. If $H_{hyp}(\cdot)$ denotes the mean curvature with respect to the metric g_{hyp}, $H_e(\cdot)$ and (N_1, N_2, N_3) denote respectively the mean curvature and the normal vector to S with respect to g_0, then*

$$H_{hyp}(S) = H_e(S) - (x_1N_1 + x_2N_2) + \frac{r^2}{2}\left(\partial_{x_1}N_1 + \partial_{x_2}N_2\right), \tag{2.31}$$

where x_1, x_2 are the cartesian coordinates on $\mathbb{D}^2$ and $r^2 = x_1 + x_2^2$.

2.9 A scaled Costa-Hoffman-Meeks type surface

In this section we shall describe the surface obtained by scaling of the Costa-Hoffmann-Meeks surface of genus $k \geqslant 1$, M_k, (see C. Costa [2], [3] and D. Hoffman and W. H. Meeks [15], [16]) and we shall study the mapping properties of its Jacobi operator. We denote by $M_{k,\varepsilon}$ the image of M_k by an homothety of parameter ε^2. We shall adapt to our situation some of the analytical tools used in [11] to show the existence of a family of minimal surfaces close to M_k with one planar end and two slightly bent catenoidal ends by an angle $\xi \in (-\xi_0, \xi_0)$, $\xi_0 > 0$ and small enough. We denote an element of this family by $M_k(\xi)$, then $M_k(\xi)|_{\xi=0} = M_k$.

2.9.1 The Costa-Hoffman-Meeks surface.

We start by giving a brief description of the surface M_k. After suitable rotation and translation, M_k enjoys the following properties.

1. It has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t and one bottom end E_b that are respectively asymptotic to the upper end and to the lower end of a catenoid with x_3-axis of revolution. The planar end E_m is located between the two catenoidal ends.

2. It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the symmetry with respect to the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

3. It intersects the $x_3 = 0$ plane in $k + 1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 = const\,(\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

We denote with X_i, with $i = t, b, m$, the parametrization of the end E_i and with $X_{i,\varepsilon}$ the parametrization of the corresponding end $E_{i,\varepsilon}$ of $M_{k,\varepsilon}$.

Now we give a local description of the surface $M_{k,\varepsilon}$ near its ends and we introduce coordinates that we shall use.

The planar end. The planar end $E_{m,\varepsilon}$ of the surface $M_{k,\varepsilon}$ can be parametrized by

$$X_{m,\varepsilon}(x) := \left(\frac{\varepsilon^2 x}{|x|^2}, \varepsilon^2 u_m(x) \right) \in \mathbb{R}^3, \tag{2.32}$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$. Here $\rho_0 > 0$ is fixed small enough. The function u_m satisfies the minimal surface equation which has the following form

$$2H_u = \frac{|x|^4}{\varepsilon^2} \operatorname{div} \left(\frac{\nabla u}{(1 + |x|^4 \, |\nabla u|^2)^{1/2}} \right) = 0. \tag{2.33}$$

It can be shown (see [11]) that the function u_m can be extended at the origin continuously using Weierstrass representation. In particular we can prove that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ and $u_m = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(|x|^{k+1})$, where the expression $\mathcal{O}_{\mathcal{C}^{n,\alpha}_b}(g)$ denotes a function that, together with its partial derivatives of order less than or equal to $n + \alpha$ is bounded by a constant times g. Furthermore, taking into account the symmetries of the surface, it is possible to show the function u_m, in polar coordinates, has to be collinear to $\cos(j(k+1)\theta)$, with $j \geqslant 1$ and odd.

If we linearize in $u = 0$ the nonlinear equation (2.33) we obtain the expression of an operator which is, up to a multiplication by ε^4, the Jacobi operator about the plane, that is $\mathcal{L}_{\mathbb{R}^2} = |x|^4\Delta_0$. To be more precise, the linearization of (2.33) gives

$$L_u\, v = \frac{|x|^4}{\varepsilon^2}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla u \frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right). \tag{2.34}$$

We shall give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is H_u. In the following we shall restrict our attention to the planar case, that is $u = 0$, on a domain of the form $\{(r,\theta) \in B_{r_0}(0) | r \in [r_1, r_2]\}$. Here we shall show that

$$2H_{u+v} = 2H_u + L_u v + \frac{|x|^4}{\varepsilon^2} Q_u(|x|^2\nabla v, |x|^2\nabla^2 v), \tag{2.35}$$

where Q_u satisfies

$$Q_u(0,0) = \nabla Q_u'(0,0).$$

To show (2.35), we start observing that:

$$\frac{1}{\sqrt{1+|x|^4|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}} + Q_{u,1}(v) \tag{2.36}$$

where the function $Q_{u,1}$ satisfies $Q_{u,1}(0) = \nabla Q_{u,1}(0) = 0$. The proof of that is very close to the one that appears in section 2.3: it's necessary only to replace F by $|x|^4$. So we can omit some details. Secondly we observe that $2H_{u+v}$ is given by

$$\frac{|x|^4}{\varepsilon^2}\operatorname{div}\left(\frac{\nabla(u+v)}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla(u+v)\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}} + \nabla(u+v)Q_{u,1}(v)\right) =$$

$$2H_u + \frac{|x|^4}{\varepsilon^2}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right) + \frac{|x|^4}{\varepsilon^2}Q_u(|x|^2\nabla v, |x|^2\nabla^2 v).$$

From this it follows the wanted expression.

Since we assume that Σ_u is a minimal surface, we shall consider $H_u = 0$.

Following what we have done in section 2.7 replacing F by $|x|^4$ we get:

$$\frac{|x|^4}{\varepsilon^2}\left(\Delta_0 v + \sqrt{1+|x|^4|\nabla u|^2}\left(\bar{L}_u v + Q_u(|x|^2\nabla v, |x|^2\nabla^2 v)\right)\right) = 0, \tag{2.37}$$

where $\bar{L}_u v$ is a second order linear operator with operator with coefficients in $\mathcal{O}_{C_b^{2,\alpha}}(|x|^{k+1})$.

It is important to remark that if the function v satisfies to equation (3.8) with $u = u_m$ then the graph of the function $\varepsilon^2(u_m + v)$ is minimal. Now we are interested in finding the equation which a function w must satisfy in such a way the surface parametrized by $X_{m,\varepsilon} + we_3$, that is the graph of w about the middle end $E_{m,\varepsilon}$, is minimal. That is equivalent to require that the graph of $\varepsilon^2 u_m + w$ is minimal. Then we can obtain the wanted equation by replacing v by w/ε^2 in (3.8). So we get

$$\frac{|x|^4}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}\Delta_0 w + \sqrt{1+|x|^4|\nabla u|^2}\left(\frac{1}{\varepsilon^2}\bar{L}_u w + Q_u\left(\frac{|x|^2}{\varepsilon^2}\nabla w, \frac{|x|^2}{\varepsilon^2}\nabla^2 w\right)\right)\right) = 0. \tag{2.38}$$

We can write it in the following way

$$\frac{|x|^4}{\varepsilon^4}\left(\Delta_0 w + \sqrt{1+|x|^4|\nabla u|^2}\left(\bar{L}_u w + \varepsilon^2 Q_u\left(\frac{|x|^2}{\varepsilon^2}\nabla w, \frac{|x|^2}{\varepsilon^2}\nabla^2 w\right)\right)\right) = 0. \tag{2.39}$$

This is the minimal surfaces equation we shall use in following sections.

The catenoidal ends. We denote by X_c the parametrization of the standard catenoid C whose axis of revolution is the x_3-axis. Its expression is

$$X_c(s,\theta) := (\cosh s\,\cos\theta, \cosh s\,\sin\theta, s) \in \mathbb{R}^3$$

where $(s,\theta) \in \mathbb{R}\times S^1$. The unit normal vector field about C is given by

$$n_c(s,\theta) := \frac{1}{\cosh s}(\cos\theta, \sin\theta, -\sinh s).$$

The catenoid C may be divided in two pieces, denoted $C_\pm$, which are defined as the image by X_c of $(\mathbb{R}^\pm \times S^1)$. For any $\varepsilon > 0$, we define the catenoid C_ε as the image of C by an homothety of parameter ε^2. We denote with $X_{c,\varepsilon} := \varepsilon^2 X_c$ its parametrization. Of course, by this transformation, to $C_\pm$, correspond two surfaces denoted $C_{\varepsilon,\pm}$.

Up to some dilation, we can assume that the two ends $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$ of $M_{k,\varepsilon}$ are asymptotic to some translated copy of the catenoid parametrized by $X_{c,\varepsilon}$ in the vertical direction. Therefore, $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$ can be parametrized, respectively, by

$$X_{t,\varepsilon} := X_{c,\varepsilon} + w_t\, n_c + \sigma_{t,\varepsilon}\, e_3 \tag{2.40}$$

for $(s,\theta) \in (s_0,\infty)\times S^1$,

$$X_{b,\varepsilon} := X_{c,\varepsilon} - w_b\, n_c - \sigma_{b,\varepsilon}\, e_3 \tag{2.41}$$

for $(s,\theta) \in (-\infty,-s_0)\times S^1$, where $\sigma_{t,\varepsilon}, \sigma_{b,\varepsilon} \in \mathbb{R}$, functions w_t, w_b tend exponentially fast to 0 as s goes to ∞ reflecting the fact that the ends are asymptotic to a catenoidal end.

Furthermore, taking into account the symmetries of the surface, it is easy to show the functions w_t, w_b, in terms of the (s,θ) coordinates, have to be collinear to $\cos(j(k+1)\theta)$, with $j \in \mathbb{N}$ and must satisfy $w_b(s,\theta) = -w_t(-s, \theta - \frac{\pi}{k+1})$. Furthermore we have $\sigma_{t,\varepsilon} = \sigma_{b,\varepsilon}$.

In section 3 of [30] it is given the expression of the mean curvature operator about of a surface close to a scaled standard catenoid. We can adapt this result to our situation. We obtain that the surface parametrized by $X_{c,\varepsilon} + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation $H_w = 0$, where

$$H_w = -\frac{1}{\varepsilon^4}\,\mathbb{L}_C w + \frac{1}{\varepsilon^2\cosh^2 s} Q_{2,\varepsilon}\left(\frac{w}{\varepsilon^2\cosh s}, \frac{\nabla w}{\varepsilon^2\cosh s}, \frac{\nabla^2 w}{\varepsilon^2\cosh s}\right) + \frac{1}{\varepsilon^2\cosh s} Q_{3,\varepsilon}\left(\frac{w}{\varepsilon^2\cosh s}, \frac{\nabla w}{\varepsilon^2\cosh s}, \frac{\nabla^2 w}{\varepsilon^2\cosh s}\right). \tag{2.42}$$

Here $\mathbb{L}_C$ is the Jacobi operator about the catenoid, that is

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\partial^2_{ss} w + \partial^2_{\theta\theta} w + \frac{2w}{\cosh^2 s}\right)$$

and $Q_{2,\varepsilon}$ and $Q_{3,\varepsilon}$ are linear second order differential operators which are bounded in $\mathcal{C}^k(\mathbb{R} \times S^1)$ for all k, uniformly in ε. They satisfy

$$Q_{2,\varepsilon}(0,0,0) = Q_{3,\varepsilon}(0,0,0) = 0 \qquad \text{and} \qquad \nabla Q_{2,\varepsilon}(0,0,0) = \nabla Q_{3,\varepsilon}(0,0,0) = 0, \tag{2.43}$$

$$\nabla^2 Q_{3,\varepsilon}(0,0,0) = 0. \tag{2.44}$$

We shall write for short

$$Q_\varepsilon(w_\Phi) = \frac{1}{\varepsilon^2\cosh^2 s} Q_{2,\varepsilon}\left(\frac{w_\Phi}{\varepsilon^2\cosh s}, \frac{\nabla w_\Phi}{\varepsilon^2\cosh s}, \frac{\nabla^2 w_\Phi}{\varepsilon^2\cosh s}\right) + \frac{1}{\varepsilon^2\cosh s} Q_{3,\varepsilon}\left(\frac{w_\Phi}{\varepsilon^2\cosh s}, \frac{\nabla w_\Phi}{\varepsilon^2\cosh s}, \frac{\nabla^2 w_\Phi}{\varepsilon^2\cosh s}\right). \tag{2.45}$$

For all $\rho < \rho_0$ and $s > s_0$, we define

$$M_{k,\varepsilon}(s,\rho) := M_{k,\varepsilon} - \left[X_{t,\varepsilon}((s,\infty)\times S^1) \cup X_{b,\varepsilon}((-\infty,-s)\times S^1) \cup X_{m,\varepsilon}(B_\rho(0))\right]. \tag{2.46}$$

The parametrizations of the three ends of $M_{k,\varepsilon}$ induce a decomposition of $M_{k,\varepsilon}$ into slightly overlapping components: a compact piece $M_{k,\varepsilon}(s_0+1, \rho_0/2)$ and three noncompact pieces $X_{t,\varepsilon}((s_0,\infty)\times S^1)$, $X_{b,\varepsilon}((-\infty,-s_0)\times S^1)$ and $X_{m,\varepsilon}(\bar{B}_{\rho_0}(0))$.

We define a weighted space of functions on $M_{k,\varepsilon}$.

Definition 2.9.1. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_{\delta}(M_{k,\varepsilon})$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_{k,\varepsilon})$ for which the following norm is finite*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_{\delta}(M_{k,\varepsilon})} := \|w\|_{\mathcal{C}^{\ell,\alpha}(M_{k,\varepsilon}(s_0+1,\rho_0/2))} + \|\, w \circ X_{m,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}$$

$$+ \sup_{s \geqslant s_0} e^{-\delta s} \, \left(\|w \circ X_{t,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)} + \|w \circ X_{b,\varepsilon}\|_{\mathcal{C}^{\ell,\alpha}([-s-1,-s]\times S^1)} \right)$$

and which are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane, that is $w(p) = w(\bar{p})$ for all $p \in M_{k,\varepsilon}$, where $\bar{p} := (x_1, -x_2, x_3)$ if $p = (x_1, x_2, x_3)$, invariant with respect to a rotation of angle $\frac{2\pi}{k+1}$ about the x_3 axis and to the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3 axis and the symmetry with respect to the $x_3 = 0$ plane.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two ended surface. We shall perturb the surface $M_{k,\varepsilon}$ by the normal graph of a function $u \in \mathcal{C}^{2,\alpha}_{\delta}(M_{k,\varepsilon})$.

2.9.2 The Jacobi operator

The Jacobi operator about $M_{k,\varepsilon}$ is

$$\mathbb{L}_{M_{k,\varepsilon}} := \Delta_{M_{k,\varepsilon}} + |A_{M_{k,\varepsilon}}|^2$$

where $|A_{M_{k,\varepsilon}}|$ is the norm of the second fundamental form on $M_{k,\varepsilon}$.

In the parametrization of the ends introduced above, the volume forms $dvol_{M_{k,\varepsilon}}$ can be written as $\gamma_t \, ds \, d\theta$ and $\gamma_b \, ds \, d\theta$ near the catenoidal type ends and as $\gamma_m \, dx_1 \, dx_2$ near the middle end. Now we can define globally on $M_{k,\varepsilon}$ a smooth function

$$\gamma : M_{k,\varepsilon} \longrightarrow [0, \infty) \tag{2.47}$$

that is identically equal to ε^4 on $M_{k,\varepsilon}(s_0 - 1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_{t,\varepsilon}$ (resp. $E_{b,\varepsilon}$, E_m). They are defined in such a way that on $X_{t,\varepsilon}((s_0, \infty) \times S^1)$ and on $X_{b,\varepsilon}((-\infty, -s_0) \times S^1)$ we have

$$\gamma \circ X_{t,\varepsilon}(s,\theta) \sim \varepsilon^4 \cosh^2 s \qquad \text{and} \qquad \gamma \circ X_{b,\varepsilon}(s,\theta) \sim \varepsilon^4 \cosh^2 s.$$

Finally on $X_{m,\varepsilon}(B_{\rho_0})$, we have

$$\gamma \circ X_m(x) \sim \frac{\varepsilon^4}{|x|^4}.$$

It is possible to check that:

$$\begin{array}{rccc} \mathcal{L}_{\varepsilon,\delta}: & \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}) \\ & w & \longmapsto & \gamma\,\mathbb{L}_{M_{k,\varepsilon}}(w) \end{array}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_{k,\varepsilon}}|}}$ in the expression of the Laplacian in the coordinates we use to parametrize the ends of the surface $M_{k,\varepsilon}$. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into the target space $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$.

To have a better grasp of what is going on, let us linearize the nonlinear equation (2.42) at $w = 0$. We get the expression of the Jacobi operator about the scaled catenoid C_ε

$$\mathbb{L}_{C_\varepsilon} := \frac{1}{\varepsilon^4\cosh^2 s}\left(\partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s}\right).$$

We can observe that the operator $\cosh^2 s\,\mathbb{L}_{C_\varepsilon}$ maps the space $(\cosh s)^\delta\,\mathcal{C}^{2,\alpha}((s_0,+\infty)\times S^1)$ into the space $(\cosh s)^\delta\,\mathcal{C}^{0,\alpha}((s_0,+\infty)\times S^1)$.

Similarly, if we linearize the nonlinear equation (2.33) at $u = 0$, we obtain (see (2.3) with $u = 0$), up to a multiplication by $1/\varepsilon^4$, the expression of the Jacobi operator about the plane.

$$\frac{1}{\varepsilon^4}\mathbb{L}_{\mathbb{R}^2} := \frac{|x|^4}{\varepsilon^4}\,\Delta_0.$$

Again, the operator $\gamma\frac{1}{\varepsilon^4}\,\mathbb{L}_{\mathbb{R}^2} = \Delta_0$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays, for the ends of the surface $M_{k,\varepsilon}$, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about $M_{k,\varepsilon}$ is asymptotic to $\frac{1}{\varepsilon^4}\mathbb{L}_{\mathbb{R}^2}$ at $E_{m,\varepsilon}$ and is asymptotic to $\mathbb{L}_{C_\varepsilon}$ at $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$, we conclude that the operator $\mathcal{L}_{\varepsilon,\delta}$ maps $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$.

Now we recall the notion of non degeneracy introduced in [11].

Definition 2.9.2. *The surface $M_{k,\varepsilon}$ is said to be non degenerate if $\mathcal{L}_{\varepsilon,\delta}$ is injective for all $\delta < -1$.*

It useful to observe that a duality argument in the weighted Lebesgue spaces, implies that

$$(\mathcal{L}_{\varepsilon,\delta}\quad \text{is injective}) \quad\Leftrightarrow\quad (\mathcal{L}_{\varepsilon,-\delta}\quad \text{is surjective})$$

if $\delta \notin \mathbb{Z}$. See [34] and [19] for more details.

The non degeneracy of $M_{k,\varepsilon}$ is related to the mapping properties of $\mathcal{L}_{\varepsilon,\delta}$ and to the kernel of this operator. From the observations made above, it follows that at the catenoidal type ends and at the middle planar end the Jacobi operators of $M_{k,\varepsilon}$ and M_k are respectively asymptotic to $\mathbb{L}_C$ and $\mathbb{L}_{C_\varepsilon}$ which coincide up to a multiplication by ε^4. So we could transpose some of the results about the surface $M_k(0)$ contained in [11] related to the study of its mean curvature operator, to the surface $M_{k,\varepsilon}$, including non degeneracy. The only difference is that here we work with spaces of functions invariant with respect to all of the symmetries of M_k.

The Jacobi fields. It is known that a smooth one parameter group of isometries containing the identity generates a Jacobi field, that is a solution of the equation $\mathbb{L}_{M_{k,\varepsilon}} u = 0$. The Jacobi fields of this type which are invariant with respect to the mirror symmetry by the $x_2 = 0$ plane, the rotation by $\frac{2\pi}{k+1}$ about the x_3 axis, the composition of the rotation by $\frac{\pi}{k+1}$ about the x_3 axis and the mirror symmetry with respect to the $x_3 = 0$ plane, are generated by dilations. Of course the Jacobi equation has other solutions which are not taken into account because they are not invariant under the action of the symmetries listed above. See [11] for details.

The Killing vector field $\Xi(p) = p$, that is associated to the one parameter group of dilations, generates the Jacobi field

$$\Phi(p) := n(p) \cdot p.$$

It is clear that $\Phi(p)$ grows linearly and so it is not bounded.

With these notations, we define the deficiency space

$$\mathcal{D} := \mathrm{Span}\{\chi_t \, \Phi, \chi_b \, \Phi \}$$

where χ_t is a cut-off function that is identically equal to 1 on $X_{t,\varepsilon}((s_0 + 1, +\infty) \times S^1)$, identically equal to 0 on $M_{k,\varepsilon} - X_{t,\varepsilon}((s_0, +\infty) \times S^1)$ and that is invariant under the action of the symmetries listed above. In particular, we agree that

$$\chi_b(\cdot) := \chi_t(- \, \cdot).$$

Clearly, if $\delta < 0$,

$$\begin{array}{rccc} \tilde{\mathcal{L}}_{\varepsilon,\delta} : & \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) \oplus \mathcal{D} & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}) \\ & w & \longmapsto & \gamma \, \mathbb{L}_{M_{k,\varepsilon}} \, w \end{array}$$

is a bounded linear operator. The linear decomposition Lemma proved in [26] for constant mean curvature surfaces (see also [19] for minimal hypersurfaces) can be adapted to our setting and thanks to the results of S. Nayatani about the dimension of the kernel of the Jacobi operator of M_k shown in [36, 37] and extended in chapter 1, we can state that there is not any bounded Jacobi field which is invariant with respect to the symmetries of $M_{k,\varepsilon}$. We get the following result

Proposition 2.9.3. *We choose $\delta \in (-2,-1)$. Then the operator $\tilde{\mathcal{L}}_{\varepsilon,\delta}$ is surjective.*

From that we get the following one about the operator $\mathcal{L}_{\varepsilon,\delta}$

Proposition 2.9.4. *We choose $\delta \in (1,2)$. Then the operator $\mathcal{L}_{\varepsilon,\delta}$ is surjective. Moreover, there exists $G_{\varepsilon,\delta}$ a right inverse for $\mathcal{L}_{\varepsilon,\delta}$ whose norm is bounded.*

2.10 An infinite dimensional family of minimal surfaces which are close to a compact part of a scaled Costa-Hoffman-Meeks type surface in $\mathbb{M}^2 \times \mathbb{R}$.

We recall that in section 2.8 we found that the mean curvature with respect to the metric g_{hyp} of a surface S in $\mathbb{M}^2 \times \mathbb{R}$ can be expressed in terms of the euclidean mean curvature of S and the components of the normal vector to the same surface with respect to the flat metric g_0.

In this section we shall apply this result to prove the existence of a family of minimal surfaces close to the surface $M_{k,\varepsilon}$ contained in a cylindrical neighbourhood of radius $r_\varepsilon = \varepsilon/2$ of $\{0,0\} \times \mathbb{R}$.

We start giving the statement of a result that can be easily obtained by [11], lemma 2.2. It describes the region of the surface $M_{k,\varepsilon}$ which can be parametrized by a graph on a annular neighbourhood of r_ε contained in the $x_3 = 0$ plane.

Lemma 2.10.1. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$ an annular part of the ends $E_{t,\varepsilon}$, $E_{b,\varepsilon}$ and $E_{m,\varepsilon}$ of $M_{k,\varepsilon}$ can be written as vertical graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ of the functions*

$$Z_t(r,\theta) = \sigma_{t,\varepsilon} + \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon^3), \tag{2.48}$$

$$Z_b(r,\theta) = -Z_t\left(r, \theta - \frac{\pi}{k+1}\right). \tag{2.49}$$

As for the parametrization of the planar end, it satisfies

$$Z_m(r,\theta) = \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}\left(\varepsilon^2\left(\frac{r}{\varepsilon^2}\right)^{-(k+1)}\right). \tag{2.50}$$

Here (r,θ) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}(\varepsilon^3)$ are defined in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ and are bounded in $\mathcal{C}_b^{2,\alpha}$ topology by a constant (independent on ε) multiplied by ε, where the partial derivatives are computed with respect to the vector fields $r\,\partial_r$ and ∂_θ.

Then $M_{k,\varepsilon}$ has two ends $E_{t,\varepsilon}$ and $E_{b,\varepsilon}$ which are graphs over the $x_3 = 0$ plane of functions Z_t and Z_b defined on the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$.

We set

$$s_\varepsilon = -\ln\varepsilon, \quad \rho_\varepsilon := 2\varepsilon$$

and we define $M^T_{k,\varepsilon}$ to be equal to $M_{k,\varepsilon}$ from which we have removed the image of $(s_\varepsilon, +\infty) \times S^1$ by $X_{t,\varepsilon}$, the image of $(-\infty, -s_\varepsilon) \times S^1$ by $X_{b,\varepsilon}$ and the image of $B_{\rho_\varepsilon}(0)$ by $X_{m,\varepsilon}$. The values of s_ε and ρ_ε have been chosen in such a way the surface $M^T_{k,\varepsilon}$ is contained in a neighbourhood of radius $r_\varepsilon = \varepsilon/2$ of $\{0,0\} \times \mathbb{R}$. In this section we shall prove the existence of a family of minimal surfaces close to $M^T_{k,\varepsilon}$. To this aim we shall use proposition 2.8.1 and we shall follow the work [11].

First, we modify the parametrization of the ends $E_{t,\varepsilon}$, $E_{b,\varepsilon}$ and $E_{m,\varepsilon}$, for appropriates values of s, so that, when $r = r_\varepsilon$ the curves corresponding to the image of

$$\theta \to (r\,\cos\theta, r\,\sin\theta, Z_b(r,\theta)), \quad \theta \to (r\,\cos\theta, r\,\sin\theta, Z_t(r,\theta)) \tag{2.51}$$

correspond, respectively, up to a vertical translation, to the horizontal curves at heights $\pm\varepsilon^2 \ln(2r_\varepsilon/\varepsilon^2)$.

The curve $\theta \to (r\,\cos\theta, r\,\sin\theta, Z_m(r,\theta))$, if $r = r_\varepsilon$, corresponds, up to a vertical translation, to an horizontal curve at height $\varepsilon^2\,(r_\varepsilon/\varepsilon^2)^{-(k+1)}$.

The second step is the modification of unit normal vector field on $M_{k,\varepsilon}$ into a transverse unit vector field $\tilde{n}_\varepsilon$ in such a way that it coincides with the normal vector field n_ε on $M_{k,\varepsilon}$, is equal to e_3 on the graph over $B_{3r_\varepsilon} - B_{3r_\varepsilon/2}$ of the functions U_t and U_b and interpolate smoothly between the different definitions of $\tilde{n}_\varepsilon$ in different subsets of $M^T_{k,\varepsilon}$.

Finally we observe that close to $E_{t,\varepsilon}$, we can give the following estimate:

$$\left|\varepsilon^4\cosh^2 s\left(\mathbb{L}_{M_{k,\varepsilon}} v - (\varepsilon^4\cosh^2 s)^{-1}\left(\partial_{ss}v + \partial_{\theta\theta}v\right)\right)\right| \leqslant c\left|(\cosh^2 s)^{-1} v\right|. \tag{2.52}$$

This follows easily from (2.42) together with the fact that w_t decays at least like $(\cosh^2 s)^{-1}$ on $E_{t,\varepsilon}$. Similar considerations hold close the bottom end $E_{b,\varepsilon}$. Near the middle planar end $E_{m,\varepsilon}$, we observe that the following estimate holds:

$$\left|\varepsilon^4|x|^{-4}\left(\mathbb{L}_{M_{k,\varepsilon}} v - |x|^4\varepsilon^{-4}\Delta_0 v\right)\right| \leqslant c\left||x|^{2k+3}\nabla v\right|. \tag{2.53}$$

This follows easily from (2.34) together with the fact that u_m decays at least like $|x|^{k+1}$ on $E_{m,\varepsilon}$.

The graph of a function u, using the vector field $\tilde{n}_\varepsilon$, is a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M^T_{k,\varepsilon}} u = \tilde{L}_\varepsilon\, u + Q_\varepsilon\,(u)$$

where $\mathbb{L}_{M^T_{k,\varepsilon}}$ is the Jacobi operator about $M^T_{k,\varepsilon}$, Q_ε is a nonlinear second order differential operator and $\tilde{L}_\varepsilon$ is a linear operator which takes into account the change of the normal vector field (only for the top and bottom ends) n_ε into $\tilde{n}_\varepsilon$ and of the change of the parametrization.

This operator is supported in a neighbourhood of $\{\pm s_\varepsilon\} \times S^1$ and of $\{\rho_\varepsilon\} \times S^1$. It is possible to show that the coefficients of $\tilde{L}_\varepsilon$ are uniformly bounded by a constant times ε^2. We start noticing that the conformal factor $(\cosh^2 s)^{-1}$ contributes with a term equal to ε^2. Furthermore the fact that $\langle \tilde{n}_\varepsilon, n_\varepsilon \rangle = 1 + \mathcal{O}_{\mathcal{C}^{2,\alpha}_b}(\varepsilon)$ in a neighbourhood of $\{\pm s_\varepsilon\} \times S^1$ and the result of [11] appendix B show that the change of vector field induces a linear operator whose coefficients are bounded by a constant times ε. The change of parametrization has consequences we can estimate as follows. As for the catenoidal ends, by (2.48) we can determine the difference between the value of s in function of r, for r in a neighbourhood of r_ε, and the height of the horizontal boundary curve (see (2.51)) We obtain that $|s - \ln\left(\frac{2r}{\varepsilon^2}\right)|$ is bounded by a constant times ε. A similar estimate holds for the planar end.

Now, we consider three functions $\varphi_t, \varphi_b, \varphi_m \in \mathcal{C}^{2,\alpha}(S^1)$ which are even, with respect to θ, φ_t is collinear to $\cos(j(k+1)\theta)$, with $j \geqslant 1$, $\varphi_b = -\varphi_t(\theta - \frac{\pi}{k+1})$, while φ_m is collinear to $\cos(l(k+1)\theta)$, with $l \geqslant 1$ and odd. Assume that they satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_m\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\, \varepsilon^2. \tag{2.54}$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

1. $\chi_+\, H_{\varphi_t}(s_\varepsilon - s, \cdot)$ on the image of $X_{t,\varepsilon}$ where χ_+ is a cut-off function equal to 0 for $s \leqslant s_0 + 1$ and identically equal to 1 for $s \in [s_0 + 2, s_\varepsilon]$
2. $\chi_-\, H_{\varphi_b}(s - s_\varepsilon, \cdot)$ on the image of $X_{b,\varepsilon}$ where χ_- is a cut-off function equal to 0 for $s \geqslant -s_0 - 1$ and identically equal to 1 for $s \in [-s_\varepsilon, -s_0 - 2]$
3. $\chi_m\, \tilde{H}_{\rho_\varepsilon, \varphi_m}(\cdot, \cdot)$ on the image of $X_{m,\varepsilon}$, where χ_m is a cut-off function equal to 0 for $\rho \geqslant \rho_0$ and identically equal to 1 for $\rho \in [\rho_\varepsilon, \rho_0/2]$
4. zero on the remaining part of the surface $M^T_{k,\varepsilon}$.

The cut-off functions just introduced must have the same symmetry properties as the functions in $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$. $\tilde{H}$ and H are harmonic extension operator introduced respectively in propositions 2.12.3 and 2.12.2.

We would like to prove that, under appropriates hypotheses, the graph Σ_u about $M^T_{k,\varepsilon}$ of the function $u = w_\Phi + v$, is a minimal surface with respect the metric g_{hyp}. We want to point out that to construct the graph of the function u, here we consider the normal vector field with respect to the euclidean metric g_0. The equation to solve is:

$$H_{hyp}(\Sigma_u) = 0.$$

If we denote by $N_u = (N_1(u), N_2(u), N_3(u))$ the unit normal vector to Σ_u, by equation (2.31) we can express $H_{hyp}(\Sigma_u)$ in terms of the euclidean mean curvature and write the equation to solve as

$$H_e(\Sigma_u) - (x_1 N_1(u) + x_2 N_2(u)) + \frac{r^2}{2}(\partial_{x_1} N_1(u) + \partial_{x_2} N_2(u)) = 0,$$

where x_1, x_2 are the coordinates on $\mathbb{D}^2$ and $r^2 = x_1^2 + x_2^2$. To simplify the notation we set $P(w_\Phi + v) = x_1 N_1(u) + x_2 N_2(u) - \frac{r^2}{2}(\partial_{x_1} N_1(u) + \partial_{x_2} N_2(u))$. Taking into account that $u = w_\Phi + v$, now the expression of equation to solve is given by

$$\mathbb{L}_{M^T_{k,\varepsilon}}(w_\Phi + v) - \tilde{L}_\varepsilon(w_\Phi + v) - Q_\varepsilon(w_\Phi + v) - P(w_\Phi + v) = 0.$$

The resolution of the previous equation is obtained by the one of the following fixed point problem:

$$v = T(\Phi, v) \tag{2.55}$$

with

$$T(\Phi, v) = G_{\varepsilon,\delta} \circ \mathcal{E}_\varepsilon \left(\gamma \left(\tilde{L}_\varepsilon(w_\Phi + v) + P(w_\Phi + v) - \mathbb{L}_{M^T_{k,\varepsilon}} w_\Phi + Q_\varepsilon(w_\Phi + v) \right) \right)$$

where $\delta \in (1, 2)$, the operator $G_{\varepsilon,\delta}$ is defined in proposition 2.9.4 and $\mathcal{E}_\varepsilon$ is a linear extension operator such that

$$\mathcal{E}_\varepsilon : \mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon}) \longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon}),$$

where $\mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon})$ denotes the space of functions of $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$ restricted to $M^T_{k,\varepsilon}$. It is defined by $\mathcal{E}_\varepsilon v = v$ in $M^T_{k,\varepsilon}$, $\mathcal{E}_\varepsilon v = 0$ in the image of $[s_\varepsilon + 1, +\infty) \times S^1$ by $X_{t,\varepsilon}$, in the image of $(-\infty, -s_\varepsilon - 1] \times S^1$ by $X_{b,\varepsilon}$ and in the image of $B_{\rho_\varepsilon/2} \times S^1$ by $X_{m,\varepsilon}$. Finally $\mathcal{E}_\varepsilon v$ is an interpolation of these values in the remaining part of $M_{k,\varepsilon}$ such that

$$(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon}(s, \theta) = (1 + s_\varepsilon - s)(v \circ X_{t,\varepsilon}(s_\varepsilon, \theta)), \quad \text{for} \quad (s, \theta) \in [s_\varepsilon, s_\varepsilon + 1] \times S^1,$$

$$(\mathcal{E}_\varepsilon v) \circ X_{b,\varepsilon}(s, \theta) = (1 + s_\varepsilon + s)(v \circ X_{b,\varepsilon}(s_\varepsilon, \theta)), \quad \text{for} \quad (s, \theta) \in [-s_\varepsilon - 1, -s_\varepsilon] \times S^1,$$

$$(\mathcal{E}_\varepsilon v) \circ X_{m,\varepsilon}(\rho, \theta) = \left(\frac{2}{\rho_\varepsilon}\rho - 1 \right)(v \circ X_{m,\varepsilon}(\rho_\varepsilon, \theta)) \quad \text{for} \quad (\rho, \theta) \in [\rho_\varepsilon/2, \rho_\varepsilon] \times S^1.$$

Remark 2.10.2. *From the definition of $\mathcal{E}_\varepsilon$, if* $supp\, v \cap (B_{\rho_\varepsilon} - B_{\rho_\varepsilon/2}) \neq \emptyset$ *then*

$$\|(\mathcal{E}_\varepsilon v) \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leqslant c\rho_\varepsilon^{-\alpha} \|v \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\varepsilon})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}([s_0,+\infty)\times S^1)} \leqslant c\|v \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}([s_0,s_\varepsilon]\times S^1)}.$$

A similar equation holds for the bottom end. In the following we shall assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_\delta(M^T_{k,\varepsilon})$ for the equation (2.55) is a consequence of the following result which proves that T is a contraction mapping.

Lemma 2.10.3. *Let $\delta \in (1, 2)$. There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|T(\Phi, 0)\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant c_\kappa\, \varepsilon^{5/2} \tag{2.56}$$

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$, $\alpha \in (0, 1/2)$

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant \frac{1}{2}\, \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})},$$

$$\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^{3/2}\, \|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}},$$

where we have set for short

$$\|\Phi_2 - \Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} = \|\varphi_{t,2} - \varphi_{t,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{b,2} - \varphi_{b,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{m,2} - \varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant 2\, c_\kappa\, \varepsilon^{5/2}$ and for all boundary data $\Phi, \Phi_1, \Phi_2 \in [\mathcal{C}^{2,\alpha}(S^1)]^3$ satisfying (2.54).

Proof. We recall that the Jacobi operator associated to $M_{k,\varepsilon}$, is asymptotic to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_{k,\varepsilon}}$ is not known: this is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ and thanks to proposition 2.9.4 we obtain the estimate

$$\|\mathcal{E}_\varepsilon\left(\gamma \mathbb{L}_{M_{k,\varepsilon}}\, w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} =$$

$$\left|\left|\left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \left(\partial^2_s + \partial^2_\theta\right)\right)(w_\Phi \circ X_{t,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} +$$

$$\left|\left|\left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \left(\partial^2_s + \partial^2_\theta\right)\right)(w_\Phi \circ X_{b,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} +$$

$$\rho_\varepsilon^{-\alpha} \left|\left|\left(\gamma \mathbb{L}_{M^T_{k,\varepsilon}} - \Delta_0\right)(w_\Phi \circ X_{m,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)}$$

$$\leqslant c\left|\left|\cosh^{-2} s\,(w_\Phi \circ X_{t,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} + c\left|\left|\cosh^{-2} s\,(w_\Phi \circ X_{b,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} +$$

$$c\varepsilon^{-\alpha}\left|\left|\rho^{2k+3}\nabla(w_\Phi \circ X_{m,\varepsilon})\right|\right|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)}$$

$$\leqslant c_\kappa\,\varepsilon^4 + c_\kappa\,\varepsilon^{\frac{5}{2}} \leqslant c_\kappa \varepsilon^{5/2}.$$

To obtain this estimate we used the following ones:

$$\sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s}\left|\cosh^{-2} s(w_\Phi \circ X_{t,\varepsilon/2})\right|_{0,\alpha;[s,s+1]} \leqslant c \sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s} e^{-2(s_\varepsilon - s)} e^{-2s}|\phi_t|_{2,\alpha} \leqslant$$

$$c\, e^{-2s_\varepsilon}|\phi_t|_{2,\alpha} \leqslant c_\kappa \varepsilon^4$$

(a similar estimate holds for the bottom end) and

$$\sup_{[\rho_\varepsilon,\rho_0]\times S^1} |\nabla(w_\Phi \circ X_{m,\varepsilon})| \leqslant c\varepsilon^{-\alpha}\|\rho^{2k+3}\nabla w_\Phi\|_{C^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)} \leqslant c_\kappa \varepsilon^{5/2}$$

together with the fact that $s_\varepsilon = -\ln\varepsilon$ and $\rho_\varepsilon = 2\varepsilon$, from which $e^{-2s_\varepsilon} = \varepsilon^2$ and $\rho_\varepsilon^{-\alpha} = \varepsilon^{-\alpha}$.

Using the properties of $\tilde{L}_\varepsilon$ and the definition of γ, we obtain

$$\|\mathcal{E}_\varepsilon\left(\gamma \tilde{L}_\varepsilon\, w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^2\|w_\Phi \circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} + c\varepsilon^2\|w_\Phi \circ X_{b,\varepsilon}\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} +$$

$$+c\varepsilon^{2-\alpha}\|w_\Phi \circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0/2]\times S^1)} \leqslant c_\kappa \varepsilon^{5/2}.$$

The estimate of $\|\mathcal{E}_\varepsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}$ is related to the estimate of the horizontal components and their derivatives of order one of the normal vector to surface and to the definition of the function γ (see (2.47)) on $M^T_{k,\varepsilon}$. It is convenient to recall that the operator $\mathcal{E}_\varepsilon$ extends smoothly a function $g \in \mathcal{C}^{0,\alpha}_\delta(M^T_{k,\varepsilon})$ to a function $\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})$ substantially letting it inchanged on $M^T_{k,\varepsilon}$ and setting it equal to the null function on the remaining part of $M_{k,\varepsilon}$. P keeps track of the difference of the mean curvatures of a same graph of a function about $M^T_{k,\varepsilon}$ computed with respect to two different metrics. It is sufficient to estimate the norm of $\gamma P(w_\Phi)$ only on $M^T_{k,\varepsilon}$, which is the compact part of $M_{k,\varepsilon}$ contained in a neighbourhood of radius $r_\varepsilon = \varepsilon/2$ of $\{0,0\}\times\mathbb{R}$. The function γ equals $\varepsilon^4\cosh^2 s$ at the catenoidal ends of $M^T_{k,\varepsilon}$, equals $\varepsilon^4/|x|^4$ at the middle end, equals ε^4 far away the ends. Furthermore it is easy to prove that the horizontal components N_1, N_2 of the normal vector to the graph of w_Φ about the middle end of $M^T_{k,\varepsilon}$, are, in absolute value, smaller than a constant times ε^2. We get $\|\mathcal{E}_\varepsilon\left(\gamma P(w_\Phi)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\,\varepsilon^{5/2}$.

As for the last term, we recall that the operator Q_ε has two different expressions if we consider the catenoidal type end and the planar end (see equation (2.42) and (2.39)). It holds that

$$\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c_\kappa \varepsilon^{5/2}.$$

In fact

$$\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})} \leqslant c\varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{t,\varepsilon}\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([s_0+1,s_\varepsilon]\times S^1)}+$$

$$c\varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{b,\varepsilon}\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([-s_\varepsilon,-s_0-1]\times S^1)}+c\varepsilon^{2(1-\alpha)}\|\frac{|x|^2}{\varepsilon^2}w_\Phi\circ X_{m,\varepsilon}\|^2_{\mathcal{C}^{2,\alpha}([\rho_\varepsilon,\rho_0/2]\times S^1)}\leqslant c_\kappa\varepsilon^{5/2}.$$

As for the second estimate, we recall that

$$T(\Phi,v):=G_{\varepsilon,\delta}\circ\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi+v)+\tilde{L}_\varepsilon(w_\Phi+v)-\mathbb{L}_{M_{k,\varepsilon}}w_\Phi+Q_\varepsilon\left(w_\Phi+v\right)\right)\right).$$

Then

$$\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}\leqslant$$

$$\leqslant\|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi+v_2)-P(w_\Phi+v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}+\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_\varepsilon(v_2-v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}+$$

$$+\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi+v_1\right)-Q_\varepsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}.$$

We observe that from the considerations above it follows that

$$\|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_\Phi+v_2)-P(w_\Phi+v_1)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}\leqslant c\varepsilon^{3/2}||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})},$$

$$\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_\varepsilon(v_2-v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}\leqslant c\varepsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}$$

and

$$\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi+v_1\right)-Q_\varepsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}$$

$$\leqslant c||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}\left(\varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{t,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{t,\varepsilon})}+\varepsilon^2\|\frac{w_\Phi}{\varepsilon^2\cosh s}\circ X_{b,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{b,\varepsilon})}+\right.$$

$$\left.+\varepsilon^{2-\alpha}\|\frac{|x|^2}{\varepsilon^2}w_\Phi\circ X_{m,\varepsilon}\|_{\mathcal{C}^{0,\alpha}(E_{m,\varepsilon})}\right)\leqslant c_\kappa\varepsilon^2||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}.$$

Then

$$\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}\leqslant c\varepsilon^{3/2}||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}.$$

To show the last estimate it is sufficient to observe that

$$\|T(\Phi_2,v)-T(\Phi_1,v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})}\leqslant\|\mathcal{E}_\varepsilon\left(\gamma\left(P(w_{\Phi_2}+v)-P(w_{\Phi_1}+v)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}+$$

$$+\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_\varepsilon(w_{\Phi_2}-w_{\Phi_1})\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}+$$

$$+\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_{\Phi_2}+v\right)-Q_\varepsilon\left(w_{\Phi_1}+v\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}\leqslant$$
$$\leqslant c\varepsilon^{3/2}\|\Phi_2-\Phi_1\|+c\|v\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\varepsilon})}\|\Phi_2-\Phi_1\|\leqslant c\varepsilon^{3/2}\|\Phi_2-\Phi_1\|.$$

□

Theorem 2.10.4. *Let be $B:=\{w\in\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})\,|\,||w||_{2,\alpha}\leqslant 2c_\kappa\varepsilon^{5/2}\}$. Then the nonlinear mapping T defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping T is a contraction mapping from the ball B of radius $2c_\kappa\varepsilon^{5/2}$ in $\mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon})$ into itself. This value follows from the estimate of the norm of $T(\Phi,0)$. Consequently thanks to Schäuder fixed point theorem, T has a unique fixed point w in this ball. □

This argument provides a minimal surface $M^T_{k,\varepsilon}(\Phi)$ which is close to $M^T_{k,\varepsilon}$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon}-B_{r_\varepsilon/2}$, with $r_\varepsilon=\varepsilon/2$, whose parametrization is, respectively, given by

$$U_t(r,\theta)=\sigma_{t,\varepsilon}+\varepsilon^2\ln\left(\frac{2r}{\varepsilon^2}\right)+H_{\varphi_t}(s_\varepsilon-\ln\frac{2r}{\varepsilon^2},\theta)+V_t(r,\theta),$$
$$U_b(r,\theta)=-U_t\left(r,\theta-\frac{\pi}{k+1}\right)$$

where $s_\varepsilon=-\ln\varepsilon$. Nearby the middle boundary the surface is a vertical graph whose parametrization is

$$U_m(r,\theta)=\tilde{H}_{\rho_\varepsilon,\varphi_m}\left(\frac{\varepsilon^2}{r},\theta\right)+V_m(r,\theta).$$

The boundaries of the surface correspond to $r=r_\varepsilon$. All the functions V_i, $i=t,m$, depend non linearly on ε,φ.

Lemma 2.10.5. *The function $V_t(\varepsilon,\varphi_t)$ satisfies $\|V_t(\varepsilon,\varphi_t)(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1-B_{1/2})}\leqslant c\varepsilon^2$ and*

$$\|V_t(\varepsilon,\varphi_{i,2})(r_\varepsilon\cdot,\cdot)-V_t(\varepsilon,\varphi_{i,1})(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1-B_{1/2})}\leqslant c\varepsilon^{3/2-\delta}\|\varphi_{i,2}-\varphi_{i,1}\|_{\mathcal{C}^{2,\alpha}}\qquad(2.57)$$

The function $V_m(\varepsilon,\varphi)$ satisfies $\|V_m(\varepsilon,\varphi)(\rho_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1-B_{1/2})}\leqslant c\varepsilon^2$ and

$$\|V_m(\varepsilon,\varphi_{m,2})(\rho_\varepsilon\cdot,\cdot)-V_m(\varepsilon,\varphi_{m,1})(\rho_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1-B_{1/2})}\leqslant c\varepsilon^{3/2}\|\varphi_{m,2}-\varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}}\qquad(2.58)$$

Proof. We start observing that the functions V_t,V_b,V_m are the restrictions to $E_{t,\varepsilon},E_{b,\varepsilon},E_{m,\varepsilon}$ of a fixed point for the operator T. Then the wanted estimates follow from

$$\|V_i(\varepsilon,\varphi_2)(\cdot,\cdot)-V_i(\varepsilon,\varphi_1)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{r_\varepsilon}-B_{r_\varepsilon/2})}\leqslant ce^{\delta s_\varepsilon}\|T(\Phi_2,V_i)-T(\Phi_1,V_i)\|_{\mathcal{C}^{2,\alpha}_\delta(E_{i,\varepsilon})},$$

for $i=t,b$ and

$$\|V_m(\varepsilon,\varphi_2)(\cdot,\cdot)-V_m(\varepsilon,\varphi_1)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{r_\varepsilon}-B_{r_\varepsilon/2})}\leqslant c\|T(\Phi_2,V_m)-T(\Phi_1,V_m)\|_{\mathcal{C}^{2,\alpha}(E_{m,\varepsilon})}$$

and the third estimate of proposition 2.10.3. □

2.11 The matching of Cauchy data

In this section we shall complete the proof of theorem 2.1.1.

Using the result of section 2.7, we obtain two minimal surfaces that are perturbations of two parts of the catenoid defined in $\mathbb{M}^2 \times \mathbb{R}$. The first surface, that we denote by $S_{t,d_t}(\varphi_t)$, after a translation by d_t along the x_3-axis, can be parameterized in $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_t(r,\theta) = \varepsilon^2 \ln \frac{2r}{\varepsilon^2} + d_t + \mathcal{H}_{r_\varepsilon,\varphi_t}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2),$$

The second surface, that we denote by $S_{b,d_b}(\varphi_b)$,where $\varphi_b(\theta) = \varphi_t(\theta - \frac{\pi}{k+1})$ and $d_b = -d_t$, can be parameterized in $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}_b(r,\theta) = -\bar{U}_t\left(r, \theta - \frac{\pi}{k+1}\right)$$

Using the result of section 2.5, we can construct the minimal graph $S_m(\varphi_m)$. It can be parameterized, in $B_{2r_\varepsilon} - B_{r_\varepsilon}$, as the vertical graph of

$$\bar{U}_m(r,\theta) = \mathcal{H}_{r_\varepsilon,\varphi_m}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2).$$

By to the result of section 2.10, we can obtain a minimal surface $M^T_{k,\varepsilon}(\Psi)$, with $\Psi = (\psi_t, \psi_b, \psi_m)$, where $\psi_b(\theta) = \psi_t(\theta - \frac{\pi}{k+1})$, which is close to a truncated and scaled genus k Costa-Hoffman-Meeks surface and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, whose parametrization is, respectively, given by

$$U_t(r,\theta) = \sigma_{t,\varepsilon} + \varepsilon^2 \ln\left(\frac{2r}{\varepsilon^2}\right) + H_{\psi_t}(s_\varepsilon - \ln \frac{2r}{\varepsilon^2}, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2),$$

$$U_b(r,\theta) = -U_t\left(r, \theta - \frac{\pi}{k+1}\right)$$

where $s_\varepsilon = -\ln\varepsilon$. Nearby the middle boundary the surface is a vertical graph whose parametrization is

$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon,\psi_m}\left(\frac{\varepsilon^2}{r}, \theta\right) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2).$$

We assume that the parameters and the boundary functions are chosen so that

$$|\eta_t| + |\eta_b| + \|\varphi_t\|_{C^{2,\alpha}(S^1)} + \|\varphi_m\|_{C^{2,\alpha}(S^1)} +$$
$$+ \|\psi_t\|_{C^{2,\alpha}(S^1)} + \|\psi_m\|_{C^{2,\alpha}(S^1)} \leqslant \kappa\varepsilon^2,$$

where $\eta_t = d_t - \sigma_{t,\varepsilon}$, the constant $\kappa > 0$ is fixed large enough. The functions $\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2)$ replace the functions $V_t, V_m, \bar{V}_t, \bar{V}_m$ that appear at the end of sections 2.5, 2.7 and 2.10. They depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant (independent of κ and ε) times ε^2 in $C_b^{2,\alpha}$ topology, where partial derivatives are taken with respect to the vector fields $r\partial_r$ and ∂_θ. It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions in such a way that the surface

$$S_{t,d_t}(\varphi_t) \cup S_{b,d_b}(\varphi_b) \cup S_m(\varphi_m) \cup \bar{M}^T_{k,\varepsilon}(\Psi)$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth and by construction it has the desired properties. This will therefore complete the proof of the main theorem.

It is necessary to fulfill the following system of equations

$$\left\{\begin{array}{c} U_t(r_\varepsilon,\cdot) = \bar{U}_t(r_\varepsilon,\cdot) \\ U_b(r_\varepsilon,\cdot) = \bar{U}_b(r_\varepsilon,\cdot) \\ U_m(r_\varepsilon,\cdot) = \bar{U}_m(r_\varepsilon,\cdot) \\ \partial_r U_b(r_\varepsilon,\cdot) = \partial_r \bar{U}_b(r_\varepsilon,\cdot) \\ \partial_r U_t(r_\varepsilon,\cdot) = \partial_r \bar{U}_t(r_\varepsilon,\cdot) \\ \partial_r U_m(r_\varepsilon,\cdot) = \partial_r \bar{U}_m(r_\varepsilon,\cdot) \end{array}\right.$$

on S^1. The first three equations lead to the system

$$\left\{\begin{array}{c} \eta_t + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2) \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2), \end{array}\right. \tag{2.59}$$

where $\eta_t = d_t - \sigma_{t,\varepsilon}$. The last three equations give the system (we applied lemma 2.12.4 and 2.12.5)

$$\left\{\begin{array}{c} \partial_\theta(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon^2) \\ \partial_\theta(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon^2). \end{array}\right. \tag{2.60}$$

Here, the functions $\mathcal{O}_{C^{l,\alpha}}(\varepsilon^2)$ in the above expansions depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant (independent of κ and ε) times ε^2 in $C^{l,\alpha}$ topology. Projecting every equation of this system over the L^2-orthogonal complement of Span$\{1\}$, we obtain the system

$$\left\{\begin{array}{c} \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2) \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon^2) \\ \partial_\theta\varphi_t + \partial_\theta\psi_t = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon^2) \\ \partial_\theta\varphi_m + \partial_\theta\psi_m = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon^2) \end{array}\right. \tag{2.61}$$

Lemma 2.11.1. *The operator h defined by*

$$
\begin{aligned}
C^{2,\alpha}(S^1) &\to C^{1,\alpha}(S^1) \\
\varphi &\to \partial_\theta \varphi
\end{aligned}
$$

acting on functions that are orthogonal to the constant function in the L^2-sense and are even, is invertible.

Proof. We observe that if we decompose $\varphi = \sum_{j\geqslant 1} \varphi_j \cos(j\theta)$, then

$$h(\varphi) = -\sum_{j\geqslant 1} \varphi_j \cos(j\theta),$$

that is clearly invertible from $H^1(S^1)$ into $L^2(S^1)$. Now elliptic regularity theory implies that this is also the case when this operator is defined between Holder spaces. □
Using this result, the system (2.61) can be rewritten as

$$(\varphi_t, \varphi_m, \psi_t, \psi_m) = \mathcal{O}_{C^{2,\alpha}}(\varepsilon^2). \tag{2.62}$$

Recall that the right hand side depends nonlinearly on $\varphi_t, \varphi_b, \psi_t, \psi_m$ and also on the parameter η_t. We look at this equation as a fixed point problem and fix κ large enough. Thanks to estimates (2.12),(2.25), (2.57) and (2.58), we can use a fixed point theorem for contraction mappings in the ball of radius $\kappa\varepsilon^2$ in $(C^{2,\alpha}(S^1))^4$ to obtain, for all ε small enough, a solution $(\varphi_t, \varphi_m, \psi_t, \psi_m)$ of (2.62). This solution being obtained a fixed point for contraction mapping and the right hand side of (2.62) being continuous with respect to all data, we see that this fixed point $(\varphi_t, \varphi_m, \psi_t, \psi_m)$ depends continuously (and in fact smoothly) on the parameter η_t. Inserting the founded solution into (2.59) and (2.60), we see that it remains to solve an equation that can be rewritten under as

$$\eta_t = \mathcal{O}(\varepsilon^2), \tag{2.63}$$

where this time, the right hand side depends nonlinearly on η_t. Now, provided κ has been fixed large enough, we can use Leray-Schäuder fixed point theorem in the ball of radius $\kappa\varepsilon^2$ in $\mathbb{R}$ to solve (2.63), for all ε small enough. This provides a set of parameters and boundary data such that (2.59) and (2.60) hold. Equivalently we have proven the existence of a solution of systems (2.59) and (2.60). So the proof of theorem 2.1.1 is complete.

2.12 Appendix

2.12.1 Harmonic extension operators

The results contained in this section are about the existence of some harmonic extension operators. The first one gives the harmonic extension of a function defined on ∂B_{r_0} to $\mathbb{D}^2 \setminus B_{r_0}$.

Proposition 2.12.1. *There exists an operator*

$$\mathcal{H}_{r_0} : C^{2,\alpha}(S^1) \longrightarrow C_1^{2,\alpha}(S^1 \times [r_0, 1]),$$

such that for every even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$*, the function* $w_\varphi = \mathcal{H}_{r_0,\varphi}$ *solves*

$$\begin{cases} \Delta_0 w_\varphi = 0 & on \quad S^1 \times [r_0, 1] \\ w_\varphi = \varphi & on \quad S^1 \times \{r_0\}. \end{cases}$$

Moreover,

$$||\mathcal{H}_{r_0,\varphi}||_{C^{2,\alpha}(S^1 \times [r_0,1])} \leqslant c\, ||\varphi||_{C^{2,\alpha}(S^1)}, \tag{2.64}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is

$$\varphi = \sum_{i=0}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(r, \theta) = \sum_{i=0}^{\infty} \left(\frac{r_0}{r}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{r_0}{r} \leqslant 1$, then $\left(\frac{r_0}{r}\right)^i \leqslant \left(\frac{r_0}{r}\right)$, we can conclude that $||w_\varphi||_{C^{2,\alpha}(S^1 \times [r_0,1])} \leqslant c||\varphi||_{C^{2,\alpha}(S^1)}$.
□

Now we give the statement of a result whose proof is contained in [6].

Proposition 2.12.2. *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}([0, +\infty) \times S^1),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$*, even function and orthogonal to* e_i*,* $i = 0, 1$ *in the* L^2*-sense, the function* $w = H(\varphi)$ *solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2)\, w &= 0 \quad in \quad S^1 \times [0, +\infty) \\ w &= \varphi \quad on \quad S^1 \times \{0\} \end{cases}$$

Moreover

$$\|H(\varphi)\|_{\mathcal{C}^{2,\alpha}_{-2}([0,+\infty)\times S^1)} \leqslant c\, \|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

for some constant $c > 0$.

The following result gives a harmonic extension of a function on $\mathbb{R}^2 \setminus D_{\bar\rho}$.

Proposition 2.12.3. *There exists an operator*

$$\tilde{H}_{\bar{\rho}} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}(S^1 \times [\bar{\rho}, +\infty)),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to the constant function then $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ solves

$$\begin{cases} \Delta w_\varphi = 0 & on \quad S^1 \times [\bar{\rho}, +\infty) \\ w_\varphi = \varphi & on \quad S^1 \times \{\bar{\rho}\}. \end{cases}$$

Moreover,

$$||\tilde{H}_{\bar{\rho},\varphi}||_{C^{2,\alpha}(S^1 \times [\bar{\rho},+\infty))} \leqslant c\, ||\varphi||_{C^{2,\alpha}(S^1)}, \tag{2.65}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(\rho, \theta) = \sum_{i=1}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar{\rho}}{\rho} \leqslant 1$, then $\left(\frac{\bar{\rho}}{\rho}\right)^i \leqslant \left(\frac{\bar{\rho}}{\rho}\right)$, we can conclude that $|w(r,\theta)| \leqslant c|\varphi(\theta)|$ and then $||w_\varphi||_{C^{2,\alpha}} \leqslant c||\varphi||_{C^{2,\alpha}}$. □

Lemma 2.12.4. *Let $u(r,\theta)$ be the harmonic extension defined on $[r_0, +\infty) \times S^1$ of the even funcion $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ and such that $u(r_0, \theta) = \varphi(\theta)$. Then*

$$\partial_\theta u(r, \theta - \pi/2)_{|r=r_0} = -r_0 \partial_r u(r,\theta)_{|r=r_0}.$$

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(r,\theta) = \sum_{i\geqslant 1} \varphi_i \left(\frac{r}{r_0}\right)^i \frac{i\cos(i\theta)}{r}$$

and

$$\partial_\theta u(r,\theta) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i i\sin(i\theta).$$

Consequently

$$\partial_\theta u(r,\theta-\pi/2) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i i\cos(i\theta)$$

from which lemma follows easily. □

Lemma 2.12.5. *Let $u(r,\theta)$ be the harmonic extension defined on $[0,r_0]\times S^1$ of the even funcion $\varphi\in\mathcal{C}^{2,\alpha}(S^1)$ and such that $u(r_0,\theta)=\varphi(\theta)$. Then*

$$\partial_\theta u(r,\theta-\pi/2)_{|r=r_0} = r_0\partial_r u(r,\theta)_{|r=r_0}.$$

Proof. If $\varphi(\theta)=\sum_{i\geqslant 0}\varphi_i\cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(r,\theta) = -\sum_{i\geqslant 1} \varphi_i \left(\frac{r_0}{r}\right)^i \frac{i\cos(i\theta)}{r}$$

and

$$\partial_\theta u(r,\theta) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i i\sin(i\theta).$$

Consequently

$$\partial_\theta u(r,\theta-\pi/2) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i i\cos(i\theta)$$

from which lemma follows easily. □

2.12.2 The proof of proposition 2.4.2

We start giving the statement of a classical result about the injectivity of Δ_0.

Lemma 2.12.6. *Given $0<r_0<r_1\leqslant 1$, let w be a solution of $\Delta_0 w=0$ on $S^1\times[r_0,r_1]$ such that $w(\cdot\,,r_0)=w(\cdot\,,r_1)=0$. Then $w=0$.*

As consequence of the lemma 2.12.6, the operator Δ_0 is injective. Hence, Fredholm alternative let us assure that there exists, an unique $w\in\mathcal{C}^{2,\alpha}(S^1\times[r_0,1])$, with $w(\theta,r)$ satisfying:

$$\begin{cases} \Delta_0\, w = f \quad \text{on } S^1\times[r_0,1] \\ w(\cdot,r_0)=w(\cdot,1)=0. \end{cases} \tag{2.66}$$

We want to prove the following assertion.

Assertion 2.12.7. *For every $0 < r_0 < 1$, $f \in \mathcal{C}^{0,\alpha}(S^1 \times [r_0, 1])$ and $w \in \mathcal{C}^{2,\alpha}(S^1 \times [r_0, 1])$ satisfying* (2.66) *there exists a constant c such that*

$$\|w\|_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])}.$$

We suppose by contradiction that the assertion 2.12.7 is false, that is it does not exist a universal constant for which the previous estimate holds. Then, for each $n \in \mathbb{N}$, there exist $r_{0,n}$ and f_n, w_n satisfying (2.66) (with $r_{0,n}, f_n, w_n$ instead of r_0, f, w) such that

$$\sup_{S^1\times[r_{0,n},1]} |f_n| = 1 \quad \text{and} \quad A_n := \sup_{S^1\times[r_{0,n},1]} |w_n| \to +\infty \quad \text{as} \quad n \to \infty.$$

Since $S^1 \times [r_{0,n}, 1]$ is a compact set, A_n is achieved at a point $(\theta_n, r_n) \in S^1 \times [r_{0,n}, 1]$.

The sequence of sets $I_n = [\frac{r_{0,n}}{r_n}, \frac{1}{r_n}]$ converges (up to some subsequence) to a set that we denote by I_∞. We shall show that it is non empty and contains 1. If $r_{0,n} < r' < r'' < 1$, elliptic estimates allow us to conclude

$$\sup_{S^1\times[r_{0,n},r']} |\nabla w_n| \leqslant c(\sup_{S^1\times[r_{0,n},r'']} |f_n| + \sup_{S^1\times[r_{0,n},r'']} |w_n|) \leqslant c(1 + A_n),$$

with c is a constant independent of n.

Then, if $n \to +\infty$, $\frac{r_{0,n}}{r_n} \to R_1 < 1$ and $\frac{1}{r_n} \to R_2 > 1$. The fact that $R_1 < 1$ follows from the above estimate for the gradient of w_n near $r = r_{0,n}$. That implies that the supremum A_n cannot be achieved at a point which is too close to $r_{0,n}$, that is the the point where w_n vanishes. In other terms the quotient $\frac{r_{0,n}}{r_n}$ remains bounded away from 1. Using similar arguments it is possible to show that ∇w_n is bounded near 1 and consequently also $\frac{1}{r_n}$ remains bounded away from 1. Then we can conclude that I_∞ is not empty. We set $I_\infty = [R_1, R_2]$ where $0 \leqslant R_1 < 1 < R_2 < +\infty \in \mathbb{R}$.

We define

$$\widetilde{w}_n(\theta, r) := \frac{1}{A_n}\, w_n(\theta, rr_n) \quad \text{and} \quad \widetilde{f}_n(\theta, r) := \frac{1}{A_n}\, f_n(\theta, rr_n),$$

for all $(\theta, r) \in S^1 \times I_n$, with $I_n = [r_{0,n}/r_n, 1/r_n]$. These functions satisfy $r_n^2 \Delta \widetilde{w}_n = \widetilde{f}_n$. From the definition of $\widetilde{w}_n$, we obtain that

$$\nabla \widetilde{w}_n = \frac{1}{A_n} \nabla w_n(\theta, rr_n),$$

then

$$|\nabla \widetilde{w}_n| \leqslant c\frac{1 + A_n}{A_n} < 2c.$$

Since the sequences $(\tilde{w}_n)_n$ and $(\nabla\tilde{w}_n)_n$ are uniformly bounded, Ascoli-Arzelà theorem assures that a subsequence of $(\tilde{w}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to a non-zero function w_∞, that vanishes on $S^1 \times \partial I_\infty$. The function w_∞ inherits the properties of $\tilde{w}_n$. In particular it holds

$$\sup_{S^1\times I_\infty} |w_\infty| = 1. \tag{2.67}$$

In the same way it's possible to prove that a subsequence of $(\tilde{f}_n)_n$ converges on compact sets of $S^1 \times I_\infty$ to the function $f_\infty \equiv 0$ since, if $n \to \infty$,

$$\sup_{S^1\times I_n} |\widetilde{f}_n| \to 0.$$

Then the limit function w_∞ must satisfy the differential equation

$$\Delta_0 w_\infty = 0$$

on $S^1 \times I_\infty$ with null boundary conditions on ∂I_∞. So we can conclude that $\forall r \in I_\infty$ $w_\infty(\theta, r) = 0$. This function does not satisfy (2.67), a contradiction. This proves the assertion 2.12.7.

The elliptic estimate

$$|\nabla w| \leqslant c\left(\sup_{S^1\times[r_0,1]} |f| + \sup_{S^1\times[r_0,1]} |w|\right),$$

allow us to get a uniform estimate of ∇w. This proves the existence of a solution of $\Delta_0 w = f$ defined on $S^1 \times [r_0, 1]$ for which it holds

$$\|w\|_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])} \leqslant c\,\|f\|_{\mathcal{C}^{0,\alpha}(S^1\times[r_0,1])}.$$

Now it is sufficient to use again elliptic estimates to obtain the estimates for the derivatives.

2.12.3 Minimal graphs in $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$

In this section, following [39], we shall find the condition to be satisfied such that the graph Σ of a function defined on $\mathbb{D}^2$ is minimal with respect to the metric $g_{hyp} = \frac{dx_1^2+dx_2^2}{F} + dx_3^2$, where $F = (1 - x_1^2 - x_2^2)^2$. We shall assume that the immersion of Σ in $\mathbb{D}^2 \times \mathbb{R}$ is given by:

$$(x_1, x_2) \to (x_1, x_2, u(x_1, x_2)).$$

The Christoffel symbols, Γ^k_{ij}, associated to g_{hyp} all vanish except

$$\Gamma^1_{11} = \Gamma^2_{21} = \Gamma^2_{12} = -\Gamma^1_{22} = \frac{2x_1}{\sqrt{F}},$$

$$\Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{21}^1 = -\Gamma_{11}^2 = \frac{2x_2}{\sqrt{F}}.$$

Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Then $\varepsilon_1 = \sqrt{F}e_1, \varepsilon_2 = \sqrt{F}e_2, \varepsilon_3 = e_3$ are an orthonormal basis for $\mathbb{M}^2 \times \mathbb{R}$. The coordinate vector fields on Σ are

$$X_1 = \frac{\varepsilon_1}{\sqrt{F}} + u'_{x_1}\varepsilon_3, \quad X_2 = \frac{\varepsilon_2}{\sqrt{F}} + u'_{x_2}\varepsilon_3, \quad N = \frac{1}{W}\left(-u'_{x_1}\sqrt{F}\varepsilon_1 - u'_{x_2}\sqrt{F}\varepsilon_2 + \varepsilon_3\right),$$

with $W = \sqrt{1 + F|\nabla u|^2}$. The induced metric on Σ is defined by

$$g_{11} = \frac{1}{F} + (u'_{x_1})^2, \quad g_{22} = \frac{1}{F} + (u'_{x_2})^2, \quad g_{12} = u''_{x_1 x_2}.$$

If $\bar{\nabla}$ denotes the riemannian connection of the metric g_{hyp}, then the coefficients of the second fundamental form are

$$b_{11} = \langle \nabla_{X_1} X_1, N\rangle = \frac{1}{W}\left(-\frac{2x_1}{\sqrt{F}}u'_{x_1} + \frac{2x_2}{\sqrt{F}}u'_{x_2} + u''_{x_1x_1}\varepsilon_3\right),$$

$$b_{22} = \langle \nabla_{X_2} X_2, N\rangle = \frac{1}{W}\left(\frac{2x_1}{\sqrt{F}}u'_{x_1} - \frac{2x_2}{\sqrt{F}}u'_{x_2} + u''_{x_2x_2}\varepsilon_3\right),$$

$$b_{12} = \langle \nabla_{X_1} X_2, N\rangle = \frac{1}{W}\left(-\frac{2x_2}{\sqrt{F}}u'_{x_1} - \frac{2x_1}{\sqrt{F}}u'_{x_2} + u''_{x_1x_2}\varepsilon_3\right),$$

where we used the following identities

$$\nabla_{X_1} X_1 = 2x_1\varepsilon_1 - 2x_2\varepsilon_2 + u''_{x_1x_1}\varepsilon_3, \quad \nabla_{X_2} X_2 = -2x_1\varepsilon_1 + 2x_2\varepsilon_2 + u''_{x_2x_2}\varepsilon_3,$$

$$\nabla_{X_1} X_2 = 2x_2\varepsilon_1 + 2x_1\varepsilon_2 + u''_{x_1x_2}\varepsilon_3.$$

The mean curvature of Σ with respect to g_{hyp} is given by

$$H(\Sigma) = \frac{1}{2}\frac{b_{11}g_{22} + b_{22}g_{11} + b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Using the expressions of the coefficients of the first and second fundamental form, we find that

$$H(\Sigma) = \frac{F}{2}\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + F|\nabla u|^2}}\right).$$

2.12.4 Minimal surfaces of rotation in $(\mathbb{D}^2 \times \mathbb{R}, g_{hyp})$

In this section, following [39], we shall find the condition to be satisfied such that a surface of revolution Σ in $\mathbb{D}^2 \times \mathbb{R}$ is minimal with respect to the metric $g_{hyp} = \frac{dx_1^2+dx_2^2}{F} + dx_3^2$, where $F = (1 - x_1^2 - x_2^2)^2$. We shall assume that the immersion of Σ in $\mathbb{D}^2 \times \mathbb{R}$ is given, in terms of the cylindrical coordinates (r, θ, z), by:

$$(\theta, z) \to (r(z), \theta, z),$$

where $r(z)$ is a function of z.
It is convenient to express the metric g_{hyp} in terms of the new coordinates. We find

$$g_{hyp} = \frac{dr^2 + r^2 d\theta^2}{F} + dz^2,$$

with $F = (1 - r^2)^2$. The Christoffel symbols, Γ_{ij}^k, associated to g_{hyp} all vanish except

$$\Gamma_{11}^1 = \frac{2r}{1-r^2}, \quad \Gamma_{22}^1 = -\frac{r(1+r^2)}{1-r^2}, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1+r^2}{r(1-r^2)}.$$

Let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$. Then the coordinate vector fields on Σ are

$$X_1 = r'(z)e_1 + e_3 \quad X_2 = e_2, \quad N = \frac{-e_1 + r'(z)e_3}{R},$$

with $R = \sqrt{\frac{1}{F} + (r'(z))^2}$. The induced metric on Σ is defined by

$$g_{11} = \frac{(r'(z))^2}{F} + 1, \quad g_{22} = \frac{r^2(z)}{F}, \quad g_{12} = 0.$$

If $\bar{\nabla}$ denotes the riemannian connection of the metric g_{hyp}, then the coefficients of the second fundamental form are

$$b_{11} = \langle \bar{\nabla}_{X_1} X_1, N \rangle = -\frac{1}{RF}\left(\frac{2r(r')^2}{1-r^2} + r''\right), \quad b_{22} = \langle \bar{\nabla}_{X_2} X_2, N \rangle = \frac{1}{RF}\left(\frac{r(1+r^2)}{1-r^2}\right),$$

$$b_{12} = \langle \bar{\nabla}_{X_1} X_2, N \rangle = 0,$$

where we used the following identities

$$\nabla_{X_1} X_1 = \left(\frac{2r(r')^2}{1-r^2} + r''\right) e_1, \quad \nabla_{X_2} X_2 = -\frac{r(1+r^2)}{1-r^2} e_1,$$

$$\nabla_{X_1} X_2 = -\frac{r'(1+r^2)}{r(1-r^2)} e_2.$$

The mean curvature of Σ with respect to g_{hyp} is given by

$$H(\Sigma) = \frac{1}{2}\frac{b_{11}g_{22} + b_{22}g_{11} + b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}.$$

Using the expressions of the coefficients of the first and second fundamental form, we find that $H(\Sigma) = 0$ if function $r(z)$ satisfies the following differential equation:

$$r(z)r''(z) - (r'(z))^2 - (1 - r(z)^4) = 0.$$

Chapter 3

An end-to-end construction for singly periodic minimal surfaces

3.1 Introduction

Besides the plane and the helicoid, the first singly periodic minimal surface was discovered by Scherk [47] in 1835. This surface, known as *Scherk's second surface*, is a properly embedded minimal surface in $\mathbb{R}^3$ invariant by one translation T we can assume along the x_2-axis, and can be seen as the desingularization of two perpendicular planes P_1 and P_2 containing the x_2-axis. We assume P_1, P_2 are symmetric with respect to the planes $\{x_1 = 0\}$ and $\{x_3 = 0\}$. By changing the angle between P_1, P_2 we obtain a 1-parameter family of properly embedded singly periodic minimal surfaces, we will refer to as Scherk surfaces. In the quotient $\mathbb{R}^3/T$ by its shortest translation T, each Scherk surface has genus zero and four ends asymptotic to flat annuli contained in $P_1/T, P_2/T$. Such ends are called Scherk-type ends.

In 1982, C. Costa [2, 3] discovered a genus one minimal surface with three embedded ends: one top catenoidal end, one middle planar end and one bottom catenoidal end. D. Hoffmann and W.H. Meeks [14, 15, 16] proved the global embeddedness for this Costa example, and generalized it for higher genus. For each $k \geqslant 1$, Costa-Hoffmann-Meeks surface M_k (we will abbreviate by saying CHM example) is a properly embedded minimal surface of genus k and three ends: two catenoidal ones and one middle planar end.

F. Martin and V. Ramos Batista [27] have recently constructed a properly embedded singly periodic minimal example which has genus one and six Scherk-type ends in the quotient $\mathbb{R}^3/T$, called Scherk-Costa surface, based on Costa surface (from now on, T will denote a translation in the x_2-direction). Roughly speaking, they remove each end of Costa surface (asymptotic to a catenoid or a plane) and replace it by two Scherk-type ends.

In this paper we obtain surfaces in the same spirit as Martin and Ramos Batista's one, but by a completely different method. We construct properly embedded singly periodic minimal surfaces with genus $k \geqslant 1$ and six Scherk-type ends in the quotient $\mathbb{R}^3/T$, by gluing (in an analytic way) a compact piece of M_k to two halves of a Scherk surface at the top and bottom catenoidal ends, and one flat horizontal annulus P/T with a disk removed at the middle planar end.

Theorem 3.1.1. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exists a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ invariant by T whose quotient in $\mathbb{R}^3/T$ has genus k and six Scherk-type ends.*

V. Ramos Batista [42] constructed a singly periodic Costa minimal surface with two catenoidal ends and two Scherk-type middle end, which has genus one in the quotient $\mathbb{R}^3/T$. This example is not embedded outside a slab in $\mathbb{R}^3/T$ which contains the topology of the surface. We observe that the surface we obtain by gluing a compact piece of M_1 (Costa surface) at its middle planar end to a flat horizontal annulus with a disk removed has the same properties of Ramos Batista's one.

In 1988, H. Karcher [22, 23] defined a family of properly embedded doubly periodic minimal surfaces, called *toroidal halfplane layers*, which has genus one and four horizontal Scherk-type end in the quotient. In 1989, W. H. Meeks and H. Rosenberg [31] developed a general theory for doubly periodic minimal surfaces having finite topology in the quotient, and used an approach of minimax type to obtain the existence of a family of properly embedded doubly periodic minimal surfaces, also with genus one and four horizontal Scherk-type ends in the quotient. These Karcher's and Meeks and Rosenberg's surfaces have been generalized in [43], constructing a 3-parameter family $\mathcal{K} = \{M_{\sigma,\alpha,\beta}\}_{\sigma,\alpha,\beta}$ of surfaces, called KMR examples (sometimes, they are also referred in the literature as toroidal halfplane layers). Such examples have been classified by J. Pérez, M. Rodríguez and M. Traizet [40] as the only properly embedded doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient. Each $M_{\sigma,\alpha,\beta}$ is invariant by a horizontal translation T (by the period vector at its ends) and a non horizontal one $\widetilde{T}$. We denote by $\widetilde{M_{\sigma,\alpha,\beta}}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$, which has genus zero, infinitely many horizontal Scherk-type ends, and two limit ends.

In 1992, F.S. Wei [49] added a handle to a KMR example $M_{\sigma,0,0}$ in a periodic way, obtaining a properly embedded doubly periodic minimal surface invariant under reflection in three orthogonal planes, which has genus two and four horizontal Scherk-type ends in the quotient. Some years later, W. Rossman, E.C. Thayer and M. Wolgemuth [45] added a different type of handle to a KMR example $M_{\sigma,0,0}$, also in a periodic way, obtaining a different minimal surfaces with the same properties as Wei's one. They also added two handles to a KMR example, getting doubly periodic examples of genus three in the quotient. L. Mazet and M. Traizet [29] have recently added N handles to a KMR example

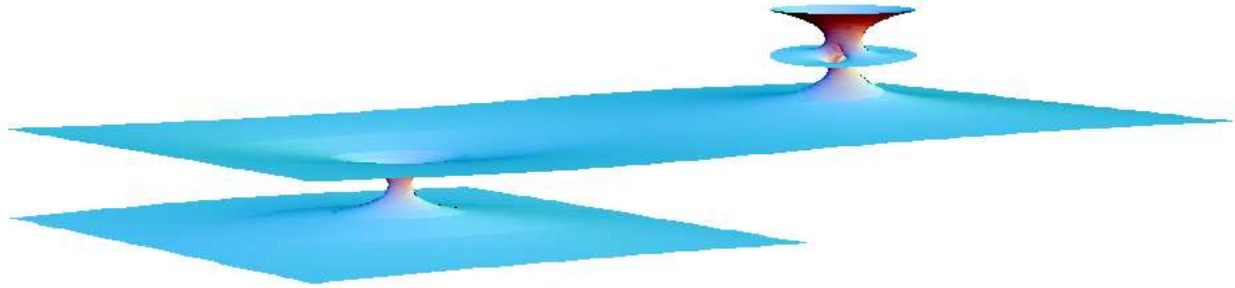

Figure 3.1: A sketch of half a KMR example $M_{\sigma,0,0}$ glued to a compact piece of Costa surface.

$M_{\sigma,0,0}$, obtaining a genus N properly embedded minimal surface in $\mathbb{R}^3/T$ with an infinite number of horizontal Scherk-type ends and two limit ends. They have also constructed a properly embedded minimal surface in $\mathbb{R}^3/T$ with infinite genus, adding handles in a quasi-periodic way to a KMR example

L. Hauswirth and F. Pacard [11] have constructed higher genus Riemann minimal examples in $\mathbb{R}^3$, by gluing two halves of a Riemann minimal example with the intersection of a conveniently chosen CHM surface with a slab. We follow their ideas to generalize Mazet and Traizet's examples by constructing properly embedded singly periodic minimal examples whose quotient to $\mathbb{R}^3/T$ has arbitrary finite genus, infinitely many horizontal Scherk-type ends and two limit ends. More precisely, we glue a compact piece of a slightly deformed CHM example M_k with tilted catenoidal ends, to two halves of a KMR example $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ (see Figure 3.1) and a periodic horizontal flat annuli with a disk removed.

Theorem 3.1.2. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exist two 1-parameter families $\mathcal{K}_1, \mathcal{K}_2$ of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus k, infinitely many horizontal Scherk-type ends and two limit ends. The surfaces in $\mathcal{K}_1$ have a vertical plane of symmetry orthogonal to the x_1-axis, and the surfaces in $\mathcal{K}_2$ have a vertical plane of symmetry orthogonal to the x_2-axis.*

L. Mazet, M. Traizet and M. Rodriguez [28] have recently constructed saddle towers with infinitely many ends: they are non-periodic properly embedded minimal surfaces in $\mathbb{R}^3/T$ with infinitely many ends and one limit end. In the present paper, we construct (non-periodic) properly embedded minimal surfaces in $\mathbb{R}^3/T$ with arbitrary finite genus $k \geqslant 0$, infinitely many ends and one limit end. With this aim, we glue half a Scherk example with half a KMR example, in the case $k = 0$; and, when $k \geqslant 1$, we glue a compact piece of the CHM example M_k to half a Scherk surface (at the bottom catenoidal end of M_k),

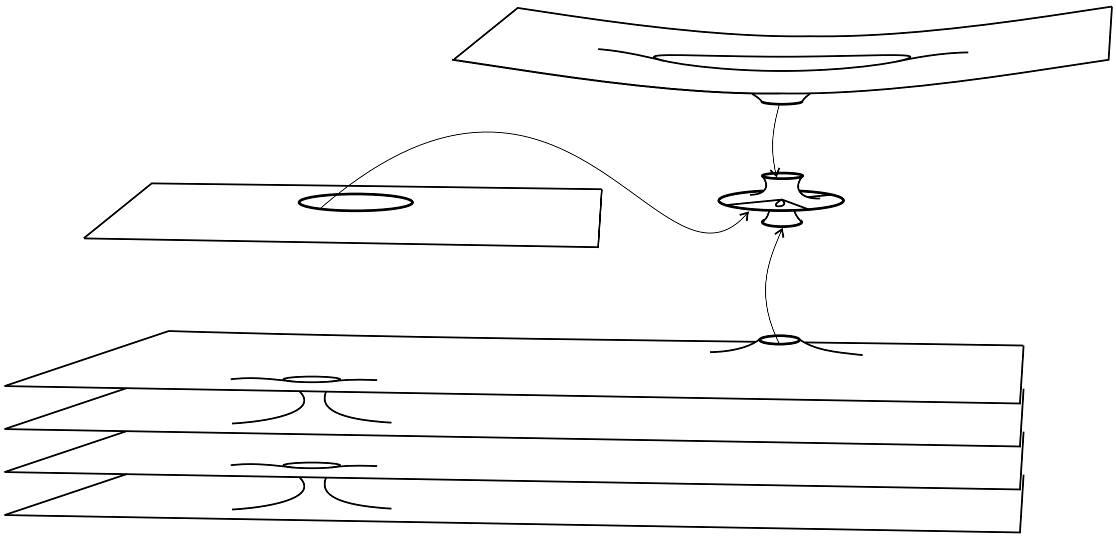

Figure 3.2: A sketch of a surface in the family of Theorem 3.1.3

a periodic horizontal flat annuli with a disk removed (at the middle planar end) and half a KMR example (at the top catenoidal end), see Figure 3.2.

Theorem 3.1.3. *Let T denote a translation in the x_2-direction. For each $k \geqslant 1$, there exist a 1-parameter family of properly embedded singly periodic minimal surfaces in $\mathbb{R}^3$ whose quotient in $\mathbb{R}^3/T$ has genus $k \geqslant 0$, infinitely many horizontal Scherk-type ends and one limit end.*

The family of KMR examples is a three parameters family which contains two subfamilies whose surfaces have a vertical plane of symmetry. In the construction of our examples, we need to have at least one vertical plane of symmetry in order to control the kernel of the Jacobi operator on each glued piece. For this reason, we are not able to produce a 3-parameter family of KMR examples with higher genus in theorem 3.1.2.

The paper is organized as follows: in section 3.2, we apply an implicit function theorem to solve the Dirichlet problem for the minimal graph equation on a horizontal flat periodic annuli with a disk B removed, prescribed the boundary data on ∂B and the asymptotic direction of the Scherk-type ends. We construct the flat annuli with a disk removed we will glue to the CHM example at its middle planar end. Varying the asymptotic direction of the ends and the flux of the surface, we obtain the pieces of Scherk example we will glue at the top and bottom catenoidal ends of the CHM example (proving theorem 3.1.1) and to half a KMR example (theorem 3.1.3).

In section 3.3 we briefly describe the CMH examples M_k and obtain, for each genus k, a 1-parameter family of surfaces $M_k(\xi)$ by bending the catenoidal ends of $M_k = M_k(0)$ keeping a vertical plane of symmetry. This is used to prescribe the flux of the deformed CHM surface, which has to be the same as the corresponding KMR example we want to glue (theorem 3.1.2). To simplify the construction of examples satisfying theorems 3.1.1 and 3.1.3, we consider a "no bent" CHM example M_k. In section 3.4 we perturb $M_k(\xi)$ using the implicit function theorem. We get an infinite dimensional family of minimal surfaces that have three boundaries.

In section 3.5, we study the KMR examples $M_{\sigma,\alpha,\beta}$ and describe a conformal parametrization of these examples on a cylinder. We also obtain an expansion of pieces of the KMR examples as the flux of $M_{\sigma,\alpha,\beta}$ becomes horizontal (i.e. near the catenoidal limit). Section 3.6 is devoted to the study of the mapping properties of the Jacobi operator about such $M_{\sigma,\alpha,\beta}$ near the catenoidal limit. And we apply in section 3.7 the implicit function theorem to perturb half a KMR example $M_{\sigma,\alpha,0}$, obtaining a family of minimal surfaces asymptotic to half a $M_{\sigma,\alpha,0}$ and whose boundary is a Jordan curve. We prescribe the boundary data of such a surface.

Finally, we do the end-to-end construction in section 2.11: we explain how the boundary data of the corresponding minimal surfaces constructed in sections 3.2, 3.4 and 3.7 can be chosen so that the union of these forms smooth minimal surfaces in the conditions of theorems 3.1.1, 3.1.2 and 3.1.3.

3.2 An infinite family of Scherk type minimal surfaces close to a horizontal periodic flat annuli

This section has two purposes. The first one is to find an infinite family of minimal surfaces close to a horizontal periodic flat annuli Σ with a disk D_s removed. The surfaces of this family have two horizontal Scherk-type ends E_1, E_2 and will be glued on the middle planar end of a CHM surface. We will prescribe the boundary data φ on ∂D_s. Assume the period T of Σ points to the x_2-direction. Then E_1, E_2 have the x_1-axis as asymptotic direction.

The second and more general purpose of this section is to show the existence of an infinite family of minimal graphs over $\Sigma - D_s$ whose ends have slightly modified asymptotic directions. When the asymptotic directions are not horizontal, these surfaces are close to half a Scherk surface, seen as a graph over $\Sigma - D_s$ (see Figure 3.2, above). A piece of one such surface will be glued to the catenoidal ends of a CHM example M_k and to an end of a KMR example $M_{\sigma,0,0}$. We will prescribe the boundary data on ∂D_s. Since we need to prescribe the flux along ∂D_s, we will modify the asymptotic direction of the ends and we will choose $|T|$ large.

The Scherk type end. We parametrize conformally the annulus $\Sigma \subset \mathbb{R}^3/T$ (with $T = |T|e_2$) on $\mathbb{C}^*$ by the mapping

$$A(w) = \left(-\frac{|T|}{2\pi}\ln(w), 0\right), \quad w \in \mathbb{C}^*.$$

A horizontal Scherk type annulus end E_1 is the graph of a function $h \in \mathcal{C}^{2,\alpha}(B_{r_0}(0))$ on a flat annulus $A(w)$ with $w \in B_{r_0}(0)$ (the function $h(w)$ is bounded and extends to the puncture (see [12])). The immersion of E_1 is given by

$$X_1(w) = A(w) + h_1(w)e_3 = \left(-\frac{|T|}{2\pi}\ln(w), h(w)\right) \in \mathbb{R}^3/T$$

in the orthonormal frame $\mathcal{F} = (e_1, e_2, e_3)$, with $w \in B_{r_0}(0) - \{0\} \subset \mathbb{R}^2$. The period T of the end is along the x_2-axis and its asymptotic direction is e_1.

The frame $\mathcal{F}$ can be changed in function of the asymptotic direction of the end. If R_θ denotes a rotation by the angle θ about the x_2-axis oriented by e_2, we consider the end E_1 having a non horizontal asymptotic direction. It can be parametrized by

$$X_{1,\theta}(w) = R_\theta \circ A(w) + h_1(w)R_\theta e_3.$$

The asymptotic direction of the end is $\cos\theta e_1 + \sin\theta e_3$. That is equivalent to consider $X(z) = (-\frac{|T|}{2\pi}\ln(z), h(z))$ in the frame $\mathcal{F}(\theta) = R_\theta\mathcal{F}$ with $z \in B^*_{r_0}(0)$. The coordinate w is called *Graph coordinate.*

The end E_2 of Σ is parametrized outside the ball $B_{r_0^{-1}}(0)$ but it admits a parametrization on the punctured disk. Actually using an inversion we can parametrize the end by

$$X_2(w) = \left(-\frac{|T|}{2\pi}\ln(w), h_2(w)\right),$$

with $w \in B^*_{r_0}(0)$, in the orthonormal frame $\mathcal{F}(\pi) = (-e_1, -e_2, e_3)$. Now the asymptotic direction of the end is $-e_1$.

In the following, for any given $\theta = (\theta_1, \theta_2) \in [0, \theta_0]^2$ we denote by A_θ the immersion of $\mathbb{C}^*$ obtained by a smooth interpolation of $R_{\theta_1} \circ A(\{|z| < r\}), R_{\theta_2} \circ A(\{|z| > r^{-1}\})$ and $A(\{r < |z| < r^{-1}\})$. Let N_θ be the vector field which equals e_3 on $\{r < |z| < r^{-1}\}$, $R_{\theta_1}e_3$ on $\{|z| < r\}$ and $R_{\theta_2}e_3$ on $\{|z| > r^{-1}\}$. We consider for a function $h \in C^{0,\alpha}(\bar{\mathbb{C}})$ the immersion

$$X_{\theta,h}(z) = A_\theta(z) + h(z)N_\theta(z).$$

At the end E_i, the mean curvature of $X_{\theta_i,h}(z) = A_{\theta_i}(z) + h(z)N_{\theta_i}$ (see [12]) is given in terms of the w-coordinate (at the puncture) by

$$2H(\theta_i, h) = \frac{4\pi|w|^2}{|T|^2}\mathrm{div}_0(P^{-1/2}\nabla_0 h),$$

where $P = 1 + \frac{4\pi|w|^2}{|T|^2}\|\nabla_0 h\|_0^2$ and the index $\cdot_0$ means that the corresponding object is computed with the respect to the flat metric of the w-plane. We denote by λ the smooth function without zeroes defined by $\lambda(w) = \frac{|T|^2}{4\pi|w|^2}$ at the puncture. Then at E_i we can write

$$2\bar{H} = 2\lambda H = P^{-1/2}\Delta_0 h - \frac{1}{2}P^{-3/2}\langle\nabla_0 P, \nabla_0 h\rangle_0.$$

So the mean curvature at the end is zero if h satisfies the equation

$$\Delta_0 h - \frac{1}{2}P^{-1}\langle\nabla_0 P, \nabla_0 h\rangle_0 = 0. \tag{3.1}$$

Definition 3.2.1. *Given $k \in \mathbb{N}$, $\alpha \in (0,1)$ we define the space of functions $C^{k,\alpha}(\bar{\mathbb{C}})$ to be the space of functions of $C^{k,\alpha}_{loc}(\bar{\mathbb{C}})$ and for which the following norm is finite*

$$||u||_{C^{k,\alpha}(\bar{\mathbb{C}})} := [u]_{k,\alpha,\bar{\mathbb{C}}},$$

where $[u]_{k,\alpha,\bar{\mathbb{C}}}$ denotes the usual $C^{k,\alpha}$ Hölder norm on the set $\bar{\mathbb{C}}$.

Let B_s the ball of radius s excised from $\mathbb{C}^*$, such that $X_{\theta,h}(B_s) = D_s$ is a geodesic ball of Σ centered at the $(0,0,0)$. We denote by $C^{k,\alpha}(\bar{\mathbb{C}} - B_s)$ the subspace of the functions of $C^{k,\alpha}(\bar{\mathbb{C}})$ restricted to $\bar{\mathbb{C}} - B_s$ and by $[C^{k,\alpha}(\bar{\mathbb{C}} - B_s)]_0$ the subspace of the functions vanishing on the boundary.

We denote by $H(\theta, h)$ the mean curvature of $X_{\theta,h}$. Lemma 4.1 of [12] shows that

$$\bar{H}(\theta, h) : \mathbb{R}^2 \times \mathcal{C}^{2,\alpha}(\bar{\mathbb{C}} - B_s) \longrightarrow \mathcal{C}^{0,\alpha}(\bar{\mathbb{C}} - B_s).$$

The Jacobi operator about A_θ is $\mathcal{L}_\theta$. We set $\bar{\mathcal{L}}_\theta = \lambda\mathcal{L}_\theta$.

Remark 3.2.2. *The operators H and $\mathcal{L}_\theta$ are the mean curvature operator and the Jacobi operator with respect to the metric $|dz|^2$ of $\bar{\mathbb{C}}$. Defining the operators $\bar{H} = \lambda H$ and $\bar{\mathcal{L}}_\theta = \lambda\mathcal{L}_\theta$ means to consider a different metric on $\bar{\mathbb{C}}$. Actually $\bar{H}$ and $\bar{\mathcal{L}}_\theta$ are the mean curvature operator and Jacobi operator with respect to the metric $|dz|^2/\lambda$. From the definition of λ at the punctures, it follows that the volume of $\bar{\mathbb{C}}$ with respect this metric is finite.*

$\bar{\mathcal{L}}_\theta$ is a second order linear elliptic operator satisfying $|\bar{\mathcal{L}}_\theta u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$ and the coefficients of $F_\theta u = (\Delta - \bar{\mathcal{L}}_\theta)u$ have compact support.

Now we fix s such that $D_s = X_{\theta,h}(B_s)$ is the geodesic disk of radius $1/2\sqrt{\varepsilon}$ and we let $|T|$ becoming large. Then s depends on values of ε and $|T|$. We choose $|T| \in [4/\sqrt{\varepsilon}, +\infty)$ such that $s \in (0, s_0)$ for a value s_0 fixed.

Proposition 3.2.3. *There exists $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and $|T| \in (\eta_0, +\infty)$, the operator*

$$G_{\varepsilon,|T|} : \mathcal{C}^{0,\alpha}(\bar{\mathbb{C}} - B_s) \longrightarrow \mathcal{C}^{2,\alpha}(\bar{\mathbb{C}} - B_s)$$

is a left inverse for Δ and for all $f \in \mathcal{C}^{0,\alpha}(\bar{\mathbb{C}} - B_s)$, $w := G_{\varepsilon,|T|}(f)$ satisfies

$$\begin{cases} \Delta w = f & \text{on} \quad \bar{\mathbb{C}} - B_s \\ w \in \mathrm{Span}\{1\} & \text{on} \quad \partial B_s \end{cases}$$

and $||w||_{\mathcal{C}^{2,\alpha}} \leqslant c||f||_{\mathcal{C}^{0,\alpha}}$ for some constant $c > 0$ which does not depend on $\varepsilon, |T|$.

Proof. Let u be a solution of $\Delta u = f$ on $\bar{\mathbb{C}} - B_s$ with $u = 0$ on ∂B_s. We set $w = u - \frac{1}{vol(\bar{\mathbb{C}} - B_s)} \int_{\bar{\mathbb{C}} - B_s} u$. We want to observe that the metric in use on $\bar{\mathbb{C}}$ is given by $|dz|^2/\lambda$. With respect to this metric $vol(\bar{\mathbb{C}} - B_s) < +\infty$ and $\int_{\bar{\mathbb{C}} - B_s} u < \infty$. So the function w is well defined and $\int_{\bar{\mathbb{C}} - B_s} w = 0$ with $w \in \mathrm{Span}\{1\}$ on ∂B_s.

If the proposition is false, there is a sequence of functions f_n, of solutions w_n and of real numbers s_n such that

$$\sup_{\bar{\mathbb{C}} - B_{s_n}} |f_n| = 1, \quad A_n := \sup_{\bar{\mathbb{C}} - B_{s_n}} |w_n| \to +\infty \text{ as } n \to +\infty,$$

where $s_n \in [0, s_0]$. Now we set $\tilde{w}_n := w_n/A_n$. Elliptic estimates imply that s_n and $\tilde{w}_n$ converge up to a subsequence, respectively, to $s_\infty \in [0, s_0]$ and to $\tilde{w}_\infty$ on $\bar{\mathbb{C}} - B_{s_\infty}$. This function satisfies

$$\Delta \tilde{w}_\infty = 0.$$

Then $\tilde{w}_\infty = \text{const}$ on $\bar{\mathbb{C}} - B_{s_\infty}$ and $\int_{\bar{\mathbb{C}} - B_{s_\infty}} \tilde{w}_\infty = 0$, a contradiction with $\sup \tilde{w}_\infty = 1$. $\square$

Now we fix $|T| \geqslant 4/\sqrt{\varepsilon}$, $\theta \in (0, \varepsilon)^2$, $s_\varepsilon = \frac{1}{2\sqrt{\varepsilon}}$ and let $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ be even (or odd) L^2-orthogonal to the function $z \to 1$, with $\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)} \leqslant \kappa\varepsilon$. Let $w_\varphi = \tilde{H}_{s_\varepsilon,\varphi}$ (see proposition 3.9.2) be the unique bounded harmonic extension of φ. We would like to solve the minimal surface equation $H(\theta, v + w_\varphi) = 0$ with fixed boundary data φ, prescribed asymptotic direction θ and with period $|T|$. Then we have to solve the equation:

$$\Delta v = F_\theta(v + w_\varphi) + Q_\theta(v + w_\varphi)$$

with Q_θ a quadratic term such that $|Q_\theta(v_1) - Q_\theta(v_2)| \leqslant c|v_1 - v_2|^2$. The resolution of the previous equation is obtained by showing the existence of a fixed point

$$v = S(\theta, \varphi, v)$$

where

$$S(\theta, \varphi, v) = G_{\varepsilon,|T|}(F_\theta(v + w_\varphi) + Q_\theta(v + w_\varphi)).$$

Proposition 3.2.4. *There exists $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that for all $|T| \geqslant 4/\sqrt{\varepsilon}$ we have*

$$||S(\theta, \varphi, 0)||_{\mathcal{C}^{2,\alpha}} \leqslant c_\kappa \varepsilon^2$$

and for all $\varepsilon \in (0, \varepsilon_\kappa)$,

$$||S(\theta, \varphi, v_1) - S(\theta, \varphi, v_2)||_{\mathcal{C}^{2,\alpha}} \leqslant \frac{1}{2}||v_2 - v_1||_{\mathcal{C}^{2,\alpha}}$$
$$||S(\theta, \varphi_1, v) - S(\theta, \varphi_2, v)||_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon||\varphi_2 - \varphi_1||_{\mathcal{C}^{2,\alpha}}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}(\bar{\mathbb{C}} - B_s)$ and satisfying $||v||_{\mathcal{C}^{2,\alpha}} \leqslant 2c_\kappa\varepsilon^2$, for all boundary data $\varphi, \varphi_1, \varphi_2 \perp 1$, whose norm is bounded by a constant κ times ε and for all $\theta = (\theta_1, \theta_2)$ such that $|\theta_1| + |\theta_2| \leqslant \varepsilon$.

Proof. Using the proposition 3.2.3, the inequality $|\bar{\mathcal{L}}_\theta u - \Delta u| \leqslant c(|\theta_1| + |\theta_2|)|u|$ and the quadratic behavior of Q_θ we derive the estimate of the proposition. The details of the proof are left to the reader. □

On the set $B_{2s_\varepsilon} - B_{s_\varepsilon}$, the function $U = v + \tilde{H}_{s_\varepsilon,\varphi}$ is the solution of the equation (3.1). Using the vertical translation $c_0 e_3$ we can fix the value at the boundary:

$$U = c_0 + \tilde{H}_{s_\varepsilon,\varphi} + v.$$

Using Schäuder estimate for the equation on a fixed bounded domain

$$||v(\varphi_1) - v(\varphi_2)||_{\mathcal{C}^{2,\alpha}(\bar{\mathbb{C}}-B_s)} \leqslant c_\kappa\varepsilon||\varphi_1 - \varphi_2||_{\mathcal{C}^{2,\alpha}(S^1)}.$$

This can be done uniformly in (θ_1, θ_2). We observe that the function U grows logarithmically close $\partial B_{s_\varepsilon}$. The hypothesis of orthogonality of φ to the constant function, implies that the function w_φ enjoys the same property and is bounded. It is not the case of v which can be seen as the sum of a bounded function and a function of the form $c\ln(r/s_\varepsilon)$, where $c = c(|T|, \theta_1, \theta_2)$, defined in a neighbourhood of $\partial B_{s_\varepsilon}$. We are able to determine c using flux formula. Let γ_1, γ_2 be two closed curves in $\bar{\Sigma}/T$ chosen in such a way to correspond by the conformal mapping to the boundaries of two circular neighbourhoods N_1, N_2 of the punctures corresponding to the ends with linear growth. Let $\mathcal{S} = \bar{\mathbb{C}} - (B_{s_\varepsilon} \cup N_1 \cup N_2)$. Now we observe that, being X the parametrization of a minimal surface, then the following equality holds:

$$\int_{\mathcal{S}} \Delta X = 0.$$

Thanks to the divergence theorem, if $\Gamma = \partial\mathcal{S}$, then

$$\int_{\mathcal{S}} \Delta X = \int_\Gamma \frac{\partial X}{\partial \eta} ds = \int_{\gamma_1} \frac{\partial X}{\partial \eta} ds + \int_{\gamma_2} \frac{\partial X}{\partial \eta} ds + \int_{\partial B_{s_\varepsilon}} \frac{\partial X}{\partial \eta} ds = 0,$$

where η denotes the conormal along Γ. This equality implies

$$\int_{\partial B_{s_\varepsilon}} \frac{\partial U}{\partial \eta} ds = -\sin\theta_1|T| - \sin\theta_2|T|.$$

By integration we can conclude that on $B_{2s_\varepsilon} - B_{s_\varepsilon}$, it holds that

$$U = -\frac{|T|}{2\pi}(\sin\theta_1 + \sin\theta_2)\ln(r/s_\varepsilon) + c_0 + w_\varphi + v^\perp.$$

with $v^\perp \perp 1$. Now we choose $|T|$ such that $\frac{|T|}{2\pi}(\sin\theta_1 + \sin\theta_2) = 1$.

We observe that, if $\theta_2 = \theta_1 = 0$, we can state that there exists an infinite family of minimal surfaces which are close to the surface $\Sigma - D_{s_\varepsilon}$. We denote by $S_m(\varphi)$ one of such surfaces. It can be seen as the graph about $B_{2s_\varepsilon} - B_{s_\varepsilon}$ of the function

$$\bar{U}_m(r,\theta) = c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \bar{V}_m$$

where $V_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$ and it satisfies

$$||\bar{V}_m(\varphi_1) - \bar{V}_m(\varphi_2)||_{\mathcal{C}^{2,\alpha}(B_{2s_\varepsilon} - B_{s_\varepsilon})} \leqslant c_\kappa \varepsilon ||\varphi_1 - \varphi_2||_{\mathcal{C}^{2,\alpha}(S^1)}, \tag{3.2}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(S^1)$.

If $(\theta_2, \theta_1) \neq 0$, we can state that there exists an infinite family of minimal surfaces which are close to the periodic Scherk type example. After a vertical translation, they can be seen as the graph about $B_{2s_\varepsilon} - B_{s_\varepsilon}$ of the function

$$\bar{U}_t(r,\theta) = -\ln(2r) + c_0 + \tilde{H}_{s_\varepsilon,\varphi}(r,\theta) + \bar{V}_t$$

where $\bar{V}_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$ and it satisfies

$$||\bar{V}_t(\varphi_1) - \bar{V}_t(\varphi_2)||_{\mathcal{C}^{2,\alpha}(B_{2s_\varepsilon} - B_{s_\varepsilon})} \leqslant c_\kappa \varepsilon ||\varphi_1 - \varphi_2||_{\mathcal{C}^{2,\alpha}(S^1)}, \tag{3.3}$$

for $\varphi_2, \varphi_1 \in C^{2,\alpha}(S^1)$.

Remark 3.2.5. *If the boundary data φ is an even function, it is clear the surfaces we have just described are symmetric with respect to the vertical plane $x_2 = 0$. Instead, if the boundary data φ is an odd function and $\theta_1 = \theta_2$ the surfaces are symmetric with respect to the vertical plane $x_1 = 0$.*

3.3 A Costa-Hoffman-Meeks type surface with bent catenoidal ends

In this section we recall the result shown in [11] about the existence of a family of minimal surfaces close to the Costa-Hoffman-Meeks surfaces of genus $k \geqslant 1$, one planar end and two slightly bent catenoidal ends by an angle ξ. We denote one member of the family by $M_k(\xi)$. Then $M_k(0)$ is the family of the Costa-Hoffman-Meeks surface of genus k.

The family of the Costa-Hoffman-Meeks surfaces. Each member of the family of surfaces $M_k(0)$, after suitable rotation and translation, enjoys the following properties.

1. It has one planar end E_m asymptotic to the $x_3 = 0$ plane, one top end E_t and one bottom end E_b that are respectively asymptotic to the upper end and to the lower end of a catenoid with x_3-axis of revolution. The planar end E_m is located between the two catenoidal ends.

2. It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the symmetry with respect to the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the symmetry with respect to the $x_3 = 0$ plane.

3. It intersects the $x_3 = 0$ plane in $k + 1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 = const\,(\neq 0)$ is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

Now we give a local description of the surfaces $M_k(0)$ near its ends and we introduce coordinates that we will use.

The planar end. The planar end E_m of the surface M_k can be parametrized by

$$X_m(x) := \left(\frac{x}{|x|^2}, u_m(x) \right) \in \mathbb{R}^3$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$. Here $\rho_0 > 0$ is fixed small enough. The minimal surface equation has the following form

$$|x|^4 \operatorname{div} \left(\frac{\nabla u}{(1 + |x|^4 \, |\nabla u|^2)^{1/2}} \right) = 0. \tag{3.4}$$

It can be shown (see [11]) that the function u_m can be extended at the origin continuously using Weierstrass representation. In particular it is possible to show that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$

and $u_m = \mathcal{O}_{C_b^{2,\alpha}}(|x|^{k+1})$, where the expression $\mathcal{O}_{C_b^{n,\alpha}}(g)$ denotes a function that, together with its partial derivatives of order less than or equal to $n+\alpha$ is bounded by a constant times g.

If we linearize in $u = 0$ the nonlinear equation (3.4) we obtain the expression of an operator which is the Jacobi operator about the plane, that is $\mathcal{L}_{\mathbb{R}^2} = |x|^4\Delta_0$. To be more precise, the linearization of (3.4) gives

$$L_u\, v = |x|^4 \mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right). \tag{3.5}$$

We will give the expression of H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of the mean curvature of Σ_u, that is H_u. Here we shall show that

$$2H_{u+v} = 2H_u + L_u v + |x|^4 Q_u(\sqrt{|x|^4}\nabla v, \sqrt{|x|^4}\nabla^2 v), \tag{3.6}$$

where Q_u satisfies

$$Q_u(0,0) = \nabla Q'_u(0,0).$$

To show (3.6), we start observing that:

$$\frac{1}{\sqrt{1+|x|^4|\nabla(u+v)|^2}} = \frac{1}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}} + Q_{u,1}(v) \tag{3.7}$$

where the function $Q_{u,1}$ satisfies $Q_{u,1}(0) = \nabla Q_{u,1}(0) = 0$. To prove (3.7) it's sufficient to set

$$f(t) = \frac{1}{\sqrt{1+|x|^4|\nabla(u+tv)|^2}}$$

and to write down the Taylor's series of order one of this function and to evaluate it in $t = 1$. That is $f(1) = f(0) + f'(0) + \frac{1}{2}f''(\bar t)$, with $\bar t \in (0,1)$. We insert (3.7) in the expression that defines $2H_{u+v}$ to get

$$|x|^4\mathrm{div}\left(\frac{\nabla(u+v)}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla(u+v)\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}} + \nabla(u+v)Q_{u,1}(v)\right) =$$

$$2H_u + |x|^4\mathrm{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right) + |x|^4 Q_u(|x|^2\nabla v, |x|^2\nabla^2 v).$$

From this it follows the wanted expression.

Since we assume that Σ_u is a minimal surface, we will consider $H_u = 0$. Then we get the expression of the minimal surfaces equation that we will use in the following sections:

$$|x|^4\left(\Delta_0 v + \sqrt{1+|x|^4|\nabla u|^2}\left(\bar L_u v + Q_u(|x|^2\nabla v, |x|^2\nabla^2 v)\right)\right) = 0, \tag{3.8}$$

where $\bar{L}_u$ is a second order linear operator whose coefficients are in $\mathcal{O}_{C^{2,\alpha}}\left(|x|^{k+1}\right)$.

The catenoidal ends. We denote by X_c the parametrization of the standard catenoid C whose axis of revolution is the x_3-axis. Its expression is

$$X_c(s,\theta) := (\cosh s\,\cos\theta, \cosh s\,\sin\theta, s) \in \mathbb{R}^3$$

where $(s,\theta) \in \mathbb{R}\times S^1$. The unit normal vector field about C is given by

$$n_c(s,\theta) := \frac{1}{\cosh s}\,(\cos\theta, \sin\theta, -\sinh s).$$

Up to some dilation, we can assume that the two ends E_t and E_b of M_k are asymptotic to some translated copy of the catenoid parametrized by X_c in the vertical direction. Therefore, E_t and E_b can be parametrized, respectively, by

$$X_t := X_c + w_t\, n_c + \sigma_t\, e_3$$

for $(s,\theta) \in (s_0,\infty)\times S^1$,

$$X_b := X_c - w_b\, n_c - \sigma_b\, e_3$$

for $(s,\theta) \in (-\infty,-s_0)\times S^1$, where $\sigma_t,\sigma_b \in \mathbb{R}$, functions w_t, w_b tend exponentially fast to 0 as s goes to ∞ reflecting the fact that the ends are asymptotic to a catenoidal end.

We recall that the surface parametrized by $X := X_c + w\, n_c$ is minimal if and only if the function w satisfies the minimal surface equation which, for normal graphs over a catenoid has the following form

$$\mathbb{L}_C w + \frac{1}{\cosh^2 s}\left(Q_2\left(\frac{w}{\cosh s}\right) + \cosh s\, Q_3\left(\frac{w}{\cosh s}\right)\right) = 0, \tag{3.9}$$

where $\mathbb{L}_C$ is the Jacobi operator about the catenoid, that is

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\partial^2_{ss} w + \partial^2_{\theta\theta} w + \frac{2w}{\cosh^2 s}\right)$$

and Q_2, Q_3 are linear second order differential operators which are bounded in $\mathcal{C}^k(\mathbb{R}\times S^1)$ for all k. These functions satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$,$\nabla^2 Q_3(0) = 0$ and then:

$$\|Q_j(v_2) - Q_j(v_1)\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)} \leqslant c\left(\sup_{i=1,2}\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right)^{j-1}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \tag{3.10}$$

for all $s\in\mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathcal{C}^{2,\alpha}[s,s+1]\times S^1)} \leqslant 1$. The constant $c>0$ does not depend on s.

The family of Costa-Hoffman-Meeks surfaces with bent catenoidal ends. Using an elaborate version of the implicit function theorem and following [19] and [26] it is possible to prove the following

Theorem 3.3.1 ([11]). *There exists $\xi_0 > 0$ and a smooth one parameter family of minimal hypersurfaces $(M_k(\xi))_\xi$, for $\xi \in (-\xi_0, \xi_0)$, with two catenoidal ends and one planar end. In particular $M_k(0) = M_k$, the upper (resp. lower) catenoidal end of $M_k(\xi)$ is, up to a translation along its axis, asymptotic to the upper (resp. lower) end of the standard catenoid whose axis of revolution is directed by $\sin\xi\, e_1 + \cos\xi\, e_3$. Moreover $M_k(\xi)$ has one horizontal planar end and is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.*

The upper (lower) end of $M_k(\xi)$ is, up to a translation, asymptotic to the upper (lower) end of the same (standard) catenoid. Then the upper end $E_t(\xi)$ and the lower end $E_b(\xi)$ of $M_k(\xi)$, if R_ξ denotes the rotation of angle ξ about the x_2 axis, can be parametrized respectively by

$$X_{t,\xi} = R_\xi\,(X_c + w_{t,\xi}\, n_c) + \sigma_{t,\xi}\, e_3 + \varsigma_{t,\xi}\, e_1 \tag{3.11}$$

$$X_{b,\xi} = R_\xi\,(X_c - w_{b,\xi}\, n_c) - \sigma_{b,\xi}\, e_3 - \varsigma_{b,\xi}\, e_1 \tag{3.12}$$

where the functions $w_{t,\xi}, w_{b,\xi}$, the numbers $\sigma_{t,\xi}, \varsigma_{t,\xi}, \sigma_{b,\xi}, \varsigma_{b,\xi} \in \mathbb{R}$ depend smoothly on ξ and satisfy

$$|\sigma_{t,\xi}-\sigma_t|+|\sigma_{b,\xi}-\sigma_b|+|\varsigma_{t,\xi}|+|\varsigma_{b,\xi}|+\|w_{t,\xi}-w_t\|_{C^{2,\alpha}_{-2}([s_0,+\infty)\times S^1)}+\|w_{b,\xi}-w_b\|_{C^{2,\alpha}_{-2}((-\infty,-s_0]\times S^1)} \leqslant c|\xi|$$

for $|\xi| < |\xi_0|$.

For all $\rho < \rho_0$ and $s > s_0$, we define

$$M_k(\xi, s, \rho) := M_k(\xi) - \left[X_{t,\xi}((s,\infty)\times S^1) \cup X_{b,\xi}((-\infty,-s)\times S^1) \cup X_m(B_\rho(0))\right]. \tag{3.13}$$

The parametrizations of the three ends of $M_k(\xi)$ induce a decomposition of $M_k(\xi)$ into slightly overlapping components: a compact piece $M_k(\xi, s_0+1, \rho_0/2)$ and three noncompact pieces $X_{t,\xi}((s_0,\infty)\times S^1)$, $X_{b,\xi}((-\infty,-s_0)\times S^1)$ and $X_m(\bar{B}_{\rho_0}(0))$.

We define the weighted space of functions on $M_k(\xi)$.

Definition 3.3.2. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_k(\xi))$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(M_k(\xi))$ for which the following norm is finite*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_k(\xi))} := \|w\|_{\mathcal{C}^{\ell,\alpha}(M_k(\xi,s_0+1,\rho_0/2))} + \|\, w\circ X_m\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}$$
$$+ \sup_{s\geqslant s_0} e^{-\delta s}\,\left(\|w\circ X_{t,\xi}\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)} + \|w\circ X_{b,\xi}\|_{\mathcal{C}^{\ell,\alpha}([-s-1,-s]\times S^1)}\right)$$

and which are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane, that is $w(p) = w(\bar{p})$ for all $p \in M_k(\xi)$, where $\bar{p} := (x_1, -x_2, x_3)$ if $p = (x_1, x_2, x_3)$.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two ended surface. We will perturb the surface $M_k(\xi)$ by the normal graph of a function $u \in \mathcal{C}^{2,\alpha}_\delta$ and the middle end E_m will be just translated in the vertical direction.

The Jacobi operator. The Jacobi operator about $M_k(\xi)$ is

$$\mathbb{L}_{M_k(\xi)} := \Delta_{M_k(\xi)} + |A_{M_k(\xi)}|^2$$

where $|A_{M_k(\xi)}|$ is the norm of the second fundamental form on $M_k(\xi)$.

In the parametrization introduced above of the ends the volume forms $dvol_{M_k(\xi)}$ can be written as $\gamma_t \, ds \, d\theta$ and $\gamma_b \, ds \, d\theta$ near the catenoidal type ends and as $\gamma_m \, dx_1 \, dx_2$ near the middle end. Now we can define globally on $M_k(\xi)$ a smooth function

$$\gamma : M_k(\xi) \longrightarrow [0, \infty)$$

that is identically equal to 1 on $M_k(\xi, s_0 - 1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_t(\xi)$ (resp. $E_b(\xi)$, E_m). Observe that, on $X_{t,\xi}((s_0, \infty) \times S^1)$ and on $X_{b,\xi}((-\infty, -s_0) \times S^1)$ we have

$$\gamma \circ X_{t,\xi}(s, \theta) \sim \cosh^2 s \qquad \text{and} \qquad \gamma \circ X_{b,\xi}(s, \theta) \sim \cosh^2 s.$$

Finally on $X_m(B_{\rho_0})$, we have

$$\gamma \circ X_m(x) \sim |x|^{-4}.$$

Granted the above defined spaces, one can check that:

$$\begin{array}{rccc} \mathcal{L}_{\xi,\delta} : & \mathcal{C}^{2,\alpha}_\delta(M_k(\xi)) & \longrightarrow & \mathcal{C}^{0,\alpha}_\delta(M_k(\xi)) \\ & w & \longmapsto & \gamma \, \mathbb{L}_{M_k(\xi)} \, (w) \end{array}$$

is a bounded linear operator. The subscript δ is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, the function γ is here to counterbalance the effect of the conformal factor $\frac{1}{\sqrt{|g_{M_k(\xi)}|}}$ in the expression of the Laplacian in the coordinates we use to parametrize the ends of the surface $M_k(\xi)$. This is precisely what is needed to have the operator defined from the space $\mathcal{C}^{2,\alpha}_\delta(M_k(\xi))$ into the target space $\mathcal{C}^{0,\alpha}_\delta(M_k(\xi))$.

To have a better grasp of what is going on, let us linearize the nonlinear equation (3.9) at $w = 0$. We get the expression of the Jacobi operator about the standard catenoid

$$\mathbb{L}_C := \tfrac{1}{\cosh^2 s} \left(\partial_s^2 + \partial_\theta^2 + \tfrac{2}{\cosh^2 s} \right).$$

We can observe that the operator $\cosh^2 s\, \mathbb{L}_C$ maps the space $(\cosh s)^\delta\, \mathcal{C}^{2,\alpha}((s_0, +\infty) \times S^1)$ into the space $(\cosh s)^\delta\, \mathcal{C}^{0,\alpha}((s_0, \infty) \times S^1)$.

Similarly, if we linearize the nonlinear equation (3.4) at $u = 0$, we obtain (see (3.5) with $u = 0$) the expression of the Jacobi operator about the plane

$$\mathbb{L}_{\mathbb{R}^2} := |x|^4\, \Delta_0.$$

Again, the operator $|x|^{-4}\, \mathbb{L}_{\mathbb{R}^2} = \Delta_0$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})$. Now, the function γ plays, for the ends of the surface $M_k(\xi)$, the role played by the function $\cosh^2 s$ for the ends of the standard catenoid and the role played by the function $|x|^{-4}$ for the plane. Since the Jacobi operator about $M_k(\xi)$ is asymptotic to $\mathbb{L}_{\mathbb{R}^2}$ at E_m and is asymptotic to $\mathbb{L}_C$ at $E_t(\xi)$ and $E_b(\xi)$, we conclude that the operator $\mathcal{L}_{\xi,\delta}$ maps $\mathcal{C}^{2,\alpha}_\delta(M_k(\xi))$ into $\mathcal{C}^{0,\alpha}_\delta(M_k(\xi))$.

We recall the notion of non degeneracy introduced in [11]:

Definition 3.3.3. *The surface $M_k(\xi)$ is said to be non degenerate if $\mathcal{L}_{\xi,\delta}$ is injective for all $\delta < -1$.*

It useful to observe that a duality argument in the weighted Lebesgue spaces, implies that

$$(\mathcal{L}_{\xi,\delta} \quad \text{is injective}) \quad \Leftrightarrow \quad (\mathcal{L}_{\xi,-\delta} \quad \text{is surjective})$$

if $\delta \notin \mathbb{Z}$. See [34] and [19] for more details.

The non degeneracy of $M_k(\xi)$ follows from the study of the kernel of $\mathcal{L}_{\xi,\delta}$.

The Jacobi fields. It is known that a smooth one parameter group of isometries containing the identity generates a Jacobi field, that is a solution of the equation $\mathbb{L}_{M_k(\xi)} u = 0$. These solutions are generated by the following one parameter groups of isometries: the vertical translations, the translations along the x_1-axis, the dilations. We refer [11] for details.

The group of vertical translations generated by the Killing vector field $\Xi(p) = e_3$ gives rise to the Jacobi field

$$\Phi^{0,+}(p) := n(p) \cdot e_3.$$

The vector field $\Xi(p) = p$ that is associated to the one parameter group of dilation generates a Jacobi fields

$$\Phi^{0,-}(p) := n(p) \cdot p.$$

The Killing vector field $\Xi(p) = e_1$ that generates the group of translations along the x_1-axis is associated to a Jacobi field

$$\Phi^{1,+}(p) := n(p) \cdot e_1.$$

Finally, we denote by

$$\Phi^{1,-}(p) := n(p) \cdot (e_2 \times p)$$

the Jacobi field associated to the Killing vector field $\Xi(p) = e_2 \times p$ that generates the group of rotations about the x_2-axis.

The Jacobi equation has other solutions which are not taken into account because in the difference with the four Jacobi fields just introduced they are not invariant under the action of the symmetry with respect to the $x_2 = 0$ plane.

With these notations, we define the deficiency space

$$\mathcal{D} := \mathrm{Span}\{\chi_t\, \Phi^{j,\pm}, \chi_b\, \Phi^{j,\pm} \;:\; j = 0, 1\}$$

where χ_t is a cutoff function that is identically equal to 1 on $X_{t,\xi}((s_0 + 1, \infty) \times S^1)$, identically equal to 0 on $M_k(\xi) - X_{t,\xi}((s_0, \infty) \times S^1)$ and that is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Also, we agree that

$$\chi_b(\cdot) := \chi_t(- \cdot).$$

Clearly for $\delta < 0$:

$$\begin{array}{rccc} \tilde{\mathcal{L}}_{\xi,\delta} : & \mathcal{C}^{2,\alpha}_{\delta}(M_k(\xi)) \oplus \mathcal{D} & \longrightarrow & \mathcal{C}^{0,\alpha}_{\delta}(M_k(\xi)) \\ & w & \longmapsto & \gamma\, \mathbb{L}_{M_k(\xi)}\,(w) \end{array}$$

is a bounded linear operator.
A result of S. Nayatani shown in [36, 37] and extended in chapter 1 assures that the bounded Jacobi fields of M_k invariant with respect to the mirror symmetry across the $x_2 = 0$ plane, is linear combination of $\Phi^{0,+}$ and $\Phi^{1,+}$. This fact together with an adaptation to our setting of the linear decomposition Lemma proved in [26] for constant mean curvature surfaces (see also [19] for minimal hypersurfaces) allows us to get the following result

Proposition 3.3.4. *We fix $\delta \in (-2, -1)$. Then (reducing ξ_0 if this is necessary) the operator $\tilde{\mathcal{L}}_{\xi,\delta}$, for $|\xi| < \xi_0$, is surjective and has a kernel of dimension 4.*

From that we get the following one about the operator $\mathcal{L}_{\xi,\delta}$

Proposition 3.3.5. *We fix $\delta \in (1, 2)$. Then (reducing ξ_0 if this is necessary) the operator $\mathcal{L}_{\xi,\delta}$ is surjective and has a kernel of dimension 4. Moreover, there exists $G_{\xi,\delta}$ a right inverse for $\mathcal{L}_{\xi,\delta}$ that depends smoothly on ξ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.*

3.4 An infinite dimensional family of minimal surfaces which are close to $M_k(\xi)$

In this section we consider a truncature of $M_k(\xi)$. First we recall a result of [11] that describes the region of the surface which can be parametrized by a graph on a $x_3 = 0$ plane.

Lemma 3.4.1 ([11]). *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and all $|\xi| \leqslant \varepsilon$ an annular part of the ends $E_t(\xi)$, $E_b(\xi)$ and E_m of $M_k(\xi)$ can be written as vertical graphs over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon/2}$ for the functions*

$$\begin{aligned} U_t(r,\theta) &= \sigma_{t,\xi} + \ln(2r) - \xi\, r\, \cos\theta + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon), \\ U_b(r,\theta) &= -\sigma_{b,\xi} - \ln(2r) - \xi\, r\, \cos\theta + \mathcal{O}_{\mathcal{C}_b^\infty}(\varepsilon), \\ U_m(r,\theta) &= \mathcal{O}_{\mathcal{C}_b^\infty}(r^{-(k+1)}). \end{aligned}$$

Here (r, θ) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}(\varepsilon)$ are defined in the annulus $B_{2\,r_\varepsilon} - B_{r_\varepsilon/2}$ and are bounded in $\mathcal{C}_b^\infty$ topology by a constant (independent on ε) multiplied by ε, where the partial derivatives are computed with respect to the vector fields $r\,\partial_r$ and ∂_θ.

Then $M_k(\varepsilon/2)$ has two ends $E_t(\varepsilon/2)$ and $E_b(\varepsilon/2)$ which are graphs over the $x_3 = 0$ plane for functions U_t and U_b defined on the annulus $B_{2\,r_\varepsilon} - B_{r_\varepsilon/2}$. We set

$$s_\varepsilon = -\frac{1}{2}\ln\varepsilon \quad \text{and} \quad \rho_\varepsilon = 2\varepsilon^{1/2}.$$

We define $M_k^T(\varepsilon/2)$ to be equal to $M_k(\varepsilon/2)$ from which we have removed the image of $(s_\varepsilon, +\infty) \times S^1$ by $X_{t,\varepsilon/2}$, the image of $(-\infty, -s_\varepsilon) \times S^1$ by $X_{b,\varepsilon/2}$ and the image of $B_{\rho_\varepsilon}(0)$ by X_m. In this section we will prove the existence of a family of surfaces close to $M_k^T(\varepsilon/2)$. We follow the work [11].

First, we modify the parametrization of the end $E_t(\varepsilon/2)$, $E_b(\varepsilon/2)$ and E_m, for appropriates values of s, so that, when $r \in [3r_\varepsilon/4, 3r_\varepsilon/2]$ the curves corresponding to the image of

$$\theta \to (r\, \cos\theta, r\, \sin\theta, U_t(r,\theta)), \quad \theta \to (r\, \cos\theta, r\, \sin\theta, U_b(r,\theta))$$

correspond to the curve $s = \pm\log(2r)$.

The curve $\theta \to (r\, \cos\theta, r\, \sin\theta, U_m(r,\theta))$ corresponds to $\rho = \frac{1}{r}$.

The second step is the modification of unit normal vector field on $M_k(\varepsilon/2)$ into a transverse unit vector field $\tilde{n}_{\varepsilon/2}$ in such a way that it coincides with the normal vector field $n_{\varepsilon/2}$ on $M_k(\varepsilon/2)$, is equal to e_3 on the graph over $B_{3r_\varepsilon/2} - B_{3r_\varepsilon/4}$ of the functions U_t and

U_b and interpolate smoothly between the different definitions of $\tilde{n}_{\varepsilon/2}$ in different subsets of $M_k^T(\varepsilon/2)$.

Finally we observe that close to $E_t(\varepsilon/2)$, we can give the following estimate:

$$\left|\cosh^2 s \left(\mathbb{L}_{M_k(\varepsilon/2)} v - \cosh^{-2} s \left(\partial^2_{ss} v + \partial^2_{\theta\theta} v\right)\right)\right| \leqslant c \left|\cosh^{-2} s\, v\right|. \tag{3.14}$$

This follows easily from (3.9) together with the fact that $w_{t,\xi}$ (see (3.11)) decays at least like $\cosh^{-2} s$ on $E_t(\varepsilon/2)$. Similar considerations hold close the bottom end $E_b(\varepsilon/2)$. Near the middle planar end E_m, we observe that the following estimate holds:

$$\left||x|^{-4} \left(\mathbb{L}_{M_k(\varepsilon/2)} v - |x|^4 \Delta_0 v\right)\right| \leqslant c \left||x|^{2k+3} \nabla v\right|. \tag{3.15}$$

This follows easily from (3.5) together with the fact that u_m decays at least like $|x|^{k+1}$ on E_m.

The graph of a function u, using the vector field $\tilde{n}_{\varepsilon/2}$, will be a minimal surface if and only if u is a solution of a second order nonlinear elliptic equation of the form

$$\mathbb{L}_{M_k^T(\varepsilon/2)}\, u = \tilde{L}_{\varepsilon/2}\, u + Q_\varepsilon\left(u\right)$$

where $\mathbb{L}_{M_k^T(\varepsilon/2)}$ is the Jacobi operator about $M_k^T(\varepsilon/2)$, Q_ε is a nonlinear second order differential operator and $\tilde{L}_{\varepsilon/2}$ is a linear operator which takes into account the change of the normal vector field (only for the top and bottom ends) and of the change of the parametrization. This operator is supported in a neighbourhood of $\{\pm s_\varepsilon\} \times S^1$ where its coefficients are uniformly bounded by a constant times ε and of $\{\rho_\varepsilon\} \times S^1$ where its coefficients are uniformly bounded by a constant times ε^2.

Now, we consider three functions $\varphi_t, \varphi_b, \varphi_m \in \mathcal{C}^{2,\alpha}(S^1)$ which are even, with respect to θ, φ_t, φ_b are L^2 orthogonal to 1 and $\cos\theta$ while φ_m is L^2 orthogonal to 1. Assume that they satisfy

$$\|\varphi_t\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_b\|_{\mathcal{C}^{2,\alpha}} + \|\varphi_m\|_{\mathcal{C}^{2,\alpha}} \leqslant \kappa\, \varepsilon. \tag{3.16}$$

We set $\Phi := (\varphi_t, \varphi_b, \varphi_m)$ and we define w_Φ to be the function equal to

1. $\chi_+ \, H_{\varphi_t}(s_\varepsilon - s, \cdot)$ on the image of $X_{t,\varepsilon/2}$ where χ_+ is a cut-off function equal to 0 for $s \leqslant s_0 + 1$ and identically equal to 1 for $s \in [s_0+2, s_\varepsilon]$,

2. $\chi_- \, H_{\varphi_b}(s - s_\varepsilon, \cdot)$ on the image of $X_{b,\varepsilon/2}$ where χ_- is a cut-off function equal to 0 for $s \geqslant -s_0 - 1$ and identically equal to 1 for $s \in [-s_\varepsilon, -s_0-2]$,

3. $\chi_m \, \tilde{H}_{\rho_\varepsilon,\varphi_m}(\cdot,\cdot)$ on the image of X_m, where χ_m is a cut-off function equal to 0 for $r \geqslant \rho_0$ and identically equal to 1 for $\rho \in [\rho_\varepsilon, \rho_0/2]$,

4. 0 on the remaining part of the surface $M_k^T(\varepsilon/2)$.

The operators $\tilde{H}$ and H are harmonic extensions, introduced in appendix A respectively in propositions 3.9.2 and 3.9.4.

We would like to prove that, under appropriates hypothesis, the graph about $M_k^T(\varepsilon/2)$ of the function $u = w_\Phi + v$ is a minimal surface. This is equivalent to solve the equation:

$$\mathbb{L}_{M_k^T(\varepsilon/2)}(w_\Phi + v) = \tilde{L}_{\varepsilon/2}(w_\Phi + v) + Q_\varepsilon(w_\Phi + v)$$

on $M_k^T(\varepsilon/2)$, so that the graph of $u = w_\Phi + v$ will be a minimal surface. The resolution of the previous equation is obtained thanks to the one of the following fixed point problem:

$$v = T(\Phi, v) \tag{3.17}$$

with

$$T(\Phi, v) = G_{\varepsilon/2,\delta} \circ \mathcal{E}_\varepsilon \left(\gamma\left(\tilde{L}_{\varepsilon/2}(w_\Phi + v) - \mathbb{L}_{M_k^T(\varepsilon/2)}\, w_\Phi + Q_\varepsilon(w_\Phi + v)\right)\right)$$

where $\delta \in (1, 2)$, the operator $G_{\varepsilon/2,\delta}$ is defined in proposition 3.3.5 and $\mathcal{E}_\varepsilon$ is a linear extension operator such that

$$\mathcal{E}_\varepsilon : \mathcal{C}_\delta^{0,\alpha}(M_k^T(\varepsilon/2)) \longrightarrow \mathcal{C}_\delta^{0,\alpha}(M_k(\varepsilon/2)),$$

where $\mathcal{C}_\delta^{0,\alpha}(M_k^T(\varepsilon/2))$ denotes the space of functions of $\mathcal{C}_\delta^{0,\alpha}(M_k(\varepsilon/2))$ restricted to $M_k^T(\varepsilon/2)$. It is defined by $\mathcal{E}_\varepsilon v = v$ in $M_k^T(\varepsilon/2)$, $\mathcal{E}_\varepsilon v = 0$ in the image of $[s_\varepsilon+1, +\infty) \times S^1$ by $X_{t,\varepsilon/2}$, in the image of $(-\infty, -s_\varepsilon - 1) \times S^1$ by $X_{b,\varepsilon/2}$ and in the image of $B_{\rho_\varepsilon/2} \times S^1$ by X_m. Finally $\mathcal{E}_\varepsilon v$ is an interpolation of these values in the remaining part of $M_k(\varepsilon/2)$ such that

$$(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon/2}(s, \theta) = (1 + s_\varepsilon - s)(v \circ X_{t,\varepsilon/2}(s_\varepsilon, \theta)), \quad \text{for} \quad (s, \theta) \in [s_\varepsilon, s_\varepsilon + 1] \times S^1,$$

$$(\mathcal{E}_\varepsilon v) \circ X_{b,\varepsilon/2}(s, \theta) = (1 + s_\varepsilon + s)(v \circ X_{b,\varepsilon/2}(s_\varepsilon, \theta)) \quad \text{for} \quad (s, \theta) \in [-s_\varepsilon - 1, -s_\varepsilon] \times S^1,$$

$$(\mathcal{E}_\varepsilon v) \circ X_m(\rho, \theta) = \left(\frac{2}{\rho_\varepsilon}\rho - 1\right)(v \circ X_m(\rho_\varepsilon, \theta)) \quad \text{for} \quad (\rho, \theta) \in [\rho_\varepsilon/2, \rho_\varepsilon] \times S^1.$$

Remark 3.4.2. *As consequence of the properties of $\mathcal{E}_\varepsilon$, if supp $v \cap \left(B_{\rho_\varepsilon} - B_{\rho_\varepsilon/2}\right) \neq \emptyset$ then*

$$\|(\mathcal{E}_\varepsilon v) \circ X_m\|_{\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leqslant c\rho_\varepsilon^{-\alpha} \|v \circ X_m\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\varepsilon})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\|(\mathcal{E}_\varepsilon v) \circ X_{t,\varepsilon/2}\|_{\mathcal{C}^{0,\alpha}([s_0,+\infty)\times S^1)} \leqslant c\|v \circ X_{t,\varepsilon/2}\|_{\mathcal{C}^{0,\alpha}([s_0,s_\varepsilon]\times S^1)}.$$

A similar equation holds for the bottom end.

In the following we will assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ for the equation (3.17) is a consequence of the following result which proves that T is a contracting mapping.

Lemma 3.4.3. *There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that*

$$\|T(\Phi,0)\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant c_\kappa\,\varepsilon^{3/2} \tag{3.18}$$

and, for all $\varepsilon \in (0,\varepsilon_\kappa)$, $0<\alpha<\frac{1}{2}$,

$$\begin{aligned}\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant \frac{1}{2}\,\|v_2-v_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))},\\ \|T(\Phi_2,v)-T(\Phi_1,v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} &\leqslant c\varepsilon\,\|\Phi_2-\Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)},\end{aligned} \tag{3.19}$$

where we have set for short

$$\|\Phi_2-\Phi_1\|_{\mathcal{C}^{2,\alpha}(S^1)} = \|\varphi_{t,2}-\varphi_{t,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{b,2}-\varphi_{b,1}\|_{\mathcal{C}^{2,\alpha}(S^1)} + \|\varphi_{m,2}-\varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}$$

for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \leqslant 2\,c_\kappa\,\varepsilon^{3/2}$ and for all boundary data $\Phi,\Phi_1,\Phi_2 \in [\mathcal{C}^{2,\alpha}(S^1)]^3$ satisfying (3.16).

Proof. We recall that the Jacobi operator associated to $M_k(\varepsilon/2)$, is asymptotic to the operator of the catenoid near the catenoidal ends, and it is asymptotic to the laplacian near of the planar end. The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_k(\varepsilon/2)}$ is not known: this is the reason of our particular choice in the definition of w_Φ. Then from the definition of w_Φ, thanks to proposition 3.3.5, to (3.14) and (3.15), we obtain the estimate

$$\begin{aligned}
&\|\mathcal{E}_\varepsilon\left(\gamma\mathbb{L}_{M_k^T(\varepsilon/2)}\,w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} = \\
&\left|\left|\left(\gamma\mathbb{L}_{M_k^T(\varepsilon/2)} - \left(\partial_s^2+\partial_\theta^2\right)\right)(w_\Phi\circ X_{t,\varepsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} + \\
&\left|\left|\left(\gamma\mathbb{L}_{M_k^T(\varepsilon/2)} - \left(\partial_s^2+\partial_\theta^2\right)\right)(w_\Phi\circ X_{b,\varepsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} + \\
&\rho_\varepsilon^{-\alpha}\left|\left|\left(\gamma\mathbb{L}_{M_k^T(\varepsilon/2)} - \Delta_0\right)(w_\Phi\circ X_m)\right|\right|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)} \\
&\leqslant c\left|\left|\cosh^{-2}s\,(w_\Phi\circ X_{t,\varepsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)} + c\left|\left|\cosh^{-2}s\,(w_\Phi\circ X_{b,\varepsilon/2})\right|\right|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)} + \\
&c\varepsilon^{-\frac{\alpha}{2}}\left|\left|\rho^{2k+3}\nabla(w_\Phi\circ X_m)\right|\right|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)} \\
&\leqslant c_\kappa\,\varepsilon^2 + c_k\,\varepsilon^{3/2} \leqslant c_\kappa\varepsilon^{3/2}.
\end{aligned}$$

To obtain this estimate we used the following ones:

$$\sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s}\left|\cosh^{-2}s\,(w_\Phi\circ X_{t,\varepsilon/2})\right|_{0,\alpha;[s,s+1]}\leqslant c\sup_{[s_0+1,s_\varepsilon]\times S^1} e^{-\delta s}e^{-2(s_\varepsilon-s)}e^{-2s}|\phi_t|_{2,\alpha}\leqslant$$

$$c\,e^{-2s_\varepsilon}|\phi_t|_{2,\alpha}\leqslant c_k\varepsilon^2$$

(a similar estimate holds for the bottom end) and

$$\sup_{[\rho_\varepsilon,\rho_0]\times S^1}|\rho^{2k+3}\nabla(w_\Phi\circ X_m)|\leqslant c\varepsilon^{-\alpha/2}\|\rho^{2k+3}w_\Phi\|_{C^{2,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)}\leqslant c_k\varepsilon^{3/2}.$$

Using the properties of $\tilde{L}_{\varepsilon/2}$, we obtain

$$\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_{\varepsilon/2}\,w_\Phi\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}\leqslant c\varepsilon\|w_\Phi\circ X_{t,\varepsilon/2}\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\varepsilon]\times S^1)}+c\varepsilon\|w_\Phi\circ X_{b,\varepsilon/2}\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\varepsilon,-s_0-1]\times S^1)}+$$

$$+c\varepsilon^{1-\alpha/2}\|w_\Phi\circ X_m\|_{\mathcal{C}^{0,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)}\leqslant c_\kappa\varepsilon^{3/2}.$$

As for the last term, we recall that the operator Q_ε has two different expressions if we consider the catenoidal type end and the planar end (see (3.9) and (3.8)). It holds that

$$\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}\leqslant c_k\varepsilon^{3/2}.$$

In fact

$$\|\mathcal{E}_\varepsilon\left(\gamma Q_\varepsilon\left(w_\Phi\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}\leqslant c\|w_\Phi\circ X_{t,\varepsilon/2}\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([s_0+1,s_\varepsilon]\times S^1)}+$$

$$c\|w_\Phi\circ X_{b,\varepsilon/2}\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([-s_\varepsilon,-s_0-1]\times S^1)}+c\varepsilon^{-\alpha}\|w_\Phi\circ X_m\|^2_{\mathcal{C}^{2,\alpha}([\rho_\varepsilon,\rho_0]\times S^1)}\leqslant c_k\varepsilon^{3/2}.$$

As for the second estimate, we recall that

$$T(\Phi,v):=G_{\varepsilon/2,\delta}\circ\mathcal{E}_\varepsilon\left(\gamma\left(\tilde{L}_{\varepsilon/2}(w_\Phi+v)-\mathbb{L}_{M_k^T(\varepsilon/2)}\,w_\Phi+Q_\varepsilon\left(w_\Phi+v\right)\right)\right).$$

Then

$$\|T(\Phi,v_2)-T(\Phi,v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))}$$

$$\leqslant c\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_{\varepsilon/2}(v_2-v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}+c\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi+v_1\right)-Q_\varepsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}.$$

We observe that from the considerations above it follows that

$$\|\mathcal{E}_\varepsilon\left(\gamma\tilde{L}_{\varepsilon/2}(v_2-v_1)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}\leqslant c\varepsilon||v_2-v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))}$$

and

$$\|\mathcal{E}_\varepsilon\left(\gamma\left(Q_\varepsilon\left(w_\Phi+v_1\right)-Q_\varepsilon\left(w_\Phi+v_2\right)\right)\right)\|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))}$$

$$\leqslant c_\kappa \varepsilon ||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))}.$$

Then

$$\|T(\Phi, v_2) - T(\Phi, v_1)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant c\varepsilon ||v_2 - v_1||_{\mathcal{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))}.$$

To show the last estimate it is sufficient to observe that

$$\|T(\Phi_2, v) - T(\Phi_1, v)\|_{\mathcal{C}^{2,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant \|\mathcal{E}_\varepsilon \left(\gamma \tilde{L}(w_{\Phi_2} - w_{\Phi_1})\right) \|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} +$$
$$+ \|\mathcal{E}_\varepsilon \left(\gamma \left(Q_\varepsilon \left(w_{\Phi_2} + v\right) - Q_\varepsilon \left(w_{\Phi_1} + v\right)\right)\right) \|_{\mathcal{C}^{0,\alpha}_\delta(M_k(\varepsilon/2))} \leqslant$$
$$\leqslant c\varepsilon \|\Phi_2 - \Phi_1\| + c\|v\|_{\mathcal{C}^{0,\alpha}_\delta(M_k^T(\varepsilon/2))} \|\Phi_2 - \Phi_1\| \leqslant c\varepsilon \|\Phi_2 - \Phi_1\|.$$

□

Theorem 3.4.4. *Let be $B := \{w \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\varepsilon}) \,|\, ||w||_{2,\alpha} \leqslant 2c_\kappa \varepsilon^{3/2}\}$. Then the nonlinear mapping T defined above has a unique fixed point v in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping T is a contraction mapping from the ball B of radius $2c_\kappa \varepsilon^{3/2}$ in $\mathcal{C}^{2,\alpha}_\delta(M_k^T(\varepsilon/2))$ into itself. This value follows from the estimate of the norm of $T(\Phi, 0)$. Consequently thanks to Schäuder fixed point theorem, T has a unique fixed point w in this ball. □

This argument provides a minimal surface $M_k^T(\varepsilon/2, \Phi)$ which is close to $M_k^T(\varepsilon/2)$ and has three boundaries. This surface is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$ whose parametrization is, respectively, given by

$$U_t(r,\theta) = \sigma_{t,\varepsilon/2} + \ln(2r) + \frac{\varepsilon}{2} r\cos\theta + H_{\varphi_t}(s_\varepsilon - \ln 2r, \theta) + V_t(r,\theta),$$
$$U_b(r,\theta) = -\sigma_{b,\varepsilon/2} - \ln(2r) + \frac{\varepsilon}{2} r\cos\theta + H_{\varphi_b}(s_\varepsilon - \ln(2r), \theta) + V_b(r,\theta),$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$. The boundaries of the surface correspond to $r_\varepsilon = \frac{1}{2}\varepsilon^{-1/2}$. Nearby the middle boundary the surface is a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$. Its parametrization is

$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon, \varphi_m}(1/r, \theta) + V_m(r,\theta).$$

All the functions V_i for $i = t, b, m$ depend non linearly on ε, φ.

Lemma 3.4.5. *The functions $V_i(\varepsilon, \varphi_i)$, for $i = t, b$, satisfy $\|V_i(\varepsilon, \varphi_i)(r_\varepsilon\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon$ and*

$$\|V_i(\varepsilon, \varphi_{i,2})(r_\varepsilon\cdot, \cdot) - V_i(\varepsilon, \varphi_{i,1})(r_\varepsilon\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon^{1-\delta/2} \|\varphi_{i,2} - \varphi_{i,1}\|_{\mathcal{C}^{2,\alpha}} \tag{3.20}$$

The function $V_m(\varepsilon, \varphi)$ satisfies $\|V_m(\varepsilon, \varphi)(\rho_\varepsilon\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon$ and

$$\|V_m(\varepsilon, \varphi_{m,2})(\rho_\varepsilon\cdot, \cdot) - V_m(\varepsilon, \varphi_{m,1})(\rho_\varepsilon\cdot, \cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_{1/2})} \leqslant c\varepsilon \|\varphi_{m,2} - \varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}} \tag{3.21}$$

Proof. We start observing that after having observed that the functions V_t, V_b, V_m are the restriction to $E_t(\varepsilon/2), E_b(\varepsilon/2), E_m$ of a fixed point for the operator T. Then the wanted estimates follow from

$$\|V_i(\varepsilon,\varphi_2)(\cdot,\cdot) - V_i(\varepsilon,\varphi_1)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{2r_\varepsilon} - B_{r_\varepsilon/2})} \leqslant c e^{\delta s_\varepsilon} \|T(\Phi_2, V_i) - T(\Phi_1, V_i)\|_{\mathcal{C}^{2,\alpha}_{\delta}(E_i(\varepsilon/2))},$$

for $i = t, b$ and

$$\|V_m(\varepsilon,\varphi_2)(\cdot,\cdot) - V_m(\varepsilon,\varphi_1)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_{2\rho_\varepsilon} - B_{\rho_\varepsilon/2})} \leqslant c \|T(\Phi_2, V_m) - T(\Phi_1, V_m)\|_{\mathcal{C}^{2,\alpha}(E_m)}$$

and the estimate (3.19) of proposition 3.4.3. □

3.5 KMR examples

In this section, we briefly present the *KMR examples* $M_{\sigma,\alpha,\beta}$ studied in [22, 23, 31, 43] (also called *toroidal halfplane layers*), which are the only properly embedded doubly periodic minimal surfaces with genus one and finitely many parallel (Scherk-type) ends in the quotient (see [40]).

For each $\sigma \in (0, \frac{\pi}{2})$, $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [0, \frac{\pi}{2}]$ with $(\alpha, \beta) \neq (0, \sigma)$, consider the rectangular torus $\Sigma_\sigma = \left\{(z, w) \in \overline{\mathbb{C}}^2 \mid w^2 = (z^2 + \lambda^2)(z^2 + \lambda^{-2})\right\}$, where $\lambda = \lambda(\sigma) = \cot\frac{\sigma}{2} > 1$. The KMR example $M_{\sigma,\alpha,\beta}$ is determined by its Gauss map g and the differential of its height function h, which are defined on Σ_σ and given by:

$$g(z, w) = \frac{az + b}{i(\overline{a} - \bar{b}z)}, \qquad dh = \mu \frac{dz}{w},$$

with

$$\begin{aligned} a &= a(\alpha,\beta) = \cos\tfrac{\alpha+\beta}{2} + i\cos\tfrac{\alpha-\beta}{2};\\ b &= b(\alpha,\beta) = \sin\tfrac{\alpha-\beta}{2} + i\sin\tfrac{\alpha+\beta}{2};\\ \mu &= \mu(\sigma) = \tfrac{\pi\csc\sigma}{\mathcal{K}(\sin^2\sigma)}, \end{aligned} \tag{3.22}$$

where $\mathcal{K}(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m\sin^2 u}}\, du$, $0 < m < 1$, is the complete elliptic integral of first kind. Such μ has been chosen so that the vertical part of the flux of $M_{\sigma,\alpha,\beta}$ along any horizontal level section equals 2π.

Remark 3.5.1. *The following statements give us a better understanding of the geometrical meaning of a, b defined above:*

(i) $b \to 0$ if and only if $\alpha \to 0$ and $\beta \to 0$, in which case $a \to 1 + i$.

(ii) $|b|^2 + |a|^2 = 2$.

(iii) *When* $\beta = 0$, *then* $a = (1+i)\cos\frac{\alpha}{2}$ *and* $b = (1+i)\sin\frac{\alpha}{2}$. *In particular,* $b = \mathcal{O}(\alpha)$.

(iv) *When* $\alpha = 0$, *then* $a = (1+i)\cos\frac{\beta}{2}$ *and* $b = (-1+i)\sin\frac{\beta}{2}$. *In particular,* $b = \mathcal{O}(\beta)$.

(v) *In general,* $\left|\frac{b}{a}\right| = \tan\frac{\varphi}{2}$, *where* φ *is the angle between the North Pole* $(0,0,1) \in \mathbb{S}^2$ *and the pole of* g *seen in* $\mathbb{S}^2$ *via the inverse of the stereographic projection.*

The KMR example $M_{\sigma,\alpha,\beta}$ can be parametrized on Σ_σ by the immersion $X = (X_1, X_2, X_3) = \Re \int \mathcal{W}$, where $\mathcal{W}$ is the Weierstrass form:

$$\mathcal{W} = \left(\frac{1}{2}\left(\frac{1}{g} - g\right) dh, \frac{i}{2}\left(\frac{1}{g} + g\right) dh, dh\right).$$

The ends of $M_{\sigma,\alpha,\beta}$ correspond to the punctures $\{A, A', A'', A'''\} = g^{-1}(\{0,\infty\})$, and the branch values of g are those with $w = 0$, i.e.

$$D = (-i\lambda, 0),\ D' = (i\lambda, 0),\ D'' = (\tfrac{i}{\lambda}, 0),\ D''' = (-\tfrac{i}{\lambda}, 0). \tag{3.23}$$

Seen in $\mathbb{S}^2$, these points form two pairs of antipodal points: $D'' = -D$ and $D''' = -D'$. (Each KMR example can be given in terms of the branch values of its Gauss map, see [43].)

In [43], it is proven that the above Weierstrass data define a properly embedded minimal surface $M_{\sigma,\alpha,\beta}$ invariant by two independent translations: the translation by the period T at its ends, and the translation by the period $\widetilde{T}$ along a homology class. Moreover, the group of isometries $\mathrm{Iso}(M_{\sigma,\alpha,\beta})$ of $M_{\sigma,\alpha,\beta}$ always contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, with generators $\mathcal{D}$ (corresponding to the deck transformation $(z,w) \mapsto (z,-w)$), which represents in $\mathbb{R}^3$ a central symmetry about any of the four branch points of the Gauss map of $M_{\sigma,\alpha,\beta}$; and $\mathcal{F}$, which consists of a translation by $\frac{1}{2}(T + \widetilde{T})$. In particular, the ends of $M_{\sigma,\alpha,\beta}$ are equally spaced.

We are going to focus on two more symmetric subfamilies of KMR examples: $\{M_{\sigma,\alpha,0} \mid 0 < \sigma < \frac{\pi}{2},\ 0 \leqslant \alpha \leqslant \frac{\pi}{2}\}$ and $\{M_{\sigma,0,\beta} \mid 0 < \sigma < \frac{\pi}{2},\ 0 \leqslant \beta < \sigma\}$ (these two families were originally studied by Karcher [22, 23]).

1. When $\alpha = \beta = 0$, $M_{\sigma,0,0}$ contains four straight lines parallel to the x_1-axis, and $\mathrm{Iso}(M_{\sigma,0,0})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ with generators S_1, S_2, S_3, R_D: S_1 is a reflection symmetry in a vertical plane orthogonal to the x_1-axis; S_2 is a reflection symmetry across a plane orthogonal to the x_2-axis; S_3 is a reflection symmetry in a horizontal plane (these three planes can be chosen meeting at a common point which is not contained in the surface); and R_D is the π-rotation around one of the four straight lines contained in the surface, see Figure 3.3 left. In this case, $T = (0, \pi\mu, 0)$.

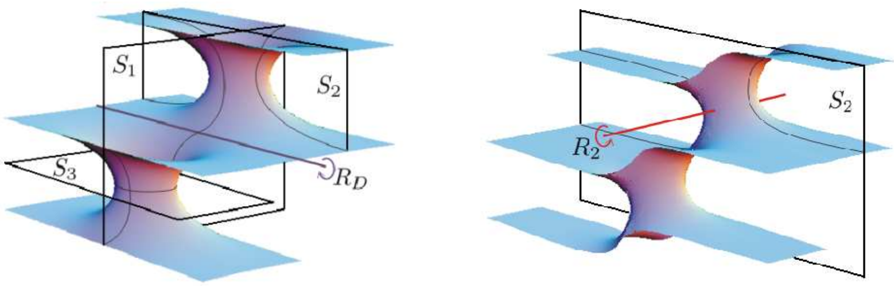

Figure 3.3: Left: $M_{\sigma,0,0}$, with $\sigma = \frac{\pi}{4}$. Right: $M_{\sigma,\alpha,0}$ for $\sigma = \alpha = \frac{\pi}{4}$.

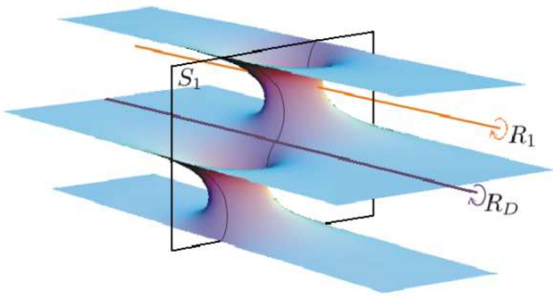

Figure 3.4: $M_{\sigma,0,\beta}$, where $\sigma = \frac{\pi}{4}$ and $\beta = \frac{\pi}{8}$.

2. When $0 < \alpha < \frac{\pi}{2}$, $\mathrm{Iso}(M_{\sigma,\alpha,0})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators $\mathcal{D}$, S_2 and R_2, where S_2 represents a reflection symmetry across a plane orthogonal to the x_2-axis, and R_2 is a π-rotation around a line parallel to the x_2-axis that cuts $M_{\sigma,\alpha,0}$ orthogonally, see Figure 3.3 right. Now $T = (0, \pi\mu t_\alpha, 0)$, with $t_\alpha = \frac{\sin\sigma}{\sqrt{\sin^2\sigma\cos^2\alpha + \sin^2\alpha}}$.

3. Suppose that $0 < \beta < \sigma$. Then $M_{\sigma,0,\beta}$ contains four straight lines parallel to the x_1-axis, and $\mathrm{Iso}(M_{\sigma,0,\beta})$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, with generators S_1, R_1 and R_D: S_1 represents a reflection symmetry across a plane orthogonal to the x_1-axis; R_1 corresponds to a π-rotation around a line parallel to the x_1-axis that cuts the surface orthogonally; and R_D is the π-rotation around any one of the straight lines contained in the surface, see Figure 3.4. Moreover, $T = (0, \pi\mu t^\beta, 0)$, where $t^\beta = \frac{\sin\sigma}{\sqrt{\sin^2\sigma - \sin^2\beta}}$.

Finally, it will be useful to see Σ_σ as a branched 2-covering of $\overline{\mathbb{C}}$ through the map $(z, w) \mapsto z$. Thus Σ_σ can be seen as two copies $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ of $\overline{\mathbb{C}}$ glued along two common cuts γ_1, γ_2, which can be taken along the imaginary axis: γ_1 from D to D' and γ_2 from D'' to D'''.

3.5.1 $M_{\sigma,\alpha,\beta}$ as a graph over $\{x_3 = 0\}/T$

The KMR examples $M_{\sigma,\alpha,\beta}$ converge as $(\sigma, \alpha, \beta) \to (0, 0, 0)$ to a vertical catenoid, since Σ_σ converges to two pinched spheres, $g(z) \to z$ and $dh \to \pm\frac{dz}{z}$ as $(\sigma, \alpha, \beta) \to (0, 0, 0)$. In fact, we can obtain two catenoids in the limit, depending on the choice of branch for w (for each copy of $\overline{\mathbb{C}}$ in Σ_σ, we obtain one catenoid in the limit). One of our aims along this paper consists of gluing KMR examples $M_{\sigma,\alpha,0}$ or $M_{\sigma,0,\beta}$ near this catenoidal limit, to a convenient compact piece of a deformed CHM surface $M_k(\varepsilon/2)$. In this subsection we express part of $M_{\sigma,\alpha,\beta}$ as a vertical graph over the $\{x_3 = 0\}$-plane when σ, α, β are small. In particular we consider $M_{\sigma,\alpha,\beta}$ near the catenoidal limit, i.e. σ, α, β close to zero. Without lost of generality, we can assume $dh \sim \frac{dz}{z}$ in $\overline{\mathbb{C}}_1$. We are studying the surface in an annulus about one of its ends, say a zero of its Gauss map.

Lemma 3.5.2. *Consider $\alpha + \beta + \sigma \leqslant \varepsilon$ small. Up to translations, $M_{\sigma,\alpha,\beta}$ can be parametrized in the annulus $\{(z, w) \in \Sigma_\sigma \mid z \in \overline{\mathbb{C}}_1,\ |\frac{b}{a}| < |z| < \nu\}$ (for $\nu > |\frac{b}{a}|$ small) as:*

$$\begin{cases} X_1 + i\,X_2 = \frac{-1}{2}\left(z + \frac{1}{\bar z}\right) - \frac{(1+i)\bar b}{4\bar z^2} + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}) \\ X_3 = \ln|z| + \mathcal{O}(\varepsilon^2 z^{-2}), \end{cases}$$

Proof. Recall we have assumed $dh \sim \frac{dz}{z}$ in the annulus we are working on. More precisely, we have

$$dh = \frac{\mu\, dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}} = \frac{\mu}{\lambda\sqrt{1+\lambda^{-2}z^2+\lambda^{-2}z^{-2}+\lambda^{-4}}}\,\frac{dz}{z}.$$

Since $\frac{\mu}{\lambda} = \frac{\pi}{(1+\cos(\sigma))\mathcal{K}(\sin^2\sigma)} = 1 + \mathcal{O}(\sigma^4)$, and $\lambda = \cot\frac{\sigma}{2} = \mathcal{O}(\varepsilon)$, we get

$$dh = \frac{dz}{z}(1 + \mathcal{O}(\varepsilon^4))(1 + \mathcal{O}(\varepsilon^2 z^2 + \varepsilon^2 z^{-2} + \varepsilon^4)).$$

Since $|z| < \nu < 1$, then

$$dh = \frac{dz}{z}(1 + \mathcal{O}(\varepsilon^2 z^{-2})).$$

Fix any point $z_0 \in \mathbb{C}_1$, $z_0 \notin \left\{\frac{-b}{a}, \frac{\bar a}{\bar b}\right\}$ (which correspond to two ends of the KMR example), and recall that $g = \frac{az+b}{i(\bar a - \bar b z)}$. Straightforward computations give us, for $|b/a| < |z| < 1$,

- $\int_{z_0}^{z} \frac{dh}{g} = -\frac{i\bar b}{a}\ln z - \frac{2i}{a^2 z} + \frac{2i\,b}{a^3 z^2} + C_1 + \mathcal{O}(\varepsilon^2 z^{-3})$,
- $\int_{z_0}^{z} g\, dh = -\frac{i\,b}{\bar a}\ln z - \frac{2i}{\bar a^2}\, z + C_2 + \mathcal{O}(\varepsilon^2 z^{-1})$,

where $C_1, C_2 \in \mathbb{C}$ verify $\overline{C_1} - C_2 = \frac{1}{2}\left(z_0 + \frac{1}{z_0}\right) + \mathcal{O}(\varepsilon)$. Taking into account that $a = (1+i) + \mathcal{O}(\varepsilon)$, we obtain

$$\begin{aligned} X_1 + i\,X_2 &= \tfrac{1}{2}\left(\overline{\int_{z_0}^{z} \tfrac{dh}{g}} - \int_{z_0}^{z} g\,dh\right) \\ &= \tfrac{i}{\overline{a}^2}\left(\overline{z} + \tfrac{1}{\overline{z}}\right) + \tfrac{i\,b}{a}\ln|z| - \tfrac{i\,\overline{b}}{\overline{a}^3\overline{z}^2} + \tfrac{1}{2}\left(\overline{z_0} + \tfrac{1}{\overline{z_0}}\right) + \mathcal{O}(\varepsilon^2 z^{-3}) \\ &= \tfrac{-1}{2}\left(\overline{z} + \tfrac{1}{\overline{z}}\right) - \tfrac{(1+i)\overline{b}}{4\overline{z}^2} + \tfrac{1}{2}\left(\overline{z_0} + \tfrac{1}{\overline{z_0}}\right) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}). \end{aligned}$$

Similarly, $\int_{z_0}^{z} dh = \ln z - \ln z_0 + \mathcal{O}(\varepsilon^2 z^{-2})$, hence

$$X_3 = \Re \int_{z_0}^{z} dh = \ln|z| - \ln|z_0| + \mathcal{O}(\varepsilon^2 z^{-2}),$$

which completes the proof of lemma 3.5.2. □

By suitably translating $M_{\sigma,\alpha,\beta}$, we can assume its coordinate functions are as in Lemma 3.5.2.

Lemma 3.5.3. *Let (r,θ) denote the polar coordinates in the $\{x_3 = 0\}$ plane. If $\alpha+\beta+\sigma \leqslant \varepsilon$ small, then a piece of $M_{\sigma,\alpha,\beta}$ can be written as a vertical graph of the function*

$$\widetilde{U}(r,\theta) = -\ln(2r) + r\big(\kappa_1\cos\theta - \kappa_2\sin\theta\big) + \mathcal{O}(\varepsilon),$$

for $(r,\theta) \in (\frac{1}{4\sqrt{\varepsilon}}, \frac{4}{\sqrt{\varepsilon}}) \times [0, 2\pi)$, where $\kappa_1 = \Re(b) + \Im(b)$ and $\kappa_2 = \Re(b) - \Im(b)$. We denote by $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ the KMR example $M_{\sigma,\alpha,\beta}$ dilated by $1+\gamma$, for $\gamma \leqslant 0$ small, and translated by a vector $\xi = (\xi_1, \xi_2, \xi_3)$. Then, $M_{\sigma,\alpha,\beta}(\gamma,\xi)$ can be written as a vertical graph of

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = -(1+\gamma)\ln\frac{2r}{1+\gamma} + r\left(\kappa_1\cos\theta - \kappa_2\sin\theta\right) - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + \xi_3 + \mathcal{O}(\varepsilon).$$

Remark 3.5.4. *Recall that $b = \sin\frac{\alpha-\beta}{2} + i\sin\frac{\alpha+\beta}{2}$. In particular:*

- *When $\beta = 0$, we have $\kappa_1 = 2\sin\frac{\alpha}{2}$ and $\kappa_2 = 0$, so*

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = -(1+\gamma)\ln\frac{2r}{1+\gamma} + r\,\kappa_1\,\cos\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + \xi_3 + \mathcal{O}(\varepsilon).$$

- *When $\alpha = 0$, $\kappa_1 = 0$ and $\kappa_2 = 2\sin\frac{\beta}{2}$, so*

$$\widetilde{U}_{\gamma,\xi}(r,\theta) = -(1+\gamma)\ln\frac{2r}{1+\gamma} - r\,\kappa_2\,\sin\theta - \frac{1+\gamma}{r}(\xi_1\cos\theta + \xi_2\sin\theta) + \xi_3 + \mathcal{O}(\varepsilon).$$

- *When $\alpha = 0$ (resp. $\beta = 0$), we will consider $\xi_1 = 0$ (resp. $\xi_2 = 0$) in Section 3.7.*

Proof. Suppose $|\frac{b}{a}| < |z| < \nu$, $\nu > |\frac{b}{a}|$ small. From Lemma 3.5.2, we know the coordinate functions (X_1, X_2, X_3) of the perturbed KMR example $M_{\sigma,\alpha,\beta}(\gamma, \xi)$ are given by

$$\begin{cases} X_1 + i\,X_2 = -\frac{1+\gamma}{2}\left(z + \frac{1}{\bar{z}}\right) + A(z) \\ \\ X_3 = (1+\gamma)\ln|z| + \xi_3 + \mathcal{O}(\varepsilon^2 z^{-2}), \end{cases} \tag{3.24}$$

where

$$\begin{aligned} A(z) &= -\frac{(1+\gamma)(1+i)\overline{b}}{4\overline{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}) \\ &= -\frac{(1+\gamma)(\kappa_1 + i\kappa_2)}{4\overline{z}^2} + (\xi_1 + i\xi_2) + \mathcal{O}(\varepsilon z^{-1} + \varepsilon^2 z^{-3}) \end{aligned}$$

If we denote $z = |z|e^{i\psi}$ and $X_1 + iX_2 = re^{i\theta}$, then $z + \frac{1}{\bar{z}} = \left(|z| + \frac{1}{|z|}\right)e^{i\psi}$ and

$$r\cos\theta = -\frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)\cos\psi + A_1,$$

$$r\sin\theta = -\frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)\sin\psi + A_2,$$

where $A_1 = \Re(A)$ and $A_2 = \Im(A)$. Therefore,

$$\begin{aligned} r^2 = \quad & \frac{(1+\gamma)^2}{4}\left(|z| + \frac{1}{|z|}\right)^2\left(1 - \frac{4|z|}{(1+\gamma)(|z|^2+1)}(A_1\cos\psi + A_2\sin\psi)\right. \\ & \left. + \frac{4|z|^2}{(1+\gamma)^2(|z|^2+1)^2}(A_1^2 + A_2^2)\right). \end{aligned} \tag{3.25}$$

When $\frac{\sqrt{\varepsilon}}{4} \leqslant |z| \leqslant 4\sqrt{\varepsilon}$, the functions A_i are bounded, and we get

$$r = \frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)(1 + \mathcal{O}(\sqrt{\varepsilon})) = \frac{1+\gamma}{2|z|} + \mathcal{O}(\sqrt{\varepsilon}). \tag{3.26}$$

In particular, $r = \mathcal{O}(1/\sqrt{\varepsilon})$. Moreover, we get $\frac{r}{\frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)} = 1 + \mathcal{O}(\sqrt{\varepsilon})$, from where

$$\frac{X_1 + iX_2}{\frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)} = e^{i\theta}(1 + \mathcal{O}(\sqrt{\varepsilon})).$$

On the other hand,

$$\frac{X_1 + iX_2}{\frac{1+\gamma}{2}\left(|z| + \frac{1}{|z|}\right)} = -e^{i\psi} + \frac{2|z|A}{(1+\gamma)(1+|z|^2)} = -e^{i\psi} + \mathcal{O}(\sqrt{\varepsilon}),$$

thus $e^{i\psi} = -e^{i\theta}(1+\mathcal{O}(\sqrt{\varepsilon}))$. From (3.25) and (3.26) we can deduce

$$\frac{(1+\gamma)^2(1+|z|^2)^2}{4|z|^2} = r^2\left(1+\frac{2}{r}\left(A_1\cos\psi + A_2\sin\psi\right)+\mathcal{O}(\varepsilon)\right),$$

from where we obtain

$$\begin{aligned}\frac{1}{|z|^2} &= \left(\frac{2r}{1+\gamma}\right)^2\left(1+\frac{2}{r}\left(A_1\cos\psi + A_2\sin\psi\right)+\mathcal{O}(\varepsilon)\right)(1+\mathcal{O}(\varepsilon))\\ &= \left(\frac{2r}{1+\gamma}\right)^2\left(1+\frac{2}{r}\left(A_1\cos\psi + A_2\sin\psi\right)+\mathcal{O}(\varepsilon)\right),\end{aligned}\tag{3.27}$$

and then

$$-\ln|z| = \ln\frac{2r}{1+\gamma}+\frac{1}{r}(A_1\cos\psi + A_2\sin\psi)+\mathcal{O}(\varepsilon).\tag{3.28}$$

Finally, it is not very difficult to prove that

$$A_1 = -\frac{1+\gamma}{4|z|^2}\left(\kappa_1\cos(2\psi)-\kappa_2\sin(2\psi)\right)+\xi_1+\mathcal{O}(\sqrt{\varepsilon})$$

$$\text{and}\qquad A_2 = -\frac{1+\gamma}{4|z|^2}\left(\kappa_1\sin(2\psi)+\kappa_2\cos(2\psi)\right)+\xi_2+\mathcal{O}(\sqrt{\varepsilon}).$$

Therefore,

$$\begin{aligned}A_1\cos\psi + A_2\sin\psi &= -\frac{1+\gamma}{4|z|^2}\left(\kappa_1\cos\psi-\kappa_2\sin\psi\right)+\xi_1\cos\psi+\xi_2\sin\psi+\mathcal{O}(\sqrt{\varepsilon})\\ &\overset{(3.27)}{=} -\frac{r^2}{1+\gamma}\left(\kappa_1\cos\theta-\kappa_2\sin\theta\right)(1+\mathcal{O}(\sqrt{\varepsilon}))+\xi_1\cos\theta+\xi_2\sin\theta+\mathcal{O}(\sqrt{\varepsilon})\\ &= -\frac{r^2}{1+\gamma}\left(\kappa_1\cos\theta-\kappa_2\sin\theta\right)+\xi_1\cos\theta+\xi_2\sin\theta+\mathcal{O}(\sqrt{\varepsilon}).\end{aligned}$$

From here, (3.28) and (3.24), Lemma 3.5.3 follows. □

3.5.2 Parametrization of the KMR example on the cylinder

In this subsection we want to parametrize the KMR example $M_{\sigma,\alpha,\beta}$ on a cylinder. Recall its conformal compactification Σ_σ only depends on σ. The parameter $\sigma \in (0, \frac{\pi}{2})$ will remains fixed along this subsection, and we will omit the dependence on σ of the functions we are introducing. Also recall that Σ_σ can be seen as a branched 2-covering of $\overline{\mathbb{C}}$, by gluing $\overline{\mathbb{C}}_1, \overline{\mathbb{C}}_2$ along two common cuts γ_1 and γ_2 along the imaginary axis joining the branch points D, D' and D'', D''' respectively (see (3.23)).

We introduce the sphero-conal coordinates (x, y) on the annulus $\mathbb{S}^2 - (\gamma_1 \cup \gamma_2)$ (see [18]): for any $(x, y) \in \mathbb{S}^1 \times (0, \pi) \equiv [0, 2\pi) \times (0, \pi)$, we define

$$F(x, y) = (\cos x \, \sin y, \sin x \, m(y), l(x) \, \cos y) \in \mathbb{S}^2 - (\gamma_1 \cup \gamma_2),$$

where

$$m(y) = \sqrt{1 - \cos^2 \sigma \cos^2 y} \quad \text{and} \quad l(x) = \sqrt{1 - \sin^2 \sigma \sin^2 x}.$$

To understand the geometrical meaning of these variables it is sufficient to remark that the lines $\{x = \textit{constant}\}$ and $\{y = \textit{constant}\}$, are two closed curves on $\mathbb{S}^2$. These two curves are the cross sections of the sphere and two elliptic cones (one with horizontal axis, the other one with vertical axis) having vertex at the center of the sphere.

If we compose $F(x, y)$ with the stereographical projection and enlarge the domain of definition of the function, we obtain the differentiable map $\mathbf{z}(x, y) : \mathbb{S}^1 \times \mathbb{S}^1 \equiv [0, 2\pi) \times [0, 2\pi) \to \overline{\mathbb{C}}$ given by

$$\mathbf{z}(x, y) = \frac{\cos x \, \sin y + i \, \sin x \, m(y)}{1 - l(x) \, \cos y}, \tag{3.29}$$

which is a branch 2-covering of $\overline{\mathbb{C}}$ with branch values in the four points whose sphero-conal coordinates are $(x, y) \in \left\{\pm \frac{\pi}{2}\right\} \times \{0, \pi\}$, which also correspond to D, D', D'', D'''. Moreover, $\mathbf{z}(x, y)$ maps $\mathbb{S}^1 \times (0, \pi)$ on $\overline{\mathbb{C}} - (\gamma_1 \cup \gamma_2)$. Hence we can parametrize the KMR example by $\mathbf{z}$, by means of its Weierstrass data

$$g(\mathbf{z}) = \frac{a\mathbf{z} + b}{i(\overline{a} - \overline{b}\mathbf{z})}, \qquad dh = \mu \, \frac{d\mathbf{z}}{\sqrt{(\mathbf{z}^2 + \lambda^2)(\mathbf{z}^2 + \lambda^{-2})}},$$

We denote by $\widetilde{M}_{\sigma,\alpha,\beta}$ the lifting of $M_{\sigma,\alpha,\beta}$ to $\mathbb{R}^3/T$ by forgetting its non horizontal period (i.e. its period in homology, $\widetilde{T}$). We can then parametrize $\widetilde{M}_{\sigma,\alpha,\beta}$ on $\mathbb{S}^1 \times \mathbb{R}$ by extending $\mathbf{z}$ to $[0, 2\pi) \times \mathbb{R}$. But such a parametrization is not conformal, since the sphero-conal coordinates $(x, y) \mapsto F(x, y)$ of the sphere are not conformal. As the stereographic projection is a conformal map, it suffices to find new conformal coordinates (u, v) of the sphere defined on the cylinder. In particular, we look for a change of variables $(x, y) \mapsto (u, v)$ for which $|\widetilde{F}_u| = |\widetilde{F}_v|$ and $\langle \widetilde{F}_u, \widetilde{F}_v \rangle = 0$, where $\widetilde{F}(u, v) = F(x(u, v), y(u, v))$. We observe that

$$|F_x| = \frac{\sqrt{k(x, y)}}{l(x)} \quad \text{and} \quad |F_y| = \frac{\sqrt{k(x, y)}}{m(y)},$$

with $k(x, y) = \sin^2 \sigma \, \cos^2 x + \cos^2 \sigma \, \sin^2 y$. Then it is natural to consider the change of variables $(x, y) \in [0, 2\pi) \times \mathbb{R} \mapsto (u, v) \in [0, U_\sigma] \times \mathbb{R}$ defined by

$$u(x) = \int_0^x \frac{1}{l(t)} dt \quad \text{and} \quad v(y) = \int_{\frac{\pi}{2}}^y \frac{1}{m(t)} dt, \tag{3.30}$$

where

$$U_\sigma = u(2\pi) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\sin^2\sigma\sin^2 t}}. \tag{3.31}$$

Note that U_σ is a function on σ that goes to 2π as σ approaches to zero, and that the above change of variables is well defined because $\sigma \in (0, \frac{\pi}{2})$. In these variables (u, v), $\mathbf{z}$ is periodic with respect the v variable with period

$$V_\sigma = v(2\pi) - v(0) = \int_0^{2\pi} \frac{dt}{\sqrt{1-\cos^2\sigma\cos^2 t}}. \tag{3.32}$$

The period V_σ goes to $+\infty$ as σ goes to zero (see lemma 3.5.5), which was clear taking into account the limit of $M_{\sigma,\alpha,\beta}$ as σ tends to zero.

From all this, we can deduce that $\widetilde{M}_{\sigma,\alpha,\beta}$ (respectively $M_{\sigma,\alpha,\beta}$) is conformally parametrized on $(u, v) \in I_\sigma \times \mathbb{R}$ ($(u, v) \in I_\sigma \times J_\sigma$), with $I_\sigma = [0, U_\sigma]$ and $J_\sigma = [v(0), v(2\pi)]$.

In section 3.6 devoted to the study of the mapping properties of the Jacobi operator about $\widetilde{M}_{\sigma,\alpha,\beta}$ we will make exclusive use of the (u, v) variables. In the proof of lemma 3.5.3 we have written as a vertical graph an appropriate piece of $\widetilde{M}_{\sigma,\alpha,\beta}$ corresponding to the annulus $\frac{\sqrt{\varepsilon}}{4} \leqslant |\mathbf{z}| \leqslant 4\sqrt{\varepsilon}$. The boundary curve along which we will glue the piece of $\widetilde{M}_{\sigma,\alpha,\beta}$ with the CHM surface corresponds to $|\mathbf{z}| = \sqrt{\varepsilon}$.

Lemma 3.5.5. *If $\frac{\sqrt{\varepsilon}}{4} \leqslant |\mathbf{z}| \leqslant 4\sqrt{\varepsilon}$ and $\sigma = \mathcal{O}(\varepsilon)$ then*

$$-\frac{1}{2}\ln\varepsilon + c_1 \leqslant v \leqslant -\frac{1}{2}\ln\varepsilon + c_2,$$

where c_1 and c_2 are constant. Under the same assumption on σ, $V_\sigma = -4\ln\varepsilon + \mathcal{O}(1)$.

Proof. The proof is articulated in two steps. Firstly, using equation (3.29), we will give the values of the y variable for which $\frac{\sqrt{\varepsilon}}{4} \leqslant |\mathbf{z}(x, y)| \leqslant 4\sqrt{\varepsilon}$ with $\sigma = \mathcal{O}(\varepsilon)$. Secondly we will determine the values of the v variable corresponding to these values of y. It is possible to show that, if $|z|$ varies in the interval specified above, then $\pi - a_1 \leqslant y \leqslant \pi - a_2$, with $a_i = d_i\sqrt{\varepsilon}$, where $d_1 > d_2 > 0$ are constants. Being v an increasing function of y, then $v(\pi - a_1) \leqslant v(y) \leqslant v(\pi - a_2)$. Now we will obtain the values of $v(y)$ for $y = \pi - a_i$, $i = 1, 2$. To this aim we observe that

$$v(y) = \int_{\frac{\pi}{2}}^{y} \frac{ds}{\sqrt{1 - \cos^2\sigma\cos^2 s}}.$$

It holds that

$$\int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma\cos^2 s}} = \int_0^y \frac{ds}{\sqrt{1-\cos^2\sigma + \cos^2\sigma\sin^2 s}} =$$

$$\frac{1}{\sin\sigma}\int_0^y \frac{ds}{\sqrt{1+\frac{\cos^2\sigma}{\sin^2\sigma}\sin^2 s}} = \frac{1}{\sin\sigma}\mathcal{F}(y, m_\sigma),$$

where $m_\sigma = -\frac{\cos^2\sigma}{\sin^2\sigma}$ and $\mathcal{F}(y,m) = \int_0^y \frac{ds}{\sqrt{1-m\sin^2 s}}$ is the incomplete elliptic integral of the first kind. $\mathcal{F}(y,m)$ is an odd function with respect to y and if $k \in \mathbb{Z}$,

$$\mathcal{F}(y+k\pi, m) = \mathcal{F}(y,m) + 2k\mathcal{K}(m),$$

where $\mathcal{K}(m) = \mathcal{F}(\frac{\pi}{2}, m)$ is the complete elliptic integral of first kind. If $y = d\sqrt{\varepsilon}$ and $\sigma = \mathcal{O}(\varepsilon)$, then

$$\frac{1}{\sin\sigma}\mathcal{F}(y, m_\sigma) = -\frac{1}{2}\ln\varepsilon + \mathcal{O}(1),$$

where the function $\mathcal{O}(\varepsilon)$ also depends on d. It is known that if m is sufficiently big then

$$\mathcal{K}(m) = \frac{1}{\sqrt{-m}}\left(\ln 4 + \frac{1}{2}\ln(-m)\right)\left(1+\mathcal{O}\left(\frac{1}{m}\right)\right).$$

It follows that

$$\frac{1}{\sin\sigma}\mathcal{K}(m_\sigma) = -\ln\varepsilon + \mathcal{O}(1).$$

Then, for $i = 1, 2$,

$$v(\pi - a_i) = \frac{1}{\sin\sigma}\left(\mathcal{F}(\pi - a_i, m_\sigma) - \mathcal{K}(m_\sigma)\right) =$$

$$\frac{1}{\sin\sigma}\left(\mathcal{F}(-a_i, m_\sigma) + 2\mathcal{K}(m_\sigma) - \mathcal{K}(m_\sigma)\right) =$$

$$\frac{1}{\sin\sigma}\left(-\mathcal{F}(d_i\sqrt{\varepsilon}, m_\sigma) + \mathcal{K}(m_\sigma)\right) = -\frac{1}{2}\ln\varepsilon + c_i.$$

The result concerning $V_\sigma = v(2\pi) - v(0)$ follows at once after having observed that $v(2\pi) = \frac{3\mathcal{K}(m_\sigma)}{\sin\sigma}$ and $v(0) = -\frac{\mathcal{K}(m_\sigma)}{\sin\sigma}$. □

From the lemma just proved it follows that the value of the v corresponding to $|\mathbf{z}| = \sqrt{\varepsilon}$, is $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + c$, where c is a constant.

3.6 The Jacobi operator about $\widetilde{M}_{\sigma,\alpha,\beta}$

The Jacobi operator for $M_{\sigma,\alpha,\beta}$ is given by $\mathcal{J} = \Delta_{ds^2} + |A|^2$, where $|A|^2$ is the squared norm of the second fundamental form on $M_{\sigma,\alpha,\beta}$ and Δ_{ds^2} is the Laplace-Beltrami operator with respect to the metric ds^2 induced on the surface by the immersion $X = (X_1, X_2, X_3)$ defined in section 3.5.1. That is

$$ds^2 = \frac{1}{4}\left(|g| + |g|^{-1}\right)^2 |dh|^2.$$

We consider the metric on the torus Σ_σ obtained as pull-back of standard metric ds_0^2 of the sphere $\mathbb{S}^2$ by the Gauss map N: $dN^*(ds_0^2) = -K\, ds^2$, where $K = -\frac{1}{2}|A|^2$ denotes the Gauss curvature of $M_{\sigma,\alpha,\beta}$. Hence $\Delta_{ds^2} = -K\,\Delta_{ds_0^2}$, and so

$$\mathcal{J} = -K\,(\Delta_{ds_0^2} + 2).$$

From [18] and taking into account the parametrization of $M_{\sigma,\alpha,\beta}$ on the cylinder given in subsection 3.5.2, we can deduce that, in the (x, y)-variables,

$$\Delta_{ds_0^2} := \frac{l(x)\,m(y)}{k(x,y)}\left[\partial_x\left(\frac{l(x)}{m(y)}\,\partial_x\right) + \partial_y\left(\frac{m(y)}{l(x)}\,\partial_y\right)\right].$$

In the (u, v)-variables defined by (3.30), we have $\mathcal{J} = \frac{-K}{k(x(u),y(v))}\mathcal{L}_\sigma$, where

$$\mathcal{L}_\sigma := \partial^2_{uu} + \partial^2_{vv} + 2\sin^2\sigma\,\cos^2(x(u)) + 2\cos^2\sigma\,\sin^2(y(v)) \tag{3.33}$$

is the Lamé operator (see [18]).

Remark 3.6.1. *In proposition 3.6.5, we will take limits as $\sigma \to 0$. For such a limit, the Riemann surface Σ_σ degenerates into a Riemann surface with nodes consisting of two spheres jointed by two common points, and the corresponding Jacobi operator equals the Legendre operator on $\mathbb{S}^2 \setminus \{p_0, p_1\}$, given by $\mathcal{L}_0 = \partial^2_{xx} + \sin y\,\partial_y\,(\sin y\,\partial_y) + 2\sin^2 y$ in the (x, y)-variables. Note that in this case the change of variables $(x, y) \mapsto (u, v)$ is not defined.*

3.6.1 The mapping properties of the Jacobi operator

From now on, we consider the conformal parametrization of $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1\times\mathbb{R} \equiv I_\sigma\times\mathbb{R}$ described in subsection 3.5.2. Our aim along this subsection is to study the mapping properties of the operator $\mathcal{J}$. It is clear that it is sufficient to study the simpler operator $\mathcal{L}_\sigma$ defined by (3.33). So we will solve the problem

$$\begin{cases} \mathcal{L}_\sigma w = f, & \text{in } I_\sigma \times [v_0, +\infty[\\ w = \varphi & \text{on } I_\sigma \times \{v_0\}, \end{cases}$$

with $v_0 \in \mathbb{R}$, considering convenient normed functional spaces for w, f and φ, so that the norm of w is bounded by the one of f.

We will work in two different functional spaces to solve the above Dirichlet problem. To explain the reason, it is convenient to recall that the isommetry group of $\widetilde{M}_{\sigma,\alpha,\beta}$ depends on the values of the three parameters σ, α, β. When $\alpha = 0$ (resp. $\beta = 0$), $\widetilde{M}_{\sigma,\alpha,\beta}$ is invariant by the reflection symmetry about the $\{x_1 = 0\}$ plane (resp. $\{x_2 = 0\}$ plane). We are interested in showing the existence of families of minimal surfaces close to $\widetilde{M}_{\sigma,0,\beta}$ and

$\widetilde{M}_{\sigma,\alpha,0}$, keeping the same properties of symmetry. Thus the surfaces in the family about $\widetilde{M}_{\sigma,0,\beta}$ (resp. $\widetilde{M}_{\sigma,\alpha,0}$) will be defined as normal graphs of functions defined in $I_\sigma \times \mathbb{R}$ which are even (resp. odd) in the first variable. We will solve the above Dirichlet problem in the first case. The second one follows similarly.

Definition 3.6.2. *Given $\sigma \in (0, \pi/2)$, $\ell \in \mathbb{N}$, $\alpha \in (0,1)$, $\mu \in \mathbb{R}$ and an interval I, we define $\mathcal{C}^{\ell,\alpha}_\mu(I_\sigma \times I)$ to be the space of functions $w = w(u,v)$ in $\mathcal{C}^{\ell,\alpha}_{loc}(I_\sigma \times I)$ which are even and U_σ periodic in the variable u and for which the following norm is finite:*

$$\|w\|_{\mathcal{C}^{\ell,\alpha}_\mu} := \sup_{v \in I} e^{-\mu v} \|w\|_{\mathcal{C}^{\ell,\alpha}(I_\sigma \times [v,v+1])}$$

We observe that the Jacobi operator $\mathcal{L}_\sigma$ becomes a Fredholm operator when restricted to the functional space just defined. Moreover $\mathcal{L}_\sigma$ has separated variables. Then we can consider the operator

$$L_\sigma = \partial^2_{uu} + 2\sin^2\sigma\cos^2(x(u)).$$

We let L_σ act on the U_σ-periodic and even functions. This operator is uniformly elliptic and self-adjoint. In particular, L_σ has discrete spectrum $(\lambda_{\sigma,i})_{i\geqslant 0}$, that we assume arranged so that $\lambda_{\sigma,i} < \lambda_{\sigma,i+1}$ for every i. Each eigenvalue $\lambda_{\sigma,i}$ is simple because we only consider even functions. We denote by $e_{\sigma,i}$ the even eigenfunction associated to $\lambda_{\sigma,i}$, normalized so that

$$\int_0^{U_\sigma} (e_{\sigma,i}(u))^2\, du = 1.$$

Lemma 3.6.3. *For every $i \geqslant 0$, the eigenvalue $\lambda_{\sigma,i}$ of the operator L_σ and its associated eigenfunctions $e_{\sigma,i}$ satisfy*

$$-2\sin^2\sigma \leqslant \lambda_{\sigma,i} - i^2 \leqslant 0, \qquad |e_{\sigma,i} - e_{0,i}|_{\mathcal{C}^2} \leqslant c_i \sin^2\sigma, \tag{3.34}$$

where $e_{0,i}(u) := \cos(ix(u))$ for every $u \in I_\sigma$, and the constant $c_i > 0$ depends only on i (it does not depend on σ).

Proof. The bound for $\lambda_{\sigma,i} - i^2$ comes from the variational characterization of the eigenvalues,

$$\lambda_{\sigma,i} = \sup_{\operatorname{codim} E = i} \inf_{\substack{e \in E \\ ||e||_{L^2} = 1}} \int_0^{U_\sigma} \left((\partial_u e)^2 - 2\sin^2\sigma\cos^2(x(u))\, e^2\right) du,$$

where E is a subset of the space of U_σ-periodic even functions in $L^2(I_\sigma)$, since it always holds $0 \leqslant 2\sin^2\sigma\cos^2(x(u)) \leqslant 2\sin^2\sigma$. The bound for the eigenfunctions follows from standard perturbation theory [24]. $\square$

The Hilbert basis $\{e_{\sigma,i}\}_{i\in\mathbb{N}}$ of the space of U_σ-periodic and even functions in $L^2(I_\sigma)$ induces the following Fourier decomposition of L^2 functions $g = g(u,v)$ which are U_σ-periodic and even in the u-variable,

$$g(u,v) = \sum_{i\geqslant 0} g_i(v)\, e_{\sigma,i}(u).$$

From this, we deduce that the operator $\mathcal{L}_\sigma$, can be decomposed as $\mathcal{L}_\sigma = \sum_{i\geqslant 0} L_{\sigma,i}$, being

$$L_{\sigma,i} = \partial^2_{vv} + 2\cos^2\sigma \sin^2(y(v)) - \lambda_{\sigma,i}\,, \quad \text{for every } i \geqslant 0.$$

Since $0 \leqslant 2\cos^2\sigma\sin^2(y(v)) \leqslant 2\cos^2\sigma = 2 - 2\sin^2\sigma$, the lemma 3.6.3 give us

$$P_{\sigma,i} := 2\cos^2\sigma\ \sin^2(y(v)) - \lambda_{\sigma,i} \leqslant 2 - i^2. \tag{3.35}$$

This fact allows us to prove the following lemma, which assures that $\mathcal{L}_\sigma$ is injective when restricted to the set of functions that are, in the variable u, even and L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$.

Lemma 3.6.4. *Given $v_0 < v_1$, let w be a solution of $\mathcal{L}_\sigma w = 0$ on $I_\sigma \times [v_0, v_1]$ such that*

(i) $w(\cdot\,, v_0) = w(\cdot\,, v_1) = 0$.

(ii) $\int_0^{U_\sigma} w(u,v) e_{\sigma,i}(u)\, du = 0$, *for every $v \in [v_0, v_1]$ and $i = 0, 1$.*

Then $w = 0$.

Proof. By *(ii)*, $w = \sum_{i\geqslant 2} w_i(v)\, e_{\sigma,i}(u)$. Since the potential $P_{\sigma,i}$ of the operator $L_{\sigma,i}$ is negative for every $i \geqslant 2$ (see (3.35)) and the operator $L_{\sigma,i}$ is elliptic, the maximum principle holds. We can then conclude the lemma 3.6.4 from *(i)*. □

To study the mapping properties of the Jacobi operator $\mathcal{L}_\sigma$ we need to give a description of the Jacobi fields of the surfaces $M_{\sigma,\alpha,0}$ and $M_{\sigma,0,\beta}$. They are defined to be the solutions of $\mathcal{L}_\sigma v = 0$.

We recall that the surfaces $M_{\sigma,\alpha,\beta}$ are invariant by the mirror symmetry about the plane $\{x_1 = 0\}$ if $\alpha = 0$, and about the plane $\{x_2 = 0\}$ if $\beta = 0$. This implies that there are only four independent Jacobi fields for each one of these surfaces.

Two Jacobi fields can be obtained by considering the one parameter families of minimal surfaces which is induced by the translations in the x_3-direction and by the translations in the x_2-direction if $\alpha = 0$, the translations in the x_1-direction if $\beta = 0$. These Jacobi fields are clearly periodic and hence bounded. A third Jacobi field can be obtained by considering the one parameter family of minimal surfaces which is induced by dilatation from the origin. The so-obtained Jacobi field is not bounded and in fact it grows linearly.

The last Jacobi field can be obtained by considering the one parameter family of minimal surfaces which is induced by changing the parameter σ. Again, this Jacobi field is not periodic and grows linearly. The Jacobi fields induced by the translation along the x_3-axis and the dilatation are the solutions of $\mathcal{L}_\sigma u = 0$ that collinear to the eigenfunction $e_{\sigma,0}$. Instead the Jacobi fields induced by the horizontal translation and by the variation of the parameter σ are collinear to $e_{\sigma,1}$.

The following proposition states that if the parameter μ is appropriately chosen, there exists a right inverse for $\mathcal{L}_\sigma$ whose norm is uniformly bounded.

Proposition 3.6.5. *Given $\mu \in (-2,-1)$, there exists a $\sigma_0 \in (0, \pi/2)$ such that, for every $\sigma \in (0, \sigma_0)$ and $v_0 \in \mathbb{R}$, there exists an operator*

$$G_{\sigma,v_0} : \ \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty)) \ \longrightarrow \ \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$$

such that for all $f \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$, the function $w := G_{\sigma,v_0}(f)$ solves

$$\begin{cases} \mathcal{L}_\sigma\, w = f & \text{on } I_\sigma \times [v_0, +\infty) \\ w \in \mathrm{Span}\{e_{\sigma,0}, e_{\sigma,1}\} & \text{on } I_\sigma \times \{v_0\}. \end{cases} \tag{3.36}$$

Moreover $||w||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\,||f||_{\mathcal{C}^{0,\alpha}_\mu}$, for some constant $c > 0$ which does not depend on $\sigma \in (0, \sigma_0)$ and $v_0 \in \mathbb{R}$.

Proof. Every $f \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$ can be decomposed as

$$f = f_0\, e_{\sigma,0} + f_1\, e_{\sigma,1} + \bar{f},$$

where $\bar{f}(\cdot, v)$ is L^2-orthogonal to $e_{\sigma,0}$ and $e_{\sigma,1}$ for each $v \in [v_0, +\infty)$.

Step 1. Firstly, let's prove the proposition 3.6.5 for the functions $\bar{f} \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, +\infty))$ that are L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$. As a consequence of the lemma 3.6.4, $\mathcal{L}_\sigma$ is injective when it acts on this set of functions. Hence, the Fredholm alternative assures that there exists, for each $v_1 > v_0 + 1$, an unique $\bar{w} \in \mathcal{C}^{2,\alpha}_\mu$, with $\bar{w}(\cdot, v)$ L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ satisfying:

$$\begin{cases} \mathcal{L}_\sigma\, \bar{w} = \bar{f} & \text{on } I_\sigma \times [v_0, v_1], \\ \bar{w}(\cdot, v_0) = \bar{w}(\cdot, v_1) = 0. \end{cases} \tag{3.37}$$

Assertion 3.6.6. *There exists a constant c and $\sigma_0 \in (0, \pi/2)$ such that for every $\sigma \in (0, \sigma_0)$, $v_0 \in \mathbb{R}$, $v_1 > v_0 + 1$, $\bar{f} \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0, v_1])$ and $\bar{w} \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_0, v_1])$ which are L^2-orthogonal to $\{e_{\sigma,0}, e_{\sigma,1}\}$ and satisfy the equation (3.37) and*

$$\sup_{I_\sigma \times [v_0, v_1]} e^{-\mu v} |\bar{w}| \leqslant c \sup_{I_\sigma \times [v_0, v_1]} e^{-\mu v} |\bar{f}|. \tag{3.38}$$

Suppose by contradiction that the assertion 3.6.6 is false. Then, for every $n \in \mathbb{N}$, there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $\bar{f}_n, \bar{w}_n$ satisfying (3.37) (for $\sigma_n, v_{0,n}, v_{1,n}$ instead of σ, v_0, v_1) such that

$$\sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} e^{-\mu v} |\bar{f}_n| = 1,$$

$$A_n := \sup_{I_{\sigma_n} \times [v_{0,n}, v_{1,n}]} e^{-\mu v} |\bar{w}_n| \to +\infty \quad \text{as} \quad n \to \infty.$$

Since $I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$ is a compact set, A_n is achieved at a point $(u_n, v_n) \in I_{\sigma_n} \times [v_{0,n}, v_{1,n}]$.

The sequence of sets $I_n = [v_{0,n} - v_n, v_{1,n} - v_n]$ converges (up to some subsequence) to a set that we denote by I_∞. We will show that I_∞ is non empty. Elliptic estimates imply that

$$\sup_{I_{\sigma,n} \times [v_{0,n}, v_{0,n} + \frac{1}{2}]} e^{-\mu v} |\nabla \bar{w}_n| \leqslant c(\sup_{I_{\sigma,n} \times [v_{0,n}, v_{0,n}+1]} e^{-\mu v} |\bar{f}_n| + \sup_{I_{\sigma,n} \times [v_{0,n}, v_{0,n}+1]} e^{-\mu v} |\bar{w}_n|) \leqslant c(1 + A_n).$$

From this estimate for the gradient of $\bar{w}_n$ near $v = v_{0,n}$ it follows that the supremum A_n cannot be achieved at a point which is too close to $v_{0,n}$, that is the points where $\bar{w}_n$ vanishes. In other terms the sequence $v_{0,n} - v_n$ remains bounded away from 0. Using similar arguments it is possible to show that $\nabla \bar{w}_n$ is bounded near $v_{1,n}$ and consequently also $v_{1,n} - v_n$ remains bounded away from 0. Then we can conclude that I_∞ is not empty. We set $I_\infty = [\bar{v}_0, \bar{v}_1]$ where $\bar{v}_0 \in [-\infty, 0)$ and $\bar{v}_1 \in (0, \infty]$.

We define

$$\widetilde{w}_n(u, v) := \frac{e^{-\mu v_n}}{A_n} \bar{w}_n(u, v + v_n),$$

for all $(u, v) \in I_{\sigma_n} \times I_n$. It follows that

$$\sup_{I_{\sigma_n} \times I_n} e^{-\mu v} |\tilde{w}_n| = 1. \tag{3.39}$$

Furthermore we observe that

$$|\widetilde{w}_n(u, v)| \leqslant e^{\mu v} \frac{e^{-\mu(v+v_n)} |\bar{w}_n(u, v + v_n)|}{A_n} \leqslant c e^{\mu v}.$$

and thanks to the estimate of $\nabla \bar{w}_n$ we obtain

$$|\nabla \widetilde{w}_n| \leqslant c \frac{1 + A_n}{A_n} e^{\mu v} < 2c \, e^{\mu v}.$$

Since the sequences $(\tilde{w}_n)_n$ and $(\nabla \tilde{w}_n)_n$ are uniformly bounded, Ascoli-Arzelà theorem assures that a subsequence of $(\tilde{w}_n)_n$ converges for $n \to \infty$ (and $\sigma_n \to 0$) on compact sets of $I_0 \times I_\infty$ to a function $\tilde{w}_\infty$, which is L^2-orthogonal to $\{e_{0,0}, e_{0,1}\}$ for each $v \in I_\infty$, and vanishes on $I_0 \times \partial I_\infty$, when $\partial I_\infty \neq \emptyset$. The function $\tilde{w}_\infty$ inherits the properties of $\tilde{w}_n$. In particular it holds

$$\sup_{I_0 \times I_\infty} e^{-\mu v} |\tilde{w}_\infty| = 1. \tag{3.40}$$

Now our aim is obtain a contradiction showing that $\tilde{w}_\infty$ cannot satisfies (3.40). That will prove the assertion 3.6.6.

If $n \to \infty$ we have $\sigma_n \to 0$, then we can conclude that, up to subsequence, the function $\tilde{w}_\infty$ satisfies $\mathcal{L}_0 \tilde{w}_\infty = 0$ and $w = 0$ on $I_0 \times I_\infty$ if $\partial I_\infty \neq \emptyset$. If I_∞ is bounded thanks to the maximum principle we can conclude that $\tilde{w}_\infty = 0$ on $I_0 \times I_\infty$ which contradicts (3.40). So in the following we can suppose I_∞ to be an unbounded interval.

Since we know the expression of $\mathcal{L}_0$ in terms of the (x, y) variables, it is convenient to work with these variables. Then the equation $\mathcal{L}_0 \tilde{w}_\infty = 0$ takes the form

$$\partial^2_{xx} \tilde{w}_\infty + \sin y \, \partial_y \left(\sin y \, \partial_y \tilde{w}_\infty \right) + 2 \sin^2 y \, \tilde{w}_\infty = 0.$$

Now we consider the eigenfunctions decomposition of $\tilde{w}_\infty$,

$$\tilde{w}_\infty(x, y) = \sum_{j \geqslant 2} a_j(y) \, \cos(j \, x).$$

Each coefficient a_j, with $j \geqslant 2$, must satisfy the associated Legendre differential equation (see appendix 3.11), that is

$$\sin y \, \partial_y \left(\sin y \, \partial_y a_j \right) - j^2 a_j + 2 \sin^2 y \, a_j = 0.$$

We obtain that $a_j(y)$, with $j \geqslant 2$ equals the associated Legendre functions of second kind, that is $a_j(y) = Q_1^j(\cos y)$, $j \geqslant 2$. We need to express $\tilde{w}_\infty$ in terms of the (u, v) variables. We observe that if $\sigma \to 0$ then from (3.30) we get

$$u \to x \quad \text{and} \quad v \to \frac{1}{2} \ln \left| \tan \frac{y}{2} \right|.$$

From the last expression we get

$$y(v) = 2 \arctan(e^{2v}) \quad \text{and} \quad e^{2v} = \left| \tan \frac{y}{2} \right|.$$

Using well kwown trigonometric formulae also we find

$$\cos y(v) = \frac{1 - e^{4v}}{1 + e^{4v}} \tag{3.41}$$

Then replacing the expressions of a_j in the eigenfunctions decomposition of $\tilde{w}_\infty$, we get

$$\tilde{w}_\infty(u,v) = \sum_{j\geqslant 2} Q_1^j(\cos y(v)) \cos(j\,u).$$

If $v \to +\infty$ (respectively $-\infty$) then $\cos(y(v))$ tends to -1 (respectively $+1$). It is possible to show that $|a_j|$ tend to $+\infty$ as the function $e^{2j|v|}$. Since I_∞ is unbounded, we can conclude that $\tilde{w}_\infty$ does not satisfy the (3.40) with $\mu \in (-2,-1)$, a contradiction.

This proves the assertion 3.6.6, that is, for every $v_1 > v_0+1$, there exists a function $\bar{w}$ satisfying (3.38). Let's take the limit as $v_1 \to \infty$. Clearly,

$$e^{-\mu\,v}|\bar{w}| \leqslant \|\bar{w}\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\,\|\bar{f}\|_{\mathcal{C}^{0,\alpha}_\mu}.$$

Schäuder estimates imply

$$e^{-\mu\,v}|\nabla\bar{w}| \leqslant \|\bar{w}\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\left(\|\bar{f}\|_{\mathcal{C}^{0,\alpha}_\mu} + \|\bar{w}\|_{\mathcal{C}^{0}_\mu}\right) \leqslant c\|\bar{f}\|_{\mathcal{C}^{0,\alpha}_\mu}.$$

Hence Ascoli-Arzelà theorem assures that a subsequence of $\{\bar{w}_{v_1}\}_{v_1>v_0+1}$ converges to a function $\bar{w}$ defined on $I_\sigma \times [v_0,\infty)$, which satisfies

$$\sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu\,v}|\bar{w}| \leqslant c \sup_{I_\sigma\times[v_0,+\infty)} e^{-\mu\,v}|\bar{f}|.$$

Using again elliptic estimates we can conclude that $\bar{w}$ satisfies the statement (iii) of proposition 3.6.5. The uniqueness of the solution follows from lemma 3.6.4.

Step 2 Let's now consider $f \in \mathcal{C}^{0,\alpha}_\mu(I_\sigma \times [v_0,+\infty))$ in $\mathrm{Span}\{e_{\sigma,0}, e_{\sigma,1}\}$, i.e.

$$f(u,v) = f_0(v)\,e_{\sigma,0}(u) + f_1(v)\,e_{\sigma,1}(u).$$

We extend the functions $f_0(v), f_1(v)$ for $v \leqslant v_0$ to be equal, respectively, to $f_0(v_0), f_1(v_0)$. Given $v_1 > v_0+1$, consider

$$\begin{cases} L_{\sigma,j}w_j = f_j, & v \in (-\infty, v_1] \\ w_j(v_1) = \partial_v w_j(v_1) = 0 \end{cases} \tag{3.42}$$

Peano theorem assures the existence and the uniqueness of the solution w_j. Our aim consists in proving the following

Assertion 3.6.7. $\sup_{(-\infty,v_1]} e^{-\mu v}|w_j| \leqslant c \sup_{(-\infty,v_1]} e^{-\mu v}|f_j|$ *for some constant c which does not depend on v_1.*

Suppose by contradiction that, for every $n \in \mathbb{N}$, there exists $\sigma_n \in (0, 1/n)$, $v_{1,n} > v_{0,n} + 1$ and $f_{j,n}, w_{j,n}$ satisfying (3.42) such that

$$\sup_{(-\infty, v_{1,n}]} e^{-\mu v} |\bar{f}_{j,n}| = 1,$$

$$A_n := \sup_{(-\infty, v_{1,n}]} e^{-\mu v} |w_{j,n}| \to +\infty \quad \text{as} \quad n \to \infty.$$

The solution $w_{j,n}$ of the previous equation is a linear combination of the two solutions of the homogeneous problem $L_{\sigma_n, j}\, w = 0$. They grow at most linearly at ∞. In fact it is known that the Jacobi fields have this rate of growth. Hence the supremum A_n is achieved in a point which we denote by $v_n \in (-\infty, v_{1,n}]$. We define on $I_n := (-\infty, v_{1,n} - v_n]$ the function $\tilde{w}_{j,n}$ by

$$\tilde{w}_{j,n}(v) := \frac{1}{A_n}\, e^{-\mu v_n}\, w_{j,n}(v_n + v).$$

As in Step 1, one shows that the sequence $(v_{1,n} - v_n)_n$ remains bounded away from 0, that is $v_{1,n} > v_n$ for each n. So, without loss of generality, we can assume that the sequence $(v_{1,n} - v_n)_n$ converges to $\bar{v}_1 \in (0, +\infty]$. We set $I_\infty = (-\infty, \bar{v}_1]$.

We can also assume that the sequence of functions $(\tilde{w}_{j,n})_n$ converges on compact subsets of I_∞ to a nontrivial function $\tilde{w}_j$ such that $\tilde{w}_j(\bar{v}_1) = \partial_v w_j(\bar{v}_1) = 0$ if $\bar{v}_1 < +\infty$, and

$$\sup_{v \in I_\infty} e^{-\mu v} |\tilde{w}_j| = 1. \tag{3.43}$$

The function $\tilde{w}_j$ is the solution of a second order ordinary differential equation, given, in terms of the (x, y), by

$$\sin y\, \partial_y \left(\sin y\, \partial_y \tilde{w}_j\right) - j^2\, \tilde{w}_j + 2 \sin^2 y\, \tilde{w}_j = 0. \tag{3.44}$$

If $\bar{v}_1 < +\infty$ then $\tilde{w}_j = 0$ and this is a contradiction with (3.43). In the case $\bar{v}_1 = +\infty$ we will try to reach a contradiction determining the solution (3.44). This is again the associated Legendre differential equation (see appendix 3.11). The solutions of the equation (3.44) are a linear combination of the associated Legendre functions of first kind, $P_1^j(\cos y)$, and second kind, $Q_1^j(\cos y)$, with $j = 0, 1$. It holds that $P_1^0(\cos y) = \cos y$ and $P_1^1(\cos y) = -\sin y$.

Now we change of coordinates to express $\tilde{w}_j$ in terms of the (u, v) variables. It is possible to show that as $v \to \pm\infty$ then $|Q_1^1(\cos y(v))|$ and $|Q_1^0(\cos y(v))|$ tend to ∞ respectively as $e^{2|v|}$ and $|v|$. We can conclude that functions $\tilde{w}_1$ and $\tilde{w}_0$ do not satisfy the equation (3.43) with $\mu \in (-2, -1)$, a contradiction.

So we have proved that

$$\sup_{(-\infty,v_1]} e^{-\mu v}\,|w_j| \leqslant c \sup_{(-\infty,v_1]} e^{-\mu v}\,|f_j|.$$

Now we pass to the limit as v_1 tends to $+\infty$ in a sequence of solutions which are defined on I_∞. This proves the existence of a solution of

$$L_{\sigma,j}\, w_j = f_j$$

which is defined in $[v_0, +\infty)$. In addition, we know that

$$\sup_{[v_0,+\infty)} e^{-\mu v}\,|w_j| \leqslant c \sup_{[v_0,+\infty)} e^{-\mu v}\,|f_j|.$$

Elliptic estimates allows us to obtain the estimates for the derivatives. To prove the uniqueness of the solution it is sufficient to observe that not any solution of $\mathcal{L}_\sigma v = 0$, that is collinear to $e_{\sigma,0}$ and $e_{\sigma,1}$ decays exponentially at ∞. This fact follows from the behaviour of the Jacobi fields. So the proof of the result is complete. □

3.7 A family of minimal surfaces close to $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$

The aim of this section is to find a family of minimal surfaces near to a translated and dilated copy of $\widetilde{M}_{\sigma,0,\beta}$ and $\widetilde{M}_{\sigma,\alpha,0}$ with given Dirichlet data on the boundary. We start recalling that in subsection 3.5.1 we observed that a translated and dilated copies of $\widetilde{M}_{\sigma,\alpha,0}$ and $\widetilde{M}_{\sigma,0,\beta}$ can be expressed as the graphs over the $x_3 = 0$ plane respectively of the functions

$$-(1+\gamma)\ln(2r) + r\kappa_1\cos\theta + \frac{1+\gamma}{r}\xi_1\cos\theta + d + \mathcal{O}(\varepsilon) \tag{3.45}$$

$$-(1+\gamma)\ln(2r) - r\kappa_2\sin\theta + \frac{1+\gamma}{r}\xi_2\sin\theta + d + \mathcal{O}(\varepsilon) \tag{3.46}$$

where $d = \xi_3 + (1+\gamma)\ln(1+\gamma)$, $\kappa_1, \kappa_2, \xi_1, \xi_2, \xi_3, \gamma \in \mathbb{R}$ small enough, $\kappa_1 = b_1 + b_2$, $\kappa_2 = b_1 - b_2$, $b_1 = \sin\frac{\alpha-\beta}{2}$, $b_2 = \sin\frac{\alpha+\beta}{2}$, and r belongs to a neighbourhood of $r_\varepsilon = \frac{1}{2\sqrt{\varepsilon}}$.

We denote by Z the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$. The following proposition, whose proof is contained in section 3.10, states that the linearized of the mean curvature operator is the Lamé operator introduced in section 3.5.2.

Proposition 3.7.1. *The surface parameterized by $Z_f := Z + f\,N$ is minimal if and only if the function f is a solution of*

$$\mathcal{L}_\sigma f = Q_\sigma(f).$$

where $\mathcal{L}_\sigma := \partial^2_{uu} + \partial^2_{vv} + 2\sin^2\sigma\,\cos^2(x(u)) + 2\cos^2\sigma\,\sin^2(y(v))$ *is the Lamé operator and* Q_σ *is a nonlinear operator which satisfies*

$$\|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathcal{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c \sup_{i=1,2} \|f_i\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\, \|f_2 - f_1\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \quad (3.47)$$

for all f_1, f_2 *such that* $\|f_i\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])} \leqslant 1$. *Here the constant* $c > 0$ *does not depend on* $v \in \mathbb{R}$, *nor on* $\sigma \in (0, \frac{\pi}{2})$.

As a consequence of the dilation of factor $1+\gamma$ of the surface the minimal surface equation becomes

$$\mathcal{L}_\sigma\, w = \frac{1}{1+\gamma} Q_\sigma\left((1+\gamma)w\right). \quad (3.48)$$

We now truncate the surfaces $\widetilde{M}_{\sigma,\alpha,0}$ and $\widetilde{M}_{\sigma,0,\beta}$ at the graph of the curve $r = \frac{1}{2\sqrt{\varepsilon}}$ of the function (3.45) and (3.46) with, respectively, $\beta = 0$ and $\alpha = 0$ and we consider only the upper half of these surfaces which we call M_1 and M_2.

We are interested in minimal normal graphs over these surfaces which are asymptotic to them. The normal graph of the function w over M_1, M_2 is minimal, if and only if w is a solution of (3.48).

In subsection 3.5.1 we have proved lemma 3.5.3, which allows us to parametrize the surfaces M_1, M_2 in a neighbourhood of r_ε as the graph of the functions given above. The surfaces M_1 and M_2 are parametrized on $I_\sigma \times [v_\varepsilon, V_\sigma]$ where v_ε denote value of the variable v corresponding to the value r_ε of the variable r. In section 3.5 we showed that $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathcal{O}(1)$.

It is important to remark that the boundary of these surfaces does not correspond to the curve $v = v_\varepsilon$. We therefore modify the above parametrization so that for $v \in [v_\varepsilon - \ln 2, v_\varepsilon + \ln 2]$ the image of the functions (3.45) and (3.46) corresponds to the horizontal curve $v = v_\varepsilon$. Furthermore we would like that the normal vector field relative to M_1, M_2 is vertical near the boundary of this surface. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}$ which agrees with the normal vector field N for all $v \geqslant v_\varepsilon + \ln 4$ and with the vector e_3 for all $v \in [v_\varepsilon, v_\varepsilon + \ln 2]$.

Now, we consider a graph over this surface for some function u, using the modified vector field $\tilde{N}$. This graph will be minimal if and only if the function u is a solution of a nonlinear elliptic equation related to (3.48). To get the new equation, we take into account the effects of the change of parameterization and the change in the vector field N into $\tilde{N}$. The new minimal surface equation is

$$\mathcal{L}_\sigma\, w = \tilde{L}_\varepsilon\, w + \tilde{Q}_\sigma\,(w). \quad (3.49)$$

Here $\tilde{Q}_\sigma$ enjoys the same properties as Q_σ, since it is obtained by a slight perturbation from it. The operator $\tilde{L}_\varepsilon$ is a linear second order operator whose coefficients are supported in $[v_\varepsilon, v_\varepsilon + \ln 4] \times S^1$ and are bounded by a constant multiplied for $\varepsilon^{1/2}$, in $\mathcal{C}^\infty$ topology, where partial derivatives are computed with respect to the vector fields ∂_u and ∂_v.

As a fact, if we take into account the effect of the change of the normal vector field, we would obtain, applying the result of Appendix B of [11], a similar formula where the coefficients of the corresponding operator $\tilde{L}_\varepsilon$ are bounded by a constant multiplied for ε since

$$\tilde{N}_\varepsilon \cdot N_\varepsilon = 1 + \mathcal{O}(\varepsilon)$$

for $t \in [t_\varepsilon, t_\varepsilon + \ln 2]$. Instead, if we take into account the effect of the change in the parameterization, we would obtain a similar formula where the coefficients of the corresponding operator $\tilde{L}_\varepsilon$ are bounded by a constant multiplied for $\varepsilon^{1/2}$. The estimate of the coefficients of $\tilde{L}_\varepsilon$ follows from these considerations.

In the following of this section we will give the detailed proof of the existence of a family of a minimal graph about M_1 and asymptotic to it. The proof relative to the case where the surface M_1 is replaced by M_2 can be obtained easily from the previous one. We recall that the surfaces M_1 and M_1 are respectively invariant with respect to two different mirror symmetries. A normal graph of the function $w = w(u, v)$ about M_1 (respectively about M_2) inherits the same property of symmetry if w is an even (respectively odd) function with respect to the variable u. Then once the proof of the result concerning M_1 will be completed, to obtain the result about M_2 it will be sufficient to replace even functions with odd functions and to repeat the same arguments.

Now, assume that we are given a function $\varphi \in \mathcal{C}^{2,\alpha}(I_\sigma)$ which is even with respect to u, L^2-orthogonal to $e_{\sigma,0}, e_{\sigma,1}$ and such that

$$\|\varphi\|_{\mathcal{C}^{2,\alpha}} \leqslant k\varepsilon. \tag{3.50}$$

We define

$$w_\varphi(\cdot,\cdot) := \bar{\mathcal{H}}_{v_\varepsilon,\varphi}(\cdot,\cdot),$$

where $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathcal{O}(1)$ and $\bar{\mathcal{H}}$ is introduced in proposition 3.9.5.
In order to solve the equation (3.49), we choose $\mu \in (-2, -1)$ and look for u of the form $w = w_\varphi + g$ where $g \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma \times [v_\varepsilon, +\infty))$ and $w = \varphi$ on $I_\sigma \times \{v_\varepsilon\}$. Using proposition 3.6.5, we can rephrase this problem as a fixed point problem

$$g = S(\varphi, g) \tag{3.51}$$

where the nonlinear mapping S which depends on ε and φ is defined by

$$S(\varphi, g) := G_{\varepsilon,v_\varepsilon}\left(\tilde{L}_\varepsilon(w_\varphi + g) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\varepsilon\,(w_\varphi + g)\right).$$

where the operator $G_{\varepsilon,v_\varepsilon}$ is defined in Proposition 3.6.5. To prove the existence of a fixed point for (3.51) we need the following

Lemma 3.7.2. *Let $\mu \in (-2,-1)$. There exist some constants $c_k > 0$ and $\varepsilon_k > 0$, such that*

$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant c_k\varepsilon^{3/2+\mu/2} \tag{3.52}$$

and, for all $\varepsilon \in (0,\varepsilon_k)$

$$\|S(\varphi,g_2) - S(\varphi,g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant \frac{1}{2}\,\|g_2 - g_1\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))},$$

$$\|S(\varphi_2,g) - S(\varphi_1,g)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant c\varepsilon^{\frac{1}{2}+\mu/2}\,\|\varphi_2-\varphi_1\|_{\mathcal{C}^{2,\alpha}} \tag{3.53}$$

for all $g, g_1, g_2 \in \mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))$ such that $\|g_i\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant 2c_k\varepsilon^{3/2+\mu/2}$ for all boundary data $\varphi,\varphi_1,\varphi_2 \in \mathcal{C}^{2,\alpha}(S^1)$ satisfying (3.50).

Proof. We know from proposition 3.6.5 that $\|G_{\varepsilon,v_\varepsilon}(f)\|_{\mathcal{C}^{2,\alpha}_\mu} \leqslant c\|f\|_{\mathcal{C}^{0,\alpha}_\mu}$, then

$$\|S(\varphi,0)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant c\|\tilde{L}_\varepsilon(w_\varphi) - \mathcal{L}_\sigma\,w_\varphi + \tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant$$

$$\leqslant c\left(\|\tilde{L}_\varepsilon(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))} + \|\mathcal{L}_\sigma\,w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))} + \|\tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))}\right).$$

So we need to find the estimates for the three terms above.
We recall that $|\varphi|_{2,\alpha} \leqslant k\varepsilon$. For all $\mu \in (-2,-1)$, thanks to proposition 3.9.5 we know that

$$|w_\varphi|_{2,\alpha;[v,v+1]} \leqslant e^{-2(v-v_\varepsilon)}|\varphi|_{2,\alpha} \tag{3.54}$$

We know that

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} = \sup_{v\in[v_\varepsilon,+\infty)} e^{-\mu v}|w_\varphi|_{2,\alpha;[v,v+1]} \leqslant \sup_{v\in[v_\varepsilon,\infty)} e^{-\mu v}e^{-2(v-v_\varepsilon)}|\varphi|_{2,\alpha} \leqslant$$

$$\leqslant e^{-\mu v_\varepsilon}|\varphi|_{2,\alpha} \leqslant c_k\varepsilon^{1+\mu/2}.$$

From this inequality and from the estimates of the coefficients of $\tilde{L}_\varepsilon$, it follows that

$$\|\tilde{L}_\varepsilon(w_\varphi)\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c\varepsilon^{1/2}\|w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu} \leqslant c_k\varepsilon^{3/2+\mu/2}.$$

As for $\mathcal{L}_\sigma$, being w_φ an harmonic function we obtain following equality

$$\mathcal{L}_\sigma\,w_\varphi = 2k\,w_\varphi,$$

where $k(u,v) = \sin^2\sigma\,\cos^2(x(u)) + \cos^2\sigma\,\sin^2(y(v)) \leqslant c\varepsilon$, because we have set $\alpha+\beta+\sigma \leqslant \varepsilon$ (see lemma 3.5.3) and $y_\varepsilon \leqslant y(v) \leqslant \pi$, with $y_\varepsilon = \pi - a_\varepsilon$, where $a_\varepsilon = \mathcal{O}(\sqrt{\varepsilon})$. Therefore, we conclude that

$$\|\mathcal{L}_\sigma\, w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))} \leqslant 2c\varepsilon\|w_\varphi\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))} \leqslant c_k\varepsilon^{2+\mu/2}.$$

The last term is estimated by

$$\|\tilde{Q}_\sigma\,(w_\varphi)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,\infty))} \leqslant c_k\varepsilon^{2+\mu/2}.$$

Putting together these estimates we get the first result. As for the second estimate, we recall that

$$S(\varphi,g) := G_{\varepsilon,v_\varepsilon}\left(\tilde{L}_\varepsilon(w_\varphi+g) - \mathcal{L}_\sigma\, w_\varphi + \tilde{Q}_\sigma\,(w_\varphi+g)\right).$$

Then

$$\|S(\varphi,g_2)-S(\varphi,g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant$$
$$\leqslant \|\tilde{L}_\varepsilon(g_2-g_1)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty)} + \|\tilde{Q}_\sigma\,(w_\varphi+g_2) - \tilde{Q}_\sigma\,(w_\varphi+g_1)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}.$$

We observe that from the considerations above it follows that

$$\|\tilde{L}_\varepsilon(g_2-g_1)\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant c_k\varepsilon^{1/2}||g_2-g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}$$

and that (3.47) implies

$$\|\tilde{Q}_\sigma\,(w_\varphi+g_2) - \tilde{Q}_\sigma\,(w_\varphi+g_1)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant$$
$$\leqslant c||g_2-g_1||_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}||w_\varphi||_{\mathcal{C}^{0,\alpha}(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant$$
$$\leqslant c_k\varepsilon||g_2-g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}.$$

We can conclude that

$$\|S(\varphi,g_2)-S(\varphi,g_1)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant c_k\varepsilon^{1/2}||g_2-g_1||_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}.$$

To show the last estimate it is sufficient to observe that

$$\|S(\varphi_2,g)-S(\varphi_1,g)\|_{\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty)} \leqslant \|\mathcal{L}_\sigma(w_{\varphi_2}-w_{\varphi_1})\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}+$$
$$\|\tilde{L}_\varepsilon(w_{\varphi_2}-w_{\varphi_1})\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} + \|\tilde{Q}_\sigma\,(w_{\varphi_2}+g) - \tilde{Q}_\sigma\,(w_{\varphi_1}+g)\,\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))} \leqslant$$
$$\leqslant c(\varepsilon^{1+\mu/2}+\varepsilon^{1/2+\mu/2})\|\varphi_2-\varphi_1\|_{\mathcal{C}^{2,\alpha}} + c\|g\|_{\mathcal{C}^{0,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))}\|\varphi_2-\varphi_1\|_{\mathcal{C}^{2,\alpha}} \leqslant c\varepsilon^{1/2+\mu/2}\|\varphi_2-\varphi_1\|_{\mathcal{C}^{2,\alpha}}.$$

$\square$

Theorem 3.7.3. *Let be $B := \{g\in\mathcal{C}^{2,\alpha}_\mu(I_\sigma\times[v_\varepsilon,+\infty))\,;\,||g||_{\mathcal{C}^{2,\alpha}_\mu} \leqslant 2c_k\varepsilon^{3/2+\mu/2}\}$. Then the nonlinear mapping S defined above has a unique fixed point g in B.*

Proof. The previous lemma shows that, if ε is chosen small enough, the nonlinear mapping S is a contraction mapping[1] from the ball B of radius $2c_k\varepsilon^{3/2+\mu/2}$ in $\mathcal{C}^{2,\alpha}_{\mu}(I_\sigma \times [v_\varepsilon, V_\sigma])$ into itself. This value comes from the estimate of the norm of $S(\varphi, 0)$. Consequently thanks to the Schauder theorem, S has a unique fixed point v in this ball. $\square$

This argument provides a minimal surface which is close to M_1 (respectively M_2) and has one boundary. This surface, denoted by $S_{t,\gamma,\xi_1,d_t}(\varphi)$, is, close to its boundary, a vertical graph over the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ whose parametrization is given, for $\beta = 0$ by

$$\bar{U}_{t,1}(r,\theta) = -(1+\gamma)\ln(2r) + r\kappa_1\cos\theta + \frac{1+\gamma}{r}\xi_1\cos\theta + d_t + \bar{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r - v_\varepsilon, \theta) + V_t(r,\theta). \tag{3.55}$$

and, for $\alpha = 0$, by

$$\bar{U}_{t,2}(r,\theta) = -(1+\gamma)\ln(2r) - r\kappa_2\sin\theta + \frac{1+\gamma}{r}\xi_2\sin\theta + d_t + \bar{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r - v_\varepsilon, \theta) + V_t(r,\theta). \tag{3.56}$$

where $v_\varepsilon = -\frac{1}{2}\ln\varepsilon + \mathcal{O}(1)$. The boundary of the surface corresponds to $r_\varepsilon = \frac{1}{2\sqrt{\varepsilon}}$. The function V_t depends non linearly on ε, ϕ. It satisfies $\|V_t(\varepsilon,\phi_i)(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon$ and

$$\|\bar{V}_t(\varepsilon,\phi)(r_\varepsilon\cdot,\cdot) - \bar{V}_t(\varepsilon,\phi')(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant c\varepsilon^{1/2}\|\phi - \phi'\|_{\mathcal{C}^{2,\alpha}}. \tag{3.57}$$

This estimate follows from

$$\|\bar{V}_t(\varepsilon,\phi)(r_\varepsilon\cdot,\cdot) - \bar{V}_t(\varepsilon,\phi')(r_\varepsilon\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_2 - B_1)} \leqslant e^{\mu v_\varepsilon}\|S(\phi,\bar{V}_t) - S(\phi',\bar{V}_t)\|_{\mathcal{C}^{2,\alpha}_{\mu}(I_\sigma\times[v_\varepsilon,+\infty))}$$

and the estimate (3.53).

3.8 The matching of Cauchy data

In this section we shall complete the proof of theorems 3.1.1, 3.1.2 and 3.1.3.

3.8.1 The proof of theorem 3.1.2

The proof of theorem 3.1.2 is articulated in two distinct parts: the proof of the existence of the family $\mathcal{K}_1$ and of the existence of the family $\mathcal{K}_2$.

We start proving the existence of the family $\mathcal{K}_2$. The proof is based on an analytical gluing procedure. The surfaces in family $\mathcal{K}_2$ are symmetric with respect to the vertical plane

[1] after the correct choise of the constant k that appears in the estimate of the norm of φ.

$x_2 = 0$. So all of the surfaces involved in the following proof must enjoy the same mirror symmetry property. We will show how to glue a compact piece of a Costa-Hoffman-Meeks type surface with bent catenoidal end to two halves of the KMR example $\widetilde{M}_{\sigma,\alpha,0}$ along the upper and lower boundaries and to a horizontal periodic flat annulus with a removed disk along the middle boundary. All of the surfaces just mentioned have the desired property of symmetry as well as the surfaces obtained by them by a slight perturbation. We will recall below the necessary results we proved in previous sections.

Using the result of section 3.7, we can obtain a minimal surface that is a perturbation of half the KMR example $\widetilde{M}_{\sigma,\alpha,0}$ and is asymptotic to it. This surface, denoted by $S_{t,\lambda_t,\xi_t,d_t}(\varphi_t)$, can be parameterized over the annular neighbourhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$, of its boundary as the vertical graph of (see (3.55))

$$\bar{U}_t(r,\theta) = -(1+\lambda_t)\ln(2r) + r\kappa_t\cos\theta - \frac{(1+\lambda_t)}{r}\xi_t\cos\theta + d_t + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r - v_\varepsilon,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

This surface will be glued to the Costa-Hoffman-Meeks example along its upper boundary. The surface that will be glued along the lower boundary, denoted by $S_{b,\lambda_b,\xi_b,d_b}(\varphi_b)$, can be parameterized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$, as the vertical graph of

$$\bar{U}_b(r,\theta) = (1+\lambda_b)\ln(2r) + r\kappa_b\cos\theta - \frac{(1+\lambda_b)}{r}\xi_b\cos\theta + d_b + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r - v_\varepsilon,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

We remark that the functions $\varphi_t, \varphi_b \in C^{2,\alpha}(S^1)$ are even and orthogonal in the L^2-sense to the constant function and to $\theta \to \cos\theta$.

Using the result of section 3.2, we can construct a minimal graph $S_m(\varphi_m)$ close to a horizontal periodic flat annulus. It can be parameterized, in the neighbourhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of its boundary, as the vertical graph of

$$\bar{U}_m(r,\theta) = \tilde{H}_{r_\varepsilon,\varphi_m}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where $\tilde{H}_{r_\varepsilon,\varphi_m}$ is the harmonic extension of $\varphi_m \in C^{2,\alpha}(S^1)$ which is an even function orthogonal in the L^2-sense to the constant function.

Thanks to the result of section 3.4, we can construct a minimal surface $\bar{M}_{k,\varepsilon}^T(\varepsilon/2,\Psi)$, with $\Psi = (\psi_t,\psi_b,\psi_m)$, which is close to a truncated genus k Costa-Hoffman-Meeks surface and has three boundaries. The functions $\psi_t,\psi_b,\psi_m \in C^{2,\alpha}(S^1)$ are even. Moreover ψ_t,ψ_b are L^2-orthogonal to the constant function and to $\theta \to \cos\theta$ and ψ_m is orthogonal to the constant function.

The surface $\bar{M}_{k,\varepsilon}^T(\varepsilon/2,\Psi)$ is, close to its upper and lower boundary, a vertical graph over the annulus $B_{r_\varepsilon} - B_{r_\varepsilon/2}$, whose parametrization is, respectively, given by

$$U_t(r,\theta) = \sigma_t - \ln(2r) - \frac{\varepsilon}{2}r\cos\theta + H_{\psi_t}(s_\varepsilon - \ln 2r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

$$U_b(r,\theta) = -\sigma_b + \ln(2r) - \frac{\varepsilon}{2} r\cos\theta + H_{\psi_b}(s_\varepsilon - \ln 2r, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$. Nearby the middle boundary the surface is a vertical graph whose parametrization is

$$U_m(r,\theta) = \tilde{H}_{\rho_\varepsilon,\psi_m}\left(\frac{1}{r},\theta\right) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

The functions $\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$ replace the functions $V_t, V_b, V_m, \bar{V}_t, \bar{V}_b, \bar{V}_m$ that appear at the end of sections 3.2, 3.4 and 3.7. They depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant times ε in $C_b^{2,\alpha}$ topology, where partial derivatives are taken with respect to the vector fields $r\partial_r$ and ∂_θ. We assume that the parameters and the boundary functions are chosen so that

$$\begin{gathered}
|\lambda_t|+|\lambda_b|+|-\lambda_t\ln\left(\varepsilon^{-1/2}\right)+\eta_t|+|\lambda_b\ln\left(\varepsilon^{-1/2}\right)+\eta_b|+\varepsilon^{-1/2}/2|\kappa_t+\varepsilon/2|+\varepsilon^{-1/2}/2|\kappa_b+\varepsilon/2|+ \\
2\varepsilon^{1/2}(|(1+\lambda_t)\xi_t|+|(1+\lambda_b)\xi_b|)+\|\varphi_t\|_{C^{2,\alpha}(S^1)}+\|\varphi_b\|_{C^{2,\alpha}(S^1)}+\|\varphi_m\|_{C^{2,\alpha}(S^1)}+ \\
+\|\psi_t\|_{C^{2,\alpha}(S^1)}+\|\psi_b\|_{C^{2,\alpha}(S^1)}+\|\psi_m\|_{C^{2,\alpha}(S^1)}\leqslant k\varepsilon,
\end{gathered} \tag{3.58}$$

where $\eta_t = d_t - \sigma_t$, $\eta_b = d_b + \sigma_b$. where the constant $k > 0$ is fixed large enough. It remains to show that, for all ε small enough, it is possible to choose the parameters and boundary functions in such a way that the surface

$$S_{t,\lambda_t,\xi_t,d_t}(\varphi_t) \cup S_{b,\lambda_b,\xi_b,d_b}(\varphi_b) \cup S_m(\varphi_m) \cup M_k^T(\varepsilon/2,\Psi)$$

is a C^1 surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth and by construction it has the desired properties. This will therefore complete the proof of the existence of the family of examples denoted by $\mathcal{K}_1$.

It is necessary to fulfill the following system of equations

$$\begin{cases}
U_t(r_\varepsilon,\cdot) = \bar{U}_t(r_\varepsilon,\cdot) \\
U_b(r_\varepsilon,\cdot) = \bar{U}_b(r_\varepsilon,\cdot) \\
U_m(r_\varepsilon,\cdot) = \bar{U}_m(r_\varepsilon,\cdot) \\
\partial_r U_b(r_\varepsilon,\cdot) = \partial_r \bar{U}_b(r_\varepsilon,\cdot) \\
\partial_r U_t(r_\varepsilon,\cdot) = \partial_r \bar{U}_t(r_\varepsilon,\cdot) \\
\partial_r U_m(r_\varepsilon,\cdot) = \partial_r \bar{U}_m(r_\varepsilon,\cdot)
\end{cases}$$

on S^1. The first three equations lead to the system

$$\begin{cases}
-\lambda_t\ln(2r_\varepsilon)+\eta_t-\left((1+\lambda_t)\frac{\xi_t}{r_\varepsilon}\right)\cos\theta+r_\varepsilon(\kappa_t+\frac{\varepsilon}{2})\cos\theta+\varphi_t-\psi_t=\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\
\lambda_b\ln(2r_\varepsilon)+\eta_b-\left((1+\lambda_b)\frac{\xi_b}{r_\varepsilon}\right)\cos\theta+r_\varepsilon(\kappa_b+\frac{\varepsilon}{2})\cos\theta+\varphi_b-\psi_b=\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\
\varphi_m-\psi_m=\mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).
\end{cases} \tag{3.59}$$

The last three equations give the system

$$\left\{\begin{array}{c} -\lambda_t + \left((1+\lambda_t)\frac{\xi_t}{r_\varepsilon}\right)\cos\theta + r_\varepsilon(\kappa_t + \frac{\varepsilon}{2})\cos\theta - \partial_\theta(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \lambda_b + \left((1+\lambda_b)\frac{\xi_b}{r_\varepsilon}\right)\cos\theta + r_\varepsilon(\kappa_b + \frac{\varepsilon}{2})\cos\theta - \partial_\theta(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{array}\right. \tag{3.60}$$

To obtain this system we applied the results of lemma 3.9.6 and 3.9.7. Here, the functions $\mathcal{O}_{C^{l,\alpha}}(\varepsilon)$ in the above expansions depend nonlinearly on the different parameters and boundary data functions but they are bounded by a constant times ε in $C^{l,\alpha}$ topology. The projection of the first two equations of each system over the L^2-orthogonal complement of $\mathrm{Span}\{1, \cos\theta\}$ and the remaining two equations gives the system

$$\left\{\begin{array}{c} \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \partial_\theta\varphi_t + \partial_\theta\psi_t = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta\varphi_b + \partial_\theta\psi_b = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta\varphi_m + \partial_\theta\psi_m = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{array}\right. \tag{3.61}$$

Lemma 3.8.1. *The operator h defined by*

$$\begin{array}{c} C^{2,\alpha}(S^1) \to C^{1,\alpha}(S^1) \\ \varphi \to \partial_\theta\varphi \end{array}$$

acting on functions that are orthogonal to the constant function in the L^2-sense and are even, is invertible.

Proof ([6]). We observe that if we decompose $\varphi = \sum_{j\geqslant 2} \varphi_j \cos(j\theta)$, then

$$h(\varphi) = -\sum_{j\geqslant 2} j\varphi_j \cos(j\theta),$$

that is clearly invertible from $H^1(S^1)$ into $L^2(S^1)$. Now elliptic regularity theory implies that this is also the case when this operator is defined between Holder spaces. □

Using this result, the system (3.61) can be rewritten as

$$(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m) = \mathcal{O}_{C^{2,\alpha}}(\varepsilon). \tag{3.62}$$

Recall that the right hand side depends nonlinearly on $\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m$ and also on the parameters $\lambda_t, \lambda_b, \eta_t, \eta_b, \xi_t, \xi_b, \kappa_t, \kappa_b$. We look at this equation as a fixed point problem and fix k large enough. Thanks to estimates (3.57), (3.2),(3.3), (3.20) and (3.21), we can use a fixed point theorem for contracting mappings in the ball of radius $k\varepsilon$ in $(C^{2,\alpha}(S^1))^6$ to obtain, for all ε small enough, a solution $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ of (3.62). This solution being obtained a fixed point for contraction mapping and the right hand side of (3.62) being continuous with respect to all data, we see that this fixed point $(\varphi_t, \varphi_b, \varphi_m, \psi_t, \psi_b, \psi_m)$ depends continuously (and in fact smoothly) on the parameters $\lambda_t, \lambda_b, \eta_t, \eta_b, \xi_t, \xi_b, \kappa_t, \kappa_b$. Inserting the founded solution into (3.59) and (3.60), we see that it remains to solve a system of the form

$$\begin{cases} -\lambda_t \ln(2r_\varepsilon) + \eta_t + \left(-(1+\lambda_t)\frac{\xi_t}{r_\varepsilon} + r_\varepsilon(\kappa_t + \frac{\varepsilon}{2})\right)\cos\theta = \mathcal{O}(\varepsilon), \\ \lambda_b \ln(2r_\varepsilon) + \eta_b + \left(-(1+\lambda_b)\frac{\xi_b}{r_\varepsilon} + r_\varepsilon(\kappa_b + \frac{\varepsilon}{2})\right)\cos\theta = \mathcal{O}(\varepsilon), \\ -\lambda_t + \left((1+\lambda_t)\frac{\xi_t}{r_\varepsilon} + r_\varepsilon(\kappa_t + \frac{\varepsilon}{2})\right)\cos\theta = \mathcal{O}(\varepsilon), \\ \lambda_b + \left((1+\lambda_b)\frac{\xi_b}{r_\varepsilon} + r_\varepsilon(\kappa_b + \frac{\varepsilon}{2})\right)\cos\theta = \mathcal{O}(\varepsilon). \end{cases} \tag{3.63}$$

where this time, the right hand sides only depend nonlinearly on $\lambda_t, \lambda_b, \eta_t, \eta_b, \xi_t, \xi_b, \kappa_t, \kappa_b$. There are eight equations that are obtained by projecting this system over the constant function and the function $\theta \to \cos\theta$. If we set

$$(\bar{\eta}_t, \bar{\eta}_b) = (-\lambda_t \ln(2r_\varepsilon) + \eta_t, \lambda_b \ln(2r_\varepsilon) + \eta_b),$$

$$(\bar{\xi}_t, \bar{\xi}_b) = r_\varepsilon^{-1}((1+\lambda_t)\xi_t, (1+\lambda_b)\xi_b), \quad (\bar{\kappa}_t, \bar{\kappa}_b) = r_\varepsilon(\kappa_t, \kappa_b),$$

the previous system can be rewritten as

$$(\lambda_t, \lambda_b, \bar{\xi}_t, \bar{\xi}_b, \bar{\eta}_t, \bar{\eta}_b, \bar{\kappa}_t, \bar{\kappa}_b) = \mathcal{O}(\varepsilon). \tag{3.64}$$

This time, provided k has been fixed large enough, we can use Leray-Schäuder fixed point theorem in the ball of radius $k\varepsilon$ in $\mathbb{R}^8$ to solve (3.64), for all ε small enough. This provides a set of parameters and a set of boundary data such that (3.59) and (3.60) hold. Equivalently we have proven the existence of a solution of systems (3.59) and (3.60). So the proof of the first part of theorem 3.1.2 is complete.

The proof of the second part of theorem 3.1.2 uses the same arguments seen above. So we will omit most of the details. Our aim is showing the existence of the family of surfaces denoted by $\mathcal{K}_1$. The surfaces in the family $\mathcal{K}_1$ are symmetric with respect to the vertical plane $x_1 = 0$. It is important to observe that in the proof, the KMR example which we deal with, is obtained by slight perturbation of $\widetilde{M}_{\sigma,0,\beta}$. The symmetry properties of this surface differ by the ones of the surface close to $\widetilde{M}_{\sigma,\alpha,0}$ involved in the gluing procedure described above. In particular $\widetilde{M}_{\sigma,0,\beta}$ is symmetric with respect to the vertical plane $x_1 = 0$. We recall that the Costa-Hoffman-Meeks type surface involved in the first gluing procedure enjoys a mirror symmetry with respect to the vertical plane $x_2 = 0$. Then it is not appropriate for the gluing with a KMR example of the type described above. To obtain the Costa-Hoffman-Meeks type surface we need, it is sufficient to rotate counterclockwise by $\pi/2$ with respect the vertical axis x_3 the Costa-Hoffman-Meeks surface with bent catenoidal ends described in section 3.4. The surface obtained in this way enjoys the mirror symmetry with respect to the vertical plane $x_1 = 0$. In the parametrizations of the top and bottom ends the cosinus function is replaced by the sinus function, that is:

$$U_t(r,\theta) = \sigma_t - \ln(2r) - \frac{\varepsilon}{2} r \sin\theta + H_{\psi_t}(s_\varepsilon - \ln(2r), \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$
$$U_b(r,\theta) = -\sigma_b + \ln(2r) - \frac{\varepsilon}{2} r \sin\theta + H_{\psi_b}(s_\varepsilon - \ln(2r), \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where $s_\varepsilon = -\frac{1}{2}\ln\varepsilon$ and $(r,\theta) \in B_{r_\varepsilon} - B_{r_\varepsilon/2}$. As for the planar middle end, the form of its parametrization remains unchanged. We refer to the first part of the proof for its expression. Another important remark to do, concerns the properties of the Dirichlet boundary data ψ_t, ψ_b, ψ_m. If before it was requested that they were even functions and orthogonal (only ψ_t, ψ_b) to the constant function and to $\theta \to \cos\theta$ to preserve the mirror symmetry property with respect to the vertical plane $x_1 = 0$, now as consequence of the above observations they must be odd functions and ψ_t, ψ_b must be orthogonal to the constant function and to $\theta \to \sin\theta$. It is clear that all the results showed in section 3.4 continue to hold.

Now we give the expressions of the parametrizations of the surface $S_{t,\lambda_t,\xi_t,d_t}(\varphi_t)$, the minimal surface obtained by perturbation from the KMR example $\widetilde{M}_{\sigma,0,\beta}$ and asymptotic to it. This surface can be parameterized in the neighbourhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$, as the vertical graph of

$$\bar{U}_t = -(1+\lambda_t)\ln(2r) + r\kappa_t \sin\theta - \frac{(1+\lambda_t)}{r}\xi_t \sin\theta + d_t + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_t}(\ln 2r - v_\varepsilon, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

The parametrization of the surface that we will glue to the Costa-Hoffman-Meeks type surface along its lower boundary is given by

$$\bar{U}_b(r,\theta) = (1+\lambda_b)\ln(2r) - \xi_b \frac{(1+\lambda_b)}{r}\sin\theta + d_b + \bar{\mathcal{H}}_{v_\varepsilon,\varphi_b}(\ln 2r - v_\varepsilon, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where $(r,\theta) \in B_{2r_\varepsilon} - B_{r_\varepsilon}$.

To prove the theorem it is necessary to show the existence of a solution of the following system of equations

$$\begin{cases} U_t(r_\varepsilon,\cdot) = \bar{U}_t(r_\varepsilon,\cdot) \\ U_b(r_\varepsilon,\cdot) = \bar{U}_b(r_\varepsilon,\cdot) \\ U_m(r_\varepsilon,\cdot) = \bar{U}_m(r_\varepsilon,\cdot) \\ \partial_r U_b(r_\varepsilon,\cdot) = \partial_r \bar{U}_b(r_\varepsilon,\cdot) \\ \partial_r U_t(r_\varepsilon,\cdot) = \partial_r \bar{U}_t(r_\varepsilon,\cdot) \\ \partial_r U_m(r_\varepsilon,\cdot) = \partial_r \bar{U}_m(r_\varepsilon,\cdot) \end{cases}$$

on S^1, under the assumption (3.58) for the parameters and the boundary functions. It is clear that the proof of the existence of a solution of this system is based on the same arguments seen before. We remark that the role played before by the functions $\theta \rightarrow \cos(i\theta)$, now is played by the functions $\theta \rightarrow \sin(i\theta)$. That completes the proof of theorem 3.1.2.

3.8.2 The proof of theorem 3.1.1

To proof the theorem 3.1.1 we will glue a compact piece of the surface $M_k^T(\xi)$, with $\xi = 0$, described in section 3.4 to two halves of a Scherk type surface along the upper and lower boundary and to a horizontal periodic flat annulus along the middle boundary. The construction of these surfaces is showed in section 3.2. In particular we showed the existence of a minimal graph close to half a Scherk type example whose ends have asymptotic directions given by $\cos\theta_1 e_1 + \sin\theta_1 e_3$ and $-\cos\theta_2 e_1 + \sin\theta_2 e_3$. These surfaces in the neighbourhood $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the boundary, admit the following parametrization

$$\bar{U}_t = d_t - \ln(2r) + \tilde{H}_{r_\varepsilon,\varphi_t}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

$$\bar{U}_b = d_b + \ln(2r) + \tilde{H}_{r_\varepsilon,\varphi_b}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

where the Dirichlet boundary data $\varphi_i \in C^{2,\alpha}(S^1)$, for $i = t, b$, are requested to be even and orthogonal to the constant function, w_{φ_i} denotes their harmonic extensions. The other surfaces involved in the gluing procedure have been described in the previous subsection.

The proof is similar to the one given for theorem 3.1.2, so we will give only the essential details. Actually to prove the theorem it is necessary to show the existence of a solution of the following system of equations

$$\left\{\begin{array}{r}
U_t(r_\varepsilon,\cdot) = \bar{U}_t(r_\varepsilon,\cdot) \\
U_b(r_\varepsilon,\cdot) = \bar{U}_b(r_\varepsilon,\cdot) \\
U_m(r_\varepsilon,\cdot) = \bar{U}_m(r_\varepsilon,\cdot) \\
\partial_r U_b(r_\varepsilon,\cdot) = \partial_r \bar{U}_b(r_\varepsilon,\cdot) \\
\partial_r U_t(r_\varepsilon,\cdot) = \partial_r \bar{U}_t(r_\varepsilon,\cdot) \\
\partial_r U_m(r_\varepsilon,\cdot) = \partial_r \bar{U}_m(r_\varepsilon,\cdot)
\end{array}\right.$$

on S^1, under a assumption similar to (3.58). We refer to subsection 3.8.1 for the expressions of $U_t, U_b, U_m, \bar{U}_m$. It is necessary to point out that in this subsection we consider the more symmetric example ($\xi = 0$) in the family $(M_k^T(\xi))_\xi$. So it is necessary to replace $\varepsilon/2$ by 0 in the expressions of the functions U_t and U_b of the top and bottom ends.

We want to remark the boundary data for the surfaces we are going to glue together do not satisfy the same hypotheses of orthogonality. All of these functions are orthogonal to the constant function, but only ψ_t, ψ_b are orthogonal to $\theta \to \cos\theta$ too. The functions denoted by $\mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(\varepsilon)$ that appear in the expressions of $\bar{U}_i$ and U_i, with $i = t, b, m$, have a Fourier series decomposition containing a term collinear to $\cos\theta$ only if the corresponding boundary data is assumed to be orthogonal only to the constant function. Furthermore the fact that $\xi = 0$ (the catenoidal ends are not bent) implies that functions which parametrize the top and bottom end of $M_k^T(0)$ are orthogonal to $\cos\theta$. In other terms, in difference with the Scherk type surfaces, we are not able to prescribe the coefficients in front of the eigenfunction $\cos\theta$ for the catenoidal ends of $M_k^T(0)$, because in this more symmetric setting are obliged to vanish.

The first three equations lead to the system

$$\left\{\begin{array}{c} \eta_t + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \eta_b + \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon), \end{array}\right. \tag{3.65}$$

where $\eta_t = d_t - \sigma_t$, $\eta_b = d_b + \sigma_b$. The last three equations give the system

$$\left\{\begin{array}{c} \partial_\theta(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{array}\right. \tag{3.66}$$

Now to complete the proof it is sufficient to use the same arguments of subsection 3.8.1.

3.8.3 The proof of theorem 3.1.3

To prove this theorem it is necessary to treat separately the case $k = 0$ and $k \geqslant 1$.

The case k=0. We will glue half a Scherk example with half a KMR example with $\alpha = \beta = 0$. We observe that this surface is symmetric with respect to the $x_1 = 0$ and $x_2 = 0$ planes. The Scherk example is symmetric with respect to the $x_2 = 0$. To preserve this property of symmetry in the surface obtained by the gluing procedure, we will consider the perturbation of $\tilde{M}_{\sigma,0,0}$ which enjoys the same mirror symmetry. That is the surface denoted by $S_{t,\lambda_t,\xi_t,d_t}(\varphi)$ with $\lambda_t = \xi_t = 0$ and $d_t = d$. It can be parametrized in the annulus $B_{2r_\varepsilon} - B_{r_\varepsilon}$ as the vertical graph of

$$\bar{U}(r,\theta) = -\ln(2r) + \bar{d} + \bar{\mathcal{H}}_{v_\varepsilon,\varphi}(\ln 2r - v_\varepsilon, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

The Scherk example is parametrized as the vertical graph of

$$U(r,\theta) = -\ln(2r) + d + \tilde{H}_{r_\varepsilon,\psi}(r,\theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon).$$

As for the Dirichlet boundary data, we assume φ to be an even function orthogonal to the constant function and to $\theta \to \cos\theta$, and ψ to be an even function orthogonal to the constant function.

To prove the theorem in the case $k = 0$, it is necessary to show the existence of a solution of the following system of equations

$$U(r_\varepsilon, \cdot) = \bar{U}(r_\varepsilon, \cdot)$$
$$\partial_r U(r_\varepsilon, \cdot) = \partial_r \bar{U}(r_\varepsilon, \cdot)$$

on S^1, under appropriate assumptions on the norms of the Dirichlet boundary data and the parameters ξ, d, $\bar{d}$. These equations lead to the system

$$\begin{cases} \eta + \varphi - \psi = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi + \psi) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon). \end{cases} \tag{3.67}$$

where $\eta = \bar{d} - d$. Now to complete the proof it is sufficient to use the same arguments of subsection 3.8.1. The details are omitted.

The case $k \geqslant 1$. The proof of the theorem in this case is similar the proof of theorem 3.1.1. In fact three of the surfaces we are going to glue are a compact piece of the Costa-Hoffman-Meeks example M_k, half a Scherk type example and a horizontal periodic flat annulus as in the proof of theorem 3.1.1. The fourth surface is half a KMR example, of the same type of the proof of theorem for $k = 0$. The surfaces are parametrized as vertical graph over $B_{2r_\varepsilon} - B_{r_\varepsilon}$ of the following functions:

$$\bar{U}_b(r, \theta) = \ln(2r) + d_b + \tilde{H}_{r_\varepsilon, \varphi_b}(r, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

for the Scherk type example,

$$\bar{U}_m(r, \theta) = \tilde{H}_{r_\varepsilon, \varphi_m}(r, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

for the horizontal periodic flat annulus,

$$\bar{U}_t(r, \theta) = -\ln(2r) + d_t + \bar{\mathcal{H}}_{v_\varepsilon, \varphi_t}(\ln 2r - v_\varepsilon, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

for the KMR example,

$$U_t(r, \theta) = \sigma_t - \ln(2r) + H_{\psi_t}(s_\varepsilon - \ln(2r), \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

$$U_b(r, \theta) = -\sigma_b + \ln(2r) + H_{\psi_b}(s_\varepsilon - \ln(2r), \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon),$$

$$U_m(r, \theta) = \tilde{H}_{\rho_\varepsilon, \varphi_m}(\frac{1}{r}, \theta) + \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon)$$

for the compact piece of the Costa-Hoffmam-Meeks example. The Dirichlet boundary data are requested to be even functions. Functions ψ_t, ψ_b are orthogonal to the constant function and to $\theta \to \cos\theta$. Functions ψ_m, φ_t, φ_b, φ_m are orthogonal only to the constant function. In this case the system of equations to solve is:

$$\left\{\begin{array}{c} \eta_t + \varphi_t - \psi_t = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \eta_b + \varphi_b - \psi_b = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \varphi_m - \psi_m = \mathcal{O}_{C_b^{2,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_t + \psi_t) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_b + \psi_b) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon) \\ \partial_\theta(\varphi_m + \psi_m) = \mathcal{O}_{C_b^{1,\alpha}}(\varepsilon), \end{array}\right. \tag{3.68}$$

where $\eta_t = d_t - \sigma_t$, $\eta_b = d_b + \sigma_b$. The details are left to the reader.

3.9 Appendix A

Definition 3.9.1. *Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\nu \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\nu(B_{\rho_0}(0))$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{loc}(B_{\rho_0}(0))$ for which the following norm is finite*

$$\|\rho^{-\nu}\, w\|_{\mathcal{C}^{\ell,\alpha}(B_{\rho_0}(0))}.$$

Now we can state the following result.

Proposition 3.9.2. *There exists an operator*

$$\tilde{H} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}_0(S^1 \times [\bar{\rho}, +\infty)),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to the constant function, then $w_\varphi = \tilde{H}_{\bar{\rho},\varphi}$ solves

$$\left\{\begin{array}{lll} \triangle w_\varphi = 0 & on & S^1 \times [\bar{\rho}, +\infty) \\ w_\varphi = \varphi & on & S^1 \times \{\bar{\rho}\}. \end{array}\right.$$

Moreover,

$$||\tilde{H}_{\bar{\rho},\varphi}||_{C^{2,\alpha}_{-1}(S^1\times[\bar{\rho},+\infty))} \leqslant c\,||\varphi||_{C^{2,\alpha}(S^1)}, \tag{3.69}$$

for some constant $c > 0$.

Remark 3.9.3. *Following the arguments of the proof below, it is possible to state a similar proposition but with the hypothesis φ odd.*

Proof. We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w_\varphi(\rho,\theta) = \sum_{i=1}^{\infty} \left(\frac{\bar\rho}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar\rho}{\rho} \leqslant 1$, then $\left(\frac{\bar\rho}{\rho}\right)^i \leqslant \left(\frac{\bar\rho}{\rho}\right)$, we can conclude that $|w(r,\theta)| \leqslant c\rho^{-1}|\varphi(\theta)|$ and then $||w_\varphi||_{C^{2,\alpha}_{-1}} \leqslant c||\varphi||_{C^{2,\alpha}}$. □

Now we give the statement of an useful result whose proof is contained in [6].

Proposition 3.9.4. *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}([0,+\infty) \times S^1),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, even function and orthogonal to 1 *and* $\cos\theta$, *in the L^2-sense, the function $w = H_\varphi$ solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2)\, w &= 0 \quad \mathit{in} \quad [0,+\infty) \times S^1 \\ w &= \varphi \quad \mathit{on} \quad \{0\} \times S^1 \end{cases}$$

Moreover

$$\|H_\varphi\|_{\mathcal{C}^{2,\alpha}_{-2}} \leqslant c\, \|\varphi\|_{\mathcal{C}^{2,\alpha}},$$

for some constant $c > 0$.

Proposition 3.9.5. *There exists an operator*

$$\bar{\mathcal{H}}_{v_0} : C^{2,\alpha}(S^1) \longrightarrow C^{2,\alpha}_{\mu}(S^1 \times [v_0,+\infty)),$$

$\mu \in (-2,-1)$, *such that for every function $\varphi(v) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to $e_{0,i}(u)$ with $i = 0,1$ and even, the function $w_\varphi = \bar{\mathcal{H}}_{v_0}(\varphi)$ solves*

$$\begin{cases} \partial^2_{uu} w_\varphi + \partial^2_{vv} w_\varphi = 0 & \mathit{on} \quad S^1 \times [v_0,+\infty) \\ w_\varphi = \varphi & \mathit{on} \quad S^1 \times \{v_0\}. \end{cases}$$

Moreover,

$$||\bar{\mathcal{H}}_{v_0}(\varphi)||_{C^{2,\alpha}_{\mu}(S^1\times[v_0,+\infty])} \leqslant c\, ||\varphi||_{C^{2,\alpha}(S^1)}, \tag{3.70}$$

for some constant $c > 0$.

Proof. We consider the decomposition of the function φ with respect to the basis $\{e_{0,i}(u)\}$, that is

$$\varphi = \sum_{i=2}^{\infty} \varphi_i e_{0,i}(u).$$

Then the solution w_φ is given by

$$w_\varphi(u,v) = \sum_{i=2}^{\infty} e^{-i(v-v_0)} \varphi_i e_{0,i}(u).$$

We recall that $\mu \in (-2,-1)$ so we have $-i \leqslant \mu$ from which it follows $|w_\varphi|_{2,\alpha;[v,v+1]} \leqslant e^{\mu(v-v_0)}|\varphi|_{2,\alpha}$ and

$$\|w_\varphi\|_{\mathcal{C}^{2,\alpha}_\mu} = \sup_{v\in[v_0,\infty]} e^{-\mu v}|w|_{2,\alpha;[v,v+1]} \leqslant \sup_{v\in[v_0,\infty]} e^{-\mu v} e^{\mu(v-v_0)}|\varphi|_{2,\alpha} \leqslant e^{-\mu v_0}|\varphi|_{2,\alpha}.$$

□

Lemma 3.9.6. *Let $u(r,\theta)$ be the harmonic extension defined on $[r_0,+\infty)\times S^1$ of the even funcion $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ and such that $u(r_0,\theta) = \varphi(\theta)$. Then*

$$\partial_\theta u(r,\theta-\pi/2)_{|r=r_0} = -r_0 \partial_r u(r,\theta)_{|r=r_0}.$$

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(r,\theta) = \sum_{i\geqslant 1} \varphi_i \left(\frac{r}{r_0}\right)^i \frac{i\cos(i\theta)}{r}$$

and

$$\partial_\theta u(r,\theta) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i i\sin(i\theta).$$

Consequently

$$\partial_\theta u(r,\theta-\pi/2) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r}{r_0}\right)^i i\cos(i\theta)$$

from which lemma follows easily. □

Lemma 3.9.7. *Let $u(r,\theta)$ be the harmonic extension defined on $[0,r_0]\times S^1$ of the even funcion $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$ and such that $u(r_0,\theta) = \varphi(\theta)$. Then*

$$\partial_\theta u(r,\theta-\pi/2)_{|r=r_0} = r_0 \partial_r u(r,\theta)_{|r=r_0}.$$

Proof. If $\varphi(\theta) = \sum_{i\geqslant 0} \varphi_i \cos(i\theta)$, then the function u is given by

$$u(r,\theta) = \sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i \cos(i\theta).$$

Then

$$\partial_r u(r,\theta) = -\sum_{i\geqslant 1} \varphi_i \left(\frac{r_0}{r}\right)^i \frac{i\cos(i\theta)}{r}$$

and

$$\partial_\theta u(r,\theta) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i i\sin(i\theta).$$

Consequently

$$\partial_\theta u(r,\theta-\pi/2) = -\sum_{i\geqslant 0} \varphi_i \left(\frac{r_0}{r}\right)^i i\cos(i\theta)$$

from which lemma follows easily. $\square$

3.10 Appendix B

Proof of proposition 3.7.1. Let Z be the immersion of the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ and N its normal vector. We want to find the differential equation to which a function f must satisfy such that the surface parametrized by $Z_f = Z + fN$ is minimal. In section 3.5.2 we parametrized the surface $\widetilde{M}_{\sigma,\alpha,\beta}$ on the cylinder $\mathbb{S}^1 \times \mathbb{R}$. We introduced the map $\mathbf{z}(x,y) : \mathbb{S}^1 \times [0,\pi[\to \bar{\mathbb{C}}$ where x, y denote the sphero-conal coordinates. We start working with the conformal variables p, q defined to be as the real and the imaginary part of $\mathbf{z}$. It holds that

$$\begin{gathered}
|Z_p|^2 = |Z_q|^2 = \Lambda, \quad |N_p|^2 = |N_q|^2 = -K\Lambda, \\
\langle N_p, N\rangle = \langle N_q, N\rangle = 0, \quad \langle Z_p, Z_q\rangle = 0, \quad \langle N_p, N_q\rangle = 0, \\
\langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle, \quad \langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle,
\end{gathered}$$

so

$$\begin{aligned}
\langle N_p, Z_p\rangle &= |N_p||Z_p|\cos\gamma_1 = \sqrt{-K}\Lambda\cos\gamma_1, \\
\langle N_p, Z_q\rangle &= |N_p||Z_q|\cos\gamma_2 = \sqrt{-K}\Lambda\cos\gamma_2.
\end{aligned}$$

Here K denotes the Gauss curvature, Z_p, Z_q and N_p, N_q denote the partial derivatives of the vectors Z and N, γ_1 is the angle between the vectors N_p and Z_p, γ_2 is the angle between the vectors N_p and Z_q.

The proof of proposition 3.7.1 is articulated in some lemmas. We denote by E_f, F_f, G_f the coefficiens of the second fundamental form for the surface parametrized by Z_f. The following lemma gives the expression of the area energy functional.

Lemma 3.10.1.
$$A(f) := \int \sqrt{E_f G_f - F_f^2}\, dp\, dq,$$
with
$$E_f G_f - F_f^2 = \Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1$$
$$-4f f_p f_q \sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4 K^2 \Lambda^2.$$

Proof. The coefficients of the second fundamental form are:

$$E_f = |\partial_p Z_f|^2 = |Z_p|^2 + f_p^2 + f^2|N_p|^2 + 2f\langle N_p, Z_p\rangle,$$

$$G_f = |\partial_q Z_f|^2 = |Z_q|^2 + f_q^2 + f^2|N_q|^2 + 2f\langle N_q, Z_q\rangle,$$
$$F_f = |\partial_p Z_f \cdot \partial_q Z_f| = f_p f_q + f(\langle Z_p, N_q\rangle + \langle Z_q, N_p\rangle).$$

Then
$$E_f G_f = |Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) +$$
$$f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2(\langle N_p Z_p\rangle)(\langle N_q Z_q\rangle) + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) +$$
$$f_p^2 f_q^2 + 2f(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2) + 2f^3(\langle N_q, Z_q\rangle|Z_p|^2 + \langle N_p, Z_p\rangle|Z_q|^2).$$

Since $\langle N_q, Z_q\rangle + \langle N_p, Z_p\rangle = 0$ and $|Z_p|^2 = |Z_q|^2$ we can conclude that the last two terms of the previous expression are zero. Since $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ we have

$$F_f = f_p f_q + 2f\langle N_p, Z_q\rangle.$$

Then
$$F_f^2 = f_p^2 f_q^2 + 4f^2(\langle N_p, Z_q\rangle)^2 + 4f f_p f_q\langle N_p, Z_q\rangle.$$

So the expression of $E_f G_f - F_f^2$ is:

$$|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) +$$

$$f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2 + 4f^2(\langle N_p, Z_p)(\langle N_q, Z_q\rangle) + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle)$$
$$-4f^2(\langle N_p, Z_q\rangle)^2 - 4f f_p f_q\langle N_p, Z_q\rangle.$$

Ordering the terms we get:

$$|Z_p|^2|Z_q|^2 + f_p^2|Z_q|^2 + f_q^2|Z_p|^2 + f^2(|N_q|^2|Z_p|^2 + |N_p|^2|Z_q|^2) - 4f^2(\langle N_p, Z_q\rangle)^2$$

$$+4f^2(\langle N_p, Z_p\rangle)(\langle N_q, Z_q\rangle) + 2f(f_p^2\langle N_q, Z_q\rangle + f_q^2\langle N_p, Z_p\rangle) - 4f f_p f_q\langle N_p, Z_q\rangle +$$
$$+f^2(f_p^2|N_q|^2 + f_q^2|N_p|^2) + f^4|N_p|^2|N_q|^2.$$

The expression of $E_fG_f - F_f^2$ becomes:

$$\Lambda^2 + \Lambda(f_p^2 + f_q^2) - 2K\Lambda^2 f^2 + 4f^2K\Lambda^2\left(\cos^2\gamma_1 + \cos^2\gamma_2\right) +$$
$$+2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1 - 4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.$$

Using the relations $\langle N_q, Z_p\rangle = \langle N_p, Z_q\rangle$ and $\langle N_q, Z_q\rangle = -\langle N_p, Z_p\rangle$, it is possible to understand that the relative positions of these vectors are such that $\gamma_2 = \frac{\pi}{2} \pm \gamma_1$. So $\cos^2\gamma_2 = \cos^2(\frac{\pi}{2} \pm \gamma_1) = \sin^2\gamma_1$ and $\cos^2\gamma_1 + \cos^2\gamma_2 = 1$. Then we can write:

$$\Lambda^2 + \Lambda(f_p^2 + f_q^2) + 2K\Lambda^2 f^2 + 2f(f_q^2 - f_p^2)\sqrt{-K}\Lambda\cos\gamma_1$$
$$-4ff_pf_q\sqrt{-K}\Lambda\cos\gamma_2 - K\Lambda f^2(f_p^2 + f_q^2) + f^4K^2\Lambda^2.$$

□

The next lemma completes the proof of the proposition 3.7.1.

Lemma 3.10.2. *The surface whose immersion is given by $Z + fN$, is minimal if and only if f satisfies*

$$\mathcal{L}_\sigma f + Q_\sigma(f) = 0,$$

where $\mathcal{L}_\sigma$ is the Lamé operator and Q_σ is a second order differential operator which satisfies

$$\|Q_\sigma(f_2) - Q_\sigma(f_1)\|_{\mathcal{C}^{0,\alpha}(I_\sigma\times[v,v+1])} \leqslant c \sup_{i=1,2}\|f_i\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])}\|f_2 - f_1\|_{\mathcal{C}^{2,\alpha}(I_\sigma\times[v,v+1])}.$$

Proof. The surface parameterized by $Z_f = Z + fN$ is minimal if and only the first variation of $A(f)$ is 0. That is

$$2\,DA_{|f}(g) = \int \frac{1}{\sqrt{(E_fG_f - F_f^2)}_{|f=0}} D_f(E_fG_f - F_f^2)\,(g)\,dp\,dq.$$

Thanks to the previous lemma it holds that

$$\frac{1}{\sqrt{(E_fG_f - F_f^2)}_{|f=0}} D_f(E_fG_f - F_f^2)(g) = \frac{1}{\Lambda}\left(2\Lambda(f_pg_p + f_qg_q) + 4K\Lambda^2 fg + \right.$$
$$+2\sqrt{-K}\Lambda\cos\gamma_1\left[2ff_qg_q + gf_q^2 - 2ff_pg_p - gf_p^2\right] +$$
$$-4\sqrt{-K}\Lambda\cos\gamma_2\left[ff_qg_p + fg_qf_p + gf_pf_q\right] +$$
$$\left.-2K\Lambda\left[fgf_p^2 + f_pg_pf^2 + fgf_q^2 + f_qg_qf^2\right] + 4K^2\Lambda^2f^3g\right).$$

Reordering the summands, we have:

$$\frac{1}{\sqrt{(E_fG_f - F_f^2)}_{|f=0}} D_f(E_fG_f - F_f^2)(g) = 2\left(f_pg_p + f_qg_q + 2K\Lambda fg + \right.$$

$$+\sqrt{-K}\cos\gamma_1\left[2f(f_q g_q - f_p g_p) + g(f_q^2 - f_p^2)\right] +$$
$$-2\sqrt{-K}\cos\gamma_2\left[f(f_q g_p + g_q f_p) + g f_p f_q\right] +$$
$$-K\left[fg(f_p^2 + f_q^2) + f^2(f_p g_p + f_q g_q)\right] + 2K^2\Lambda f^3 g\big) .$$

In the next computation we can skip the factor 2 in front of the last expression.

$$f_p g_p + f_q g_q + 2K\Lambda f g + Q_1(f, f_p, f_q)g - Q_2(f, f_p, f_q)g_p - Q_3(f, f_p, f_q)g_q = 0,$$

where

$$Q_1(f, f_p, f_q) = -(f_p^2 - f_q^2)\sqrt{-K}\cos\gamma_1 - 2f_p f_q\sqrt{-K}\cos\gamma_2 - Kf(f_p^2 + f_q^2) + 2K^2\Lambda f^3,$$

$$Q_2(f, f_p, f_q) = 2ff_p\sqrt{-K}\cos\gamma_1 + 2ff_q\sqrt{-K}\cos\gamma_2 + Kf^2 f_p,$$
$$Q_3(f, f_p, f_q) = -2ff_q\sqrt{-K}\cos\gamma_1 + 2ff_p\sqrt{-K}\cos\gamma_2 + Kf^2 f_q.$$

An integration by parts and a change of sign give us the equation:

$$(f_{pp} + f_{qq} - 2K\Lambda f - Q_1(f, f_p, f_q)+$$
$$+P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) + P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}))\, g = 0,$$

where

$$P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \partial_p Q_2(f, f_p, f_q)$$

and

$$P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = \partial_q Q_3(f, f_p, f_q).$$

That is

$$P_2(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = 2(f_p^2 + ff_{pp})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + ff_{pq})\sqrt{-K}\cos\gamma_2 +$$
$$+K(2ff_p^2 + f^2 f_{pp}) + 2f(f_p(\sqrt{-K}\cos\gamma_1)_p + f_q(\sqrt{-K}\cos\gamma_2)_p) + f^2 f_p K_p$$

and

$$P_3(f, f_p, f_q, f_{pp}, f_{pq}, f_{qq}) = -2(f_q^2 + ff_{qq})\sqrt{-K}\cos\gamma_1 + 2(f_p f_q + ff_{pq})\sqrt{-K}\cos\gamma_2 +$$
$$+K(2ff_q^2 + f^2 f_{qq}) + 2f(-f_q(\sqrt{-K}\cos\gamma_1)_q + f_p(\sqrt{-K}\cos\gamma_2)_q) + f^2 f_q K_q.$$

Now we change the variables, passing from the (p, q) variables to the (u, v) variables. Then we want to understand how above differential equation changes. We recall that p and q are the real and imaginary part of the variable $\mathbf{z}$ of which we know the expression in terms of the spheroconal coordinates x, y (see (3.29)). It is known that the metric $\bar{g}$ induced on a surface whose immersion Z is given by the Weierstrass representation on a domain of the complex $\mathbf{z}$-plane, can be expressed in terms of the metric $d\bar{s}^2 = dp^2 + dq^2$, by $\bar{g} = \Lambda(dp^2 + dq^2)$, where $\Lambda = |Z_p|^2 = |Z_q|^2$. The Laplace-Beltrami operators written with respect to the metrics $d\bar{s}^2$ and $\bar{g}$ are related by:

$$\Delta_{d\bar{s}^2} = \frac{1}{\Lambda}\Delta_{\bar{g}}.$$

That is they differ by the conformal factor $1/\Lambda$. In section 3.7.1 we observed that the conformal factor related to the change of coordinates $(x, y) \to (u, v)$ is $-K/k$. So the conformal factor due to the change of coordinates $(p, q) \to (u, v)$ is obtained by multiplication of the conformal factors described above. Summarizing it holds that

$$f_{pp} + f_{qq} = \frac{-K\Lambda}{k}(f_{uu} + f_{vv}).$$

So we can write

$$\frac{-K\Lambda}{k}(f_{uu} + f_{vv}) + 2(-K\Lambda) f + R_1 + R_2 + R_3 = 0,$$

where

$$R_1(f, f_u, f_v) = -\frac{-K\Lambda}{k}\left[-(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 - 2f_uf_v\sqrt{-K}\cos\gamma_2 - Kf(f_u^2 + f_v^2)\right] - 2K^2\Lambda f^3$$

$$= \frac{-K\Lambda}{k}\left[(f_u^2 - f_v^2)\sqrt{-K}\cos\gamma_1 + 2f_uf_v\sqrt{-K}\cos\gamma_2 + Kf(f_u^2 + f_v^2) - 2Kkf^3\right] =$$

$$\frac{-K\Lambda}{k}\bar{P}_1(f, f_u, f_v),$$

$$R_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) = \frac{-K\Lambda}{k}P_2(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv})$$

and

$$R_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}) = \frac{-K\Lambda}{k}P_3(f, f_u, f_v, f_{uu}, f_{uv}, f_{vv}).$$

We can write, simplifying the notation:

$$\frac{-K\Lambda}{k}\left[f_{uu} + f_{vv} + 2k(u, v)f + \bar{P}_1(f) + P_2(f) + P_3(f)\right] = 0.$$

We can recognize the Lamé operator,

$$\mathcal{L}_\sigma f = f_{uu} + f_{vv} + 2(\sin^2\sigma\cos^2 x(u) + \cos^2\sigma\sin^2 y(v))f,$$

then, if we set $Q_\sigma = \bar{P}_1(f) + P_2(f) + P_3(f)$, the equation can be written

$$\mathcal{L}_\sigma f + Q_\sigma(f) = 0.$$

To show the estimate about Q_σ is sufficient to show that all its coefficients are bounded. In particular we will show that the Gauss curvature K and its derivatives K_u, K_v are bounded. We start observing that $\frac{-K}{k(x(u),y(v))}$ is bounded. It is well known that the Gauss curvature has the following expression in terms of the Weierstrass data g, dh:

$$K = -16\left(|g| + \frac{1}{|g|}\right)^{-4} \left|\frac{dg}{g}\right|^2 |dh|^{-2}$$

We recall that $dh = \frac{\mu\, dz}{\sqrt{(z^2+\lambda^2)(z^2+\lambda^{-2})}}$. Since $|z^2+\lambda^2||z^2+\lambda^{-2}|$ and $k(x,y) = \sin^2\sigma\, \cos^2 x(u) + \cos^2\sigma\, \sin^2 y(v)$ have the same zeroes, that is the points D, D', D'', D''' given by (3.23), then $-K/k$ is bounded as well as its derivatives.

We can give an estimate of the derivatives of K and $\sqrt{-K}$. We can write $\sqrt{-K} = \sqrt{k}\sqrt{\frac{-K}{k}}$. From the observations made above it follows that it is sufficient to study the derivatives of $\sqrt{k}$ to show that the derivatives of $\sqrt{-K}$ are bounded.

We recall that

$$l(x) = \sqrt{1 - \sin^2\sigma \sin^2 x} \quad m(y) = \sqrt{1 - \cos^2\sigma\cos^2 y}.$$

From the expression of k, using (3.30) it is easy to get:

$$\frac{\partial}{\partial u}\sqrt{k} = -\frac{\sin^2\sigma \sin 2x(u)}{2\sqrt{k}} l(x(u)),$$

$$\frac{\partial}{\partial v}\sqrt{k} = \frac{\cos^2\sigma \sin 2y(v)}{2\sqrt{k}} m(y(v)).$$

Then

$$\left|\frac{\partial}{\partial u}\sqrt{k}\right| = \frac{\sin^2\sigma |\sin 2x(u)| l(x(u))}{2\sqrt{\sin^2\sigma\cos^2 x(u) + \cos^2\sigma \sin^2 y(v)}} \leqslant \frac{\sin^2\sigma|\sin 2x(u)|}{2\sin\sigma|\cos x(u)|} \leqslant \sin\sigma,$$

$$\left|\frac{\partial}{\partial v}\sqrt{k}\right| = \frac{\cos^2\sigma |\sin 2y(v)| m(y(v))}{2\sqrt{\sin^2\sigma\cos^2 x(u) + \cos^2\sigma \sin^2 y(v)}} \leqslant \frac{\cos^2\sigma|\sin 2y(v)|}{2\cos\sigma|\sin y(v)|} \leqslant \cos\sigma.$$

That is the derivatives of $\sqrt{k}$ (and consequently the ones of $\sqrt{-K}$) are bounded. That completes the proof.

□

3.11 Appendix C

The differential equation

$$\sin y\, \partial_y\,(\sin y\, \partial_y f) - j^2 f + 2 \sin^2 y\, f = 0 \tag{3.71}$$

is a particular case ($l = 1$) of the associated Legendre differential equation, that is given by

$$\sin y\, \partial_y\,(\sin y\, \partial_y f) - j^2 f + l(l+1) \sin^2 y\, f = 0,$$

where $l, j \in \mathbb{N}$. The family of the solutions of equation (3.71) (see [1]) is

$$c_1 P_l^j(\cos y) + c_2 Q_l^j(\cos y),$$

for $l = 1$, where $P_l^j(t)$ and $Q_l^j(t)$ are respectively the associated Legendre functions of first and second kind. If $l = 1$ these functions are defined as follows:

$$P_1^j(t) = \begin{cases} t & \text{if} \quad j = 0 \\ -\sqrt{1-t^2} & \text{if} \quad j = 1 \\ 0, & \text{if} \quad j \geqslant 2, \end{cases}$$

$$Q_1^j(t) = (-1)^j \sqrt{(1-t^2)^j}\, \frac{d^j Q_1^0(t)}{dt^j} \quad \text{and} \quad Q_1^0(t) = \frac{t}{2} \ln\left(\frac{1+t}{1-t}\right) - 1.$$

Bibliography

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Series, 55.

[2] C. J. Costa, *Imersões minimas en $\mathbb{R}^3$ de gênero un e curvatura total finita*, PhD thesis, IMPA, Rio de Janeiro, Brasil, 1982.

[3] C. J. Costa, *Example of a complete minimal immersion in $\mathbb{R}^3$ of genus one and three embedded ends*, Bol. Soc. Brasil. Mat. 15 (1984), no. 1-2, 47-54.

[4] V.I. Danilov, V.V. Shokurov, *Algebraic curves, algebraic manifolds and schemes*, Springer Verlag, (1998).

[5] N. Ejiri, M. Kotani, *Index and flat ends of minimal surfaces*, Tokyo J. Math., 16 (1993), no. 1, 37-48.

[6] S. Fakhi and F. Pacard,*Existence result for minimal hypersurfaces with a prescribed finite number of planar ends*, Manuscripta Math., 103 (2000), no. 4, 465-512.

[7] D.Fischer-Colbrie, *On complete minimal surfaces with Morse index in three manifold*, Inventiones Mathematicae, 82 (1985), 121-132.

[8] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag 1998.

[9] R. Gulliver, H.B. Lawson, Jr., *The structure of stable minimal hypersurfaces near a singularity*, Proc. Symp. Pure Mathematics, 44 (1986), 213-237

[10] L. Hauswirth, *Minimal surfaces of Riemann type in three-dimensional product manifolds*, Pacific J. Math. 224 (2006), no. 1, 91-117.

[11] L. Hauswirth, F. Pacard, *Higher genus Riemann minimal surfaces*, Invent. Math. 169 (2007), no. 3, 569-620.

[12] L. Hauswirth, M. Traizet, *The space of embedded doubly-periodic minimal surfaces*, Indiana University Mathematics Journal, 51 (2002), n. 5, 1041-1079.

[13] D. Hoffman, H. Karcher, *Complete embedded minimal surfaces of finite total curvature*, Geometry, V, 5-93, 267-272, Encyclopaedia Math. Sci., 90, Springer, Berlin, 1997.

[14] D. Hoffman, W.H. Meeks III, *A Complete Embedded Minimal Surface in* $\mathbb{R}^3$ *with Genus One and Three Ends*, Journal of Differential Geometry, 21 (1985), 109-127.

[15] D. Hoffman, W.H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, Journal of the AMS, (2) 4 (1989), 667-681.

[16] D. Hoffman, W.H. Meeks III, *Embedded minimal surfaces of finite topology*, Annals of Mathematics, 131 (1990), 1-34.

[17] D. Hoffmann, W.H. Meeks III, *Limits of minimal surfaces and Scherk's fifth surface*, Arch. Rational Mech. Anal. 111 (1990) 181-195.

[18] J. K. M. Jansen, *Simple-periodic and non-periodic Lamé functions*, Mathematisch Centrum, Amsterdam 1977.

[19] M. Jleli, *Constant mean curvature hypersurfaces*, PhD Thesis, University Paris 12 (2004).

[20] N. Kapouleas, *Complete constant mean curvature surfaces in euclidean three-space,* Annals of Math. 131, 1990, 239-330.

[21] N. Kapouleas, *Complete embedded minimal surfaces of finite total curvature*, J. Differential Geometry, (49) 1997, 95-169.

[22] H. Karcher, *Embedded minimal surfaces derived from Scherk's examples*, Manuscripta Math., 62, 83-114, 1988.

[23] H. Karcher, *Construction of minimal surfaces*, Surveys in Geometry, 1 - 96, University of Tokyo, 1989 and Lecture Notes n. 12, SFB256, Bonn, 1989.

[24] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin 1980.

[25] N. Korevaar, R. Kusner, B. Solomon, *The Structure of Complete Embedded n. Surfaces with Constant Mean Curvature*, Jour. of Differential Geometry, 30 (1989), 465-503.

[26] R. Kusner, R. Mazzeo, D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces,* Geom. Funct. Anal., 6 (1996) 120 - 137.

[27] F. Martín, V. Ramos Batista, *The embedded syngly periodic Scherk-Costa surfaces*, Math. Ann., 336 (2006) 155–189.

[28] L. Mazet, M. M. Rodríguez, M. Traizet, *Saddle towers with infinitely many ends*, Indiana Univ. Math. J.l, 56:6 (2007).

[29] L. Mazet, M. Traizet, *A quasi-periodic minimal surface*, Preprint.

[30] R. Mazzeo, F. Pacard, D. Pollack, *Connected sums of constant mean curvature surfaces in Euclidean 3 space*, J. Reine Angew. Math., 536, (2001), 115-165.

[31] W. H. Meeks III, H. Rosenberg, *The Global Theory of Doubly Periodic Minimal Surfaces*, Invent. Math., 97 (1989) 351-379.

[32] W. H. Meeks III, H. Rosenberg, *Stable minimal surfaces in in $M \times \mathbb{R}$*, J. Differential Geom., 68, (2004), no. 3, 515-534.

[33] W. H. Meeks III, H. Rosenberg, *The theory of minimal surfaces in $M \times \mathbb{R}$*, Comment. Math. Helv., 80, (2005), no. 4, 811-858.

[34] R. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, (1993).

[35] S. Montiel, A. Ros, *Schrödinger operators associated to a holomorphic map*, Global differential geometry and global analysis, Lecture Notes in Mathematics n. 1481, Springer, Berlin, (1991), 147-174.

[36] S. Nayatani, *Morse index of complete minimal surfaces*, The problem of Plateau, ed. Th. M. Rassias, (1992), 181-189.

[37] S. Nayatani, *Morse Index and Gauss maps of complete minimal surfaces in Euclidean 3-space*, Comment. Math. Helv., 68, (1993), Number 4, 511-537.

[38] S. Nayatani, *Personal communication.*

[39] B. Nelli, H. Rosenberg, *Minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$*, Bull. Braz. Math. Soc., 33, (2002), 263-292.

[40] J. Pérez, M. M. Rodríguez, M. Traizet, *The classification of doubly periodic minimal tori with parallel ends*, Journal of Differential Geometry, 69, no. 3, 523-577, 2005.

[41] J. Pérez, A. Ros, *The space of properly embedded minimal surfaces with finite total curvature*, Indiana Univ. Math. Journal, 45, (1996), no. 1, 177-204.

[42] V. Ramos Batista, *Singly periodic Costa surfaces*, J. London Math. Soc. 2, 72(2), 478 - 496, 2005.

[43] M. M. Rodríguez *The space of doubly periodic minimal tori with parallel ends: the standard examples*, Michigan Math. Journal, 55, (2007), 103-122.

[44] H. Rosenberg, *Minimal surfaces in $M \times \mathbb{R}$*, Illinois J. Math., 46, (2002), no. 4, 1177-1195.

[45] W. Rossman, E.C. Thayer, M. Wohlgemuth, *Doubly periodic minimal surfaces*, Experiment. Math., 9:2 (2000), 197-219.

[46] R. Sa Earp, E. Toubiana, *Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Illinois J. Math., 49, (2005), no. 4, 1323-1362.

[47] H.F. Scherk, *Bemerkungen über die kleinste Fläche innerhalb gegebener Grenzen,* J. R. Angew. Math., 13, 185-208, 1935.

[48] M. Traizet, *Construction de surfaces minimales en recollant des surfaces de Scherk,* Annales Institut Fourier, 46, (1996), n.5, 1385-1442.

[49] F.S. Wei, *Some existence and uniqueness theorems for doubly periodic minimal surfaces*, Invent. Math., 109 (1992), 113-136.